Completeness: 3COLOR

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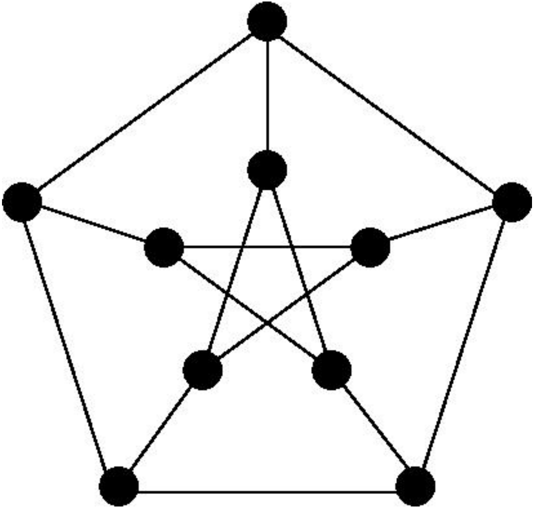
March 14, 2016

Outline

Description

$\textit{3COLOR} = \{\langle G \rangle \mid$ the nodes of G can be colored with three colors such that no two adjacent nodes are the same color $\}$

Example



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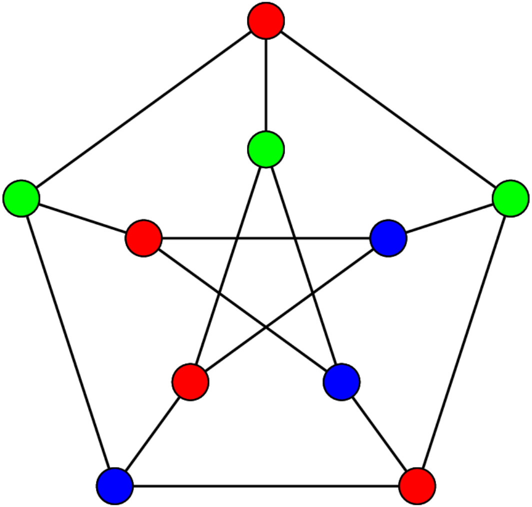

The Problem

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- This is tough to decide, but easy to verify!

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- Step 3 has largest time complexity of $O(n^2)$. 3COLOR is in NP because it can be verified in polynomial time.

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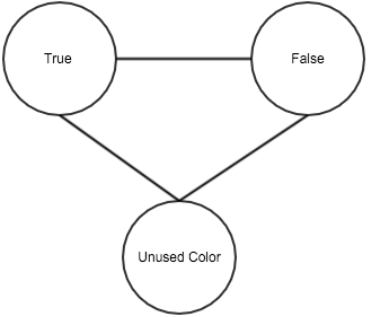
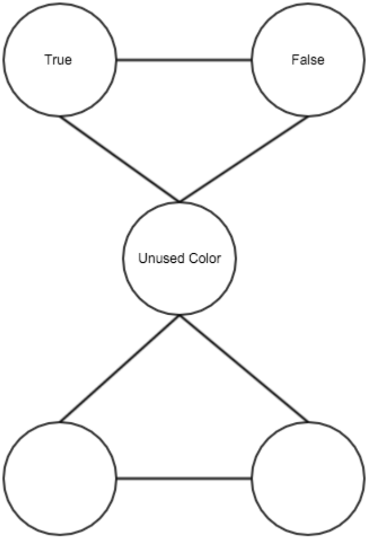
Construct a transformation T from $3SAT$ to $3COLOR$.

1. Establish Truthiness
2. Force variables to be true or false
3. Use these subgraphs to create a graph that is 3 colorable iff variables are satisfiable

Constructing the Reduction - Truthiness



Constructing the Reduction - Variables



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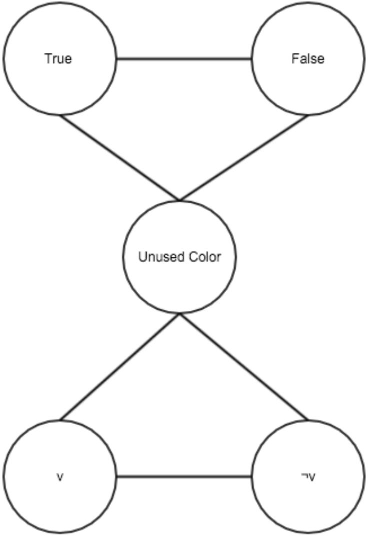
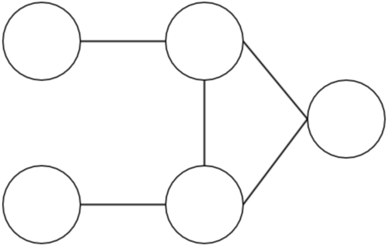
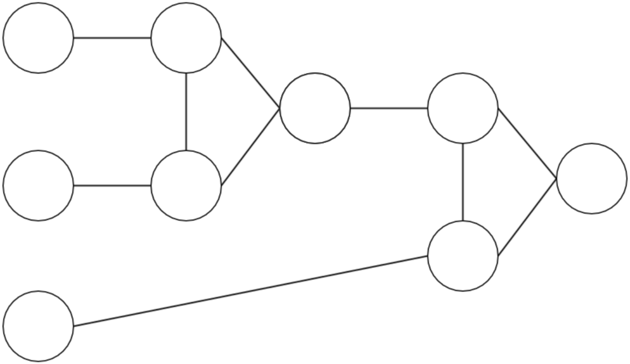


Constructing the Reduction - OR

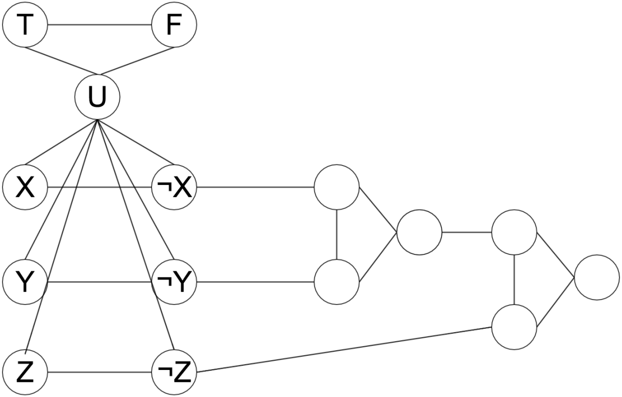


$x \lor y$

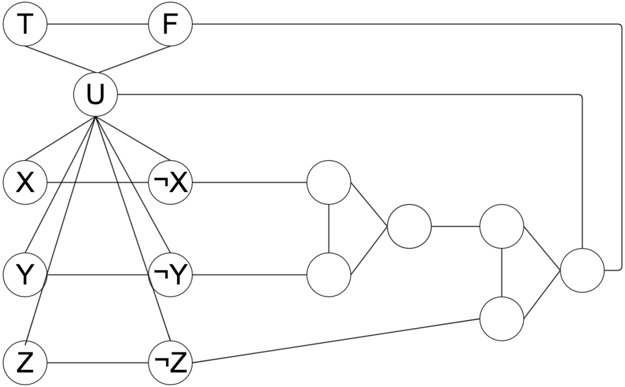
Constructing the Reduction - OR

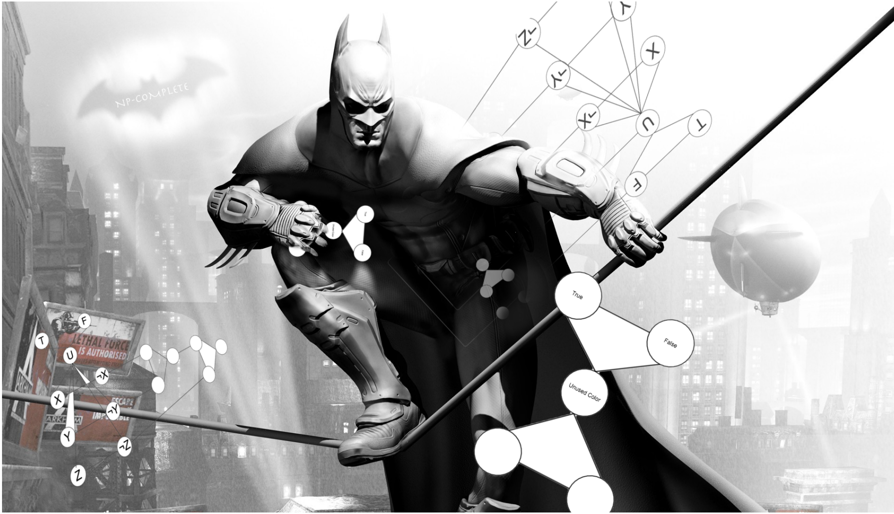


Constructing the Reduction - Clause



Constructing the Reduction - Clause




NP-COMPLETE
True
False
Unused Color
LETHAL FORCE
IS AUTHORISED

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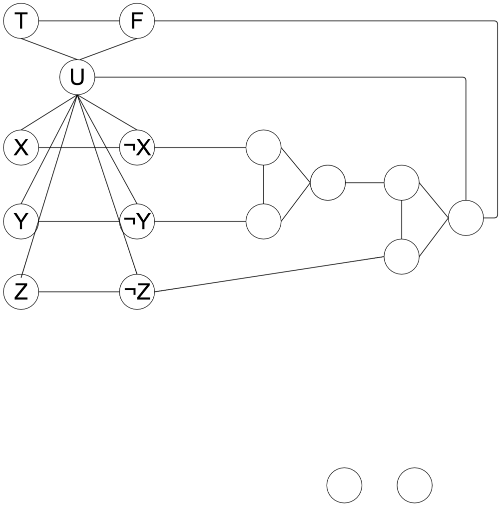
- Add nodes v and v_0 connected by an edge
- Connect nodes v and v_0 to the "unused" end of t
- Connect node v_0 to one input of the clause's 3 way OR gate O_i"

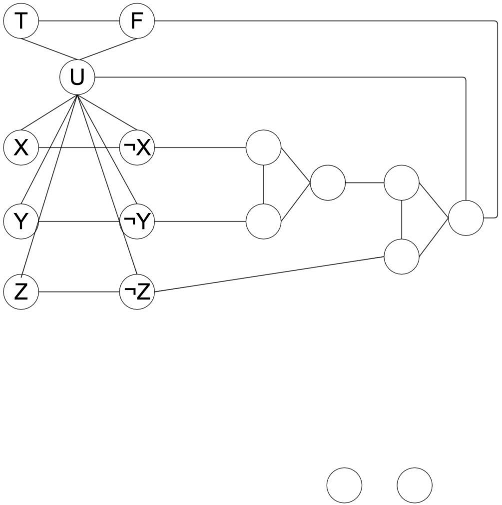
Example

$$(u \vee \neg v \vee w) \wedge (v \vee x \vee \neg y)$$

Transformation - Forward



Transformation - Backward



Transformation - Polynomial Time

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- Variable T/F nodes - $O(1)$
- $O(n)$ for n clauses
- Overall - $O(n)$

Sources

http://web.stanford.edu/class/archive/cs/cs103/cs103.1132/lectures/27/Small27.pdf

http://www.cs.princeton.edu/courses/archive/spring07/cos226/lectures/23Reductions.pdf