Coherent Structures in Isotropic Turbulence

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April 17, 2025

Contents

Abstract

Your abstract.

1 Introduction

Turbulence is a phenomenon known for its chaotic and seemingly random nature. Despite this apparent disorder, turbulent flows often contain regions of striking organization: structures such as swirling vortices or aligned "jets" of velocity. These coherent structures play a central role in the energy transport and dynamics of turbulent systems.

It has been shown that such structures can be identified using correlation measures applied directly to the velocity fields. These tools have revealed that certain regions of turbulence, especially those with intermediate vorticity, exhibit surprising levels of internal organization, giving rise to what appears to be emergent behavior in an otherwise chaotic field. My main source of inspiration and knowledge comes from two master theses.

Frank Groen's thesis explores the structure of turbulence through spatial correlations of force fields derived from the Navier-Stokes equations, revealing how different forces contribute to the dynamics of jets and swirls in turbulence.

Abinash Mishra's thesis takes a functional perspective using the Biot–Savart law to reconstruct velocity and vorticity structures, showing how jets and swirls emerge from organized regions in the otherwise chaotic background field. In this thesis, we combine these two perspectives by analyzing the time derivative of the velocity field through Biot–Savart reconstruction, aiming to uncover new insights into the generation and spatial organization of relevant structures in turbulence.

We aim to investigate coherent structures in homogeneous isotropic turbulence by applying correlation-based structure identification to the time derivative of the velocity field ($\partial u/\partial t$), and then analyze these structures using Biot–Savart reconstruction techniques. Ultimately, this approach provides a new lens through which to understand the dynamics of turbulence from both a structural and functional standpoint.

2 Theory

2.1 Structure and Correlation

Many have an intuition of what "structure" really is. One could define it just as an "order". Think about something that seems random, like the forming of a snowflake. However, if one zooms in on a snowflake, there is a clear structure. No snowflakes are the same, but all snowflakes kind of have the same structure. The same with the universe, which is huge, dark and expanding. In this "chaos", one can clearly find all kinds of structures, from galaxies to the Solar System to the rings of Saturn. In the context of turbulence, a good example is vortices. A vortex is a swirling motion of fluid; for example, a tornado is one big vortex. Within turbulence, one can find many types of swirling motions, including clear vortices. This vortex could be defined as a structure in turbulence. According to Mukherjee et al., "structure in a field can be defined as a certain distribution of the properties of the field in a region, characterised by a small number of parameters, which can be described (deterministically) in a 'simple way'." This implies that a structure in turbulence is not necessarily a visible pattern, but could be a set of points that could mathematically be described in a "simple way".

A tool we use to find these structures is "correlation measures". These are mathematical tools that quantify the organization of a set of points. In order to analyze the presence and behavior of coherent structures in turbulence, it is useful to define a measure that captures how the velocity at one point in the flow relates to the velocity at nearby points. One effective approach is to consider the instantaneous correlation of the velocity field along arbitrary directions in space. This provides a directional view of how aligned or related fluid motions are across short distances, and serves as a means of identifying patterns or structures that might not be immediately obvious from the raw velocity field alone.

Mukherjee et al. proposed a specific implementation of this idea by introducing an integration over a finite distance Λ centered at each point in the flow. For their analysis, they selected Λ to be equal to the Taylor microscale λ, which represents a characteristic length scale in turbulence. Using $\Lambda = \lambda$ ensures that the correlation measure remains sensitive to the small-scale structure of the flow without being overwhelmed by noise or dominated by large-scale flow features.

To simplify the interpretation of the velocity correlation, the measure is reduced to a three-component quantity, $L_i(x, \Lambda)$, which can be understood as a vector:

$$L_i(x, \Lambda) = \int_{-\Lambda}^{\Lambda} u(x) \cdot u(x + r_i)\, \mathrm{d}r_i. \tag{2.1}$$

Here, $u(x)$ is the velocity vector at a point x, and r_i represents a directional offset along the i-th axis. This integral captures the degree to which the velocity at a given point is correlated with velocities in its immediate neighborhood along each spatial direction. A strong correlation in a particular direction may indicate the presence of a coherent flow structure, such as a vortex. Even though $u(x)$ is usually referred to as velocity, one can choose any property as $u(x)$, as we will do later on.

2.2 Force fields

The velocity field for an incompressible fluid with no external forces is described by the Navier-Stokes equations:

$$\nabla \cdot \mathbf{u} = 0 \tag{2.2}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u} \tag{2.3}$$

Here, [2.1] describes the continuity, or incompressibility. It states that no matter can be created nor destroyed. [2.2] is the force balance equation. We will investigate each force field individually:

$$\mathbf{f}_t + \mathbf{f}_\rho = \mathbf{f}_p + \mathbf{f}_\mu \tag{2.4}$$

- $\mathbf{f}_t = \rho \frac{\partial \mathbf{u}}{\partial t}$ is the rate of change. This is the ma part in $F = ma$. This force will be of our greatest interest in this research.

- $\mathbf{f}_\rho = \rho(\mathbf{u} \cdot \nabla)\mathbf{u}$ is the inertial force. This is due to the fact the material derivative of $\mathbf{u}$ is taken, and not the time derivative. The nonlinearity in this force is the main reason why turbulence is chaotic.
- $\mathbf{f}_p = -\nabla p$ is the pressure force. For us, this force will not be of great interest, since it disappears in equation [2.5].
- $\mathbf{f}_\mu = \mu \nabla^2 \mathbf{u}$ is the viscous force. This can be seen as the internal friction between the fluid particles. The μ is the reason why honey sticks together, but water runs more smoothly.

For us, it will be of great interest to take the curl of equation [2.2]. Our motives will become clear in the next subsection. By definition, the vorticity is given as:

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{u} \tag{2.5}$$

Now, by taking the curl of equation [2.2], we will arrive at the vorticity equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega} \tag{2.6}$$

Note that, due to the curl-free pressure gradient, the pressure disappears. By defining the lamb vector, $\boldsymbol{\ell} \equiv \boldsymbol{\omega} \times \mathbf{u}$, we can write the vorticity equation in a more useful way:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times \boldsymbol{\ell} + \nu \nabla^2 \boldsymbol{\omega}, \tag{2.7}$$

since

$$\nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} \tag{2.8}$$

Here, $\nu = \mu/\rho$. These two components, $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$ and $(\mathbf{u} \cdot \nabla)\boldsymbol{\omega}$, are called the stretching and convection term respectively. The stretching term describes how the vorticity changes when the flow field changes, like stretching a swirling motion. The convection term could be interpreted as how vorticity is carried with the flow.

2.3 Biot-Savart

In fluid dynamics, it is often useful to understand how local flow quantities like velocity or acceleration are influenced by the rest of the flow field. One powerful tool to do this is the **Biot–Savart law**, which originates in electromagnetism but applies directly to fluids through the Helmholtz decomposition of vector fields.

According to the *Helmholtz theorem*, any sufficiently smooth vector field (such as velocity or force) can be decomposed into three parts:

- An irrotational part, derived from the field's divergence (like pressure effects),
- A solenoidal part, derived from the field's curl (like vorticity),
- And a harmonic part, which accounts for boundary effects.

In our case, we are working with flows that are *incompressible* and set in a *periodic domain*. This means:

$$\nabla \cdot \mathbf{u} = 0 \quad \text{and} \quad \text{boundary terms vanish.}$$

Under these conditions, the velocity (or other relevant vector fields) can be reconstructed solely from the vorticity:

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi} \int_V \frac{\boldsymbol{\omega}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \, dV'.$$

This is the **Biot–Savart law**, where:

- $\mathbf{u}(\mathbf{x})$ is the velocity at point $\mathbf{x}$,
- $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity at source point $\mathbf{x}'$,

- and the kernel $\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$ defines the spatial influence.

In this thesis, we apply the same idea to reconstruct and study the time derivative of the velocity field ($\partial u/\partial t$), effectively analyzing the acceleration or force per unit mass using a Biot–Savart-like formulation. The goal is to understand which parts of the flow generate this force and how they are spatially organized.

The linearity of the time derivative and nabla operator makes it possible to analyze $\partial u/\partial t$ the same way as u is analyzed, since

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{u}) = \nabla \times \frac{\partial \mathbf{u}}{\partial t}$$

Because we use periodic boundary conditions and a large enough domain, the harmonic (boundary) terms vanish, and the reconstruction is dominated by the organized structures nearby. This makes the Biot–Savart approach especially well-suited for isolating and analyzing structures, such as vortices and jets, in turbulent flows.

3 Results

3.1 High L of $\mathbf{f}_t$

3.2 Biot-Savart reconstruction

4 Conclusion

References