Name: ___________________, ___________________

PHYSICS 401 : SPRING SEMESTER 2020

Project #5: Magnetic field and dissipative trajectories

For this project, you can use either the methods of Project 2 or Project 3 to hand in graphs of the trajectory. No animation is necessary and no Apps to download.

1) An electron's trajectory in a magnetic field can be solve by the second order algorithm

$$\mathcal{T}_{2b} = \mathrm{e}^{\frac{1}{2}\Delta t T}\mathrm{e}^{\Delta t V}\mathrm{e}^{\frac{1}{2}\Delta t T}$$

where

$$\mathrm{e}^{\Delta t T}\begin{pmatrix}\mathbf{r}\\ \mathbf{v}\end{pmatrix} = \begin{pmatrix}\mathbf{r}+\Delta t\mathbf{v}\\ \mathbf{v}\end{pmatrix}$$

$$\mathrm{e}^{\Delta t V}\begin{pmatrix}\mathbf{r}\\ \mathbf{v}\end{pmatrix} = \begin{pmatrix}\mathbf{r}\\ \mathbf{v}_B(\mathbf{r},\mathbf{v},\Delta t)\end{pmatrix}$$

and

$$\mathbf{v}_B(\mathbf{r},\mathbf{v},\Delta t) \equiv \mathbf{v} + \sin\theta(\hat{\mathbf{B}}\times\mathbf{v}) + (1-\cos\theta)\hat{\mathbf{B}}\times(\hat{\mathbf{B}}\times\mathbf{v})$$

with

$$\theta = \omega(\mathbf{r})\Delta t \quad \text{and} \quad \omega(\mathbf{r}) = \frac{eB(\mathbf{r})}{m}$$

Let the magnetic field be in the z direction

$$\hat{\mathbf{B}} = \hat{\mathbf{z}} \quad \text{and} \quad \omega(\mathbf{r}) = \omega(x,y,z) = \frac{1}{x^2}.$$

Consider only the planar motion perpendicular to the field with $\mathbf{r} = \mathbf{r}_\perp = (x,y)$ and $\mathbf{v} = \mathbf{v}_\perp = (v_x, v_y)$.

a) Start the motion at $\mathbf{v}_0 = (0, 0.5)$, $\mathbf{r}_0 = (1,0)$ with $\Delta t = 0.4$ and plot the resulting trajectory for 5 cyclotron motion. Repeat the calculation at $\Delta t = 0.2$ and $\Delta t = 0.1$. Hand in the three trajectories in one plot. Is the $\Delta t = 0.2$ small enough to produce a consistent trajectory?

b) Now apply the fourth-order FR algorithm with sym2b using the same three time steps and hand in the three trajectories in one plot.

The damped harmonic oscillator

2) Consider the damped harmonic oscillator with equations of motion

$$\dot{q} = v \quad \text{and} \quad \dot{v} = -\omega_0^2 q - 2\gamma v.$$

The three elementary updating steps are

$$\begin{aligned} q' &= q + v\Delta t,\\ v' &= v - \omega_0^2 q\Delta t,\\ v' &= \mathrm{e}^{-2\gamma\Delta t}v \end{aligned}$$

The exact solution to the under-damped case is

$$q(t) = \mathrm{e}^{-\gamma t}\left[q_0\cos(\omega t) + \frac{v_0+\gamma q_0}{\omega}\sin(\omega t)\right], \quad \text{where} \quad \omega = \sqrt{\omega_0^2-\gamma^2}.$$

a) For the case of $\omega_0 = 1$, $\gamma = 0.6$, $q_0 = 1$, $v_0 = 0$, compare the trajectories of a first and a second-order algorithm with that of the exact trajectory at a not too small Δt you have chosen (so that the first and second-order trajectories are distinguishable). Hand in this plot. How would you measure the error in this case when we do not have a Hamiltonian?

b) Devise a fourth-order algorithm to solve this problem. Hand in the resulting trajectory.