DSP: Underlying Concepts

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What is a signal?

- For our purposes: a time-varying function $f(t)$.
- Analogue or digital?
- Discrete or continuous?

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# Essential Concepts

* A **sum**: \begin{equation} \sum_{n=0}^{M} f(nT)\end{equation}

* An **integral**: \begin{equation} \int_{-\pi}^{\pi} f(t) dt \end{equation}

* A **sum** which (**under some conditions**) _converges_ to an **integral**
<sup>1</sup>: \begin{equation} \lim_{\delta T \to 0}( \sum_{n} f(n \delta T) \delt
T ) = \int f(t) dt \end{equation}
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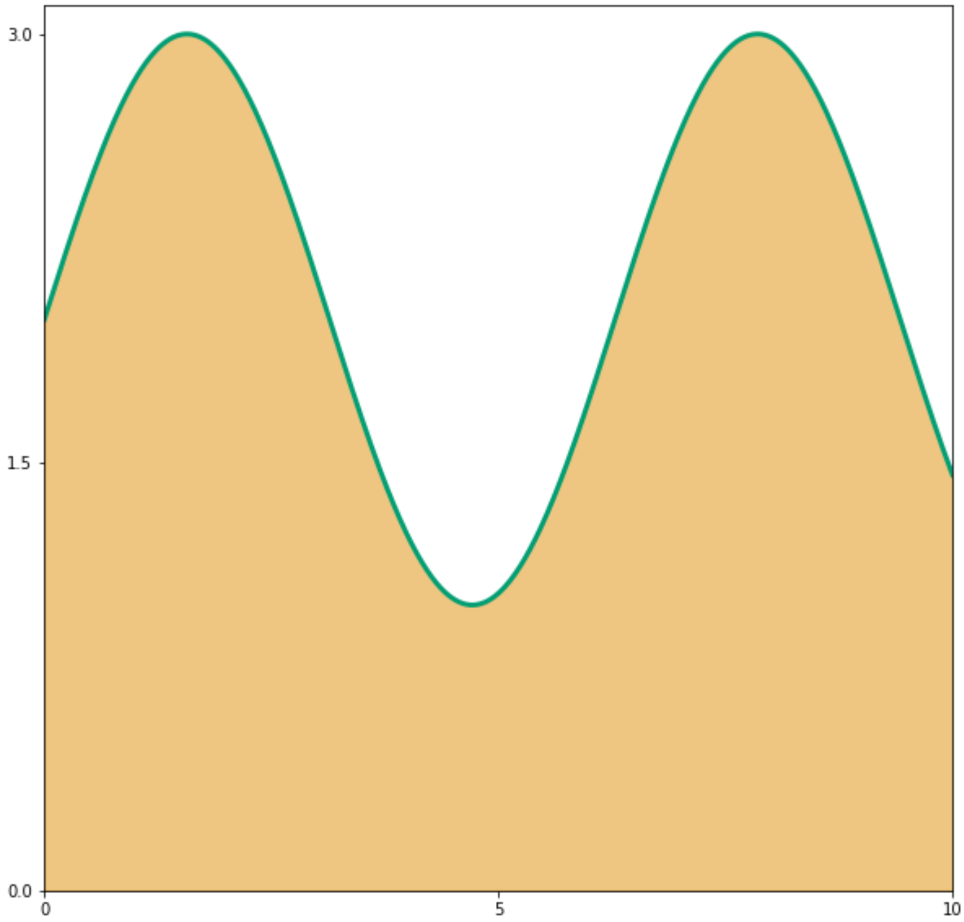
[1]This is a Riemann integral. Thankfully this is all most of us will ever need...

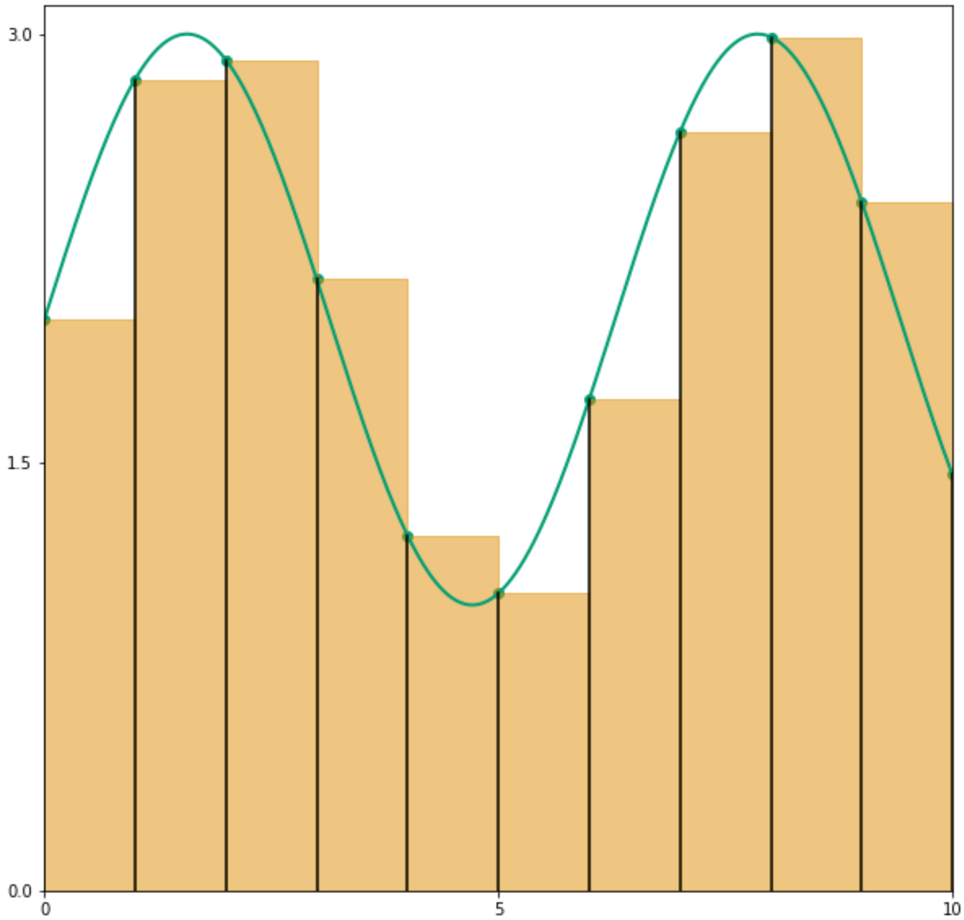


$$f(t) = sin(t) + 2$$

The integral:

$$\int_0^{10} f(t)dt$$

is just the area under this curve. We can also approximate this by a sum:



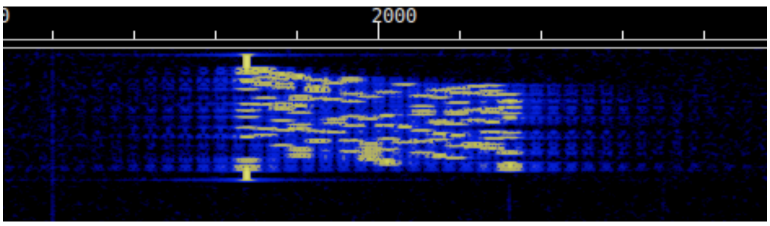
(with $\delta T = 1$)

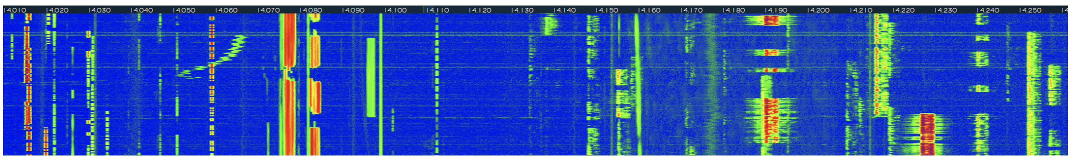
$$\sum_{n=0}^{10} 1 \times f(n \times 1)$$

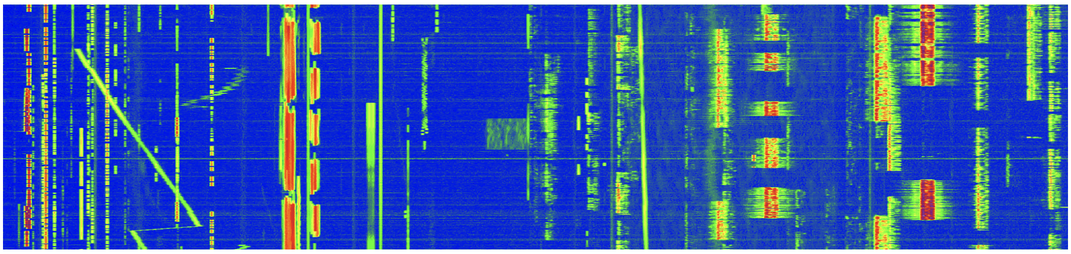
Now, intuitively it seems that as $\delta T \to 0$, we get a better estimate of the area: when we get to an infinitesimal δT, we obtain the integral.

Waterfalls & Hotplates

- Probably all have heard of the Fourier Transform and seen a waterfall display like this; do you know why it works?







(images from George M1GEO and John M5ET respectively)

- A common goal in computing the Fourier **transform** is to determine how much energy/signal there is at a given frequency. We will fudge our way into this by starting from the basics...

Towards Fourier Space

- Ways of approximating a function: Taylor Series
$$f(x) \approx a_0 + a_1x + a_2x^2 + a_3x^3 + ... \text{for some } \mathrm{a_i}$$
- But what if we want to represent a **periodic** function? We might try [1]

$$f(t) \approx c + a_1 sin(t) + b_1 cos(t) + a_2 sin(2t) + b_2 cos(2t) + ...$$

[1] Advanced students should explain to their neighbours in the next break why we need both sine and cosine terms here.

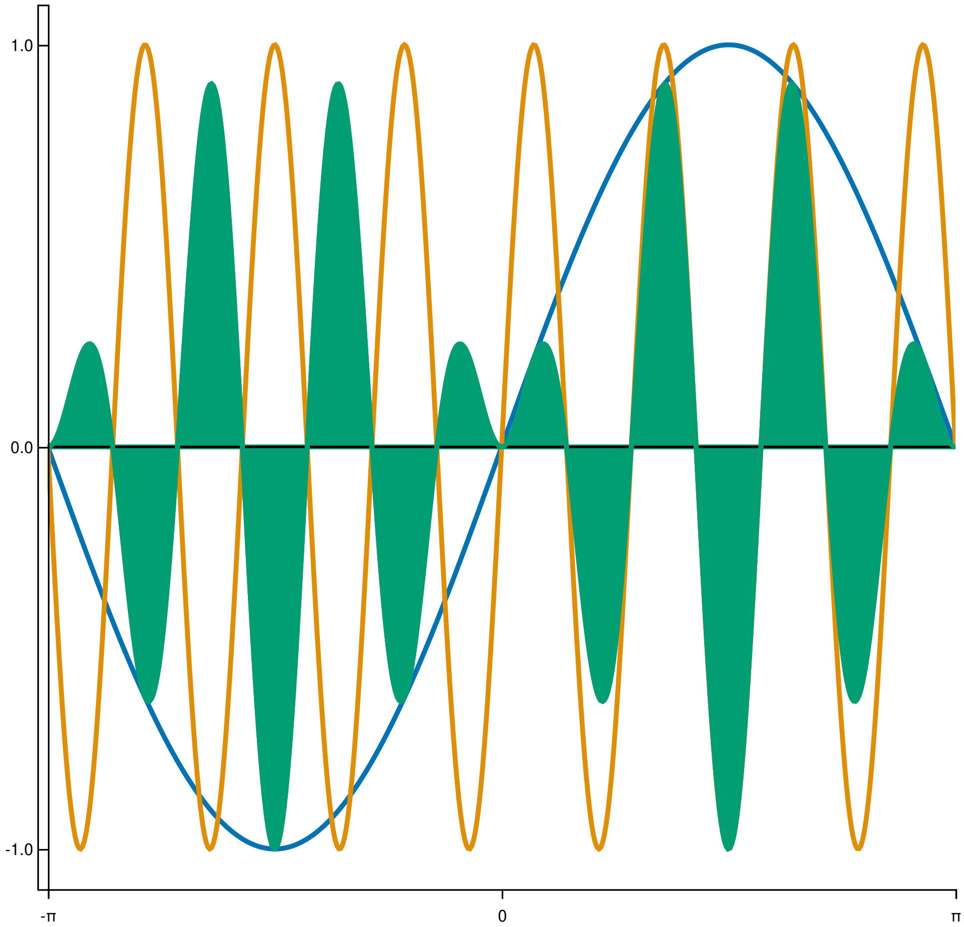
This is a *Fourier Series*. How do we find a_i and b_i? **Orthogonality of sines**

$$\int_{-\pi}^{\pi} sin(mx)sin(nx)dx = \begin{cases} \pi & \forall m = n : m \in \mathbb{Z} \\ 0 & m \neq n \end{cases}$$

(and the same is true for cos, I promise!)

$$\int_{-\pi}^{\pi} sin(mx)cos(nx)dx = 0 \; \forall m, n \in \mathbb{Z}$$

Orthogonality of Sines



- Why does this help us?

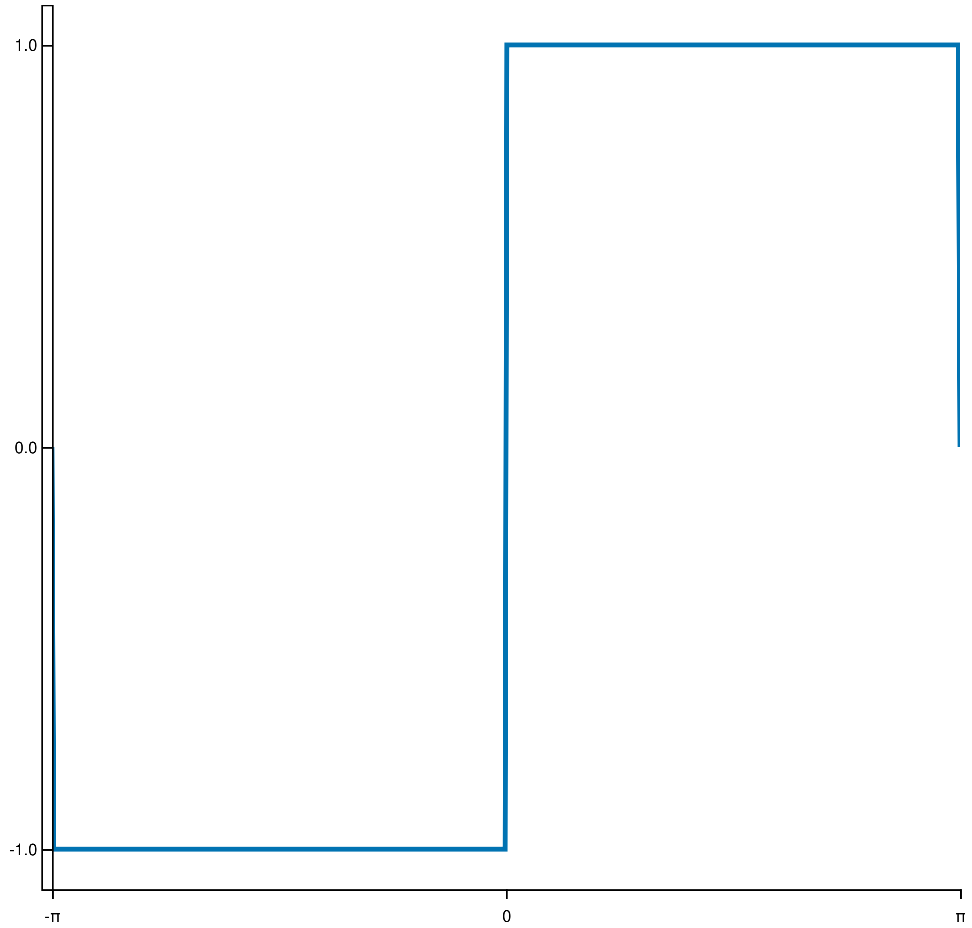
$$f(t) \approx c + a_1 sin(t) + b_1 cos(t) + a_2 sin(2t) + b_2 cos(2t) + ...$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)sin(nt)dt$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)cos(nt)dt$$

(plus a DC offset...)

$$c = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$$

An example: the square wave

- This is a carefully chosen example: why?

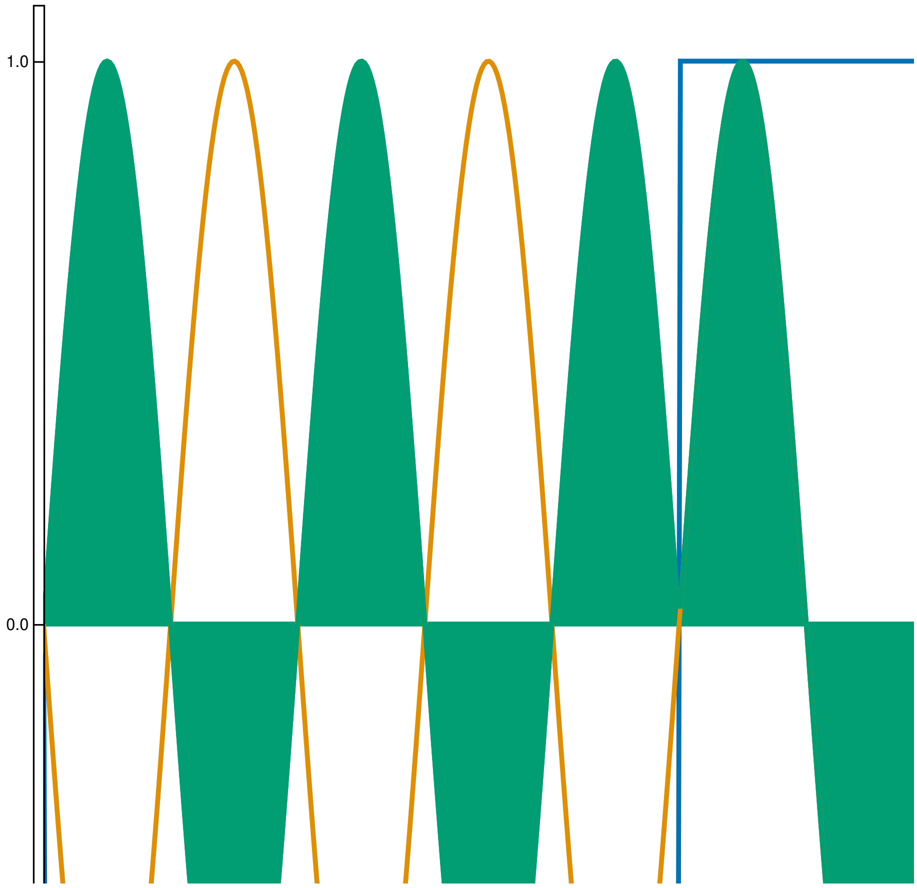


- Sine series only: there is **no** DC offset and **no** cosine term. (those advanced students can discuss this later too)

Calculating the coefficients

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)sin(t)dt$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -1sin(t)dt + \int_{0}^{\pi} 1sin(t)dt\right] = \frac{4}{\pi}$$

m 5



$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)sin(2t)dt$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -1sin(2t)dt + \int_{0}^{\pi} 1sin(2t)dt\right] = 0$$

$$a_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)sin(3t)dt$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -1sin(3t)dt + \int_{0}^{\pi} 1sin(3t)dt\right] = \frac{4}{3\pi}$$

(...many hours later...)

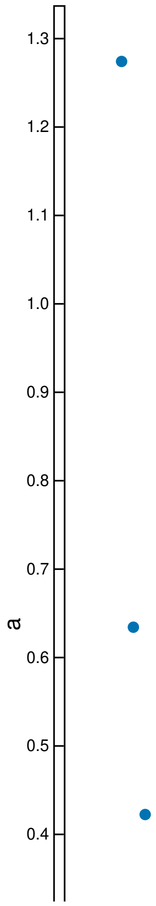
(an identity that makes it much easier!)

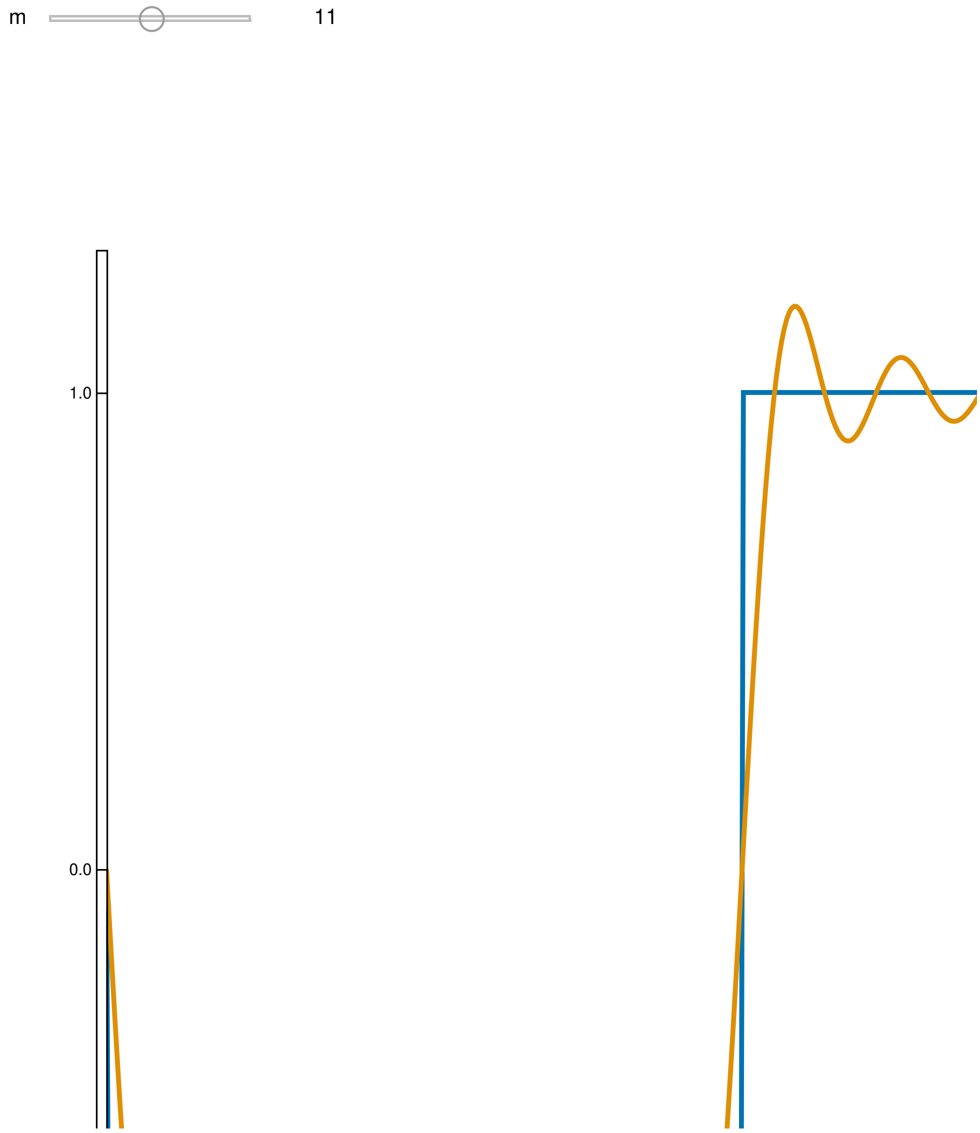
$$\int_a^b sin(nt)dt = -\frac{1}{n}[cos(nt)]_a^b$$

$$cos(n\pi) = \begin{cases} 1 & \text{even } n \\ -1 & \text{odd } n \end{cases}$$

Skipping some algebra (if you enjoy that sort of thing, fair enough...)

$\{ 0 \quad \text{even n}$





Something I never told you

(I'm sorry)

$$i = \sqrt{-1}$$
$$i^2 = -1$$

$$e^{i\theta} = cos(\theta) + isin(\theta)$$

We might *intuitively* suspect that, if $e^{i\theta}$ is a mixture of sines and cosines, we can represent our old friend the Fourier Series in this fashion. Sure enough, (without proof),

$$f(t) = \sum_{n=0}^{\infty} a_n e^{int}$$

$$a_n = \frac{1}{\quad} \int^{\pi} f(t)e^{-int}\,dt$$

Arbitrary Periods

- Why limit ourselves to the interval $[-\pi, \pi]$?
- Use simple algebra to make the period T instead of 2π...
- In short, make t go through a range of 2π radians in T seconds: multiply by $\frac{2\pi}{T}$ and iron out the constants!

$$f(t) = \sum_{n=0}^{\infty} a_n e^{in \times \frac{2\pi t}{T}}$$

$$a_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t)e^{-in \times \frac{2\pi t}{T}}\,dt$$

- This is **all** we need to do before we can move to the Fourier **transform**!

Fourier Transform

Problem

- We want to find the frequency content of an aperiodic function
- We do not expect the function to be representable by a sum of discrete frequency trig functions/complex exponentials...
- We will define the time function $f(t)$ and its corresponding Fourier transform $\tilde{f}(k)$ (where k is the frequency in units of $[\frac{1}{[t]}]$)
- To represent the operation of taking the Fourier transform, we will write
$$\tilde{f}(k) = \mathcal{F}\left[f(t)\right]$$
- There also exists an *inverse* Fourier transform:
$$f(t) = \mathcal{F}^{-1}\left[\tilde{f}(k)\right]$$

The algebra in this slide is fiddly: ignore the details unless you find them obvious and work them out later. Focus on the concepts.

Begin with the Fourier Series for a function of period T.

$$f(t) = \sum_{n=0}^{\infty} a_n e^{int \times \frac{2\pi}{T}}$$

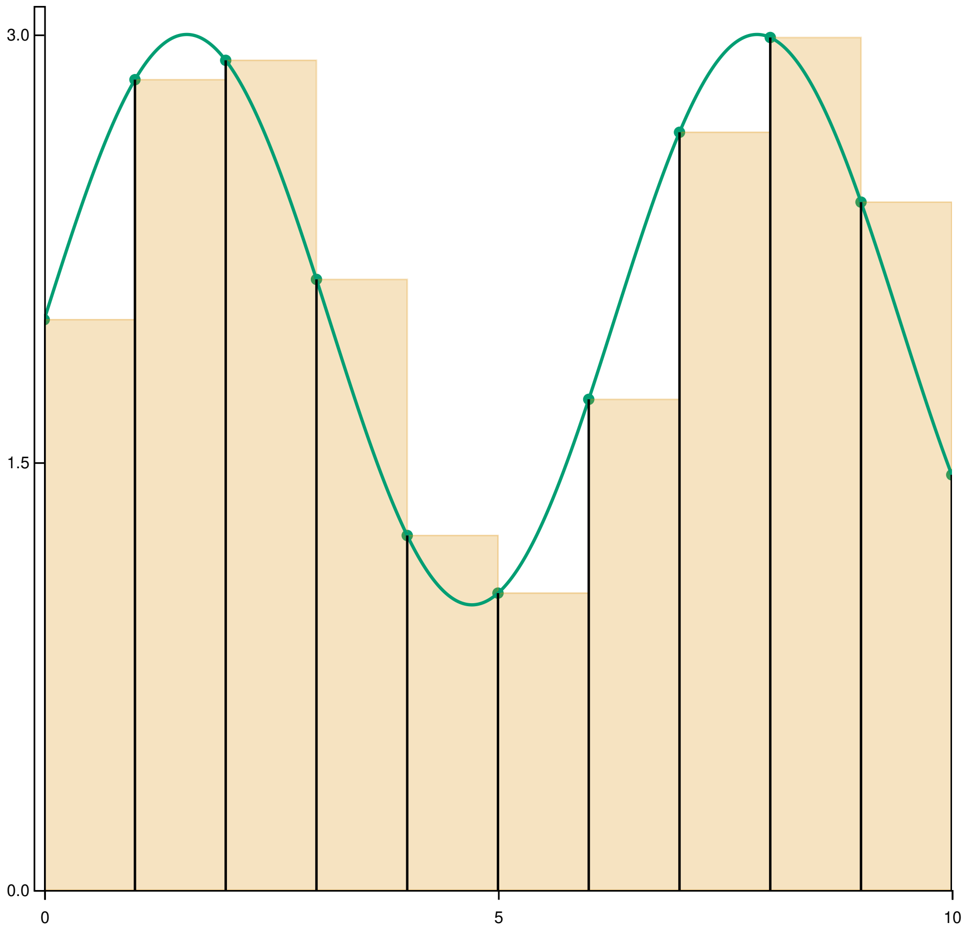
$$a_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t)e^{-int \times \frac{2\pi}{T}}\,dt$$

- Define a new variable $k = \frac{n}{T}$.
- It follows that $n = kT$.

(there is no key concept in this - just a constant relationship defined for algebraic convenience)

- Replace a_n with some function that yields the same coefficients: let's call it $\tilde{f}(k)$.
- The sum incremented n by 1 each pass. Incrementing k by 1 is equivalent to incrementing n by T, so we write $a_n = \tilde{f}(k)\delta k$. For equivalence, $\delta k = \frac{1}{T}$

We're subtly preparing for a move from discrete 'frequencies' n to continuous values k. For now, we 'discretise' the continuous value by multiplying by δk and fiddle with the algebra so that it is exactly equivalent to the Fourier series. Remember the Riemann integral earlier?



$$f(t) = \sum_{k=0}^{\infty} \tilde{f}(k)\delta k \; e^{it \times kT \times \frac{2\pi}{T}}$$

$$= \sum_{k=0}^{\infty} \tilde{f}(k)\delta k \; e^{2\pi i t k}$$

- Remember that $\delta k = \frac{1}{T}$.
- What about an aperiodic function: $T \to \infty$??
- We can take the *limit* of this, as $\delta k \to 0$.

*Concept: as the function becomes aperiodic, the increment in frequency becomes infinitessimal: we end up with a **continuous frequency spectrum**.*

$$f(t) = \lim_{\delta k \to 0} \left(\sum_{k=0}^{\infty} \tilde{f}(k) \delta k e^{2\pi i t k} \right)$$

- This is just like the sum converging to an integral from earlier!

$$f(t) = \int_{-\infty}^{\infty} \tilde{f}(k) e^{2\pi i t k} dk$$

This is the inverse Fourier Transform: what we've just written is equivalent to $f(t) = \mathcal{F}^{-1}(\tilde{f}(k))$

Fourier, finally!

Forward

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i t k} dk$$

Inverse

$$f(t) = \int_{-\infty}^{\infty} \tilde{f}(k) e^{2\pi i t k} dk$$

But there's always a catch.

- Can we integrate over infinite time? Rather an expensive operation...
- What about sampled signals? Then we need a sum again!
- No time to look at the detail of the solutions.

Discrete Fourier Transform

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