Completeness: 3COLOR

Weston Dransfield

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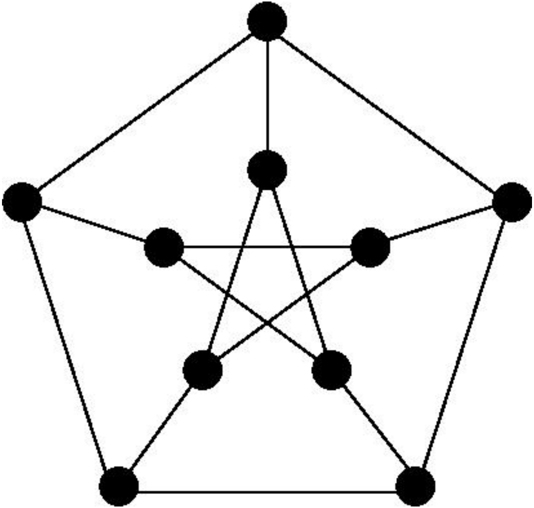
Outline

Description

$\textit{\textbf{3COLOR}} = \{\langle G \rangle \mid$ the nodes of G can be colored with three colors such that no two adjacent nodes are the same color $\}$

Example



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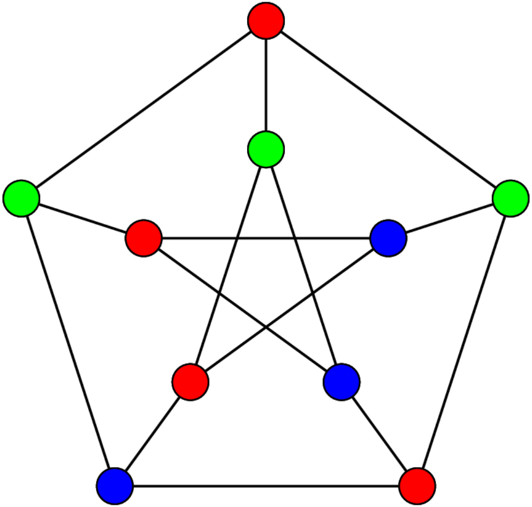

The Problem

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- This is tough to decide, but easy to verify!

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- Step 3 has largest time complexity of $O(n^2)$. 3COLOR is in NP because it can be verified in polynomial time.

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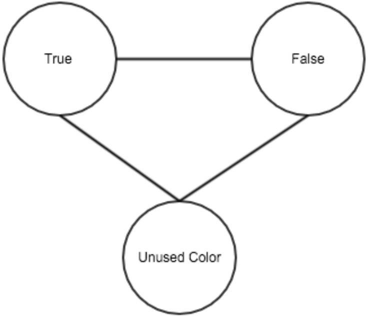
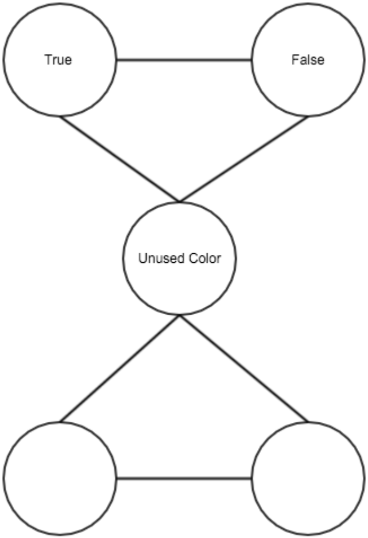
2. Force variables to be true or false
3. Use these subgraphs to create a graph that is 3 colorable iff the statement is satisfiable

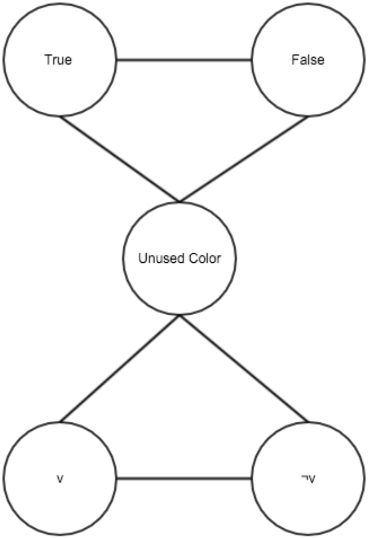
Constructing the Reduction - Truthiness



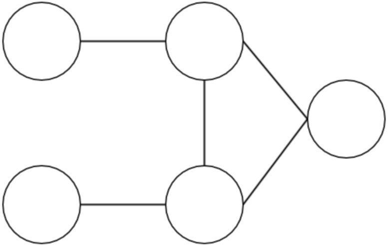
Constructing the Reduction - Variables



Constructing the Reduction - Variables

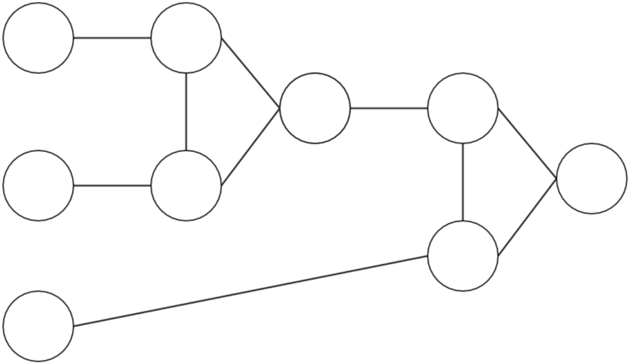


Constructing the Reduction - OR

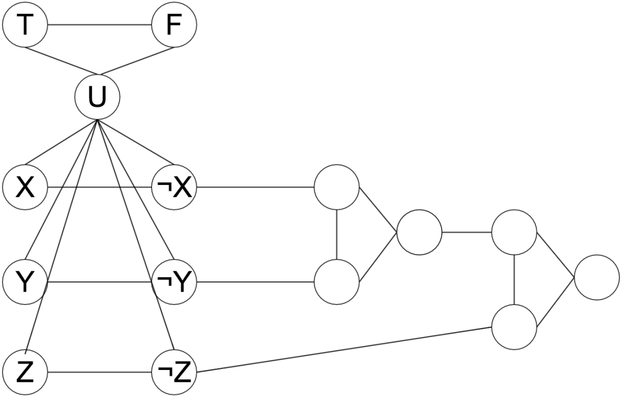


$x \vee y$

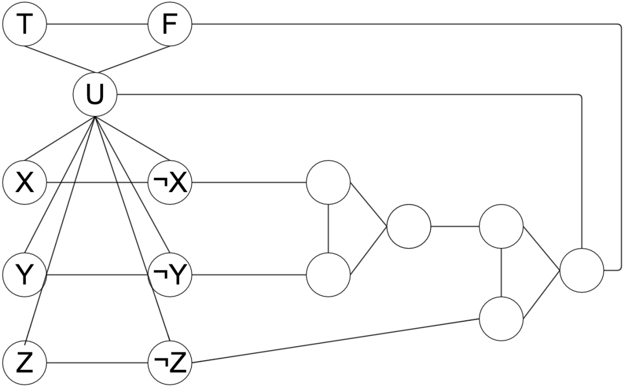
Constructing the Reduction - OR

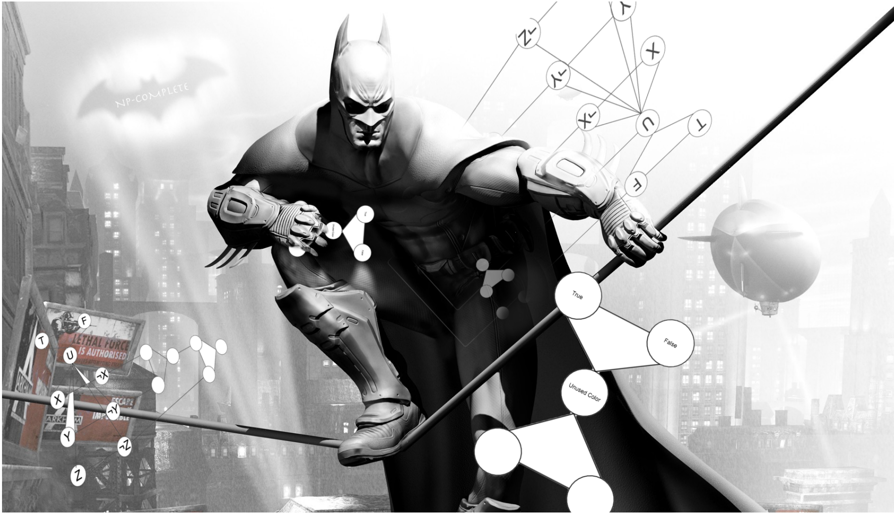


Constructing the Reduction - Clause



Constructing the Reduction - Clause




NP-COMPLETE
LETHAL FORCE
IS AUTHORISED
True
False
Unused Color

Transformation

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- Add nodes v and v_0 connected by an edge
- Connect nodes v and v_0 to the "unused" end of t
- Connect the corresponding node (v_0 or v_v) to one input of the clause's 3 way OR gate O_i"

Example

$$(x \vee y \vee \neg z) \vee (\neg x \vee \neg y \vee z)$$

Transformation - Forward

If boolean expression S is satisfiable, G_s is 3 colorable

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- If S is satisfialbe at least one literal in each clause is colored true.
- Output of 3-way OR gate can be colored with true as the output. This leads to a valid 3 coloring

Transformation - Backward

If graph G_s is 3 colorable, S is satisfiable

- A coloring of the graph forces the output of the 3-way OR gades to be colored true

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If graph G_s is 3 colorable, S is satisfiable

- A coloring of the graph forces the output of the 3-way OR gades to be colored true
- For each clause in S there must be at least one varaible colored true

Transformation - Polynomial Time

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- Variable T/F nodes - $O(1)$
- $O(n)$ for n clauses
- Overall - $O(n)$

Sources

http://web.stanford.edu/class/archive/cs/cs103/cs103.1132/lectures/27/Small27.pdf
http://www.cs.princeton.edu/courses/archive/spring07/cos226/lectures/23Reductions.pdf