# PSALTer v 1.0.0 Documentation

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# Introduction

## How to use this documentation

Welcome to the documentation file for the PSALTer package. Commentary is provided in this green text throughout. There will also be some displayed math.

$$1 + 2 X + X^2 == 0 \qquad (1)$$

**Key observation:** Occasionally, more important points will be highlighted in boxes like this.

All the important samples of code are presented in these initialization cells, together with their expected output:

```
Print["Hello world."];
```

```
Hello world.
```

These cells must be run in order from the top of this notebook (to do this automatically, select Evaluation from the notebook menu and then Evaluate Initialization Cells). You can also copy the code into your own notebook.

Basic familiarity with xAct, in particular xTensor, is assumed.

## Loading the package

**Key observation:** Please make sure that you have read README.md carefully, ideally as it appears on the GitHub website (i.e. with markdown formatting). There, you can find installation instructions specific to your operating system.

If not already installed, you will probably require the following dependency:

```
PacletInstall["JasonB/RectanglePacking"];
```

**Key observation:** As explained in README.md, the only reason why we need RectanglePacking is to improve the formatting of the final output graphic. If you are having problems with the production of this graphic (this especially applies to macOS or Microsoft Windows users), you should set the following global variable equal to "True" (this is a temporary fix in the current version of the software):

```
xAct`PSALTer`Private`$NoExport = False;
```

The first step after installing is to load the PSALTer package:

```
Get["xAct`PSALTer`"];
```

```
------------------------------------------------------------
Package xAct`SymManipulator`  version 0.9.5, {2021, 9, 14}
CopyRight (C) 2011-2021, Thomas Bäckdahl, under the General Public License.
------------------------------------------------------------
Package xAct`xPert`  version 1.0.6, {2018, 2, 28}
CopyRight (C) 2005-2020, David Brizuela, Jose M. Martin-Garcia and Guillermo A. Mena Marugan, under the General Public License.
** Variable $CovDFormat changed from Prefix to Postfix
** Option AllowUpperDerivatives of ContractMetric changed from False to True
** Option MetricOn of MakeRule changed from None to All
** Option ContractMetrics of MakeRule changed from False to True
------------------------------------------------------------
Package xAct`Invar`  version 2.0.5, {2013, 7, 1}
CopyRight (C) 2006-2020, J. M. Martin-Garcia, D. Yllanes and R. Portugal, under the General Public License.
** DefConstantSymbol: Defining constant symbol sigma.
** DefConstantSymbol: Defining constant symbol dim.
** Option CurvatureRelations of DefCovD changed from True to False
** Variable $CommuteCovDsOnScalars changed from True to False
------------------------------------------------------------
Package xAct`xCoba`  version 0.8.6, {2021, 2, 28}
CopyRight (C) 2005-2021, David Yllanes and Jose M. Martin-Garcia, under the General Public License.
------------------------------------------------------------
Package xAct`xTras`  version 1.4.2, {2014, 10, 30}
CopyRight (C) 2012-2014, Teake Nutma, under the General Public License.
** Variable $CovDFormat changed from Postfix to Prefix
** Option CurvatureRelations of DefCovD changed from False to True
------------------------------------------------------------
Package xAct`PSALTer` version 1.0.0, {2024, 6, 1}
CopyRight © 2022, Will Barker, Carlo Marzo and Claire Rigouzzo, under the General Public License.
```

```
------------------------------------------------------------
These packages come with ABSOLUTELY NO WARRANTY; for details type Disclaimer[]. This is free software, and you are welcome to redistribute it under certain conditions. See the General Public License for details.
------------------------------------------------------------
```

PSALTer is now loaded. It is helpful to briefly review all the symbols which are provided by the package:

```
Print[Names["xAct`PSALTer`*"]];
```

```
{a, a$, b, b$, c, cartesian, CD, ChristoffelCD, ChristoffelPDcartesian, c$, d, Def, DefField, DetG, e, EinsteinCCCD, EinsteinCCPDcartesian, EinsteinCD, En, Eps, epsilonG, etaDowncartesian, etaUpcartesian, f, g, G, h, i, i$, j, j$, k, KretschmannCD, l, m, M4, MaxLaurentDepth, Mo, n, o, p, P, ParticleSpectrum, PDcartesian, PrintSourceAs, q, r, RicciCD, RicciPDcartesian, RicciScalarCD,
 RicciScalarPDcartesian, RiemannCD, RiemannPDcartesian, s, SchoutenCCCD, SchoutenCCPDcartesian, SchoutenCD, SchoutenPDcartesian, SymRiemannCD, SymRiemannPDcartesian, t, TangentM4, TetraG, TetraG†, TFRicciCD, TheoryName, TorsionCD, TorsionPDcartesian, u, v, V, w, WeylCD, x, y, z, z$9039, z$9040, z$9041, z$9042, z$9043, z$9044, z$9045, z$9046, z$9047, z$9048, z$9063, z$9064, z$9069, z$9073, z$9141, z$9148, z$9153, z$9154, z$9207, z$9208, $ReadOnly}
```

To use the package we only need to know about a handful of these pre-defined symbols.

## Pre-defined geometry

We can see that the geometry of the spacetime has been pre-defined:

```
Print /@ {Information["M4", LongForm → False], G[-m, -n], Information["G", LongForm → False], Defer[CD[-a][G[-m, -n]]], Information["CD", LongForm → False]};
```

Symbol

M4 is the flat, four–dimensional Lorentzian spacetime manifold.

$\eta_{\mu\nu}$

Symbol

G[–a,–b] is the Minkowski spacetime metric in rectilinear Cartesian coordinates on M4, with the West Coast signature (1,–1,–1,–1).

$\partial_\alpha \eta_{\mu\nu}$

Symbol

CD[–a] is the partial derivative in rectilinear Cartesian coordinates on M4.

All lower-case letters in the Latin alphabet are valid spacetime indices, but they will automatically format as lower-case Greek letters when used inside tensors.

```
Print[IndicesOfVBundle[TangentM4]];
```

```
{{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z}, {}}
```

Strictly, xAct views all these indices as abstract indices. The interpretation of the indices as being associated with a Cartesian chart is a convenient abuse of notation.

## Pre-defined physics

Massive particles are associated with a rest-frame defined by their timelike momenta:

```
Print /@ {V[-m], Information["V", LongForm → False], P[-m], Information["P", LongForm → False], Def, Information["Def", LongForm → False], En, Information["En", LongForm → False], Mo, Information["Mo", LongForm → False]};
```

$n_{\mu}$

Symbol

V[−a] is a unit timelike vector V[−a]V[a]=1, which is assumed to be proportional to the momentum P[−a], and which functions as the four−velocity of an observer in whose rest frame all massive particles in the spectrum are also taken to be at rest.

$k_{\mu}$

Symbol

P[−a] is the timelike momentum used in the massive particle analysis, which approaches the null cone in the limit of the massless analysis.

$k$

Symbol

Def is the constant symbol which represents the positive square root of the norm of the timelike momentum.

$\mathcal{E}$

Symbol

En is the constant symbol which represents the energy, i.e. the time component of the timelike momentum.

$p$

Symbol

Mo is the constant symbol which represents the relativistic momentum, i.e. the z−component of the timelike momentum.

**Key observation:** Unlike for the case of the pre-defined geometry symbols, the user should not have to interact with these pre-defined physics symbols directly (we mention them only because they appear in the output).

## Provided functions

The package provides only two functions:

```
Print /@ {Information["DefField", LongForm → False], Information["ParticleSpectrum", LongForm → False]};
```

Symbol

DefField[F[Inds],SymmExpr,Options] defines a tensorial field F with indices Inds and index symmetries given by SymmExpr. Options are PrintAs and PrintSourceAs.

Symbol

ParticleSpectrum[L,Options] performs the whole propagator analysis on a scalar Lagrangian density L, which is quadratic in the perturbed fields and their derivatives, and linear in the couplings. Options are TheoryName, Method and MaxLaurentDepth.

Some of the options that must be passed to these functions are defined already in xAct, but some are new:

```
Print /@ {Information["PrintSourceAs", LongForm → False], Information["TheoryName", LongForm → False], Information["MaxLaurentDepth", LongForm → False]};
```

Symbol

PrintSourceAs is an option for DefField which acts as the PrintAs option for the conjugate source.

Symbol

TheoryName is a mandatory option for ParticleSpectrum which associates a name with the linearised Lagrangian density. The option must be passed as a (string) name for the new theory.

Symbol

MaxLaurentDepth is an option for ParticleSpectrum which sets the maximum positive integer n for which the 1/k^(2n) null pole residues are requested. The default is 1, from which the massless spectrum can be obtained. Setting higher n naturally leads to longer runtimes, but also allows potential (pathological) higher−order/non−simple propagator poles to be identified, down to the requested depth.

## Global variables and settings

We will impose some "personal" settings which make xAct easier to use (you may have seen these in other example notebooks for xAct, which can be found on the internet):

```
SetOptions[$FrontEndSession, EvaluationCompletionAction → "ScrollToOutput"]; $DefInfoQ = False; Unprotect[AutomaticRules]; Options[AutomaticRules] = {Verbose → False}; Protect[AutomaticRules];
```

We will also impose a global setting for PSALTer:

```
Print[Information["$ReadOnly", LongForm → False]]; $ReadOnly = False;
```

Symbol

$ReadOnly is a boolean variable which controls whether the analysis is actually performed or simply read in from a binary file. Default is False.

With these settings in place, the kernel is ready for science operations.

# Examples with code

## A single scalar

Let's define a scalar field:

```
DefField[ScalarField[], PrintAs → "ϕ", PrintSourceAs → "ρ"];
```

| Fundamental field | Symmetries | Decomposition into SO(3) irrep(s) | Source |
|---|---|---|---|
| $\phi$ | Symmetry[0, $\phi$, {}, StrongGenSet[{}, GenSet[]]] | $\phi^{\#1}_{0^+}$ | $\rho$ |
| SO(3) irrep | Symmetries | Expansion in terms of the fundamental field | Source SO(3) irrep |
| $\phi^{\#1}_{0^+}$ | Symmetry[0, $\phi^{\#1}_{0^+}$, {}, StrongGenSet[{}, GenSet[]]] | $\phi$ | $\rho^{\#1}_{0^+}$ |

Also, we define various coupling constants:

```
DefConstantSymbol[Coupling1, PrintAs → "α"]; DefConstantSymbol[Coupling2, PrintAs → "β"]; DefConstantSymbol[Coupling3, PrintAs → "γ"];
```

### Massless scalar (shift-symmetric field)

A massless scalar field theory.

$$\alpha\, \partial_{\alpha}\phi\, \partial^{\alpha}\phi \tag{2}$$

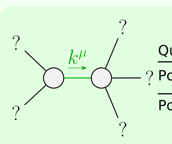
```
ParticleSpectrum[Coupling1 * CD[-a][ScalarField[]] * CD[a][ScalarField[]], TheoryName → "MasslessScalarTheory", Method → "Hard", MaxLaurentDepth → 3];
```

| Lagrangian density |
|---|
| $\alpha\,\partial_{\alpha}\phi\,\partial^{\alpha}\phi$ |
| Added source term: $\phi\,\rho$ |

| | $\rho^{\#1}_{0^+}$ |
|---|---|
| $\rho^{\#1\,\dagger}_{0^+}$ | $\boxed{\frac{1}{\alpha k^2}}$ |

| | $\phi^{\#1}_{0^+}$ |
|---|---|
| $\phi^{\#1\,\dagger}_{0^+}$ | $\boxed{\alpha k^2}$ |

(No source constraints)




| Quadratic pole | |
|---|---|
| Pole residue: | $\frac{1}{\alpha} > 0$ |
| Polarisations: | 1 |

Unitarity conditions

$\alpha > 0$

(No massive particles)

There is one massless polarisation, supported by a no-ghost condition.

### Massive scalar (Higgs field, pions)

A massive scalar field theory.

$$-\beta\,\phi^2 + \alpha\,\partial_{\alpha}\phi\,\partial^{\alpha}\phi \tag{3}$$

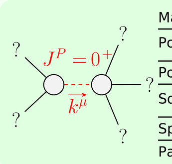
```
ParticleSpectrum[-(Coupling2 * ScalarField[]^2) + Coupling1 * CD[-a][ScalarField[]] * CD[a][ScalarField[]], TheoryName → "MassiveScalarTheory", Method → "Hard", MaxLaurentDepth → 3];
```

| Lagrangian density |
|---|
| $-\beta\,\phi^2 + \alpha\,\partial_{\alpha}\phi\,\partial^{\alpha}\phi$ |
| Added source term: $\phi\,\rho$ |

| | $\phi^{\#1}_{0^+}$ |
|---|---|
| $\phi^{\#1\,\dagger}_{0^+}$ | $\boxed{-\beta + \alpha k^2}$ |

| | $\rho^{\#1}_{0^+}$ |
|---|---|
| $\rho^{\#1\,\dagger}_{0^+}$ | $\boxed{\frac{1}{-\beta + \alpha k^2}}$ |

(No source constraints)




| Massive particle | |
|---|---|
| Pole residue: | $\frac{1}{\alpha} > 0$ |
| Polarisations: | 1 |
| Square mass: | $\frac{\beta}{\alpha} > 0$ |
| Spin: | 0 |
| Parity: | Even |

Unitarity conditions

$\alpha > 0\ \&\&\ \beta > 0$

(No massless particles)

We find that the massless eigenvalue has disappeared, but the propagator develops a massive pole whose no-ghost condition is equivalent. There is an additional no-tachyon condition on the Klein-Gordon mass.

## A single vector

Let's define a vector field:

```
DefField[VectorField[-a], PrintAs → "ℬ", PrintSourceAs → "𝒥"];
```

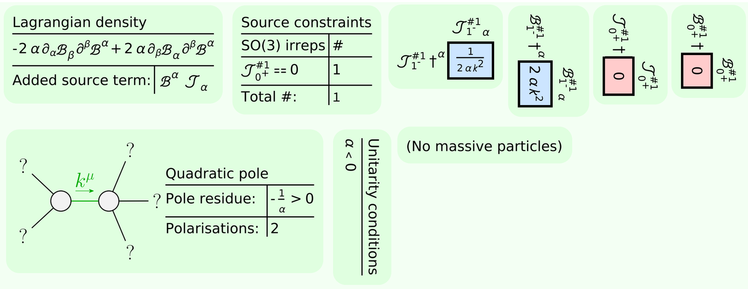
| Fundamental field | Symmetries | Decomposition into SO(3) irrep(s) | Source |
|---|---|---|---|
| $\mathcal{B}_{\alpha}$ | Symmetry[1, $\mathcal{B}^{\bullet 1}$, {●1 → -a}, StrongGenSet[{}, GenSet[]]] | $\mathcal{B}^{\#1}_{1^-}{}_{\alpha} + \mathcal{B}^{\#1}_{0^+}\, n_{\alpha}$ | $\mathcal{J}_{\alpha}$ |
| SO(3) irrep | Symmetries | Expansion in terms of the fundamental field | Source SO(3) irrep |
| $\mathcal{B}^{\#1}_{0^+}$ | Symmetry[0, $\mathcal{B}^{\#1}_{0^+}$, {}, StrongGenSet[{}, GenSet[]]] | $\mathcal{B}^{\alpha}\, n_{\alpha}$ | $\mathcal{J}^{\#1}_{0^+}$ |
| $\mathcal{B}^{\#1}_{1^-}{}_{\alpha}$ | Symmetry[1, $\mathcal{B}^{\#1}_{1^-}{}^{\bullet 1}$, {●1 → -a}, StrongGenSet[{}, GenSet[]]] | $\mathcal{B}_{\alpha} - \mathcal{B}^{\beta}\, n_{\alpha}\, n_{\beta}$ | $\mathcal{J}^{\#1}_{1^-}{}_{\alpha}$ |

# Maxwell field (quantum electrodynamics)

If we contract the square of the Maxwell tensor, we get a viable kinetic term which propagates the two massless photon polarisations.

$$\alpha\left(\partial_{\alpha}\mathcal{B}_{\beta} - \partial_{\beta}\mathcal{B}_{\alpha}\right)\left(\partial^{\alpha}\mathcal{B}^{\beta} - \partial^{\beta}\mathcal{B}^{\alpha}\right) \tag{4}$$

```
ParticleSpectrum[-2 * Coupling1 * CD[-a][VectorField[-b]] * CD[b][VectorField[a]] + 2 * Coupling1 * CD[-b][VectorField[-a]] * CD[b][VectorField[a]], TheoryName → "MaxwellTheory", Method → "Easy", MaxLaurentDepth → 3];
```



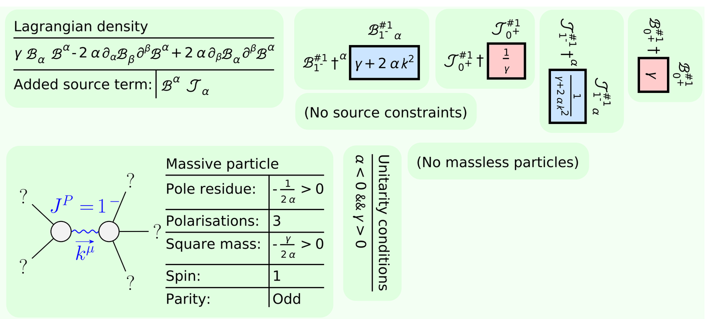
There are no mass terms in our Lagrangian Eq. (4), and hence no massive poles in the propagator. Instead, there are two massless eigenvalues which suggest that the vector part of the theory propagates two massless polarisations. The no-ghost condition of this massless vector requires that our kinetic coupling be negative. There is only one gauge constraint on the source current, which tells us that the positive-parity scalar part of the current must vanish. Reverse-engineering this condition from momentum to position space, we see that the four-divergence of the source must vanish: this implies charge conservation.

# Proca field (electroweak bosons)

Having investigated the massless theory in Eq. (4) we keep the same kinetic term but now add a mass term. This is the Proca theory.

$$\gamma\, \mathcal{B}_{\alpha}\, \mathcal{B}^{\alpha} + \alpha\left(\partial_{\alpha}\mathcal{B}_{\beta} - \partial_{\beta}\mathcal{B}_{\alpha}\right)\left(\partial^{\alpha}\mathcal{B}^{\beta} - \partial^{\beta}\mathcal{B}^{\alpha}\right) \tag{5}$$

```
ParticleSpectrum[Coupling3 * VectorField[-a] * VectorField[a] - 2 * Coupling1 * CD[-a][VectorField[-b]] * CD[b][VectorField[a]] + 2 * Coupling1 * CD[-b][VectorField[-a]] * CD[b][VectorField[a]], TheoryName → "ProcaTheory", Method → "Easy", MaxLaurentDepth → 3];
```



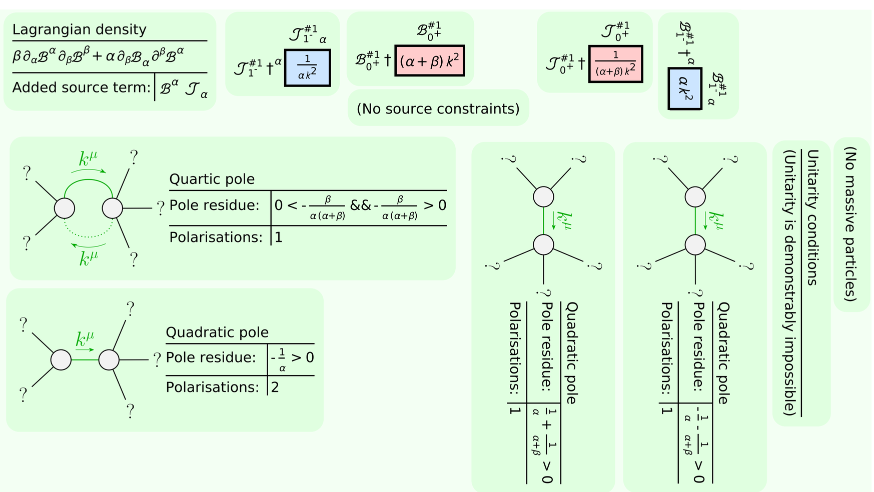
The divergence of the Proca equation of motion restricts the vector to be divergence-free, which is another way of saying that the helicity-0 mode vanishes on-shell. This is not a gauge condition (evidenced by the fact that the wave operator matrices are non-singular), but it does mean that in common with Maxwell's theory, we have the parity-odd vector mode. The theory is now massive, and so there is a massive pole in the propagator. There are now two unitarity conditions: the original no-ghost condition and a new no-tachyon condition which protects the Proca mass from becoming imaginary.

# Sickly quantum electrodynamics

Up to surface terms, there are two kinetic terms which are consistent with the basic requirement of Lorentz invariance in a vector theory.

$$\alpha\, \partial_{\alpha}\mathcal{B}_{\beta}\, \partial^{\alpha}\mathcal{B}^{\beta} + \beta\, \partial_{\alpha}\mathcal{B}^{\alpha}\, \partial_{\beta}\mathcal{B}^{\beta} \tag{6}$$

```
ParticleSpectrum[Coupling2 * CD[-a][VectorField[a]] * CD[-b][VectorField[b]] + Coupling1 * CD[-b][VectorField[-a]] * CD[b][VectorField[a]], TheoryName → "SickMaxwellTheory", Method → "Easy", MaxLaurentDepth → 3];
```



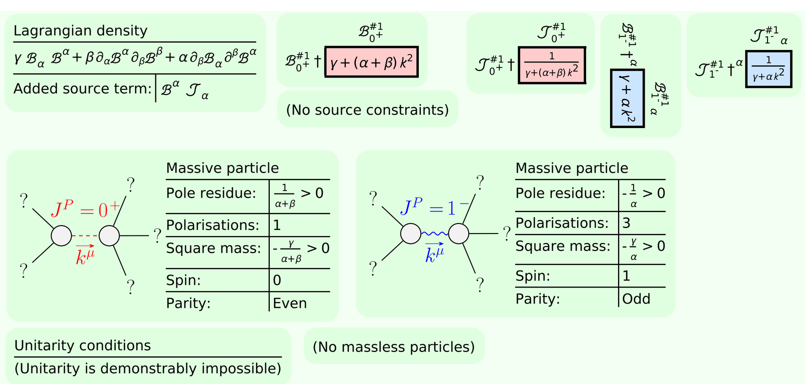
Notice the appearance of two extra massless eigenvalues, alongside the familiar photon polarisations. These carry different signs, and thus cannot be positive-definite: the theory is immutably sick. What has happened here is a result of the Ostrogradsky theorem. Our kinetic structure in Eq. (6) has destroyed the gauge invariance of the theory, and so the helicity-0 part of the field (the divergence of some scalar superpotential) has begun to move. Because the helicity-0 part contains an implicit divergence, that part of the theory now contains four implicit derivatives, and is a sickly higher-derivative model.

# Sickly Proca field

We extend our analysis of Eq. (6) to the general massive case.

$$\gamma\, \mathcal{B}_{\alpha}\, \mathcal{B}^{\alpha} + \alpha\, \partial_{\alpha}\mathcal{B}_{\beta}\, \partial^{\alpha}\mathcal{B}^{\beta} + \beta\, \partial_{\alpha}\mathcal{B}^{\alpha}\, \partial_{\beta}\mathcal{B}^{\beta} \tag{7}$$

```
ParticleSpectrum[Coupling3 * VectorField[-a] * VectorField[a] + Coupling2 * CD[-a][VectorField[a]] * CD[-b][VectorField[b]] + Coupling1 * CD[-b][VectorField[-a]] * CD[b][VectorField[a]], TheoryName → "SickProcaTheory", Method → "Easy", MaxLaurentDepth → 3];
```



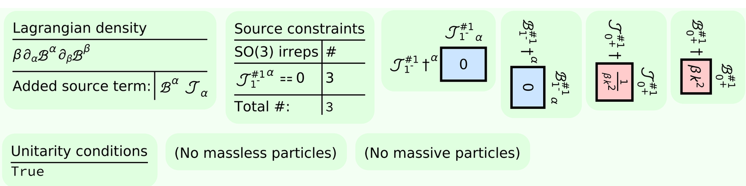
Once again, the theory is sick in the helicity-0 sector. In case the massive parity-odd vector is unitary, then the helicity-0 mode must either be a ghost or a tachyon.

# Pure longitudinal massless

There is another special case of Eq. (6) distinct from Eq. (4), which is not pathological.

$$\beta\, \partial_{\alpha}\mathcal{B}^{\alpha}\, \partial_{\beta}\mathcal{B}^{\beta} \tag{8}$$

```
ParticleSpectrum[Coupling2 * CD[-a][VectorField[a]] * CD[-b][VectorField[b]], TheoryName → "LongitudinalMassless", Method → "Easy", MaxLaurentDepth → 3];
```



The spectrum in this case is entirely empty.

# Pure longitudinal massive

Similarly, there is a healthy special case of Eq. (7), which is distinct from Eq. (5), and which is just the massive case of Eq. (8).

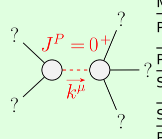
$$\gamma\, \mathcal{B}_{\alpha}\, \mathcal{B}^{\alpha} + \beta\, \partial_{\alpha}\mathcal{B}^{\alpha}\, \partial_{\beta}\mathcal{B}^{\beta} \tag{9}$$

```
ParticleSpectrum[Coupling3 * VectorField[-a] * VectorField[a] + Coupling2 * CD[-a][VectorField[a]] * CD[-b][VectorField[b]], TheoryName → "LongitudinalMassive", Method → "Easy", MaxLaurentDepth → 3];
```

| Lagrangian density | |
|---|---|
| $\gamma\, \mathcal{B}_{\alpha}\, \mathcal{B}^{\alpha} + \beta\, \partial_{\alpha}\mathcal{B}^{\alpha}\, \partial_{\beta}\mathcal{B}^{\beta}$ | |
| Added source term: | $\mathcal{B}^{\alpha}\, \mathcal{J}_{\alpha}$ |

$$\begin{array}{c|c} & \mathcal{J}^{\#1}_{1^-\,\alpha} \\ \hline \mathcal{J}^{\#1}_{1^-}{}^{\dagger\alpha} & \frac{1}{\gamma} \end{array} \qquad \begin{array}{c|c} & \mathcal{B}^{\#1}_{1^-\,\alpha} \\ \hline \mathcal{B}^{\#1}_{1^-}{}^{\dagger\alpha} & \gamma \end{array} \qquad \begin{array}{c|c} & \mathcal{J}^{\#1}_{0^+} \\ \hline \mathcal{J}^{\#1}_{0^+}{}^{\dagger} & \frac{1}{\gamma+\beta k^2} \end{array} \qquad \begin{array}{c|c} & \mathcal{B}^{\#1}_{0^+} \\ \hline \mathcal{B}^{\#1}_{0^+}{}^{\dagger} & \gamma+\beta k^2 \end{array}$$

(No source constraints)




| Massive particle | |
|---|---|
| Pole residue: | $\frac{1}{\beta} > 0$ |
| Polarisations: | 1 |
| Square mass: | $-\frac{\gamma}{\beta} > 0$ |
| Spin: | 0 |
| Parity: | Even |

Unitarity conditions

$\beta > 0\ \&\&\ \gamma < 0$

(No massless particles)

The theory propagates a single, healthy massive scalar.

# A single antisymmetric rank-two tensor

Let's define an antisymmetric tensor field:

```
DefField[TwoFormField[-a, -b], Antisymmetric[{-a, -b}], PrintAs → "𝓑", PrintSourceAs → "𝓙"];
```

| Fundamental field | Symmetries | Decomposition into SO(3) irrep(s) | Source |
|---|---|---|---|
| $\mathcal{B}_{\alpha\beta}$ | Symmetry[2, $\mathcal{B}^{\bullet1\bullet2}$, {●1 → -a, ●2 → -b}, StrongGenSet[{1, 2}, GenSet[-(1,2)]]] | $\mathcal{B}^{\#1}_{1^+\,\alpha\beta} - \mathcal{B}^{\#1}_{1^-\,\beta}\, n_{\alpha} + \mathcal{B}^{\#1}_{1^-\,\alpha}\, n_{\beta}$ | $\mathcal{J}_{\alpha\beta}$ |
| SO(3) irrep | Symmetries | Expansion in terms of the fundamental field | Source SO(3) irrep |
| $\mathcal{B}^{\#1}_{1^+\,\alpha\beta}$ | Symmetry[2, $\mathcal{B}^{\#1\,\bullet1\bullet2}_{1^+}$, {●1 → -a, ●2 → -b}, StrongGenSet[{1, 2}, GenSet[-(1,2)]]] | $\mathcal{B}_{\alpha\beta} + \mathcal{B}_{\beta\chi}\, n_{\alpha}\, n^{\chi} - \mathcal{B}_{\alpha\chi}\, n_{\beta}\, n^{\chi}$ | $\mathcal{J}^{\#1}_{1^+\,\alpha\beta}$ |
| $\mathcal{B}^{\#1}_{1^-\,\alpha}$ | Symmetry[1, $\mathcal{B}^{\#1\,\bullet1}_{1^-}$, {●1 → -a}, StrongGenSet[{}, GenSet[]]] | $\mathcal{B}_{\alpha\beta}\, n^{\beta}$ | $\mathcal{J}^{\#1}_{1^-\,\alpha}$ |

## Massless two-form theory

We now examine two-form electrodynamics.

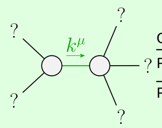
$$\frac{1}{6}\, \alpha\, \partial^{\alpha}\mathcal{B}^{\beta\chi} \left(\partial_{\alpha}\mathcal{B}_{\beta\chi} - \partial_{\alpha}\mathcal{B}_{\chi\beta} - \partial_{\beta}\mathcal{B}_{\alpha\chi} + \partial_{\beta}\mathcal{B}_{\chi\alpha} + \partial_{\chi}\mathcal{B}_{\alpha\beta} - \partial_{\chi}\mathcal{B}_{\beta\alpha}\right) \tag{10}$$

```
ParticleSpectrum[(-2 * Coupling1 * CD[-b][TwoFormField[-a, -c]] * CD[c][TwoFormField[a, b]]) / 3 + (Coupling1 * CD[-c][TwoFormField[-a, -b]] * CD[c][TwoFormField[a, b]]) / 3, TheoryName → "TwoFormEectrodynamics", Method → "Easy", MaxLaurentDepth → 1];
```

| Lagrangian density | |
|---|---|
| $-\frac{2}{3}\, \alpha\, \partial_{\beta}\mathcal{B}_{\alpha\chi}\, \partial^{\chi}\mathcal{B}^{\alpha\beta} + \frac{1}{3}\, \alpha\, \partial_{\chi}\mathcal{B}_{\alpha\beta}\, \partial^{\chi}\mathcal{B}^{\alpha\beta}$ | |
| Added source term: | $\mathcal{B}^{\alpha\beta}\, \mathcal{J}_{\alpha\beta}$ |

$$\begin{array}{c|cc} & \mathcal{J}^{\#1}_{1^+\,\alpha\beta} & \mathcal{J}^{\#1}_{1^-\,\alpha} \\ \hline \mathcal{J}^{\#1}_{1^+}{}^{\dagger\alpha\beta} & \frac{3}{\alpha k^2} & 0 \\ \mathcal{J}^{\#1}_{1^-}{}^{\dagger\alpha} & 0 & 0 \end{array} \qquad \begin{array}{c|cc} & \mathcal{B}^{\#1}_{1^+\,\alpha\beta} & \mathcal{B}^{\#1}_{1^-\,\alpha} \\ \hline \mathcal{B}^{\#1}_{1^+}{}^{\dagger\alpha\beta} & \frac{\alpha k^2}{3} & 0 \\ \mathcal{B}^{\#1}_{1^-}{}^{\dagger\alpha} & 0 & 0 \end{array}$$

| Source constraints | |
|---|---|
| SO(3) irreps | # |
| $\mathcal{J}^{\#1\,\alpha}_{1^-} == 0$ | 3 |
| Total #: | 3 |




| Quadratic pole | |
|---|---|
| Pole residue: | $\frac{1}{\alpha} > 0$ |
| Polarisations: | 1 |

Unitarity conditions

$\alpha > 0$

(No massive particles)

This result matches the literature.

## Massive two-form theory

We now add a mass-term.

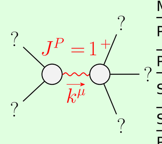
$$\beta\, \mathcal{B}_{\alpha\beta}\, \mathcal{B}^{\alpha\beta} + \frac{1}{6}\, \alpha\, \partial^{\alpha}\mathcal{B}^{\beta\chi} \left(\partial_{\alpha}\mathcal{B}_{\beta\chi} - \partial_{\alpha}\mathcal{B}_{\chi\beta} - \partial_{\beta}\mathcal{B}_{\alpha\chi} + \partial_{\beta}\mathcal{B}_{\chi\alpha} + \partial_{\chi}\mathcal{B}_{\alpha\beta} - \partial_{\chi}\mathcal{B}_{\beta\alpha}\right) \tag{11}$$

```
ParticleSpectrum[Coupling2 * TwoFormField[-a, -b] * TwoFormField[a, b] - (2 * Coupling1 * CD[-b][TwoFormField[-a, -c]] * CD[c][TwoFormField[a, b]]) / 3 + (Coupling1 * CD[-c][TwoFormField[-a, -b]] * CD[c][TwoFormField[a, b]]) / 3, TheoryName → "TwoFormEectrodynamicsMassive", Method → "Easy", MaxLaurentDepth → 1];
```

| Lagrangian density | |
|---|---|
| $\beta\, \mathcal{B}_{\alpha\beta}\, \mathcal{B}^{\alpha\beta} - \frac{2}{3}\, \alpha\, \partial_{\beta}\mathcal{B}_{\alpha\chi}\, \partial^{\chi}\mathcal{B}^{\alpha\beta} + \frac{1}{3}\, \alpha\, \partial_{\chi}\mathcal{B}_{\alpha\beta}\, \partial^{\chi}\mathcal{B}^{\alpha\beta}$ | |
| Added source term: | $\mathcal{B}^{\alpha\beta}\, \mathcal{J}_{\alpha\beta}$ |

(No source constraints)

$$\begin{array}{c|cc} & \mathcal{J}^{\#1}_{1^+\,\alpha\beta} & \mathcal{J}^{\#1}_{1^-\,\alpha} \\ \hline \mathcal{J}^{\#1}_{1^+}{}^{\dagger\alpha\beta} & \frac{1}{\beta+\frac{\alpha k^2}{3}} & 0 \\ \mathcal{J}^{\#1}_{1^-}{}^{\dagger\alpha} & 0 & \frac{1}{\beta} \end{array} \qquad \begin{array}{c|cc} & \mathcal{B}^{\#1}_{1^+\,\alpha\beta} & \mathcal{B}^{\#1}_{1^-\,\alpha} \\ \hline \mathcal{B}^{\#1}_{1^+}{}^{\dagger\alpha\beta} & \beta+\frac{\alpha k^2}{3} & 0 \\ \mathcal{B}^{\#1}_{1^-}{}^{\dagger\alpha} & 0 & \beta \end{array}$$




| Massive particle | |
|---|---|
| Pole residue: | $\frac{3}{\alpha} > 0$ |
| Polarisations: | 3 |
| Square mass: | $-\frac{3\beta}{\alpha} > 0$ |
| Spin: | 1 |
| Parity: | Even |

Unitarity conditions

$\alpha > 0\ \&\&\ \beta < 0$

(No massless particles)

This result matches the literature.

# A single symmetric rank-two tensor field

Let's define a symmetric tensor field, which we want to interpret as the metric perturbation:

```
DefField[MetricPerturbation[-a, -b], Symmetric[{-a, -b}], PrintAs → "𝒽", PrintSourceAs → "𝒯"];
```

| Fundamental field | Symmetries | Decomposition into SO(3) irrep(s) | Source |
|---|---|---|---|
| $h_{\alpha\beta}$ | Symmetry[2, $h^{\bullet1\bullet2}$, {●1 → -a, ●2 → -b}, StrongGenSet[{1, 2}, GenSet[(1,2)]]] | $\frac{1}{3}\, \eta_{\alpha\beta}\, h^{\#1}_{0^+} + h^{\#1}_{2^+\,\alpha\beta} + h^{\#1}_{1^-\,\beta}\, n_{\alpha} + h^{\#1}_{1^-\,\alpha}\, n_{\beta} - \frac{1}{3}\, h^{\#1}_{0^+}\, n_{\alpha}\, n_{\beta} + h^{\#2}_{0^+}\, n_{\alpha}\, n_{\beta}$ | $\mathcal{T}_{\alpha\beta}$ |
| SO(3) irrep | Symmetries | Expansion in terms of the fundamental field | Source SO(3) irrep |
| $h^{\#1}_{0^+}$ | Symmetry[0, $h^{\#1}_{0^+}$, {}, StrongGenSet[{}, GenSet[]]] | $h^{\alpha}{}_{\alpha} - h_{\alpha\beta}\, n^{\alpha}\, n^{\beta}$ | $\mathcal{T}^{\#1}_{0^+}$ |
| $h^{\#2}_{0^+}$ | Symmetry[0, $h^{\#2}_{0^+}$, {}, StrongGenSet[{}, GenSet[]]] | $h_{\alpha\beta}\, n^{\alpha}\, n^{\beta}$ | $\mathcal{T}^{\#2}_{0^+}$ |
| $h^{\#1}_{2^+\,\alpha\beta}$ | Symmetry[2, $h^{\#1\,\bullet1\bullet2}_{2^+}$, {●1 → -a, ●2 → -b}, StrongGenSet[{1, 2}, GenSet[(1,2)]]] | $h_{\alpha\beta} - \frac{1}{3}\, \eta_{\alpha\beta}\, h^{\chi}{}_{\chi} + \frac{1}{3}\, h^{\chi}{}_{\chi}\, n_{\alpha}\, n_{\beta} - h_{\beta\chi}\, n_{\alpha}\, n^{\chi} - h_{\alpha\chi}\, n_{\beta}\, n^{\chi} + \frac{1}{3}\, \eta_{\alpha\beta}\, h_{\chi\delta}\, n^{\chi}\, n^{\delta} + \frac{2}{3}\, h_{\chi\delta}\, n_{\alpha}\, n_{\beta}\, n^{\chi}\, n^{\delta}$ | $\mathcal{T}^{\#1}_{2^+\,\alpha\beta}$ |
| $h^{\#1}_{1^-\,\alpha}$ | Symmetry[1, $h^{\#1\,\bullet1}_{1^-}$, {●1 → -a}, StrongGenSet[{}, GenSet[]]] | $h_{\alpha\beta}\, n^{\beta} - h_{\beta\chi}\, n_{\alpha}\, n^{\beta}\, n^{\chi}$ | $\mathcal{T}^{\#1}_{1^-\,\alpha}$ |

## Fierz-Pauli (linear gravity)

The natural theory to check will be the Fierz-Pauli theory.

$$\alpha\left(-\partial^{\alpha}h_{\alpha\beta}\, \partial^{\beta}h^{\chi}{}_{\chi} + \frac{1}{2}\, \partial_{\beta}h^{\alpha}{}_{\alpha}\, \partial^{\beta}h^{\chi}{}_{\chi} - \frac{1}{2}\, \partial_{\chi}h^{\alpha\beta}\, \partial^{\chi}h_{\alpha\beta} + \partial_{\beta}h^{\alpha\beta}\, \partial^{\chi}h_{\alpha\chi}\right) \tag{12}$$

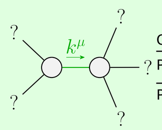
```
ParticleSpectrum[(Coupling1 * CD[-b][MetricPerturbation[c, -c]] * CD[b][MetricPerturbation[a, -a]]) / 2 + Coupling1 * CD[-a][MetricPerturbation[a, b]] * CD[-c][MetricPerturbation[-b, c]] - Coupling1 * CD[b][MetricPerturbation[a, -a]] * CD[-c][MetricPerturbation[-b, c]] - (Coupling1 * CD[-c][MetricPerturbation[-a, -b]] * CD[c][MetricPerturbation[a, b]]) / 2, TheoryName → "FierzPauliTheory", Method → "Hard", MaxLaurentDepth → 3];
```

| Lagrangian density | |
|---|---|
| $\frac{1}{2}\, \alpha\, \partial_{\beta}h^{\chi}{}_{\chi}\, \partial^{\beta}h^{\alpha}{}_{\alpha} + \alpha\, \partial_{\alpha}h^{\alpha\beta}\, \partial_{\chi}h_{\beta}{}^{\chi} - \alpha\, \partial^{\beta}h^{\alpha}{}_{\alpha}\, \partial_{\chi}h_{\beta}{}^{\chi} - \frac{1}{2}\, \alpha\, \partial_{\chi}h_{\alpha\beta}\, \partial^{\chi}h^{\alpha\beta}$ | |
| Added source term: | $h^{\alpha\beta}\, \mathcal{T}_{\alpha\beta}$ |

| Source constraints | |
|---|---|
| SO(3) irreps | # |
| $\mathcal{T}^{\#2}_{0^+} == 0$ | 1 |
| $\mathcal{T}^{\#1\,\alpha}_{1^-} == 0$ | 3 |
| Total #: | 4 |

$$\begin{array}{c|cc} & \mathcal{T}^{\#1}_{0^+} & \mathcal{T}^{\#2}_{0^+} \\ \hline \mathcal{T}^{\#1}_{0^+}{}^{\dagger} & \frac{1}{\alpha k^2} & 0 \\ \mathcal{T}^{\#2}_{0^+}{}^{\dagger} & 0 & 0 \end{array} \qquad \begin{array}{c|cc} & h^{\#1}_{0^+} & h^{\#2}_{0^+} \\ \hline h^{\#1}_{0^+}{}^{\dagger} & \alpha k^2 & 0 \\ h^{\#2}_{0^+}{}^{\dagger} & 0 & 0 \end{array} \qquad \begin{array}{c|c} & h^{\#1}_{1^-\,\alpha} \\ \hline h^{\#1}_{1^-}{}^{\dagger\alpha} & 0 \end{array}$$

$$\begin{array}{c|c} & \mathcal{T}^{\#1}_{2^+\,\alpha\beta} \\ \hline \mathcal{T}^{\#1}_{2^+}{}^{\dagger\alpha\beta} & -\frac{2}{\alpha k^2} \end{array} \qquad \begin{array}{c|c} & h^{\#1}_{2^+\,\alpha\beta} \\ \hline h^{\#1}_{2^+}{}^{\dagger\alpha\beta} & -\frac{\alpha k^2}{2} \end{array} \qquad \begin{array}{c|c} & \mathcal{T}^{\#1}_{1^-\,\alpha} \\ \hline \mathcal{T}^{\#1}_{1^-}{}^{\dagger\alpha} & 0 \end{array}$$




| Quadratic pole | |
|---|---|
| Pole residue: | $-\frac{1}{\alpha} > 0$ |
| Polarisations: | 2 |

Unitarity conditions

$\alpha < 0$

(No massive particles)

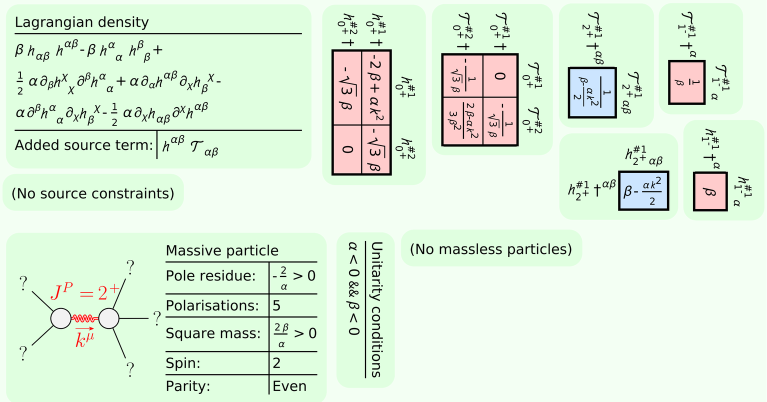
The Fierz-Pauli theory in Eq. (12) propagates two massless polarisations, and the no-ghost condition is consistent with a positive Einstein or Newton-Cavendish constant, or a positive square Planck mass. The diffeomorphism invariance of the theory is manifest as a gauge symmetry, whose constraints on the source currents are commensurate with the conservation of the matter stress-energy tensor.

## Massive gravity

We now add to Eq. (12) the unique mass term which corresponds to massive gravity (i.e. Fierz-Pauli tuning).

$$\beta\left(h_{\alpha\beta}\, h^{\alpha\beta} - h^{\alpha}{}_{\alpha}\, h^{\beta}{}_{\beta}\right) + \alpha\left(-\partial^{\alpha}h_{\alpha\beta}\, \partial^{\beta}h^{\chi}{}_{\chi} + \frac{1}{2}\, \partial_{\beta}h^{\alpha}{}_{\alpha}\, \partial^{\beta}h^{\chi}{}_{\chi} - \frac{1}{2}\, \partial_{\chi}h^{\alpha\beta}\, \partial^{\chi}h_{\alpha\beta} + \partial_{\beta}h^{\alpha\beta}\, \partial^{\chi}h_{\alpha\chi}\right) \tag{13}$$

```
ParticleSpectrum[Coupling2 * MetricPerturbation[-a, -b] * MetricPerturbation[a, b] - Coupling2 * MetricPerturbation[a, -a] * MetricPerturbation[b, -b] + (Coupling1 * CD[-b][MetricPerturbation[c, -c]] * CD[b][MetricPerturbation[a, -a]]) / 2 + Coupling1 * CD[-a][MetricPerturbation[a, b]] * CD[-c][MetricPerturbation[-b, c]] - Coupling1 * CD[b][MetricPerturbation[a, -a]] * CD[-c][MetricPerturbation[-b, c]] - (Coupling1 * CD[-c][MetricPerturbation[-a, -b]] * CD[c][MetricPerturbation[a, b]]) / 2, TheoryName → "MassiveGravity", Method → "Easy", MaxLaurentDepth → 3];
```




There is no massless sector. The propagator develops a massive pole in the positive-parity tensor sector. The no-ghost condition is as before, but now a no-tachyon condition protects the graviton mass.

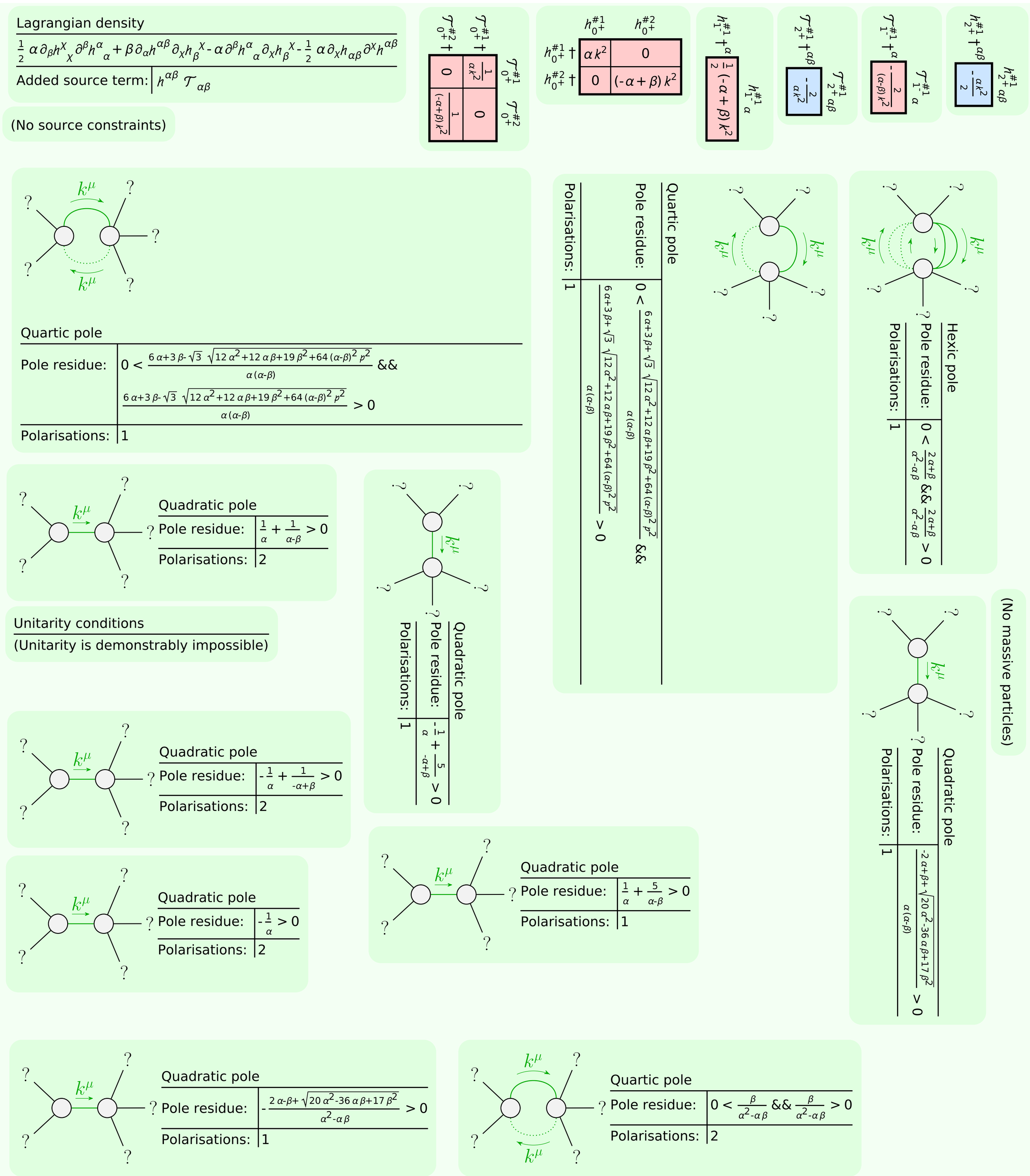
Returning to the viable case without any mass terms in Eq. (12), we consider some deviations from that model.

## Sick Fierz-Pauli (first variation)

We allow the fourth term in Eq. (12) to float.

$$\alpha\left(-\partial^{\alpha}h_{\alpha\beta}\,\partial^{\beta}h^{\chi}{}_{\chi}+\frac{1}{2}\,\partial_{\beta}h^{\alpha}{}_{\alpha}\,\partial^{\beta}h^{\chi}{}_{\chi}-\frac{1}{2}\,\partial_{\chi}h^{\alpha\beta}\,\partial^{\chi}h_{\alpha\beta}\right)+\beta\,\partial_{\beta}h^{\alpha\beta}\,\partial^{\chi}h_{\alpha\chi} \tag{14}$$

```
ParticleSpectrum[(Coupling1 * CD[-b][MetricPerturbation[c, -c]] * CD[b][MetricPerturbation[a, -a]]) / 2 + Coupling2 * CD[-a][MetricPerturbation[a, b]] * CD[-c][MetricPerturbation[-b, c]] - Coupling1 * CD[b][MetricPerturbation[a, -a]] * CD[-c][MetricPerturbation[-b, c]] - (Coupling1 * CD[-c][MetricPerturbation[-a, -b]] * CD[c][MetricPerturbation[a, b]]) / 2, TheoryName → "FirstSickFierzPauliTheory", Method → "Easy", MaxLaurentDepth → 3];
```



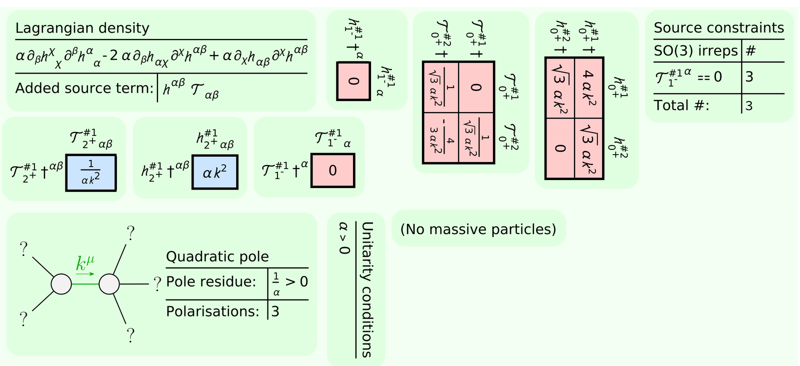

So this variation has no gauge symmetries, too many propagating species and no hope of unitarity.

## Marzo theory

There is a specific deformation of Eq. (12), which also appears to constitute a viable linear model. This model was first proposed by Carlo Marzo in unpublished correspondence.

$$-\alpha\left(2\,\partial_{\beta}h_{\alpha}{}^{\chi}\,\partial_{\chi}h^{\alpha\beta}-\partial_{\chi}h^{\alpha\beta}\,\partial^{\chi}h_{\alpha\beta}-\partial_{\chi}h^{\alpha}{}_{\alpha}\,\partial^{\chi}h^{\beta}{}_{\beta}\right) \tag{15}$$

```
ParticleSpectrum[Coupling1 * CD[-b][MetricPerturbation[c, -c]] * CD[b][MetricPerturbation[a, -a]] - 2 * Coupling1 * CD[-b][MetricPerturbation[-a, -c]] * CD[c][MetricPerturbation[a, b]] + Coupling1 * CD[-c][MetricPerturbation[-a, -b]] * CD[c][MetricPerturbation[a, b]], TheoryName → "MarzoTheory", Method → "Easy", MaxLaurentDepth → 3];
```



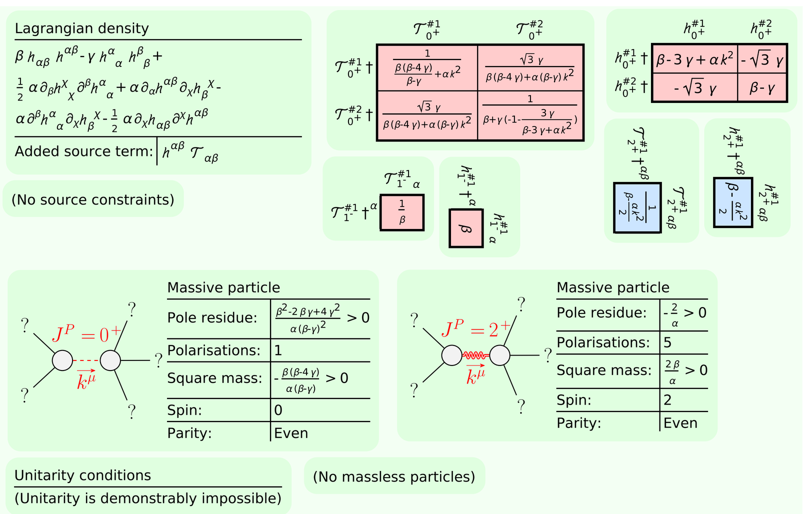

This model appears to be viable.

## Sick massive gravity

We now break the Fierz-Pauli tuning in Eq. (13).

$$\beta\,h_{\alpha\beta}\,h^{\alpha\beta}-\gamma\,h^{\alpha}{}_{\alpha}\,h^{\beta}{}_{\beta}+\alpha\left(-\partial^{\alpha}h_{\alpha\beta}\,\partial^{\beta}h^{\chi}{}_{\chi}+\frac{1}{2}\,\partial_{\beta}h^{\alpha}{}_{\alpha}\,\partial^{\beta}h^{\chi}{}_{\chi}-\frac{1}{2}\,\partial_{\chi}h^{\alpha\beta}\,\partial^{\chi}h_{\alpha\beta}+\partial_{\beta}h^{\alpha\beta}\,\partial^{\chi}h_{\alpha\chi}\right) \tag{16}$$

```
ParticleSpectrum[Coupling2 * MetricPerturbation[-a, -b] * MetricPerturbation[a, b] - Coupling3 * MetricPerturbation[a, -a] * MetricPerturbation[b, -b] + (Coupling1 * CD[-b][MetricPerturbation[c, -c]] * CD[b][MetricPerturbation[a, -a]]) / 2 + Coupling1 * CD[-a][MetricPerturbation[a, b]] * CD[-c][MetricPerturbation[-b, c]] - Coupling1 * CD[b][MetricPerturbation[a, -a]] * CD[-c][MetricPerturbation[-b, c]] - (Coupling1 * CD[-c][MetricPerturbation[-a, -b]] * CD[c][MetricPerturbation[a, b]]) / 2, TheoryName → "SickMassiveGravity", Method → "Easy", MaxLaurentDepth → 3];
```



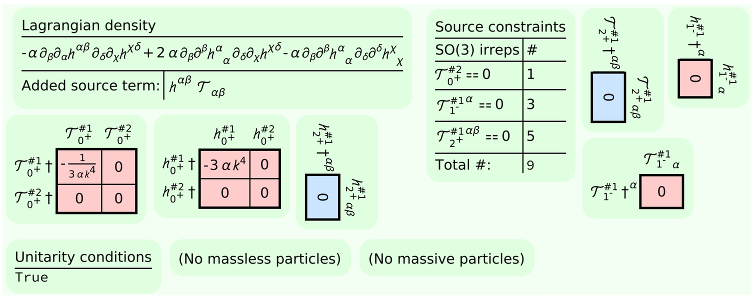

The consequence is seen in the positive-parity scalar sector, which develops a massive pole. This is the Boulware-Deser ghost, which always spoils the unitarity of the theory.

## Pure Ricci-square theory

We build a theory out of the square of the Ricci tensor.

$$-\alpha\left(-\partial_\alpha\partial^\alpha h_\beta{}^\beta+\partial_\alpha\partial_\beta h^{\alpha\beta}\right)\left(-\partial_\chi\partial^\chi h_\delta{}^\delta+\partial_\chi\partial_\delta h^{\chi\delta}\right) \tag{17}$$

```
ParticleSpectrum[-(Coupling1 * CD[-b][CD[-a][MetricPerturbation[a, b]]] * CD[-d][CD[-c][MetricPerturbation[c, d]]]) + 2 * Coupling1 * CD[-b][CD[b][MetricPerturbation[a, -a]]] * CD[-d][CD[-c][MetricPerturbation[c, d]]] - Coupling1 * CD[-b][CD[b][MetricPerturbation[a, -a]]] * CD[-d][CD[d][MetricPerturbation[c, -c]]], TheoryName → "SquareRicciTheory", Method → "Easy", MaxLaurentDepth → 3];
```



The empty spectrum is consistent with the literature.

# A scalar and a symmetric rank-two tensor

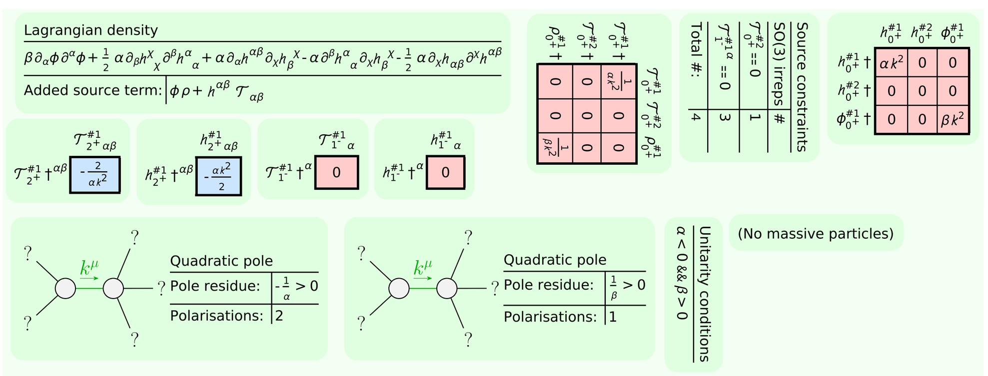
For the first time we will combine two kinds of field in one theory.

## Einstein-Klein-Gordon theory

We check the Fierz-Pauli theory in Eq. (12) accompanied by (but not coupled to) the massless scalar in Eq. (2).

$$\beta\,\partial_\alpha\phi\,\partial^\alpha\phi+\alpha\left(-\partial^\alpha h_{\alpha\beta}\,\partial^\beta h^\chi{}_\chi+\frac{1}{2}\,\partial_\beta h^\alpha{}_\alpha\,\partial^\beta h^\chi{}_\chi-\frac{1}{2}\,\partial_\chi h^{\alpha\beta}\,\partial^\chi h_{\alpha\beta}+\partial_\beta h^{\alpha\beta}\,\partial^\chi h_{\alpha\chi}\right) \tag{18}$$

```
ParticleSpectrum[Coupling2 * CD[-a][ScalarField[]] * CD[a][ScalarField[]] + (Coupling1 * CD[-b][MetricPerturbation[c, -c]] * CD[b][MetricPerturbation[a, -a]]) / 2 + Coupling1 * CD[-a][MetricPerturbation[a, b]] * CD[-c][MetricPerturbation[-b, c]] - Coupling1 * CD[b][MetricPerturbation[a, -a]] * CD[-c][MetricPerturbation[-b, c]] - (Coupling1 * CD[-c][MetricPerturbation[-a, -b]] * CD[c][MetricPerturbation[a, b]]) / 2, TheoryName → "ScalarFierzPauliTheory", Method → "Easy", MaxLaurentDepth → 3];
```



We obtain the graviton and a massless scalar.

# An asymmetric rank-two tensor and a pair-antisymmetric rank-three tensor

This is the kinematic setup which is used in Poincaré gauge theory (PGT).

We will set up an antisymmetric rank-three tensor field (the perturbation of the spin connection).

```
DefField[SpinConnection[-a, -b, -c], Antisymmetric[{-a, -b}], PrintAs → "ω", PrintSourceAs → "σ"];
```

$\omega^{\#1}_{2^-}{}_{\alpha\beta\chi}$

$\frac{2}{3}\,\omega^{\#1}_{2^-}{}_{\alpha\beta\chi}+\frac{1}{3}\,\omega^{\#1}_{2^-}{}_{\alpha\chi\beta}-\frac{1}{3}\,\omega^{\#1}_{2^-}{}_{\beta\chi\alpha}$

DefField: The reduced–index mode Rank3AntisymmetricPara2m appears not to be invertible using the provided rules.

| Fundamental field | Symmetries | Decomposition into SO(3) irrep(s) | Source |
|---|---|---|---|
| $\omega_{\alpha\beta\chi}$ | Symmetry[3, $\omega^{\bullet1\bullet2\bullet3}$, {●1 → -a, ●2 → -b, ●3 → -c}, StrongGenSet[{1, 2}, GenSet[-(1,2)]]] | $-\frac{1}{2}\,\eta_{\beta\chi}\,\omega^{\#1}_{1^-}{}_\alpha+\frac{1}{2}\,\eta_{\alpha\chi}\,\omega^{\#1}_{1^-}{}_\beta+\frac{4}{3}\,\omega^{\#1}_{2^-}{}_{\alpha\beta\chi}+\frac{1}{3}\,\eta_{\beta\chi}\,\omega^{\#1}_{0^+}\,n_\alpha+\omega^{\#1}_{1^+}{}_{\beta\chi}\,n_\alpha+\omega^{\#1}_{2^+}{}_{\beta\chi}\,n_\alpha-\frac{1}{3}\,\eta_{\alpha\chi}\,\omega^{\#1}_{0^+}\,n_\beta-\omega^{\#1}_{1^+}{}_{\alpha\chi}\,n_\beta-\omega^{\#1}_{2^+}{}_{\alpha\chi}\,n_\beta+\omega^{\#2}_{1^+}{}_{\alpha\beta}\,n_\chi-\frac{1}{2}\,\omega^{\#1}_{1^-}{}_\beta\,n_\alpha\,n_\chi-\omega^{\#2}_{1^-}{}_\beta\,n_\alpha\,n_\chi+\frac{1}{2}\,\omega^{\#1}_{1^-}{}_\alpha\,n_\beta\,n_\chi+\omega^{\#2}_{1^-}{}_\alpha\,n_\beta\,n_\chi-\frac{1}{6}\,\epsilon\eta_{\alpha\beta\chi\delta}\,\omega^{\#1}_{0^-}\,n^\delta$ | $\sigma_{\alpha\beta\chi}$ |
| SO(3) irrep | Symmetries | Expansion in terms of the fundamental field | Source SO(3) irrep |
| $\omega^{\#1}_{0^+}$ | Symmetry[0, $\omega^{\#1}_{0^+}$, {}, StrongGenSet[{}, GenSet[]]] | $\omega_\alpha{}^\beta{}_\beta\,n^\alpha$ | $\sigma^{\#1}_{0^+}$ |
| $\omega^{\#1}_{0^-}$ | Symmetry[0, $\omega^{\#1}_{0^-}$, {}, StrongGenSet[{}, GenSet[]]] | $-\epsilon\eta_{\alpha\beta\chi\delta}\,\omega^{\beta\chi\delta}\,n^\alpha$ | $\sigma^{\#1}_{0^-}$ |
| $\omega^{\#1}_{1^+}{}_{\alpha\beta}$ | Symmetry[2, $\omega^{\#1\bullet1\bullet2}_{1^+}$, {●1 → -a, ●2 → -b}, StrongGenSet[{1, 2}, GenSet[-(1,2)]]] | $-\frac{1}{2}\,\omega_{\alpha\chi\beta}\,n^\chi+\frac{1}{2}\,\omega_{\beta\chi\alpha}\,n^\chi-\frac{1}{2}\,\omega_{\beta\chi\delta}\,n_\alpha\,n^\chi\,n^\delta+\frac{1}{2}\,\omega_{\alpha\chi\delta}\,n_\beta\,n^\chi\,n^\delta$ | $\sigma^{\#1}_{1^+}{}_{\alpha\beta}$ |
| $\omega^{\#2}_{1^+}{}_{\alpha\beta}$ | Symmetry[2, $\omega^{\#2\bullet1\bullet2}_{1^+}$, {●1 → -a, ●2 → -b}, StrongGenSet[{1, 2}, GenSet[-(1,2)]]] | $\omega_{\alpha\beta\chi}\,n^\chi+\omega_{\beta\chi\delta}\,n_\alpha\,n^\chi\,n^\delta-\omega_{\alpha\chi\delta}\,n_\beta\,n^\chi\,n^\delta$ | $\sigma^{\#2}_{1^+}{}_{\alpha\beta}$ |
| $\omega^{\#1}_{1^-}{}_\alpha$ | Symmetry[1, $\omega^{\#1\bullet1}_{1^-}$, {●1 → -a}, StrongGenSet[{}, GenSet[]]] | $-\omega_\alpha{}^\beta{}_\beta+\omega_\beta{}^\chi{}_\chi\,n_\alpha\,n^\beta+\omega_{\alpha\beta\chi}\,n^\beta\,n^\chi$ | $\sigma^{\#1}_{1^-}{}_\alpha$ |
| $\omega^{\#2}_{1^-}{}_\alpha$ | Symmetry[1, $\omega^{\#2\bullet1}_{1^-}$, {●1 → -a}, StrongGenSet[{}, GenSet[]]] | $\omega_{\alpha\beta\chi}\,n^\beta\,n^\chi$ | $\sigma^{\#2}_{1^-}{}_\alpha$ |
| $\omega^{\#1}_{2^+}{}_{\alpha\beta}$ | Symmetry[2, $\omega^{\#1\bullet1\bullet2}_{2^+}$, {●1 → -a, ●2 → -b}, StrongGenSet[{1, 2}, GenSet[(1,2)]]] | $-\frac{1}{2}\,\omega_{\alpha\chi\beta}\,n^\chi-\frac{1}{2}\,\omega_{\beta\chi\alpha}\,n^\chi-\frac{1}{3}\,\eta_{\alpha\beta}\,\omega_\chi{}^\delta{}_\delta\,n^\chi+\frac{1}{3}\,\omega_\chi{}^\delta{}_\delta\,n_\alpha\,n_\beta\,n^\chi+\frac{1}{2}\,\omega_{\beta\chi\delta}\,n_\alpha\,n^\chi\,n^\delta+\frac{1}{2}\,\omega_{\alpha\chi\delta}\,n_\beta\,n^\chi\,n^\delta$ | $\sigma^{\#1}_{2^+}{}_{\alpha\beta}$ |
| $\omega^{\#1}_{2^-}{}_{\alpha\beta\chi}$ | Symmetry[3, $\omega^{\#1\bullet1\bullet2\bullet3}_{2^-}$, {●1 → -a, ●2 → -b, ●3 → -c}, StrongGenSet[{1, 2}, GenSet[-(1,2)]]] | $\frac{1}{2}\,\omega_{\alpha\beta\chi}+\frac{1}{4}\,\omega_{\alpha\chi\beta}-\frac{3}{8}\,\eta_{\beta\chi}\,\omega_\alpha{}^\delta{}_\delta-\frac{1}{4}\,\omega_{\beta\chi\alpha}+\frac{3}{8}\,\eta_{\alpha\chi}\,\omega_\beta{}^\delta{}_\delta-\frac{3}{8}\,\omega_\beta{}^\delta{}_\delta\,n_\alpha\,n_\chi+\frac{3}{8}\,\omega_\alpha{}^\delta{}_\delta\,n_\beta\,n_\chi+\frac{1}{4}\,\omega_{\beta\chi\delta}\,n_\alpha\,n^\delta+\frac{1}{2}\,\omega_{\beta\delta\chi}\,n_\alpha\,n^\delta+\frac{1}{4}\,\omega_{\chi\delta\beta}\,n_\alpha\,n^\delta+\frac{3}{8}\,\eta_{\beta\chi}\,\omega_\delta{}^\epsilon{}_\epsilon\,n_\alpha\,n^\delta-\frac{1}{4}\,\omega_{\alpha\chi\delta}\,n_\beta\,n^\delta-\frac{1}{2}\,\omega_{\alpha\delta\chi}\,n_\beta\,n^\delta-\frac{1}{4}\,\omega_{\chi\delta\alpha}\,n_\beta\,n^\delta-\frac{3}{8}\,\eta_{\alpha\chi}\,\omega_\delta{}^\epsilon{}_\epsilon\,n_\beta\,n^\delta-\frac{1}{2}\,\omega_{\alpha\beta\delta}\,n_\chi\,n^\delta-\frac{1}{4}\,\omega_{\alpha\delta\beta}\,n_\chi\,n^\delta+\frac{1}{4}\,\omega_{\beta\delta\alpha}\,n_\chi\,n^\delta+\frac{3}{8}\,\eta_{\beta\chi}\,\omega_{\alpha\delta\epsilon}\,n^\delta\,n^\epsilon-\frac{3}{8}\,\eta_{\alpha\chi}\,\omega_{\beta\delta\epsilon}\,n^\delta\,n^\epsilon-\frac{3}{8}\,\omega_{\beta\delta\epsilon}\,n_\alpha\,n_\chi\,n^\delta\,n^\epsilon+\frac{3}{8}\,\omega_{\alpha\delta\epsilon}\,n_\beta\,n_\chi\,n^\delta\,n^\epsilon$ | $\sigma^{\#1}_{2^-}{}_{\alpha\beta\chi}$ |

**Key observation:** Note the error message from the above. In fact, there is nothing wrong. As part of its internal self-consistency checks, DefField will check that expansion followed by decomposition returns the original quantity, and vice versa. However, ToCanonical is not capable of taking into account the fact that the negative-parity spin-two mode has an extra cyclic symmetry on its indices, which makes it look like decomposition is not the inverse of expansion. The inability to handle multi-term symmetries is a well-known limitation of xAct.

We will also set up an asymmetric tensor field (the perturbation of the tetrad).

```
DefField[TetradPerturbation[-a, -b], PrintAs → "f", PrintSourceAs → "τ"];
```

| Fundamental field | Symmetries | Decomposition into SO(3) irrep(s) | Source |
|---|---|---|---|
| $f_{\alpha\beta}$ | Symmetry[2, $f^{\bullet1\bullet2}$, {●1 → -a, ●2 → -b}, StrongGenSet[{}, GenSet[]]] | $\frac{1}{3}\,\eta_{\alpha\beta}\,f^{\#1}_{0^+}+f^{\#1}_{1^+}{}_{\alpha\beta}+f^{\#1}_{2^+}{}_{\alpha\beta}+f^{\#1}_{1^-}{}_\beta\,n_\alpha+f^{\#2}_{1^-}{}_\alpha\,n_\beta-\frac{1}{3}\,f^{\#1}_{0^+}\,n_\alpha\,n_\beta+f^{\#2}_{0^+}\,n_\alpha\,n_\beta$ | $\tau_{\alpha\beta}$ |
| SO(3) irrep | Symmetries | Expansion in terms of the fundamental field | Source SO(3) irrep |
| $f^{\#1}_{0^+}$ | Symmetry[0, $f^{\#1}_{0^+}$, {}, StrongGenSet[{}, GenSet[]]] | $f^\alpha{}_\alpha-f^{\alpha\beta}\,n_\alpha\,n_\beta$ | $\tau^{\#1}_{0^+}$ |
| $f^{\#2}_{0^+}$ | Symmetry[0, $f^{\#2}_{0^+}$, {}, StrongGenSet[{}, GenSet[]]] | $f^{\alpha\beta}\,n_\alpha\,n_\beta$ | $\tau^{\#2}_{0^+}$ |
| $f^{\#1}_{1^+}{}_{\alpha\beta}$ | Symmetry[2, $f^{\#1\bullet1\bullet2}_{1^+}$, {●1 → -a, ●2 → -b}, StrongGenSet[{1, 2}, GenSet[-(1,2)]]] | $\frac{f_{\alpha\beta}}{2}-\frac{f_{\beta\alpha}}{2}+\frac{1}{2}\,f_\beta{}^\chi\,n_\alpha\,n_\chi-\frac{1}{2}\,f^\chi{}_\beta\,n_\alpha\,n_\chi-\frac{1}{2}\,f_\alpha{}^\chi\,n_\beta\,n_\chi+\frac{1}{2}\,f^\chi{}_\alpha\,n_\beta\,n_\chi$ | $\tau^{\#1}_{1^+}{}_{\alpha\beta}$ |
| $f^{\#1}_{1^-}{}_\alpha$ | Symmetry[1, $f^{\#1\bullet1}_{1^-}$, {●1 → -a}, StrongGenSet[{}, GenSet[]]] | $f^\beta{}_\alpha\,n_\beta-f^{\beta\chi}\,n_\alpha\,n_\beta\,n_\chi$ | $\tau^{\#1}_{1^-}{}_\alpha$ |
| $f^{\#2}_{1^-}{}_\alpha$ | Symmetry[1, $f^{\#2\bullet1}_{1^-}$, {●1 → -a}, StrongGenSet[{}, GenSet[]]] | $f_\alpha{}^\beta\,n_\beta-f^{\beta\chi}\,n_\alpha\,n_\beta\,n_\chi$ | $\tau^{\#2}_{1^-}{}_\alpha$ |
| $f^{\#1}_{2^+}{}_{\alpha\beta}$ | Symmetry[2, $f^{\#1\bullet1\bullet2}_{2^+}$, {●1 → -a, ●2 → -b}, StrongGenSet[{1, 2}, GenSet[(1,2)]]] | $\frac{f_{\alpha\beta}}{2}+\frac{f_{\beta\alpha}}{2}-\frac{1}{3}\,\eta_{\alpha\beta}\,f^\chi{}_\chi+\frac{1}{3}\,f^\chi{}_\chi\,n_\alpha\,n_\beta-\frac{1}{2}\,f_\beta{}^\chi\,n_\alpha\,n_\chi-\frac{1}{2}\,f^\chi{}_\beta\,n_\alpha\,n_\chi-\frac{1}{2}\,f_\alpha{}^\chi\,n_\beta\,n_\chi-\frac{1}{2}\,f^\chi{}_\alpha\,n_\beta\,n_\chi+\frac{1}{3}\,\eta_{\alpha\beta}\,f^{\chi\delta}\,n_\chi\,n_\delta+\frac{2}{3}\,f^{\chi\delta}\,n_\alpha\,n_\beta\,n_\chi\,n_\delta$ | $\tau^{\#1}_{2^+}{}_{\alpha\beta}$ |

Here is the inverse translational gauge field, or tetrad.

$$h_\alpha{}^\chi \tag{19}$$

Here is the translational gauge field, or inverse tetrad.

$$b^\alpha{}_\chi \tag{20}$$

Here is the Riemann-Cartan tensor.

$$\mathcal{R}^{\alpha\beta}{}_{\delta\epsilon} \tag{21}$$

$$h_\delta{}^\chi\,h_\epsilon{}^\phi\,\omega^{\alpha\gamma}{}_\phi\,\omega^\beta{}_{\gamma\chi}-h_\delta{}^\chi\,h_\epsilon{}^\phi\,\omega^{\alpha\gamma}{}_\chi\,\omega^\beta{}_{\gamma\phi}+h_\delta{}^\chi\,h_\epsilon{}^\phi\,\partial_\chi\omega^{\alpha\beta}{}_\phi-h_\delta{}^\chi\,h_\epsilon{}^\phi\,\partial_\phi\omega^{\alpha\beta}{}_\chi \tag{22}$$

Here is the torsion tensor.

$$\mathcal{T}^\alpha{}_{\beta\chi} \tag{23}$$

$$-h_\chi{}^\delta\,\omega^\alpha{}_{\beta\delta}+h_\beta{}^\delta\,\omega^\alpha{}_{\chi\delta}+h_\beta{}^\delta\,h_\chi{}^\epsilon\,\partial_\delta b^\alpha{}_\epsilon-h_\beta{}^\delta\,h_\chi{}^\epsilon\,\partial_\epsilon b^\alpha{}_\delta \tag{24}$$

Now we set up the general Lagrangian. In the first instance we will do this with some coupling constants which are proportional to those used by Hayashi and Shirafuji in Prog. Theor. Phys. 64 (1980) 2222, and identical to those used in arXiv:2205.13534 and (up to re-labelling) arXiv:gr-qc/9902032.

```
DefConstantSymbol[Alp0, PrintAs → "\!\(\*SubscriptBox[\(α\), \(0\)]\)"]; DefConstantSymbol[Alp1, PrintAs → "\!\(\*SubscriptBox[\(α\), \(1\)]\)"]; DefConstantSymbol[Alp2, PrintAs → "\!\(\*SubscriptBox[\(α\), \(2\)]\)"]; DefConstantSymbol[Alp3, PrintAs → "\!\(\*SubscriptBox[\(α\), \(3\)]\)"]; DefConstantSymbol[Alp4, PrintAs → "\!\(\*SubscriptBox[\(α\), \(4\)]\)"];
DefConstantSymbol[Alp5, PrintAs → "\!\(\*SubscriptBox[\(α\), \(5\)]\)"]; DefConstantSymbol[Alp6, PrintAs → "\!\(\*SubscriptBox[\(α\), \(6\)]\)"]; DefConstantSymbol[Bet1, PrintAs → "\!\(\*SubscriptBox[\(β\), \(1\)]\)"]; DefConstantSymbol[Bet2, PrintAs → "\!\(\*SubscriptBox[\(β\), \(2\)]\)"]; DefConstantSymbol[Bet3, PrintAs → "\!\(\*SubscriptBox[\(β\), \(3\)]\)"];
```

$$-\frac{1}{2}\,\alpha_0\,\eta^{\alpha\chi}\,\eta^{\beta\delta}\,\mathcal{R}_{\alpha\beta\chi\delta}+\left(\alpha_1\,\hat{\mathcal{P}}_{\mathcal{R}}\,1_{\iota\theta\gamma\eta}{}^{\alpha\beta\chi\delta}+\alpha_2\,\hat{\mathcal{P}}_{\mathcal{R}}\,2_{\iota\theta\gamma\eta}{}^{\alpha\beta\chi\delta}+\alpha_3\,\hat{\mathcal{P}}_{\mathcal{R}}\,3_{\iota\theta\gamma\eta}{}^{\alpha\beta\chi\delta}+\alpha_4\,\hat{\mathcal{P}}_{\mathcal{R}}\,4_{\iota\theta\gamma\eta}{}^{\alpha\beta\chi\delta}+\alpha_5\,\hat{\mathcal{P}}_{\mathcal{R}}\,5_{\iota\theta\gamma\eta}{}^{\alpha\beta\chi\delta}+\alpha_6\,\hat{\mathcal{P}}_{\mathcal{R}}\,6_{\iota\theta\gamma\eta}{}^{\alpha\beta\chi\delta}\right)\mathcal{R}_{\alpha\beta\chi\delta}\,\mathcal{R}^{\iota\theta\gamma\eta}+\left(\beta_1\,\hat{\mathcal{P}}_{\mathcal{T}}\,1_{\iota\gamma\eta}{}^{\alpha\chi\delta}+\beta_2\,\hat{\mathcal{P}}_{\mathcal{T}}\,2_{\iota\gamma\eta}{}^{\alpha\chi\delta}+\beta_3\,\hat{\mathcal{P}}_{\mathcal{T}}\,3_{\iota\gamma\eta}{}^{\alpha\chi\delta}\right)\mathcal{T}_{\alpha\chi\delta}\,\mathcal{T}^{\iota\gamma\eta} \tag{25}$$

In Eq. (25) we are using projectors to extract the Lorentz irreducible representations of the fields. Next we will expand these.

$$-\frac{1}{2}\alpha_0\,\mathcal{R}^{\alpha\beta}{}_{\alpha\beta}+\frac{1}{6}(2\alpha_1+3\alpha_2+\alpha_3)\,\mathcal{R}_{\alpha\beta\chi\delta}\,\mathcal{R}^{\alpha\beta\chi\delta}+\frac{2}{3}(\alpha_1-\alpha_3)\,\mathcal{R}_{\alpha\chi\beta\delta}\,\mathcal{R}^{\alpha\beta\chi\delta}+(-\alpha_1-\alpha_2+\alpha_4+\alpha_5)\,\mathcal{R}^{\alpha\beta}{}_{\alpha}{}^{\chi}\,\mathcal{R}_{\beta}{}^{\delta}{}_{\chi\delta}+\frac{1}{6}(2\alpha_1-3\alpha_2+\alpha_3)\,\mathcal{R}^{\alpha\beta\chi\delta}\,\mathcal{R}_{\chi\delta\alpha\beta}+(-\alpha_1+\alpha_2+\alpha_4-\alpha_5)\,\mathcal{R}^{\alpha\beta}{}_{\alpha}{}^{\chi}\,\mathcal{R}_{\chi}{}^{\delta}{}_{\beta\delta}+\frac{1}{6}(2\alpha_1-3\alpha_4+\alpha_6)\,\mathcal{R}^{\alpha\beta}{}_{\alpha\beta}\,\mathcal{R}^{\chi\delta}{}_{\chi\delta}+\frac{1}{3}(2\beta_1+\beta_3)\,\mathcal{T}_{\alpha\beta\chi}\,\mathcal{T}^{\alpha\beta\chi}+\frac{2}{3}(\beta_1-\beta_3)\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\alpha\chi}+\frac{2}{3}(\beta_1-\beta_2)\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\,\mathcal{T}^{\chi}{}_{\beta\chi} \tag{26}$$

We can also use a different set of coupling coefficients to those in Eq. (26), as developed by Karananas and used in e.g. arXiv:1910.14197.

```
DefConstantSymbol[kLambda, PrintAs → "λ"]; DefConstantSymbol[kR1, PrintAs → "\!\(\*SubscriptBox[\(r\), \(1\)]\)"]; DefConstantSymbol[kR2, PrintAs → "\!\(\*SubscriptBox[\(r\), \(2\)]\)"]; DefConstantSymbol[kR3, PrintAs → "\!\(\*SubscriptBox[\(r\), \(3\)]\)"]; DefConstantSymbol[kR4, PrintAs → "\!\(\*SubscriptBox[\(r\), \(4\)]\)"];
DefConstantSymbol[kR5, PrintAs → "\!\(\*SubscriptBox[\(r\), \(5\)]\)"]; DefConstantSymbol[kR6, PrintAs → "\!\(\*SubscriptBox[\(r\), \(6\)]\)"]; DefConstantSymbol[kT1, PrintAs → "\!\(\*SubscriptBox[\(t\), \(1\)]\)"]; DefConstantSymbol[kT2, PrintAs → "\!\(\*SubscriptBox[\(t\), \(2\)]\)"]; DefConstantSymbol[kT3, PrintAs → "\!\(\*SubscriptBox[\(t\), \(3\)]\)"];
```

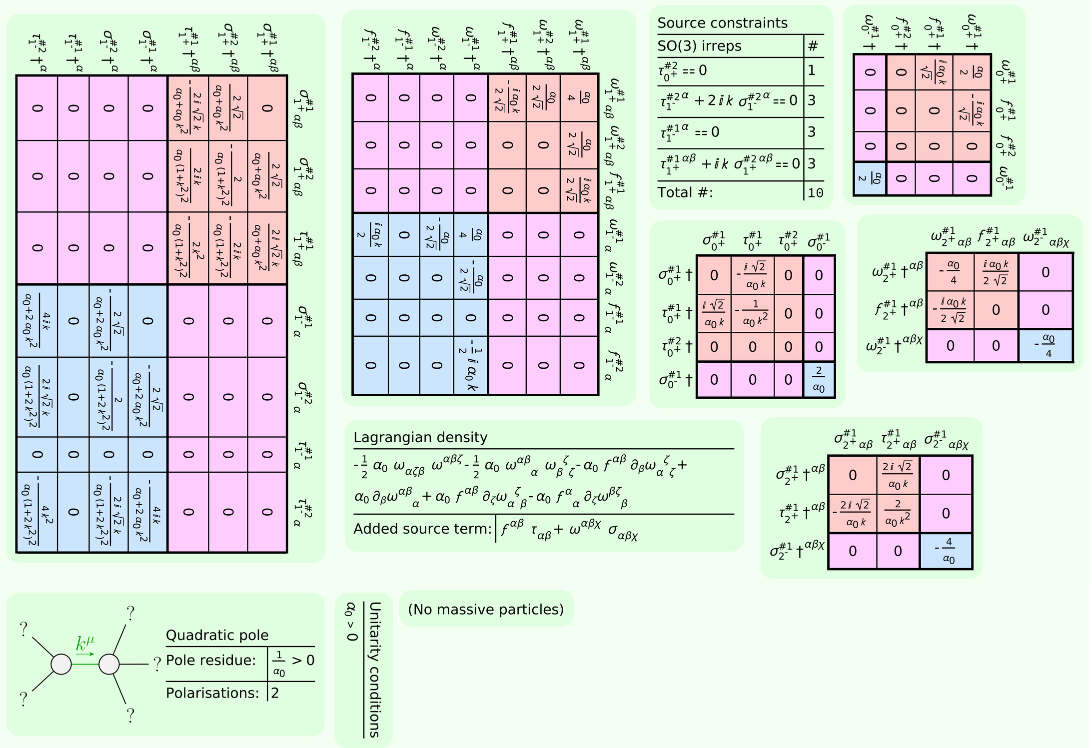
$$-\lambda\,\mathcal{R}^{\iota\theta}{}_{\iota\theta}+\left(\frac{r_1}{3}+\frac{r_2}{6}\right)\mathcal{R}_{\iota\theta\kappa\lambda}\,\mathcal{R}^{\iota\theta\kappa\lambda}+\left(\frac{2r_1}{3}-\frac{2r_2}{3}\right)\mathcal{R}_{\iota\kappa\theta\lambda}\,\mathcal{R}^{\iota\theta\kappa\lambda}+(r_4+r_5)\,\mathcal{R}_{\iota}{}^{\lambda}{}_{\theta\lambda}\,\mathcal{R}^{\iota\kappa\theta}{}_{\kappa}+(r_4-r_5)\,\mathcal{R}^{\iota\kappa\theta}{}_{\kappa}\,\mathcal{R}_{\theta}{}^{\lambda}{}_{\iota\lambda}+\left(\frac{r_1}{3}+\frac{r_2}{6}-r_3\right)\mathcal{R}^{\iota\theta\kappa\lambda}\,\mathcal{R}_{\kappa\lambda\iota\theta}+\left(\frac{\lambda}{4}+\frac{t_1}{3}+\frac{t_2}{12}\right)\mathcal{T}_{\iota\theta\kappa}\,\mathcal{T}^{\iota\theta\kappa}+\left(-\frac{\lambda}{2}-\frac{t_1}{3}+\frac{t_2}{6}\right)\mathcal{T}^{\iota\theta\kappa}\,\mathcal{T}_{\theta\kappa\iota}+\left(-\lambda-\frac{t_1}{3}+\frac{2t_3}{3}\right)\mathcal{T}_{\iota}{}^{\theta\iota}\,\mathcal{T}_{\kappa\theta}{}^{\kappa} \tag{27}$$

# Einstein-Cartan theory (ECT)

We would like to check the basic Einstein-Cartan theory.

$$-\frac{1}{2}\alpha_0\,\mathcal{R}^{\alpha\beta}{}_{\alpha\beta} \tag{28}$$

```
ParticleSpectrum[-1/2*(Alp0*SpinConnection[-a, -z, -b]*SpinConnection[a, b, z]) - (Alp0*SpinConnection[a, b, -a]*SpinConnection[-b, z, -z])/2 - Alp0*TetradPerturbation[a, b]*CD[-b][SpinConnection[-a, z, -z]] + Alp0*CD[-b][SpinConnection[a, b, -a]] + Alp0*TetradPerturbation[a, b]*CD[-z][SpinConnection[-a, z, -b]] - Alp0*TetradPerturbation[a, -a]*CD[-z][SpinConnection[b, z, -b]], TheoryName → "EinsteinCartanTheory", Method → "Hard", MaxLaurentDepth → 3];
```



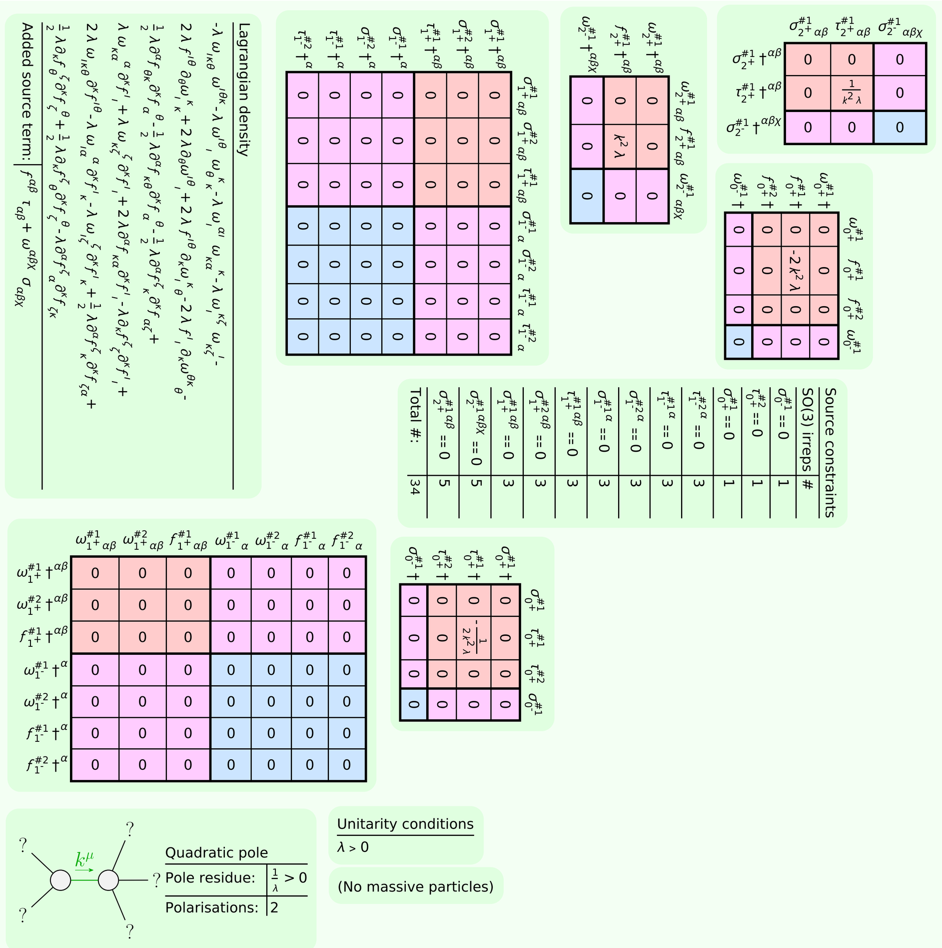

What we find are no propagating massive modes, but instead two degrees of freedom in the massive sector. The no-ghost conditions on these massless d.o.f restrict the sign in front of the Einstein-Hilbert term to be negative (which is what we expect for our conventions). Note that this results is essentially the same as Eq. (12).

# General relativity (GR)

Using the coupling coefficients in Eq. (27), it is particularly easy to also look at GR, instead of Einstein-Cartan theory. The difference here is that the quadratic torsion coefficients are manually removed.

$$-\lambda\,\mathcal{R}^{\iota\theta}{}_{\iota\theta}+\frac{1}{4}\lambda\,\mathcal{T}_{\iota\theta\kappa}\,\mathcal{T}^{\iota\theta\kappa}+\frac{1}{2}\lambda\,\mathcal{T}^{\iota\theta\kappa}\,\mathcal{T}_{\theta\iota\kappa}+\lambda\,\mathcal{T}^{\iota}{}_{\iota}{}^{\theta}\,\mathcal{T}^{\kappa}{}_{\theta\kappa} \tag{29}$$

```
ParticleSpectrum[-(kLambda*SpinConnection[-i, -k, -j]*SpinConnection[i, j, k]) - kLambda*SpinConnection[i, j, -i]*SpinConnection[-j, k, -k] - kLambda*SpinConnection[-i, a, i]*SpinConnection[-k, -a, k] - kLambda*SpinConnection[-i, k, z]*SpinConnection[-k, -z, i] - 2*kLambda*TetradPerturbation[i, j]*CD[-j][SpinConnection[-i, k, -k]] + 2*kLambda*CD[-j][SpinConnection[i, j, -i]] + 2*kLambda*TetradPerturbation[i, j]*CD[-k][SpinConnection[-i, k, -j]] - 2*kLambda*TetradPerturbation[i, -i]*CD[-k][SpinConnection[j, k, -j]] - (kLambda*CD[a][TetradPerturbation[-j, -k]]*CD[k][TetradPerturbation[-a, j]])/2 -
  (kLambda*CD[a][TetradPerturbation[-k, -j]]*CD[k][TetradPerturbation[-a, j]])/2 - (kLambda*CD[a][TetradPerturbation[z, -k]]*CD[k][TetradPerturbation[-a, -z]])/2 + kLambda*SpinConnection[-k, -a, a]*CD[k][TetradPerturbation[i, -i]] + kLambda*SpinConnection[-k, -z, z]*CD[k][TetradPerturbation[i, -i]] + 2*kLambda*CD[a][TetradPerturbation[-k, -a]]*CD[k][TetradPerturbation[i, -i]] - kLambda*CD[-k][TetradPerturbation[z, -z]]*CD[k][TetradPerturbation[i, -i]] + 2*kLambda*SpinConnection[-i, -k, -j]*CD[k][TetradPerturbation[i, j]] -
  kLambda*SpinConnection[-i, -a, a]*CD[k][TetradPerturbation[i, -k]] - kLambda*SpinConnection[-i, -z, z]*CD[k][TetradPerturbation[i, -k]] + (kLambda*CD[a][TetradPerturbation[z, -k]]*CD[k][TetradPerturbation[-z, -a]])/2 + (kLambda*CD[-k][TetradPerturbation[-j, z]]*CD[k][TetradPerturbation[-z, j]])/2 + (kLambda*CD[-k][TetradPerturbation[z, -j]]*CD[k][TetradPerturbation[-z, j]])/2 - kLambda*CD[a][TetradPerturbation[z, -a]]*CD[k][TetradPerturbation[-z, -k]], TheoryName → "GeneralRelativity", Method → "Easy", MaxLaurentDepth → 3];
```




The spectra of Eqs. (28), and (29) are identical, as expected.

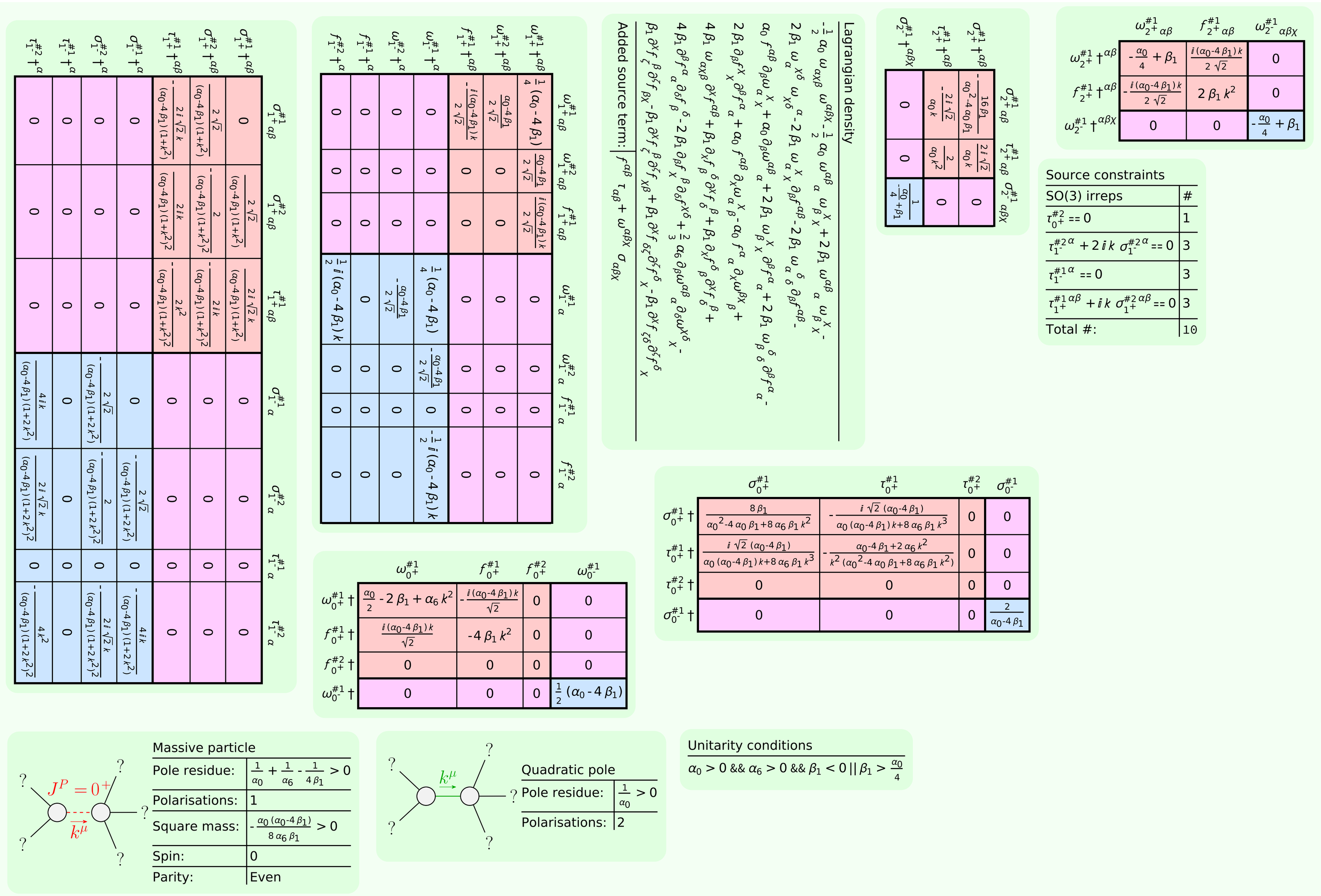
We will study the minimal models of arXiv:9902032. We will do this using the general coupling coefficients defined in Eq. (26).

# Minimal even-parity scalar model

We will study the minimal model set out in Eq. (4.1) of arXiv:9902032.

$$-\frac{1}{2}\alpha_0\,\mathcal{R}^{\alpha\beta}{}_{\alpha\beta}+\frac{1}{6}\alpha_6\,\mathcal{R}^{\alpha\beta}{}_{\alpha\beta}\,\mathcal{R}^{\chi\delta}{}_{\chi\delta}+\frac{1}{2}\beta_1\,\mathcal{T}_{\alpha\beta\chi}\,\mathcal{T}^{\alpha\beta\chi}+\beta_1\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\alpha\chi}+2\beta_1\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\,\mathcal{T}^{\chi}{}_{\beta\chi} \tag{30}$$

```
ParticleSpectrum[-1/2*(Alp0*SpinConnection[-a, -c, -b]*SpinConnection[a, b, c]) - (Alp0*SpinConnection[a, b, -a]*SpinConnection[-b, c, -c])/2 + 2*Bet1*SpinConnection[a, b, -a]*SpinConnection[-b, c, -c] - 2*Bet1*SpinConnection[-a, c, d]*SpinConnection[-c, -d, a] - Alp0*TetradPerturbation[a, b]*CD[-b][SpinConnection[-a, c, -c]] + Alp0*CD[-b][SpinConnection[a, b, -a]] - 2*Bet1*SpinConnection[-a, c, -c]*CD[-b][TetradPerturbation[a, b]] - 2*Bet1*SpinConnection[-a, d, -d]*CD[-b][TetradPerturbation[a, b]] + 2*Bet1*SpinConnection[-b, c, -c]*CD[b][TetradPerturbation[a, -a]] +
    2*Bet1*SpinConnection[-b, d, -d]*CD[b][TetradPerturbation[a, -a]] - 2*Bet1*CD[-b][TetradPerturbation[c, -c]]*CD[b][TetradPerturbation[a, -a]] + Alp0*TetradPerturbation[a, b]*CD[-c][SpinConnection[-a, c, -b]] - Alp0*TetradPerturbation[a, -a]*CD[-c][SpinConnection[b, c, -b]] + 4*Bet1*SpinConnection[-a, -c, -b]*CD[c][TetradPerturbation[a, b]] + Bet1*CD[-c][TetradPerturbation[-b, d]]*CD[c][TetradPerturbation[-d, b]] + Bet1*CD[-c][TetradPerturbation[d, -b]]*CD[c][TetradPerturbation[-d, b]] + (2*Alp6*CD[-b][SpinConnection[a, b, -a]]*CD[-d][SpinConnection[c, d, -c]])/3 +
    4*Bet1*CD[b][TetradPerturbation[a, -a]]*CD[-d][TetradPerturbation[-b, d]] - 2*Bet1*CD[-b][TetradPerturbation[-c, b]]*CD[-d][TetradPerturbation[c, d]] - Bet1*CD[c][TetradPerturbation[-z, b]]*CD[z][TetradPerturbation[-b, -c]] - Bet1*CD[c][TetradPerturbation[-z, b]]*CD[z][TetradPerturbation[-c, -b]] + Bet1*CD[c][TetradPerturbation[-d, -z]]*CD[z][TetradPerturbation[d, -c]] - Bet1*CD[c][TetradPerturbation[-z, -d]]*CD[z][TetradPerturbation[d, -c]], TheoryName → "MinimalEvenScalar", Method → "Hard", MaxLaurentDepth → 3];
```



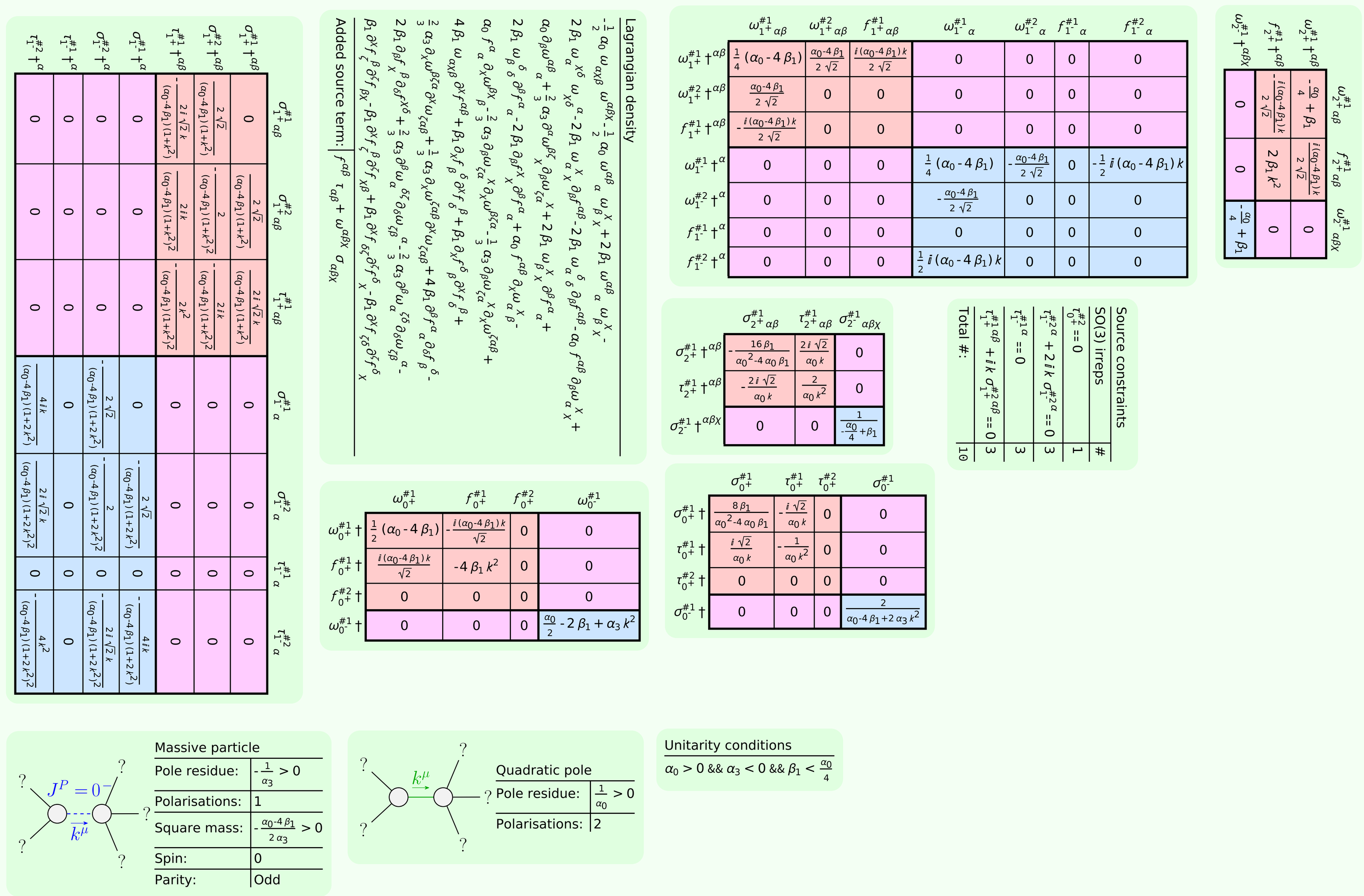
Thus we see that only the odd-parity scalar mode is moving with a mass, as claimed.

# Minimal massive odd-parity scalar model

We will study the minimal model set out in Eq. (4.25) of arXiv:9902032.

$$-\frac{1}{2}\alpha_0\,\mathcal{R}^{\alpha\beta}{}_{\alpha\beta}+\frac{1}{6}\alpha_3\,\mathcal{R}_{\alpha\beta\chi\delta}\,\mathcal{R}^{\alpha\beta\chi\delta}-\frac{2}{3}\alpha_3\,\mathcal{R}_{\alpha\chi\beta\delta}\,\mathcal{R}^{\alpha\beta\chi\delta}+\frac{1}{6}\alpha_3\,\mathcal{R}^{\alpha\beta\chi\delta}\,\mathcal{R}_{\chi\delta\alpha\beta}+\frac{1}{2}\beta_1\,\mathcal{T}_{\alpha\beta\chi}\,\mathcal{T}^{\alpha\beta\chi}+\beta_1\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\alpha\chi}+2\beta_1\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\,\mathcal{T}^{\chi}{}_{\beta\chi} \tag{31}$$

```
ParticleSpectrum[-1/2*(Alp0*SpinConnection[-a, -c, -b]*SpinConnection[a, b, c]) - (Alp0*SpinConnection[a, b, -a]*SpinConnection[-b, c, -c])/2 + 2*Bet1*SpinConnection[a, b, -a]*SpinConnection[-b, c, -c] - 2*Bet1*SpinConnection[-a, c, d]*SpinConnection[-c, -d, a] - Alp0*TetradPerturbation[a, b]*CD[-b][SpinConnection[-a, c, -c]] + Alp0*CD[-b][SpinConnection[a, b, -a]] + (2*Alp3*CD[a][SpinConnection[b, z, -c]]*CD[-b][SpinConnection[-z, -a, c]])/3 - 2*Bet1*SpinConnection[-a, c, -c]*CD[-b][TetradPerturbation[a, b]] -
    2*Bet1*SpinConnection[-a, d, -d]*CD[-b][TetradPerturbation[a, b]] + 2*Bet1*SpinConnection[-b, c, -c]*CD[b][TetradPerturbation[a, -a]] + 2*Bet1*SpinConnection[-b, d, -d]*CD[b][TetradPerturbation[a, -a]] - 2*Bet1*CD[-b][TetradPerturbation[c, -c]]*CD[b][TetradPerturbation[a, -a]] + Alp0*TetradPerturbation[a, b]*CD[-c][SpinConnection[-a, c, -b]] - Alp0*TetradPerturbation[a, -a]*CD[-c][SpinConnection[b, c, -b]] - (2*Alp3*CD[-b][SpinConnection[-z, -a, c]]*CD[-c][SpinConnection[b, z, a]])/3 - (Alp3*CD[-b][SpinConnection[-z, -a, c]]*CD[-c][SpinConnection[z, a, b]])/3 +
    (2*Alp3*CD[-c][SpinConnection[b, z, a]]*CD[c][SpinConnection[-z, -a, -b]])/3 + (Alp3*CD[-c][SpinConnection[z, a, b]]*CD[c][SpinConnection[-z, -a, -b]])/3 + 4*Bet1*SpinConnection[-a, -c, -b]*CD[c][TetradPerturbation[a, b]] + Bet1*CD[-c][TetradPerturbation[-b, d]]*CD[c][TetradPerturbation[-d, b]] + Bet1*CD[-c][TetradPerturbation[d, -b]]*CD[c][TetradPerturbation[-d, b]] + (2*Alp3*CD[b][SpinConnection[-a, d, z]]*CD[-d][SpinConnection[-z, -b, a]])/3 - (2*Alp3*CD[b][SpinConnection[-a, z, d]]*CD[-d][SpinConnection[-z, -b, a]])/3 +
    4*Bet1*CD[b][TetradPerturbation[a, -a]]*CD[-d][TetradPerturbation[-b, d]] - 2*Bet1*CD[-b][TetradPerturbation[-c, b]]*CD[-d][TetradPerturbation[c, d]] - Bet1*CD[c][TetradPerturbation[-z, b]]*CD[z][TetradPerturbation[-b, -c]] - Bet1*CD[c][TetradPerturbation[-z, b]]*CD[z][TetradPerturbation[-c, -b]] + Bet1*CD[c][TetradPerturbation[-d, -z]]*CD[z][TetradPerturbation[d, -c]] - Bet1*CD[c][TetradPerturbation[-z, -d]]*CD[z][TetradPerturbation[d, -c]], TheoryName → "MinimalMassiveOddScalar", Method → "Hard", MaxLaurentDepth → 3];
```



Thus we see that only the even-parity scalar mode is moving with a mass, as claimed.

# Minimal massless odd-parity scalar model

We will study the minimal model set out between Eqs. (4.47) and (4.48) of arXiv:9902032.

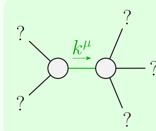
$$-2\beta_1\,\mathcal{R}^{\alpha\beta}{}_{\alpha\beta}+\frac{1}{6}\alpha_3\,\mathcal{R}_{\alpha\beta\chi\delta}\,\mathcal{R}^{\alpha\beta\chi\delta}-\frac{2}{3}\alpha_3\,\mathcal{R}_{\alpha\chi\beta\delta}\,\mathcal{R}^{\alpha\beta\chi\delta}+\frac{1}{6}\alpha_3\,\mathcal{R}^{\alpha\beta\chi\delta}\,\mathcal{R}_{\chi\delta\alpha\beta}+\frac{1}{2}\beta_1\,\mathcal{T}_{\alpha\beta\chi}\,\mathcal{T}^{\alpha\beta\chi}+\beta_1\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\alpha\chi}+2\beta_1\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\,\mathcal{T}^{\chi}{}_{\beta\chi} \tag{32}$$

```
ParticleSpectrum[-2 * Bet1 * SpinConnection[-a, -c, -b] * SpinConnection[a, b, c] - 2 * Bet1 * SpinConnection[-a, c, d] * SpinConnection[-c, -d, a] - 4 * Bet1 * TetradPerturbation[a, b] * CD[-b][SpinConnection[-a, c, -c]] + 4 * Bet1 * CD[-b][SpinConnection[a, b, -a]] + (2 * Alp3 * CD[a][SpinConnection[b, z, -c]] * CD[-b][SpinConnection[-z, -a, c]]) / 3 - 2 * Bet1 * SpinConnection[-a, c, -c] * CD[-b][TetradPerturbation[a, b]] - 2 * Bet1 * SpinConnection[-a, d, -d] * CD[-b][TetradPerturbation[a, b]] +
  2 * Bet1 * SpinConnection[-b, c, -c] * CD[b][TetradPerturbation[a, -a]] + 2 * Bet1 * SpinConnection[-b, d, -d] * CD[b][TetradPerturbation[a, -a]] - 2 * Bet1 * CD[-b][TetradPerturbation[c, -c]] * CD[b][TetradPerturbation[a, -a]] - 4 * Bet1 * TetradPerturbation[a, -a] * CD[-c][SpinConnection[b, c, -b]] - (2 * Alp3 * CD[-b][SpinConnection[-z, -a, c]] * CD[-c][SpinConnection[b, z, a]]) / 3 - (Alp3 * CD[-b][SpinConnection[-z, -a, c]] * CD[-c][SpinConnection[z, a, b]]) / 3 +
  (2 * Alp3 * CD[-c][SpinConnection[b, z, a]] * CD[c][SpinConnection[-z, -a, -b]]) / 3 + (Alp3 * CD[-c][SpinConnection[z, a, b]] * CD[c][SpinConnection[-z, -a, -b]]) / 3 + 4 * Bet1 * SpinConnection[-a, -c, -b] * CD[c][TetradPerturbation[a, b]] + Bet1 * CD[-c][TetradPerturbation[-b, d]] * CD[c][TetradPerturbation[d, -b]] + Bet1 * CD[c][TetradPerturbation[-d, b]] * CD[c][TetradPerturbation[d, -b]] + (2 * Alp3 * CD[b][SpinConnection[-a, z, d]] * CD[-d][SpinConnection[-z, -b, a]]) / 3 + (2 * Alp3 * CD[d][SpinConnection[-a, z, d]] * CD[-d][SpinConnection[-z, -b, a]]) / 3 +
  4 * Bet1 * CD[b][TetradPerturbation[a, -a]] * CD[-d][TetradPerturbation[-b, d]] - 2 * Bet1 * CD[-b][TetradPerturbation[-c, b]] * CD[-d][TetradPerturbation[c, d]] - Bet1 * CD[c][TetradPerturbation[-z, b]] * CD[z][TetradPerturbation[-c, -b]] - Bet1 * CD[c][TetradPerturbation[-z, b]] * CD[z][TetradPerturbation[-c, -b]] + Bet1 * CD[c][TetradPerturbation[-d, -z]] * CD[z][TetradPerturbation[d, -c]] - Bet1 * CD[c][TetradPerturbation[-z, -d]] * CD[z][TetradPerturbation[d, -c]], TheoryName → "MinimalMasslessOddScalar", Method → "Hard", MaxLaurentDepth → 3];
```

Lagrangian density [illegible]

Added source term: [illegible]

Source constraints: [illegible]



| Quadratic pole | |
|---|---|
| Pole residue: | $\frac{1}{\beta_1} > 0$ |
| Polarisations: | 2 |

Unitarity conditions

$\beta_1 > 0$

(No massive particles)

Thus we see that only the Einstein graviton is moving, with no extra species whatever. This might seem strange, since we have taken the massless limit of the pseudoscalar mode, but the fact that the spectrum empties discontinuously in this scenario is already predicted in the Hamiltonian analysis, see the second paragraph on page 20 of arXiv:9902032.

## Most general PGT

We want to study the most general PGT. We will do this using the general coupling coefficients defined in Eq. (26).

$$-\frac{1}{2}\alpha_0 \mathcal{R}^{\alpha\beta}{}_{\alpha\beta} + \frac{1}{6}(2\alpha_1 + 3\alpha_2 + \alpha_3)\mathcal{R}_{\alpha\beta\chi\delta}\mathcal{R}^{\alpha\beta\chi\delta} + \frac{2}{3}(\alpha_1 - \alpha_3)\mathcal{R}_{\alpha\chi\beta\delta}\mathcal{R}^{\alpha\beta\chi\delta} + (-\alpha_1 - \alpha_2 + \alpha_4 + \alpha_5)\mathcal{R}^{\alpha}{}_{\beta}\mathcal{R}^{\beta}{}_{\alpha} + \frac{1}{6}(2\alpha_1 - 3\alpha_2 + \alpha_3)\mathcal{R}^{\alpha\beta\chi\delta}\mathcal{R}_{\chi\delta\alpha\beta} + (-\alpha_1 + \alpha_2 + \alpha_4 - \alpha_5)\mathcal{R}^{\alpha\beta}\mathcal{R}_{\beta\alpha} + \frac{1}{6}(2\alpha_1 - 3\alpha_4 + \alpha_6)\mathcal{R}^{2} + \frac{1}{2}(2\beta_1 + \beta_3)\mathcal{T}_{\alpha\beta\chi}\mathcal{T}^{\alpha\beta\chi} + \frac{2}{3}(\beta_1 - \beta_3)\mathcal{T}^{\alpha\beta\chi}\mathcal{T}_{\beta\alpha\chi} + \frac{2}{3}(\beta_1 - \beta_2)\mathcal{T}^{\alpha}\mathcal{T}_{\alpha} \quad (33)$$

```
ParticleSpectrum[-1 / 2 * (Alp0 * SpinConnection[-a, -c, -b] * SpinConnection[a, b, c]) - (Alp0 * SpinConnection[a, b, -a] * SpinConnection[-b, c, -c]) / 2 + (2 * Bet1 * SpinConnection[a, b, -a] * SpinConnection[-b, c, -c]) / 3 - (2 * Bet2 * SpinConnection[a, b, -a] * SpinConnection[-b, c, -c]) / 3 - (2 * Bet1 * SpinConnection[-a, c, d] * SpinConnection[-c, -d, a]) / 3 + (8 * Bet3 * SpinConnection[-a, c, d] * SpinConnection[-c, -d, a]) / 3 + (2 * Bet1 * SpinConnection[-c, -d, a] * SpinConnection[c, d, -a]) / 3 + (4 * Bet3 * SpinConnection[-c, -d, a] * SpinConnection[c, d, -a]) / 3 + Alp1 * CD[-a][SpinConnection[c, d, -c]] * CD[a][SpinConnection[-d, z, -z]] +
  Alp2 * CD[-a][SpinConnection[c, d, -c]] * CD[a][SpinConnection[-d, z, -z]] - Alp4 * CD[-a][SpinConnection[c, d, -c]] * CD[a][SpinConnection[-d, z, -z]] - Alp5 * CD[-a][SpinConnection[c, d, -c]] * CD[a][SpinConnection[-d, z, -z]] - Alp0 * TetradPerturbation[a, b] * CD[-b][SpinConnection[-a, c, -c]] + Alp0 * CD[-b][SpinConnection[a, b, -a]] - (2 * Alp1 * CD[a][SpinConnection[b, z, -c]] * CD[-b][SpinConnection[-z, -a, c]]) / 3 + (2 * Alp3 * CD[a][SpinConnection[b, z, -c]] * CD[-b][SpinConnection[-z, -a, c]]) / 3 - (2 * Bet1 * SpinConnection[-a, c, -c] * CD[-b][TetradPerturbation[a, b]]) / 3 +
  (2 * Bet2 * SpinConnection[-a, c, -c] * CD[-b][TetradPerturbation[a, b]]) / 3 - (2 * Bet1 * SpinConnection[-a, d, -d] * CD[-b][TetradPerturbation[a, b]]) / 3 + (2 * Bet2 * SpinConnection[-a, d, -d] * CD[-b][TetradPerturbation[a, b]]) / 3 + (2 * Bet1 * SpinConnection[-b, c, -c] * CD[b][TetradPerturbation[a, -a]]) / 3 - (2 * Bet2 * SpinConnection[-b, c, -c] * CD[b][TetradPerturbation[a, -a]]) / 3 + (2 * Bet1 * SpinConnection[-b, d, -d] * CD[b][TetradPerturbation[a, -a]]) / 3 - (2 * Bet2 * SpinConnection[-b, d, -d] * CD[b][TetradPerturbation[a, -a]]) / 3 - (2 * Bet1 * CD[-b][TetradPerturbation[c, -c]] * CD[b][TetradPerturbation[a, -a]]) / 3 +
  (2 * Bet2 * CD[-b][TetradPerturbation[c, -c]] * CD[b][TetradPerturbation[a, -a]]) / 3 + Alp0 * TetradPerturbation[a, -a] * CD[-c][SpinConnection[b, c, -b]] - Alp0 * TetradPerturbation[a, -a] * CD[-c][SpinConnection[b, c, -b]] + Alp1 * CD[-b][SpinConnection[-d, z, -z]] * CD[-c][SpinConnection[b, c, d]] - Alp2 * CD[-b][SpinConnection[-d, z, -z]] * CD[-c][SpinConnection[b, c, d]] - Alp4 * CD[-b][SpinConnection[-d, z, -z]] * CD[-c][SpinConnection[b, c, d]] + Alp5 * CD[-b][SpinConnection[-d, z, -z]] * CD[-c][SpinConnection[b, c, d]] + (2 * Alp1 * CD[-b][SpinConnection[-z, -a, c]] * CD[-c][SpinConnection[z, a, b]]) / 3 -
  (2 * Alp3 * CD[-b][SpinConnection[-z, -a, c]] * CD[-c][SpinConnection[z, a, b]]) / 3 - 2 * Alp1 * CD[-b][SpinConnection[-d, z, -z]] * CD[c][SpinConnection[c, d, b]] - 2 * Alp2 * CD[-b][SpinConnection[-d, z, -z]] * CD[c][SpinConnection[c, d, b]] + 2 * Alp4 * CD[-b][SpinConnection[-d, z, -z]] * CD[c][SpinConnection[c, d, b]] + 2 * Alp5 * CD[-b][SpinConnection[-d, z, -z]] * CD[-c][SpinConnection[c, d, b]] - (2 * Alp1 * CD[-b][SpinConnection[-z, -a, c]] * CD[-c][SpinConnection[z, a, b]]) / 3 - Alp2 * CD[-b][SpinConnection[-z, -a, c]] * CD[-c][SpinConnection[z, a, b]] - (Alp3 * CD[-b][SpinConnection[-z, -a, c]] * CD[-c][SpinConnection[z, a, b]]) / 3 -
  (2 * Alp1 * CD[-c][SpinConnection[b, z, a]] * CD[c][SpinConnection[-z, -a, -b]]) / 3 + (2 * Alp3 * CD[-c][SpinConnection[b, z, a]] * CD[c][SpinConnection[-z, -a, -b]]) / 3 + (2 * Alp1 * CD[-c][SpinConnection[z, a, b]] * CD[c][SpinConnection[-z, -a, -b]]) / 3 + Alp2 * CD[-c][SpinConnection[z, a, b]] * CD[c][SpinConnection[-z, -a, -b]] + (Alp3 * CD[-c][SpinConnection[z, a, b]] * CD[c][SpinConnection[-z, -a, -b]]) / 3 + (2 * Bet1 * SpinConnection[-a, -c, -b] * CD[c][TetradPerturbation[a, b]]) / 3 + (8 * Bet1 * SpinConnection[-a, -c, -b] * CD[c][TetradPerturbation[a, b]]) / 3 +
  (8 * Bet3 * SpinConnection[-a, -c, -b] * CD[c][TetradPerturbation[a, b]]) / 3 - (2 * Bet1 * SpinConnection[-b, -a, -c] * CD[c][TetradPerturbation[a, b]]) / 3 - (4 * Bet3 * SpinConnection[-b, -a, -c] * CD[c][TetradPerturbation[a, b]]) / 3 + (4 * Bet1 * SpinConnection[-b, -c, -a] * CD[c][TetradPerturbation[a, b]]) / 3 + (8 * Bet3 * SpinConnection[-b, -c, -a] * CD[c][TetradPerturbation[a, b]]) / 3 + (2 * Bet1 * CD[-c][TetradPerturbation[-b, d]] * CD[c][TetradPerturbation[d, -b]]) / 3 + (4 * Bet1 * CD[-c][TetradPerturbation[d, -b]] * CD[c][TetradPerturbation[-d, b]]) / 3 -
  (2 * Bet3 * CD[-c][TetradPerturbation[d, -b]] * CD[c][TetradPerturbation[-d, b]]) / 3 + (4 * Alp1 * CD[-b][SpinConnection[a, b, -a]] * CD[-d][SpinConnection[c, d, -c]]) / 3 - 2 * Alp4 * CD[-b][SpinConnection[a, b, -a]] * CD[-d][SpinConnection[c, d, -c]] + (2 * Alp6 * CD[-b][SpinConnection[a, b, -a]] * CD[-d][SpinConnection[c, d, -c]]) / 3 + (4 * Alp1 * CD[b][SpinConnection[-a, d, z]] * CD[-d][SpinConnection[-z, -b, a]]) / 3 - 2 * Alp2 * CD[b][SpinConnection[-a, d, z]] * CD[-d][SpinConnection[-z, -b, a]] + (2 * Alp3 * CD[b][SpinConnection[-a, d, z]] * CD[-d][SpinConnection[-z, -b, a]]) / 3 +
  (2 * Alp1 * CD[b][SpinConnection[-a, z, d]] * CD[-d][SpinConnection[-z, -b, a]]) / 3 - (2 * Alp3 * CD[b][SpinConnection[-a, z, d]] * CD[-d][SpinConnection[-z, -b, a]]) / 3 + (4 * Bet1 * CD[b][TetradPerturbation[a, -a]] * CD[-d][TetradPerturbation[-b, d]]) / 3 - (4 * Bet2 * CD[b][TetradPerturbation[a, -a]] * CD[-d][TetradPerturbation[-b, d]]) / 3 - (2 * Bet1 * CD[-b][TetradPerturbation[-c, b]] * CD[-d][TetradPerturbation[c, d]]) / 3 + (2 * Bet2 * CD[-b][TetradPerturbation[-c, b]] * CD[-d][TetradPerturbation[c, d]]) / 3 - Alp1 * CD[-b][SpinConnection[c, d, -c]] * CD[-z][SpinConnection[-d, z, -b]] + Alp2 * CD[-b][SpinConnection[c, d, -c]] * CD[-z][SpinConnection[-d, z, -b]] +
  Alp4 * CD[-b][SpinConnection[-d, z, -z]] * CD[d][SpinConnection[b, c, -c]] - Alp5 * CD[-b][SpinConnection[-d, z, -z]] * CD[d][SpinConnection[b, c, -c]] - Alp1 * CD[-c][SpinConnection[b, c, d]] * CD[-z][SpinConnection[-d, z, -b]] + Alp2 * CD[-c][SpinConnection[b, c, d]] * CD[-z][SpinConnection[-d, z, -b]] + Alp4 * CD[-c][SpinConnection[b, c, d]] * CD[-z][SpinConnection[-d, z, -b]] - Alp5 * CD[-c][SpinConnection[b, c, d]] * CD[-z][SpinConnection[-d, z, -b]] + Alp1 * CD[-c][SpinConnection[c, d, b]] * CD[-z][SpinConnection[-d, z, -b]] + Alp2 * CD[-c][SpinConnection[c, d, b]] * CD[-z][SpinConnection[-d, z, -b]] -
  Alp4 * CD[-c][SpinConnection[c, d, b]] * CD[-z][SpinConnection[-d, z, -b]] - Alp5 * CD[-c][SpinConnection[c, d, b]] * CD[-z][SpinConnection[-d, z, -b]] + Alp1 * CD[d][SpinConnection[b, c, -c]] * CD[-z][SpinConnection[-d, z, -b]] - Alp2 * CD[d][SpinConnection[b, c, -c]] * CD[-z][SpinConnection[-d, z, -b]] - Alp4 * CD[d][SpinConnection[b, c, -c]] * CD[-z][SpinConnection[-d, z, -b]] + Alp5 * CD[d][SpinConnection[b, c, -c]] * CD[-z][SpinConnection[-d, z, -b]] - (2 * Bet1 * CD[c][TetradPerturbation[-z, b]] * CD[z][TetradPerturbation[-c, -b]]) / 3 + (2 * Bet3 * CD[c][TetradPerturbation[-z, b]] * CD[z][TetradPerturbation[-c, -b]]) / 3 -
  (4 * Bet1 * CD[c][TetradPerturbation[-z, b]] * CD[z][TetradPerturbation[-c, -b]]) / 3 - (2 * Bet3 * CD[c][TetradPerturbation[-z, b]] * CD[z][TetradPerturbation[-c, -b]]) / 3 + (2 * Bet1 * CD[c][TetradPerturbation[-d, -z]] * CD[z][TetradPerturbation[d, -c]]) / 3 - (2 * Bet3 * CD[c][TetradPerturbation[-d, -z]] * CD[z][TetradPerturbation[d, -c]]) / 3 - (2 * Bet1 * CD[c][TetradPerturbation[-z, -d]] * CD[z][TetradPerturbation[d, -c]]) / 3 + (2 * Bet3 * CD[c][TetradPerturbation[-z, -d]] * CD[z][TetradPerturbation[d, -c]]) / 3, TheoryName → "GeneralPGT", Method → "Hard", MaxLaurentDepth → 3];
```

| | $\sigma^{\#1}_{1^+\,\alpha\beta}$ | $\sigma^{\#2}_{1^+\,\alpha\beta}$ | $\tau^{\#1}_{1^+\,\alpha\beta}$ | $\sigma^{\#1}_{1^-\,\alpha}$ | $\sigma^{\#2}_{1^-\,\alpha}$ | $\tau^{\#1}_{1^-\,\alpha}$ | $\tau^{\#2}_{1^-\,\alpha}$ |
|---|---|---|---|---|---|---|---|
| $\sigma^{\#1}_{1^+}\dagger^{\alpha\beta}$ | $\frac{1}{-\frac{3(\alpha_0-4\beta_1)(\alpha_0+8\beta_3)}{16(\beta_1+2\beta_3)}+(\alpha_2+\alpha_5)k^2}$ | $-\frac{2\sqrt{2}(3\alpha_0-4\beta_1+16\beta_3)}{(1+k^2)(-3(\alpha_0-4\beta_1)(\alpha_0+8\beta_3)+16(\alpha_2+\alpha_5)(\beta_1+2\beta_3)k^2)}$ | $-\frac{2i\sqrt{2}(3\alpha_0-4\beta_1+16\beta_3)k}{(1+k^2)(-3(\alpha_0-4\beta_1)(\alpha_0+8\beta_3)+16(\alpha_2+\alpha_5)(\beta_1+2\beta_3)k^2)}$ | 0 | 0 | 0 | 0 |
| $\sigma^{\#2}_{1^+}\dagger^{\alpha\beta}$ | $-\frac{2\sqrt{2}(3\alpha_0-4\beta_1+16\beta_3)}{(1+k^2)(-3(\alpha_0-4\beta_1)(\alpha_0+8\beta_3)+16(\alpha_2+\alpha_5)(\beta_1+2\beta_3)k^2)}$ | $\frac{6\alpha_0+8(\beta_1+8\beta_3+3(\alpha_2+\alpha_5)k^2)}{(1+k^2)^2(-3(\alpha_0-4\beta_1)(\alpha_0+8\beta_3)+16(\alpha_2+\alpha_5)(\beta_1+2\beta_3)k^2)}$ | $\frac{2ik(3\alpha_0+4(\beta_1+8\beta_3+3(\alpha_2+\alpha_5)k^2))}{(1+k^2)^2(-3(\alpha_0-4\beta_1)(\alpha_0+8\beta_3)+16(\alpha_2+\alpha_5)(\beta_1+2\beta_3)k^2)}$ | 0 | 0 | 0 | 0 |
| $\tau^{\#1}_{1^+}\dagger^{\alpha\beta}$ | $\frac{2i\sqrt{2}(3\alpha_0-4\beta_1+16\beta_3)k}{(1+k^2)(-3(\alpha_0-4\beta_1)(\alpha_0+8\beta_3)+16(\alpha_2+\alpha_5)(\beta_1+2\beta_3)k^2)}$ | $-\frac{2ik(3\alpha_0+4(\beta_1+8\beta_3+3(\alpha_2+\alpha_5)k^2))}{(1+k^2)^2(-3(\alpha_0-4\beta_1)(\alpha_0+8\beta_3)+16(\alpha_2+\alpha_5)(\beta_1+2\beta_3)k^2)}$ | $\frac{2k^2(3\alpha_0+4(\beta_1+8\beta_3+3(\alpha_2+\alpha_5)k^2))}{(1+k^2)^2(-3(\alpha_0-4\beta_1)(\alpha_0+8\beta_3)+16(\alpha_2+\alpha_5)(\beta_1+2\beta_3)k^2)}$ | 0 | 0 | 0 | 0 |
| $\sigma^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{1}{-\frac{3(\alpha_0-4\beta_1)(\alpha_0+2\beta_2)}{8(2\beta_1+\beta_2)}+(\alpha_4+\alpha_5)k^2}$ | $\frac{2\sqrt{2}(3\alpha_0-4\beta_1+4\beta_2)}{(1+2k^2)(-3(\alpha_0-4\beta_1)(\alpha_0+2\beta_2)+8(\alpha_4+\alpha_5)(2\beta_1+\beta_2)k^2)}$ | 0 | $\frac{4i(3\alpha_0-4\beta_1+4\beta_2)k}{(1+2k^2)(-3(\alpha_0-4\beta_1)(\alpha_0+2\beta_2)+8(\alpha_4+\alpha_5)(2\beta_1+\beta_2)k^2)}$ |
| $\sigma^{\#2}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{2\sqrt{2}(3\alpha_0-4\beta_1+4\beta_2)}{(1+2k^2)(-3(\alpha_0-4\beta_1)(\alpha_0+2\beta_2)+8(\alpha_4+\alpha_5)(2\beta_1+\beta_2)k^2)}$ | $\frac{6\alpha_0+8(\beta_1+2\beta_2+3(\alpha_4+\alpha_5)k^2)}{(1+2k^2)^2(-3(\alpha_0-4\beta_1)(\alpha_0+2\beta_2)+8(\alpha_4+\alpha_5)(2\beta_1+\beta_2)k^2)}$ | 0 | $\frac{2i\sqrt{2}k(3\alpha_0+4(\beta_1+2\beta_2+3(\alpha_4+\alpha_5)k^2))}{(1+2k^2)^2(-3(\alpha_0-4\beta_1)(\alpha_0+2\beta_2)+8(\alpha_4+\alpha_5)(2\beta_1+\beta_2)k^2)}$ |
| $\tau^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\tau^{\#2}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $-\frac{4i(3\alpha_0-4\beta_1+4\beta_2)k}{(1+2k^2)(-3(\alpha_0-4\beta_1)(\alpha_0+2\beta_2)+8(\alpha_4+\alpha_5)(2\beta_1+\beta_2)k^2)}$ | $-\frac{2i\sqrt{2}k(3\alpha_0+4(\beta_1+2\beta_2+3(\alpha_4+\alpha_5)k^2))}{(1+2k^2)^2(-3(\alpha_0-4\beta_1)(\alpha_0+2\beta_2)+8(\alpha_4+\alpha_5)(2\beta_1+\beta_2)k^2)}$ | 0 | $\frac{4k^2(3\alpha_0+4(\beta_1+2\beta_2+3(\alpha_4+\alpha_5)k^2))}{(1+2k^2)^2(-3(\alpha_0-4\beta_1)(\alpha_0+2\beta_2)+8(\alpha_4+\alpha_5)(2\beta_1+\beta_2)k^2)}$ |

Lagrangian density [illegible]

Added source term: [illegible]

| | $\sigma^{\#1}_{0^+}$ | $\tau^{\#1}_{0^+}$ | $\tau^{\#2}_{0^+}$ | $\sigma^{\#1}_{0^-}$ |
|---|---|---|---|---|
| $\sigma^{\#1}_{0^+}\dagger$ | $-\frac{4\beta_2}{\alpha_0^2+2\alpha_0\beta_2-4(\alpha_4+\alpha_6)\beta_2k^2}$ | $\frac{i\sqrt{2}(\alpha_0+2\beta_2)}{-\alpha_0(\alpha_0+2\beta_2)k+4(\alpha_4+\alpha_6)\beta_2k^3}$ | 0 | 0 |
| $\tau^{\#1}_{0^+}\dagger$ | $\frac{i\sqrt{2}(\alpha_0+2\beta_2)}{\alpha_0(\alpha_0+2\beta_2)k-4(\alpha_4+\alpha_6)\beta_2k^3}$ | $\frac{\frac{\alpha_0}{2}+\beta_2+(\alpha_4+\alpha_6)k^2}{-\frac{1}{2}\alpha_0(\alpha_0+2\beta_2)k^2+2(\alpha_4+\alpha_6)\beta_2k^4}$ | 0 | 0 |
| $\tau^{\#2}_{0^+}\dagger$ | 0 | 0 | 0 | 0 |
| $\sigma^{\#1}_{0^-}\dagger$ | 0 | 0 | 0 | $\frac{2}{\alpha_0+8\beta_3+2(\alpha_2+\alpha_3)k^2}$ |

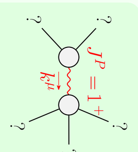
| | $\sigma^{\#1}_{2^+\,\alpha\beta}$ | $\tau^{\#1}_{2^+\,\alpha\beta}$ | $\sigma^{\#1}_{2^-\,\alpha\beta\chi}$ |
|---|---|---|---|
| $\sigma^{\#1}_{2^+}\dagger^{\alpha\beta}$ | $\frac{16\beta_1}{-\alpha_0^2+4\alpha_0\beta_1+16(\alpha_1+\alpha_4)\beta_1k^2}$ | $\frac{2i\sqrt{2}(\alpha_0-4\beta_1)}{\alpha_0(\alpha_0-4\beta_1)k-16(\alpha_1+\alpha_4)\beta_1k^3}$ | 0 |
| $\tau^{\#1}_{2^+}\dagger^{\alpha\beta}$ | $-\frac{2i\sqrt{2}(\alpha_0-4\beta_1)}{\alpha_0(\alpha_0-4\beta_1)k-16(\alpha_1+\alpha_4)\beta_1k^3}$ | $\frac{2(\alpha_0-4(\beta_1+(\alpha_1+\alpha_4)k^2))}{k^2(\alpha_0^2-4\alpha_0\beta_1-16(\alpha_1+\alpha_4)\beta_1k^2)}$ | 0 |
| $\sigma^{\#1}_{2^-}\dagger^{\alpha\beta\chi}$ | 0 | 0 | $\frac{1}{-\frac{\alpha_0}{4}+\beta_1+(\alpha_1+\alpha_2)k^2}$ |

Source constraints

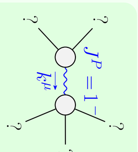
| SO(3) irreps | # |
|---|---|
| $\tau^{\#2}_{0^+} == 0$ | 1 |
| $\tau^{\#1}_{1^+}{}^{\alpha\beta} + 2ik\,\sigma^{\#2}_{1^-}{}^{\alpha} == 0$ | 3 |
| $\tau^{\#1}_{1^-}{}^{\alpha} == 0$ | 3 |
| $\tau^{\#1}_{1^+}{}^{\alpha\beta} + ik\,\sigma^{\#2}_{1^+}{}^{\alpha\beta} == 0$ | 3 |
| Total #: | 10 |

[illegible]

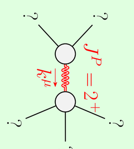
| | $\omega^{\#1}_{1^+\,\alpha\beta}$ | $\omega^{\#2}_{1^+\,\alpha\beta}$ | $f^{\#1}_{1^+\,\alpha\beta}$ | $\omega^{\#1}_{1^-\,\alpha}$ | $\omega^{\#2}_{1^-\,\alpha}$ | $f^{\#1}_{1^-\,\alpha}$ | $f^{\#2}_{1^-\,\alpha}$ |
|---|---|---|---|---|---|---|---|
| $\omega^{\#1}_{1^+}\dagger^{\alpha\beta}$ | $\frac{\alpha_0}{4}+\frac{1}{3}(\beta_1+8\beta_3)+(\alpha_2+\alpha_5)k^2$ | $\frac{3\alpha_0-4\beta_1+16\beta_3}{6\sqrt{2}}$ | $\frac{i(3\alpha_0-4\beta_1+16\beta_3)k}{6\sqrt{2}}$ | 0 | 0 | 0 | 0 |
| $\omega^{\#2}_{1^+}\dagger^{\alpha\beta}$ | $\frac{3\alpha_0-4\beta_1+16\beta_3}{6\sqrt{2}}$ | $\frac{2}{3}(\beta_1+2\beta_3)$ | $\frac{2}{3}i(\beta_1+2\beta_3)k$ | 0 | 0 | 0 | 0 |
| $f^{\#1}_{1^+}\dagger^{\alpha\beta}$ | $-\frac{i(3\alpha_0-4\beta_1+16\beta_3)k}{6\sqrt{2}}$ | $-\frac{2}{3}i(\beta_1+2\beta_3)k$ | $\frac{2}{3}(\beta_1+2\beta_3)k^2$ | 0 | 0 | 0 | 0 |
| $\omega^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{\alpha_0}{4}+\frac{1}{3}(\beta_1+2\beta_2)+(\alpha_4+\alpha_5)k^2$ | $-\frac{3\alpha_0-4\beta_1+4\beta_2}{6\sqrt{2}}$ | 0 | $-\frac{1}{6}i(3\alpha_0-4\beta_1+4\beta_2)k$ |
| $\omega^{\#2}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $-\frac{3\alpha_0-4\beta_1+4\beta_2}{6\sqrt{2}}$ | $\frac{1}{3}(2\beta_1+\beta_2)$ | 0 | $\frac{1}{3}i\sqrt{2}(2\beta_1+\beta_2)k$ |
| $f^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $f^{\#2}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{1}{6}i(3\alpha_0-4\beta_1+4\beta_2)k$ | $-\frac{1}{3}i\sqrt{2}(2\beta_1+\beta_2)k$ | 0 | $\frac{2}{3}(2\beta_1+\beta_2)k^2$ |



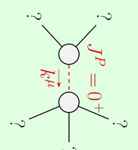

| Massive particle | |
|---|---|
| Pole residue: | [illegible] |
| Polarisations: | 3 |
| Square mass: | [illegible] |
| Spin: | 1 |
| Parity: | Even |



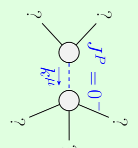

| Massive particle | |
|---|---|
| Pole residue: | [illegible] |
| Polarisations: | 3 |
| Square mass: | [illegible] |
| Spin: | 1 |
| Parity: | Odd |



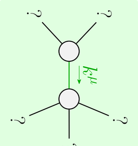

| Massive particle | |
|---|---|
| Pole residue: | [illegible] |
| Polarisations: | 5 |
| Square mass: | [illegible] |
| Spin: | 2 |
| Parity: | Even |




| Massive particle | |
|---|---|
| Pole residue: | $\frac{1}{\alpha_0} + \frac{\alpha_4+\alpha_6+2\beta_2}{2\alpha_4\beta_2+2\alpha_6\beta_2} > 0$ |
| Polarisations: | 1 |
| Square mass: | [illegible] |
| Spin: | 0 |
| Parity: | Even |



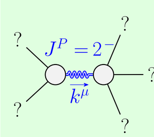

| Massive particle | |
|---|---|
| Pole residue: | $-\frac{1}{\alpha_2+\alpha_3} > 0$ |
| Polarisations: | 1 |
| Square mass: | $-\frac{\alpha_0+8\beta_3}{2(\alpha_2+\alpha_3)} > 0$ |
| Spin: | 0 |
| Parity: | Odd |



| Quadratic pole | |
|---|---|
| Pole residue: | $\frac{1}{\alpha_0} > 0$ |
| Polarisations: | 2 |

Unitarity conditions

(Unitarity is demonstrably impossible)




| Massive particle | |
|---|---|
| Pole residue: | $-\frac{1}{\alpha_1+\alpha_2} > 0$ |
| Polarisations: | 5 |
| Square mass: | $\frac{\alpha_0-4\beta_1}{4(\alpha_1+\alpha_2)} > 0$ |
| Spin: | 2 |
| Parity: | Odd |

These results should be compared with the Hayashi and Shirafuji papers, in particular Eqs. (4.11) in Prog. Theor. Phys. 64 (1980) 2222.

## Singh-Hagen theory

We want to study the theories which are considered in Phys. Rev. D. 9, 898 (1974) and arXiv:1902.05118

We define the higher-spin field:

```
DefField[HigherSpinField[-a, -b, -c], Symmetric[{-a, -b, -c}], PrintAs → "h", PrintSourceAs → "𝓕"];
```

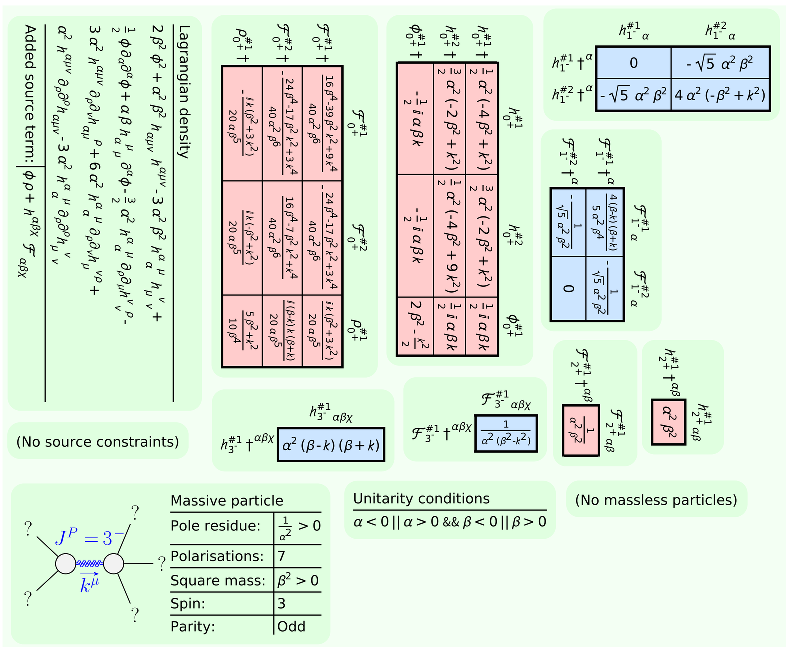
| Fundamental field | Symmetries | Decomposition into SO(3) irrep(s) | Source |
|---|---|---|---|
| $h_{\alpha\beta\chi}$ | Symmetry[3, $h^{\bullet1\bullet2\bullet3}$, {●1 → -a, ●2 → -b, ●3 → -c}, StrongGenSet[{1, 2, 3}, GenSet[(1,2), (2,3)]]] | $h^{\#1}_{3^-}{}_{\alpha\beta\chi} + \frac{1}{15}\eta_{\beta\chi}h^{\#2}_{1^-}{}_{\alpha} + \frac{1}{15}\eta_{\alpha\chi}h^{\#2}_{1^-}{}_{\beta} + \frac{1}{15}\eta_{\alpha\beta}h^{\#2}_{1^-}{}_{\chi} + \frac{1}{9}\eta_{\beta\chi}h^{\#2}_{0^+}n_{\alpha} + \frac{1}{3}h^{\#1}_{2^+}{}_{\beta\chi}n_{\alpha} + \frac{1}{9}\eta_{\alpha\chi}h^{\#2}_{0^+}n_{\beta} + \frac{1}{3}h^{\#1}_{2^+}{}_{\alpha\chi}n_{\beta} - \frac{1}{15}h^{\#2}_{1^-}{}_{\chi}n_{\alpha}n_{\beta} + \frac{1}{3}h^{\#1}_{1^-}{}_{\chi}n_{\alpha}n_{\beta} + \frac{1}{9}\eta_{\alpha\beta}h^{\#2}_{0^+}n_{\chi} + \frac{1}{3}h^{\#1}_{2^+}{}_{\alpha\beta}n_{\chi} - \frac{1}{15}h^{\#2}_{1^-}{}_{\beta}n_{\alpha}n_{\chi} + \frac{1}{3}h^{\#1}_{1^-}{}_{\beta}n_{\alpha}n_{\chi} - \frac{1}{15}h^{\#2}_{1^-}{}_{\alpha}n_{\beta}n_{\chi} + \frac{1}{3}h^{\#1}_{1^-}{}_{\alpha}n_{\beta}n_{\chi} - \frac{1}{3}h^{\#2}_{0^+}n_{\alpha}n_{\beta}n_{\chi} + h^{\#1}_{0^+}n_{\alpha}n_{\beta}n_{\chi}$ | $\mathcal{F}_{\alpha\beta\chi}$ |
| SO(3) irrep | Symmetries | Expansion in terms of the fundamental field | Source SO(3) irrep |
| $h^{\#1}_{0^+}$ | Symmetry[0, $h^{\#1}_{0^+}$, {}, StrongGenSet[{}, GenSet[]]] | $h_{\alpha\beta\chi}n^{\alpha}n^{\beta}n^{\chi}$ | $\mathcal{F}^{\#1}_{0^+}$ |
| $h^{\#2}_{0^+}$ | Symmetry[0, $h^{\#2}_{0^+}$, {}, StrongGenSet[{}, GenSet[]]] | $3h_{\alpha}{}^{\beta}{}_{\beta}n^{\alpha} - 3h_{\alpha\beta\chi}n^{\alpha}n^{\beta}n^{\chi}$ | $\mathcal{F}^{\#2}_{0^+}$ |
| $h^{\#1}_{1^-}{}_{\alpha}$ | Symmetry[1, $h^{\#1\bullet1}_{1^-}$, {●1 → -a}, StrongGenSet[{}, GenSet[]]] | $3h_{\alpha\beta\chi}n^{\beta}n^{\chi} - 3h_{\beta\chi\delta}n_{\alpha}n^{\beta}n^{\chi}n^{\delta}$ | $\mathcal{F}^{\#1}_{1^-}{}_{\alpha}$ |
| $h^{\#2}_{1^-}{}_{\alpha}$ | Symmetry[1, $h^{\#2\bullet1}_{1^-}$, {●1 → -a}, StrongGenSet[{}, GenSet[]]] | $3h_{\alpha}{}^{\beta}{}_{\beta} - 3h_{\beta}{}^{\chi}{}_{\chi}n_{\alpha}n^{\beta} - 3h_{\alpha\beta\chi}n^{\beta}n^{\chi} + 3h_{\beta\chi\delta}n_{\alpha}n^{\beta}n^{\chi}n^{\delta}$ | $\mathcal{F}^{\#2}_{1^-}{}_{\alpha}$ |
| $h^{\#1}_{2^+}{}_{\alpha\beta}$ | Symmetry[2, $h^{\#1\bullet1\bullet2}_{2^+}$, {●1 → -a, ●2 → -b}, StrongGenSet[{1, 2}, GenSet[(1,2)]]] | $3h_{\alpha\beta\chi}n^{\chi} - \eta_{\alpha\beta}h_{\chi}{}^{\delta}{}_{\delta}n^{\chi} + h_{\chi}{}^{\delta}{}_{\delta}n_{\alpha}n_{\beta}n^{\chi} - 3h_{\beta\chi\delta}n_{\alpha}n^{\chi}n^{\delta} - 3h_{\alpha\chi\delta}n_{\beta}n^{\chi}n^{\delta} + \eta_{\alpha\beta}h_{\chi\delta\epsilon}n^{\chi}n^{\delta}n^{\epsilon} + 2h_{\chi\delta\epsilon}n_{\alpha}n_{\beta}n^{\chi}n^{\delta}n^{\epsilon}$ | $\mathcal{F}^{\#1}_{2^+}{}_{\alpha\beta}$ |
| $h^{\#1}_{3^-}{}_{\alpha\beta\chi}$ | Symmetry[3, $h^{\#1\bullet1\bullet2\bullet3}_{3^-}$, {●1 → -a, ●2 → -b, ●3 → -c}, StrongGenSet[{1, 2, 3}, GenSet[(1,2), (2,3)]]] | $h_{\alpha\beta\chi} - \frac{1}{5}\eta_{\beta\chi}h_{\alpha}{}^{\delta}{}_{\delta} - \frac{1}{5}\eta_{\alpha\chi}h_{\beta}{}^{\delta}{}_{\delta} - \frac{1}{5}\eta_{\alpha\beta}h_{\chi}{}^{\delta}{}_{\delta} + \frac{1}{5}h_{\chi}{}^{\delta}{}_{\delta}n_{\alpha}n_{\beta} + \frac{1}{5}h_{\beta}{}^{\delta}{}_{\delta}n_{\alpha}n_{\chi} + \frac{1}{5}h_{\alpha}{}^{\delta}{}_{\delta}n_{\beta}n_{\chi} - h_{\beta\chi\delta}n_{\alpha}n^{\delta} + \frac{1}{5}\eta_{\beta\chi}h_{\delta}{}^{\epsilon}{}_{\epsilon}n_{\alpha}n^{\delta} - h_{\alpha\chi\delta}n_{\beta}n^{\delta} + \frac{1}{5}\eta_{\alpha\chi}h_{\delta}{}^{\epsilon}{}_{\epsilon}n_{\beta}n^{\delta} - h_{\alpha\beta\delta}n_{\chi}n^{\delta} + \frac{1}{5}\eta_{\alpha\beta}h_{\delta}{}^{\epsilon}{}_{\epsilon}n_{\chi}n^{\delta} - \frac{3}{5}h_{\delta}{}^{\epsilon}{}_{\epsilon}n_{\alpha}n_{\beta}n_{\chi}n^{\delta} + \frac{1}{5}\eta_{\beta\chi}h_{\alpha\delta\epsilon}n^{\delta}n^{\epsilon} + \frac{1}{5}\eta_{\alpha\chi}h_{\beta\delta\epsilon}n^{\delta}n^{\epsilon} + \frac{1}{5}\eta_{\alpha\beta}h_{\chi\delta\epsilon}n^{\delta}n^{\epsilon} + \frac{4}{5}h_{\chi\delta\epsilon}n_{\alpha}n_{\beta}n^{\delta}n^{\epsilon} + \frac{4}{5}h_{\beta\delta\epsilon}n_{\alpha}n_{\chi}n^{\delta}n^{\epsilon} + \frac{4}{5}h_{\alpha\delta\epsilon}n_{\beta}n_{\chi}n^{\delta}n^{\epsilon} - \frac{1}{5}\eta_{\beta\chi}h_{\delta\epsilon\phi}n_{\alpha}n^{\delta}n^{\epsilon}n^{\phi} - \frac{1}{5}\eta_{\alpha\chi}h_{\delta\epsilon\phi}n_{\beta}n^{\delta}n^{\epsilon}n^{\phi} - \frac{1}{5}\eta_{\alpha\beta}h_{\delta\epsilon\phi}n_{\chi}n^{\delta}n^{\epsilon}n^{\phi} - \frac{2}{5}h_{\delta\epsilon\phi}n_{\alpha}n_{\beta}n_{\chi}n^{\delta}n^{\epsilon}n^{\phi}$ | $\mathcal{F}^{\#1}_{3^-}{}_{\alpha\beta\chi}$ |

Then we define the Singh-Hagen model.

$$\alpha^2\beta^2 h_{\alpha\mu\nu}h^{\alpha\mu\nu} - 3\alpha^2\beta^2 h^{\alpha}{}_{\alpha}{}^{\mu}h_{\mu}{}^{\nu}{}_{\nu} + 2\beta^2\phi^2 + \frac{1}{2}\phi\,\partial_{\alpha}\partial^{\alpha}\phi + \alpha\beta\, h_{\alpha}{}^{\mu}{}_{\mu}\partial^{\alpha}\phi - \frac{3}{2}\alpha^2 h^{\alpha}{}_{\alpha}{}^{\mu}\partial_{\rho}\partial_{\mu}h^{\nu}{}_{\nu}{}^{\rho} - 3\alpha^2 h^{\alpha\mu\nu}\partial_{\rho}\partial_{\nu}h_{\alpha\mu}{}^{\rho} + 6\alpha^2 h^{\alpha}{}_{\alpha}{}^{\mu}\partial_{\rho}\partial_{\nu}h_{\mu}{}^{\nu\rho} + \alpha^2 h^{\alpha\mu\nu}\partial_{\rho}\partial^{\rho}h_{\alpha\mu\nu} - 3\alpha^2 h^{\alpha}{}_{\alpha}{}^{\mu}\partial_{\rho}\partial^{\rho}h_{\mu}{}^{\nu}{}_{\nu} \tag{34}$$

Now we try to compute the particle spectrum:

```
ParticleSpectrum[Coupling1^2 * Coupling2^2 * HigherSpinField[-a, -m, -n] * HigherSpinField[a, m, n] - 3 * Coupling1^2 * Coupling2^2 * HigherSpinField[a, -a, m] * HigherSpinField[-m, n, -n] + 2 * Coupling2^2 * ScalarField[]^2 + (ScalarField[] * CD[-a][CD[a][ScalarField[]]]) / 2 + Coupling1 * Coupling2 * HigherSpinField[-a, m, -m] * CD[a][ScalarField[]] -
  (3 * Coupling1^2 * HigherSpinField[a, -a, m] * CD[-r][CD[-m][HigherSpinField[n, -n, r]]]) / 2 - 3 * Coupling1^2 * HigherSpinField[a, m, n] * CD[-r][CD[-n][HigherSpinField[-a, -m, r]]] + 6 * Coupling1^2 * HigherSpinField[a, -a, m] * CD[-r][CD[-n][HigherSpinField[-m, n, r]]] + Coupling1^2 * HigherSpinField[a, m, n] * CD[-r][CD[r][HigherSpinField[-a, -m, -n]]] - 3 * Coupling1^2 * HigherSpinField[a, -a, m] * CD[-r][CD[r][HigherSpinField[-m, n, -n]]], TheoryName → "SinghHagenTheory", Method → "Easy", MaxLaurentDepth → 3];
```




# A symmetric rank-two tensor and an asymmetric rank-three tensor

This is the kinematic setup which is used in metric-affine gravity (MAG).

Let's define a connection field:

```
DefField[Connection[-a, -b, -c], PrintAs → "Γ", PrintSourceAs → "Δ"];
```

$\Gamma^{\#1}_{2^-}{}_{\alpha\beta\chi}$

$$\frac{2}{3}\Gamma^{\#1}_{2^-}{}_{\alpha\beta\chi} + \frac{1}{3}\Gamma^{\#1}_{2^-}{}_{\alpha\chi\beta} - \frac{1}{3}\Gamma^{\#1}_{2^-}{}_{\beta\chi\alpha}$$

··· DefField: The reduced–index mode Rank3AntiPara2m appears not to be invertible using the provided rules.

$\Gamma^{\#2}_{2^-\alpha\beta\chi}$

$\frac{2}{3}\Gamma^{\#2}_{2^-\alpha\beta\chi}+\frac{1}{3}\Gamma^{\#2}_{2^-\alpha\chi\beta}-\frac{1}{3}\Gamma^{\#2}_{2^-\beta\chi\alpha}$

··· DefField: The reduced–index mode Rank3SymmPara2m appears not to be invertible using the provided rules.

| Fundamental field | Symmetries | Decomposition into SO(3) irrep(s) | Source |
|---|---|---|---|
| $\Gamma_{\alpha\beta\chi}$ | Symmetry[3, $\Gamma^{\bullet1\bullet2\bullet3}$, {●1 → -a, ●2 → -b, ●3 → -c}, StrongGenSet[{}, GenSet[]]] | $-\frac{1}{2}\eta_{\alpha\chi}\Gamma^{\#1}_{1^-\beta}+\frac{1}{2}\eta_{\alpha\beta}\Gamma^{\#1}_{1^-\chi}+\frac{4}{3}\Gamma^{\#1}_{2^-\beta\chi\alpha}+\frac{1}{2}\Gamma^{\#2}_{2^-\alpha\beta\chi}+\frac{1}{2}\Gamma^{\#2}_{2^-\alpha\chi\beta}+\Gamma^{\#1}_{3^-\alpha\beta\chi}+\frac{1}{3}\eta_{\beta\chi}\Gamma^{\#6}_{1^-\alpha}-\frac{1}{6}\eta_{\alpha\chi}\Gamma^{\#6}_{1^-\beta}-\frac{1}{6}\eta_{\alpha\beta}\Gamma^{\#6}_{1^-\chi}+\frac{1}{15}\eta_{\beta\chi}\Gamma^{\#4}_{1^-\alpha}+\frac{1}{15}\eta_{\alpha\chi}\Gamma^{\#4}_{1^-\beta}+\frac{1}{15}\eta_{\alpha\beta}\Gamma^{\#4}_{1^-\chi}+\Gamma^{\#2}_{1^+\beta\chi}n_\alpha+\frac{1}{9}\eta_{\beta\chi}\Gamma^{\#3}_{0^+}n_\alpha+\frac{1}{3}\Gamma^{\#2}_{2^+\beta\chi}n_\alpha+\frac{2}{3}\Gamma^{\#3}_{2^+\beta\chi}n_\alpha+\frac{2}{9}\eta_{\beta\chi}\Gamma^{\#4}_{0^+}n_\alpha+\frac{1}{3}\eta_{\alpha\chi}\Gamma^{\#1}_{0^+}n_\beta-\Gamma^{\#1}_{1^+\alpha\chi}n_\beta+\Gamma^{\#1}_{2^+\alpha\chi}n_\beta+\frac{1}{9}\eta_{\alpha\chi}\Gamma^{\#3}_{0^+}n_\beta-\frac{1}{2}\Gamma^{\#3}_{1^+\alpha\chi}n_\beta+\frac{1}{3}\Gamma^{\#2}_{2^+\alpha\chi}n_\beta-\frac{1}{3}\Gamma^{\#3}_{2^+\alpha\chi}n_\beta-\frac{1}{9}\eta_{\alpha\chi}\Gamma^{\#4}_{0^+}n_\beta-\frac{1}{2}\Gamma^{\#1}_{1^-\chi}n_\alpha n_\beta-\Gamma^{\#2}_{1^-\chi}n_\alpha n_\beta+\frac{1}{6}\Gamma^{\#6}_{1^-\chi}n_\alpha n_\beta-\frac{1}{15}\Gamma^{\#4}_{1^-\chi}n_\alpha n_\beta-\frac{1}{3}\Gamma^{\#5}_{1^-\chi}n_\alpha n_\beta+\frac{1}{3}\Gamma^{\#3}_{1^-\chi}n_\alpha n_\beta-\frac{1}{3}\eta_{\alpha\beta}\Gamma^{\#1}_{0^+}n_\chi+\Gamma^{\#1}_{1^+\alpha\beta}n_\chi-\Gamma^{\#1}_{2^+\alpha\beta}n_\chi+\frac{1}{9}\eta_{\alpha\beta}\Gamma^{\#3}_{0^+}n_\chi-\frac{1}{2}\Gamma^{\#3}_{1^+\alpha\beta}n_\chi+\frac{1}{3}\Gamma^{\#2}_{2^+\alpha\beta}n_\chi-\frac{1}{3}\Gamma^{\#3}_{2^+\alpha\beta}n_\chi-\frac{1}{9}\eta_{\alpha\beta}\Gamma^{\#4}_{0^+}n_\chi+\frac{1}{2}\Gamma^{\#1}_{1^-\beta}n_\alpha n_\chi+\Gamma^{\#2}_{1^-\beta}n_\alpha n_\chi+\frac{1}{6}\Gamma^{\#6}_{1^-\beta}n_\alpha n_\chi-\frac{1}{15}\Gamma^{\#4}_{1^-\beta}n_\alpha n_\chi-\frac{1}{3}\Gamma^{\#5}_{1^-\beta}n_\alpha n_\chi+\frac{1}{3}\Gamma^{\#3}_{1^-\beta}n_\alpha n_\chi-\frac{1}{3}\Gamma^{\#6}_{1^-\alpha}n_\beta n_\chi-\frac{1}{15}\Gamma^{\#4}_{1^-\alpha}n_\beta n_\chi+\frac{2}{3}\Gamma^{\#5}_{1^-\alpha}n_\beta n_\chi+\frac{1}{3}\Gamma^{\#3}_{1^-\alpha}n_\beta n_\chi-\frac{1}{3}\Gamma^{\#3}_{0^+}n_\alpha n_\beta n_\chi+\Gamma^{\#2}_{0^+}n_\alpha n_\beta n_\chi-\frac{1}{6}\epsilon\eta_{\alpha\beta\chi\delta}\Gamma^{\#1}_{0^-}n^\delta$ | $\Delta_{\alpha\beta\chi}$ |
| **SO(3) irrep** | **Symmetries** | **Expansion in terms of the fundamental field** | **Source SO(3) irrep** |
| $\Gamma^{\#1}_{0^+}$ | Symmetry[0, $\Gamma^{\#1}_{0^+}$, {}, StrongGenSet[{}, GenSet[]]] | $-\frac{1}{2}\Gamma^{\alpha}{}_{\alpha}{}^{\beta}n_\beta+\frac{1}{2}\Gamma^{\alpha\beta}{}_{\alpha}n_\beta$ | $\Delta^{\#1}_{0^+}$ |
| $\Gamma^{\#2}_{0^+}$ | Symmetry[0, $\Gamma^{\#2}_{0^+}$, {}, StrongGenSet[{}, GenSet[]]] | $\Gamma^{\alpha\beta\chi}n_\alpha n_\beta n_\chi$ | $\Delta^{\#2}_{0^+}$ |
| $\Gamma^{\#3}_{0^+}$ | Symmetry[0, $\Gamma^{\#3}_{0^+}$, {}, StrongGenSet[{}, GenSet[]]] | $\Gamma^{\alpha\beta}{}_{\beta}n_\alpha+\Gamma^{\alpha}{}_{\alpha}{}^{\beta}n_\beta+\Gamma^{\alpha\beta}{}_{\alpha}n_\beta-3\Gamma^{\alpha\beta\chi}n_\alpha n_\beta n_\chi$ | $\Delta^{\#3}_{0^+}$ |
| $\Gamma^{\#4}_{0^+}$ | Symmetry[0, $\Gamma^{\#4}_{0^+}$, {}, StrongGenSet[{}, GenSet[]]] | $\Gamma^{\alpha\beta}{}_{\beta}n_\alpha-\frac{1}{2}\Gamma^{\alpha}{}_{\alpha}{}^{\beta}n_\beta-\frac{1}{2}\Gamma^{\alpha\beta}{}_{\alpha}n_\beta$ | $\Delta^{\#4}_{0^+}$ |
| $\Gamma^{\#1}_{0^-}$ | Symmetry[0, $\Gamma^{\#1}_{0^-}$, {}, StrongGenSet[{}, GenSet[]]] | $\epsilon\eta_{\alpha\beta\chi\delta}\Gamma^{\alpha\beta\chi}n^\delta$ | $\Delta^{\#1}_{0^-}$ |
| $\Gamma^{\#1}_{1^+\alpha\beta}$ | Symmetry[2, $\Gamma^{\#1\bullet1\bullet2}_{1^+}$, {●1 → -a, ●2 → -b}, StrongGenSet[{1, 2}, GenSet[-(1,2)]]] | $\frac{1}{4}\Gamma_{\alpha\beta}{}^{\chi}n_\chi-\frac{1}{4}\Gamma_{\alpha}{}^{\chi}{}_{\beta}n_\chi-\frac{1}{4}\Gamma_{\beta\alpha}{}^{\chi}n_\chi+\frac{1}{4}\Gamma_{\beta}{}^{\chi}{}_{\alpha}n_\chi-\frac{1}{4}\Gamma^{\chi}{}_{\beta}{}^{\delta}n_\alpha n_\chi n_\delta+\frac{1}{4}\Gamma^{\chi\delta}{}_{\beta}n_\alpha n_\chi n_\delta+\frac{1}{4}\Gamma^{\chi}{}_{\alpha}{}^{\delta}n_\beta n_\chi n_\delta-\frac{1}{4}\Gamma^{\chi\delta}{}_{\alpha}n_\beta n_\chi n_\delta$ | $\Delta^{\#1}_{1^+\alpha\beta}$ |
| $\Gamma^{\#2}_{1^+\alpha\beta}$ | Symmetry[2, $\Gamma^{\#2\bullet1\bullet2}_{1^+}$, {●1 → -a, ●2 → -b}, StrongGenSet[{1, 2}, GenSet[-(1,2)]]] | $\frac{1}{2}\Gamma^{\chi}{}_{\alpha\beta}n_\chi-\frac{1}{2}\Gamma^{\chi}{}_{\beta\alpha}n_\chi+\frac{1}{2}\Gamma^{\chi}{}_{\beta}{}^{\delta}n_\alpha n_\chi n_\delta-\frac{1}{2}\Gamma^{\chi\delta}{}_{\beta}n_\alpha n_\chi n_\delta-\frac{1}{2}\Gamma^{\chi}{}_{\alpha}{}^{\delta}n_\beta n_\chi n_\delta+\frac{1}{2}\Gamma^{\chi\delta}{}_{\alpha}n_\beta n_\chi n_\delta$ | $\Delta^{\#2}_{1^+\alpha\beta}$ |
| $\Gamma^{\#3}_{1^+\alpha\beta}$ | Symmetry[2, $\Gamma^{\#3\bullet1\bullet2}_{1^+}$, {●1 → -a, ●2 → -b}, StrongGenSet[{1, 2}, GenSet[-(1,2)]]] | $-\frac{1}{2}\Gamma_{\alpha\beta}{}^{\chi}n_\chi-\frac{1}{2}\Gamma_{\alpha}{}^{\chi}{}_{\beta}n_\chi+\frac{1}{2}\Gamma_{\beta\alpha}{}^{\chi}n_\chi+\frac{1}{2}\Gamma_{\beta}{}^{\chi}{}_{\alpha}n_\chi-\Gamma_{\beta}{}^{\chi\delta}n_\alpha n_\chi n_\delta+\frac{1}{2}\Gamma^{\chi}{}_{\beta}{}^{\delta}n_\alpha n_\chi n_\delta+\frac{1}{2}\Gamma^{\chi\delta}{}_{\beta}n_\alpha n_\chi n_\delta+\Gamma_{\alpha}{}^{\chi\delta}n_\beta n_\chi n_\delta-\frac{1}{2}\Gamma^{\chi}{}_{\alpha}{}^{\delta}n_\beta n_\chi n_\delta-\frac{1}{2}\Gamma^{\chi\delta}{}_{\alpha}n_\beta n_\chi n_\delta$ | $\Delta^{\#3}_{1^+\alpha\beta}$ |
| $\Gamma^{\#1}_{1^-\alpha}$ | Symmetry[1, $\Gamma^{\#1\bullet1}_{1^-}$, {●1 → -a}, StrongGenSet[{}, GenSet[]]] | $-\frac{1}{2}\Gamma^{\beta}{}_{\alpha\beta}+\frac{1}{2}\Gamma^{\beta}{}_{\beta\alpha}-\frac{1}{2}\Gamma^{\beta}{}_{\beta}{}^{\chi}n_\alpha n_\chi+\frac{1}{2}\Gamma^{\beta\chi}{}_{\beta}n_\alpha n_\chi+\frac{1}{2}\Gamma^{\beta}{}_{\alpha}{}^{\chi}n_\beta n_\chi-\frac{1}{2}\Gamma^{\beta\chi}{}_{\alpha}n_\beta n_\chi$ | $\Delta^{\#1}_{1^-\alpha}$ |
| $\Gamma^{\#2}_{1^-\alpha}$ | Symmetry[1, $\Gamma^{\#2\bullet1}_{1^-}$, {●1 → -a}, StrongGenSet[{}, GenSet[]]] | $\frac{1}{2}\Gamma^{\beta}{}_{\alpha}{}^{\chi}n_\beta n_\chi-\frac{1}{2}\Gamma^{\beta\chi}{}_{\alpha}n_\beta n_\chi$ | $\Delta^{\#2}_{1^-\alpha}$ |
| $\Gamma^{\#3}_{1^-\alpha}$ | Symmetry[1, $\Gamma^{\#3\bullet1}_{1^-}$, {●1 → -a}, StrongGenSet[{}, GenSet[]]] | $\Gamma_{\alpha}{}^{\beta\chi}n_\beta n_\chi+\Gamma^{\beta}{}_{\alpha}{}^{\chi}n_\beta n_\chi+\Gamma^{\beta\chi}{}_{\alpha}n_\beta n_\chi-3\Gamma^{\beta\chi\delta}n_\alpha n_\beta n_\chi n_\delta$ | $\Delta^{\#3}_{1^-\alpha}$ |
| $\Gamma^{\#4}_{1^-\alpha}$ | Symmetry[1, $\Gamma^{\#4\bullet1}_{1^-}$, {●1 → -a}, StrongGenSet[{}, GenSet[]]] | $\Gamma_{\alpha}{}^{\beta}{}_{\beta}+\Gamma^{\beta}{}_{\alpha\beta}+\Gamma^{\beta}{}_{\beta\alpha}-\Gamma^{\beta\chi}{}_{\chi}n_\alpha n_\beta-\Gamma^{\beta}{}_{\beta}{}^{\chi}n_\alpha n_\chi-\Gamma^{\beta\chi}{}_{\beta}n_\alpha n_\chi-\Gamma_{\alpha}{}^{\beta\chi}n_\beta n_\chi-\Gamma^{\beta}{}_{\alpha}{}^{\chi}n_\beta n_\chi-\Gamma^{\beta\chi}{}_{\alpha}n_\beta n_\chi+3\Gamma^{\beta\chi\delta}n_\alpha n_\beta n_\chi n_\delta$ | $\Delta^{\#4}_{1^-\alpha}$ |
| $\Gamma^{\#5}_{1^-\alpha}$ | Symmetry[1, $\Gamma^{\#5\bullet1}_{1^-}$, {●1 → -a}, StrongGenSet[{}, GenSet[]]] | $\Gamma_{\alpha}{}^{\beta\chi}n_\beta n_\chi-\frac{1}{2}\Gamma^{\beta}{}_{\alpha}{}^{\chi}n_\beta n_\chi-\frac{1}{2}\Gamma^{\beta\chi}{}_{\alpha}n_\beta n_\chi$ | $\Delta^{\#5}_{1^-\alpha}$ |
| $\Gamma^{\#6}_{1^-\alpha}$ | Symmetry[1, $\Gamma^{\#6\bullet1}_{1^-}$, {●1 → -a}, StrongGenSet[{}, GenSet[]]] | $\Gamma_{\alpha}{}^{\beta}{}_{\beta}-\frac{1}{2}\Gamma^{\beta}{}_{\alpha\beta}-\frac{1}{2}\Gamma^{\beta}{}_{\beta\alpha}-\Gamma^{\beta\chi}{}_{\chi}n_\alpha n_\beta+\frac{1}{2}\Gamma^{\beta}{}_{\beta}{}^{\chi}n_\alpha n_\chi+\frac{1}{2}\Gamma^{\beta\chi}{}_{\beta}n_\alpha n_\chi-\Gamma_{\alpha}{}^{\beta\chi}n_\beta n_\chi+\frac{1}{2}\Gamma^{\beta}{}_{\alpha}{}^{\chi}n_\beta n_\chi+\frac{1}{2}\Gamma^{\beta\chi}{}_{\alpha}n_\beta n_\chi$ | $\Delta^{\#6}_{1^-\alpha}$ |
| $\Gamma^{\#1}_{2^+\alpha\beta}$ | Symmetry[2, $\Gamma^{\#1\bullet1\bullet2}_{2^+}$, {●1 → -a, ●2 → -b}, StrongGenSet[{1, 2}, GenSet[(1,2)]]] | $-\frac{1}{4}\Gamma_{\alpha\beta}{}^{\chi}n_\chi+\frac{1}{4}\Gamma_{\alpha}{}^{\chi}{}_{\beta}n_\chi-\frac{1}{4}\Gamma_{\beta\alpha}{}^{\chi}n_\chi+\frac{1}{4}\Gamma_{\beta}{}^{\chi}{}_{\alpha}n_\chi+\frac{1}{6}\eta_{\alpha\beta}\Gamma^{\chi}{}_{\chi}{}^{\delta}n_\delta-\frac{1}{6}\eta_{\alpha\beta}\Gamma^{\chi\delta}{}_{\chi}n_\delta-\frac{1}{6}\Gamma^{\chi}{}_{\chi}{}^{\delta}n_\alpha n_\beta n_\delta+\frac{1}{6}\Gamma^{\chi\delta}{}_{\chi}n_\alpha n_\beta n_\delta+\frac{1}{4}\Gamma^{\chi}{}_{\beta}{}^{\delta}n_\alpha n_\chi n_\delta-\frac{1}{4}\Gamma^{\chi\delta}{}_{\beta}n_\alpha n_\chi n_\delta+\frac{1}{4}\Gamma^{\chi}{}_{\alpha}{}^{\delta}n_\beta n_\chi n_\delta-\frac{1}{4}\Gamma^{\chi\delta}{}_{\alpha}n_\beta n_\chi n_\delta$ | $\Delta^{\#1}_{2^+\alpha\beta}$ |
| $\Gamma^{\#2}_{2^+\alpha\beta}$ | Symmetry[2, $\Gamma^{\#2\bullet1\bullet2}_{2^+}$, {●1 → -a, ●2 → -b}, StrongGenSet[{1, 2}, GenSet[(1,2)]]] | $\frac{1}{2}\Gamma_{\alpha\beta}{}^{\chi}n_\chi+\frac{1}{2}\Gamma_{\alpha}{}^{\chi}{}_{\beta}n_\chi+\frac{1}{2}\Gamma_{\beta\alpha}{}^{\chi}n_\chi+\frac{1}{2}\Gamma_{\beta}{}^{\chi}{}_{\alpha}n_\chi+\frac{1}{2}\Gamma^{\chi}{}_{\alpha\beta}n_\chi+\frac{1}{2}\Gamma^{\chi}{}_{\beta\alpha}n_\chi-\frac{1}{3}\eta_{\alpha\beta}\Gamma^{\chi\delta}{}_{\delta}n_\chi+\frac{1}{3}\Gamma^{\chi\delta}{}_{\delta}n_\alpha n_\beta n_\chi-\frac{1}{3}\eta_{\alpha\beta}\Gamma^{\chi}{}_{\chi}{}^{\delta}n_\delta-\frac{1}{3}\eta_{\alpha\beta}\Gamma^{\chi\delta}{}_{\chi}n_\delta+\frac{1}{3}\Gamma^{\chi}{}_{\chi}{}^{\delta}n_\alpha n_\beta n_\delta+\frac{1}{3}\Gamma^{\chi\delta}{}_{\chi}n_\alpha n_\beta n_\delta-\Gamma_{\beta}{}^{\chi\delta}n_\alpha n_\chi n_\delta-\Gamma^{\chi}{}_{\beta}{}^{\delta}n_\alpha n_\chi n_\delta-\Gamma^{\chi\delta}{}_{\beta}n_\alpha n_\chi n_\delta-\Gamma_{\alpha}{}^{\chi\delta}n_\beta n_\chi n_\delta-\Gamma^{\chi}{}_{\alpha}{}^{\delta}n_\beta n_\chi n_\delta-\Gamma^{\chi\delta}{}_{\alpha}n_\beta n_\chi n_\delta+\eta_{\alpha\beta}\Gamma^{\chi\delta\epsilon}n_\chi n_\delta n_\epsilon+2\Gamma^{\chi\delta\epsilon}n_\alpha n_\beta n_\chi n_\delta n_\epsilon$ | $\Delta^{\#2}_{2^+\alpha\beta}$ |
| $\Gamma^{\#3}_{2^+\alpha\beta}$ | Symmetry[2, $\Gamma^{\#3\bullet1\bullet2}_{2^+}$, {●1 → -a, ●2 → -b}, StrongGenSet[{1, 2}, GenSet[(1,2)]]] | $-\frac{1}{4}\Gamma_{\alpha\beta}{}^{\chi}n_\chi-\frac{1}{4}\Gamma_{\alpha}{}^{\chi}{}_{\beta}n_\chi-\frac{1}{4}\Gamma_{\beta\alpha}{}^{\chi}n_\chi-\frac{1}{4}\Gamma_{\beta}{}^{\chi}{}_{\alpha}n_\chi+\frac{1}{2}\Gamma^{\chi}{}_{\alpha\beta}n_\chi+\frac{1}{2}\Gamma^{\chi}{}_{\beta\alpha}n_\chi-\frac{1}{3}\eta_{\alpha\beta}\Gamma^{\chi\delta}{}_{\delta}n_\chi+\frac{1}{3}\Gamma^{\chi\delta}{}_{\delta}n_\alpha n_\beta n_\chi+\frac{1}{6}\eta_{\alpha\beta}\Gamma^{\chi}{}_{\chi}{}^{\delta}n_\delta+\frac{1}{6}\eta_{\alpha\beta}\Gamma^{\chi\delta}{}_{\chi}n_\delta-\frac{1}{6}\Gamma^{\chi}{}_{\chi}{}^{\delta}n_\alpha n_\beta n_\delta-\frac{1}{6}\Gamma^{\chi\delta}{}_{\chi}n_\alpha n_\beta n_\delta+\frac{1}{2}\Gamma_{\beta}{}^{\chi\delta}n_\alpha n_\chi n_\delta-\frac{1}{4}\Gamma^{\chi}{}_{\beta}{}^{\delta}n_\alpha n_\chi n_\delta-\frac{1}{4}\Gamma^{\chi\delta}{}_{\beta}n_\alpha n_\chi n_\delta+\frac{1}{2}\Gamma_{\alpha}{}^{\chi\delta}n_\beta n_\chi n_\delta-\frac{1}{4}\Gamma^{\chi}{}_{\alpha}{}^{\delta}n_\beta n_\chi n_\delta-\frac{1}{4}\Gamma^{\chi\delta}{}_{\alpha}n_\beta n_\chi n_\delta$ | $\Delta^{\#3}_{2^+\alpha\beta}$ |
| $\Gamma^{\#1}_{2^-\alpha\beta\chi}$ | Symmetry[3, $\Gamma^{\#1\bullet1\bullet2\bullet3}_{2^-}$, {●1 → -a, ●2 → -b, ●3 → -c}, StrongGenSet[{1, 2}, GenSet[-(1,2)]]] | $-\frac{1}{8}\Gamma_{\alpha\beta\chi}+\frac{1}{8}\Gamma_{\alpha\chi\beta}+\frac{1}{8}\Gamma_{\beta\alpha\chi}-\frac{1}{8}\Gamma_{\beta\chi\alpha}+\frac{1}{4}\Gamma_{\chi\alpha\beta}-\frac{1}{4}\Gamma_{\chi\beta\alpha}-\frac{3}{16}\eta_{\beta\chi}\Gamma^{\delta}{}_{\alpha\delta}+\frac{3}{16}\eta_{\alpha\chi}\Gamma^{\delta}{}_{\beta\delta}+\frac{3}{16}\eta_{\beta\chi}\Gamma^{\delta}{}_{\delta\alpha}-\frac{3}{16}\eta_{\alpha\chi}\Gamma^{\delta}{}_{\delta\beta}-\frac{3}{16}\Gamma^{\delta}{}_{\beta\delta}n_\alpha n_\chi+\frac{3}{16}\Gamma^{\delta}{}_{\delta\beta}n_\alpha n_\chi+\frac{3}{16}\Gamma^{\delta}{}_{\alpha\delta}n_\beta n_\chi-\frac{3}{16}\Gamma^{\delta}{}_{\delta\alpha}n_\beta n_\chi+\frac{1}{8}\Gamma_{\beta\chi}{}^{\delta}n_\alpha n_\delta-\frac{1}{8}\Gamma_{\beta}{}^{\delta}{}_{\chi}n_\alpha n_\delta+\frac{1}{4}\Gamma_{\chi\beta}{}^{\delta}n_\alpha n_\delta-\frac{1}{4}\Gamma_{\chi}{}^{\delta}{}_{\beta}n_\alpha n_\delta+\frac{1}{8}\Gamma^{\delta}{}_{\beta\chi}n_\alpha n_\delta-\frac{1}{8}\Gamma^{\delta}{}_{\chi\beta}n_\alpha n_\delta-\frac{1}{8}\Gamma_{\alpha\chi}{}^{\delta}n_\beta n_\delta+\frac{1}{8}\Gamma_{\alpha}{}^{\delta}{}_{\chi}n_\beta n_\delta-\frac{1}{4}\Gamma_{\chi\alpha}{}^{\delta}n_\beta n_\delta+\frac{1}{4}\Gamma_{\chi}{}^{\delta}{}_{\alpha}n_\beta n_\delta-\frac{1}{8}\Gamma^{\delta}{}_{\alpha\chi}n_\beta n_\delta+\frac{1}{8}\Gamma^{\delta}{}_{\chi\alpha}n_\beta n_\delta+\frac{1}{8}\Gamma_{\alpha\beta}{}^{\delta}n_\chi n_\delta-\frac{1}{8}\Gamma_{\alpha}{}^{\delta}{}_{\beta}n_\chi n_\delta-\frac{1}{8}\Gamma_{\beta\alpha}{}^{\delta}n_\chi n_\delta+\frac{1}{8}\Gamma_{\beta}{}^{\delta}{}_{\alpha}n_\chi n_\delta-\frac{1}{4}\Gamma^{\delta}{}_{\alpha\beta}n_\chi n_\delta+\frac{1}{4}\Gamma^{\delta}{}_{\beta\alpha}n_\chi n_\delta-\frac{3}{16}\eta_{\beta\chi}\Gamma^{\delta}{}_{\delta}{}^{\epsilon}n_\alpha n_\epsilon+\frac{3}{16}\eta_{\beta\chi}\Gamma^{\delta\epsilon}{}_{\delta}n_\alpha n_\epsilon+\frac{3}{16}\eta_{\alpha\chi}\Gamma^{\delta}{}_{\delta}{}^{\epsilon}n_\beta n_\epsilon-\frac{3}{16}\eta_{\alpha\chi}\Gamma^{\delta\epsilon}{}_{\delta}n_\beta n_\epsilon+\frac{3}{16}\eta_{\beta\chi}\Gamma^{\delta}{}_{\alpha}{}^{\epsilon}n_\delta n_\epsilon-\frac{3}{16}\eta_{\alpha\chi}\Gamma^{\delta}{}_{\beta}{}^{\epsilon}n_\delta n_\epsilon-\frac{3}{16}\eta_{\beta\chi}\Gamma^{\delta\epsilon}{}_{\alpha}n_\delta n_\epsilon+\frac{3}{16}\eta_{\alpha\chi}\Gamma^{\delta\epsilon}{}_{\beta}n_\delta n_\epsilon-\frac{3}{16}\Gamma^{\delta}{}_{\beta}{}^{\epsilon}n_\alpha n_\chi n_\delta n_\epsilon+\frac{3}{16}\Gamma^{\delta\epsilon}{}_{\beta}n_\alpha n_\chi n_\delta n_\epsilon+\frac{3}{16}\Gamma^{\delta}{}_{\alpha}{}^{\epsilon}n_\beta n_\chi n_\delta n_\epsilon-\frac{3}{16}\Gamma^{\delta\epsilon}{}_{\alpha}n_\beta n_\chi n_\delta n_\epsilon$ | $\Delta^{\#1}_{2^-\alpha\beta\chi}$ |
| $\Gamma^{\#2}_{2^-\alpha\beta\chi}$ | Symmetry[3, $\Gamma^{\#2\bullet1\bullet2\bullet3}_{2^-}$, {●1 → -a, ●2 → -b, ●3 → -c}, StrongGenSet[{1, 2}, GenSet[-(1,2)]]] | $\frac{1}{3}\Gamma_{\alpha\beta\chi}+\frac{1}{3}\Gamma_{\alpha\chi\beta}-\frac{1}{3}\eta_{\beta\chi}\Gamma_{\alpha}{}^{\delta}{}_{\delta}-\frac{1}{3}\Gamma_{\beta\alpha\chi}-\frac{1}{3}\Gamma_{\beta\chi\alpha}+\frac{1}{3}\eta_{\alpha\chi}\Gamma_{\beta}{}^{\delta}{}_{\delta}+\frac{1}{6}\eta_{\beta\chi}\Gamma^{\delta}{}_{\alpha\delta}-\frac{1}{6}\eta_{\alpha\chi}\Gamma^{\delta}{}_{\beta\delta}+\frac{1}{6}\eta_{\beta\chi}\Gamma^{\delta}{}_{\delta\alpha}-\frac{1}{6}\eta_{\alpha\chi}\Gamma^{\delta}{}_{\delta\beta}-\frac{1}{3}\Gamma_{\beta}{}^{\delta}{}_{\delta}n_\alpha n_\chi+\frac{1}{6}\Gamma^{\delta}{}_{\beta\delta}n_\alpha n_\chi+\frac{1}{6}\Gamma^{\delta}{}_{\delta\beta}n_\alpha n_\chi+\frac{1}{3}\Gamma_{\alpha}{}^{\delta}{}_{\delta}n_\beta n_\chi-\frac{1}{6}\Gamma^{\delta}{}_{\alpha\delta}n_\beta n_\chi-\frac{1}{6}\Gamma^{\delta}{}_{\delta\alpha}n_\beta n_\chi+\frac{1}{3}\Gamma_{\beta\chi}{}^{\delta}n_\alpha n_\delta+\frac{1}{3}\Gamma_{\beta}{}^{\delta}{}_{\chi}n_\alpha n_\delta-\frac{1}{3}\Gamma^{\delta}{}_{\beta\chi}n_\alpha n_\delta-\frac{1}{3}\Gamma^{\delta}{}_{\chi\beta}n_\alpha n_\delta+\frac{1}{3}\eta_{\beta\chi}\Gamma^{\delta\epsilon}{}_{\epsilon}n_\alpha n_\delta-\frac{1}{3}\Gamma_{\alpha\chi}{}^{\delta}n_\beta n_\delta-\frac{1}{3}\Gamma_{\alpha}{}^{\delta}{}_{\chi}n_\beta n_\delta+\frac{1}{3}\Gamma^{\delta}{}_{\alpha\chi}n_\beta n_\delta+\frac{1}{3}\Gamma^{\delta}{}_{\chi\alpha}n_\beta n_\delta-\frac{1}{3}\eta_{\alpha\chi}\Gamma^{\delta\epsilon}{}_{\epsilon}n_\beta n_\delta-\frac{1}{3}\Gamma_{\alpha\beta}{}^{\delta}n_\chi n_\delta-\frac{1}{3}\Gamma_{\alpha}{}^{\delta}{}_{\beta}n_\chi n_\delta+\frac{1}{3}\Gamma_{\beta\alpha}{}^{\delta}n_\chi n_\delta+\frac{1}{3}\Gamma_{\beta}{}^{\delta}{}_{\alpha}n_\chi n_\delta-\frac{1}{6}\eta_{\beta\chi}\Gamma^{\delta}{}_{\delta}{}^{\epsilon}n_\alpha n_\epsilon-\frac{1}{6}\eta_{\beta\chi}\Gamma^{\delta\epsilon}{}_{\delta}n_\alpha n_\epsilon+\frac{1}{6}\eta_{\alpha\chi}\Gamma^{\delta}{}_{\delta}{}^{\epsilon}n_\beta n_\epsilon+\frac{1}{6}\eta_{\alpha\chi}\Gamma^{\delta\epsilon}{}_{\delta}n_\beta n_\epsilon+\frac{1}{3}\eta_{\beta\chi}\Gamma_{\alpha}{}^{\delta\epsilon}n_\delta n_\epsilon-\frac{1}{3}\eta_{\alpha\chi}\Gamma_{\beta}{}^{\delta\epsilon}n_\delta n_\epsilon-\frac{1}{6}\eta_{\beta\chi}\Gamma^{\delta}{}_{\alpha}{}^{\epsilon}n_\delta n_\epsilon+\frac{1}{6}\eta_{\alpha\chi}\Gamma^{\delta}{}_{\beta}{}^{\epsilon}n_\delta n_\epsilon-\frac{1}{6}\eta_{\beta\chi}\Gamma^{\delta\epsilon}{}_{\alpha}n_\delta n_\epsilon+\frac{1}{6}\eta_{\alpha\chi}\Gamma^{\delta\epsilon}{}_{\beta}n_\delta n_\epsilon-\frac{1}{3}\Gamma_{\beta}{}^{\delta\epsilon}n_\alpha n_\chi n_\delta n_\epsilon+\frac{1}{6}\Gamma^{\delta}{}_{\beta}{}^{\epsilon}n_\alpha n_\chi n_\delta n_\epsilon+\frac{1}{6}\Gamma^{\delta\epsilon}{}_{\beta}n_\alpha n_\chi n_\delta n_\epsilon+\frac{1}{3}\Gamma_{\alpha}{}^{\delta\epsilon}n_\beta n_\chi n_\delta n_\epsilon-\frac{1}{6}\Gamma^{\delta}{}_{\alpha}{}^{\epsilon}n_\beta n_\chi n_\delta n_\epsilon-\frac{1}{6}\Gamma^{\delta\epsilon}{}_{\alpha}n_\beta n_\chi n_\delta n_\epsilon$ | $\Delta^{\#2}_{2^-\alpha\beta\chi}$ |
| $\Gamma^{\#1}_{3^-\alpha\beta\chi}$ | Symmetry[3, $\Gamma^{\#1\bullet1\bullet2\bullet3}_{3^-}$, {●1 → -a, ●2 → -b, ●3 → -c}, StrongGenSet[{1, 2, 3}, GenSet[(1,2), (2,3)]]] | $\frac{1}{6}\Gamma_{\alpha\beta\chi}+\frac{1}{6}\Gamma_{\alpha\chi\beta}-\frac{1}{15}\eta_{\beta\chi}\Gamma_{\alpha}{}^{\delta}{}_{\delta}+\frac{1}{6}\Gamma_{\beta\alpha\chi}+\frac{1}{6}\Gamma_{\beta\chi\alpha}-\frac{1}{15}\eta_{\alpha\chi}\Gamma_{\beta}{}^{\delta}{}_{\delta}+\frac{1}{6}\Gamma_{\chi\alpha\beta}+\frac{1}{6}\Gamma_{\chi\beta\alpha}-\frac{1}{15}\eta_{\alpha\beta}\Gamma_{\chi}{}^{\delta}{}_{\delta}-\frac{1}{15}\eta_{\beta\chi}\Gamma^{\delta}{}_{\alpha\delta}-\frac{1}{15}\eta_{\alpha\chi}\Gamma^{\delta}{}_{\beta\delta}-\frac{1}{15}\eta_{\alpha\beta}\Gamma^{\delta}{}_{\chi\delta}-\frac{1}{15}\eta_{\beta\chi}\Gamma^{\delta}{}_{\delta\alpha}-\frac{1}{15}\eta_{\alpha\chi}\Gamma^{\delta}{}_{\delta\beta}-\frac{1}{15}\eta_{\alpha\beta}\Gamma^{\delta}{}_{\delta\chi}+\frac{1}{15}\Gamma_{\chi}{}^{\delta}{}_{\delta}n_\alpha n_\beta+\frac{1}{15}\Gamma^{\delta}{}_{\chi\delta}n_\alpha n_\beta+\frac{1}{15}\Gamma^{\delta}{}_{\delta\chi}n_\alpha n_\beta+\frac{1}{15}\Gamma_{\beta}{}^{\delta}{}_{\delta}n_\alpha n_\chi+\frac{1}{15}\Gamma^{\delta}{}_{\beta\delta}n_\alpha n_\chi+\frac{1}{15}\Gamma^{\delta}{}_{\delta\beta}n_\alpha n_\chi+\frac{1}{15}\Gamma_{\alpha}{}^{\delta}{}_{\delta}n_\beta n_\chi+\frac{1}{15}\Gamma^{\delta}{}_{\alpha\delta}n_\beta n_\chi+\frac{1}{15}\Gamma^{\delta}{}_{\delta\alpha}n_\beta n_\chi-\frac{1}{6}\Gamma_{\beta\chi}{}^{\delta}n_\alpha n_\delta-\frac{1}{6}\Gamma_{\beta}{}^{\delta}{}_{\chi}n_\alpha n_\delta-\frac{1}{6}\Gamma_{\chi\beta}{}^{\delta}n_\alpha n_\delta-\frac{1}{6}\Gamma_{\chi}{}^{\delta}{}_{\beta}n_\alpha n_\delta-\frac{1}{6}\Gamma^{\delta}{}_{\beta\chi}n_\alpha n_\delta-\frac{1}{6}\Gamma^{\delta}{}_{\chi\beta}n_\alpha n_\delta+\frac{1}{15}\eta_{\beta\chi}\Gamma^{\delta\epsilon}{}_{\epsilon}n_\alpha n_\delta-\frac{1}{6}\Gamma_{\alpha\chi}{}^{\delta}n_\beta n_\delta-\frac{1}{6}\Gamma_{\alpha}{}^{\delta}{}_{\chi}n_\beta n_\delta-\frac{1}{6}\Gamma_{\chi\alpha}{}^{\delta}n_\beta n_\delta-\frac{1}{6}\Gamma_{\chi}{}^{\delta}{}_{\alpha}n_\beta n_\delta-\frac{1}{6}\Gamma^{\delta}{}_{\alpha\chi}n_\beta n_\delta-\frac{1}{6}\Gamma^{\delta}{}_{\chi\alpha}n_\beta n_\delta+\frac{1}{15}\eta_{\alpha\chi}\Gamma^{\delta\epsilon}{}_{\epsilon}n_\beta n_\delta-\frac{1}{6}\Gamma_{\alpha\beta}{}^{\delta}n_\chi n_\delta-\frac{1}{6}\Gamma_{\alpha}{}^{\delta}{}_{\beta}n_\chi n_\delta-\frac{1}{6}\Gamma_{\beta\alpha}{}^{\delta}n_\chi n_\delta-\frac{1}{6}\Gamma_{\beta}{}^{\delta}{}_{\alpha}n_\chi n_\delta-\frac{1}{6}\Gamma^{\delta}{}_{\alpha\beta}n_\chi n_\delta-\frac{1}{6}\Gamma^{\delta}{}_{\beta\alpha}n_\chi n_\delta+\frac{1}{15}\eta_{\alpha\beta}\Gamma^{\delta\epsilon}{}_{\epsilon}n_\chi n_\delta-\frac{1}{5}\Gamma^{\delta\epsilon}{}_{\epsilon}n_\alpha n_\beta n_\chi n_\delta+\frac{1}{15}\eta_{\beta\chi}\Gamma^{\delta}{}_{\delta}{}^{\epsilon}n_\alpha n_\epsilon+\frac{1}{15}\eta_{\beta\chi}\Gamma^{\delta\epsilon}{}_{\delta}n_\alpha n_\epsilon+\frac{1}{15}\eta_{\alpha\chi}\Gamma^{\delta}{}_{\delta}{}^{\epsilon}n_\beta n_\epsilon+\frac{1}{15}\eta_{\alpha\chi}\Gamma^{\delta\epsilon}{}_{\delta}n_\beta n_\epsilon+\frac{1}{15}\eta_{\alpha\beta}\Gamma^{\delta}{}_{\delta}{}^{\epsilon}n_\chi n_\epsilon+\frac{1}{15}\eta_{\alpha\beta}\Gamma^{\delta\epsilon}{}_{\delta}n_\chi n_\epsilon-\frac{1}{5}\Gamma^{\delta}{}_{\delta}{}^{\epsilon}n_\alpha n_\beta n_\chi n_\epsilon-\frac{1}{5}\Gamma^{\delta\epsilon}{}_{\delta}n_\alpha n_\beta n_\chi n_\epsilon+\frac{1}{15}\eta_{\beta\chi}\Gamma_{\alpha}{}^{\delta\epsilon}n_\delta n_\epsilon+\frac{1}{15}\eta_{\alpha\chi}\Gamma_{\beta}{}^{\delta\epsilon}n_\delta n_\epsilon+\frac{1}{15}\eta_{\alpha\beta}\Gamma_{\chi}{}^{\delta\epsilon}n_\delta n_\epsilon+\frac{1}{15}\eta_{\beta\chi}\Gamma^{\delta}{}_{\alpha}{}^{\epsilon}n_\delta n_\epsilon+\frac{1}{15}\eta_{\alpha\chi}\Gamma^{\delta}{}_{\beta}{}^{\epsilon}n_\delta n_\epsilon+\frac{1}{15}\eta_{\alpha\beta}\Gamma^{\delta}{}_{\chi}{}^{\epsilon}n_\delta n_\epsilon+\frac{1}{15}\eta_{\beta\chi}\Gamma^{\delta\epsilon}{}_{\alpha}n_\delta n_\epsilon+\frac{1}{15}\eta_{\alpha\chi}\Gamma^{\delta\epsilon}{}_{\beta}n_\delta n_\epsilon+\frac{1}{15}\eta_{\alpha\beta}\Gamma^{\delta\epsilon}{}_{\chi}n_\delta n_\epsilon+\frac{4}{15}\Gamma_{\chi}{}^{\delta\epsilon}n_\alpha n_\beta n_\delta n_\epsilon+\frac{4}{15}\Gamma^{\delta}{}_{\chi}{}^{\epsilon}n_\alpha n_\beta n_\delta n_\epsilon+\frac{4}{15}\Gamma^{\delta\epsilon}{}_{\chi}n_\alpha n_\beta n_\delta n_\epsilon+\frac{4}{15}\Gamma_{\beta}{}^{\delta\epsilon}n_\alpha n_\chi n_\delta n_\epsilon+\frac{4}{15}\Gamma^{\delta}{}_{\beta}{}^{\epsilon}n_\alpha n_\chi n_\delta n_\epsilon+\frac{4}{15}\Gamma^{\delta\epsilon}{}_{\beta}n_\alpha n_\chi n_\delta n_\epsilon+\frac{4}{15}\Gamma_{\alpha}{}^{\delta\epsilon}n_\beta n_\chi n_\delta n_\epsilon+\frac{4}{15}\Gamma^{\delta}{}_{\alpha}{}^{\epsilon}n_\beta n_\chi n_\delta n_\epsilon+\frac{4}{15}\Gamma^{\delta\epsilon}{}_{\alpha}n_\beta n_\chi n_\delta n_\epsilon-\frac{1}{5}\eta_{\beta\chi}\Gamma^{\delta\epsilon\phi}n_\alpha n_\delta n_\epsilon n_\phi-\frac{1}{5}\eta_{\alpha\chi}\Gamma^{\delta\epsilon\phi}n_\beta n_\delta n_\epsilon n_\phi-\frac{1}{5}\eta_{\alpha\beta}\Gamma^{\delta\epsilon\phi}n_\chi n_\delta n_\epsilon n_\phi-\frac{2}{5}\Gamma^{\delta\epsilon\phi}n_\alpha n_\beta n_\chi n_\delta n_\epsilon n_\phi$ | $\Delta^{\#1}_{3^-\alpha\beta\chi}$ |

## Key observation: This time there are two negative-parity spin-two modes, so there are two error messages. As before, these do not signal a real problem.

We will define some more coupling constants:

```
DefConstantSymbol[A0, PrintAs → "\!\(\*SubscriptBox[\(a\), \(0\)]\)"]; DefConstantSymbol[A1, PrintAs → "\!\(\*SubscriptBox[\(a\), \(1\)]\)"]; DefConstantSymbol[A2, PrintAs → "\!\(\*SubscriptBox[\(a\), \(2\)]\)"]; DefConstantSymbol[A3, PrintAs → "\!\(\*SubscriptBox[\(a\), \(3\)]\)"]; DefConstantSymbol[A4, PrintAs → "\!\(\*SubscriptBox[\(a\), \(4\)]\)"]; DefConstantSymbol[A5, PrintAs → "\!\(\*SubscriptBox[\(a\), \(5\)]\)"]; DefConstantSymbol[A6, PrintAs → "\!\(\*SubscriptBox[\(a\), \(6\)]\)"];
DefConstantSymbol[A7, PrintAs → "\!\(\*SubscriptBox[\(a\), \(7\)]\)"]; DefConstantSymbol[A8, PrintAs → "\!\(\*SubscriptBox[\(a\), \(8\)]\)"]; DefConstantSymbol[A9, PrintAs → "\!\(\*SubscriptBox[\(a\), \(9\)]\)"]; DefConstantSymbol[A10, PrintAs → "\!\(\*SubscriptBox[\(a\), \(10\)]\)"]; DefConstantSymbol[A11, PrintAs → "\!\(\*SubscriptBox[\(a\), \(11\)]\)"]; DefConstantSymbol[C1, PrintAs → "\!\(\*SubscriptBox[\(c\), \(1\)]\)"]; DefConstantSymbol[C2, PrintAs → "\!\(\*SubscriptBox[\(c\), \(2\)]\)"];
DefConstantSymbol[C3, PrintAs → "\!\(\*SubscriptBox[\(c\), \(3\)]\)"]; DefConstantSymbol[C4, PrintAs → "\!\(\*SubscriptBox[\(c\), \(4\)]\)"]; DefConstantSymbol[C5, PrintAs → "\!\(\*SubscriptBox[\(c\), \(5\)]\)"]; DefConstantSymbol[C6, PrintAs → "\!\(\*SubscriptBox[\(c\), \(6\)]\)"]; DefConstantSymbol[C7, PrintAs → "\!\(\*SubscriptBox[\(c\), \(7\)]\)"]; DefConstantSymbol[C8, PrintAs → "\!\(\*SubscriptBox[\(c\), \(8\)]\)"]; DefConstantSymbol[C9, PrintAs → "\!\(\*SubscriptBox[\(c\), \(9\)]\)"];
DefConstantSymbol[C10, PrintAs → "\!\(\*SubscriptBox[\(c\), \(10\)]\)"]; DefConstantSymbol[C11, PrintAs → "\!\(\*SubscriptBox[\(c\), \(11\)]\)"]; DefConstantSymbol[C12, PrintAs → "\!\(\*SubscriptBox[\(c\), \(12\)]\)"]; DefConstantSymbol[C13, PrintAs → "\!\(\*SubscriptBox[\(c\), \(13\)]\)"]; DefConstantSymbol[C14, PrintAs → "\!\(\*SubscriptBox[\(c\), \(14\)]\)"]; DefConstantSymbol[C15, PrintAs → "\!\(\*SubscriptBox[\(c\), \(15\)]\)"]; DefConstantSymbol[C16, PrintAs → "\!\(\*SubscriptBox[\(c\), \(16\)]\)"];
```

Our conventions closely follow arXiv:1912.01023, and also arXiv:2110.14788.

First we write out rules which define the field strength tensors.

We want to define the curvature in Equation (2.1) on page 4 of arXiv:1912.01023.

$$\mathcal{F}_{\mu\nu}{}^{\rho}{}_{\sigma} \tag{35}$$

$$\Gamma^{\rho}_{\mu}{}_{\alpha}\Gamma^{\alpha}_{\nu}{}_{\sigma}-\Gamma^{\alpha}_{\mu}{}_{\sigma}\Gamma^{\rho}_{\nu}{}_{\alpha}+\partial_\mu\Gamma^{\rho}_{\nu}{}_{\sigma}-\partial_\nu\Gamma^{\rho}_{\mu}{}_{\sigma} \tag{36}$$

Next want to define the torsion in Equation (2.2) on page 5 of arXiv:1912.01023.

$$\mathcal{T}_{\mu}{}^{\alpha}{}_{\nu} \tag{37}$$

$$\Gamma_{\mu}{}^{\alpha}{}_{\nu}-\Gamma_{\nu}{}^{\alpha}{}_{\mu} \tag{38}$$

And finally the non-metricity in Equation (2.3) on page 5 of arXiv:1912.01023. Watch out for the trivial misprint in the arXiv paper. Note also, that since the non-metricity only appears via quadratic invariants we don't need to worry about perturbing the metric here.

$$Q_{\lambda\mu\nu} \tag{39}$$

$$\Gamma_{\lambda\mu\nu}+\Gamma_{\lambda\nu\mu}-\partial_\lambda h_{\mu\nu} \tag{40}$$

Now we compute the seven contractions defined in Equation (2.5) on page 5 of arXiv:1912.01023. Most of these contractions only appear in quadratic invariants, so again we only need these formulae to be accurate to first order in small quantities.

First comes the torsion contraction.

$$\mathcal{T}_{\mu} \tag{41}$$

$$\mathcal{T}_{\alpha}{}^{\alpha}{}_{\mu} \tag{42}$$

Next the (standard) non-metricity contraction.

$$Q_{\mu} \tag{43}$$

$$Q_{\mu\alpha}{}^{\alpha} \tag{44}$$

Next the (tilde) non-metricity contraction.

$$\tilde{Q}_{\mu} \tag{45}$$

$$Q_{\alpha}{}^{\alpha}{}_{\mu} \tag{46}$$

Next the homothetic curvature.

$$\mathcal{F}_{\mu\nu} \tag{47}$$

$$\mathcal{F}_{\mu\nu\alpha}{}^{\alpha} \tag{48}$$

Next the first of the pseudo-Ricci tensors.

$$\mathcal{F}^{(14)}{}_{\mu\nu} \tag{49}$$

$$\mathcal{F}_{\mu\alpha\nu}{}^{\alpha} \tag{50}$$

Next the second of the pseudo-Ricci tensors.

$$\mathcal{F}^{(13)}{}_{\mu\nu} \tag{51}$$

$$\mathcal{F}_{\alpha\mu}{}^{\alpha}{}_{\nu} \tag{52}$$

Now we move on to computing the (conventional) Ricci scalar. This time we need to be careful to retain contributions up to second order in smallness, since this is the only invariant which appears on its own.

$$\mathcal{F} \tag{53}$$

$$\left(\mathcal{F}_{\alpha\beta}{}^{\alpha}{}_{\chi}\right)\left(g^{\beta\chi}-h^{\beta\chi}\right) \tag{54}$$

Now we define a further two contractions by analogy to arXiv:2212.09820.

First the traceless, asymmetric average of the Ricci and co-Ricci tensors similar to that defined in Eq. (2.10) on page 4 of arXiv:2212.09820.

$$\hat{\mathcal{F}}_{\mu\nu} \tag{55}$$

$$\frac{1}{2}\mathcal{F}_{\alpha\mu}{}^{\alpha}{}_{\nu}+\frac{1}{2}\mathcal{F}_{\mu\alpha\nu}{}^{\alpha} \tag{56}$$

Next the antisymmetric projectively-invariant combination derived from Eq. (2.17) on page 5 of arXiv:2212.09820, and subject also to the parameter conditions set out in the paragraph at the top of page 6 in arXiv:2212.09820.

$$\check{\mathcal{F}}_{\mu\nu} \tag{57}$$

$$\frac{1}{4}\mathcal{F}_{\alpha\mu}{}^{\alpha}{}_{\nu}-\frac{1}{4}\mathcal{F}_{\mu\alpha\nu}{}^{\alpha}-\frac{1}{4}\mathcal{F}_{\alpha\nu}{}^{\alpha}{}_{\mu}+\frac{1}{4}\mathcal{F}_{\nu\alpha\mu}{}^{\alpha} \tag{58}$$

In order to check that we've defined the correct Ricci in Eqs. (57), and (58), we need to attempt a projective transformation.

Define a vector to generate the transformation.

$$\xi_{\mu} \tag{59}$$

Define also a perturbative parameter.

$$\epsilon \tag{60}$$

Now define the projective transformation itself. This is defined e.g. in Eq. (2.16) on page 7 of arXiv:1912.01023.

$$\Gamma_{\mu}{}^{\rho}{}_{\nu} \tag{61}$$

$$\Gamma_{\mu}{}^{\rho}{}_{\nu}+\epsilon\,\delta_{\nu}{}^{\rho}\,\xi_{\mu} \tag{62}$$

Now we have defined (both infinitesimal and finite, if we ignore the perturbative parameter) projective transformation in Eqs. (61), and (62), we check the transformation properties.

$$\check{\mathcal{F}}_{\mu\nu} \tag{63}$$

$$-\frac{1}{4}\Gamma_{\alpha\beta\nu}\Gamma_{\mu}{}^{\alpha\beta}-\frac{1}{4}\Gamma_{\beta\nu\alpha}\Gamma_{\mu}{}^{\alpha\beta}+\frac{1}{4}\Gamma^{\beta}{}_{\beta\alpha}\Gamma_{\mu}{}^{\alpha}{}_{\nu}+\frac{1}{4}\Gamma^{\beta}{}_{\alpha\beta}\Gamma_{\mu\nu}{}^{\alpha}+\frac{1}{4}\Gamma_{\alpha\beta\mu}\Gamma_{\nu}{}^{\alpha\beta}+\frac{1}{4}\Gamma_{\beta\mu\alpha}\Gamma_{\nu}{}^{\alpha\beta}-\frac{1}{4}\Gamma^{\beta}{}_{\beta\alpha}\Gamma_{\nu}{}^{\alpha}{}_{\mu}-\frac{1}{4}\Gamma^{\beta}{}_{\alpha\beta}\Gamma_{\nu\mu}{}^{\alpha}+\frac{1}{4}\partial_{\alpha}\Gamma_{\mu}{}^{\alpha}{}_{\nu}-\frac{1}{4}\partial_{\alpha}\Gamma_{\mu\nu}{}^{\alpha}-\frac{1}{4}\partial_{\alpha}\Gamma_{\nu}{}^{\alpha}{}_{\mu}+\frac{1}{4}\partial_{\alpha}\Gamma_{\nu\mu}{}^{\alpha}-\frac{1}{4}\partial_{\mu}\Gamma^{\alpha}{}_{\alpha\nu}+\frac{1}{4}\partial_{\mu}\Gamma^{\alpha}{}_{\nu\alpha}+\frac{1}{4}\partial_{\nu}\Gamma^{\alpha}{}_{\alpha\mu}-\frac{1}{4}\partial_{\nu}\Gamma^{\alpha}{}_{\mu\alpha} \tag{64}$$

$$\frac{1}{4}\left(\Gamma^{\beta}{}_{\beta\alpha}+\epsilon\,\eta_{\alpha\beta}\,\xi^{\beta}\right)\left(\Gamma_{\mu}{}^{\alpha}{}_{\nu}+\epsilon\,\delta_{\nu}{}^{\alpha}\,\xi_{\mu}\right)+\frac{1}{4}\left(\Gamma^{\beta}{}_{\alpha\beta}+\epsilon\,\eta_{\beta\alpha}\,\xi^{\beta}\right)\left(\Gamma_{\mu\nu}{}^{\alpha}+\epsilon\,\delta_{\nu}{}^{\alpha}\,\xi_{\mu}\right)-\frac{1}{4}\left(\Gamma_{\alpha\beta\nu}+\epsilon\,\eta_{\nu\beta}\,\xi_{\alpha}\right)\left(\Gamma_{\mu}{}^{\alpha\beta}+\epsilon\,\eta^{\alpha\beta}\,\xi_{\mu}\right)-\frac{1}{4}\left(\Gamma_{\beta\nu\alpha}+\epsilon\,\eta_{\alpha\nu}\,\xi_{\beta}\right)\left(\Gamma_{\mu}{}^{\alpha\beta}+\epsilon\,\eta^{\alpha\beta}\,\xi_{\mu}\right)-\frac{1}{4}\left(\Gamma^{\beta}{}_{\beta\alpha}+\epsilon\,\eta_{\alpha\beta}\,\xi^{\beta}\right)\left(\Gamma_{\nu}{}^{\alpha}{}_{\mu}+\epsilon\,\delta_{\mu}{}^{\alpha}\,\xi_{\nu}\right)-\frac{1}{4}\left(\Gamma^{\beta}{}_{\alpha\beta}+\epsilon\,\eta_{\beta\alpha}\,\xi^{\beta}\right)\left(\Gamma_{\nu\mu}{}^{\alpha}+\epsilon\,\delta_{\mu}{}^{\alpha}\,\xi_{\nu}\right)+\frac{1}{4}\left(\Gamma_{\alpha\beta\mu}+\epsilon\,\eta_{\mu\beta}\,\xi_{\alpha}\right)\left(\Gamma_{\nu}{}^{\alpha\beta}+\epsilon\,\eta^{\alpha\beta}\,\xi_{\nu}\right)+\frac{1}{4}\left(\Gamma_{\beta\mu\alpha}+\epsilon\,\eta_{\alpha\mu}\,\xi_{\beta}\right)\left(\Gamma_{\nu}{}^{\alpha\beta}+\epsilon\,\eta^{\alpha\beta}\,\xi_{\nu}\right)+\frac{1}{4}\left(\partial_{\alpha}\Gamma_{\mu}{}^{\alpha}{}_{\nu}+\epsilon\,\eta_{\nu\alpha}\,\partial_{\mu}\xi^{\alpha}\right)+\frac{1}{4}\left(-\partial_{\alpha}\Gamma_{\mu\nu}{}^{\alpha}-\epsilon\,\eta_{\nu\alpha}\,\partial_{\mu}\xi^{\alpha}\right)+\frac{1}{4}\left(-\partial_{\alpha}\Gamma_{\nu}{}^{\alpha}{}_{\mu}-\epsilon\,\partial_{\nu}\xi_{\mu}\right)+\frac{1}{4}\left(\partial_{\alpha}\Gamma_{\nu\mu}{}^{\alpha}+\epsilon\,\partial_{\nu}\xi_{\mu}\right)+\frac{1}{4}\left(-\partial_{\mu}\Gamma^{\alpha}{}_{\alpha\nu}-\epsilon\,\eta_{\nu\alpha}\,\partial_{\mu}\xi^{\alpha}\right)+\frac{1}{4}\left(\partial_{\mu}\Gamma^{\alpha}{}_{\nu\alpha}+\epsilon\,\eta_{\alpha\nu}\,\partial_{\mu}\xi^{\alpha}\right)+\frac{1}{4}\left(-\partial_{\nu}\Gamma^{\alpha}{}_{\mu\alpha}-\epsilon\,\partial_{\nu}\xi_{\mu}\right)+\frac{1}{4}\left(\partial_{\nu}\Gamma^{\alpha}{}_{\alpha\mu}+\epsilon\,\partial_{\nu}\xi_{\mu}\right) \tag{65}$$

$$-\frac{1}{4}\Gamma_{\alpha\beta\nu}\Gamma_{\mu}{}^{\alpha\beta}-\frac{1}{4}\Gamma_{\beta\nu\alpha}\Gamma_{\mu}{}^{\alpha\beta}+\frac{1}{4}\Gamma^{\beta}{}_{\beta\alpha}\Gamma_{\mu}{}^{\alpha}{}_{\nu}+\frac{1}{4}\Gamma^{\beta}{}_{\alpha\beta}\Gamma_{\mu\nu}{}^{\alpha}+\frac{1}{4}\Gamma_{\alpha\beta\mu}\Gamma_{\nu}{}^{\alpha\beta}+\frac{1}{4}\Gamma_{\beta\mu\alpha}\Gamma_{\nu}{}^{\alpha\beta}-\frac{1}{4}\Gamma^{\beta}{}_{\beta\alpha}\Gamma_{\nu}{}^{\alpha}{}_{\mu}-\frac{1}{4}\Gamma^{\beta}{}_{\alpha\beta}\Gamma_{\nu\mu}{}^{\alpha}+\frac{1}{4}\partial_{\alpha}\Gamma_{\mu}{}^{\alpha}{}_{\nu}-\frac{1}{4}\partial_{\alpha}\Gamma_{\mu\nu}{}^{\alpha}-\frac{1}{4}\partial_{\alpha}\Gamma_{\nu}{}^{\alpha}{}_{\mu}+\frac{1}{4}\partial_{\alpha}\Gamma_{\nu\mu}{}^{\alpha}-\frac{1}{4}\partial_{\mu}\Gamma^{\alpha}{}_{\alpha\nu}+\frac{1}{4}\partial_{\mu}\Gamma^{\alpha}{}_{\nu\alpha}+\frac{1}{4}\partial_{\nu}\Gamma^{\alpha}{}_{\alpha\mu}-\frac{1}{4}\partial_{\nu}\Gamma^{\alpha}{}_{\mu\alpha} \tag{66}$$

So because Eqs. (64), and (66) are identical, we can have confidence that the tensor Eqs. (57), and (58) is indeed (proportional to) the object considered in arXiv:2212.09820.

Now all the generally-covariant contractions of the field strength tensors have been defined, so we construct the general, parity-preserving Lagrangian proposed in Equation (2.4) on page 5 of arXiv:1912.01023.

$$\frac{1}{2}\left(-\mathcal{F}^{\mu\nu\rho\sigma}\left(c_1\,\mathcal{F}_{\mu\nu\rho\sigma}+c_2\,\mathcal{F}_{\mu\nu\sigma\rho}+c_4\,\mathcal{F}_{\mu\rho\nu\sigma}+c_5\,\mathcal{F}_{\mu\sigma\nu\rho}+c_6\,\mathcal{F}_{\mu\sigma\rho\nu}+c_3\,\mathcal{F}_{\rho\sigma\mu\nu}\right)-\left(a_5\,Q_{\nu\mu\rho}+a_4\,Q_{\rho\mu\nu}\right)Q^{\rho\mu\nu}-a_6\,Q_{\mu}\,Q^{\mu}-a_8\,Q_{\mu}\,\tilde{Q}^{\mu}-a_7\,\tilde{Q}_{\mu}\,\tilde{Q}^{\mu}+a_0\,\mathcal{F}-c_{16}\,\mathcal{F}^{2}-\mathcal{F}^{(13)\mu\nu}\left(c_7\,\mathcal{F}^{(13)}{}_{\mu\nu}+c_8\,\mathcal{F}^{(13)}{}_{\nu\mu}\right)-\mathcal{F}^{\mu\nu}\left(c_{13}\,\mathcal{F}_{\mu\nu}+c_{14}\,\mathcal{F}^{(13)}{}_{\mu\nu}+c_{15}\,\mathcal{F}^{(14)}{}_{\mu\nu}\right)-\left(c_{11}\,\mathcal{F}^{(13)}{}_{\mu\nu}+c_{12}\,\mathcal{F}^{(13)}{}_{\nu\mu}\right)\mathcal{F}^{(14)\mu\nu}-\mathcal{F}^{(14)\mu\nu}\left(c_9\,\mathcal{F}^{(14)}{}_{\mu\nu}+c_{10}\,\mathcal{F}^{(14)}{}_{\nu\mu}\right)-a_9\,Q_{\mu\rho\nu}\,\mathcal{T}^{\mu\rho\nu}-\left(a_2\,\mathcal{T}_{\mu\nu\rho}+a_1\,\mathcal{T}_{\mu\rho\nu}\right)\mathcal{T}^{\mu\rho\nu}-\left(a_{10}\,Q_{\mu}+a_{11}\,\tilde{Q}_{\mu}\right)\mathcal{T}^{\mu}-a_3\,\mathcal{T}_{\mu}\,\mathcal{T}^{\mu}\right) \tag{67}$$

This general Lagrangian is something that we must linearize. First, we need the linearized measure, otherwise the Einstein--Hilbert term (which has first-order perturbed contributions) won't have the right linearization.

$$1+\frac{h^{\alpha}{}_{\alpha}}{2} \tag{68}$$

Now we attempt the linearization.

$$\ldots\;[illegible]\;\ldots+\frac{1}{2}c_{12}\,\partial_{\beta}\Gamma^{\alpha\beta\chi}\,\partial_{\delta}\Gamma_{\chi\alpha}{}^{\delta}+\frac{1}{2}c_{11}\,\partial^{\chi}\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\,\partial_{\delta}\Gamma_{\chi\beta}{}^{\delta}+\frac{1}{2}c_{9}\,\partial^{\chi}\Gamma^{\alpha\beta}{}_{\alpha}\,\partial_{\delta}\Gamma_{\chi\beta}{}^{\delta}-\frac{1}{2}c_{8}\,\partial_{\beta}\Gamma^{\alpha\beta\chi}\,\partial_{\delta}\Gamma_{\chi}{}^{\delta}{}_{\alpha}+\frac{1}{2}c_{11}\,\partial^{\chi}\Gamma_{\beta\alpha}{}^{\beta}\,\partial_{\delta}\Gamma_{\chi}{}^{\delta\alpha}+c_{7}\,\partial^{\chi}\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\,\partial_{\delta}\Gamma_{\chi}{}^{\delta}{}_{\beta}-\frac{1}{2}c_{16}\,\partial_{\beta}\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\,\partial_{\delta}\Gamma^{\chi}{}_{\chi}{}^{\delta}+c_{16}\,\partial_{\beta}\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\,\partial_{\delta}\Gamma^{\chi\delta}{}_{\chi}-\frac{1}{2}c_{16}\,\partial_{\beta}\Gamma^{\alpha\beta}{}_{\alpha}\,\partial_{\delta}\Gamma^{\chi\delta}{}_{\chi}+\frac{1}{2}c_{5}\,\partial_{\alpha}\Gamma_{\beta\chi\delta}\,\partial^{\delta}\Gamma^{\alpha\beta\chi}+c_{4}\,\partial_{\alpha}\Gamma_{\beta\delta\chi}\,\partial^{\delta}\Gamma^{\alpha\beta\chi}+c_{6}\,\partial_{\alpha}\Gamma_{\chi\beta\delta}\,\partial^{\delta}\Gamma^{\alpha\beta\chi}+\frac{1}{2}c_{5}\,\partial_{\alpha}\Gamma_{\chi\delta\beta}\,\partial^{\delta}\Gamma^{\alpha\beta\chi}+c_{1}\,\partial_{\alpha}\Gamma_{\delta\beta\chi}\,\partial^{\delta}\Gamma^{\alpha\beta\chi}+c_{2}\,\partial_{\alpha}\Gamma_{\delta\chi\beta}\,\partial^{\delta}\Gamma^{\alpha\beta\chi}-\frac{1}{2}c_{5}\,\partial_{\beta}\Gamma_{\alpha\chi\delta}\,\partial^{\delta}\Gamma^{\alpha\beta\chi}-\frac{1}{2}c_{4}\,\partial_{\beta}\Gamma_{\alpha\delta\chi}\,\partial^{\delta}\Gamma^{\alpha\beta\chi}-\ldots\;[illegible] \tag{69}$$

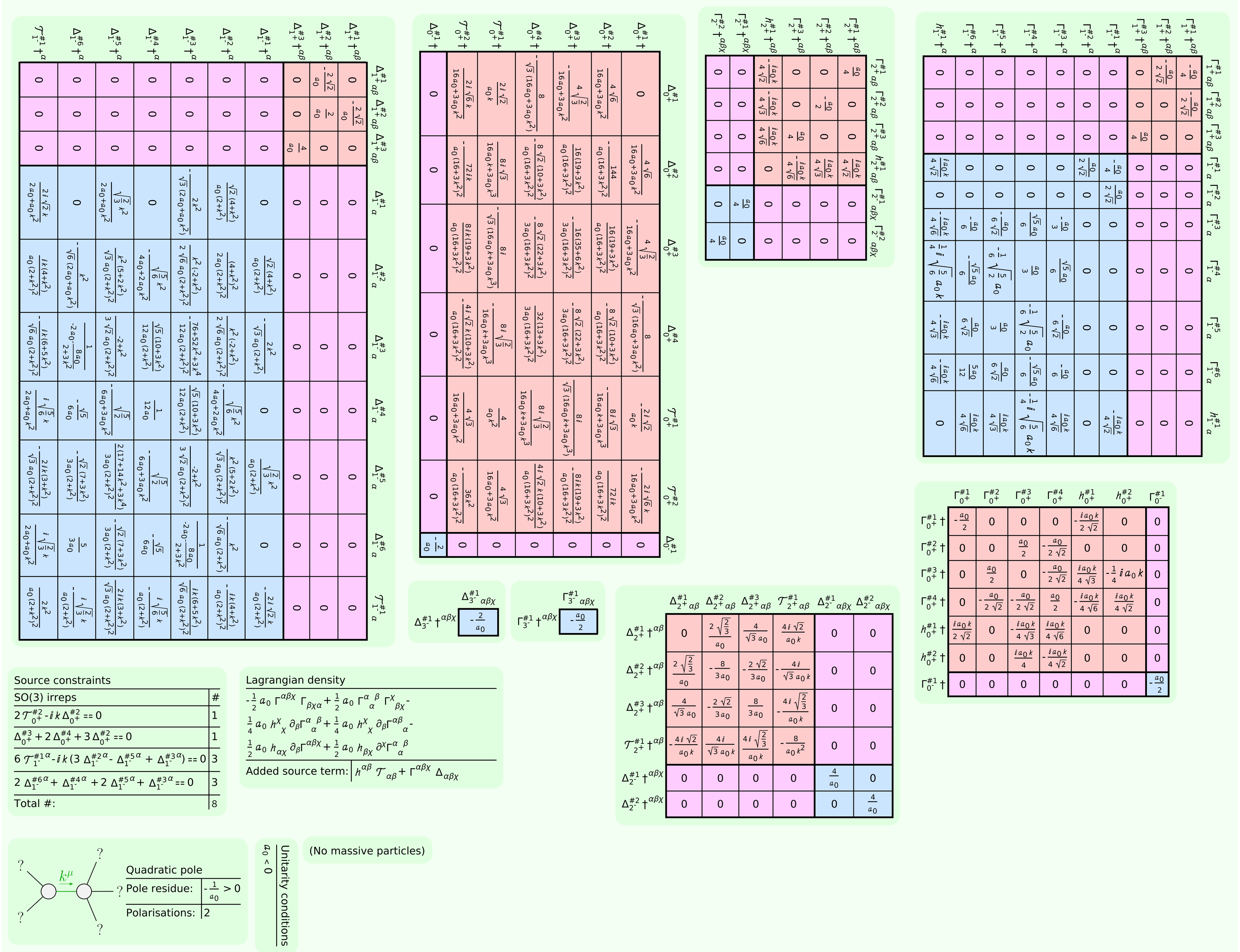
We see that Eq. (69) is generically quite a complicated expression, and there is not a very high degree of degeneracy among the coupling constants.

## Einstein-Hilbert theory

The first theory we will look at is the simple Einstein-Hilbert case.

$$\frac{1}{2}a_0\,\mathcal{F}\left(1+\frac{h^{\alpha}{}_{\alpha}}{2}\right) \tag{70}$$

```
ParticleSpectrum[-1/2*(A0*Connection[a, b, c]*Connection[-b, -c, -a]) + (A0*Connection[a, -a, b]*Connection[c, -b, -c])/2 - (A0*MetricPerturbation[c, -c]*CD[-b][Connection[a, -a, b]])/4 + (A0*MetricPerturbation[c, -c]*CD[-b][Connection[a, b, -a]])/4 - (A0*MetricPerturbation[-a, -c]*CD[-b][Connection[a, b, c]])/2 + (A0*MetricPerturbation[-b, -c]*CD[c][Connection[a, -a, b]])/2, TheoryName → "MetricAffineEinsteinHilbertTheory", Method → "Hard", MaxLaurentDepth → 3];
```




We find that there are no massive poles, and hence no massive gravitons. There are however two massless degrees of freedom which we take to be the graviton polarisations. The unitarity conditions of these polarisations just make sure that the Einstein-Hilbert coupling carries the right sign, i.e. that the square of the Planck mass is positive. The spectra Eqs. (70), (28), and (29) are all the same.

Now we want to try this using the second order formalism.

$$\frac{1}{2} a_0 \mathcal{F}\left(1+\frac{h^\alpha{}_\alpha}{2}\right) \tag{71}$$

$$-\frac{1}{2} a_0 \Gamma^{\alpha\beta\chi} \Gamma_{\beta\chi\alpha}+\frac{1}{2} a_0 \Gamma^{\alpha}{}_{\alpha}{}^{\beta} \Gamma^{\chi}{}_{\beta\chi}-\frac{1}{4} a_0 h^{\chi}{}_{\chi} \partial_\beta\Gamma^{\alpha}{}_{\alpha}{}^{\beta}+\frac{1}{4} a_0 h^{\chi}{}_{\chi} \partial_\beta\Gamma^{\alpha\beta}{}_{\alpha}-\frac{1}{2} a_0 h_{\alpha\chi} \partial_\beta\Gamma^{\alpha\beta\chi}-\frac{1}{2} a_0 \Gamma^{\alpha\beta\chi} \partial_\beta h_{\alpha\chi}-\frac{1}{4} a_0 \Gamma^{\alpha}{}_{\alpha}{}^{\beta} \partial_\beta h^{\chi}{}_{\chi}+\frac{1}{4} a_0 \Gamma^{\alpha\beta}{}_{\alpha} \partial_\beta h^{\chi}{}_{\chi}+\frac{1}{2} a_0 h^{\alpha\beta} \partial_\beta\partial_\alpha h^{\chi}{}_{\chi}-\frac{1}{8} a_0 \partial_\beta h^{\chi}{}_{\chi}\partial^\beta h^{\alpha}{}_{\alpha}+\frac{1}{2} a_0 \Gamma^{\alpha}{}_{\alpha}{}^{\beta} \partial_\chi h_\beta{}^\chi+\frac{1}{2} a_0 \partial_\alpha h^{\alpha\beta}\partial_\chi h_\beta{}^\chi+\frac{1}{2} a_0 \partial^\beta h^{\alpha}{}_{\alpha}\partial_\chi h_\beta{}^\chi-a_0 h^{\alpha\beta}\partial_\chi\partial_\beta h_\alpha{}^\chi+\frac{1}{4} a_0 h^{\alpha}{}_{\alpha}\partial_\chi\partial_\beta h^{\beta\chi}+\frac{1}{2} a_0 h^{\alpha\beta}\partial_\chi\partial^\chi h_{\alpha\beta}-\frac{1}{4} a_0 h^{\alpha}{}_{\alpha}\partial_\chi\partial^\chi h^{\beta}{}_{\beta}+\frac{1}{2} a_0 h_{\beta\chi}\partial^\chi\Gamma^{\alpha}{}_{\alpha}{}^{\beta}-\frac{1}{4} a_0 \partial_\beta h_{\alpha\chi}\partial^\chi h^{\alpha\beta}+\frac{3}{8} a_0 \partial_\chi h_{\alpha\beta}\partial^\chi h^{\alpha\beta} \tag{72}$$

```
ParticleSpectrum[-1/2*(A0*Connection[a, b, c]*Connection[-b, -c, -a]) + (A0*Connection[a, -a, b]*Connection[c, -b, -c])/2 - (A0*MetricPerturbation[c, -c]*CD[-b][Connection[a, -a, b]])/4 + (A0*MetricPerturbation[c, -c]*CD[-b][Connection[a, b, -a]])/4 - (A0*MetricPerturbation[-a, -c]*CD[-b][Connection[a, b, c]])/2 - (A0*Connection[a, b, c]*CD[-b][MetricPerturbation[-a, -c]])/2 - (A0*Connection[a, -a, b]*CD[-b][MetricPerturbation[c, -c]])/4 +
   (A0*Connection[a, b, -a]*CD[-b][MetricPerturbation[c, -c]])/4 + (A0*MetricPerturbation[a, b]*CD[-b][CD[-a][MetricPerturbation[c, -c]]])/2 - (A0*CD[-b][MetricPerturbation[c, -c]]*CD[b][MetricPerturbation[a, -a]])/8 + (A0*Connection[a, -a, b]*CD[-c][MetricPerturbation[-b, c]])/2 + (A0*CD[-a][MetricPerturbation[a, b]]*CD[-c][MetricPerturbation[-b, c]])/2 + (A0*CD[b][MetricPerturbation[a, -a]]*CD[-c][MetricPerturbation[-b, c]])/2 - A0*MetricPerturbation[a, b]*CD[-c][CD[-b][MetricPerturbation[-a, c]]] +
   (A0*MetricPerturbation[a, -a]*CD[-c][CD[-b][MetricPerturbation[b, c]]])/4 + (A0*MetricPerturbation[a, b]*CD[-c][CD[c][MetricPerturbation[-a, -b]]])/2 - (A0*MetricPerturbation[a, -a]*CD[-c][CD[c][MetricPerturbation[b, -b]]])/4 + (A0*MetricPerturbation[-b, -c]*CD[c][Connection[a, -a, b]])/2 - (A0*CD[-b][MetricPerturbation[-a, -c]]*CD[c][MetricPerturbation[a, b]])/4 + (3*A0*CD[-c][MetricPerturbation[-a, -b]]*CD[c][MetricPerturbation[a, b]])/8, TheoryName -> "MetricAffineEinsteinHilbertTheorySecondOrder", Method -> "Hard", MaxLaurentDepth -> 3];
```

| | $\Delta^{\#1}_{3^-\,\alpha\beta\chi}$ |
|---|---|
| $\Delta^{\#1}_{3^-}\dagger^{\alpha\beta\chi}$ | $-\frac{2}{a_0}$ |

| | $\Gamma^{\#1}_{3^-\,\alpha\beta\chi}$ |
|---|---|
| $\Gamma^{\#1}_{3^-}\dagger^{\alpha\beta\chi}$ | $-\frac{a_0}{2}$ |

| | $\Gamma^{\#1}_{1^+\,\alpha\beta}$ | $\Gamma^{\#2}_{1^+\,\alpha\beta}$ | $\Gamma^{\#3}_{1^+\,\alpha\beta}$ | $\Gamma^{\#1}_{1^-\,\alpha}$ | $\Gamma^{\#2}_{1^-\,\alpha}$ | $\Gamma^{\#3}_{1^-\,\alpha}$ | $\Gamma^{\#4}_{1^-\,\alpha}$ | $\Gamma^{\#5}_{1^-\,\alpha}$ | $\Gamma^{\#6}_{1^-\,\alpha}$ | $h^{\#1}_{1^-\,\alpha}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\Gamma^{\#1}_{1^+}\dagger^{\alpha\beta}$ | $-\frac{a_0}{4}$ | $-\frac{a_0}{2\sqrt{2}}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#2}_{1^+}\dagger^{\alpha\beta}$ | $-\frac{a_0}{2\sqrt{2}}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#3}_{1^+}\dagger^{\alpha\beta}$ | 0 | 0 | $\frac{a_0}{4}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $-\frac{a_0}{4}$ | $\frac{a_0}{2\sqrt{2}}$ | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#2}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{a_0}{2\sqrt{2}}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#3}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | $-\frac{a_0}{3}$ | $\frac{\sqrt{5}\,a_0}{6}$ | $-\frac{a_0}{6\sqrt{2}}$ | $-\frac{a_0}{6}$ | 0 |
| $\Gamma^{\#4}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | $\frac{\sqrt{5}\,a_0}{6}$ | $\frac{a_0}{3}$ | $-\frac{1}{6}\sqrt{\frac{5}{2}}\,a_0$ | $-\frac{\sqrt{5}\,a_0}{6}$ | 0 |
| $\Gamma^{\#5}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | $-\frac{a_0}{6\sqrt{2}}$ | $-\frac{1}{6}\sqrt{\frac{5}{2}}\,a_0$ | $\frac{a_0}{3}$ | $\frac{a_0}{6\sqrt{2}}$ | 0 |
| $\Gamma^{\#6}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | $-\frac{a_0}{6}$ | $-\frac{\sqrt{5}\,a_0}{6}$ | $\frac{a_0}{6\sqrt{2}}$ | $\frac{5a_0}{12}$ | 0 |
| $h^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| | $\Gamma^{\#1}_{0^+}$ | $\Gamma^{\#2}_{0^+}$ | $\Gamma^{\#3}_{0^+}$ | $\Gamma^{\#4}_{0^+}$ | $h^{\#1}_{0^+}$ | $h^{\#2}_{0^+}$ | $\Gamma^{\#1}_{0^-}$ |
|---|---|---|---|---|---|---|---|
| $\Gamma^{\#1}_{0^+}\dagger$ | $-\frac{a_0}{2}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#2}_{0^+}\dagger$ | 0 | 0 | $\frac{a_0}{2}$ | $-\frac{a_0}{2\sqrt{2}}$ | 0 | 0 | 0 |
| $\Gamma^{\#3}_{0^+}\dagger$ | 0 | $\frac{a_0}{2}$ | 0 | $-\frac{a_0}{2\sqrt{2}}$ | 0 | 0 | 0 |
| $\Gamma^{\#4}_{0^+}\dagger$ | 0 | $-\frac{a_0}{2\sqrt{2}}$ | $-\frac{a_0}{2\sqrt{2}}$ | $\frac{a_0}{2}$ | 0 | 0 | 0 |
| $h^{\#1}_{0^+}\dagger$ | 0 | 0 | 0 | 0 | $\frac{a_0 k^2}{4}$ | 0 | 0 |
| $h^{\#2}_{0^+}\dagger$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#1}_{0^-}\dagger$ | 0 | 0 | 0 | 0 | 0 | 0 | $-\frac{a_0}{2}$ |

| | $\Delta^{\#1}_{0^+}$ | $\Delta^{\#2}_{0^+}$ | $\Delta^{\#3}_{0^+}$ | $\Delta^{\#4}_{0^+}$ | $\mathcal{T}^{\#1}_{0^+}$ | $\mathcal{T}^{\#2}_{0^+}$ | $\Delta^{\#1}_{0^-}$ |
|---|---|---|---|---|---|---|---|
| $\Delta^{\#1}_{0^+}\dagger$ | $-\frac{2}{a_0}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#2}_{0^+}\dagger$ | 0 | $-\frac{3}{4a_0}$ | $\frac{5}{4a_0}$ | $-\frac{1}{2\sqrt{2}\,a_0}$ | 0 | 0 | 0 |
| $\Delta^{\#3}_{0^+}\dagger$ | 0 | $\frac{5}{4a_0}$ | $-\frac{3}{4a_0}$ | $-\frac{1}{2\sqrt{2}\,a_0}$ | 0 | 0 | 0 |
| $\Delta^{\#4}_{0^+}\dagger$ | 0 | $-\frac{1}{2\sqrt{2}\,a_0}$ | $-\frac{1}{2\sqrt{2}\,a_0}$ | $\frac{1}{2a_0}$ | 0 | 0 | 0 |
| $\mathcal{T}^{\#1}_{0^+}\dagger$ | 0 | 0 | 0 | 0 | $\frac{4}{a_0 k^2}$ | 0 | 0 |
| $\mathcal{T}^{\#2}_{0^+}\dagger$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#1}_{0^-}\dagger$ | 0 | 0 | 0 | 0 | 0 | 0 | $-\frac{2}{a_0}$ |

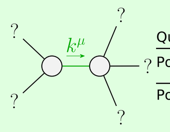
| Source constraints | |
|---|---|
| SO(3) irreps | # |
| $\mathcal{T}^{\#2}_{0^+} == 0$ | 1 |
| $\Delta^{\#3}_{0^+}+2\,\Delta^{\#4}_{0^+}+3\,\Delta^{\#2}_{0^+} == 0$ | 1 |
| $\mathcal{T}^{\#1\,\alpha}_{1^-} == 0$ | 3 |
| $2\,\Delta^{\#6\,\alpha}_{1^-}+\Delta^{\#4\,\alpha}_{1^-}+2\,\Delta^{\#5\,\alpha}_{1^-}+\Delta^{\#3\,\alpha}_{1^-} == 0$ | 3 |
| Total #: | 8 |

| | $\Gamma^{\#1}_{2^+\,\alpha\beta}$ | $\Gamma^{\#2}_{2^+\,\alpha\beta}$ | $\Gamma^{\#3}_{2^+\,\alpha\beta}$ | $h^{\#1}_{2^+\,\alpha\beta}$ | $\Gamma^{\#1}_{2^-\,\alpha\beta\chi}$ | $\Gamma^{\#2}_{2^-\,\alpha\beta\chi}$ |
|---|---|---|---|---|---|---|
| $\Gamma^{\#1}_{2^+}\dagger^{\alpha\beta}$ | $\frac{a_0}{4}$ | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#2}_{2^+}\dagger^{\alpha\beta}$ | 0 | $-\frac{a_0}{2}$ | 0 | 0 | 0 | 0 |
| $\Gamma^{\#3}_{2^+}\dagger^{\alpha\beta}$ | 0 | 0 | $\frac{a_0}{4}$ | 0 | 0 | 0 |
| $h^{\#1}_{2^+}\dagger^{\alpha\beta}$ | 0 | 0 | 0 | $-\frac{a_0 k^2}{8}$ | 0 | 0 |
| $\Gamma^{\#1}_{2^-}\dagger^{\alpha\beta\chi}$ | 0 | 0 | 0 | 0 | $\frac{a_0}{4}$ | 0 |
| $\Gamma^{\#2}_{2^-}\dagger^{\alpha\beta\chi}$ | 0 | 0 | 0 | 0 | 0 | $\frac{a_0}{4}$ |

| | $\Delta^{\#1}_{1^+\,\alpha\beta}$ | $\Delta^{\#2}_{1^+\,\alpha\beta}$ | $\Delta^{\#3}_{1^+\,\alpha\beta}$ | $\Delta^{\#1}_{1^-\,\alpha}$ | $\Delta^{\#2}_{1^-\,\alpha}$ | $\Delta^{\#3}_{1^-\,\alpha}$ | $\Delta^{\#4}_{1^-\,\alpha}$ | $\Delta^{\#5}_{1^-\,\alpha}$ | $\Delta^{\#6}_{1^-\,\alpha}$ | $\mathcal{T}^{\#1}_{1^-\,\alpha}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta^{\#1}_{1^+}\dagger^{\alpha\beta}$ | 0 | $-\frac{2\sqrt{2}}{a_0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#2}_{1^+}\dagger^{\alpha\beta}$ | $-\frac{2\sqrt{2}}{a_0}$ | $\frac{2}{a_0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#3}_{1^+}\dagger^{\alpha\beta}$ | 0 | 0 | $\frac{4}{a_0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | $\frac{2\sqrt{2}}{a_0}$ | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#2}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{2\sqrt{2}}{a_0}$ | $\frac{2}{a_0}$ | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#3}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | $-\frac{19}{12a_0}$ | $\frac{5\sqrt{5}}{12a_0}$ | $-\frac{1}{6\sqrt{2}\,a_0}$ | $-\frac{1}{6a_0}$ | 0 |
| $\Delta^{\#4}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | $\frac{5\sqrt{5}}{12a_0}$ | $\frac{1}{12a_0}$ | $-\frac{\sqrt{\frac{5}{2}}}{6a_0}$ | $-\frac{\sqrt{5}}{6a_0}$ | 0 |
| $\Delta^{\#5}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | $-\frac{1}{6\sqrt{2}\,a_0}$ | $-\frac{\sqrt{\frac{5}{2}}}{6a_0}$ | $\frac{17}{6a_0}$ | $-\frac{7}{3\sqrt{2}\,a_0}$ | 0 |
| $\Delta^{\#6}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | $-\frac{1}{6a_0}$ | $-\frac{\sqrt{5}}{6a_0}$ | $-\frac{7}{3\sqrt{2}\,a_0}$ | $\frac{5}{3a_0}$ | 0 |
| $\mathcal{T}^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| Lagrangian density |
|---|
| $-\frac{1}{2} a_0 \Gamma^{\alpha\beta\chi} \Gamma_{\beta\chi\alpha}+\frac{1}{2} a_0 \Gamma^{\alpha}{}_{\alpha}{}^{\beta} \Gamma^{\chi}{}_{\beta\chi}-\frac{1}{2} a_0 \Gamma^{\alpha\beta\chi} \partial_\beta h_{\alpha\chi}-\frac{1}{4} a_0 \Gamma^{\alpha}{}_{\alpha}{}^{\beta} \partial_\beta h^{\chi}{}_{\chi}+\frac{1}{4} a_0 \Gamma^{\alpha\beta}{}_{\alpha} \partial_\beta h^{\chi}{}_{\chi}-\frac{1}{4} a_0 h^{\chi}{}_{\chi} \partial_\beta\Gamma^{\alpha}{}_{\alpha}{}^{\beta}+\frac{1}{4} a_0 h^{\chi}{}_{\chi} \partial_\beta\Gamma^{\alpha\beta}{}_{\alpha}-\frac{1}{2} a_0 h_{\alpha\chi} \partial_\beta\Gamma^{\alpha\beta\chi}+\frac{1}{2} a_0 h^{\alpha\beta}\partial_\beta\partial_\alpha h^{\chi}{}_{\chi}-\frac{1}{8} a_0 \partial_\beta h^{\chi}{}_{\chi}\partial^\beta h^{\alpha}{}_{\alpha}+\frac{1}{2} a_0 \Gamma^{\alpha}{}_{\alpha}{}^{\beta}\partial_\chi h_\beta{}^\chi+\frac{1}{2} a_0 \partial_\alpha h^{\alpha\beta}\partial_\chi h_\beta{}^\chi+\frac{1}{2} a_0 \partial^\beta h^{\alpha}{}_{\alpha}\partial_\chi h_\beta{}^\chi-a_0 h^{\alpha\beta}\partial_\chi\partial_\beta h_\alpha{}^\chi+\frac{1}{4} a_0 h^{\alpha}{}_{\alpha}\partial_\chi\partial_\beta h^{\beta\chi}+\frac{1}{2} a_0 h^{\alpha\beta}\partial_\chi\partial^\chi h_{\alpha\beta}-\frac{1}{4} a_0 h^{\alpha}{}_{\alpha}\partial_\chi\partial^\chi h^{\beta}{}_{\beta}-\frac{1}{4} a_0 \partial_\beta h_{\alpha\chi}\partial^\chi h^{\alpha\beta}+\frac{3}{8} a_0 \partial_\chi h_{\alpha\beta}\partial^\chi h^{\alpha\beta}+\frac{1}{2} a_0 h_{\beta\chi}\partial^\chi\Gamma^{\alpha}{}_{\alpha}{}^{\beta}$ |
| Added source term: $h^{\alpha\beta}\mathcal{T}_{\alpha\beta}+\Gamma^{\alpha\beta\chi}\Delta_{\alpha\beta\chi}$ |

| | $\Delta^{\#1}_{2^+\,\alpha\beta}$ | $\Delta^{\#2}_{2^+\,\alpha\beta}$ | $\Delta^{\#3}_{2^+\,\alpha\beta}$ | $\mathcal{T}^{\#1}_{2^+\,\alpha\beta}$ | $\Delta^{\#1}_{2^-\,\alpha\beta\chi}$ | $\Delta^{\#2}_{2^-\,\alpha\beta\chi}$ |
|---|---|---|---|---|---|---|
| $\Delta^{\#1}_{2^+}\dagger^{\alpha\beta}$ | $\frac{4}{a_0}$ | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#2}_{2^+}\dagger^{\alpha\beta}$ | 0 | $-\frac{2}{a_0}$ | 0 | 0 | 0 | 0 |
| $\Delta^{\#3}_{2^+}\dagger^{\alpha\beta}$ | 0 | 0 | $\frac{4}{a_0}$ | 0 | 0 | 0 |
| $\mathcal{T}^{\#1}_{2^+}\dagger^{\alpha\beta}$ | 0 | 0 | 0 | $-\frac{8}{a_0 k^2}$ | 0 | 0 |
| $\Delta^{\#1}_{2^-}\dagger^{\alpha\beta\chi}$ | 0 | 0 | 0 | 0 | $\frac{4}{a_0}$ | 0 |
| $\Delta^{\#2}_{2^-}\dagger^{\alpha\beta\chi}$ | 0 | 0 | 0 | 0 | 0 | $\frac{4}{a_0}$ |




| Quadratic pole | |
|---|---|
| Pole residue: | $-\frac{1}{a_0} > 0$ |
| Polarisations: | 2 |

| Unitarity conditions |
|---|
| $a_0 < 0$ |

(No massive particles)

We note that the conclusions are identical.

We will next study the four theories with physically interesting vector symmetries.

## The iso-Weyl theory

We will study the iso-Weyl symmetry.

$$\frac{1}{2}\left(-\left(\left(a_1 Q_{\lambda\mu\nu}+\left(\frac{a_0}{8}-\frac{a_1}{4}-\frac{9a_2}{4}\right)Q_{\mu\nu\lambda}\right)Q^{\mu\nu\lambda}\right)-a_0 Q_\mu Q^\mu-\left(\frac{a_0}{4}-\frac{a_1}{2}-\frac{a_2}{2}\right)Q_\mu \hat{Q}^\mu-a_2 \hat{Q}_\mu \hat{Q}^\mu+a_0\mathcal{F}-c_{13}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}-a_0 Q_{\mu\nu\lambda}\mathcal{T}^{\mu\nu\lambda}-\left(a_2\mathcal{T}_{\mu\nu\lambda}+a_1\mathcal{T}_{\nu\lambda\mu}\right)\mathcal{T}^{\mu\nu\lambda}-\left(\left(-\frac{2a_0}{3}+\frac{a_2}{3}\right)Q_\mu+\left(\frac{2a_0}{3}-\frac{a_2}{3}\right)\hat{Q}_\mu\right)\mathcal{T}^\mu-\left(-\frac{2a_0}{3}-\frac{2a_1}{3}-\frac{a_2}{3}\right)\mathcal{T}_\mu\mathcal{T}^\mu\right) \tag{73}$$

We study Eq. (73) in the second-order formulation:

```
ParticleSpectrum[((-2*A0 + 2*A1 + A2 - 12*A6 + 2*A9)*Connection[-a, m, -m]*Connection[a, b, -b])/6 + ((-A0 + 2*(-4*A1 + A5 + 9*A7 - 2*A9))*Connection[-a, -b, -m]*Connection[a, b, m])/8 + ((-A0 + 2*(-2*A2 + A5 + 9*A7 - 2*A9))*Connection[-a, -m, -b]*Connection[a, b, m])/8 + ((-A2 - A5)*Connection[a, b, m]*Connection[-b, -a, -m])/2 + ((-A0 + 2*A2 - 2*A5 + A9)*Connection[a, b, m]*Connection[-b, -m, -a])/2 + ((A0 - 8*A1 - 4*A2 + 6*A5 + 6*A7 - 6*A9)*Connection[a, -a, b]*Connection[-b, m, -m])/12 + ((A0 + 6*A5 + 6*A7 - 2*A9)*Connection[a, b, -a]*Connection[-b, m, -m])/12 +
  ((2*A1 - A5 + A9)*Connection[a, b, m]*Connection[-m, -b, -a])/2 + ((A0 - 6*A7 + A9)*Connection[a, -a, b]*Connection[m, -b, -m])/6 - (A7*Connection[a, b, -a]*Connection[m, -b, -m])/2 + ((2*A1 + A2 - 3*A7 + A9)*Connection[a, -a, b]*Connection[m, -m, -b])/6 - (A0*MetricPerturbation[m, -m]*CD[-b][Connection[a, -a, b]])/4 + (A0*MetricPerturbation[m, -m]*CD[-b][Connection[a, b, -a]])/4 - (A0*MetricPerturbation[a, -m]*CD[-b][Connection[a, b, m]])/2 - (A0*Connection[a, b, m]*CD[-b][MetricPerturbation[-a, -m]])/2 -
  (A0*Connection[a, -a, b]*CD[-b][MetricPerturbation[m, -m]])/4 + (A0*Connection[a, b, -a]*CD[-b][MetricPerturbation[m, -m]])/4 + (A0*MetricPerturbation[a, b]*CD[-b][CD[-a][MetricPerturbation[m, -m]]])/4 - (A0*MetricPerturbation[a, b]*CD[-m][CD[-b][MetricPerturbation[-a, m]]])/2 + (A0*Connection[a, -a, b]*CD[-m][MetricPerturbation[-b, m]])/2 - (A0*MetricPerturbation[a, b]*CD[-m][CD[m][MetricPerturbation[-a, -b]]])/4 +
  (A0*MetricPerturbation[a, b]*CD[-m][CD[m][MetricPerturbation[-a, -b]]])/4 - (A0*MetricPerturbation[a, -a]*CD[-m][CD[m][MetricPerturbation[b, -b]]])/4 + (A0*MetricPerturbation[-b, -m]*CD[m][Connection[a, -a, b]])/2 + C13*CD[-a][Connection[a, m, -m]]*CD[n][Connection[a, b, -b]] - C13*CD[-m][Connection[-a, n, -n]]*CD[m][Connection[a, b, -b]] - (A0*CD[-b][MetricPerturbation[-a, -m]]*CD[m][MetricPerturbation[a, b]])/4 + (A0*CD[-m][MetricPerturbation[a, b]]*CD[m][MetricPerturbation[-a, -b]])/8, TheoryName → "IsoWeylTheorySecondOrder", Method → "Hard", MaxLaurentDepth → 1];
```

| | $\Delta^{\#1}_{1^+\alpha\beta}$ | $\Delta^{\#2}_{1^+\alpha\beta}$ | $\Delta^{\#3}_{1^+\alpha\beta}$ | $\Delta^{\#1}_{1^-\alpha}$ | $\Delta^{\#2}_{1^-\alpha}$ | $\Delta^{\#3}_{1^-\alpha}$ | $\Delta^{\#4}_{1^-\alpha}$ | $\Delta^{\#5}_{1^-\alpha}$ | $\Delta^{\#6}_{1^-\alpha}$ | $\mathcal{T}^{\#1}_{1^-\alpha}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta^{\#1}_{1^+}\dagger^{\alpha\beta}$ | $\frac{4}{3}\left(-\frac{1}{a_0+4a_1-4a_2}+\frac{2a_1+a_2-2a_5-6a_7+2a_9}{2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9)}\right)$ | $\frac{2}{3}\sqrt{2}\left(-\frac{1}{a_0+4a_1-4a_2}-\frac{2(2a_1+a_2-2a_5-6a_7+2a_9)}{2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9)}\right)$ | $-\frac{4(2a_1+a_2+a_9)}{3(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#2}_{1^+}\dagger^{\alpha\beta}$ | $\frac{2}{3}\sqrt{2}\left(-\frac{1}{a_0+4a_1-4a_2}-\frac{2(2a_1+a_2-2a_5-6a_7+2a_9)}{2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9)}\right)$ | $-\frac{2}{3(a_0+4a_1-4a_2)}+\frac{8(2a_1+a_2-2a_5-6a_7+2a_9)}{3(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | $\frac{4\sqrt{2}(2a_1+a_2+a_9)}{3(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#3}_{1^+}\dagger^{\alpha\beta}$ | $-\frac{4(2a_1+a_2+a_9)}{3(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | $\frac{4\sqrt{2}(2a_1+a_2+a_9)}{3(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | $-\frac{4(a_0-2a_1-a_2)}{3(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{4(2a_1+a_2-2a_5-6a_7+2a_9)}{3(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | $\frac{4\sqrt{2}(2a_1+a_2-2a_5-6a_7+2a_9)}{3(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | 0 | 0 | $-\frac{4\sqrt{\frac{2}{3}}(2a_1+a_2+a_9)}{3(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | $\frac{4(2a_1+a_2+a_9)}{3\sqrt{3}(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | 0 |
| $\Delta^{\#2}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{4\sqrt{2}(2a_1+a_2-2a_5-6a_7+2a_9)}{3(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | $\frac{8(2a_1+a_2-2a_5-6a_7+2a_9)}{3(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | 0 | 0 | $-\frac{8(2a_1+a_2+a_9)}{3\sqrt{3}(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | $\frac{4\sqrt{\frac{2}{3}}(2a_1+a_2+a_9)}{3(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | 0 |
| $\Delta^{\#3}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | $-\frac{10}{9(a_0+2a_5-6a_7)}-\frac{1}{6(3a_0-2(a_5-8a_6+5a_7-4c_{13}k^2))}$ | $\frac{1}{18}\sqrt{5}\left(\frac{4}{a_0+2a_5-6a_7}-\frac{3}{3a_0-2a_5+16a_6-10a_7+8c_{13}k^2}\right)$ | $-\frac{1}{\sqrt{2}(9a_0-6(a_5-8a_6+5a_7-4c_{13}k^2))}$ | $-\frac{1}{9a_0-6(a_5-8a_6+5a_7-4c_{13}k^2)}$ | 0 |
| $\Delta^{\#4}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | $\frac{1}{18}\sqrt{5}\left(\frac{4}{a_0+2a_5-6a_7}-\frac{3}{3a_0-2a_5+16a_6-10a_7+8c_{13}k^2}\right)$ | $-\frac{2}{9(a_0+2a_5-6a_7)}-\frac{5}{6(3a_0-2(a_5-8a_6+5a_7-4c_{13}k^2))}$ | $-\frac{\sqrt{\frac{5}{2}}}{9a_0-6(a_5-8a_6+5a_7-4c_{13}k^2)}$ | $-\frac{\sqrt{5}}{9a_0-6(a_5-8a_6+5a_7-4c_{13}k^2)}$ | 0 |
| $\Delta^{\#5}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $-\frac{4\sqrt{\frac{2}{3}}(2a_1+a_2+a_9)}{3(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | $-\frac{8(2a_1+a_2+a_9)}{3\sqrt{3}(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | $-\frac{1}{\sqrt{2}(9a_0-6(a_5-8a_6+5a_7-4c_{13}k^2))}$ | $-\frac{\sqrt{\frac{5}{2}}}{9a_0-6(a_5-8a_6+5a_7-4c_{13}k^2)}$ | $\frac{8(-a_0+2a_1+a_2)}{9(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}-\frac{1}{9a_0-6(a_5-8a_6+5a_7-4c_{13}k^2)}$ | $(\sqrt{2}(12a_0^2-3a_9^2-a_0(30a_1+15a_2+2a_5-64a_6+22a_7+6a_9-32c_{13}k^2)+2(2a_1+a_2)(a_5-32a_6+11a_7-16c_{13}k^2)))/(9(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))(3a_0-2(a_5-8a_6+5a_7-4c_{13}k^2)))$ | 0 |
| $\Delta^{\#6}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{4(2a_1+a_2+a_9)}{3\sqrt{3}(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | $\frac{4\sqrt{\frac{2}{3}}(2a_1+a_2+a_9)}{3(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | $-\frac{1}{9a_0-6(a_5-8a_6+5a_7-4c_{13}k^2)}$ | $-\frac{\sqrt{5}}{9a_0-6(a_5-8a_6+5a_7-4c_{13}k^2)}$ | $(\sqrt{2}(12a_0^2-3a_9^2-a_0(30a_1+15a_2+2a_5-64a_6+22a_7+6a_9-32c_{13}k^2)+2(2a_1+a_2)(a_5-32a_6+11a_7-16c_{13}k^2)))/(9(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))(3a_0-2(a_5-8a_6+5a_7-4c_{13}k^2)))$ | $\frac{-4a_0+8a_1+4a_2}{9(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}-\frac{2}{9a_0-6(a_5-8a_6+5a_7-4c_{13}k^2)}$ | 0 |
| $\mathcal{T}^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| | $\Gamma^{\#1}_{1^+\alpha\beta}$ | $\Gamma^{\#2}_{1^+\alpha\beta}$ | $\Gamma^{\#3}_{1^+\alpha\beta}$ | $\Gamma^{\#1}_{1^-\alpha}$ | $\Gamma^{\#2}_{1^-\alpha}$ | $\Gamma^{\#3}_{1^-\alpha}$ | $\Gamma^{\#4}_{1^-\alpha}$ | $\Gamma^{\#5}_{1^-\alpha}$ | $\Gamma^{\#6}_{1^-\alpha}$ | $h^{\#1}_{1^-\alpha}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\Gamma^{\#1}_{1^+}\dagger^{\alpha\beta}$ | $\frac{1}{4}(-a_0-6a_1+5a_2)$ | $-\frac{a_0+2a_1-3a_2}{2\sqrt{2}}$ | $\frac{1}{4}(-2a_1-a_2-a_9)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#2}_{1^+}\dagger^{\alpha\beta}$ | $-\frac{a_0+2a_1-3a_2}{2\sqrt{2}}$ | $\frac{1}{2}(-2a_1+a_2)$ | $\frac{2a_1+a_2+a_9}{2\sqrt{2}}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#3}_{1^+}\dagger^{\alpha\beta}$ | $\frac{1}{4}(-2a_1-a_2-a_9)$ | $\frac{2a_1+a_2+a_9}{2\sqrt{2}}$ | $-\frac{1}{4}(2a_1+a_2-2a_5-6a_7+2a_9)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{1}{12}(a_0-2a_1-a_2)$ | $\frac{a_0-2a_1+a_2}{6\sqrt{2}}$ | 0 | 0 | $-\frac{2a_1+a_2+a_9}{2\sqrt{6}}$ | $\frac{2a_1+a_2+a_9}{4\sqrt{3}}$ | 0 |
| $\Gamma^{\#2}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{a_0-2a_1+a_2}{6\sqrt{2}}$ | $\frac{1}{6}(a_0-2a_1-a_2)$ | 0 | 0 | $-\frac{2a_1+a_2+a_9}{2\sqrt{3}}$ | $\frac{2a_1+a_2+a_9}{2\sqrt{6}}$ | 0 |
| $\Gamma^{\#3}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | $\frac{1}{12}(-9a_0-14a_5-8a_6+50a_7-4c_{13}k^2)$ | $\frac{1}{3}\sqrt{5}(a_5-2a_6-a_7-c_{13}k^2)$ | $-\frac{3a_0+2(a_5-8a_6+5a_7-4c_{13}k^2)}{12\sqrt{2}}$ | $-\frac{5a_0}{4}+\frac{1}{4}(a_5-8a_6+5a_7-4c_{13}k^2)$ | 0 |
| $\Gamma^{\#4}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | $\frac{1}{3}\sqrt{5}(a_5-2a_6-a_7-c_{13}k^2)$ | $\frac{1}{12}(-9a_0+2a_5-40a_6+34a_7-20c_{13}k^2)$ | $\frac{1}{12}\sqrt{\frac{5}{2}}(-3a_0+2(a_5-8a_6+5a_7-4c_{13}k^2))$ | $\frac{1}{12}\sqrt{5}(-3a_0+2(a_5-8a_6+5a_7-4c_{13}k^2))$ | 0 |
| $\Gamma^{\#5}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $-\frac{2a_1+a_2+a_9}{2\sqrt{6}}$ | $-\frac{2a_1+a_2+a_9}{2\sqrt{3}}$ | $-\frac{3a_0+2(a_5-8a_6+5a_7-4c_{13}k^2)}{12\sqrt{2}}$ | $\frac{1}{12}\sqrt{\frac{5}{2}}(-3a_0+2(a_5-8a_6+5a_7-4c_{13}k^2))$ | $\frac{1}{12}(-3a_0-2(6a_1+3a_2-7a_5+8a_6-23a_7+6a_9+4c_{13}k^2))$ | $\frac{3a_0-6a_1-3a_2+4a_5+16a_6+8a_7-6a_9+8c_{13}k^2}{6\sqrt{2}}$ | 0 |
| $\Gamma^{\#6}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{2a_1+a_2+a_9}{4\sqrt{3}}$ | $\frac{2a_1+a_2+a_9}{2\sqrt{6}}$ | $-\frac{5a_0}{4}+\frac{1}{4}(a_5-8a_6+5a_7-4c_{13}k^2)$ | $\frac{1}{12}\sqrt{5}(-3a_0+2(a_5-8a_6+5a_7-4c_{13}k^2))$ | $\frac{3a_0-6a_1-3a_2+4a_5+16a_6+8a_7-6a_9+8c_{13}k^2}{6\sqrt{2}}$ | $\frac{1}{12}(-6a_0-6a_1-3a_2+10a_5-32a_6+38a_7-6a_9-16c_{13}k^2)$ | 0 |
| $h^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Lagrangian density

$-\frac{1}{3}a_0\,\Gamma_\alpha{}^\mu{}_\mu\,\Gamma^{\alpha\beta}{}_\beta+\frac{1}{3}a_1\,\Gamma_\alpha{}^\mu{}_\mu\,\Gamma^{\alpha\beta}{}_\beta+\frac{1}{6}a_2\,\Gamma_\alpha{}^\mu{}_\mu\,\Gamma^{\alpha\beta}{}_\beta-2a_6\,\Gamma_\alpha{}^\mu{}_\mu\,\Gamma^{\alpha\beta}{}_\beta+\frac{1}{3}a_9\,\Gamma_\alpha{}^\mu{}_\mu\,\Gamma^{\alpha\beta}{}_\beta-\frac{1}{8}a_0\,\Gamma_{\alpha\beta\mu}\,\Gamma^{\alpha\beta\mu}-a_1\,\Gamma_{\alpha\beta\mu}\,\Gamma^{\alpha\beta\mu}+\frac{1}{4}a_5\,\Gamma_{\alpha\beta\mu}\,\Gamma^{\alpha\beta\mu}+\frac{9}{4}a_7\,\Gamma_{\alpha\beta\mu}\,\Gamma^{\alpha\beta\mu}-\frac{1}{2}a_9\,\Gamma_{\alpha\beta\mu}\,\Gamma^{\alpha\beta\mu}-\frac{1}{8}a_0\,\Gamma_{\alpha\mu\beta}\,\Gamma^{\alpha\beta\mu}-\ldots$

$\ldots+c_{13}\,\partial_\beta\Gamma_\alpha{}^\mu{}_\mu\,\partial^\beta\Gamma^{\alpha\beta}{}_\beta-c_{13}\,\partial_\alpha\Gamma_\mu{}^\alpha{}_\beta\,\partial^\beta\Gamma^{\mu\beta}{}_\beta$

Added source term: $\Delta^{\alpha\beta}\,\mathcal{T}_{\alpha\beta}+\Gamma^{\alpha\beta\chi}\,\Delta_{\alpha\beta\chi}$

| Source constraints | |
|---|---|
| SO(3) irreps | # |
| $\mathcal{T}^{\#1}_{0^+}==0$ | 1 |
| $\Delta^{\#1}_{0^+}==0$ | 1 |
| $\Delta^{\#2}_{0^+}+3\Delta^{\#3}_{0^+}==2\Delta^{\#4}_{0^+}$ | 1 |
| $\mathcal{T}^{\#1}_{1^-}{}^{\alpha}==0$ | 3 |
| $\Delta^{\#1}_{1^-}{}^{\alpha}==0$ | 3 |
| $2\Delta^{\#3}_{1^-}{}^{\alpha}+\Delta^{\#5}_{1^-}{}^{\alpha}==\Delta^{\#4}_{1^-}{}^{\alpha}+\Delta^{\#6}_{1^-}{}^{\alpha}$ | 3 |
| Total #: | 12 |

| | $\Delta^{\#1}_{2^+\alpha\beta}$ | $\Delta^{\#2}_{2^+\alpha\beta}$ | $\Delta^{\#3}_{2^+\alpha\beta}$ | $\mathcal{T}^{\#1}_{2^+\alpha\beta}$ | $\Delta^{\#1}_{2^-\alpha\beta\chi}$ | $\Delta^{\#2}_{2^-\alpha\beta\chi}$ |
|---|---|---|---|---|---|---|
| $\Delta^{\#1}_{2^+}\dagger^{\alpha\beta}$ | $\frac{4(2a_1+a_2-2a_5-6a_7+2a_9)}{2(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | 0 | $\frac{4(2a_1+a_2+a_9)}{\sqrt{3}(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | 0 | 0 | 0 |
| $\Delta^{\#2}_{2^+}\dagger^{\alpha\beta}$ | 0 | $-\frac{4}{3(a_0+2a_5-6a_7)}$ | 0 | 0 | 0 | 0 |
| $\Delta^{\#3}_{2^+}\dagger^{\alpha\beta}$ | $\frac{4(2a_1+a_2+a_9)}{\sqrt{3}(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | 0 | $\frac{4(a_0-2a_1-a_2)}{3(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | 0 | 0 | 0 |
| $\mathcal{T}^{\#1}_{2^+}\dagger^{\alpha\beta}$ | 0 | 0 | 0 | $\frac{8}{a_0k^2}$ | 0 | 0 |
| $\Delta^{\#1}_{2^-}\dagger^{\alpha\beta\chi}$ | 0 | 0 | 0 | 0 | $\frac{4(2a_1+a_2-2a_5-6a_7+2a_9)}{2(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | $\frac{4(2a_1+a_2+a_9)}{\sqrt{3}(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ |
| $\Delta^{\#2}_{2^-}\dagger^{\alpha\beta\chi}$ | 0 | 0 | 0 | 0 | $\frac{4(2a_1+a_2+a_9)}{\sqrt{3}(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ | $\frac{4(a_0-2a_1-a_2)}{3(2(2a_1+a_2)(a_5+3a_7)+a_9^2+a_0(2a_1+a_2-2a_5-6a_7+2a_9))}$ |

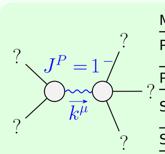
| | $\Gamma^{\#1}_{2^+\alpha\beta}$ | $\Gamma^{\#2}_{2^+\alpha\beta}$ | $\Gamma^{\#3}_{2^+\alpha\beta}$ | $h^{\#1}_{2^+\alpha\beta}$ | $\Gamma^{\#1}_{2^-\alpha\beta\chi}$ | $\Gamma^{\#2}_{2^-\alpha\beta\chi}$ |
|---|---|---|---|---|---|---|
| $\Gamma^{\#1}_{2^+}\dagger^{\alpha\beta}$ | $\frac{1}{4}(a_0-2a_1-a_2)$ | 0 | $-\frac{1}{4}\sqrt{3}(2a_1+a_2+a_9)$ | 0 | 0 | 0 |
| $\Gamma^{\#2}_{2^+}\dagger^{\alpha\beta}$ | 0 | $-\frac{3}{4}(a_0+2a_5-6a_7)$ | 0 | 0 | 0 | 0 |
| $\Gamma^{\#3}_{2^+}\dagger^{\alpha\beta}$ | $-\frac{1}{4}\sqrt{3}(2a_1+a_2+a_9)$ | 0 | $-\frac{3}{4}(2a_1+a_2-2a_5-6a_7+2a_9)$ | 0 | 0 | 0 |
| $h^{\#1}_{2^+}\dagger^{\alpha\beta}$ | 0 | 0 | 0 | $-\frac{a_0k^2}{8}$ | 0 | 0 |
| $\Gamma^{\#1}_{2^-}\dagger^{\alpha\beta\chi}$ | 0 | 0 | 0 | 0 | $\frac{1}{4}(a_0-2a_1-a_2)$ | $-\frac{1}{4}\sqrt{3}(2a_1+a_2+a_9)$ |
| $\Gamma^{\#2}_{2^-}\dagger^{\alpha\beta\chi}$ | 0 | 0 | 0 | 0 | $-\frac{1}{4}\sqrt{3}(2a_1+a_2+a_9)$ | $-\frac{3}{4}(2a_1+a_2-2a_5-6a_7+2a_9)$ |

| | $\Gamma^{\#1}_{0^+}$ | $\Gamma^{\#2}_{0^+}$ | $\Gamma^{\#3}_{0^+}$ | $\Gamma^{\#4}_{0^+}$ | $h^{\#1}_{0^+}$ | $h^{\#2}_{0^+}$ | $\Gamma^{\#1}_{0^-}$ |
|---|---|---|---|---|---|---|---|
| $\Gamma^{\#1}_{0^+}\dagger$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#2}_{0^+}\dagger$ | 0 | $\frac{1}{4}(-3a_0-2(a_5+4a_6-7a_7))$ | $a_5-2a_6-a_7$ | $\frac{-3a_0+2(a_5-8a_6+5a_7)}{4\sqrt{2}}$ | 0 | 0 | 0 |
| $\Gamma^{\#3}_{0^+}\dagger$ | 0 | $a_5-2a_6-a_7$ | $\frac{1}{4}(-3a_0-2(a_5+4a_6-7a_7))$ | $\frac{-3a_0+2(a_5-8a_6+5a_7)}{4\sqrt{2}}$ | 0 | 0 | 0 |
| $\Gamma^{\#4}_{0^+}\dagger$ | 0 | $\frac{-3a_0+2(a_5-8a_6+5a_7)}{4\sqrt{2}}$ | $\frac{-3a_0+2(a_5-8a_6+5a_7)}{4\sqrt{2}}$ | $\frac{1}{4}(-3a_0+2(a_5-8a_6+5a_7))$ | 0 | 0 | 0 |
| $h^{\#1}_{0^+}\dagger$ | 0 | 0 | 0 | 0 | $\frac{a_0k^2}{4}$ | 0 | 0 |
| $h^{\#2}_{0^+}\dagger$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#1}_{0^-}\dagger$ | 0 | 0 | 0 | 0 | 0 | 0 | $-\frac{a_0}{2}-2a_1+2a_2$ |

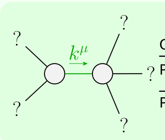
| | $\Delta^{\#1}_{0^+}$ | $\Delta^{\#2}_{0^+}$ | $\Delta^{\#3}_{0^+}$ | $\Delta^{\#4}_{0^+}$ | $\mathcal{T}^{\#1}_{0^+}$ | $\mathcal{T}^{\#2}_{0^+}$ | $\Delta^{\#1}_{0^-}$ |
|---|---|---|---|---|---|---|---|
| $\Delta^{\#1}_{0^+}\dagger$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#2}_{0^+}\dagger$ | 0 | $-\frac{2}{3(a_0+2a_5-6a_7)}-\frac{1}{6a_0-4(a_5-8a_6+5a_7)}$ | $\frac{2}{3(a_0+2a_5-6a_7)}-\frac{1}{6a_0-4(a_5-8a_6+5a_7)}$ | $-\frac{1}{\sqrt{2}(3a_0-2(a_5-8a_6+5a_7))}$ | 0 | 0 | 0 |
| $\Delta^{\#3}_{0^+}\dagger$ | 0 | $\frac{2}{3(a_0+2a_5-6a_7)}-\frac{1}{6a_0-4(a_5-8a_6+5a_7)}$ | $-\frac{2}{3(a_0+2a_5-6a_7)}-\frac{1}{6a_0-4(a_5-8a_6+5a_7)}$ | $-\frac{1}{\sqrt{2}(3a_0-2(a_5-8a_6+5a_7))}$ | 0 | 0 | 0 |
| $\Delta^{\#4}_{0^+}\dagger$ | 0 | $-\frac{1}{\sqrt{2}(3a_0-2(a_5-8a_6+5a_7))}$ | $-\frac{1}{\sqrt{2}(3a_0-2(a_5-8a_6+5a_7))}$ | $\frac{1}{-3a_0+2(a_5-8a_6+5a_7)}$ | 0 | 0 | 0 |
| $\mathcal{T}^{\#1}_{0^+}\dagger$ | 0 | 0 | 0 | 0 | $\frac{4}{a_0k^2}$ | 0 | 0 |
| $\mathcal{T}^{\#2}_{0^+}\dagger$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#1}_{0^-}\dagger$ | 0 | 0 | 0 | 0 | 0 | 0 | $-\frac{2}{a_0+4a_1-4a_2}$ |

| | $\Gamma^{\#1}_{3^-\alpha\beta\chi}$ |
|---|---|
| $\Gamma^{\#1}_{3^-}\dagger^{\alpha\beta\chi}$ | $-\frac{3}{4}(a_0+2a_5-6a_7)$ |

| | $\Delta^{\#1}_{3^-\alpha\beta\chi}$ |
|---|---|
| $\Delta^{\#1}_{3^-}\dagger^{\alpha\beta\chi}$ | $-\frac{4}{3(a_0+2a_5-6a_7)}$ |




| Massive particle | |
|---|---|
| Pole residue: | $\frac{1}{c_{13}}>0$ |
| Polarisations: | 3 |
| Square mass: | $\frac{-3a_0+2(a_5-8a_6+5a_7)}{4c_{13}}>0$ |
| Spin: | 1 |
| Parity: | Odd |




| Quadratic pole | |
|---|---|
| Pole residue: | $-\frac{1}{a_0}>0$ |
| Polarisations: | 2 |

| Unitarity conditions |
|---|
| $a_0<0\,\&\&\,a_7>\frac{1}{30}(3a_0-2a_5+16a_6)\,\&\&\,c_{13}>0$ |

A massive vector accompanies the graviton, and the whole theory may be made to be unitary.

## The extended-projective theory

We will study the extended-projective symmetry.

$$\frac{1}{2}\Big(-\mathcal{F}^{\alpha\chi\rho\sigma}\left(c_1\mathcal{F}_{\alpha\chi\rho\sigma}-c_1\mathcal{F}_{\alpha\chi\sigma\rho}-2c_1\mathcal{F}_{\rho\sigma\alpha\chi}+4c_1\mathcal{F}_{\alpha\rho\chi\sigma}+2c_1\mathcal{F}_{\chi\rho\alpha\sigma}-2c_1\mathcal{F}_{\rho\chi\alpha\sigma}\right)-\left(\left(a_0-\frac{a_1}{2}-\frac{a_2}{4}-\frac{3a_3}{4}+8a_6-5a_7\right)Q_{\alpha\chi\rho}+\left(\frac{a_0}{4}+\frac{a_1}{2}+\frac{a_2}{4}+\frac{3a_3}{4}-2a_6-a_7\right)Q_{\rho\chi\alpha}\right)Q^{\alpha\chi\rho}-a_6\,Q_\alpha\,Q^\alpha-\left(-\frac{a_0}{2}-4a_6+2a_7\right)Q_\alpha\,\hat{Q}^\alpha-a_7\,\hat{Q}_\alpha\,\hat{Q}^\alpha+a_9\,\mathcal{T}^\alpha-a_9\,Q_{\rho\alpha\chi}\,\mathcal{T}^{\alpha\chi\rho}-\left(a_1\,\mathcal{T}_{\alpha\chi\rho}+a_1\,\mathcal{T}_{\chi\alpha\rho}\right)\mathcal{T}^{\alpha\chi\rho}-\left(\left(\frac{2a_1}{3}+\frac{a_2}{3}+a_3+\frac{a_9}{3}\right)Q_\alpha+\left(-\frac{2a_1}{3}-\frac{a_2}{3}-a_3-\frac{a_9}{3}\right)\hat{Q}_\alpha\right)\mathcal{T}^\alpha-a_3\,\mathcal{T}_\alpha\,\mathcal{T}^\alpha\Big) \quad (74)$$

We study Eq. (74) in the second-order formulation:

```
ParticleSpectrum[(4∗A1 + 2∗A2 + 3∗A3 - 12∗A6 + 2∗A9)∗Connection[a, c, -c]∗Connection[a, b, -b]) / 6 + ((-A0 - 6∗A1 - A2 - 3∗A3 + 8∗A6 + 4∗A7 - 2∗A9)∗Connection[-a, -b, -c]∗Connection[a, b, c]) / 4 + ((-A0 - 2∗A1 - 3∗A2 - 3∗A3 + 8∗A6 + 4∗A7 - 2∗A9)∗Connection[-a, -c, -b]∗Connection[a, b, c]) / 4 + ((-4∗A0 + 2∗A1 - 3∗A2 + 3∗A3 - 32∗A6 + 20∗A7)∗Connection[a, b, c]∗Connection[-b, -a, -c]) / 8 + ((-6∗A0 + 2∗A1 + 5∗A2 + 3∗A3 - 32∗A6 + 20∗A7 + 2∗A9)∗Connection[a, b, c]∗Connection[-b, -c, -a]) / 4 + ((A0 - 2∗A1 - A2 - A3 + 8∗A6 - 4∗A7 - A9)∗Connection[a, -a, b]∗Connection[-b, c, -c]) / 2 +
   ((3∗A0 - 2∗A1 - A2 - 3∗A3 + 24∗A6 - 12∗A7 - A9)∗Connection[a, b, -a]∗Connection[-b, c, -c]) / 6 + ((-4∗A0 + 10∗A1 + A2 + 3∗A3 - 32∗A6 + 20∗A7 + 4∗A9)∗Connection[a, b, c]∗Connection[-c, -b, -a]) / 8 + ((3∗A0 + 2∗A1 + A2 + 3∗A3 - 6∗A7 + A9)∗Connection[a, -a, b]∗Connection[c, -b, -c]) / 6 - (A7∗Connection[a, b, -a]∗Connection[c, -b, -c]) / 2 + ((2∗A1 + A2 - 3∗A7 + A9)∗Connection[a, -a, b]∗Connection[c, -c, -b]) / 6 - (A0∗MetricPerturbation[c, -c]∗CD[-b][Connection[a, -a, b]]) / 4 - (A0∗MetricPerturbation[c, -c]∗CD[-b][Connection[a, b, -a]]) / 4 - (A0∗MetricPerturbation[a, -c]∗CD[-b][Connection[a, b, c]]) / 2 -
   (A0∗Connection[a, b, c]∗CD[-b][MetricPerturbation[-a, -c]]) / 2 - (A0∗Connection[a, -a, b]∗CD[-b][MetricPerturbation[c, -c]]) / 4 + (A0∗Connection[a, b, -a]∗CD[-b][MetricPerturbation[c, -c]]) / 4 + (A0∗MetricPerturbation[a, b]∗CD[-b][CD[-a][MetricPerturbation[c, -c]]]) / 4 - (A0∗CD[-b][MetricPerturbation[c, -c]]∗CD[b][MetricPerturbation[a, -a]]) / 8 + (A0∗Connection[a, -a, b]∗CD[-c][MetricPerturbation[-b, c]]) / 2 + (A0∗CD[b][MetricPerturbation[a, -a]]∗CD[-c][MetricPerturbation[-b, c]]) / 4 - (A0∗MetricPerturbation[a, b]∗CD[-c][CD[-b][MetricPerturbation[-a, c]]]) / 2 +
   (A0∗MetricPerturbation[a, -a]∗CD[-c][CD[c][MetricPerturbation[b, -b]]]) / 4 + (A0∗MetricPerturbation[a, b]∗CD[-c][CD[c][MetricPerturbation[-a, -b]]]) / 4 - (A0∗CD[-b][MetricPerturbation[-a, -c]]∗CD[c][MetricPerturbation[a, b]]) / 4 + (A0∗CD[-c][MetricPerturbation[-a, -b]]∗CD[c][MetricPerturbation[a, b]]) / 8 + 2∗C1∗CD[-a][Connection[-b, -c, -m]]∗CD[m][Connection[a, b, c]] - 2∗C1∗CD[-a][Connection[-b, -m, -c]]∗CD[m][Connection[a, b, c]] -
   2∗C1∗CD[-a][Connection[-c, -b, -m]]∗CD[m][Connection[a, b, c]] + 2∗C1∗CD[-a][Connection[-c, -m, -b]]∗CD[m][Connection[a, b, c]] + C1∗CD[-a][Connection[-m, -b, -c]]∗CD[m][Connection[a, b, c]] - C1∗CD[-a][Connection[-m, -c, -b]]∗CD[m][Connection[a, b, c]] - 2∗C1∗CD[-b][Connection[-a, -c, -m]]∗CD[m][Connection[a, b, c]] + C1∗CD[-b][Connection[-a, -m, -c]]∗CD[m][Connection[a, b, c]] - C1∗CD[-b][Connection[-c, -m, -a]]∗CD[m][Connection[a, b, c]] + C1∗CD[-c][Connection[-b, -a, -m]]∗CD[m][Connection[a, b, c]] - C1∗CD[-c][Connection[-b, -m, -a]]∗CD[m][Connection[a, b, c]] + 2∗C1∗CD[-c][Connection[-b, -m, -a]]∗CD[m][Connection[a, b, c]] -
   C1∗CD[-m][Connection[-a, -b, -c]]∗CD[m][Connection[a, b, c]] + C1∗CD[-m][Connection[-a, -c, -b]]∗CD[m][Connection[a, b, c]] + C1∗CD[-m][Connection[-b, -a, -c]]∗CD[m][Connection[a, b, c]] - 2∗C1∗CD[-m][Connection[-b, -c, -a]]∗CD[m][Connection[a, b, c]] + C1∗CD[-m][Connection[-c, -b, -a]]∗CD[m][Connection[a, b, c]] + C1∗CD[-c][CD[-b][MetricPerturbation[-a, -m]]]∗CD[m][CD[c][MetricPerturbation[a, b]]] - C1∗CD[-m][CD[-b][MetricPerturbation[-a, -c]]]∗CD[m][CD[c][MetricPerturbation[a, b]]], TheoryName → "ExtendedProjectiveTheorySecondOrder", Method → "Hard", MaxLaurentDepth → 1];
```

| | $\Delta^{\#1}_{1^+\,\alpha\beta}$ | $\Delta^{\#2}_{1^+\,\alpha\beta}$ | $\Delta^{\#3}_{1^+\,\alpha\beta}$ | $\Delta^{\#1}_{1^-\,\alpha}$ | $\Delta^{\#2}_{1^-\,\alpha}$ | $\Delta^{\#3}_{1^-\,\alpha}$ | $\Delta^{\#4}_{1^-\,\alpha}$ | $\Delta^{\#5}_{1^-\,\alpha}$ | $\Delta^{\#6}_{1^-\,\alpha}$ | $\mathcal{T}^{\#1}_{1^-\,\alpha}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta^{\#1}_{1^+}\dagger^{\alpha\beta}$ | $\frac{4}{3}\left(-\frac{1}{a_0+a_1-4a_7}+(a_0-4a_1-2a_2-3a_3+16a_6-4a_7-2a_9)/(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))\right)$ | $\frac{4}{3}\sqrt{2}\left(-\frac{1}{a_0+a_1-4a_7}+(-2a_0+8a_1+4a_2+6a_3-32a_6+8a_7+4a_9)/(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))\right)$ | $\frac{4(2a_1+a_2+a_9)}{3(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#2}_{1^+}\dagger^{\alpha\beta}$ | $\frac{4}{3}\sqrt{2}\left(-\frac{1}{a_0+a_1-4a_7}+(-2a_0+8a_1+4a_2+6a_3-32a_6+8a_7+4a_9)/(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))\right)$ | $-\frac{2}{3(a_0+a_1-4a_7)}+(8(a_0-4a_1-2a_2-3a_3+16a_6-4a_7-2a_9))/(3(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9)))$ | $-((4\sqrt{2}(2a_1+a_2+a_9))/(3(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))))$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#3}_{1^+}\dagger^{\alpha\beta}$ | $\frac{4(2a_1+a_2+a_9)}{3(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))}$ | $-((4\sqrt{2}(2a_1+a_2+a_9))/(3(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))))$ | $\frac{4(a_0-2a_1-a_2)}{3(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{4}{3}\left(-\frac{1}{2a_0+2a_1+a_2+3a_3}+(a_0-4a_1-2a_2-3a_3+16a_6-4a_7-2a_9)/(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))\right)$ | $(4\sqrt{2}(3a_0^2-4a_1^2-a_2^2-3a_3(3a_1+4(-4a_6+a_7))-6a_1a_9-a_9^2-2a_2(3a_3+a_9)-4a_1(2a_2+3a_3+a_9)))/(3(2a_0+2a_1+a_2+3a_3)(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9)))$ | 0 | 0 | $\frac{4\sqrt{\frac{2}{3}}(2a_1+a_2+a_9)}{3(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))}$ | $-((4(2a_1+a_2+a_9))/(3\sqrt{3}(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))))$ | 0 |
| $\Delta^{\#2}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $(4\sqrt{2}(3a_0^2-4a_1^2-a_2^2-3a_3(3a_1+4(-4a_6+a_7))-6a_1a_9-a_9^2-2a_2(3a_3+a_9)-4a_1(2a_2+3a_3+a_9)))/(3(2a_0+2a_1+a_2+3a_3)(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9)))$ | $-\frac{8}{3(2a_0+2a_1+a_2+3a_3)}+(8(a_0-4a_1-2a_2-3a_3+16a_6-4a_7-2a_9))/(3(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9)))$ | 0 | 0 | $(8(2a_1+a_2+a_9))/(3\sqrt{3}(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9)))$ | $-((4\sqrt{\frac{2}{3}}(2a_1+a_2+a_9))/(3(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))))$ | 0 |
| $\Delta^{\#3}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | $-\frac{5}{18(a_0+4a_6-4a_7)}$ | $\frac{\sqrt{5}}{18(a_0+4a_6-4a_7)}$ | 0 | 0 | 0 |
| $\Delta^{\#4}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | $\frac{\sqrt{5}}{18(a_0+4a_6-4a_7)}$ | $\frac{1}{-18(a_0+4a_6)+72a_7}$ | 0 | 0 | 0 |
| $\Delta^{\#5}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{4\sqrt{\frac{2}{3}}(2a_1+a_2+a_9)}{3(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))}$ | $(8(2a_1+a_2+a_9))/(3\sqrt{3}(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9)))$ | 0 | 0 | $\frac{8(a_0-2a_1-a_2)}{9(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))}$ | $\frac{4\sqrt{2}(a_0-2a_1-a_2)}{9(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))}$ | 0 |
| $\Delta^{\#6}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $-((4(2a_1+a_2+a_9))/(3\sqrt{3}(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))))$ | $-((4\sqrt{\frac{2}{3}}(2a_1+a_2+a_9))/(3(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))))$ | 0 | 0 | $\frac{4\sqrt{2}(a_0-2a_1-a_2)}{9(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))}$ | $\frac{4(a_0-2a_1-a_2)}{9(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))}$ | 0 |
| $\mathcal{T}^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| | $\Gamma^{\#1}_{1^+\,\alpha\beta}$ | $\Gamma^{\#2}_{1^+\,\alpha\beta}$ | $\Gamma^{\#3}_{1^+\,\alpha\beta}$ | $\Gamma^{\#1}_{1^-\,\alpha}$ | $\Gamma^{\#2}_{1^-\,\alpha}$ | $\Gamma^{\#3}_{1^-\,\alpha}$ | $\Gamma^{\#4}_{1^-\,\alpha}$ | $\Gamma^{\#5}_{1^-\,\alpha}$ | $\Gamma^{\#6}_{1^-\,\alpha}$ | $h^{\#1}_{1^-\,\alpha}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\Gamma^{\#1}_{1^+}\dagger^{\alpha\beta}$ | $\frac{1}{4}(-a_0-6a_1+5a_2)$ | $-\frac{a_0+2a_1-3a_2}{2\sqrt{2}}$ | $\frac{1}{4}(-2a_1-a_2-a_9)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#2}_{1^+}\dagger^{\alpha\beta}$ | $-\frac{a_0+2a_1-3a_2}{2\sqrt{2}}$ | $\frac{1}{2}(-2a_1+a_2)$ | $\frac{2a_1+a_2+a_9}{2\sqrt{2}}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#3}_{1^+}\dagger^{\alpha\beta}$ | $\frac{1}{4}(-2a_1-a_2-a_9)$ | $\frac{2a_1+a_2+a_9}{2\sqrt{2}}$ | $\frac{3}{4}(a_0-4a_1-2a_2-3a_3+16a_6-4a_7-2a_9)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{1}{4}(-a_0-2a_1-a_2-2a_3)$ | $\frac{a_0+a_3}{2\sqrt{2}}$ | 0 | 0 | $-\frac{2a_1+a_2+a_9}{2\sqrt{6}}$ | $\frac{2a_1+a_2+a_9}{4\sqrt{3}}$ | 0 |
| $\Gamma^{\#2}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{a_0+a_3}{2\sqrt{2}}$ | $\frac{1}{4}(-2a_1-a_2-a_3)$ | 0 | 0 | $-\frac{2a_1+a_2+a_9}{2\sqrt{3}}$ | $\frac{2a_1+a_2+a_9}{2\sqrt{6}}$ | 0 |
| $\Gamma^{\#3}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | $-\frac{5}{2}(a_0+4a_6-4a_7)$ | $\frac{1}{2}\sqrt{5}(a_0+4a_6-4a_7)$ | 0 | 0 | 0 |
| $\Gamma^{\#4}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | $\frac{1}{2}\sqrt{5}(a_0+4a_6-4a_7)$ | $-\frac{a_0}{2}-2a_6+2a_7$ | 0 | 0 | 0 |
| $\Gamma^{\#5}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $-\frac{2a_1+a_2+a_9}{2\sqrt{6}}$ | $-\frac{2a_1+a_2+a_9}{2\sqrt{3}}$ | 0 | 0 | $\frac{1}{2}(a_0-4a_1-2a_2-3a_3+16a_6-4a_7-2a_9)$ | $\frac{-a_0+4a_1+2a_2+3a_3-16a_6+4a_7+2a_9}{2\sqrt{2}}$ | 0 |
| $\Gamma^{\#6}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{2a_1+a_2+a_9}{4\sqrt{3}}$ | $\frac{2a_1+a_2+a_9}{2\sqrt{6}}$ | 0 | 0 | $\frac{-a_0+4a_1+2a_2+3a_3-16a_6+4a_7+2a_9}{2\sqrt{2}}$ | $\frac{1}{4}(a_0-4a_1-2a_2-3a_3+16a_6-4a_7-2a_9)$ | 0 |
| $h^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| | $\Delta^{\#1}_{0^+}$ | $\Delta^{\#2}_{0^+}$ | $\Delta^{\#3}_{0^+}$ | $\Delta^{\#4}_{0^+}$ | $\mathcal{T}^{\#1}_{0^+}$ | $\mathcal{T}^{\#2}_{0^+}$ | $\Delta^{\#1}_{0^-}$ |
|---|---|---|---|---|---|---|---|
| $\Delta^{\#1}_{0^+}\dagger$ | $-\frac{4}{2a_0+2a_1+a_2+3a_3}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#2}_{0^+}\dagger$ | 0 | $\frac{1}{-6(a_0+4a_6)+24a_7}$ | $\frac{1}{6(a_0+4a_6-4a_7)}$ | 0 | 0 | 0 | 0 |
| $\Delta^{\#3}_{0^+}\dagger$ | 0 | $\frac{1}{6a_0+24a_6-24a_7}$ | $\frac{1}{-6(a_0+4a_6)+24a_7}$ | 0 | 0 | 0 | 0 |
| $\Delta^{\#4}_{0^+}\dagger$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathcal{T}^{\#1}_{0^+}\dagger$ | 0 | 0 | 0 | 0 | $\frac{4}{a_0k^2}$ | 0 | 0 |
| $\mathcal{T}^{\#2}_{0^+}\dagger$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#1}_{0^-}\dagger$ | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{2}{a_0+4(a_1+a_2)+3c_1k^2}$ |

Lagrangian density

$$\frac{2}{3}a_1\Gamma_{\alpha}{}^{\chi}{}_{\chi}\Gamma^{\alpha\beta}{}_{\beta}+\frac{1}{3}a_2\Gamma_{\alpha}{}^{\chi}{}_{\chi}\Gamma^{\alpha\beta}{}_{\beta}+\frac{1}{2}a_3\Gamma_{\alpha}{}^{\chi}{}_{\chi}\Gamma^{\alpha\beta}{}_{\beta}-2a_6\Gamma_{\alpha}{}^{\chi}{}_{\chi}\Gamma^{\alpha\beta}{}_{\beta}+\frac{1}{3}a_9\Gamma_{\alpha}{}^{\chi}{}_{\chi}\Gamma^{\alpha\beta}{}_{\beta}-\frac{1}{4}a_0\Gamma_{\alpha\beta\chi}\Gamma^{\alpha\beta\chi}-\frac{3}{2}a_1\Gamma_{\alpha\beta\chi}\Gamma^{\alpha\beta\chi}-\frac{1}{4}a_2\Gamma_{\alpha\beta\chi}\Gamma^{\alpha\beta\chi}-\frac{3}{4}a_3\Gamma_{\alpha\beta\chi}\Gamma^{\alpha\beta\chi}+2a_6\Gamma_{\alpha\beta\chi}\Gamma^{\alpha\beta\chi}+a_7\Gamma_{\alpha\beta\chi}\Gamma^{\alpha\beta\chi}-\frac{1}{2}a_9\Gamma_{\alpha\beta\chi}\Gamma^{\alpha\beta\chi}-\frac{1}{4}a_0\Gamma_{\alpha\chi\beta}\Gamma^{\alpha\beta\chi}-\frac{1}{2}a_1\Gamma_{\alpha\chi\beta}\Gamma^{\alpha\beta\chi}-\frac{3}{4}a_2\Gamma_{\alpha\chi\beta}\Gamma^{\alpha\beta\chi}-\frac{3}{4}a_3\Gamma_{\alpha\chi\beta}\Gamma^{\alpha\beta\chi}+2a_6\Gamma_{\alpha\chi\beta}\Gamma^{\alpha\beta\chi}+a_7\Gamma_{\alpha\chi\beta}\Gamma^{\alpha\beta\chi}-\frac{1}{2}a_9\Gamma_{\alpha\chi\beta}\Gamma^{\alpha\beta\chi}-\frac{1}{2}a_0\Gamma^{\alpha\beta\chi}\Gamma_{\beta\alpha\chi}+\frac{1}{4}a_1\Gamma^{\alpha\beta\chi}\Gamma_{\beta\alpha\chi}-\frac{3}{8}a_2\Gamma^{\alpha\beta\chi}\Gamma_{\beta\alpha\chi}+\frac{3}{8}a_3\Gamma^{\alpha\beta\chi}\Gamma_{\beta\alpha\chi}-4a_6\Gamma^{\alpha\beta\chi}\Gamma_{\beta\alpha\chi}+\frac{5}{2}a_7\Gamma^{\alpha\beta\chi}\Gamma_{\beta\alpha\chi}-\frac{3}{2}a_0\Gamma^{\alpha\beta\chi}\Gamma_{\beta\chi\alpha}+\frac{1}{2}a_1\Gamma^{\alpha\beta\chi}\Gamma_{\beta\chi\alpha}+\frac{5}{4}a_2\Gamma^{\alpha\beta\chi}\Gamma_{\beta\chi\alpha}+\frac{3}{4}a_3\Gamma^{\alpha\beta\chi}\Gamma_{\beta\chi\alpha}-8a_6\Gamma^{\alpha\beta\chi}\Gamma_{\beta\chi\alpha}+5a_7\Gamma^{\alpha\beta\chi}\Gamma_{\beta\chi\alpha}+\frac{1}{2}a_9\Gamma^{\alpha\beta\chi}\Gamma_{\beta\chi\alpha}+\frac{1}{2}a_0\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma_{\beta}{}^{\chi}{}_{\chi}-a_1\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma_{\beta}{}^{\chi}{}_{\chi}-\frac{1}{2}a_2\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma_{\beta}{}^{\chi}{}_{\chi}-\frac{1}{2}a_3\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma_{\beta}{}^{\chi}{}_{\chi}+4a_6\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma_{\beta}{}^{\chi}{}_{\chi}-2a_7\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma_{\beta}{}^{\chi}{}_{\chi}-\frac{1}{2}a_9\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma_{\beta}{}^{\chi}{}_{\chi}+\frac{1}{2}a_0\Gamma^{\alpha\beta}{}_{\alpha}\Gamma_{\beta}{}^{\chi}{}_{\chi}-\frac{1}{3}a_1\Gamma^{\alpha\beta}{}_{\alpha}\Gamma_{\beta}{}^{\chi}{}_{\chi}-\frac{1}{6}a_2\Gamma^{\alpha\beta}{}_{\alpha}\Gamma_{\beta}{}^{\chi}{}_{\chi}-\frac{1}{2}a_3\Gamma^{\alpha\beta}{}_{\alpha}\Gamma_{\beta}{}^{\chi}{}_{\chi}+4a_6\Gamma^{\alpha\beta}{}_{\alpha}\Gamma_{\beta}{}^{\chi}{}_{\chi}-2a_7\Gamma^{\alpha\beta}{}_{\alpha}\Gamma_{\beta}{}^{\chi}{}_{\chi}-\frac{1}{6}a_9\Gamma^{\alpha\beta}{}_{\alpha}\Gamma_{\beta}{}^{\chi}{}_{\chi}-\frac{1}{2}a_0\Gamma^{\alpha\beta\chi}\Gamma_{\chi\beta\alpha}+\frac{5}{4}a_1\Gamma^{\alpha\beta\chi}\Gamma_{\chi\beta\alpha}+\frac{1}{8}a_2\Gamma^{\alpha\beta\chi}\Gamma_{\chi\beta\alpha}+\frac{3}{8}a_3\Gamma^{\alpha\beta\chi}\Gamma_{\chi\beta\alpha}-4a_6\Gamma^{\alpha\beta\chi}\Gamma_{\chi\beta\alpha}+\frac{5}{2}a_7\Gamma^{\alpha\beta\chi}\Gamma_{\chi\beta\alpha}+\frac{1}{2}a_9\Gamma^{\alpha\beta\chi}\Gamma_{\chi\beta\alpha}+\frac{1}{2}a_0\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma^{\chi}{}_{\beta\chi}+\frac{1}{3}a_1\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma^{\chi}{}_{\beta\chi}+\frac{1}{6}a_2\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma^{\chi}{}_{\beta\chi}+\frac{1}{2}a_3\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma^{\chi}{}_{\beta\chi}-a_7\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma^{\chi}{}_{\beta\chi}+\frac{1}{6}a_9\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma^{\chi}{}_{\beta\chi}-\frac{1}{2}a_7\Gamma^{\alpha\beta}{}_{\alpha}\Gamma^{\chi}{}_{\beta\chi}+\frac{1}{3}a_1\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma^{\chi}{}_{\chi\beta}+\frac{1}{6}a_2\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma^{\chi}{}_{\chi\beta}-\frac{1}{2}a_7\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma^{\chi}{}_{\chi\beta}+\frac{1}{6}a_9\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma^{\chi}{}_{\chi\beta}-\frac{1}{2}a_0\Gamma^{\alpha\beta\chi}\partial_{\beta}h_{\alpha\chi}-\frac{1}{4}a_0\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\partial_{\beta}h^{\chi}{}_{\chi}+\frac{1}{4}a_0\Gamma^{\alpha\beta}{}_{\alpha}\partial_{\beta}h^{\chi}{}_{\chi}-\frac{1}{4}a_0h^{\chi}{}_{\chi}\partial_{\beta}\Gamma^{\alpha}{}_{\alpha}{}^{\beta}+\frac{1}{4}a_0h^{\chi}{}_{\chi}\partial_{\beta}\Gamma^{\alpha\beta}{}_{\alpha}-\frac{1}{2}a_0h_{\alpha\chi}\partial_{\beta}\Gamma^{\alpha\beta\chi}+\frac{1}{4}a_0h^{\alpha\beta}\partial_{\beta}\partial_{\alpha}h^{\chi}{}_{\chi}-\frac{1}{8}a_0\partial_{\beta}h^{\chi}{}_{\chi}\partial^{\beta}h^{\alpha}{}_{\alpha}+\frac{1}{2}a_0\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\partial_{\chi}h_{\beta}{}^{\chi}+\frac{1}{4}a_0\partial^{\beta}h^{\alpha}{}_{\alpha}\partial_{\chi}h_{\beta}{}^{\chi}-\frac{1}{2}a_0h^{\alpha\beta}\partial_{\chi}\partial_{\beta}h_{\alpha}{}^{\chi}+\frac{1}{4}a_0h^{\alpha}{}_{\alpha}\partial_{\chi}\partial_{\beta}h^{\beta\chi}+\frac{1}{4}a_0h^{\alpha\beta}\partial_{\chi}\partial^{\chi}h_{\alpha\beta}-\frac{1}{4}a_0h^{\alpha}{}_{\alpha}\partial_{\chi}\partial^{\chi}h^{\beta}{}_{\beta}-\frac{1}{4}a_0\partial_{\beta}h_{\alpha\chi}\partial^{\chi}h^{\alpha\beta}+\frac{1}{8}a_0\partial_{\chi}h_{\alpha\beta}\partial^{\chi}h^{\alpha\beta}+\frac{1}{2}a_0h_{\beta\chi}\partial^{\chi}\Gamma^{\alpha}{}_{\alpha}{}^{\beta}+2c_1\partial_{\alpha}\Gamma_{\beta\chi\mu}\partial^{\mu}\Gamma^{\alpha\beta\chi}-2c_1\partial_{\alpha}\Gamma_{\beta\mu\chi}\partial^{\mu}\Gamma^{\alpha\beta\chi}-2c_1\partial_{\alpha}\Gamma_{\chi\beta\mu}\partial^{\mu}\Gamma^{\alpha\beta\chi}+2c_1\partial_{\alpha}\Gamma_{\chi\mu\beta}\partial^{\mu}\Gamma^{\alpha\beta\chi}+c_1\partial_{\alpha}\Gamma_{\mu\beta\chi}\partial^{\mu}\Gamma^{\alpha\beta\chi}-c_1\partial_{\alpha}\Gamma_{\mu\chi\beta}\partial^{\mu}\Gamma^{\alpha\beta\chi}-2c_1\partial_{\beta}\Gamma_{\alpha\chi\mu}\partial^{\mu}\Gamma^{\alpha\beta\chi}+c_1\partial_{\beta}\Gamma_{\alpha\mu\chi}\partial^{\mu}\Gamma^{\alpha\beta\chi}-c_1\partial_{\beta}\Gamma_{\chi\mu\alpha}\partial^{\mu}\Gamma^{\alpha\beta\chi}+c_1\partial_{\chi}\Gamma_{\alpha\beta\mu}\partial^{\mu}\Gamma^{\alpha\beta\chi}-c_1\partial_{\chi}\Gamma_{\beta\alpha\mu}\partial^{\mu}\Gamma^{\alpha\beta\chi}+2c_1\partial_{\chi}\Gamma_{\beta\mu\alpha}\partial^{\mu}\Gamma^{\alpha\beta\chi}-c_1\partial_{\mu}\Gamma_{\alpha\beta\chi}\partial^{\mu}\Gamma^{\alpha\beta\chi}+c_1\partial_{\mu}\Gamma_{\alpha\chi\beta}\partial^{\mu}\Gamma^{\alpha\beta\chi}+c_1\partial_{\mu}\Gamma_{\beta\alpha\chi}\partial^{\mu}\Gamma^{\alpha\beta\chi}-2c_1\partial_{\mu}\Gamma_{\beta\chi\alpha}\partial^{\mu}\Gamma^{\alpha\beta\chi}+c_1\partial_{\mu}\Gamma_{\chi\beta\alpha}\partial^{\mu}\Gamma^{\alpha\beta\chi}+c_1\partial_{\chi}\partial_{\beta}h_{\alpha\mu}\partial^{\mu}\partial^{\chi}h^{\alpha\beta}-c_1\partial_{\mu}\partial_{\beta}h_{\alpha\chi}\partial^{\mu}\partial^{\chi}h^{\alpha\beta}$$

Added source term: $h^{\alpha\beta}\,\mathcal{T}_{\alpha\beta}+\Gamma^{\alpha\beta\chi}\,\Delta_{\alpha\beta\chi}$

| | $\Gamma^{\#1}_{0^+}$ | $\Gamma^{\#2}_{0^+}$ | $\Gamma^{\#3}_{0^+}$ | $\Gamma^{\#4}_{0^+}$ | $h^{\#1}_{0^+}$ | $h^{\#2}_{0^+}$ | $\Gamma^{\#1}_{0^-}$ |
|---|---|---|---|---|---|---|---|
| $\Gamma^{\#1}_{0^+}\dagger$ | [illegible] | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#2}_{0^+}\dagger$ | 0 | [illegible] | [illegible] | 0 | 0 | 0 | 0 |
| $\Gamma^{\#3}_{0^+}\dagger$ | 0 | [illegible] | [illegible] | 0 | 0 | 0 | 0 |
| $\Gamma^{\#4}_{0^+}\dagger$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $h^{\#1}_{0^+}\dagger$ | 0 | 0 | 0 | 0 | [illegible] | 0 | 0 |
| $h^{\#2}_{0^+}\dagger$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#1}_{0^-}\dagger$ | 0 | 0 | 0 | 0 | 0 | 0 | [illegible] |

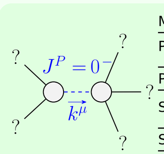
| | $\Delta^{\#1}_{2^+\,\alpha\beta}$ | $\Delta^{\#2}_{2^+\,\alpha\beta}$ | $\Delta^{\#3}_{2^+\,\alpha\beta}$ | $\mathcal{T}^{\#1}_{2^+\,\alpha\beta}$ | $\Delta^{\#1}_{2^-\,\alpha\beta\chi}$ | $\Delta^{\#2}_{2^-\,\alpha\beta\chi}$ |
|---|---|---|---|---|---|---|
| $\Delta^{\#1}_{2^+}\dagger^{\alpha\beta}$ | $\frac{4(a_0-4a_1-2a_2-3a_3+16a_6-4a_7-2a_9)}{a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9)}$ | 0 | $\frac{4(2a_1+a_2+a_9)}{\sqrt{3}(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))}$ | 0 | 0 | 0 |
| $\Delta^{\#2}_{2^+}\dagger^{\alpha\beta}$ | 0 | $\frac{1}{-3(a_0+4a_6)+12a_7}$ | 0 | 0 | 0 | 0 |
| $\Delta^{\#3}_{2^+}\dagger^{\alpha\beta}$ | $\frac{4(2a_1+a_2+a_9)}{\sqrt{3}(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))}$ | 0 | $\frac{4(a_0-2a_1-a_2)}{3(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))}$ | 0 | 0 | 0 |
| $\mathcal{T}^{\#1}_{2^+}\dagger^{\alpha\beta}$ | 0 | 0 | 0 | $-\frac{8}{a_0k^2}$ | 0 | 0 |
| $\Delta^{\#1}_{2^-}\dagger^{\alpha\beta\chi}$ | 0 | 0 | 0 | 0 | $\frac{4(a_0-4a_1-2a_2-3a_3+16a_6-4a_7-2a_9)}{a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9)}$ | $\frac{4(2a_1+a_2+a_9)}{\sqrt{3}(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))}$ |
| $\Delta^{\#2}_{2^-}\dagger^{\alpha\beta\chi}$ | 0 | 0 | 0 | 0 | $\frac{4(2a_1+a_2+a_9)}{\sqrt{3}(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))}$ | $\frac{4(a_0-2a_1-a_2)}{3(a_0^2+(2a_1+a_2)(2a_1+a_2+3a_3-16a_6+4a_7)-a_9^2-a_0(6a_1+3a_2+3a_3-16a_6+4a_7+2a_9))}$ |

| | $\Gamma^{\#1}_{2^+\,\alpha\beta}$ | $\Gamma^{\#2}_{2^+\,\alpha\beta}$ | $\Gamma^{\#3}_{2^+\,\alpha\beta}$ | $h^{\#1}_{2^+\,\alpha\beta}$ | $\Gamma^{\#1}_{2^-\,\alpha\beta\chi}$ | $\Gamma^{\#2}_{2^-\,\alpha\beta\chi}$ |
|---|---|---|---|---|---|---|
| $\Gamma^{\#1}_{2^+}\dagger^{\alpha\beta}$ | $\frac{1}{4}(a_0-2a_1-a_2)$ | 0 | $-\frac{1}{4}\sqrt{3}(2a_1+a_2+a_9)$ | 0 | 0 | 0 |
| $\Gamma^{\#2}_{2^+}\dagger^{\alpha\beta}$ | 0 | $-3(a_0+4a_6-4a_7)$ | 0 | 0 | 0 | 0 |
| $\Gamma^{\#3}_{2^+}\dagger^{\alpha\beta}$ | $-\frac{1}{4}\sqrt{3}(2a_1+a_2+a_9)$ | 0 | $\frac{3}{4}(a_0-4a_1-2a_2-3a_3+16a_6-4a_7-2a_9)$ | 0 | 0 | 0 |
| $h^{\#1}_{2^+}\dagger^{\alpha\beta}$ | 0 | 0 | 0 | $-\frac{a_0k^2}{8}$ | 0 | 0 |
| $\Gamma^{\#1}_{2^-}\dagger^{\alpha\beta\chi}$ | 0 | 0 | 0 | 0 | $\frac{1}{4}(a_0-2a_1-a_2)$ | $-\frac{1}{4}\sqrt{3}(2a_1+a_2+a_9)$ |
| $\Gamma^{\#2}_{2^-}\dagger^{\alpha\beta\chi}$ | 0 | 0 | 0 | 0 | $-\frac{1}{4}\sqrt{3}(2a_1+a_2+a_9)$ | $\frac{3}{4}(a_0-4a_1-2a_2-3a_3+16a_6-4a_7-2a_9)$ |

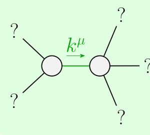
| Source constraints | |
|---|---|
| SO(3) irreps | # |
| $\mathcal{T}^{\#2}_{0^+}==0$ | 1 |
| $\Delta^{\#4}_{0^+}==0$ | 1 |
| $\Delta^{\#3}_{0^+}+3\Delta^{\#2}_{0^+}==0$ | 1 |
| $\mathcal{T}^{\#1\alpha}_{1^-}==0$ | 3 |
| $\Delta^{\#6\alpha}_{1^-}+\Delta^{\#5\alpha}_{1^-}==0$ | 3 |
| $\Delta^{\#4\alpha}_{1^-}+\Delta^{\#3\alpha}_{1^-}==0$ | 3 |
| Total #: | 12 |

| | $\Gamma^{\#1}_{3^-\,\alpha\beta\chi}$ |
|---|---|
| $\Gamma^{\#1}_{3^-}\dagger^{\alpha\beta\chi}$ | $-3(a_0+4a_6-4a_7)$ |

| | $\Delta^{\#1}_{3^-\,\alpha\beta\chi}$ |
|---|---|
| $\Delta^{\#1}_{3^-}\dagger^{\alpha\beta\chi}$ | $\frac{1}{-3(a_0+4a_6)+12a_7}$ |




| Massive particle | |
|---|---|
| Pole residue: | $-\frac{1}{a_0c_1}>0$ |
| Polarisations: | 1 |
| Square mass: | $\frac{a_0+2a_1+a_2}{12c_1}>0$ |
| Spin: | 0 |
| Parity: | Odd |




| Quadratic pole | |
|---|---|
| Pole residue: | $-\frac{1}{a_0}>0$ |
| Polarisations: | 2 |

Unitarity conditions

$a_0<0\,\&\&\,a_2>\frac{1}{4}(a_0+4a_1)\,\&\&\,c_1>0$

A massive pseudoscalar accompanies the graviton, and the whole theory may be made to be unitary.

## The projective theory

We will study the projective symmetry.

$$\frac{1}{2}\Big(-\mathcal{F}^{\mu\nu\rho\sigma}\left(c_1\mathcal{F}_{\mu\nu\rho\sigma}+c_2\mathcal{F}_{\mu\nu\sigma\rho}+c_4\mathcal{F}_{\mu\rho\nu\sigma}+c_5\mathcal{F}_{\mu\sigma\nu\rho}+c_6\mathcal{F}_{\mu\sigma\rho\nu}+c_3\mathcal{F}_{\rho\sigma\mu\nu}\right)-\left(a_5Q_{\nu\mu\rho}+a_4Q_{\rho\mu\nu}\right)Q^{\rho\mu\nu}-a_6Q_{\mu}Q^{\mu}-\left(\frac{3a_1}{8}+\frac{3a_2}{16}+\frac{9a_3}{16}-a_4-\frac{a_5}{4}-4a_6-\frac{a_7}{4}\right)Q_{\mu}\tilde{Q}^{\mu}-a_7\tilde{Q}_{\mu}\tilde{Q}^{\mu}+a_0\mathcal{F}-c_{16}\mathcal{F}^2-\mathcal{F}^{(13)\mu\nu}\left(c_7\mathcal{F}^{(13)}{}_{\mu\nu}+c_8\mathcal{F}^{(13)}{}_{\nu\mu}\right)-\mathcal{F}^{\mu\nu}\left(c_{13}\mathcal{F}_{\mu\nu}+c_{14}\mathcal{F}^{(13)}{}_{\mu\nu}+(2c_1+8c_{13}-c_{14}+2c_2)\mathcal{F}^{(14)}{}_{\mu\nu}\right)-\left(c_{11}\mathcal{F}^{(13)}{}_{\mu\nu}+(c_{11}-4c_{14}+c_{12}+2c_5+2c_1-2c_9)\mathcal{F}^{(13)}{}_{\nu\mu}\right)\mathcal{F}^{(14)\mu\nu}-\mathcal{F}^{(14)\mu\nu}\left(c_9\mathcal{F}^{(14)}{}_{\mu\nu}+(-4c_1-16c_{13}+4c_{14}-4c_2+c_4-c_5-c_7+c_8+c_9)\mathcal{F}^{(14)}{}_{\nu\mu}\right)-a_5Q_{\mu\rho\nu}\tilde{Q}^{\rho\mu\nu}-\left(a_2\mathcal{T}_{\nu\rho\mu}+a_3\mathcal{T}_{\rho\nu\mu}\right)\mathcal{T}^{\mu\nu\rho}-\left(\left(\frac{a_1}{4}+\frac{a_2}{8}+\frac{3a_3}{8}+\frac{2a_4}{3}-\frac{a_5}{6}+\frac{8a_6}{3}-\frac{a_7}{6}+\frac{a_9}{3}\right)Q_{\mu}+\left(a_1+\frac{a_2}{2}+\frac{3a_3}{2}-\frac{8a_4}{3}+\frac{2a_5}{3}-\frac{32a_6}{3}+\frac{2a_7}{3}-\frac{a_9}{3}\right)\tilde{Q}_{\mu}\right)\mathcal{T}^{\mu}-a_1\mathcal{T}_{\mu}\mathcal{T}^{\mu}\Big) \tag{75}$$

In the most general form Eq. (75), the theory has a symmetry which is cumbersome to express in terms of the remaining couplings. We therefore take an arbitrary case of the theory.

$$\frac{1}{2}\left(-\mathcal{F}^{\mu\nu\rho\sigma}\left(c_1\mathcal{F}_{\mu\nu\rho\sigma}+c_1\mathcal{F}_{\mu\nu\sigma\rho}+c_1\mathcal{F}_{\mu\rho\nu\sigma}+c_1\mathcal{F}_{\mu\sigma\nu\rho}+c_1\mathcal{F}_{\mu\sigma\rho\nu}+c_1\mathcal{F}_{\rho\sigma\mu\nu}\right)+a_0\mathcal{F}-c_1\mathcal{F}^2-\mathcal{F}^{(13)\mu\nu}\left(c_1\mathcal{F}^{(13)}{}_{\mu\nu}+c_1\mathcal{F}^{(13)}{}_{\nu\mu}\right)-\mathcal{F}^{\mu\nu}\left(c_1\mathcal{F}_{\mu\nu}+c_1\mathcal{F}^{(13)}{}_{\mu\nu}+11c_1\mathcal{F}^{(14)}{}_{\nu\mu}\right)-c_1\mathcal{F}^{(13)}{}_{\mu\nu}\mathcal{F}^{(14)\mu\nu}-\mathcal{F}^{(14)\mu\nu}\left(c_1\mathcal{F}^{(14)}{}_{\mu\nu}-19c_1\mathcal{F}^{(14)}{}_{\nu\mu}\right)\right) \tag{76}$$

First we study Eq. (76) in the first-order formulation:

```
ParticleSpectrum[-1/2 (A0 * Connection[a, b, c] * Connection[-b, -c, -a]) + (A0 * Connection[a, -a, b] * Connection[c, -b, -c])/2 - (A0 * MetricPerturbation[c, -c] * CD[-b][Connection[a, -a, b]])/4 + (A0 * MetricPerturbation[c, -c] * CD[-b][Connection[a, b, -a]])/4 - (A0 * MetricPerturbation[-a, -c] * CD[-b][Connection[a, b, c]])/2 + (11 * C1 * CD[a][Connection[c, d, -d]] * CD[-b][Connection[-c, -a, b]])/2 + (C1 * CD[a][Connection[-c, -a, b]] * CD[-b][Connection[c, d, -d]])/2 - 19 * C1 * CD[a][Connection[c, d, -c]] * CD[-b][Connection[-d, -a, b]] + (A0 * MetricPerturbation[-b, -c] * CD[c][Connection[a, -a, b]])/2 -
  (C1 * CD[-b][Connection[-c, d, -d]] * CD[c][Connection[a, -a, b]])/2 - (C1 * CD[-b][Connection[d, -d, -c]] * CD[c][Connection[a, -a, b]])/2 + (C1 * CD[-c][Connection[-b, d, -d]] * CD[c][Connection[a, -a, b]])/2 - (C1 * CD[-c][Connection[d, -b, -d]] * CD[c][Connection[a, -a, b]])/2 - (11 * C1 * CD[-b][Connection[-c, d, -d]] * CD[c][Connection[a, b, -a]])/2 + (19 * C1 * CD[-b][Connection[d, -c, -d]] * CD[c][Connection[a, b, -a]])/2 + (11 * C1 * CD[-c][Connection[-b, d, -d]] * CD[c][Connection[a, b, -a]])/2 - (C1 * CD[-c][Connection[d, -b, -d]] * CD[c][Connection[a, b, -a]])/2 +
  C1 * CD[-a][Connection[-c, d, -d]] * CD[c][Connection[a, b, -b]] - C1 * CD[-c][Connection[-a, d, -d]] * CD[c][Connection[a, b, -b]] - (C1 * CD[-b][Connection[a, b, c]] * CD[-d][Connection[-a, -c, d]])/2 - (C1 * CD[-b][Connection[a, b, c]] * CD[-d][Connection[-a, d, -c]])/2 + (19 * C1 * CD[-c][Connection[a, b, c]] * CD[-d][Connection[-b, -a, d]])/2 + (C1 * CD[-c][Connection[a, b, c]] * CD[-d][Connection[-b, d, -a]])/2 -
  (C1 * CD[-b][Connection[a, -a, b]] * CD[-d][Connection[c, -c, d]])/2 + C1 * CD[-b][Connection[a, -a, b]] * CD[-d][Connection[c, d, -c]] - (C1 * CD[-b][Connection[a, b, -a]] * CD[-d][Connection[c, d, -c]])/2 + (C1 * CD[-c][Connection[a, b, c]] * CD[d][Connection[-a, -b, -d]])/2 + C1 * CD[-a][Connection[a, b, c]] * CD[d][Connection[-b, -c, -d]] +
  (C1 * CD[-a][Connection[-c, -d, -b]] * CD[d][Connection[a, b, c]])/2 + C1 * CD[-a][Connection[-d, -b, -c]] * CD[d][Connection[a, b, c]] + (C1 * CD[-a][Connection[-c, -b, -d]] * CD[d][Connection[a, b, c]])/2 - (C1 * CD[-b][Connection[-a, -d, -c]] * CD[d][Connection[a, b, c]])/2 - (C1 * CD[-b][Connection[-c, -d, -a]] * CD[d][Connection[a, b, c]])/2 - (C1 * CD[-c][Connection[-b, -d, -a]] * CD[d][Connection[a, b, c]])/2 +
  C1 * CD[-d][Connection[-a, -b, -c]] * CD[d][Connection[a, b, c]] - C1 * CD[-d][Connection[-a, -c, -b]] * CD[d][Connection[a, b, c]] - (C1 * CD[-d][Connection[-b, -a, -c]] * CD[d][Connection[a, b, c]])/2 - (C1 * CD[-d][Connection[-b, -c, -a]] * CD[d][Connection[a, b, c]])/2 - (C1 * CD[-d][Connection[-c, -b, -a]] * CD[d][Connection[a, b, c]])/2 - (11 * C1 * CD[-b][Connection[-d, -a, b]] * CD[d][Connection[a, c, -c]])/2 - (C1 * CD[a][Connection[-d, -a, b]] * CD[d][Connection[-b, c, -c]])/2 + (C1 * CD[-b][Connection[-d, -a, b]] * CD[d][Connection[c, a, -c]])/2, TheoryName → "ProjectiveTheory", Method → "Hard", MaxLaurentDepth → 1];
```

| | $\Delta^{\#1}_{1^+\,\alpha\beta}$ | $\Delta^{\#2}_{1^+\,\alpha\beta}$ | $\Delta^{\#3}_{1^+\,\alpha\beta}$ | $\Delta^{\#1}_{1^-\,\alpha}$ | $\Delta^{\#2}_{1^-\,\alpha}$ | $\Delta^{\#3}_{1^-\,\alpha}$ | $\Delta^{\#4}_{1^-\,\alpha}$ | $\Delta^{\#5}_{1^-\,\alpha}$ | $\Delta^{\#6}_{1^-\,\alpha}$ | $\mathcal{T}^{\#1}_{1^-\,\alpha}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta^{\#1}_{1^+}\dagger^{\alpha\beta}$ | 0 | $-\frac{2\sqrt{2}}{a_0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#2}_{1^+}\dagger^{\alpha\beta}$ | $-\frac{2\sqrt{2}}{a_0}$ | $\frac{2(a_0^2-16a_0c_1k^2-35c_1^2k^4)}{a_0^2(a_0-29c_1k^2)}$ | $\frac{40\sqrt{2}c_1k^2}{a_0^2-29a_0c_1k^2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#3}_{1^+}\dagger^{\alpha\beta}$ | 0 | $\frac{40\sqrt{2}c_1k^2}{a_0^2-29a_0c_1k^2}$ | $\frac{4}{a_0-29c_1k^2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | $\frac{\sqrt{2}(4+k^2)}{a_0(2+k^2)}$ | $-\frac{2k^2}{\sqrt{3}a_0(2+k^2)}$ | 0 | $\frac{\sqrt{\frac{2}{3}}k^2}{a_0(2+k^2)}$ | 0 | $-\frac{2i\sqrt{2}k}{a_0(2+k^2)}$ |
| $\Delta^{\#2}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{\sqrt{2}(4+k^2)}{a_0(2+k^2)}$ | $\frac{a_0^2(4+k^2)^2-30a_0c_1k^4(4+k^2)+c_1^2k^4(6416+7928k^2+1901k^4)}{2a_0^2(2+k^2)^2(a_0-33c_1k^2)}$ | $\frac{k^2(a_0^2(-2+k^2)+a_0c_1(560+302k^2+71k^4)-2c_1^2k^2(9440+1901k^2(4+k^2)))}{2\sqrt{6}a_0^2(2+k^2)^2(a_0-33c_1k^2)}$ | $-\frac{\sqrt{\frac{2}{3}}k^2(a_0+c_1(40-31k^2))}{a_0(2+k^2)(a_0-33c_1k^2)}$ | $\frac{k^2(2a_0^2(5+2k^2)-a_0c_1(880+778k^2+199k^4)+c_1^2k^2(9440+1901k^2(4+k^2)))}{2\sqrt{3}a_0^2(2+k^2)^2(a_0-33c_1k^2)}$ | $\frac{k^2(-a_0+c_1(200+43k^2))}{\sqrt{6}a_0(2+k^2)(a_0-33c_1k^2)}$ | $-\frac{ik(-30a_0c_1k^4+a_0^2(4+k^2)+27c_1^2k^4(-28+3k^2))}{a_0^2(2+k^2)^2(a_0-33c_1k^2)}$ |
| $\Delta^{\#3}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $-\frac{2k^2}{\sqrt{3}a_0(2+k^2)}$ | $\frac{k^2(a_0^2(-2+k^2)+a_0c_1(560+302k^2+71k^4)-2c_1^2k^2(9440+1901k^2(4+k^2)))}{2\sqrt{6}a_0^2(2+k^2)^2(a_0-33c_1k^2)}$ | $\frac{a_0^2(76+52k^2+3k^4)+4a_0c_1k^2(472+214k^2+19k^4)+4c_1^2k^4(5120+7280k^2+1901k^4)}{12a_0^2(2+k^2)^2(a_0-33c_1k^2)}$ | $\frac{\sqrt{2}(10a_0+3a_0k^2-328c_1k^2-62c_1k^4)}{12a_0(2+k^2)(a_0-33c_1k^2)}$ | $\frac{2a_0^2(-2+k^2)+a_0c_1k^2(472+934k^2+289k^4)-2c_1^2k^4(5120+7280k^2+1901k^4)}{6\sqrt{2}a_0^2(2+k^2)^2(a_0-33c_1k^2)}$ | $-\frac{2a_0+(3a_0-56c_1)k^2+86c_1k^4}{6a_0(2+k^2)(a_0-33c_1k^2)}$ | $\frac{ik(54c_1^2k^4(40+3k^2)+a_0^2(6+5k^2)-3a_0c_1k^2(86+23k^2))}{\sqrt{6}a_0^2(2+k^2)^2(a_0-33c_1k^2)}$ |
| $\Delta^{\#4}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | $-\frac{\sqrt{\frac{2}{3}}k^2(a_0+c_1(40-31k^2))}{a_0(2+k^2)(a_0-33c_1k^2)}$ | $\frac{\sqrt{2}(10a_0+3a_0k^2-328c_1k^2-62c_1k^4)}{12a_0(2+k^2)(a_0-33c_1k^2)}$ | $\frac{1}{12a_0-396c_1k^2}$ | $\frac{\sqrt{\frac{2}{3}}(-3a_0+c_1k^2(164+31k^2))}{6a_0(2+k^2)(a_0-33c_1k^2)}$ | $-\frac{\sqrt{3}}{6(a_0-33c_1k^2)}$ | $\frac{i\sqrt{\frac{2}{3}}k(a_0-51c_1k^2)}{a_0(2+k^2)(a_0-33c_1k^2)}$ |
| $\Delta^{\#5}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{\sqrt{\frac{2}{3}}k^2}{a_0(2+k^2)}$ | $\frac{k^2(2a_0^2(5+2k^2)-a_0c_1(880+778k^2+199k^4)+c_1^2k^2(9440+1901k^2(4+k^2)))}{2\sqrt{3}a_0^2(2+k^2)^2(a_0-33c_1k^2)}$ | $\frac{2a_0^2(-2+k^2)+a_0c_1k^2(472+934k^2+289k^4)-2c_1^2k^4(5120+7280k^2+1901k^4)}{6\sqrt{2}a_0^2(2+k^2)^2(a_0-33c_1k^2)}$ | $\frac{\sqrt{\frac{2}{3}}(-3a_0+c_1k^2(164+31k^2))}{6a_0(2+k^2)(a_0-33c_1k^2)}$ | $\frac{4a_0^2(17+14k^2+3k^4)-4a_0c_1k^2(236+287k^2+77k^4)+c_1^2k^4(5120+7280k^2+1901k^4)}{6a_0^2(2+k^2)^2(a_0-33c_1k^2)}$ | $-\frac{c_1k^2(28-43k^2)+2a_0(7+3k^2)}{3\sqrt{2}a_0(2+k^2)(a_0-33c_1k^2)}$ | $\frac{ik(2a_0^2(3+k^2)-27c_1^2k^4(40+3k^2)+3a_0c_1k^2(34+7k^2))}{\sqrt{3}a_0^2(2+k^2)^2(a_0-33c_1k^2)}$ |
| $\Delta^{\#6}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | 0 | $\frac{k^2(-a_0+c_1(200+43k^2))}{\sqrt{6}a_0(2+k^2)(a_0-33c_1k^2)}$ | $-\frac{2a_0+(3a_0-56c_1)k^2+86c_1k^4}{6a_0(2+k^2)(a_0-33c_1k^2)}$ | $-\frac{\sqrt{3}}{6(a_0-33c_1k^2)}$ | $-\frac{c_1k^2(28-43k^2)+2a_0(7+3k^2)}{3\sqrt{2}a_0(2+k^2)(a_0-33c_1k^2)}$ | $\frac{5}{3(a_0-33c_1k^2)}$ | $-\frac{i\sqrt{\frac{2}{3}}k(a_0+57c_1k^2)}{a_0(2+k^2)(a_0-33c_1k^2)}$ |
| $\mathcal{T}^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{2i\sqrt{2}k}{2a_0+a_0k^2}$ | $\frac{i(-30a_0c_1k^5+a_0^2k(4+k^2)+27c_1^2k^5(-28+3k^2))}{a_0^2(2+k^2)^2(a_0-33c_1k^2)}$ | $-\frac{i(54c_1^2k^5(40+3k^2)+a_0^2k(6+5k^2)-3a_0c_1k^3(86+23k^2))}{\sqrt{6}a_0^2(2+k^2)^2(a_0-33c_1k^2)}$ | $-\frac{i\sqrt{\frac{2}{3}}k(a_0-51c_1k^2)}{a_0(2+k^2)(a_0-33c_1k^2)}$ | $-\frac{i(2a_0^2k(3+k^2)-27c_1^2k^5(40+3k^2)+3a_0c_1k^3(34+7k^2))}{\sqrt{3}a_0^2(2+k^2)^2(a_0-33c_1k^2)}$ | $\frac{i\sqrt{\frac{2}{3}}k(a_0+57c_1k^2)}{a_0(2+k^2)(a_0-33c_1k^2)}$ | $\frac{2k^2(a_0^2+30a_0c_1k^2-459c_1^2k^4)}{a_0^2(2+k^2)^2(a_0-33c_1k^2)}$ |

| | $\Gamma^{\#1}_{1^+\,\alpha\beta}$ | $\Gamma^{\#2}_{1^+\,\alpha\beta}$ | $\Gamma^{\#3}_{1^+\,\alpha\beta}$ | $\Gamma^{\#1}_{1^-\,\alpha}$ | $\Gamma^{\#2}_{1^-\,\alpha}$ | $\Gamma^{\#3}_{1^-\,\alpha}$ | $\Gamma^{\#4}_{1^-\,\alpha}$ | $\Gamma^{\#5}_{1^-\,\alpha}$ | $\Gamma^{\#6}_{1^-\,\alpha}$ | $h^{\#1}_{1^-\,\alpha}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\Gamma^{\#1}_{1^+}\dagger^{\alpha\beta}$ | $\frac{1}{4}(-a_0-15c_1k^2)$ | $-\frac{a_0}{2\sqrt{2}}$ | $5c_1k^2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#2}_{1^+}\dagger^{\alpha\beta}$ | $-\frac{a_0}{2\sqrt{2}}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#3}_{1^+}\dagger^{\alpha\beta}$ | $5c_1k^2$ | 0 | $\frac{1}{4}(a_0-29c_1k^2)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{1}{4}(-a_0-3c_1k^2)$ | $\frac{a_0}{2\sqrt{2}}$ | $\frac{5}{2}\sqrt{3}c_1k^2$ | $-\frac{5}{2}\sqrt{\frac{5}{3}}c_1k^2$ | $5\sqrt{\frac{3}{2}}c_1k^2$ | $-\frac{5c_1k^2}{\sqrt{3}}$ | $-\frac{ia_0k}{4\sqrt{2}}$ |
| $\Gamma^{\#2}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{a_0}{2\sqrt{2}}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Gamma^{\#3}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{5}{2}\sqrt{3}c_1k^2$ | 0 | $-\frac{a_0}{3}$ | $\frac{1}{6}\sqrt{5}(a_0-8c_1k^2)$ | $-\frac{a_0}{6\sqrt{2}}$ | $\frac{1}{6}(-a_0+20c_1k^2)$ | $\frac{ia_0k}{4\sqrt{6}}$ |
| $\Gamma^{\#4}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $-\frac{5}{2}\sqrt{\frac{5}{3}}c_1k^2$ | 0 | $\frac{1}{6}\sqrt{5}(a_0-8c_1k^2)$ | $\frac{1}{3}(a_0+7c_1k^2)$ | $-\frac{1}{6}\sqrt{\frac{5}{2}}(a_0+16c_1k^2)$ | $-\frac{1}{6}\sqrt{5}(a_0-5c_1k^2)$ | $-\frac{1}{4}i\sqrt{\frac{5}{6}}a_0k$ |
| $\Gamma^{\#5}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $5\sqrt{\frac{3}{2}}c_1k^2$ | 0 | $-\frac{a_0}{6\sqrt{2}}$ | $-\frac{1}{6}\sqrt{\frac{5}{2}}(a_0+16c_1k^2)$ | $\frac{a_0}{3}$ | $\frac{a_0+40c_1k^2}{6\sqrt{2}}$ | $\frac{ia_0k}{4\sqrt{3}}$ |
| $\Gamma^{\#6}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $-\frac{5c_1k^2}{\sqrt{3}}$ | 0 | $\frac{1}{6}(-a_0+20c_1k^2)$ | $-\frac{1}{6}\sqrt{5}(a_0-5c_1k^2)$ | $\frac{a_0+40c_1k^2}{6\sqrt{2}}$ | $\frac{5}{12}(a_0-17c_1k^2)$ | $\frac{ia_0k}{4\sqrt{6}}$ |
| $h^{\#1}_{1^-}\dagger^{\alpha}$ | 0 | 0 | 0 | $\frac{ia_0k}{4\sqrt{2}}$ | 0 | $-\frac{ia_0k}{4\sqrt{6}}$ | $\frac{1}{4}i\sqrt{\frac{5}{6}}a_0k$ | $-\frac{ia_0k}{4\sqrt{3}}$ | $-\frac{ia_0k}{4\sqrt{6}}$ | 0 |

| | $\Delta^{\#1}_{2^+\,\alpha\beta}$ | $\Delta^{\#2}_{2^+\,\alpha\beta}$ | $\Delta^{\#3}_{2^+\,\alpha\beta}$ | $\mathcal{T}^{\#1}_{2^+\,\alpha\beta}$ | $\Delta^{\#1}_{2^-\,\alpha\beta\chi}$ | $\Delta^{\#2}_{2^-\,\alpha\beta\chi}$ |
|---|---|---|---|---|---|---|
| $\Delta^{\#1}_{2^+}\dagger^{\alpha\beta}$ | 0 | $\frac{2\sqrt{\frac{2}{3}}}{a_0}$ | $\frac{4}{\sqrt{3}a_0}$ | $\frac{4i\sqrt{2}}{a_0k}$ | 0 | 0 |
| $\Delta^{\#2}_{2^+}\dagger^{\alpha\beta}$ | $\frac{2\sqrt{\frac{2}{3}}}{a_0}$ | $-\frac{8(a_0+13c_1k^2)}{3a_0^2}$ | $-\frac{2\sqrt{2}(a_0+52c_1k^2)}{3a_0^2}$ | $-\frac{4i(a_0+31c_1k^2)}{\sqrt{3}a_0^2k}$ | 0 | 0 |
| $\Delta^{\#3}_{2^+}\dagger^{\alpha\beta}$ | $\frac{4}{\sqrt{3}a_0}$ | $-\frac{2\sqrt{2}(a_0+52c_1k^2)}{3a_0^2}$ | $\frac{8(a_0-26c_1k^2)}{3a_0^2}$ | $-\frac{4i\sqrt{\frac{2}{3}}(a_0+31c_1k^2)}{a_0^2k}$ | 0 | 0 |
| $\mathcal{T}^{\#1}_{2^+}\dagger^{\alpha\beta}$ | $-\frac{4i\sqrt{2}}{a_0k}$ | $\frac{4i(a_0+31c_1k^2)}{\sqrt{3}a_0^2k}$ | $\frac{4i\sqrt{\frac{2}{3}}(a_0+31c_1k^2)}{a_0^2k}$ | $-\frac{8(a_0+11c_1k^2)}{a_0^2k^2}$ | 0 | 0 |
| $\Delta^{\#1}_{2^-}\dagger^{\alpha\beta\chi}$ | 0 | 0 | 0 | 0 | $\frac{4}{a_0-c_1k^2}$ | 0 |
| $\Delta^{\#2}_{2^-}\dagger^{\alpha\beta\chi}$ | 0 | 0 | 0 | 0 | 0 | $\frac{4}{a_0-5c_1k^2}$ |

| Source constraints | |
|---|---|
| SO(3) irreps | # |
| $2\mathcal{T}^{\#2}_{0^+}-ik\Delta^{\#2}_{0^+}==0$ | 1 |
| $\Delta^{\#3}_{0^+}+2\Delta^{\#4}_{0^+}+3\Delta^{\#2}_{0^+}==0$ | 1 |
| $6\mathcal{T}^{\#1\alpha}_{1^-}-ik(3\Delta^{\#2\alpha}_{1^-}-\Delta^{\#5\alpha}_{1^-}+\Delta^{\#3\alpha}_{1^-})==0$ | 3 |
| $2\Delta^{\#6\alpha}_{1^-}+\Delta^{\#4\alpha}_{1^-}+2\Delta^{\#5\alpha}_{1^-}+\Delta^{\#3\alpha}_{1^-}==0$ | 3 |
| Total #: | 8 |

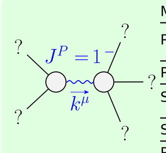
| | $\Gamma^{\#1}_{3^-\,\alpha\beta\chi}$ |
|---|---|
| $\Gamma^{\#1}_{3^-}\dagger^{\alpha\beta\chi}$ | $\frac{1}{2}(-a_0-7c_1k^2)$ |

| | $\Delta^{\#1}_{3^-\,\alpha\beta\chi}$ |
|---|---|
| $\Delta^{\#1}_{3^-}\dagger^{\alpha\beta\chi}$ | $-\frac{2}{a_0+7c_1k^2}$ |

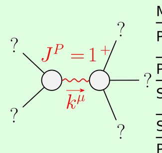
Lagrangian density

$$-\tfrac{1}{2}a_0\Gamma^{\alpha\beta\chi}\Gamma_{\beta\chi\alpha}+\tfrac{1}{2}a_0\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\Gamma^{\chi}{}_{\beta\chi}-\tfrac{1}{4}a_0h^{\chi}{}_{\chi}\partial_\beta\Gamma^{\alpha}{}_{\alpha}{}^{\beta}+\tfrac{1}{4}a_0h^{\chi}{}_{\chi}\partial_\beta\Gamma^{\alpha\beta}{}_{\alpha}-\tfrac{1}{2}a_0h_{\alpha\chi}\partial_\beta\Gamma^{\alpha\beta\chi}+\tfrac{11}{2}c_1\partial^\alpha\Gamma^{\chi\delta}{}_{\delta}\partial_\beta\Gamma_{\chi\alpha}{}^{\beta}+\tfrac{1}{2}c_1\partial^\alpha\Gamma_{\chi\alpha}{}^{\beta}\partial_\beta\Gamma^{\chi\delta}{}_{\delta}-19c_1\partial^\alpha\Gamma^{\chi\delta}{}_{\chi}\partial_\beta\Gamma_{\delta\alpha}{}^{\beta}+\tfrac{1}{2}a_0h_{\beta\chi}\partial^\chi\Gamma^{\alpha}{}_{\alpha}{}^{\beta}-\tfrac{1}{2}c_1\partial_\beta\Gamma_{\chi}{}^{\delta}{}_{\delta}\partial^\chi\Gamma^{\alpha}{}_{\alpha}{}^{\beta}-\tfrac{1}{2}c_1\partial_\beta\Gamma^{\delta}{}_{\delta\chi}\partial^\chi\Gamma^{\alpha}{}_{\alpha}{}^{\beta}+\tfrac{1}{2}c_1\partial_\chi\Gamma_{\beta}{}^{\delta}{}_{\delta}\partial^\chi\Gamma^{\alpha}{}_{\alpha}{}^{\beta}-\tfrac{1}{2}c_1\partial_\chi\Gamma^{\delta}{}_{\beta\delta}\partial^\chi\Gamma^{\alpha}{}_{\alpha}{}^{\beta}-\tfrac{1}{2}c_1\partial_\chi\Gamma^{\delta}{}_{\delta\beta}\partial^\chi\Gamma^{\alpha}{}_{\alpha}{}^{\beta}-\tfrac{11}{2}c_1\partial_\beta\Gamma_{\chi}{}^{\delta}{}_{\delta}\partial^\chi\Gamma^{\alpha\beta}{}_{\alpha}+\tfrac{19}{2}c_1\partial_\beta\Gamma^{\delta}{}_{\chi\delta}\partial^\chi\Gamma^{\alpha\beta}{}_{\alpha}+\tfrac{11}{2}c_1\partial_\chi\Gamma_{\beta}{}^{\delta}{}_{\delta}\partial^\chi\Gamma^{\alpha\beta}{}_{\alpha}-\tfrac{1}{2}c_1\partial_\chi\Gamma^{\delta}{}_{\beta\delta}\partial^\chi\Gamma^{\alpha\beta}{}_{\alpha}+c_1\partial_\alpha\Gamma_{\chi}{}^{\delta}{}_{\delta}\partial^\chi\Gamma^{\alpha\beta}{}_{\beta}-c_1\partial_\chi\Gamma_{\alpha}{}^{\delta}{}_{\delta}\partial^\chi\Gamma^{\alpha\beta}{}_{\beta}-\tfrac{1}{2}c_1\partial_\chi\Gamma^{\alpha\beta\chi}\partial_\delta\Gamma_{\alpha\beta}{}^{\delta}-\tfrac{1}{2}c_1\partial_\beta\Gamma^{\alpha\beta\chi}\partial_\delta\Gamma_{\alpha\chi}{}^{\delta}-\tfrac{1}{2}c_1\partial_\beta\Gamma^{\alpha\beta\chi}\partial_\delta\Gamma_{\alpha}{}^{\delta}{}_{\chi}+\tfrac{19}{2}c_1\partial_\chi\Gamma^{\alpha\beta\chi}\partial_\delta\Gamma_{\beta\alpha}{}^{\delta}+c_1\partial^\chi\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\partial_\delta\Gamma_{\beta}{}^{\delta}{}_{\chi}+\tfrac{1}{2}c_1\partial^\chi\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\partial_\delta\Gamma_{\chi\beta}{}^{\delta}+\tfrac{1}{2}c_1\partial^\chi\Gamma^{\alpha\beta}{}_{\alpha}\partial_\delta\Gamma_{\chi\beta}{}^{\delta}-\tfrac{1}{2}c_1\partial_\beta\Gamma^{\alpha\beta\chi}\partial_\delta\Gamma_{\chi}{}^{\delta}{}_{\alpha}+\tfrac{1}{2}c_1\partial^\chi\Gamma_{\beta\alpha}{}^{\beta}\partial_\delta\Gamma_{\chi}{}^{\delta\alpha}+c_1\partial^\chi\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\partial_\delta\Gamma_{\chi}{}^{\delta}{}_{\beta}-\tfrac{1}{2}c_1\partial_\beta\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\partial_\delta\Gamma^{\chi}{}_{\chi}{}^{\delta}+c_1\partial_\beta\Gamma^{\alpha}{}_{\alpha}{}^{\beta}\partial_\delta\Gamma^{\chi\delta}{}_{\chi}-\tfrac{1}{2}c_1\partial_\beta\Gamma^{\alpha\beta}{}_{\alpha}\partial_\delta\Gamma^{\chi\delta}{}_{\chi}+\tfrac{1}{2}c_1\partial_\alpha\Gamma_{\beta\chi\delta}\partial^\delta\Gamma^{\alpha\beta\chi}+c_1\partial_\alpha\Gamma_{\beta\delta\chi}\partial^\delta\Gamma^{\alpha\beta\chi}+c_1\partial_\alpha\Gamma_{\chi\beta\delta}\partial^\delta\Gamma^{\alpha\beta\chi}+\tfrac{1}{2}c_1\partial_\alpha\Gamma_{\chi\delta\beta}\partial^\delta\Gamma^{\alpha\beta\chi}+c_1\partial_\alpha\Gamma_{\delta\beta\chi}\partial^\delta\Gamma^{\alpha\beta\chi}+c_1\partial_\alpha\Gamma_{\delta\chi\beta}\partial^\delta\Gamma^{\alpha\beta\chi}-\tfrac{1}{2}c_1\partial_\beta\Gamma_{\alpha\chi\delta}\partial^\delta\Gamma^{\alpha\beta\chi}-\tfrac{1}{2}c_1\partial_\beta\Gamma_{\alpha\delta\chi}\partial^\delta\Gamma^{\alpha\beta\chi}-\tfrac{1}{2}c_1\partial_\beta\Gamma_{\chi\delta\alpha}\partial^\delta\Gamma^{\alpha\beta\chi}-\tfrac{1}{2}c_1\partial_\chi\Gamma_{\alpha\beta\delta}\partial^\delta\Gamma^{\alpha\beta\chi}-\tfrac{1}{2}c_1\partial_\chi\Gamma_{\beta\alpha\delta}\partial^\delta\Gamma^{\alpha\beta\chi}+c_1\partial_\chi\Gamma_{\beta\delta\alpha}\partial^\delta\Gamma^{\alpha\beta\chi}-c_1\partial_\delta\Gamma_{\alpha\beta\chi}\partial^\delta\Gamma^{\alpha\beta\chi}-c_1\partial_\delta\Gamma_{\alpha\chi\beta}\partial^\delta\Gamma^{\alpha\beta\chi}-\tfrac{1}{2}c_1\partial_\delta\Gamma_{\beta\alpha\chi}\partial^\delta\Gamma^{\alpha\beta\chi}-\tfrac{1}{2}c_1\partial_\delta\Gamma_{\beta\chi\alpha}\partial^\delta\Gamma^{\alpha\beta\chi}-\tfrac{1}{2}c_1\partial_\delta\Gamma_{\chi\beta\alpha}\partial^\delta\Gamma^{\alpha\beta\chi}-\tfrac{11}{2}c_1\partial_\beta\Gamma_{\delta\alpha}{}^{\beta}\partial^\delta\Gamma^{\alpha\chi}{}_{\chi}-\tfrac{1}{2}c_1\partial^\alpha\Gamma_{\delta\alpha}{}^{\beta}\partial^\delta\Gamma_{\beta}{}^{\chi}{}_{\chi}+\tfrac{1}{2}c_1\partial_\beta\Gamma_{\delta\alpha}{}^{\beta}\partial^\delta\Gamma^{\chi\alpha}{}_{\chi}$$

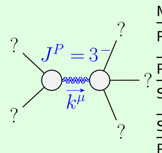
| Added source term: | $h^{\alpha\beta}\mathcal{T}_{\alpha\beta}+\Gamma^{\alpha\beta\chi}\Delta_{\alpha\beta\chi}$ |
|---|---|



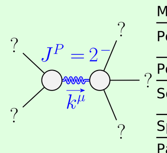

| Massive particle | |
|---|---|
| Pole residue: | $\frac{3287a_0+323862c_1}{35937c_1(a_0+66c_1)}>0$ |
| Polarisations: | 3 |
| Square mass: | $\frac{a_0}{33c_1}>0$ |
| Spin: | 1 |
| Parity: | Odd |



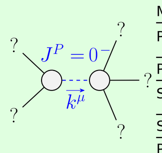

| Massive particle | |
|---|---|
| Pole residue: | $-\frac{4164}{24389c_1}>0$ |
| Polarisations: | 3 |
| Square mass: | $\frac{a_0}{29c_1}>0$ |
| Spin: | 1 |
| Parity: | Even |



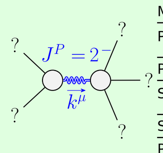

| Massive particle | |
|---|---|
| Pole residue: | $\frac{2}{7c_1}>0$ |
| Polarisations: | 7 |
| Square mass: | $-\frac{a_0}{7c_1}>0$ |
| Spin: | 3 |
| Parity: | Odd |



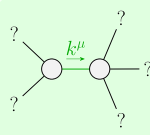

| Massive particle | |
|---|---|
| Pole residue: | $\frac{4}{5c_1}>0$ |
| Polarisations: | 5 |
| Square mass: | $\frac{a_0}{5c_1}>0$ |
| Spin: | 2 |
| Parity: | Odd |




| Massive particle | |
|---|---|
| Pole residue: | $-\frac{2}{c_1}>0$ |
| Polarisations: | 1 |
| Square mass: | $\frac{a_0}{c_1}>0$ |
| Spin: | 0 |
| Parity: | Odd |




| Massive particle | |
|---|---|
| Pole residue: | $\frac{4}{c_1}>0$ |
| Polarisations: | 5 |
| Square mass: | $\frac{a_0}{c_1}>0$ |
| Spin: | 2 |
| Parity: | Odd |



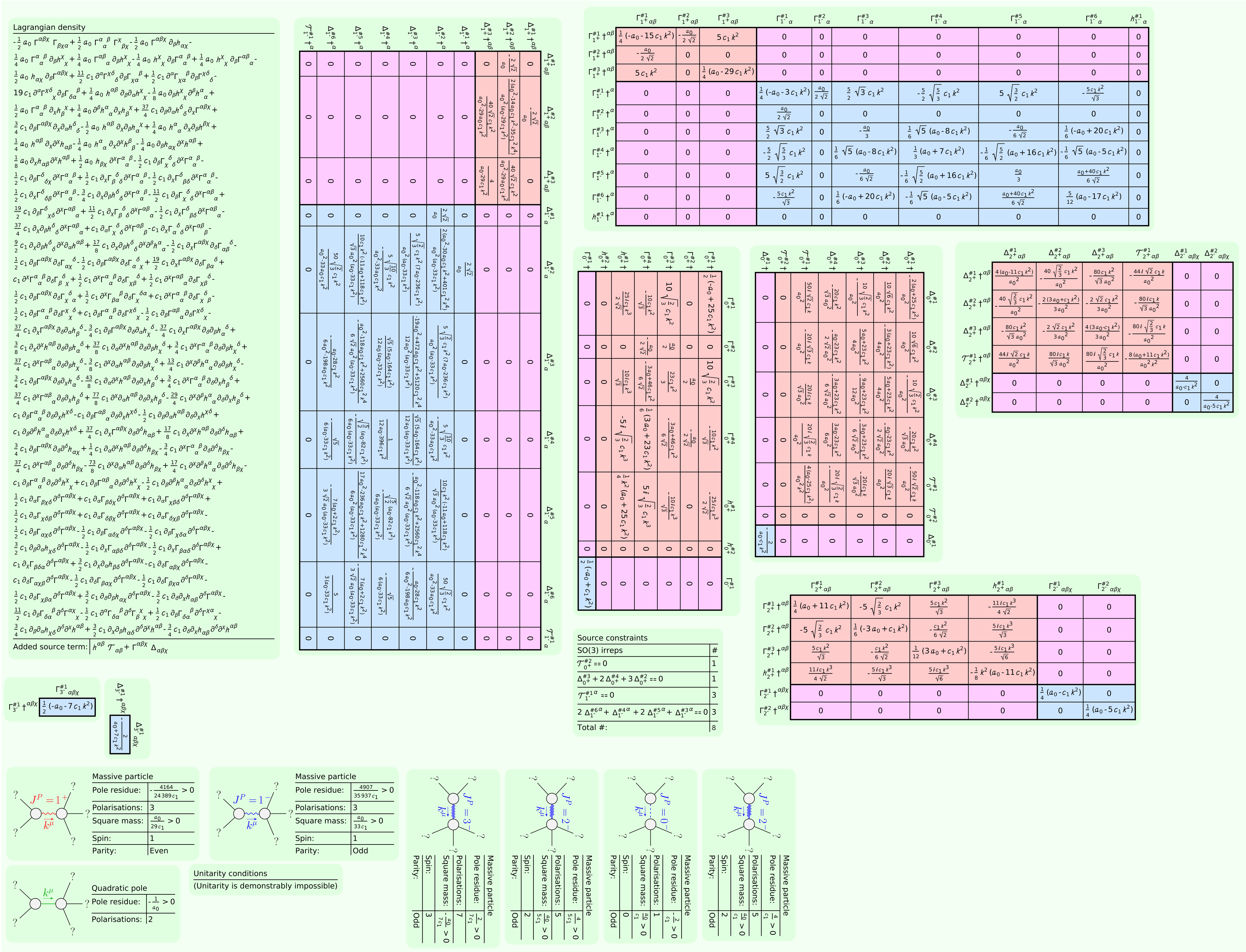

| Quadratic pole | |
|---|---|
| Pole residue: | $-\frac{1}{a_0}>0$ |
| Polarisations: | 2 |

Unitarity conditions

(Unitarity is demonstrably impossible)

Next we study Eq. (75) in the second-order formulation:

```
ParticleSpectrum[-1/2*(A0*Connection[a, b, c]*Connection[-b, -c, -a]) + (A0*Connection[a, -a, b]*Connection[c, -b, -c])/2 - (A0*MetricPerturbation[c, -c]*CD[-b][Connection[a, -a, b]])/4 + (A0*MetricPerturbation[c, -c]*CD[-b][Connection[a, b, -a]])/4 - (A0*MetricPerturbation[-a, -c]*CD[-b][Connection[a, b, c]])/2 + (11*C1*CD[a][Connection[c, d, -d]]*CD[-b][Connection[-c, -a, b]])/2 + (C1*CD[a][Connection[-c, -a, b]]*CD[-b][Connection[c, d, -d]])/2 - 19*C1*CD[a][Connection[c, d, -c]]*CD[-b][Connection[-d, -a, b]] - (A0*Connection[a, b, c]*CD[-b][MetricPerturbation[-a, -c]])/2 -
  (A0*Connection[a, -a, b]*CD[-b][MetricPerturbation[c, -c]])/4 + (A0*Connection[a, b, -a]*CD[-b][MetricPerturbation[c, -c]])/4 + (A0*MetricPerturbation[a, b]*CD[-b][CD[-a][MetricPerturbation[c, -c]]])/4 - (A0*CD[-b][MetricPerturbation[c, -c]]*CD[b][MetricPerturbation[a, -a]])/8 + (37*C1*CD[-b][CD[-a][MetricPerturbation[d, -d]]]*CD[-c][Connection[a, b, c]])/4 + (A0*Connection[a, -a, b]*CD[-c][MetricPerturbation[-b, c]])/2 + (A0*CD[b][MetricPerturbation[a, -a]]*CD[-c][MetricPerturbation[-b, c]])/4 + (3*C1*CD[-b][Connection[a, b, c]]*CD[-c][CD[-a][MetricPerturbation[d, -d]]])/4 -
  (A0*MetricPerturbation[a, b]*CD[-c][CD[-b][MetricPerturbation[-a, c]]])/2 + (A0*MetricPerturbation[a, -a]*CD[-c][CD[-b][MetricPerturbation[b, c]]])/4 + (A0*MetricPerturbation[a, b]*CD[-c][CD[c][MetricPerturbation[-a, -b]]])/4 - (A0*MetricPerturbation[a, -a]*CD[-c][CD[c][MetricPerturbation[b, -b]]])/4 + (A0*MetricPerturbation[-b, -c]*CD[c][Connection[a, -a, b]])/2 - (C1*CD[-b][Connection[-c, d, -d]]*CD[c][Connection[a, -a, b]])/2 - (C1*CD[-b][Connection[d, -d, -c]]*CD[c][Connection[a, -a, b]])/2 + (C1*CD[-c][Connection[-b, d, -d]]*CD[c][Connection[a, -a, b]])/2 -
  (C1*CD[-c][Connection[d, -b, -d]]*CD[c][Connection[a, -a, b]])/2 - (C1*CD[-c][Connection[d, -d, -b]]*CD[c][Connection[a, -a, b]])/2 - (3*C1*CD[-c][CD[-b][MetricPerturbation[d, -d]]]*CD[c][Connection[a, -a, b]])/4 - (11*C1*CD[-b][Connection[-c, d, -d]]*CD[c][Connection[a, b, -a]])/2 + (19*C1*CD[-b][Connection[d, -c, -d]]*CD[c][Connection[a, b, -a]])/2 + (11*C1*CD[-c][Connection[-b, d, -d]]*CD[c][Connection[a, b, -a]])/2 - (C1*CD[-c][Connection[d, -b, -d]]*CD[c][Connection[a, b, -a]])/2 - (37*C1*CD[-c][CD[-b][MetricPerturbation[d, -d]]]*CD[c][Connection[a, b, -a]])/4 +
  C1*CD[-a][Connection[-c, d, -d]]*CD[c][Connection[a, b, -b]] - C1*CD[-c][Connection[-a, d, -d]]*CD[c][Connection[a, b, -b]] - (A0*CD[-b][MetricPerturbation[-a, -c]]*CD[c][MetricPerturbation[a, b]])/4 + (A0*CD[-c][MetricPerturbation[-a, -b]]*CD[c][MetricPerturbation[a, b]])/8 - (9*C1*CD[-c][CD[-b][MetricPerturbation[d, -d]]]*CD[c][CD[-a][MetricPerturbation[a, b]]])/2 + (17*C1*CD[-c][CD[-b][MetricPerturbation[d, -d]]]*CD[c][CD[b][MetricPerturbation[a, -a]]])/8 - (C1*CD[-c][Connection[a, b, c]]*CD[-d][Connection[-a, -b, d]])/2 - (C1*CD[-b][Connection[a, b, c]]*CD[-d][Connection[-a, -c, d]])/2 -
  (C1*CD[-b][Connection[a, b, c]]*CD[-d][Connection[-a, d, -c]])/2 + (19*C1*CD[-c][Connection[a, b, c]]*CD[-d][Connection[-b, -a, d]])/2 + C1*CD[c][Connection[a, -a, b]]*CD[-d][Connection[-b, d, -c]] + (C1*CD[c][Connection[a, -a, b]]*CD[-d][Connection[-c, -b, d]])/2 + (C1*CD[c][Connection[a, b, -a]]*CD[-d][Connection[-c, -b, d]])/2 - (C1*CD[-b][Connection[a, b, c]]*CD[-d][Connection[-c, d, -a]])/2 + (C1*CD[c][Connection[-b, -a, b]]*CD[-d][Connection[-c, d, a]])/2 + C1*CD[c][Connection[a, -a, b]]*CD[-d][Connection[-c, d, -b]] - (C1*CD[-b][Connection[a, -a, b]]*CD[-d][Connection[c, -c, d]])/2 +
  C1*CD[-b][Connection[a, -a, b]]*CD[-d][Connection[c, d, -c]] - (C1*CD[-b][Connection[a, b, -a]]*CD[-d][Connection[c, d, -c]])/2 - (37*C1*CD[-c][Connection[a, b, c]]*CD[-d][CD[-a][MetricPerturbation[-b, d]]])/4 - (3*C1*CD[-b][Connection[a, b, c]]*CD[-d][CD[-a][MetricPerturbation[-c, d]]])/4 - (37*C1*CD[-c][Connection[a, b, c]]*CD[-d][CD[-b][MetricPerturbation[-a, d]]])/4 + (3*C1*CD[-c][CD[c][MetricPerturbation[a, b]]]*CD[-d][CD[-b][MetricPerturbation[-a, d]]])/8 + (37*C1*CD[-a][CD[c][MetricPerturbation[a, b]]]*CD[-d][CD[-b][MetricPerturbation[-c, d]]])/8 +
  (3*C1*CD[c][Connection[a, -a, b]]*CD[-d][CD[-b][MetricPerturbation[-c, d]]])/4 + (37*C1*CD[c][Connection[a, b, -a]]*CD[-d][CD[-b][MetricPerturbation[-c, d]]])/4 - (3*C1*CD[c][CD[-a][MetricPerturbation[a, b]]]*CD[-d][CD[-b][MetricPerturbation[-c, d]]])/8 + (13*C1*CD[c][CD[b][MetricPerturbation[a, -a]]]*CD[-d][CD[-b][MetricPerturbation[-c, d]]])/4 - (3*C1*CD[-b][Connection[a, b, c]]*CD[-d][CD[-c][MetricPerturbation[-a, d]]])/4 - (43*C1*CD[-a][CD[c][MetricPerturbation[a, b]]]*CD[-d][CD[-c][MetricPerturbation[-b, d]]])/8 + (3*C1*CD[c][Connection[a, -a, b]]*CD[-d][CD[-c][MetricPerturbation[-b, d]]])/4 +
  (37*C1*CD[c][Connection[a, b, -a]]*CD[-d][CD[-c][MetricPerturbation[-b, d]]])/4 + (77*C1*CD[c][CD[-a][MetricPerturbation[a, b]]]*CD[-d][CD[-c][MetricPerturbation[-b, d]]])/8 - (29*C1*CD[c][CD[b][MetricPerturbation[a, -a]]]*CD[-d][CD[-c][MetricPerturbation[-b, d]]])/4 + C1*CD[-b][Connection[a, -a, b]]*CD[-d][CD[-c][MetricPerturbation[c, d]]] - C1*CD[-b][Connection[a, b, -a]]*CD[-d][CD[-c][MetricPerturbation[c, d]]] - (C1*CD[-b][CD[-a][MetricPerturbation[a, b]]]*CD[-d][CD[-c][MetricPerturbation[c, d]]])/2 + C1*CD[-b][CD[b][MetricPerturbation[a, -a]]]*CD[-d][CD[-c][MetricPerturbation[c, d]]] +
  (37*C1*CD[-c][Connection[a, b, c]]*CD[-d][CD[d][MetricPerturbation[-a, -b]]])/4 + (17*C1*CD[-c][CD[c][MetricPerturbation[a, b]]]*CD[-d][CD[d][MetricPerturbation[-a, -b]]])/8 + (3*C1*CD[-b][Connection[a, b, c]]*CD[-d][CD[d][MetricPerturbation[-a, -c]]])/4 + (C1*CD[-a][CD[c][MetricPerturbation[a, b]]]*CD[-d][CD[d][MetricPerturbation[-b, -c]]])/4 - (3*C1*CD[c][Connection[a, -a, b]]*CD[-d][CD[d][MetricPerturbation[-b, -c]]])/4 - (37*C1*CD[c][Connection[a, b, -a]]*CD[-d][CD[d][MetricPerturbation[-b, -c]]])/4 - (73*C1*CD[c][CD[-a][MetricPerturbation[a, b]]]*CD[-d][CD[d][MetricPerturbation[-b, -c]]])/8 +
  (17*C1*CD[c][CD[b][MetricPerturbation[a, -a]]]*CD[-d][CD[d][MetricPerturbation[-b, -c]]])/4 - C1*CD[-b][Connection[a, -a, b]]*CD[-d][CD[d][MetricPerturbation[c, -c]]] + C1*CD[-b][Connection[a, b, -a]]*CD[-d][CD[d][MetricPerturbation[c, -c]]] - (C1*CD[-b][CD[b][MetricPerturbation[a, -a]]]*CD[-d][CD[d][MetricPerturbation[c, -c]]])/2 + (C1*CD[-a][Connection[-b, -c, -d]]*CD[d][Connection[a, b, c]])/2 + C1*CD[-a][Connection[-b, -d, -c]]*CD[d][Connection[a, b, c]] + C1*CD[-a][Connection[-c, -b, -d]]*CD[d][Connection[a, b, c]] + (C1*CD[-a][Connection[-c, -d, -b]]*CD[d][Connection[a, b, c]])/2 +
  C1*CD[-a][Connection[-d, -b, -c]]*CD[d][Connection[a, b, c]] + C1*CD[-a][Connection[-d, -c, -b]]*CD[d][Connection[a, b, c]] - (C1*CD[-b][Connection[-a, -c, -d]]*CD[d][Connection[a, b, c]])/2 - (C1*CD[-b][Connection[-a, -d, -c]]*CD[d][Connection[a, b, c]])/2 - (C1*CD[-b][Connection[-c, -d, -a]]*CD[d][Connection[a, b, c]])/2 - (3*C1*CD[-b][CD[-a][MetricPerturbation[-c, -d]]]*CD[d][Connection[a, b, c]])/2 - (C1*CD[-c][Connection[-a, -b, -d]]*CD[d][Connection[a, b, c]])/2 - (C1*CD[-c][Connection[-b, -a, -d]]*CD[d][Connection[a, b, c]])/2 + C1*CD[-c][Connection[-b, -d, -a]]*CD[d][Connection[a, b, c]] +
  (3*C1*CD[-c][CD[-a][MetricPerturbation[-b, -d]]]*CD[d][Connection[a, b, c]])/2 - C1*CD[-d][Connection[-a, -b, -c]]*CD[d][Connection[a, b, c]] - C1*CD[-d][Connection[-a, -c, -b]]*CD[d][Connection[a, b, c]] - (C1*CD[-d][Connection[-b, -a, -c]]*CD[d][Connection[a, b, c]])/2 - (C1*CD[-d][Connection[-b, -c, -a]]*CD[d][Connection[a, b, c]])/2 - (C1*CD[-d][Connection[-c, -b, -a]]*CD[d][Connection[a, b, c]])/2 + (3*C1*CD[-d][CD[-b][MetricPerturbation[-a, -c]]]*CD[d][Connection[a, b, c]])/2 - (3*C1*CD[-d][CD[-c][MetricPerturbation[-a, -b]]]*CD[d][Connection[a, b, c]])/2 -
  (11*C1*CD[-b][Connection[-d, -a, b]]*CD[d][Connection[a, c, -c]])/2 - (C1*CD[a][Connection[-d, -a, b]]*CD[d][Connection[-b, c, -c]])/2 + (C1*CD[-b][Connection[-d, -a, b]]*CD[d][Connection[c, a, -c]])/2 - (3*C1*CD[-b][CD[-a][MetricPerturbation[-c, -d]]]*CD[d][CD[c][MetricPerturbation[a, b]]])/4 + (3*C1*CD[-c][CD[-b][MetricPerturbation[-a, -d]]]*CD[d][CD[c][MetricPerturbation[a, b]]])/2 - (3*C1*CD[-d][CD[-c][MetricPerturbation[-a, -b]]]*CD[d][CD[c][MetricPerturbation[a, b]]])/4, TheoryName → "ProjectiveTheorySecondOrder", Method → "Hard", MaxLaurentDepth → 1];
```



Unlike the theory with concurrent symmetry, we see that the massive spectrum of the projective-symmetric theory is quite rich. In particular, the spin-three mode is active.

# Examples of bad usage

We will test the error messages.

Try passing a theory name that is not a string:

```
ParticleSpectrum[Coupling1*CD[-a][ScalarField[]]*CD[a][ScalarField[]], TheoryName → NotStringTheory, Method → "Easy", MaxLaurentDepth → 3];
```

ParticleSpectrum: You must pass a string to the option TheoryName.

Try passing a method that is not implemented:

```
ParticleSpectrum[Coupling1*CD[-a][ScalarField[]]*CD[a][ScalarField[]], TheoryName → "MyVeryCoolTheory", Method → "InfiniteMonkeys", MaxLaurentDepth → 3];
```

··· ParticleSpectrum: The method "InfiniteMonkeys" for evaluating the source constraints and matrix pseudoinverses appears not to be either of the strings "Easy" or "Hard".

Try passing a maximum Laurent depth that is too great:

```
ParticleSpectrum[Coupling1*CD[-a][ScalarField[]]*CD[a][ScalarField[]], TheoryName → "MyVeryCoolTheory", Method → "Hard", MaxLaurentDepth → 4];
```

··· ParticleSpectrum: The maximum requested depth n of the 1/k^(2n) residue n=4 appears not to be a natural number 1, 2 or 3.

Try passing a Lagrangian density which is not quadratic in the fields:

$$\beta\,\phi + \gamma\,\phi^3 + \alpha\,\partial_a\phi\,\partial^a\phi \tag{77}$$

```
ParticleSpectrum[Coupling2*ScalarField[] + Coupling3*ScalarField[]^3 + Coupling1*CD[-a][ScalarField[]]*CD[a][ScalarField[]], TheoryName → "MyVeryCoolTheory", Method → "Hard", MaxLaurentDepth → 3];
```

··· ParticleSpectrum: The Lagrangian density contains the monomial γ ScalarField$^3$ which is not quadratic in fields or their derivatives.

Try passing a Lagrangian density with a field which was not defined using DefField:

```
DefTensor[Psi[], M4, PrintAs → "ψ"];
```

$$\alpha\,\partial_a\phi\,\partial^a\psi \tag{78}$$

```
ParticleSpectrum[Coupling1*CD[-a][ScalarField[]]*CD[a][Psi[]], TheoryName → "MyVeryCoolTheory", Method → "Hard", MaxLaurentDepth → 3];
```

··· ParticleSpectrum: The Lagrangian density contains a tensor Psi which was not defined using DefField.

Try passing a Lagrangian density which is not parity-even:

$$\alpha\,\epsilon^{abcd}\,\partial_a\phi\,\partial_b B_{cd} \tag{79}$$

```
ParticleSpectrum[Coupling1*epsilonG[a, b, c, d]*CD[-a][ScalarField[]]*CD[-b][TwoFormField[-c, -d]], TheoryName → "MyVeryCoolTheory", Method → "Hard", MaxLaurentDepth → 3];
```

··· ParticleSpectrum: The Lagrangian density contains at least one term with an odd power of totally antisymmetric tensors.

**Key observation:** We have now reached the end of the documentation.