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Supplemental materials: Multipliers are not trivial

Introduction

How to use this document

These calculations are designed to accompany our manuscript in the form of supplemental materials, for the sake of reproducibility. Throughout, commentary by the authors takes the form of green text. Citations, where needed, will be managed by direct reference to arXiv numbers, and all such references are already provided in full within the body of our manuscript. One exception is the source referred to throughout as `Blagojević'; this pertains to the book `Gravitation and Gauge Symmetries', which is also referenced within the manuscript.

Concrete relation to manuscript: In boxes like this, we will make specific connections between a result which is obtained in the supplemental material and a claim which is made in the manuscript. These points of contact are not always numbered equations, they could be textual.

Note that a programmatical session in the Wolfram language does not really correspond to the clean flow of thoughts in a LaTeX document: there are differences that can't (and shouldn't) be ignored. Thus, whilst this document should be at least readable in standalone format, the reader is encouraged to follow it in tandem with the Wolfram language source codes, so as to avoid ambiguities. In this way, the specific operations and manipulations of quantities will become absolutely clear.

Loading HiGGS and GeoHiGGS

For these calculations, we will use the HiGGS and GeoHiGGS packages. Note that GeoHiGGS was not developed for public release, and so is not documented. The versions of HiGGS and GeoHiGGS used for the computations here are both developer-only, and so we include copies of the sources with these supplemental materials.

Define a few constant parameters for the Lagrangian

Define a Planck mass.

$$\mathcal{M}_{\mathrm{Pl}} \tag{1}$$

Define the (dimensionless) coupling in front of the Ricci squared term.

$$\alpha \tag{2}$$

Define the (dimensionless) mass parameter in front of the vector.

$$\overset{2}{\cdot}\mu \tag{3}$$

Define the (dimensionless) mass parameter in front of the axial vector.

$$\overset{3}{\cdot}\mu \tag{4}$$

Post-Riemannian decomposition of curvature

First set up the post-Riemannian expansion of the Ricci scalar.

$$\mathrm{R}[\nabla] \tag{5}$$

$$\mathrm{R}\left[\mathring{\nabla}\right] + \frac{1}{4}\,\mathcal{T}_{\alpha\alpha'\beta}\;\mathcal{T}^{\alpha\alpha'\beta} + \frac{1}{2}\,\mathcal{T}^{\alpha\alpha'\beta}\;\mathcal{T}_{\alpha'\alpha\beta} + \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\;\mathcal{T}^{\beta}{}_{\alpha'\beta} - 2\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right) \tag{6}$$

Next set up the post-Riemannian expansion of the Ricci tensor.

$$\mathrm{R}[\nabla]_{\mu\nu} \tag{7}$$

$$\begin{aligned}&\mathrm{R}\left[\mathring{\nabla}\right]_{\mu\nu} - \frac{1}{2}\,\mathcal{T}^{\alpha}{}_{\mu\nu}\;\mathcal{T}^{\alpha'}{}_{\alpha\alpha'} + \frac{1}{2}\,\mathcal{T}^{\alpha'}{}_{\alpha\alpha'}\;\mathcal{T}_{\mu\nu}{}^{\alpha} + \frac{1}{4}\,\mathcal{T}_{\mu}{}^{\alpha\alpha'}\;\mathcal{T}_{\nu\alpha\alpha'} + \frac{1}{2}\,\mathcal{T}_{\alpha\mu\alpha'}\;\mathcal{T}_{\nu}{}^{\alpha\alpha'} + \\ &\frac{1}{2}\,\mathcal{T}^{\alpha'}{}_{\alpha\alpha'}\;\mathcal{T}_{\nu\mu}{}^{\alpha} + \frac{1}{2}\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\alpha}{}_{\mu\nu}\right) - \frac{1}{2}\left(\mathring{\nabla}_{\alpha}\mathcal{T}_{\mu\nu}{}^{\alpha}\right) - \frac{1}{2}\left(\mathring{\nabla}_{\alpha}\mathcal{T}_{\nu\mu}{}^{\alpha}\right) + \mathring{\nabla}_{\mu}\mathcal{T}^{\alpha}{}_{\nu\alpha}\end{aligned} \tag{8}$$

Irrep decomposition of the torsion

We will also present our conventions for the tensor, axial vector and vector parts of the torsion field.

$$\mathcal{T}^{\sigma}{}_{\mu\nu} \tag{9}$$

$$\frac{2}{3}\;\overset{1}{\cdot}\mathcal{T}_{\mu}{}^{\sigma}{}_{\nu} - \frac{2}{3}\;\overset{1}{\cdot}\mathcal{T}_{\nu}{}^{\sigma}{}_{\mu} - \frac{1}{3}\;\delta_{\nu}{}^{\sigma}\;\overset{2}{\cdot}\mathcal{T}_{\mu} + \frac{1}{3}\;\delta_{\mu}{}^{\sigma}\;\overset{2}{\cdot}\mathcal{T}_{\nu} + \epsilon\,\mathring{g}_{\mu\nu}{}^{\sigma}{}_{\alpha}\;\overset{3}{\cdot}\mathcal{T}^{\alpha} \tag{10}$$

Torsion Maxwell tensors

It is natural to define some Maxwell tensors $\overset{2}{\cdot}\mathcal{F}_{\mu\nu}$ and $\overset{3}{\cdot}\mathcal{F}_{\mu\nu}$ for the vector parts of the torsion, such as follows.

$$\overset{2}{\cdot}\mathcal{F}_{\mu\nu} \tag{11}$$

$$\mathring{\nabla}_{\mu}\overset{2}{\cdot}\mathcal{T}_{\nu} - \mathring{\nabla}_{\nu}\overset{2}{\cdot}\mathcal{T}_{\mu} \tag{12}$$

The 2-form field

Now we define a general 2-form field.

$$\mathcal{B}_{\alpha\beta} \tag{13}$$

$$-\frac{\overset{2}{\cdot}\mathcal{F}_{\alpha\beta}}{\mathcal{M}_{\mathrm{Pl}}}+\frac{3\,\epsilon\,\mathring{g}_{\alpha\beta\alpha'\beta'}\ \overset{3}{\cdot}\mathcal{F}^{\alpha'\beta'}}{4\,\mathcal{M}_{\mathrm{Pl}}}+\frac{\mathring{\nabla}_{\alpha'}\,\overset{1}{\cdot}\mathcal{T}_{\alpha}{}^{\alpha'}{}_{\beta}}{\mathcal{M}_{\mathrm{Pl}}}-\frac{\mathring{\nabla}_{\alpha'}\,\overset{1}{\cdot}\mathcal{T}_{\beta}{}^{\alpha'}{}_{\alpha}}{\mathcal{M}_{\mathrm{Pl}}} \tag{14}$$

Note that in Eq. (14) and moving forward, the symbol $\epsilon\,\mathring{g}^{\alpha\beta\chi\delta}$ denotes the totally antisymmetric tensor, covariantised with respect to $\mathring{g}_{\mu\nu}$.

Summary of findings: when we want the vector to propagate

The next few sections will be focussed on the following class of torsion theory, in which the Einstein-Hilbert term is augmented by the square of the antisymmetric Ricci tensor, and by a vector torsion mass term, and by a multiplier which switches off the axial vector torsion.

$$\alpha\ R[\nabla]_{\alpha\beta}\ R[\nabla]^{\alpha\beta}-\alpha\ R[\nabla]^{\alpha\beta}\ R[\nabla]_{\beta\alpha}-\frac{\mathcal{M}_{\mathrm{Pl}}{}^{2}\,R[\nabla]}{2}+\mathcal{M}_{\mathrm{Pl}}{}^{2}\ \overset{2}{\cdot}\mu\ \overset{2}{\cdot}\mathcal{T}_{\alpha}\ \overset{2}{\cdot}\mathcal{T}^{\alpha}+\overset{3}{\cdot}\mathcal{T}^{\alpha}\ \overset{3}{\cdot}\lambda_{\mathcal{T}\alpha} \tag{15}$$

By setting to zero either or both of the mass and multiplier terms in Eq. (15), we find some interesting effects in the linear spectra. These are summarised in the following table. Note that there are always an extra +2 d.o.f from Einstein's graviton.

	(No conditions)	$\overset{2}{\cdot}\mu == 0$	$\overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\mu} == 0$	$\overset{2}{\cdot}\mu == \overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\mu} == 0$
Linear d.o.f	6 + 2	3 + 2	3 + 2	0 + 2
Nonlinear d.o.f	6 + 2	6 + 2	6 + 2	6 + 2

(16)

Concrete relation to manuscript: Note that we can now compare Eq. (16) with Table (1) in the manuscript. Not all the information is present: we are focussing here specifically on the variations of the model which we would naively think to try, i.e. those for which we do not invoke the tensor multiplier $\overset{1}{\cdot}\mathcal{T}^{\alpha}{}_{\mu\nu}$. Note that the order of columns in these two tables differs.

Concrete relation to manuscript: The various columns in this table appear to cause some surprise, such as `how does the multiplier not kill off 3 d.o.f' and `how can the spectrum be completely empty'. We will now explore how these effects come to pass; in short it is all due to the interesting mediating effect of the tensor part of the torsion. We will focus mostly on the linear spectrum, but for the case which is thoroughly discussed in the manuscript (i.e. the 2nd column), we will also use Lagrangian techniques to examine the nonlinear spectrum. Later, we will exchange the vector for its axial counterpart (for which $\alpha > 0$). In that version of the theory, we will again tabulate and study the possible combinations of multipliers and masses, but for each case we will explicitly demonstrate the full nonlinear spectrum using Hamiltonian techniques.

Column 1 of Eq. (16): Vector mass and axial multiplier

Setting up the Lagrangian

We define the Lagrangian. It contains both the nonvanishing vector mass parameter ${}^{2}\mu$ and a multiplier field ${}^{3}\lambda_{\mathcal{T}}{}^{\mu}$ to disable the axial torsion.

$$\alpha\, R[\nabla]_{\alpha\beta}\, R[\nabla]^{\alpha\beta} - \alpha\, R[\nabla]^{\alpha\beta}\, R[\nabla]_{\beta\alpha} - \frac{\mathcal{M}_{Pl}{}^{2}\, R[\nabla]}{2} + \mathcal{M}_{Pl}{}^{2}\, {}^{2}\mu\, {}^{2}\mathcal{T}_{\alpha}\, {}^{2}\mathcal{T}^{\alpha} + {}^{3}\mathcal{T}^{\alpha}\, {}^{3}\lambda_{\mathcal{T}\alpha} \tag{17}$$

Now we would like to have the post-Riemannian decomposition of the Lagrangian Eq. (17).

$$\begin{aligned}
&-\frac{\mathcal{M}_{Pl}{}^{2}\, R[\mathring{\nabla}]}{2} - \frac{1}{8}\mathcal{M}_{Pl}{}^{2}\, \mathcal{T}_{\alpha\alpha'\beta}\, \mathcal{T}^{\alpha\alpha'\beta} - \frac{1}{4}\mathcal{M}_{Pl}{}^{2}\, \mathcal{T}^{\alpha\alpha'\beta}\, \mathcal{T}_{\alpha'\alpha\beta} - \\
&\frac{1}{2}\mathcal{M}_{Pl}{}^{2}\, \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\, \mathcal{T}^{\beta}{}_{\alpha'\beta} - \mathcal{M}_{Pl}{}^{2}\, {}^{2}\mu\, \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\, \mathcal{T}^{\beta}{}_{\alpha'\beta} + \frac{1}{4}\alpha\, \mathcal{T}_{\alpha}{}^{\beta'\chi}\, \mathcal{T}^{\alpha\alpha'\beta}\, \mathcal{T}_{\alpha'\beta}{}^{\prime}\, \mathcal{T}_{\beta'\chi'} + \\
&\frac{1}{4}\alpha\, \mathcal{T}^{\alpha\alpha'\beta}\, \mathcal{T}_{\alpha'\beta}{}^{\beta'}\, \mathcal{T}_{\beta'}{}^{\chi'}\, \mathcal{T}_{\chi\alpha'} - \alpha\, \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\, \mathcal{T}_{\alpha'}{}^{\beta\beta'}\, \mathcal{T}_{\beta}{}^{\chi'}\, \mathcal{T}_{\chi\beta''} - \\
&\frac{1}{2}\alpha\, \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\, \mathcal{T}_{\alpha'}{}^{\beta\beta'}\, \mathcal{T}^{\chi}{}_{\beta\beta'}\, \mathcal{T}^{\prime}{}_{\chi'} + \frac{1}{6}\epsilon\, \mathring{g}_{\alpha\alpha'\beta\beta'}\, \mathcal{T}^{\alpha'\beta\beta'}\, {}^{3}\lambda_{\mathcal{T}}{}^{\alpha} + \\
&\alpha\, \mathcal{T}^{\alpha\alpha'\beta}\, \mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\chi}{}_{\beta'\chi}\right) + \mathcal{M}_{Pl}{}^{2}\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right) + \alpha\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\beta'}{}_{\beta\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right) - \\
&\alpha\left(\mathring{\nabla}_{\beta}\mathcal{T}^{\beta'}{}_{\alpha'\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right) + \frac{1}{2}\alpha\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right) + 2\,\alpha\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right) - \\
&\alpha\, \mathcal{T}^{\alpha\alpha'\beta}\, \mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\chi}{}_{\alpha\chi}\right) - 2\,\alpha\, \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\, \mathcal{T}_{\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\chi}{}_{\beta\chi}\right) + \\
&\alpha\, \mathcal{T}^{\alpha\alpha'\beta}\, \mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\mathring{\nabla}_{\chi}\mathcal{T}^{\chi}{}_{\alpha\beta'}\right) + \alpha\, \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\, \mathcal{T}_{\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}_{\chi}\mathcal{T}^{\chi}{}_{\beta\beta'}\right)
\end{aligned} \tag{18}$$

We want to study the theory when it is linearised. As an intermediate step in order to do this, we just keep in Eq. (18) the second-order terms in torsion and no higher. Also from this point onwards we completely neglect factors of the curvature which may arise in the field equations by commuting covariant derivatives.

$$-\frac{\mathcal{M}_{Pl}{}^{2}\, R[\mathring{\nabla}]}{2} - \frac{1}{8}\mathcal{M}_{Pl}{}^{2}\, \mathcal{T}_{\alpha\alpha'\beta}\, \mathcal{T}^{\alpha\alpha'\beta} - \frac{1}{4}\mathcal{M}_{Pl}{}^{2}\, \mathcal{T}^{\alpha\alpha'\beta}\, \mathcal{T}_{\alpha'\alpha\beta} -$$

$$\frac{1}{2}\mathcal{M}_{\mathrm{Pl}}^{2}\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\;\mathcal{T}^{\beta}{}_{\alpha'\beta}-\mathcal{M}_{\mathrm{Pl}}^{2}\,{}^{2}_{\cdot}\mu\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\;\mathcal{T}^{\beta}{}_{\alpha'\beta}+\frac{1}{6}\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\;\mathcal{T}^{\alpha'\beta\beta'}\;{}^{3}_{\cdot}\lambda_{\mathcal{T}}{}^{\alpha}+$$
$$\mathcal{M}_{\mathrm{Pl}}^{2}\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)+\alpha\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\beta'}{}_{\beta\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)-\alpha\left(\mathring{\nabla}_{\beta}\mathcal{T}^{\beta'}{}_{\alpha'\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)+$$
$$\frac{1}{2}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right)+2\,\alpha\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right)$$

Now we decompose the torsion in Eq. (19) into the Lorentz irreps.

$$-\frac{\mathcal{M}_{\mathrm{Pl}}^{2}\,\mathrm{R}[\mathring{\nabla}]}{2}-\frac{2}{9}\mathcal{M}_{\mathrm{Pl}}^{2}\;{}^{1}_{\cdot}\mathcal{T}_{\alpha\alpha'\beta}\;{}^{1}_{\cdot}\mathcal{T}^{\alpha\alpha'\beta}+\frac{2}{9}\mathcal{M}_{\mathrm{Pl}}^{2}\;{}^{1}_{\cdot}\mathcal{T}_{\alpha\beta\alpha'}\;{}^{1}_{\cdot}\mathcal{T}^{\alpha\alpha'\beta}+\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}^{2}\;{}^{2}_{\cdot}\mathcal{T}_{\alpha}\;{}^{2}_{\cdot}\mathcal{T}^{\alpha}+$$
$$\mathcal{M}_{\mathrm{Pl}}^{2}\,{}^{2}_{\cdot}\mu\;{}^{2}_{\cdot}\mathcal{T}_{\alpha}\;{}^{2}_{\cdot}\mathcal{T}^{\alpha}-\frac{3}{4}\mathcal{M}_{\mathrm{Pl}}^{2}\;{}^{3}_{\cdot}\mathcal{T}_{\alpha}\;{}^{3}_{\cdot}\mathcal{T}^{\alpha}+{}^{3}_{\cdot}\mathcal{T}^{\alpha}\;{}^{3}_{\cdot}\lambda_{\mathcal{T}\alpha}+\mathcal{M}_{\mathrm{Pl}}^{2}\left(\mathring{\nabla}_{\alpha}{}^{2}_{\cdot}\mathcal{T}^{\alpha}\right)-$$
$$\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}{}^{2}_{\cdot}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha'}\;{}^{2}_{\cdot}\mathcal{T}^{\alpha}\right)+\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha'}{}^{2}_{\cdot}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha'}\;{}^{2}_{\cdot}\mathcal{T}^{\alpha}\right)+\alpha\left(\mathring{\nabla}_{\alpha}{}^{3}_{\cdot}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha'}\;{}^{3}_{\cdot}\mathcal{T}^{\alpha}\right)- \tag{20}$$
$$\alpha\left(\mathring{\nabla}_{\alpha'}{}^{3}_{\cdot}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha'}\;{}^{3}_{\cdot}\mathcal{T}^{\alpha}\right)+\frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha'}\;{}^{2}_{\cdot}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}{}^{1}_{\cdot}\mathcal{T}_{\alpha}{}^{\beta}{}_{\alpha'}\right)-\frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha'}\;{}^{2}_{\cdot}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}{}^{1}_{\cdot}\mathcal{T}_{\alpha'}{}^{\beta}{}_{\alpha}\right)-$$
$$\frac{4}{3}\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta'\chi}\left(\mathring{\nabla}^{\alpha'}\;{}^{3}_{\cdot}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}{}^{1}_{\cdot}\mathcal{T}^{\beta\beta'\chi}\right)+\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}{}^{1}_{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}{}^{1}_{\cdot}\mathcal{T}_{\alpha'}{}^{\beta'}{}_{\beta}\right)-$$
$$\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}{}^{1}_{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}{}^{1}_{\cdot}\mathcal{T}_{\beta}{}^{\beta'}{}_{\alpha'}\right)-\frac{4}{3}\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\left(\mathring{\nabla}^{\alpha'}\;{}^{2}_{\cdot}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\beta'}\;{}^{3}_{\cdot}\mathcal{T}^{\beta}\right)$$

Manipulating the field equations

Here is the tensor field equation.

$$-\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}^{2}\;{}^{1}_{\cdot}\mathcal{T}^{\iota}{}_{\theta\kappa}+\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}^{2}\;{}^{1}_{\cdot}\mathcal{T}^{\iota}{}_{\kappa\theta}-\frac{2}{9}\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha}{}^{2}_{\cdot}\mathcal{F}_{\theta}{}^{\alpha}\right)+$$
$$\frac{2}{9}\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha}{}^{2}_{\cdot}\mathcal{F}_{\kappa}{}^{\alpha}\right)-\frac{2}{9}\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}{}^{1}_{\cdot}\mathcal{T}^{\alpha\alpha'}{}_{\theta}\right)+\frac{2}{9}\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}{}^{1}_{\cdot}\mathcal{T}^{\alpha\alpha'}{}_{\kappa}\right)+$$
$$\frac{2}{9}\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}{}^{1}_{\cdot}\mathcal{T}_{\theta}{}^{\alpha\alpha'}\right)-\frac{2}{9}\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}{}^{1}_{\cdot}\mathcal{T}_{\kappa}{}^{\alpha\alpha'}\right)-\frac{1}{3}\alpha\,\epsilon\,\mathring{g}_{\theta\kappa\alpha\alpha'}\left(\mathring{\nabla}^{\iota}\;{}^{3}_{\cdot}\mathcal{F}^{\alpha\alpha'}\right)-$$
$$\frac{4}{9}\alpha\left(\mathring{\nabla}^{\iota}\mathring{\nabla}_{\alpha}{}^{1}_{\cdot}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\right)+\frac{4}{9}\alpha\left(\mathring{\nabla}^{\iota}\mathring{\nabla}_{\alpha}{}^{1}_{\cdot}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\right)+\frac{1}{3}\alpha\left(\mathring{\nabla}_{\theta}{}^{2}_{\cdot}\mathcal{F}^{\iota}{}_{\kappa}\right)-\frac{1}{6}\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\kappa\alpha\alpha'}\left(\mathring{\nabla}_{\theta}{}^{3}_{\cdot}\mathcal{F}^{\alpha\alpha'}\right)- \tag{21}$$
$$\frac{2}{9}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}{}^{1}_{\cdot}\mathcal{T}^{\iota\alpha}{}_{\kappa}\right)+\frac{2}{9}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}{}^{1}_{\cdot}\mathcal{T}_{\kappa}{}^{\alpha\iota}\right)+\frac{1}{3}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\iota}\;{}^{2}_{\cdot}\mathcal{T}_{\kappa}\right)-\frac{1}{3}\alpha\left(\mathring{\nabla}_{\kappa}{}^{2}_{\cdot}\mathcal{F}^{\iota}{}_{\theta}\right)+$$
$$\frac{1}{6}\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\theta\alpha\alpha'}\left(\mathring{\nabla}_{\kappa}{}^{3}_{\cdot}\mathcal{F}^{\alpha\alpha'}\right)+\frac{2}{9}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}{}^{1}_{\cdot}\mathcal{T}^{\iota\alpha}{}_{\theta}\right)-\frac{2}{9}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}{}^{1}_{\cdot}\mathcal{T}_{\theta}{}^{\alpha\iota}\right)-\frac{1}{3}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\iota}\;{}^{2}_{\cdot}\mathcal{T}_{\theta}\right)==0$$

There is some nuance to how we obtain Eq. (21), in that we can't just vary with respect to the tensor irrep. If we do that, then the resulting equation can have traces which are not true on-shell. It is safest to actually vary with respect to the whole torsion tensor and then project out the (traceless by construction) tensor irrep from that equation.

Now for the vector equation.

$$\frac{2}{3}\mathcal{M}_{\mathrm{Pl}}^{2}\,\overset{2}{\cdot}\mathcal{T}^{\mu}+2\,\mathcal{M}_{\mathrm{Pl}}^{2}\,\overset{2}{\cdot}\mu\,\overset{2}{\cdot}\mathcal{T}^{\mu}+\frac{8}{9}\,\alpha\left(\mathring{\nabla}_{\alpha}\overset{2}{\cdot}\mathcal{F}^{\mu\alpha}\right)+\frac{8}{9}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\mu}\right)-\frac{8}{9}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\mu\alpha\alpha'}\right)==0 \tag{22}$$

And for the axial vector equation.

$$-\frac{3}{2}\mathcal{M}_{\mathrm{Pl}}^{2}\,\overset{3}{\cdot}\mathcal{T}^{\mu}+\overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\mu}-2\,\alpha\left(\mathring{\nabla}_{\alpha}\overset{3}{\cdot}\mathcal{F}^{\mu\alpha}\right)+\frac{4}{3}\,\alpha\,\epsilon\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\left(\mathring{\nabla}^{\beta'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\right)==0 \tag{23}$$

Now we can also take the dual of Eq. (23), and so we write this out for completeness.

$$\begin{aligned}&\frac{3}{2}\mathcal{M}_{\mathrm{Pl}}^{2}\,\epsilon\mathring{g}^{\alpha\beta\chi}{}_{\alpha'}\,\overset{3}{\cdot}\mathcal{T}^{\alpha'}-\epsilon\mathring{g}^{\alpha\beta\chi}{}_{\alpha'}\,\overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\alpha'}-2\,\alpha\,\epsilon\mathring{g}^{\alpha\beta\chi}{}_{\mu}\left(\mathring{\nabla}_{\alpha'}\overset{3}{\cdot}\mathcal{F}^{\alpha'\mu}\right)-\\&\quad\frac{4}{3}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha}\overset{1}{\cdot}\mathcal{T}^{\beta\alpha'\chi}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha}\overset{1}{\cdot}\mathcal{T}^{\chi\alpha'\beta}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\chi}\right)-\\&\quad\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\overset{1}{\cdot}\mathcal{T}^{\chi\alpha'\alpha}\right)-\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha'}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha'}\overset{1}{\cdot}\mathcal{T}^{\beta\alpha'\alpha}\right)==0\end{aligned} \tag{24}$$

Finally we have the equation for the multiplier itself.

$$\overset{3}{\cdot}\mathcal{T}_{\mu}==0 \tag{25}$$

With the effective 2-form field in Eq. (14), we are ready to simplify the equations of motion.

Here is the vector equation Eq. (22).

$$\frac{2}{3}\mathcal{M}_{\mathrm{Pl}}^{2}\left(1+3\,\overset{2}{\cdot}\mu\right)\overset{2}{\cdot}\mathcal{T}^{\mu}-\frac{8}{9}\,\alpha\,\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\mu\alpha}\right)==0 \tag{26}$$

Here is the axial vector equation Eq. (23).

$$\overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\mu}+\frac{2}{3}\,\alpha\,\mathcal{M}_{\mathrm{Pl}}\,\epsilon\mathring{g}^{\mu}{}_{\alpha\alpha'\beta}\left(\mathring{\nabla}^{\beta}\mathcal{B}^{\alpha\alpha'}\right)==0 \tag{27}$$

Again here is the dual part of Eq. (26)

$$\begin{aligned}&-\epsilon\mathring{g}_{\iota\theta\kappa\alpha}\,\overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\alpha}-2\,\alpha\,\epsilon\mathring{g}_{\iota\theta\kappa\alpha'}\left(\mathring{\nabla}_{\alpha}\overset{3}{\cdot}\mathcal{F}^{\alpha\alpha'}\right)-2\,\alpha\,\epsilon\mathring{g}_{\theta\kappa\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\overset{3}{\cdot}\mathcal{F}_{\iota}{}^{\alpha}\right)+\\&\quad 2\,\alpha\,\epsilon\mathring{g}_{\iota\kappa\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\overset{3}{\cdot}\mathcal{F}_{\theta}{}^{\alpha}\right)-2\,\alpha\,\epsilon\mathring{g}_{\iota\theta\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\overset{3}{\cdot}\mathcal{F}_{\kappa}{}^{\alpha}\right)-\frac{4}{3}\,\alpha\,\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\iota}\mathcal{B}_{\theta\kappa}\right)-\frac{4}{3}\,\alpha\left(\mathring{\nabla}_{\iota}\overset{2}{\cdot}\mathcal{F}_{\theta\kappa}\right)+\\&\quad\frac{4}{3}\,\alpha\,\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\theta}\mathcal{B}_{\iota\kappa}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}_{\theta}\overset{2}{\cdot}\mathcal{F}_{\iota\kappa}\right)-\frac{4}{3}\,\alpha\,\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\kappa}\mathcal{B}_{\iota\theta}\right)-\frac{4}{3}\,\alpha\left(\mathring{\nabla}_{\kappa}\overset{2}{\cdot}\mathcal{F}_{\iota\theta}\right)==0\end{aligned} \tag{28}$$

Note that Eqs. (26), and (27) allow us to solve for $\overset{2}{\cdot}\mathcal{T}_{\mu}$ and $\overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\mu}$ purely in terms of the 2-form.

And here is the tensor equation Eq. (21).

$$-\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}^{2}\;\overset{1}{\cdot}\mathcal{T}^{\prime}{}_{\theta\kappa}+\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}^{2}\;\overset{1}{\cdot}\mathcal{T}^{\prime}{}_{\kappa\theta}+\frac{2}{9}\alpha\mathcal{M}_{\mathrm{Pl}}\;\delta^{\prime}{}_{\kappa}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)-\frac{2}{9}\alpha\mathcal{M}_{\mathrm{Pl}}\;\delta^{\prime}{}_{\theta}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{4}{9}\alpha\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}^{\prime}\mathcal{B}_{\theta\kappa}\right)-\frac{4}{9}\alpha\left(\mathring{\nabla}^{\prime}\overset{2}{\cdot}\mathcal{F}_{\theta\kappa}\right)-\frac{2}{9}\alpha\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\theta}\mathcal{B}^{\prime}{}_{\kappa}\right)+\frac{5}{18}\alpha\left(\mathring{\nabla}_{\theta}\overset{2}{\cdot}\mathcal{F}^{\prime}{}_{\kappa}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\prime}\overset{2}{\cdot}\mathcal{T}_{\kappa}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}\overset{2}{\cdot}\mathcal{T}^{\prime}\right)+\frac{2}{9}\alpha\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\kappa}\mathcal{B}^{\prime}{}_{\theta}\right)-\frac{5}{18}\alpha\left(\mathring{\nabla}_{\kappa}\overset{2}{\cdot}\mathcal{F}^{\prime}{}_{\theta}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\prime}\overset{2}{\cdot}\mathcal{T}_{\theta}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\overset{2}{\cdot}\mathcal{T}^{\prime}\right)==0$$

Now the next thing we do is to take the divergence of Eq. (29).

$$-\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}^{2}\left(\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\right)+\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}^{2}\left(\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\right)-\frac{4}{9}\alpha\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)-\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\overset{2}{\cdot}\mathcal{F}_{\theta\kappa}\right)+\frac{2}{9}\alpha\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\overset{2}{\cdot}\mathcal{F}_{\kappa}{}^{\alpha}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha}\overset{2}{\cdot}\mathcal{T}_{\kappa}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}\overset{2}{\cdot}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right)+\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\overset{2}{\cdot}\mathcal{F}_{\theta}{}^{\alpha}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha}\overset{2}{\cdot}\mathcal{T}_{\theta}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\overset{2}{\cdot}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{2}{9}\alpha\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)==0 \tag{30}$$

Next we substitute into Eq. (30) for the 2-form field in Eq. (14) again.

$$-\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}^{3}\;\mathcal{B}_{\theta\kappa}-\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}^{2}\;\overset{2}{\cdot}\mathcal{F}_{\theta\kappa}+\frac{1}{4}\mathcal{M}_{\mathrm{Pl}}^{2}\;\epsilon\,\mathring{g}_{\theta\kappa\alpha\alpha'}\;\overset{3}{\cdot}\mathcal{F}^{\alpha\alpha'}-\frac{4}{9}\alpha\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)-\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\overset{2}{\cdot}\mathcal{F}_{\theta\kappa}\right)+\frac{2}{9}\alpha\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\overset{2}{\cdot}\mathcal{F}_{\kappa}{}^{\alpha}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha}\overset{2}{\cdot}\mathcal{T}_{\kappa}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}\overset{2}{\cdot}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right)+\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\overset{2}{\cdot}\mathcal{F}_{\theta}{}^{\alpha}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha}\overset{2}{\cdot}\mathcal{T}_{\theta}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\overset{2}{\cdot}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{2}{9}\alpha\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)==0 \tag{31}$$

Next we will take Eq. (31) and expand the vector and axial Maxwell tensors back into derivatives, and substitute for the solutions in terms of the 2-form field that we obtained from Eqs. (26), and (27).

$$-\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}^{3}\;\mathcal{B}_{\theta\kappa}-\frac{4}{9}\alpha\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)-\frac{4\,\alpha\,\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)}{9+27\;\overset{2}{\cdot}\mu}+\frac{4\,\alpha\,\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)}{9+27\;\overset{2}{\cdot}\mu}==0 \tag{32}$$

So now the equation in Eq. (32) contains all the dynamical information about the linear spectrum of the theory Eq. (17), since Eqs. (26), (27), and (29) serve only to determine the torsion in terms of the 2-form. So the key question is how much of the 2-form does Eq. (32) propagate?

Let's first take the divergence of Eq. (32).

$$\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}^{3}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{4\,\alpha\,\mathcal{M}_{\mathrm{Pl}}\;\overset{2}{\cdot}\mu\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)}{3+9\;\overset{2}{\cdot}\mu}==0 \tag{33}$$

So Eq. (33) still looks like a propagating equation.

Let's now take the dual of the gradient of Eq. (32).

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{3}\ \epsilon\overset{\circ}{g}{}^{\psi}{}_{\alpha\theta\kappa}\left(\overset{\circ}{\nabla}{}^{\kappa}\mathcal{B}^{\alpha\theta}\right)-\frac{4}{9}\alpha\mathcal{M}_{\text{Pl}}\ \epsilon\overset{\circ}{g}{}^{\psi}{}_{\alpha\theta\omega}\left(\overset{\circ}{\nabla}{}^{\omega}\overset{\circ}{\nabla}_{\kappa}\overset{\circ}{\nabla}{}^{\kappa}\mathcal{B}^{\alpha\theta}\right)==0 \tag{34}$$

So Eq. (34) still looks like a propagating equation.

Concrete relation to manuscript: Since we are unable to find non-propagating parts in the field equation, we conclude that the whole 2-form propagates, i.e. 6 propagating d.o.f in the linear spectrum.

Column 2 of Eq. (16): Axial multiplier, but no vector mass

Setting up the Lagrangian

We define the Lagrangian. It contains vanishing vector mass parameter $\overset{2}{\cdot}\mu$ and a multiplier field $\overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\mu}$ to disable the axial torsion.

$$\alpha\ \text{R}[\nabla]_{\alpha\beta}\ \text{R}[\nabla]^{\alpha\beta}-\alpha\ \text{R}[\nabla]^{\alpha\beta}\ \text{R}[\nabla]_{\beta\alpha}-\frac{\mathcal{M}_{\text{Pl}}^{2}\ \text{R}[\nabla]}{2}+\overset{3}{\cdot}\mathcal{T}^{\alpha}\ \overset{3}{\cdot}\lambda_{\mathcal{T}\alpha} \tag{35}$$

Now we would like to have the post-Riemannian decomposition of the Lagrangian Eq. (35).

$$\begin{aligned}&-\frac{\mathcal{M}_{\text{Pl}}^{2}\ \text{R}[\overset{\circ}{\nabla}]}{2}-\frac{1}{8}\mathcal{M}_{\text{Pl}}^{2}\ \mathcal{T}_{\alpha\alpha'\beta}\ \mathcal{T}^{\alpha\alpha'\beta}-\frac{1}{4}\mathcal{M}_{\text{Pl}}^{2}\ \mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\alpha\beta}-\frac{1}{2}\mathcal{M}_{\text{Pl}}^{2}\ \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\ \mathcal{T}^{\beta}{}_{\alpha'\beta}+\\&\frac{1}{4}\alpha\ \mathcal{T}_{\alpha}{}^{\beta'\chi}\ \mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\beta}{}^{\mu}\ \mathcal{T}_{\beta'\chi\mu}+\frac{1}{4}\alpha\ \mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\beta}{}^{\beta'}\ \mathcal{T}_{\beta'}{}^{\chi\mu}\ \mathcal{T}_{\chi\alpha\mu}-\\&\alpha\ \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\ \mathcal{T}_{\alpha'}{}^{\beta\beta'}\ \mathcal{T}_{\beta}{}^{\chi\mu}\ \mathcal{T}_{\chi\beta'\mu}-\frac{1}{2}\alpha\ \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\ \mathcal{T}_{\alpha'}{}^{\beta\beta'}\ \mathcal{T}^{\chi}{}_{\beta\beta'}\ \mathcal{T}^{\mu}{}_{\chi\mu}+\\&\frac{1}{6}\ \epsilon\overset{\circ}{g}_{\alpha\alpha'\beta\beta'}\ \mathcal{T}^{\alpha'\beta\beta'}\ \overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\alpha}+\alpha\ \mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\overset{\circ}{\nabla}_{\alpha}\mathcal{T}^{\chi}{}_{\beta'\chi}\right)+\mathcal{M}_{\text{Pl}}^{2}\left(\overset{\circ}{\nabla}_{\alpha'}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)+\\&\alpha\left(\overset{\circ}{\nabla}_{\alpha'}\mathcal{T}^{\beta'}{}_{\beta\beta'}\right)\left(\overset{\circ}{\nabla}{}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)-\alpha\left(\overset{\circ}{\nabla}_{\beta}\mathcal{T}^{\beta'}{}_{\alpha'\beta'}\right)\left(\overset{\circ}{\nabla}{}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)+\frac{1}{2}\alpha\left(\overset{\circ}{\nabla}_{\alpha}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\overset{\circ}{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right)+\\&2\alpha\left(\overset{\circ}{\nabla}{}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)\left(\overset{\circ}{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right)-\alpha\ \mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\overset{\circ}{\nabla}_{\beta'}\mathcal{T}^{\chi}{}_{\alpha\chi}\right)-\\&2\alpha\ \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\ \mathcal{T}_{\alpha'}{}^{\beta\beta'}\left(\overset{\circ}{\nabla}_{\beta'}\mathcal{T}^{\chi}{}_{\beta\chi}\right)+\alpha\ \mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\overset{\circ}{\nabla}_{\chi}\mathcal{T}^{\chi}{}_{\alpha\beta'}\right)+\alpha\ \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\ \mathcal{T}_{\alpha'}{}^{\beta\beta'}\left(\overset{\circ}{\nabla}_{\chi}\mathcal{T}^{\chi}{}_{\beta\beta'}\right)\end{aligned} \tag{36}$$

We want to study the theory when it is linearised. As an intermediate step in order to do this, we just keep in Eq. (36) the second-order terms in torsion and no higher. Also from this point onwards we completely neglect factors of the curvature which may arise in the field equations by commuting covariant derivatives.

$$-\frac{\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\mathrm{R}[\mathring{\nabla}]}{2}-\frac{1}{8}\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\mathcal{T}_{\alpha\alpha'\beta}\,\mathcal{T}^{\alpha\alpha'\beta}-\frac{1}{4}\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\mathcal{T}^{\alpha\alpha'\beta}\,\mathcal{T}_{\alpha'\alpha\beta}-\frac{1}{2}\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\,\mathcal{T}^{\beta}{}_{\alpha'\beta}+$$
$$\frac{1}{6}\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\,\mathcal{T}^{\alpha'\beta\beta'}\,\overset{3}{\lambda}_{\mathcal{T}}{}^{\alpha}+\mathcal{M}_{\mathrm{Pl}}{}^{2}\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)+\alpha\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\beta'}{}_{\beta\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)-$$
$$\alpha\left(\mathring{\nabla}_{\beta}\mathcal{T}^{\beta'}{}_{\alpha'\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)+\frac{1}{2}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right)+2\,\alpha\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right) \tag{37}$$

Now we decompose the torsion in Eq. (37) into the Lorentz irreps.

$$-\frac{\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\mathrm{R}[\mathring{\nabla}]}{2}-\frac{2}{9}\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\overset{1}{\mathcal{T}}_{\alpha\alpha'\beta}\,\overset{1}{\mathcal{T}}{}^{\alpha\alpha'\beta}+\frac{2}{9}\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\overset{1}{\mathcal{T}}_{\alpha\beta\alpha'}\,\overset{1}{\mathcal{T}}{}^{\alpha\alpha'\beta}+$$
$$\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\overset{2}{\mathcal{T}}_{\alpha}\,\overset{2}{\mathcal{T}}{}^{\alpha}-\frac{3}{4}\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\overset{3}{\mathcal{T}}_{\alpha}\,\overset{3}{\mathcal{T}}{}^{\alpha}+\overset{3}{\mathcal{T}}{}^{\alpha}\,\overset{3}{\lambda}_{\mathcal{T}\alpha}+\mathcal{M}_{\mathrm{Pl}}{}^{2}\left(\mathring{\nabla}_{\alpha}\overset{2}{\mathcal{T}}{}^{\alpha}\right)-$$
$$\frac{4}{9}\,\alpha\left(\mathring{\nabla}_{\alpha}\overset{2}{\mathcal{T}}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha}\right)+\frac{4}{9}\,\alpha\left(\mathring{\nabla}_{\alpha'}\overset{2}{\mathcal{T}}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha}\right)+\alpha\left(\mathring{\nabla}_{\alpha}\overset{3}{\mathcal{T}}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha'}\overset{3}{\mathcal{T}}{}^{\alpha}\right)-$$
$$\alpha\left(\mathring{\nabla}_{\alpha'}\overset{3}{\mathcal{T}}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha'}\overset{3}{\mathcal{T}}{}^{\alpha}\right)+\frac{8}{9}\,\alpha\left(\mathring{\nabla}^{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}\overset{1}{\mathcal{T}}_{\alpha}{}^{\beta}{}_{\alpha'}\right)-\frac{8}{9}\,\alpha\left(\mathring{\nabla}^{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}\overset{1}{\mathcal{T}}_{\alpha'}{}^{\beta}{}_{\alpha}\right)-$$
$$\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta'\chi}\left(\mathring{\nabla}^{\alpha'}\overset{3}{\mathcal{T}}{}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}\overset{1}{\mathcal{T}}{}^{\beta\beta'\chi}\right)+\frac{4}{9}\,\alpha\left(\mathring{\nabla}_{\alpha}\overset{1}{\mathcal{T}}{}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}\overset{1}{\mathcal{T}}_{\alpha'}{}^{\beta'}{}_{\beta}\right)-$$
$$\frac{4}{9}\,\alpha\left(\mathring{\nabla}_{\alpha}\overset{1}{\mathcal{T}}{}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}\overset{1}{\mathcal{T}}_{\beta}{}^{\beta'}{}_{\alpha'}\right)-\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\left(\mathring{\nabla}^{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha}\right)\left(\mathring{\nabla}^{\beta'}\overset{3}{\mathcal{T}}{}^{\beta}\right) \tag{38}$$

Manipulating the field equations

Here is the tensor field equation.

$$-\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\overset{1}{\mathcal{T}}{}^{\iota}{}_{\theta\kappa}+\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\overset{1}{\mathcal{T}}{}^{\iota}{}_{\kappa\theta}-\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha}\overset{2}{\mathcal{F}}_{\theta}{}^{\alpha}\right)+$$
$$\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha}\overset{2}{\mathcal{F}}_{\kappa}{}^{\alpha}\right)-\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\mathcal{T}}{}^{\alpha\alpha'}{}_{\theta}\right)+\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\mathcal{T}}{}^{\alpha\alpha'}{}_{\kappa}\right)+$$
$$\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\mathcal{T}}_{\theta}{}^{\alpha\alpha'}\right)-\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\mathcal{T}}_{\kappa}{}^{\alpha\alpha'}\right)-\frac{1}{3}\,\alpha\,\epsilon\,\mathring{g}_{\theta\kappa\alpha\alpha'}\left(\mathring{\nabla}^{\iota}\overset{3}{\mathcal{F}}{}^{\alpha\alpha'}\right)-$$
$$\frac{4}{9}\,\alpha\left(\mathring{\nabla}^{\iota}\mathring{\nabla}_{\alpha}\overset{1}{\mathcal{T}}_{\theta}{}^{\alpha}{}_{\kappa}\right)+\frac{4}{9}\,\alpha\left(\mathring{\nabla}^{\iota}\mathring{\nabla}_{\alpha}\overset{1}{\mathcal{T}}_{\kappa}{}^{\alpha}{}_{\theta}\right)+\frac{1}{3}\,\alpha\left(\mathring{\nabla}_{\theta}\overset{2}{\mathcal{F}}{}^{\iota}{}_{\kappa}\right)-\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\kappa\alpha\alpha'}\left(\mathring{\nabla}_{\theta}\overset{3}{\mathcal{F}}{}^{\alpha\alpha'}\right)-$$
$$\frac{2}{9}\,\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\overset{1}{\mathcal{T}}{}^{\iota\alpha}{}_{\kappa}\right)+\frac{2}{9}\,\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\overset{1}{\mathcal{T}}_{\kappa}{}^{\alpha\iota}\right)+\frac{1}{3}\,\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\iota}\overset{2}{\mathcal{T}}_{\kappa}\right)-\frac{1}{3}\,\alpha\left(\mathring{\nabla}_{\kappa}\overset{2}{\mathcal{F}}{}^{\iota}{}_{\theta}\right)+$$
$$\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\theta\alpha\alpha'}\left(\mathring{\nabla}_{\kappa}\overset{3}{\mathcal{F}}{}^{\alpha\alpha'}\right)+\frac{2}{9}\,\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\overset{1}{\mathcal{T}}{}^{\iota\alpha}{}_{\theta}\right)-\frac{2}{9}\,\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\overset{1}{\mathcal{T}}_{\theta}{}^{\alpha\iota}\right)-\frac{1}{3}\,\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\iota}\overset{2}{\mathcal{T}}_{\theta}\right)==0 \tag{39}$$

There is some nuance to how we obtain Eq. (39), in that we can't just vary with respect to the tensor irrep. If we do that, then the resulting equation can have traces which are not true on-shell. It is safest to actually vary with respect to the whole torsion tensor and then project out the (traceless by construction) tensor irrep from that equation.

Now for the vector equation.

$$\frac{2}{3}\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{2}{\cdot}\mathcal{T}^{\mu}+\frac{8}{9}\,\alpha\left(\mathring{\nabla}_{\alpha}\overset{2}{\cdot}\mathcal{F}^{\mu\alpha}\right)+\frac{8}{9}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\mu}\right)-\frac{8}{9}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\mu\alpha\alpha'}\right)==0 \tag{40}$$

And for the axial vector equation.

$$-\frac{3}{2}\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{3}{\cdot}\mathcal{T}^{\mu}+\overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\mu}-2\,\alpha\left(\mathring{\nabla}_{\alpha}\overset{3}{\cdot}\mathcal{F}^{\mu\alpha}\right)+\frac{4}{3}\,\alpha\,\epsilon\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\left(\mathring{\nabla}^{\beta'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\right)==0 \tag{41}$$

Now we can also take the dual of Eq. (41), and so we write this out for completeness.

$$\begin{aligned}&\frac{3}{2}\mathcal{M}_{\text{Pl}}{}^{2}\,\epsilon\mathring{g}^{\alpha\beta\chi}{}_{\alpha'}\,\overset{3}{\cdot}\mathcal{T}^{\alpha'}-\epsilon\mathring{g}^{\alpha\beta\chi}{}_{\alpha'}\,\overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\alpha'}-2\,\alpha\,\epsilon\mathring{g}^{\alpha\beta\chi}{}_{\mu}\left(\mathring{\nabla}_{\alpha'}\overset{3}{\cdot}\mathcal{F}^{\alpha'\mu}\right)-\\&\quad\frac{4}{3}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha}\overset{1}{\cdot}\mathcal{T}^{\beta\alpha'\chi}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha}\overset{1}{\cdot}\mathcal{T}^{\chi\alpha'\beta}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\chi}\right)-\\&\quad\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\overset{1}{\cdot}\mathcal{T}^{\chi\alpha'\alpha}\right)-\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha'}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha'}\overset{1}{\cdot}\mathcal{T}^{\beta\alpha'\alpha}\right)==0\end{aligned} \tag{42}$$

Finally we have the equation for the multiplier itself.

$$\overset{3}{\cdot}\mathcal{T}_{\mu}==0 \tag{43}$$

With the effective 2-form field in Eq. (14), we are ready to simplify the equations of motion.

Here is the vector equation Eq. (40).

$$\frac{2}{3}\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{2}{\cdot}\mathcal{T}^{\mu}-\frac{8}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\mu\alpha}\right)==0 \tag{44}$$

Concrete relation to manuscript: Compare Eq. (44) with (the linear part of) Eq. (5b) in our manuscript.

Here is the axial vector equation Eq. (41).

$$\overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\mu}+\frac{2}{3}\,\alpha\,\mathcal{M}_{\text{Pl}}\,\epsilon\mathring{g}^{\mu}{}_{\alpha\alpha'\beta}\left(\mathring{\nabla}^{\beta}\mathcal{B}^{\alpha\alpha'}\right)==0 \tag{45}$$

Again here is the dual part of Eq. (45)

$$-\epsilon\mathring{g}_{\iota\theta\kappa\alpha}\,\overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\alpha}-\frac{4}{3}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\iota}\mathcal{B}_{\theta\kappa}\right)+\frac{4}{3}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathcal{B}_{\iota\kappa}\right)-\frac{4}{3}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathcal{B}_{\iota\theta}\right)==0 \tag{46}$$

Note that Eqs. (44), and (45) allow us to solve for $\overset{2}{\cdot}\mathcal{T}_{\mu}$ and $\overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\mu}$ purely in terms of the 2-form.

And here is the tensor equation Eq. (39).

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\;{}^{1}_{\cdot}\mathcal{T}'{}_{\theta\kappa}+\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\;{}^{1}_{\cdot}\mathcal{T}'{}_{\kappa\theta}+\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\;\delta'{}_{\kappa}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)-\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\;\delta'{}_{\theta}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{4}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}{}'\mathcal{B}_{\theta\kappa}\right)-\frac{4}{9}\alpha\left(\mathring{\nabla}{}'\,{}^{2}_{\cdot}\mathcal{F}_{\theta\kappa}\right)-\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathcal{B}'{}_{\kappa}\right)+\frac{5}{18}\alpha\left(\mathring{\nabla}_{\theta}\,{}^{2}_{\cdot}\mathcal{F}'{}_{\kappa}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}{}'\,{}^{2}_{\cdot}\mathcal{T}_{\kappa}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}\,{}^{2}_{\cdot}\mathcal{T}'\right)+\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathcal{B}'{}_{\theta}\right)-\frac{5}{18}\alpha\left(\mathring{\nabla}_{\kappa}\,{}^{2}_{\cdot}\mathcal{F}'{}_{\theta}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}{}'\,{}^{2}_{\cdot}\mathcal{T}_{\theta}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\,{}^{2}_{\cdot}\mathcal{T}'\right)==0$$

Concrete relation to manuscript: Compare Eq. (47) with (the linear part of) Eq. (5a) in our manuscript (note that many higher-order terms will cancel after expanding the third term using Eq. (14)).

Now the next thing we do is to take the divergence of Eq. (47).

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\left(\mathring{\nabla}_{\alpha}\,{}^{1}_{\cdot}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\right)+\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\left(\mathring{\nabla}_{\alpha}\,{}^{1}_{\cdot}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\right)-\frac{4}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)-\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,{}^{2}_{\cdot}\mathcal{F}_{\theta\kappa}\right)+\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\,{}^{2}_{\cdot}\mathcal{F}_{\kappa}{}^{\alpha}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha}\,{}^{2}_{\cdot}\mathcal{T}_{\kappa}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}\,{}^{2}_{\cdot}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right)+\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\,{}^{2}_{\cdot}\mathcal{F}_{\theta}{}^{\alpha}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha}\,{}^{2}_{\cdot}\mathcal{T}_{\theta}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\,{}^{2}_{\cdot}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)==0 \tag{48}$$

Next we substitute into Eq. (48) for the 2-form field in Eq. (14) again.

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{3}\,\mathcal{B}_{\theta\kappa}-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\;{}^{2}_{\cdot}\mathcal{F}_{\theta\kappa}+\frac{1}{4}\mathcal{M}_{\text{Pl}}^{2}\;\epsilon\mathring{g}_{\theta\kappa\alpha\alpha'}\;{}^{3}_{\cdot}\mathcal{F}^{\alpha\alpha'}-\frac{4}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)-\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,{}^{2}_{\cdot}\mathcal{F}_{\theta\kappa}\right)+\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\,{}^{2}_{\cdot}\mathcal{F}_{\kappa}{}^{\alpha}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha}\,{}^{2}_{\cdot}\mathcal{T}_{\kappa}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}\,{}^{2}_{\cdot}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right)+\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\,{}^{2}_{\cdot}\mathcal{F}_{\theta}{}^{\alpha}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha}\,{}^{2}_{\cdot}\mathcal{T}_{\theta}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\,{}^{2}_{\cdot}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)==0 \tag{49}$$

Next we will take Eq. (49) and expand the vector and axial Maxwell tensors back into derivatives, and substitute for the solutions in terms of the 2-form field that we obtained from Eqs. (44), and (45).

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{3}\,\mathcal{B}_{\theta\kappa}-\frac{4}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)-\frac{4}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{4}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)==0 \tag{50}$$

So now the equation in Eq. (50) contains all the dynamical information about the linear spectrum of the theory Eq. (35), since Eqs. (44), (45), and (47) serve only to determine the torsion in terms of the 2-form. So the key question is how much of the 2-form does Eq. (50) propagate?

Let's first take the divergence of Eq. (50).

$$\frac{1}{3}\mathcal{M}_{\text{Pl}}{}^{3}\left(\overset{\circ}{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right) == 0 \tag{51}$$

So Eq. (51) is just a constraint, which knocks out 3 d.o.f from the 2-form.

Let's now take the dual of the gradient of Eq. (50).

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}{}^{3}\ \epsilon\overset{\circ}{g}{}^{\psi}{}_{\alpha\theta\kappa}\left(\overset{\circ}{\nabla}{}^{\kappa}\mathcal{B}^{\alpha\theta}\right)-\frac{4}{9}\alpha\mathcal{M}_{\text{Pl}}\ \epsilon\overset{\circ}{g}{}^{\psi}{}_{\alpha\theta\omega}\left(\overset{\circ}{\nabla}{}^{\omega}\overset{\circ}{\nabla}_{\kappa}\overset{\circ}{\nabla}{}^{\kappa}\mathcal{B}^{\alpha\theta}\right) == 0 \tag{52}$$

So Eq. (52) still looks like a propagating equation.

Concrete relation to manuscript: Since we lose 3 d.o.f to a constraint, we conclude that half the 2-form propagates, i.e. 3 propagating d.o.f in the linear spectrum. Note that we may directly integrate Eq. (50) to produce (in flat space) the first line in the effective theory Eq. (6) in our manuscript. What about the second line? To explore this, we have to deal with the extension of the theory into the nonlinear regime.

Column 2 of Eq. (16): Axial multiplier, but no vector mass (nonlinear extension)

Setting up the Lagrangian

We define the Lagrangian. It contains vanishing vector mass parameter $\overset{2}{\cdot}\mu$ and a multiplier field $\overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\mu}$ to disable the axial torsion.

$$\alpha\ \text{R}[\nabla]_{\alpha\beta}\ \text{R}[\nabla]^{\alpha\beta}-\alpha\ \text{R}[\nabla]^{\alpha\beta}\ \text{R}[\nabla]_{\beta\alpha}-\frac{\mathcal{M}_{\text{Pl}}{}^{2}\ \text{R}[\nabla]}{2}+\overset{3}{\cdot}\mathcal{T}^{\alpha}\ \overset{3}{\cdot}\lambda_{\mathcal{T}\alpha} \tag{53}$$

Now we would like to have the post-Riemannian decomposition of the Lagrangian Eq. (53).

$$\begin{aligned}
&-\frac{\mathcal{M}_{\text{Pl}}{}^{2}\ \text{R}[\overset{\circ}{\nabla}]}{2}-\frac{1}{8}\mathcal{M}_{\text{Pl}}{}^{2}\ \mathcal{T}_{\alpha\alpha'\beta}\ \mathcal{T}^{\alpha\alpha'\beta}-\frac{1}{4}\mathcal{M}_{\text{Pl}}{}^{2}\ \mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\alpha\beta}-\frac{1}{2}\mathcal{M}_{\text{Pl}}{}^{2}\ \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\ \mathcal{T}^{\beta}{}_{\alpha'\beta}+\\
&\frac{1}{4}\alpha\,\mathcal{T}_{\alpha}{}^{\beta'\chi}\ \mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\beta}{}^{\mu}\ \mathcal{T}_{\beta'\chi\mu}+\frac{1}{4}\alpha\,\mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\beta}{}^{\beta'}\ \mathcal{T}_{\beta'}{}^{\chi\mu}\ \mathcal{T}_{\chi\alpha\mu}-\\
&\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\ \mathcal{T}_{\alpha'}{}^{\beta\beta'}\ \mathcal{T}_{\beta}{}^{\chi\mu}\ \mathcal{T}_{\chi\beta'\mu}-\frac{1}{2}\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\ \mathcal{T}_{\alpha'}{}^{\beta\beta'}\ \mathcal{T}^{\chi}{}_{\beta\beta'}\ \mathcal{T}^{\mu}{}_{\chi\mu}+\\
&\frac{1}{6}\ \epsilon\overset{\circ}{g}_{\alpha\alpha'\beta\beta'}\ \mathcal{T}^{\alpha'\beta\beta'}\ \overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\alpha}+\alpha\,\mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\overset{\circ}{\nabla}_{\alpha}\mathcal{T}^{\chi}{}_{\beta'\chi}\right)+\mathcal{M}_{\text{Pl}}{}^{2}\left(\overset{\circ}{\nabla}_{\alpha'}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)+\\
&\alpha\left(\overset{\circ}{\nabla}_{\alpha'}\mathcal{T}^{\beta'}{}_{\beta\beta'}\right)\left(\overset{\circ}{\nabla}{}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)-\alpha\left(\overset{\circ}{\nabla}_{\beta}\mathcal{T}^{\beta'}{}_{\alpha'\beta'}\right)\left(\overset{\circ}{\nabla}{}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)+\frac{1}{2}\alpha\left(\overset{\circ}{\nabla}_{\alpha}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\overset{\circ}{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right)+\\
&2\,\alpha\left(\overset{\circ}{\nabla}{}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)\left(\overset{\circ}{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right)-\alpha\,\mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\overset{\circ}{\nabla}_{\beta'}\mathcal{T}^{\chi}{}_{\alpha\chi}\right)-
\end{aligned} \tag{54}$$

$$2\,\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\,\mathcal{T}_{\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\chi}{}_{\beta\chi}\right)+\alpha\,\mathcal{T}^{\alpha\alpha'\beta}\,\mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\mathring{\nabla}_{\chi}\mathcal{T}^{\chi}{}_{\alpha\beta'}\right)+\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\,\mathcal{T}_{\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}_{\chi}\mathcal{T}^{\chi}{}_{\beta\beta'}\right)$$

We want to study the theory when it is nonlinear. As an intermediate step in order to do this, we just keep in Eq. (54) the third-order terms in torsion and no higher: this differs obviously from Eqs. (36), and (37) above. Also from this point onwards we completely neglect factors of the curvature which may arise in the field equations by commuting covariant derivatives.

$$\begin{aligned}
&-\frac{\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\mathrm{R}[\mathring{\nabla}]}{2}-\frac{1}{8}\,\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\mathcal{T}_{\alpha\alpha'\beta}\,\mathcal{T}^{\alpha\alpha'\beta}-\frac{1}{4}\,\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\mathcal{T}^{\alpha\alpha'\beta}\,\mathcal{T}_{\alpha'\alpha\beta}-\\
&\frac{1}{2}\,\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\,\mathcal{T}^{\beta}{}_{\alpha'\beta}+\frac{1}{6}\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\,\mathcal{T}^{\alpha'\beta\beta'}\,{}^{3}_{\bullet}\lambda_{\mathcal{T}}{}^{\alpha}+\alpha\,\mathcal{T}^{\alpha\alpha'\beta}\,\mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\chi}{}_{\beta'\chi}\right)+\\
&\mathcal{M}_{\mathrm{Pl}}{}^{2}\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)+\alpha\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\beta'}{}_{\beta\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)-\alpha\left(\mathring{\nabla}_{\beta}\mathcal{T}^{\beta'}{}_{\alpha'\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)+\\
&\frac{1}{2}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right)+2\,\alpha\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right)-\alpha\,\mathcal{T}^{\alpha\alpha'\beta}\,\mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\chi}{}_{\alpha\chi}\right)-\\
&2\,\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\,\mathcal{T}_{\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\chi}{}_{\beta\chi}\right)+\alpha\,\mathcal{T}^{\alpha\alpha'\beta}\,\mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\mathring{\nabla}_{\chi}\mathcal{T}^{\chi}{}_{\alpha\beta'}\right)+\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\,\mathcal{T}_{\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}_{\chi}\mathcal{T}^{\chi}{}_{\beta\beta'}\right)
\end{aligned}\tag{55}$$

Now we decompose the torsion in Eq. (55) into the Lorentz irreps.

$$\begin{aligned}
&-\frac{\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\mathrm{R}[\mathring{\nabla}]}{2}-\frac{2}{9}\,\mathcal{M}_{\mathrm{Pl}}{}^{2}\,{}^{1}_{\bullet}\mathcal{T}_{\alpha\alpha'\beta}\,{}^{1}_{\bullet}\mathcal{T}^{\alpha\alpha'\beta}+\frac{2}{9}\,\mathcal{M}_{\mathrm{Pl}}{}^{2}\,{}^{1}_{\bullet}\mathcal{T}_{\alpha\beta\alpha'}\,{}^{1}_{\bullet}\mathcal{T}^{\alpha\alpha'\beta}+\\
&\frac{1}{3}\,\mathcal{M}_{\mathrm{Pl}}{}^{2}\,{}^{2}_{\bullet}\mathcal{T}_{\alpha}\,{}^{2}_{\bullet}\mathcal{T}^{\alpha}-\frac{3}{4}\,\mathcal{M}_{\mathrm{Pl}}{}^{2}\,{}^{3}_{\bullet}\mathcal{T}_{\alpha}\,{}^{3}_{\bullet}\mathcal{T}^{\alpha}+{}^{3}_{\bullet}\mathcal{T}^{\alpha}\,{}^{3}_{\bullet}\lambda_{\mathcal{T}\,\alpha}+\\
&\mathcal{M}_{\mathrm{Pl}}{}^{2}\left(\mathring{\nabla}_{\alpha}{}^{2}_{\bullet}\mathcal{T}^{\alpha}\right)-\frac{4}{9}\,\alpha\left(\mathring{\nabla}_{\alpha}{}^{2}_{\bullet}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha'}\,{}^{2}_{\bullet}\mathcal{T}^{\alpha}\right)+\frac{4}{9}\,\alpha\left(\mathring{\nabla}_{\alpha'}{}^{2}_{\bullet}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha'}\,{}^{2}_{\bullet}\mathcal{T}^{\alpha}\right)+\\
&\alpha\left(\mathring{\nabla}_{\alpha}{}^{3}_{\bullet}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha'}\,{}^{3}_{\bullet}\mathcal{T}^{\alpha}\right)-\alpha\left(\mathring{\nabla}_{\alpha'}{}^{3}_{\bullet}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha'}\,{}^{3}_{\bullet}\mathcal{T}^{\alpha}\right)+\frac{8}{9}\,\alpha\left(\mathring{\nabla}^{\alpha'}\,{}^{2}_{\bullet}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}{}^{1}_{\bullet}\mathcal{T}_{\alpha}{}^{\beta}{}_{\alpha'}\right)-\\
&\frac{8}{9}\,\alpha\left(\mathring{\nabla}^{\alpha'}\,{}^{2}_{\bullet}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}{}^{1}_{\bullet}\mathcal{T}_{\alpha'}{}^{\beta}{}_{\alpha}\right)-\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta'\chi}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}_{\bullet}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}{}^{1}_{\bullet}\mathcal{T}^{\beta\beta'\chi}\right)+\\
&\frac{8}{9}\,\alpha\,{}^{1}_{\bullet}\mathcal{T}_{\alpha\alpha'\beta}\,{}^{2}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{2}_{\bullet}\mathcal{T}^{\alpha'}\right)-\frac{8}{9}\,\alpha\,{}^{1}_{\bullet}\mathcal{T}_{\alpha\beta\alpha'}\,{}^{2}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{2}_{\bullet}\mathcal{T}^{\alpha'}\right)-\\
&\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}_{\alpha\beta\beta'\chi}\,{}^{1}_{\bullet}\mathcal{T}_{\alpha'}{}^{\beta'\chi}\,{}^{3}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{2}_{\bullet}\mathcal{T}^{\alpha'}\right)+\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta'\chi}\,{}^{1}_{\bullet}\mathcal{T}_{\beta}{}^{\beta'\chi}\,{}^{3}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{2}_{\bullet}\mathcal{T}^{\alpha'}\right)-\\
&\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}_{\alpha'\beta\beta'\chi}\,{}^{1}_{\bullet}\mathcal{T}_{\alpha}{}^{\beta'\chi}\,{}^{2}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{3}_{\bullet}\mathcal{T}^{\alpha'}\right)-2\,\alpha\,{}^{1}_{\bullet}\mathcal{T}_{\alpha\alpha'\beta}\,{}^{3}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{3}_{\bullet}\mathcal{T}^{\alpha'}\right)+\\
&2\,\alpha\,{}^{1}_{\bullet}\mathcal{T}_{\alpha\beta\alpha'}\,{}^{3}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{3}_{\bullet}\mathcal{T}^{\alpha'}\right)+\frac{8}{9}\,\alpha\,{}^{1}_{\bullet}\mathcal{T}_{\alpha}{}^{\alpha'\beta}\,{}^{2}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\beta'}{}^{1}_{\bullet}\mathcal{T}_{\alpha'}{}^{\beta'}{}_{\beta}\right)+\\
&\frac{4}{9}\,\alpha\left(\mathring{\nabla}_{\alpha}{}^{1}_{\bullet}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}{}^{1}_{\bullet}\mathcal{T}_{\alpha'}{}^{\beta'}{}_{\beta}\right)-\frac{8}{9}\,\alpha\,{}^{1}_{\bullet}\mathcal{T}_{\alpha}{}^{\alpha'\beta}\,{}^{2}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\beta'}{}^{1}_{\bullet}\mathcal{T}_{\beta}{}^{\beta'}{}_{\alpha'}\right)-\\
&\frac{4}{9}\,\alpha\left(\mathring{\nabla}_{\alpha}{}^{1}_{\bullet}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}{}^{1}_{\bullet}\mathcal{T}_{\beta}{}^{\beta'}{}_{\alpha'}\right)-\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}_{\bullet}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\beta'}\,{}^{3}_{\bullet}\mathcal{T}^{\beta}\right)-
\end{aligned}\tag{56}$$

$$\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}_{\alpha\beta\beta'\chi'}\;{}^{1}\mathcal{T}^{\alpha'\beta\beta'}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\chi}{}^{1}\mathcal{T}_{\alpha'}{}^{\chi\chi'}\right)+\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}_{\alpha\beta\beta'\chi'}\;{}^{1}\mathcal{T}^{\alpha'\beta\beta'}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\chi}{}^{1}\mathcal{T}^{\chi\chi'}{}_{\alpha'}\right)$$

Manipulating the field equations

Here is the tensor field equation.

$$-\frac{1}{6}\,\alpha\,\delta^{\iota}{}_{\kappa}\;\epsilon\,\mathring{g}_{\theta\alpha'\beta\beta'}\;{}^{3}\mathcal{F}^{\alpha\alpha'}\;{}^{1}\mathcal{T}_{\alpha}{}^{\beta\beta'}+\frac{1}{6}\,\alpha\,\delta^{\iota}{}_{\theta}\;\epsilon\,\mathring{g}_{\kappa\alpha'\beta\beta'}\;{}^{3}\mathcal{F}^{\alpha\alpha'}\;{}^{1}\mathcal{T}_{\alpha}{}^{\beta\beta'}+$$

$$\frac{1}{9}\,\alpha\;{}^{2}\mathcal{F}_{\kappa}{}^{\alpha}\;{}^{1}\mathcal{T}^{\iota}{}_{\alpha\theta}-\frac{1}{9}\,\alpha\;{}^{2}\mathcal{F}_{\theta}{}^{\alpha}\;{}^{1}\mathcal{T}^{\iota}{}_{\alpha\kappa}-\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}_{\kappa\alpha\alpha'\beta}\;{}^{3}\mathcal{F}_{\theta}{}^{\alpha}\;{}^{1}\mathcal{T}^{\iota\alpha'\beta}+$$

$$\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;{}^{3}\mathcal{F}_{\kappa}{}^{\alpha}\;{}^{1}\mathcal{T}^{\iota\alpha'\beta}-\frac{1}{3}\,\mathcal{M}_{\mathrm{Pl}}{}^{2}\;{}^{1}\mathcal{T}^{\iota}{}_{\theta\kappa}+\frac{1}{3}\,\mathcal{M}_{\mathrm{Pl}}{}^{2}\;{}^{1}\mathcal{T}^{\iota}{}_{\kappa\theta}-$$

$$\frac{1}{9}\,\alpha\,\delta^{\iota}{}_{\kappa}\;{}^{2}\mathcal{F}^{\alpha\alpha'}\;{}^{1}\mathcal{T}_{\theta\alpha\alpha'}-\frac{1}{9}\,\alpha\;{}^{2}\mathcal{F}_{\kappa}{}^{\alpha}\;{}^{1}\mathcal{T}_{\theta\alpha}{}^{\iota}-\frac{2}{9}\,\alpha\;{}^{2}\mathcal{F}^{\iota\alpha}\;{}^{1}\mathcal{T}_{\theta\alpha\kappa}-$$

$$\frac{1}{3}\,\alpha\,\epsilon\,\mathring{g}_{\kappa\alpha\alpha'\beta}\;{}^{3}\mathcal{F}^{\iota\alpha}\;{}^{1}\mathcal{T}_{\theta}{}^{\alpha'\beta}-\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\alpha\alpha'\beta}\;{}^{3}\mathcal{F}_{\kappa}{}^{\alpha}\;{}^{1}\mathcal{T}_{\theta}{}^{\alpha'\beta}+$$

$$\frac{1}{6}\,\alpha\,\delta^{\iota}{}_{\kappa}\;\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\;{}^{3}\mathcal{F}^{\alpha\alpha'}\;{}^{1}\mathcal{T}_{\theta}{}^{\beta\beta'}+\frac{1}{9}\,\alpha\,\delta^{\iota}{}_{\theta}\;{}^{2}\mathcal{F}^{\alpha\alpha'}\;{}^{1}\mathcal{T}_{\kappa\alpha\alpha'}+\frac{1}{9}\,\alpha\;{}^{2}\mathcal{F}_{\theta}{}^{\alpha}\;{}^{1}\mathcal{T}_{\kappa\alpha}{}^{\iota}+$$

$$\frac{2}{9}\,\alpha\;{}^{2}\mathcal{F}^{\iota\alpha}\;{}^{1}\mathcal{T}_{\kappa\alpha\theta}+\frac{1}{3}\,\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;{}^{3}\mathcal{F}^{\iota\alpha}\;{}^{1}\mathcal{T}_{\kappa}{}^{\alpha'\beta}+\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\alpha\alpha'\beta}\;{}^{3}\mathcal{F}_{\theta}{}^{\alpha}\;{}^{1}\mathcal{T}_{\kappa}{}^{\alpha'\beta}-$$

$$\frac{1}{6}\,\alpha\,\delta^{\iota}{}_{\theta}\;\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\;{}^{3}\mathcal{F}^{\alpha\alpha'}\;{}^{1}\mathcal{T}_{\kappa}{}^{\beta\beta'}+\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\kappa}\;{}^{2}\mathcal{F}_{\theta\alpha}\;{}^{2}\mathcal{T}^{\alpha}-\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\theta}\;{}^{2}\mathcal{F}_{\kappa\alpha}\;{}^{2}\mathcal{T}^{\alpha}-$$

$$\frac{1}{6}\,\alpha\,\delta^{\iota}{}_{\kappa}\;\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;{}^{3}\mathcal{F}^{\alpha'\beta}\;{}^{2}\mathcal{T}^{\alpha}+\frac{1}{6}\,\alpha\,\delta^{\iota}{}_{\theta}\;\epsilon\,\mathring{g}_{\kappa\alpha\alpha'\beta}\;{}^{3}\mathcal{F}^{\alpha'\beta}\;{}^{2}\mathcal{T}^{\alpha}-\frac{4}{9}\,\alpha\;{}^{2}\mathcal{F}_{\theta\kappa}\;{}^{2}\mathcal{T}^{\iota}+$$

$$\frac{1}{3}\,\alpha\,\epsilon\,\mathring{g}_{\theta\kappa\alpha\alpha'}\;{}^{3}\mathcal{F}^{\alpha\alpha'}\;{}^{2}\mathcal{T}^{\iota}-\frac{2}{9}\,\alpha\;{}^{2}\mathcal{F}^{\iota}{}_{\kappa}\;{}^{2}\mathcal{T}_{\theta}+\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\kappa\alpha\alpha'}\;{}^{3}\mathcal{F}^{\alpha\alpha'}\;{}^{2}\mathcal{T}_{\theta}+$$

$$\frac{2}{9}\,\alpha\;{}^{2}\mathcal{F}^{\iota}{}_{\theta}\;{}^{2}\mathcal{T}_{\kappa}-\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\theta\alpha\alpha'}\;{}^{3}\mathcal{F}^{\alpha\alpha'}\;{}^{2}\mathcal{T}_{\kappa}+\frac{1}{3}\,\alpha\,\delta^{\iota}{}_{\kappa}\;\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;{}^{2}\mathcal{F}^{\alpha'\beta}\;{}^{3}\mathcal{T}^{\alpha}-$$

$$\frac{1}{3}\,\alpha\,\delta^{\iota}{}_{\theta}\;\epsilon\,\mathring{g}_{\kappa\alpha\alpha'\beta}\;{}^{2}\mathcal{F}^{\alpha'\beta}\;{}^{3}\mathcal{T}^{\alpha}+\frac{2}{3}\,\alpha\,\epsilon\,\mathring{g}_{\theta\kappa\alpha\alpha'}\;{}^{2}\mathcal{F}^{\iota\alpha'}\;{}^{3}\mathcal{T}^{\alpha}+$$

$$\frac{1}{3}\,\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\kappa\alpha\alpha'}\;{}^{2}\mathcal{F}_{\theta}{}^{\alpha'}\;{}^{3}\mathcal{T}^{\alpha}-\frac{1}{3}\,\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\theta\alpha\alpha'}\;{}^{2}\mathcal{F}_{\kappa}{}^{\alpha'}\;{}^{3}\mathcal{T}^{\alpha}-\frac{1}{2}\,\alpha\,\delta^{\iota}{}_{\kappa}\;{}^{3}\mathcal{F}_{\theta\alpha}\;{}^{3}\mathcal{T}^{\alpha}+$$

$$\frac{1}{2}\,\alpha\,\delta^{\iota}{}_{\theta}\;{}^{3}\mathcal{F}_{\kappa\alpha}\;{}^{3}\mathcal{T}^{\alpha}+\alpha\;{}^{3}\mathcal{F}_{\theta\kappa}\;{}^{3}\mathcal{T}^{\iota}+\frac{1}{2}\,\alpha\;{}^{3}\mathcal{F}^{\iota}{}_{\kappa}\;{}^{3}\mathcal{T}_{\theta}-\frac{1}{2}\,\alpha\;{}^{3}\mathcal{F}^{\iota}{}_{\theta}\;{}^{3}\mathcal{T}_{\kappa}-$$

$$\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{F}_{\theta}{}^{\alpha}\right)+\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{F}_{\kappa}{}^{\alpha}\right)-\frac{2}{9}\,\alpha\;{}^{2}\mathcal{T}_{\kappa}\left(\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}^{\iota\alpha}{}_{\theta}\right)+\frac{2}{9}\,\alpha\;{}^{2}\mathcal{T}_{\theta}\left(\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}^{\iota\alpha}{}_{\kappa}\right)+$$

$$\frac{2}{9}\,\alpha\;{}^{2}\mathcal{T}_{\kappa}\left(\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}_{\theta}{}^{\alpha\iota}\right)+\frac{4}{9}\,\alpha\;{}^{2}\mathcal{T}^{\iota}\left(\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\right)-\frac{2}{9}\,\alpha\;{}^{2}\mathcal{T}_{\theta}\left(\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}_{\kappa}{}^{\alpha\iota}\right)-$$

$$\frac{4}{9}\,\alpha\;{}^{2}\mathcal{T}^{\iota}\left(\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\right)-\frac{2}{9}\,\alpha\;{}^{1}\mathcal{T}_{\theta\alpha\kappa}\left(\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}^{\iota}\right)+\frac{2}{9}\,\alpha\;{}^{1}\mathcal{T}_{\kappa\alpha\theta}\left(\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}^{\iota}\right)-$$

$$\frac{1}{9}\,\alpha\;{}^{1}\mathcal{T}^{\iota}{}_{\alpha\kappa}\left(\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}_{\theta}\right)+\frac{1}{9}\,\alpha\;{}^{1}\mathcal{T}_{\kappa\alpha}{}^{\iota}\left(\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}_{\theta}\right)+\frac{1}{9}\,\alpha\;{}^{1}\mathcal{T}^{\iota}{}_{\alpha\theta}\left(\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}_{\kappa}\right)-$$

$$\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\theta\alpha}{}^{\iota}\left(\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}_{\kappa}\right)-\frac{1}{3}\alpha\,\epsilon\,\mathring{g}_{\kappa\alpha\alpha'\beta}\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha'\beta}\left(\mathring{\nabla}^{\alpha}\,{}^{3}\mathcal{T}^{\iota}\right)+\frac{1}{3}\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha'\beta}\left(\mathring{\nabla}^{\alpha}\,{}^{3}\mathcal{T}^{\iota}\right)-$$

$$\frac{1}{6}\alpha\,\epsilon\,\mathring{g}_{\kappa\alpha\alpha'\beta}\,{}^{1}\mathcal{T}^{\iota\alpha'\beta}\left(\mathring{\nabla}^{\alpha}\,{}^{3}\mathcal{T}_{\theta}\right)+\frac{1}{6}\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\alpha\alpha'\beta}\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha'\beta}\left(\mathring{\nabla}^{\alpha}\,{}^{3}\mathcal{T}_{\theta}\right)+$$

$$\frac{1}{6}\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\,{}^{1}\mathcal{T}^{\iota\alpha'\beta}\left(\mathring{\nabla}^{\alpha}\,{}^{3}\mathcal{T}_{\kappa}\right)-\frac{1}{6}\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\alpha\alpha'\beta}\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha'\beta}\left(\mathring{\nabla}^{\alpha}\,{}^{3}\mathcal{T}_{\kappa}\right)-$$

$$\alpha\,\delta^{\iota}{}_{\kappa}\,\epsilon\,\mathring{g}_{\theta\alpha\beta\beta'}\,{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}^{\alpha'\beta\beta'}\right)+\alpha\,\delta^{\iota}{}_{\theta}\,\epsilon\,\mathring{g}_{\kappa\alpha\beta\beta'}\,{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}^{\alpha'\beta\beta'}\right)+$$

$$\frac{2}{3}\alpha\,\epsilon\,\mathring{g}_{\theta\kappa\alpha\beta}\,{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}^{\alpha'\beta\iota}\right)+\frac{1}{3}\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\kappa\alpha\beta}\,{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}^{\alpha'\beta}{}_{\theta}\right)-$$

$$\frac{1}{3}\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\theta\alpha\beta}\,{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}^{\alpha'\beta}{}_{\kappa}\right)-\frac{2}{3}\alpha\,\epsilon\,\mathring{g}_{\theta\kappa\alpha\beta}\,{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}^{\iota\alpha'\beta}\right)+$$

$$\frac{2}{9}\alpha\,\delta^{\iota}{}_{\kappa}\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}_{\theta\alpha}{}^{\alpha'}\right)-\frac{2}{9}\alpha\,\delta^{\iota}{}_{\kappa}\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha'}{}_{\alpha}\right)-$$

$$\frac{1}{3}\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\kappa\alpha\beta}\,{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha'\beta}\right)-\frac{2}{9}\alpha\,\delta^{\iota}{}_{\theta}\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}_{\kappa\alpha}{}^{\alpha'}\right)+$$

$$\frac{2}{9}\alpha\,\delta^{\iota}{}_{\theta}\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha'}{}_{\alpha}\right)+\frac{1}{3}\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\theta\alpha\beta}\,{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha'\beta}\right)-\frac{2}{9}\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}^{\alpha\alpha'}{}_{\theta}\right)+$$

$$\frac{2}{9}\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}^{\alpha\alpha'}{}_{\kappa}\right)+\frac{2}{9}\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha\alpha'}\right)-\frac{2}{9}\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha\alpha'}\right)-$$

$$\frac{2}{9}\alpha\,\delta^{\iota}{}_{\kappa}\,{}^{1}\mathcal{T}_{\alpha\alpha'\theta}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)+\frac{2}{9}\alpha\,\delta^{\iota}{}_{\theta}\,{}^{1}\mathcal{T}_{\alpha\alpha'\kappa}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)+\frac{1}{9}\alpha\,\delta^{\iota}{}_{\kappa}\,{}^{1}\mathcal{T}_{\theta\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)+$$

$$\frac{1}{9}\alpha\,\delta^{\iota}{}_{\kappa}\,{}^{1}\mathcal{T}_{\theta\alpha'\alpha}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)-\frac{1}{9}\alpha\,\delta^{\iota}{}_{\theta}\,{}^{1}\mathcal{T}_{\kappa\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)-\frac{1}{9}\alpha\,\delta^{\iota}{}_{\theta}\,{}^{1}\mathcal{T}_{\kappa\alpha'\alpha}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)-$$

$$\frac{1}{6}\alpha\,\delta^{\iota}{}_{\kappa}\,\epsilon\,\mathring{g}_{\theta\alpha'\beta\beta'}\,{}^{1}\mathcal{T}_{\alpha}{}^{\beta\beta'}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)+\frac{1}{6}\alpha\,\delta^{\iota}{}_{\theta}\,\epsilon\,\mathring{g}_{\kappa\alpha'\beta\beta'}\,{}^{1}\mathcal{T}_{\alpha}{}^{\beta\beta'}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)-$$

$$\frac{1}{6}\alpha\,\delta^{\iota}{}_{\kappa}\,\epsilon\,\mathring{g}_{\theta\alpha\beta\beta'}\,{}^{1}\mathcal{T}_{\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)+\frac{1}{6}\alpha\,\delta^{\iota}{}_{\theta}\,\epsilon\,\mathring{g}_{\kappa\alpha\beta\beta'}\,{}^{1}\mathcal{T}_{\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)-$$

$$\frac{1}{3}\alpha\,\delta^{\iota}{}_{\kappa}\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\,{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta'}\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha'\beta}\right)+\frac{1}{3}\alpha\,\delta^{\iota}{}_{\theta}\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\,{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta'}\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha'\beta}\right)-$$

$$\frac{1}{3}\alpha\,\epsilon\,\mathring{g}_{\theta\kappa\alpha\alpha'}\left(\mathring{\nabla}^{\iota}\,{}^{3}\mathcal{F}^{\alpha\alpha'}\right)-\frac{4}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\iota}\,{}^{1}\mathcal{T}_{\theta\alpha\kappa}\right)-\frac{2}{3}\alpha\,\epsilon\,\mathring{g}_{\kappa\alpha\alpha'\beta}\,{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\iota}\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha'\beta}\right)+$$

$$\frac{4}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\iota}\,{}^{1}\mathcal{T}_{\kappa\alpha\theta}\right)+\frac{2}{3}\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\,{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\iota}\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha'\beta}\right)-\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\theta\alpha\kappa}\left(\mathring{\nabla}^{\iota}\,{}^{2}\mathcal{T}^{\alpha}\right)+$$

$$\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa\alpha\theta}\left(\mathring{\nabla}^{\iota}\,{}^{2}\mathcal{T}^{\alpha}\right)-\frac{1}{3}\alpha\,\epsilon\,\mathring{g}_{\kappa\alpha\alpha'\beta}\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha'\beta}\left(\mathring{\nabla}^{\iota}\,{}^{3}\mathcal{T}^{\alpha}\right)+\frac{1}{3}\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha'\beta}\left(\mathring{\nabla}^{\iota}\,{}^{3}\mathcal{T}^{\alpha}\right)-$$

$$\frac{4}{9}\alpha\left(\mathring{\nabla}^{\iota}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\right)+\frac{4}{9}\alpha\left(\mathring{\nabla}^{\iota}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\right)+\frac{1}{3}\alpha\left(\mathring{\nabla}_{\theta}\,{}^{2}\mathcal{F}^{\iota}{}_{\kappa}\right)-\frac{1}{6}\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\kappa\alpha\alpha'}\left(\mathring{\nabla}_{\theta}\,{}^{3}\mathcal{F}^{\alpha\alpha'}\right)-$$

$$\frac{2}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}\,{}^{1}\mathcal{T}^{\iota}{}_{\alpha\kappa}\right)-\frac{1}{3}\alpha\,\epsilon\,\mathring{g}_{\kappa\alpha\alpha'\beta}\,{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}\,{}^{1}\mathcal{T}^{\iota\alpha'\beta}\right)+\frac{2}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}\,{}^{1}\mathcal{T}_{\kappa\alpha}{}^{\iota}\right)+$$

$$\frac{1}{3}\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\alpha\alpha'\beta}\,{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha'\beta}\right)-\frac{1}{9}\alpha\,{}^{1}\mathcal{T}^{\iota}{}_{\alpha\kappa}\left(\mathring{\nabla}_{\theta}\,{}^{2}\mathcal{T}^{\alpha}\right)+\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa\alpha}{}^{\iota}\left(\mathring{\nabla}_{\theta}\,{}^{2}\mathcal{T}^{\alpha}\right)-$$

$$
\begin{aligned}
&\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}_{\kappa\alpha\alpha'\beta}\;{}^{1}\mathcal{T}^{\iota\alpha'\beta}\left(\mathring{\nabla}_{\theta}{}^{3}\mathcal{T}^{\alpha}\right)+\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\alpha\alpha'\beta}\;{}^{1}\mathcal{T}_{\kappa}{}^{\alpha'\beta}\left(\mathring{\nabla}_{\theta}{}^{3}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\,\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}^{\iota\alpha}{}_{\kappa}\right)+\\
&\frac{2}{9}\,\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}_{\kappa}{}^{\alpha\iota}\right)+\frac{1}{3}\,\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\iota}\,{}^{2}\mathcal{T}_{\kappa}\right)-\frac{1}{3}\,\alpha\left(\mathring{\nabla}_{\kappa}{}^{2}\mathcal{F}^{\iota}{}_{\theta}\right)+\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\theta\alpha\alpha'}\left(\mathring{\nabla}_{\kappa}{}^{3}\mathcal{F}^{\alpha\alpha'}\right)+\\
&\frac{2}{9}\,\alpha\;{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\kappa}{}^{1}\mathcal{T}^{\iota}{}_{\alpha\theta}\right)+\frac{1}{3}\,\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\kappa}{}^{1}\mathcal{T}^{\iota\alpha'\beta}\right)-\frac{2}{9}\,\alpha\;{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\kappa}{}^{1}\mathcal{T}_{\theta\alpha}{}^{\iota}\right)-\\
&\frac{1}{3}\,\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\alpha\alpha'\beta}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\kappa}{}^{1}\mathcal{T}_{\theta}{}^{\alpha'\beta}\right)+\frac{1}{9}\,\alpha\;{}^{1}\mathcal{T}^{\iota}{}_{\alpha\theta}\left(\mathring{\nabla}_{\kappa}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{1}{9}\,\alpha\;{}^{1}\mathcal{T}_{\theta\alpha}{}^{\iota}\left(\mathring{\nabla}_{\kappa}{}^{2}\mathcal{T}^{\alpha}\right)+\\
&\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;{}^{1}\mathcal{T}^{\iota\alpha'\beta}\left(\mathring{\nabla}_{\kappa}{}^{3}\mathcal{T}^{\alpha}\right)-\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\alpha\alpha'\beta}\;{}^{1}\mathcal{T}_{\theta}{}^{\alpha'\beta}\left(\mathring{\nabla}_{\kappa}{}^{3}\mathcal{T}^{\alpha}\right)+\\
&\frac{2}{9}\,\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}^{\iota\alpha}{}_{\theta}\right)-\frac{2}{9}\,\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}_{\theta}{}^{\alpha\iota}\right)-\frac{1}{3}\,\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\iota}\,{}^{2}\mathcal{T}_{\theta}\right)==0
\end{aligned}
$$

There is some nuance to how we obtain Eq. (57), in that we can't just vary with respect to the tensor irrep. If we do that, then the resulting equation can have traces which are not true on-shell. It is safest to actually vary with respect to the whole torsion tensor and then project out the (traceless by construction) tensor irrep from that equation.

Now for the vector equation.

$$
\begin{aligned}
&\frac{2}{3}\,\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\;{}^{3}\mathcal{F}^{\alpha\alpha'}\;{}^{1}\mathcal{T}_{\alpha}{}^{\beta\beta'}-\frac{4}{9}\,\alpha\;{}^{2}\mathcal{F}^{\alpha\alpha'}\;{}^{1}\mathcal{T}^{\mu}{}_{\alpha\alpha'}+\frac{2}{3}\,\mathcal{M}_{\mathrm{Pl}}{}^{2}\;{}^{2}\mathcal{T}^{\mu}+\\
&\frac{8}{9}\,\alpha\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{F}^{\mu\alpha}\right)+\frac{8}{9}\,\alpha\;{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\alpha}{}^{\alpha'\mu}\right)+\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha\beta\beta'}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}^{\alpha'\beta\beta'}\right)-\\
&\frac{8}{9}\,\alpha\;{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}^{\mu}{}_{\alpha}{}^{\alpha'}\right)+\frac{8}{9}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}^{\alpha\alpha'\mu}\right)-\frac{8}{9}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}^{\mu\alpha\alpha'}\right)+\\
&\frac{8}{9}\,\alpha\;{}^{1}\mathcal{T}_{\alpha\alpha'}{}^{\mu}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)-\frac{4}{9}\,\alpha\;{}^{1}\mathcal{T}^{\mu}{}_{\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)-\frac{4}{9}\,\alpha\;{}^{1}\mathcal{T}^{\mu}{}_{\alpha'\alpha}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)+\\
&\frac{2}{3}\,\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\;{}^{1}\mathcal{T}_{\alpha}{}^{\beta\beta'}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)+\frac{2}{3}\,\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha\beta\beta'}\;{}^{1}\mathcal{T}_{\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)+\\
&\frac{8}{9}\,\alpha\;{}^{1}\mathcal{T}^{\mu\alpha\alpha'}\left(\mathring{\nabla}_{\beta}{}^{1}\mathcal{T}_{\alpha}{}^{\beta}{}_{\alpha'}\right)-\frac{8}{9}\,\alpha\;{}^{1}\mathcal{T}^{\mu\alpha\alpha'}\left(\mathring{\nabla}_{\beta}{}^{1}\mathcal{T}_{\alpha'}{}^{\beta}{}_{\alpha}\right)+\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta'}\,{}^{1}\mathcal{T}^{\mu\alpha'\beta}\right)==0
\end{aligned}
\tag{58}
$$

And for the axial vector equation.

$$
\begin{aligned}
&\frac{2}{3}\,\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\;{}^{2}\mathcal{F}^{\alpha\alpha'}\;{}^{1}\mathcal{T}_{\alpha}{}^{\beta\beta'}+\alpha\;{}^{3}\mathcal{F}^{\alpha\alpha'}\;{}^{1}\mathcal{T}^{\mu}{}_{\alpha\alpha'}-\frac{3}{2}\,\mathcal{M}_{\mathrm{Pl}}{}^{2}\;{}^{3}\mathcal{T}^{\mu}+\\
&{}^{3}\lambda_{\mathcal{T}}{}^{\mu}-2\,\alpha\left(\mathring{\nabla}_{\alpha}{}^{3}\mathcal{F}^{\mu\alpha}\right)-2\,\alpha\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\alpha}{}^{\alpha'\mu}\right)+2\,\alpha\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}^{\mu}{}_{\alpha}{}^{\alpha'}\right)+\\
&\frac{2}{3}\,\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\;{}^{1}\mathcal{T}_{\alpha}{}^{\beta\beta'}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)+\frac{2}{3}\,\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha\beta\beta'}\;{}^{1}\mathcal{T}_{\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)-\\
&2\,\alpha\;{}^{1}\mathcal{T}_{\alpha\alpha'}{}^{\mu}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)+\alpha\;{}^{1}\mathcal{T}^{\mu}{}_{\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)+\alpha\;{}^{1}\mathcal{T}^{\mu}{}_{\alpha'\alpha}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)-
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
&\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\chi}\;\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}_{\alpha}{}^{\beta'\chi}\right)+\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\chi}\;\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}^{\beta'\chi}{}_{\alpha}\right)+\\
&\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\;\overset{2}{\cdot}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta'}\overset{1}{\cdot}\mathcal{T}_{\alpha}{}^{\alpha'\beta}\right)+\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\left(\mathring{\nabla}^{\beta'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\right)==0
\end{aligned}
$$

Now we can also take the dual of Eq. (59), and so we write this out for completeness.

$$
\begin{aligned}
&\frac{2}{3}\,\alpha\,\overset{2}{\cdot}\mathcal{F}^{\chi\alpha'}\;\overset{1}{\cdot}\mathcal{T}^{\alpha}{}_{\alpha'}{}^{\beta}-\frac{2}{3}\,\alpha\,\overset{2}{\cdot}\mathcal{F}^{\beta\alpha'}\;\overset{1}{\cdot}\mathcal{T}^{\alpha}{}_{\alpha'}{}^{\chi}-\alpha\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\mu}\;\overset{3}{\cdot}\mathcal{F}^{\alpha'\beta'}\;\overset{1}{\cdot}\mathcal{T}_{\alpha'}{}^{\mu}{}_{\beta'}-\\
&\frac{2}{3}\,\alpha\,\overset{2}{\cdot}\mathcal{F}^{\chi\alpha'}\;\overset{1}{\cdot}\mathcal{T}^{\beta}{}_{\alpha'}{}^{\alpha}+\frac{2}{3}\,\alpha\,\overset{2}{\cdot}\mathcal{F}^{\alpha\alpha'}\;\overset{1}{\cdot}\mathcal{T}^{\beta}{}_{\alpha'}{}^{\chi}+\frac{2}{3}\,\alpha\,\overset{2}{\cdot}\mathcal{F}^{\beta\alpha'}\;\overset{1}{\cdot}\mathcal{T}^{\chi}{}_{\alpha'}{}^{\alpha}-\\
&\frac{2}{3}\,\alpha\,\overset{2}{\cdot}\mathcal{F}^{\alpha\alpha'}\;\overset{1}{\cdot}\mathcal{T}^{\chi}{}_{\alpha'}{}^{\beta}+\frac{3}{2}\,\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\alpha'}\;\overset{3}{\cdot}\mathcal{T}^{\alpha'}-\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\alpha'}\;\overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\alpha'}-\\
&\frac{4}{3}\,\alpha\,\overset{2}{\cdot}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}^{\alpha}\overset{1}{\cdot}\mathcal{T}^{\beta}{}_{\alpha'}{}^{\chi}\right)+\frac{4}{3}\,\alpha\,\overset{2}{\cdot}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}^{\alpha}\overset{1}{\cdot}\mathcal{T}^{\chi}{}_{\alpha'}{}^{\beta}\right)-\frac{2}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\beta}{}_{\alpha'}{}^{\chi}\left(\mathring{\nabla}^{\alpha}\overset{2}{\cdot}\mathcal{T}^{\alpha'}\right)+\\
&\frac{2}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\chi}{}_{\alpha'}{}^{\beta}\left(\mathring{\nabla}^{\alpha}\overset{2}{\cdot}\mathcal{T}^{\alpha'}\right)-2\,\alpha\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\beta'}\left(\mathring{\nabla}_{\alpha'}\overset{3}{\cdot}\mathcal{F}^{\alpha'\beta'}\right)-\frac{4}{3}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha}\overset{1}{\cdot}\mathcal{T}^{\beta\alpha'\chi}\right)+\\
&\frac{4}{3}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha}\overset{1}{\cdot}\mathcal{T}^{\chi\alpha'\beta}\right)-\frac{2}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\beta}{}_{\alpha'}{}^{\chi}\left(\mathring{\nabla}^{\alpha'}\overset{2}{\cdot}\mathcal{T}^{\alpha}\right)+\frac{2}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\chi}{}_{\alpha'}{}^{\beta}\left(\mathring{\nabla}^{\alpha'}\overset{2}{\cdot}\mathcal{T}^{\alpha}\right)+\\
&\frac{2}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\alpha}{}_{\alpha'}{}^{\chi}\left(\mathring{\nabla}^{\alpha'}\overset{2}{\cdot}\mathcal{T}^{\beta}\right)-\frac{2}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\chi}{}_{\alpha'}{}^{\alpha}\left(\mathring{\nabla}^{\alpha'}\overset{2}{\cdot}\mathcal{T}^{\beta}\right)-\frac{2}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\alpha}{}_{\alpha'}{}^{\beta}\left(\mathring{\nabla}^{\alpha'}\overset{2}{\cdot}\mathcal{T}^{\chi}\right)+\\
&\frac{2}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\beta}{}_{\alpha'}{}^{\alpha}\left(\mathring{\nabla}^{\alpha'}\overset{2}{\cdot}\mathcal{T}^{\chi}\right)+\frac{4}{3}\,\alpha\,\overset{2}{\cdot}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}^{\beta}\overset{1}{\cdot}\mathcal{T}^{\alpha}{}_{\alpha'}{}^{\chi}\right)-\frac{4}{3}\,\alpha\,\overset{2}{\cdot}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}^{\beta}\overset{1}{\cdot}\mathcal{T}^{\chi}{}_{\alpha'}{}^{\alpha}\right)+\\
&\frac{2}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\alpha}{}_{\alpha'}{}^{\chi}\left(\mathring{\nabla}^{\beta}\overset{2}{\cdot}\mathcal{T}^{\alpha'}\right)-\frac{2}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\chi}{}_{\alpha'}{}^{\alpha}\left(\mathring{\nabla}^{\beta}\overset{2}{\cdot}\mathcal{T}^{\alpha'}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\chi}\right)-\\
&\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\overset{1}{\cdot}\mathcal{T}^{\chi\alpha'\alpha}\right)-\frac{4}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\beta\alpha'\chi}\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}^{\alpha\beta'}{}_{\alpha'}\right)+\frac{4}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\chi\alpha'\beta}\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}^{\alpha\beta'}{}_{\alpha'}\right)+\\
&\frac{4}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\beta\alpha'\chi}\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}_{\alpha'}{}^{\beta'\alpha}\right)-\frac{4}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\chi\alpha'\beta}\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}_{\alpha'}{}^{\beta'\alpha}\right)-\frac{4}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\chi}\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}_{\alpha'}{}^{\beta'\beta}\right)+\\
&\frac{4}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\chi\alpha'\alpha}\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}_{\alpha'}{}^{\beta'\beta}\right)+\frac{4}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}_{\alpha'}{}^{\beta'\chi}\right)-\frac{4}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\beta\alpha'\alpha}\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}_{\alpha'}{}^{\beta'\chi}\right)+\\
&2\,\alpha\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\mu}\;\overset{3}{\cdot}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}_{\alpha'}{}^{\beta'\mu}\right)+\frac{4}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\chi}\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}^{\beta\beta'}{}_{\alpha'}\right)-\frac{4}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\chi\alpha'\alpha}\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}^{\beta\beta'}{}_{\alpha'}\right)-\\
&\frac{4}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}^{\chi\beta'}{}_{\alpha'}\right)+\frac{4}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\beta\alpha'\alpha}\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}^{\chi\beta'}{}_{\alpha'}\right)+2\,\alpha\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\mu}\;\overset{1}{\cdot}\mathcal{T}_{\alpha'\beta'}{}^{\mu}\left(\mathring{\nabla}^{\beta'}\overset{3}{\cdot}\mathcal{T}^{\alpha'}\right)-\\
&\alpha\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\mu}\;\overset{1}{\cdot}\mathcal{T}_{\alpha'}{}^{\mu}{}_{\beta'}\left(\mathring{\nabla}^{\beta'}\overset{3}{\cdot}\mathcal{T}^{\alpha'}\right)-\alpha\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\mu}\;\overset{1}{\cdot}\mathcal{T}_{\beta'}{}^{\mu}{}_{\alpha'}\left(\mathring{\nabla}^{\beta'}\overset{3}{\cdot}\mathcal{T}^{\alpha'}\right)-\frac{4}{3}\,\alpha\,\overset{2}{\cdot}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}^{\chi}\overset{1}{\cdot}\mathcal{T}^{\alpha}{}_{\alpha'}{}^{\beta}\right)+\\
&\frac{4}{3}\,\alpha\,\overset{2}{\cdot}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}^{\chi}\overset{1}{\cdot}\mathcal{T}^{\beta}{}_{\alpha'}{}^{\alpha}\right)-\frac{2}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\alpha}{}_{\alpha'}{}^{\beta}\left(\mathring{\nabla}^{\chi}\overset{2}{\cdot}\mathcal{T}^{\alpha'}\right)+\frac{2}{3}\,\alpha\,\overset{1}{\cdot}\mathcal{T}^{\beta}{}_{\alpha'}{}^{\alpha}\left(\mathring{\nabla}^{\chi}\overset{2}{\cdot}\mathcal{T}^{\alpha'}\right)-\\
&\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha'}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha'}\overset{1}{\cdot}\mathcal{T}^{\beta\alpha'\alpha}\right)-2\,\alpha\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\beta'}\;\overset{3}{\cdot}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}_{\mu}\overset{1}{\cdot}\mathcal{T}_{\alpha'}{}^{\beta'\mu}\right)==0
\end{aligned}
\tag{60}
$$

Finally we have the equation for the multiplier itself.

$$ {}^{3}\mathcal{T}_{\mu} == 0 \tag{61}$$

With the effective 2-form field in Eq. (14), we are ready to simplify the equations of motion.

Here is the vector equation Eq. (58).

$$\begin{aligned}
&\frac{8}{9}\alpha\mathcal{M}_{\text{Pl}}\mathcal{B}^{\alpha\alpha'}\,{}^{1}\mathcal{T}^{\mu}{}_{\alpha\alpha'}+\frac{4}{9}\alpha\,{}^{2}\mathcal{F}^{\alpha\alpha'}\,{}^{1}\mathcal{T}^{\mu}{}_{\alpha\alpha'}-\\
&\quad\frac{4}{9}\alpha\mathcal{M}_{\text{Pl}}\mathcal{B}^{\mu}{}_{\alpha}\,{}^{2}\mathcal{T}^{\alpha}-\frac{4}{9}\alpha\,{}^{2}\mathcal{F}^{\mu}{}_{\alpha}\,{}^{2}\mathcal{T}^{\alpha}+\frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\,{}^{2}\mathcal{T}^{\mu}-\frac{8}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\mu\alpha}\right)+\\
&\quad\frac{4}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\alpha}{}^{\alpha'\mu}\right)-\frac{8}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}^{\mu}{}_{\alpha}{}^{\alpha'}\right)+\frac{4}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}^{\mu\alpha'}{}_{\alpha}\right)+\\
&\quad\frac{8}{9}\alpha\,{}^{1}\mathcal{T}_{\alpha\alpha'}{}^{\mu}\left(\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{4}{9}\alpha\,{}^{1}\mathcal{T}^{\mu}{}_{\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{4}{9}\alpha\,{}^{1}\mathcal{T}^{\mu}{}_{\alpha'\alpha}\left(\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)==0
\end{aligned}\tag{62}$$

Concrete relation to manuscript: Compare Eq. (62) with Eq. (5b) in our manuscript (note that many higher-order terms will cancel after expanding the third term using Eq. (14)).

Here is the axial vector equation Eq. (59).

$$\begin{aligned}
&-\frac{4}{3}\alpha\mathcal{M}_{\text{Pl}}\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\mathcal{B}^{\alpha\alpha'}\,{}^{1}\mathcal{T}_{\alpha}{}^{\beta\beta'}-\frac{2}{3}\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\,{}^{2}\mathcal{F}^{\alpha\alpha'}\,{}^{1}\mathcal{T}_{\alpha}{}^{\beta\beta'}+\\
&\quad{}^{3}\lambda_{\mathcal{T}}{}^{\mu}+\frac{2}{3}\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\,{}^{1}\mathcal{T}_{\alpha}{}^{\beta\beta'}\left(\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{2}{3}\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha\beta\beta'}\,{}^{1}\mathcal{T}_{\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)+\\
&\quad\frac{2}{3}\alpha\mathcal{M}_{\text{Pl}}\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha\alpha'\beta}\left(\mathring{\nabla}^{\beta}\mathcal{B}^{\alpha\alpha'}\right)+\frac{4}{3}\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta'}{}^{1}\mathcal{T}_{\alpha}{}^{\alpha'\beta}\right)==0
\end{aligned}\tag{63}$$

Again here is the dual part of Eq. (63)

$$\begin{aligned}
&-\frac{4}{3}\alpha\mathcal{M}_{\text{Pl}}\mathcal{B}_{\kappa}{}^{\alpha}\,{}^{1}\mathcal{T}_{\iota\alpha\theta}-\frac{2}{3}\alpha\,{}^{2}\mathcal{F}_{\kappa}{}^{\alpha}\,{}^{1}\mathcal{T}_{\iota\alpha\theta}+\frac{4}{3}\alpha\mathcal{M}_{\text{Pl}}\mathcal{B}_{\theta}{}^{\alpha}\,{}^{1}\mathcal{T}_{\iota\alpha\kappa}+\frac{2}{3}\alpha\,{}^{2}\mathcal{F}_{\theta}{}^{\alpha}\,{}^{1}\mathcal{T}_{\iota\alpha\kappa}+\\
&\quad\frac{4}{3}\alpha\mathcal{M}_{\text{Pl}}\mathcal{B}_{\kappa}{}^{\alpha}\,{}^{1}\mathcal{T}_{\theta\alpha\iota}+\frac{2}{3}\alpha\,{}^{2}\mathcal{F}_{\kappa}{}^{\alpha}\,{}^{1}\mathcal{T}_{\theta\alpha\iota}-\frac{4}{3}\alpha\mathcal{M}_{\text{Pl}}\mathcal{B}_{\iota}{}^{\alpha}\,{}^{1}\mathcal{T}_{\theta\alpha\kappa}-\frac{2}{3}\alpha\,{}^{2}\mathcal{F}_{\iota}{}^{\alpha}\,{}^{1}\mathcal{T}_{\theta\alpha\kappa}-\\
&\quad\frac{4}{3}\alpha\mathcal{M}_{\text{Pl}}\mathcal{B}_{\theta}{}^{\alpha}\,{}^{1}\mathcal{T}_{\kappa\alpha\iota}-\frac{2}{3}\alpha\,{}^{2}\mathcal{F}_{\theta}{}^{\alpha}\,{}^{1}\mathcal{T}_{\kappa\alpha\iota}+\frac{4}{3}\alpha\mathcal{M}_{\text{Pl}}\mathcal{B}_{\iota}{}^{\alpha}\,{}^{1}\mathcal{T}_{\kappa\alpha\theta}+\frac{2}{3}\alpha\,{}^{2}\mathcal{F}_{\iota}{}^{\alpha}\,{}^{1}\mathcal{T}_{\kappa\alpha\theta}-\\
&\quad\epsilon\,\mathring{g}_{\iota\theta\kappa\alpha}\,{}^{3}\lambda_{\mathcal{T}}{}^{\alpha}-\frac{2}{3}\alpha\,{}^{1}\mathcal{T}_{\theta\alpha\kappa}\left(\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\iota}\right)+\frac{2}{3}\alpha\,{}^{1}\mathcal{T}_{\kappa\alpha\theta}\left(\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\iota}\right)+\frac{2}{3}\alpha\,{}^{1}\mathcal{T}_{\iota\alpha\kappa}\left(\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\theta}\right)-\\
&\quad\frac{2}{3}\alpha\,{}^{1}\mathcal{T}_{\kappa\alpha\iota}\left(\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\theta}\right)-\frac{2}{3}\alpha\,{}^{1}\mathcal{T}_{\iota\alpha\theta}\left(\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\kappa}\right)+\frac{2}{3}\alpha\,{}^{1}\mathcal{T}_{\theta\alpha\iota}\left(\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\kappa}\right)-\\
&\quad\frac{4}{3}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\iota}\mathcal{B}_{\theta\kappa}\right)-\frac{4}{3}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\iota}{}^{1}\mathcal{T}_{\theta\alpha\kappa}\right)+\frac{4}{3}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\iota}{}^{1}\mathcal{T}_{\kappa\alpha\theta}\right)-\frac{2}{3}\alpha\,{}^{1}\mathcal{T}_{\theta\alpha\kappa}\left(\mathring{\nabla}_{\iota}{}^{2}\mathcal{T}^{\alpha}\right)+\\
&\quad\frac{2}{3}\alpha\,{}^{1}\mathcal{T}_{\kappa\alpha\theta}\left(\mathring{\nabla}_{\iota}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{4}{3}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathcal{B}_{\iota\kappa}\right)+\frac{4}{3}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}{}^{1}\mathcal{T}_{\iota\alpha\kappa}\right)-\frac{4}{3}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}{}^{1}\mathcal{T}_{\kappa\alpha\iota}\right)+
\end{aligned}\tag{64}$$

$$\frac{2}{3}\alpha\,{}^{1}\mathcal{T}_{\iota\alpha\kappa}\left(\mathring{\nabla}_{\theta}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{2}{3}\alpha\,{}^{1}\mathcal{T}_{\kappa\alpha\iota}\left(\mathring{\nabla}_{\theta}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{4}{3}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathcal{B}_{\iota\theta}\right)-\frac{4}{3}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\kappa}{}^{1}\mathcal{T}_{\iota\alpha\theta}\right)+\frac{4}{3}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\kappa}{}^{1}\mathcal{T}_{\theta\alpha\iota}\right)-\frac{2}{3}\alpha\,{}^{1}\mathcal{T}_{\iota\alpha\theta}\left(\mathring{\nabla}_{\kappa}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{2}{3}\alpha\,{}^{1}\mathcal{T}_{\theta\alpha\iota}\left(\mathring{\nabla}_{\kappa}{}^{2}\mathcal{T}^{\alpha}\right)==0$$

Note that Eqs. (62), and (63) allow us to solve for ${}^{2}\mathcal{T}_{\mu}$ and ${}^{3}\lambda_{\mathcal{T}}{}^{\mu}$ in terms of the 2-form to linear order, and the (general) torsion at higher orders.

And here is the tensor equation Eq. (57).

$$\begin{aligned}
&\frac{1}{9}\alpha\,{}^{2}\mathcal{F}_{\kappa}{}^{\alpha}\,{}^{1}\mathcal{T}^{\iota}{}_{\alpha\theta}-\frac{1}{9}\alpha\,{}^{2}\mathcal{F}_{\theta}{}^{\alpha}\,{}^{1}\mathcal{T}^{\iota}{}_{\alpha\kappa}-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\,{}^{1}\mathcal{T}^{\iota}{}_{\theta\kappa}+\\
&\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\,{}^{1}\mathcal{T}^{\iota}{}_{\kappa\theta}-\frac{1}{9}\alpha\,\delta^{\iota}{}_{\kappa}\,{}^{2}\mathcal{F}^{\alpha\alpha'}\,{}^{1}\mathcal{T}_{\theta\alpha\alpha'}-\frac{1}{9}\alpha\,{}^{2}\mathcal{F}_{\kappa}{}^{\alpha}\,{}^{1}\mathcal{T}_{\theta\alpha}{}^{\iota}-\\
&\frac{2}{9}\alpha\,{}^{2}\mathcal{F}^{\iota\alpha}\,{}^{1}\mathcal{T}_{\theta\alpha\kappa}+\frac{1}{9}\alpha\,\delta^{\iota}{}_{\theta}\,{}^{2}\mathcal{F}^{\alpha\alpha'}\,{}^{1}\mathcal{T}_{\kappa\alpha\alpha'}+\frac{1}{9}\alpha\,{}^{2}\mathcal{F}_{\theta}{}^{\alpha}\,{}^{1}\mathcal{T}_{\kappa\alpha}{}^{\iota}+\\
&\frac{2}{9}\alpha\,{}^{2}\mathcal{F}^{\iota\alpha}\,{}^{1}\mathcal{T}_{\kappa\alpha\theta}-\frac{1}{9}\alpha\,\mathcal{M}_{\text{Pl}}\,\delta^{\iota}{}_{\kappa}\,\mathcal{B}_{\theta\alpha}\,{}^{2}\mathcal{T}^{\alpha}+\frac{1}{9}\alpha\,\mathcal{M}_{\text{Pl}}\,\delta^{\iota}{}_{\theta}\,\mathcal{B}_{\kappa\alpha}\,{}^{2}\mathcal{T}^{\alpha}+\\
&\frac{1}{9}\alpha\,\delta^{\iota}{}_{\kappa}\,{}^{2}\mathcal{F}_{\theta\alpha}\,{}^{2}\mathcal{T}^{\alpha}-\frac{1}{9}\alpha\,\delta^{\iota}{}_{\theta}\,{}^{2}\mathcal{F}_{\kappa\alpha}\,{}^{2}\mathcal{T}^{\alpha}+\frac{4}{9}\alpha\,\mathcal{M}_{\text{Pl}}\,\mathcal{B}_{\theta\kappa}\,{}^{2}\mathcal{T}^{\iota}+\\
&\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\,\mathcal{B}^{\iota}{}_{\kappa}\,{}^{2}\mathcal{T}_{\theta}-\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\,\mathcal{B}^{\iota}{}_{\theta}\,{}^{2}\mathcal{T}_{\kappa}+\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)-\\
&\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\theta\alpha\kappa}\left(\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}^{\iota}\right)+\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa\alpha\theta}\left(\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}^{\iota}\right)-\\
&\frac{1}{9}\alpha\,{}^{1}\mathcal{T}^{\iota}{}_{\alpha\kappa}\left(\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}_{\theta}\right)+\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa\alpha}{}^{\iota}\left(\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}_{\theta}\right)+\frac{1}{9}\alpha\,{}^{1}\mathcal{T}^{\iota}{}_{\alpha\theta}\left(\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}_{\kappa}\right)-\\
&\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\theta\alpha}{}^{\iota}\left(\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}_{\kappa}\right)-\frac{1}{9}\alpha\,\delta^{\iota}{}_{\kappa}\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\alpha}{}^{\alpha'}{}_{\theta}\right)+\frac{1}{9}\alpha\,\delta^{\iota}{}_{\theta}\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\alpha}{}^{\alpha'}{}_{\kappa}\right)+\\
&\frac{2}{9}\alpha\,\delta^{\iota}{}_{\kappa}\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\theta\alpha}{}^{\alpha'}\right)-\frac{1}{9}\alpha\,\delta^{\iota}{}_{\kappa}\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\theta}{}^{\alpha'}{}_{\alpha}\right)-\frac{2}{9}\alpha\,\delta^{\iota}{}_{\theta}\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\kappa\alpha}{}^{\alpha'}\right)+\\
&\frac{1}{9}\alpha\,\delta^{\iota}{}_{\theta}\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\kappa}{}^{\alpha'}{}_{\alpha}\right)-\frac{2}{9}\alpha\,\delta^{\iota}{}_{\kappa}\,{}^{1}\mathcal{T}_{\alpha\alpha'\theta}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)+\frac{2}{9}\alpha\,\delta^{\iota}{}_{\theta}\,{}^{1}\mathcal{T}_{\alpha\alpha'\kappa}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)+\\
&\frac{1}{9}\alpha\,\delta^{\iota}{}_{\kappa}\,{}^{1}\mathcal{T}_{\theta\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)+\frac{1}{9}\alpha\,\delta^{\iota}{}_{\kappa}\,{}^{1}\mathcal{T}_{\theta\alpha'\alpha}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)-\frac{1}{9}\alpha\,\delta^{\iota}{}_{\theta}\,{}^{1}\mathcal{T}_{\kappa\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)-\\
&\frac{1}{9}\alpha\,\delta^{\iota}{}_{\theta}\,{}^{1}\mathcal{T}_{\kappa\alpha'\alpha}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)-\frac{4}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}^{\iota}\mathcal{B}_{\theta\kappa}\right)-\frac{4}{9}\alpha\left(\mathring{\nabla}^{\iota}\,{}^{2}\mathcal{F}_{\theta\kappa}\right)-\frac{4}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\iota}\,{}^{1}\mathcal{T}_{\theta\alpha\kappa}\right)+\\
&\frac{4}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\iota}\,{}^{1}\mathcal{T}_{\kappa\alpha\theta}\right)-\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\theta\alpha\kappa}\left(\mathring{\nabla}^{\iota}\,{}^{2}\mathcal{T}^{\alpha}\right)+\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa\alpha\theta}\left(\mathring{\nabla}^{\iota}\,{}^{2}\mathcal{T}^{\alpha}\right)-\\
&\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathcal{B}^{\iota}{}_{\kappa}\right)+\frac{5}{18}\alpha\left(\mathring{\nabla}_{\theta}{}^{2}\mathcal{F}^{\iota}{}_{\kappa}\right)-\frac{2}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}{}^{1}\mathcal{T}^{\iota}{}_{\alpha\kappa}\right)+\frac{2}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}{}^{1}\mathcal{T}_{\kappa\alpha}{}^{\iota}\right)-\\
&\frac{1}{9}\alpha\,{}^{1}\mathcal{T}^{\iota}{}_{\alpha\kappa}\left(\mathring{\nabla}_{\theta}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa\alpha}{}^{\iota}\left(\mathring{\nabla}_{\theta}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\iota}\,{}^{2}\mathcal{T}_{\kappa}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}{}^{2}\mathcal{T}^{\iota}\right)+
\end{aligned}\qquad(65)$$

$$
\begin{aligned}
&\frac{2}{9}\,\alpha\,\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\kappa}\mathcal{B}'_{\theta}\right)-\frac{5}{18}\,\alpha\left(\mathring{\nabla}_{\kappa}{}^{2}\mathcal{F}'_{\theta}\right)+\frac{2}{9}\,\alpha\;{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\kappa}{}^{1}\mathcal{T}'_{\alpha\theta}\right)-\frac{2}{9}\,\alpha\;{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\kappa}{}^{1}\mathcal{T}_{\theta\alpha}{}'\right)+\\
&\frac{1}{9}\,\alpha\;{}^{1}\mathcal{T}'_{\alpha\theta}\left(\mathring{\nabla}_{\kappa}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{1}{9}\,\alpha\;{}^{1}\mathcal{T}_{\theta\alpha}{}'\left(\mathring{\nabla}_{\kappa}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{1}{6}\,\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}'\,{}^{2}\mathcal{T}_{\theta}\right)-\frac{1}{6}\,\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}{}^{2}\mathcal{T}'\right)==0
\end{aligned}
$$

Concrete relation to manuscript: Compare Eq. (65) with Eq. (5a) in our manuscript.

Now we want to be able to solve for the tensor in terms of the 2-form field.

$$
\begin{aligned}
&\frac{1}{18}\Big(9\,\mathcal{M}_{\mathrm{Pl}}^{2}\;{}^{1}\mathcal{T}_{\kappa\lambda\theta}+4\,\alpha\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\kappa\lambda}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)-2\,\alpha\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\theta\lambda}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\\
&\qquad 2\,\alpha\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\theta\kappa}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\lambda}{}^{\alpha}\right)-6\,\alpha\,\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta\lambda}\right)-6\,\alpha\,\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\lambda}\mathcal{B}_{\theta\kappa}\right)\Big)-\\
&\quad\frac{1}{81\,\mathcal{M}_{\mathrm{Pl}}^{2}}\,2\Big(-18\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}^{2}\,\mathcal{B}_{\theta\lambda}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)-18\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}^{2}\,\mathcal{B}_{\theta\kappa}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\lambda}{}^{\alpha}\right)-\\
&\qquad 3\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}^{2}\,\mathcal{B}_{\lambda}{}^{\alpha}\,\mathring{g}_{\theta\kappa}\left(\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\alpha'}\right)-3\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}^{2}\,\mathcal{B}_{\kappa}{}^{\alpha}\,\mathring{g}_{\theta\lambda}\left(\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\alpha'}\right)+\\
&\qquad 6\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}^{2}\,\mathcal{B}_{\theta}{}^{\alpha}\,\mathring{g}_{\kappa\lambda}\left(\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\alpha'}\right)-6\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\kappa\lambda}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\beta}{}^{1}\mathcal{T}_{\alpha'}{}^{\beta}{}_{\theta}\right)+\\
&\qquad 3\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\theta\lambda}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\beta}{}^{1}\mathcal{T}_{\alpha'}{}^{\beta}{}_{\kappa}\right)+3\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\theta\kappa}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\beta}{}^{1}\mathcal{T}_{\alpha'}{}^{\beta}{}_{\lambda}\right)+\\
&\qquad 12\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\kappa\lambda}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\beta}{}^{1}\mathcal{T}_{\theta\alpha'}{}^{\beta}\right)-6\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\kappa\lambda}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\beta}{}^{1}\mathcal{T}_{\theta}{}^{\beta}{}_{\alpha'}\right)-\\
&\qquad 6\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\theta\lambda}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\beta}{}^{1}\mathcal{T}_{\kappa\alpha'}{}^{\beta}\right)+3\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\theta\lambda}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\beta}{}^{1}\mathcal{T}_{\kappa}{}^{\beta}{}_{\alpha'}\right)-\\
&\qquad 6\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\theta\kappa}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\beta}{}^{1}\mathcal{T}_{\lambda\alpha'}{}^{\beta}\right)+3\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\theta\kappa}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\beta}{}^{1}\mathcal{T}_{\lambda}{}^{\beta}{}_{\alpha'}\right)+\\
&\qquad 12\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\kappa\lambda}\;{}^{1}\mathcal{T}^{\alpha\alpha'}{}_{\theta}\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\beta}\right)-6\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\theta\lambda}\;{}^{1}\mathcal{T}^{\alpha\alpha'}{}_{\kappa}\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\beta}\right)-\\
&\qquad 6\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\theta\kappa}\;{}^{1}\mathcal{T}^{\alpha\alpha'}{}_{\lambda}\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\beta}\right)-12\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\kappa\lambda}\;{}^{1}\mathcal{T}_{\theta}{}^{\alpha\alpha'}\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\beta}\right)+\\
&\qquad 6\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\theta\lambda}\;{}^{1}\mathcal{T}_{\kappa}{}^{\alpha\alpha'}\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\beta}\right)+6\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\,\mathring{g}_{\theta\kappa}\;{}^{1}\mathcal{T}_{\lambda}{}^{\alpha\alpha'}\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\beta}\right)-\\
&\qquad 8\,\alpha^{3}\,\mathring{g}_{\kappa\lambda}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\theta}{}^{\beta}\right)+4\,\alpha^{3}\,\mathring{g}_{\theta\lambda}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\kappa}{}^{\beta}\right)+\\
&\qquad 4\,\alpha^{3}\,\mathring{g}_{\theta\kappa}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\lambda}{}^{\beta}\right)+8\,\alpha^{3}\,\mathring{g}_{\kappa\lambda}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\beta}\mathcal{B}_{\alpha'}{}^{\beta}\right)-\\
&\qquad 18\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\kappa}{}^{1}\mathcal{T}_{\theta\alpha'\lambda}\right)+18\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\kappa}{}^{1}\mathcal{T}_{\lambda\alpha'\theta}\right)+\\
&\qquad 18\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\;{}^{1}\mathcal{T}_{\theta}{}^{\alpha}{}_{\lambda}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\alpha'}\right)-18\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\;{}^{1}\mathcal{T}_{\lambda}{}^{\alpha}{}_{\theta}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\alpha'}\right)-\\
&\qquad 4\,\alpha^{3}\,\mathring{g}_{\theta\lambda}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\beta}\mathcal{B}_{\alpha'}{}^{\beta}\right)-18\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\lambda}{}^{1}\mathcal{T}_{\theta\alpha'\kappa}\right)+\\
&\qquad 18\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\lambda}{}^{1}\mathcal{T}_{\kappa\alpha'\theta}\right)+18\,\alpha^{2}\,\mathcal{M}_{\mathrm{Pl}}\;{}^{1}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\left(\mathring{\nabla}_{\lambda}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\alpha'}\right)-
\end{aligned}
\tag{66}
$$

$$18\,\alpha^2\,\mathcal{M}_{\rm Pl}\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\left(\mathring{\nabla}_{\lambda}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\alpha'}\right)-4\,\alpha^3\,\mathring{g}_{\theta\kappa}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\lambda}\mathring{\nabla}_{\beta}\mathcal{B}_{\alpha'}{}^{\beta}\right)==0$$

Note that Eq. (66) allows us to solve for ${}^{1}\mathcal{T}^{\alpha}{}_{\mu\nu}$ in terms of the 2-form field to linear order, and in terms of itself at higher orders: clearly this facilitates a perturbative solution approach.

Now the next thing we do is to take the divergence of Eq. (65).

$$\begin{aligned}
&\frac{2}{9}\alpha\mathcal{M}_{\rm Pl}\,{}^{2}\mathcal{T}_{\kappa}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)+\frac{4}{9}\alpha\mathcal{M}_{\rm Pl}\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta\kappa}\right)-\frac{2}{9}\alpha\mathcal{M}_{\rm Pl}\,{}^{2}\mathcal{T}_{\theta}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\\
&\frac{1}{3}\mathcal{M}_{\rm Pl}^2\left(\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\right)+\frac{1}{3}\mathcal{M}_{\rm Pl}^2\left(\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\right)+\frac{4}{9}\alpha\mathcal{M}_{\rm Pl}\,\mathcal{B}_{\theta\kappa}\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{4}{9}\alpha\mathcal{M}_{\rm Pl}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)-\\
&\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}{}^{2}\mathcal{F}_{\theta\kappa}\right)-\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\theta\alpha'\kappa}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa\alpha'\theta}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)+\\
&\frac{2}{9}\alpha\mathcal{M}_{\rm Pl}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}{}^{2}\mathcal{F}_{\kappa}{}^{\alpha}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\kappa}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}{}^{2}\mathcal{T}^{\alpha}\right)-\\
&\frac{2}{9}\alpha\mathcal{M}_{\rm Pl}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right)+\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}{}^{2}\mathcal{F}_{\theta}{}^{\alpha}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\theta}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}{}^{2}\mathcal{T}^{\alpha}\right)-\\
&\frac{2}{9}\alpha\mathcal{M}_{\rm Pl}\,\mathcal{B}_{\kappa\alpha}\left(\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\theta}\right)+\frac{2}{9}\alpha\mathcal{M}_{\rm Pl}\,\mathcal{B}_{\theta\alpha}\left(\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\kappa}\right)+\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}{}^{2}\mathcal{F}_{\alpha}{}^{\alpha'}\right)-\\
&\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}{}^{2}\mathcal{F}_{\alpha}{}^{\alpha'}\right)-\frac{1}{9}\alpha\,{}^{1}\mathcal{T}^{\alpha\alpha'}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}{}^{2}\mathcal{F}_{\theta\alpha}\right)+\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha\alpha'}\left(\mathring{\nabla}_{\alpha'}{}^{2}\mathcal{F}_{\theta\alpha}\right)+\\
&\frac{1}{9}\alpha\,{}^{1}\mathcal{T}^{\alpha\alpha'}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}{}^{2}\mathcal{F}_{\kappa\alpha}\right)-\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha\alpha'}\left(\mathring{\nabla}_{\alpha'}{}^{2}\mathcal{F}_{\kappa\alpha}\right)+\frac{1}{9}\alpha\,{}^{2}\mathcal{F}_{\kappa}{}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\alpha}{}^{\alpha'}{}_{\theta}\right)+\\
&\frac{1}{9}\alpha\left(\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\kappa}\right)\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\alpha}{}^{\alpha'}{}_{\theta}\right)-\frac{1}{9}\alpha\,{}^{2}\mathcal{F}_{\theta}{}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\alpha}{}^{\alpha'}{}_{\kappa}\right)-\frac{1}{9}\alpha\left(\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\theta}\right)\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\alpha}{}^{\alpha'}{}_{\kappa}\right)-\\
&\frac{1}{9}\alpha\,{}^{2}\mathcal{F}_{\kappa}{}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\theta\alpha}{}^{\alpha'}\right)-\frac{1}{9}\alpha\left(\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\kappa}\right)\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\theta\alpha}{}^{\alpha'}\right)+\frac{2}{9}\alpha\,{}^{2}\mathcal{F}^{\alpha\alpha'}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\theta\alpha\kappa}\right)+\\
&\frac{1}{9}\alpha\,{}^{2}\mathcal{F}_{\theta}{}^{\alpha}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\kappa\alpha}{}^{\alpha'}\right)+\frac{1}{9}\alpha\left(\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\theta}\right)\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\kappa\alpha}{}^{\alpha'}\right)-\frac{2}{9}\alpha\,{}^{2}\mathcal{F}^{\alpha\alpha'}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\kappa\alpha\theta}\right)-\\
&\frac{4}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha'}{}^{1}\mathcal{T}_{\theta\alpha\kappa}\right)+\frac{4}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha'}{}^{1}\mathcal{T}_{\kappa\alpha\theta}\right)-\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\theta\alpha\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)+\\
&\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa\alpha\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\theta}{}^{1}\mathcal{T}_{\alpha}{}^{\alpha'}{}_{\kappa}\right)+\frac{2}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\theta}{}^{1}\mathcal{T}_{\kappa\alpha}{}^{\alpha'}\right)+\\
&\frac{2}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\kappa}{}^{1}\mathcal{T}_{\alpha}{}^{\alpha'}{}_{\theta}\right)-\frac{2}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\kappa}{}^{1}\mathcal{T}_{\theta\alpha}{}^{\alpha'}\right)-\frac{2}{9}\alpha\left(\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}_{\theta\alpha'\kappa}\right)\left(\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)+\\
&\frac{2}{9}\alpha\left(\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}_{\kappa\alpha'\theta}\right)\left(\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{2}{3}\alpha\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\theta\alpha\kappa}\right)\left(\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{2}{3}\alpha\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\kappa\alpha\theta}\right)\left(\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)-\\
&\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\alpha\alpha'\kappa}\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\theta}\right)+\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\theta}\right)+\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\alpha\alpha'\theta}\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\kappa}\right)-\\
&\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\theta\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\kappa}\right)-\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\alpha\alpha'\kappa}\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}_{\theta}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}_{\theta}{}^{2}\mathcal{T}^{\alpha}\right)+
\end{aligned}\tag{67}$$

$$\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\alpha\alpha'\theta}\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}_{\kappa}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\theta\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}_{\kappa}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{1}{9}\alpha\,\mathcal{M}_{\rm Pl}\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa\alpha}\right)+$$
$$\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha\alpha'}\left(\mathring{\nabla}_{\theta}{}^{2}\mathcal{F}_{\alpha\alpha'}\right)-\frac{1}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}{}^{2}\mathcal{F}_{\kappa\alpha}\right)+\frac{1}{9}\alpha\,{}^{2}\mathcal{F}^{\alpha\alpha'}\left(\mathring{\nabla}_{\theta}{}^{1}\mathcal{T}_{\kappa\alpha\alpha'}\right)+$$
$$\frac{1}{9}\alpha\left(\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\theta}{}^{1}\mathcal{T}_{\kappa\alpha\alpha'}\right)-\frac{1}{9}\alpha\left(\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\theta}{}^{1}\mathcal{T}_{\kappa\alpha'\alpha}\right)+\frac{1}{9}\alpha\,\mathcal{M}_{\rm Pl}\,\mathcal{B}_{\kappa\alpha}\left(\mathring{\nabla}_{\theta}{}^{2}\mathcal{T}^{\alpha}\right)-$$
$$\frac{1}{9}\alpha\,{}^{2}\mathcal{F}_{\kappa\alpha}\left(\mathring{\nabla}_{\theta}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{1}{9}\alpha\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\kappa\alpha}{}^{\alpha'}\right)\left(\mathring{\nabla}_{\theta}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{1}{9}\alpha\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\kappa}{}^{\alpha'}{}_{\alpha}\right)\left(\mathring{\nabla}_{\theta}{}^{2}\mathcal{T}^{\alpha}\right)-$$
$$\frac{2}{9}\alpha\,\mathcal{M}_{\rm Pl}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{1}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\alpha}{}^{\alpha'}{}_{\kappa}\right)-\frac{2}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\kappa\alpha}{}^{\alpha'}\right)+$$
$$\frac{1}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\kappa}{}^{\alpha'}{}_{\alpha}\right)+\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\alpha\alpha'\kappa}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa\alpha\alpha'}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)-$$
$$\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa\alpha'\alpha}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{1}{9}\alpha\,\mathcal{M}_{\rm Pl}\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta\alpha}\right)-\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha\alpha'}\left(\mathring{\nabla}_{\kappa}{}^{2}\mathcal{F}_{\alpha\alpha'}\right)+$$
$$\frac{1}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\kappa}{}^{2}\mathcal{F}_{\theta\alpha}\right)-\frac{1}{9}\alpha\,{}^{2}\mathcal{F}^{\alpha\alpha'}\left(\mathring{\nabla}_{\kappa}{}^{1}\mathcal{T}_{\theta\alpha\alpha'}\right)-\frac{1}{9}\alpha\left(\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\kappa}{}^{1}\mathcal{T}_{\theta\alpha\alpha'}\right)+$$
$$\frac{1}{9}\alpha\left(\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\kappa}{}^{1}\mathcal{T}_{\theta\alpha'\alpha}\right)-\frac{1}{9}\alpha\,\mathcal{M}_{\rm Pl}\,\mathcal{B}_{\theta\alpha}\left(\mathring{\nabla}_{\kappa}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{1}{9}\alpha\,{}^{2}\mathcal{F}_{\theta\alpha}\left(\mathring{\nabla}_{\kappa}{}^{2}\mathcal{T}^{\alpha}\right)+$$
$$\frac{1}{9}\alpha\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\theta\alpha}{}^{\alpha'}\right)\left(\mathring{\nabla}_{\kappa}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{1}{9}\alpha\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\theta}{}^{\alpha'}{}_{\alpha}\right)\left(\mathring{\nabla}_{\kappa}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{2}{9}\alpha\,\mathcal{M}_{\rm Pl}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)-$$
$$\frac{1}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\alpha}{}^{\alpha'}{}_{\theta}\right)+\frac{2}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\theta\alpha}{}^{\alpha'}\right)-\frac{1}{9}\alpha\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\theta}{}^{\alpha'}{}_{\alpha}\right)-$$
$$\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\alpha\alpha'\theta}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\theta\alpha\alpha'}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\theta\alpha'\alpha}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)==0$$

Next we substitute into Eq. (67) for the 2-form field in Eq. (14) again.

$$-\frac{1}{3}\mathcal{M}_{\rm Pl}^{3}\,\mathcal{B}_{\theta\kappa}+\frac{2}{9}\alpha\,\mathcal{M}_{\rm Pl}\,\mathcal{B}_{\kappa}{}^{\alpha}\,{}^{2}\mathcal{F}_{\theta\alpha}-\frac{1}{3}\mathcal{M}_{\rm Pl}^{2}\,{}^{2}\mathcal{F}_{\theta\kappa}-\frac{2}{9}\alpha\,\mathcal{M}_{\rm Pl}\,\mathcal{B}_{\theta}{}^{\alpha}\,{}^{2}\mathcal{F}_{\kappa\alpha}+$$
$$\frac{2}{9}\alpha\,\mathcal{M}_{\rm Pl}\,{}^{2}\mathcal{T}_{\kappa}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)+\frac{4}{9}\alpha\,\mathcal{M}_{\rm Pl}\,{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta\kappa}\right)-\frac{2}{9}\alpha\,\mathcal{M}_{\rm Pl}\,{}^{2}\mathcal{T}_{\theta}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+$$
$$\frac{4}{9}\alpha\,\mathcal{M}_{\rm Pl}\,\mathcal{B}_{\theta\kappa}\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{4}{9}\alpha\,\mathcal{M}_{\rm Pl}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)-\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}{}^{2}\mathcal{F}_{\theta\kappa}\right)-$$
$$\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\theta\alpha'\kappa}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa\alpha'\theta}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha'}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{2}{9}\alpha\,\mathcal{M}_{\rm Pl}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right)-$$
$$\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}{}^{2}\mathcal{F}_{\kappa}{}^{\alpha}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\kappa}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\,\mathcal{M}_{\rm Pl}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right)+$$
$$\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}{}^{2}\mathcal{F}_{\theta}{}^{\alpha}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha}{}^{2}\mathcal{T}_{\theta}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}{}^{2}\mathcal{F}_{\alpha}{}^{\alpha'}\right)-$$
$$\frac{2}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}{}^{2}\mathcal{F}_{\alpha}{}^{\alpha'}\right)-\frac{1}{9}\alpha\,{}^{1}\mathcal{T}^{\alpha\alpha'}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}{}^{2}\mathcal{F}_{\theta\alpha}\right)+\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha\alpha'}\left(\mathring{\nabla}_{\alpha'}{}^{2}\mathcal{F}_{\theta\alpha}\right)+$$
$$\frac{1}{9}\alpha\,{}^{1}\mathcal{T}^{\alpha\alpha'}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}{}^{2}\mathcal{F}_{\kappa\alpha}\right)-\frac{1}{9}\alpha\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha\alpha'}\left(\mathring{\nabla}_{\alpha'}{}^{2}\mathcal{F}_{\kappa\alpha}\right)+\frac{4}{9}\alpha\,{}^{2}\mathcal{F}^{\alpha\alpha'}\left(\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}_{\theta\alpha\kappa}\right)-$$

$$
\begin{aligned}
&\frac{4}{9}\alpha\,\overset{2}{\mathcal{F}}{}^{\alpha\alpha'}\left(\mathring{\nabla}_{\alpha'}\overset{1}{\mathcal{T}}_{\kappa\alpha\theta}\right)-\frac{4}{9}\alpha\,\overset{2}{\mathcal{T}}{}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha'}\overset{1}{\mathcal{T}}_{\theta\alpha\kappa}\right)+\frac{4}{9}\alpha\,\overset{2}{\mathcal{T}}{}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha'}\overset{1}{\mathcal{T}}_{\kappa\alpha\theta}\right)-\\
&\frac{2}{9}\alpha\,\overset{1}{\mathcal{T}}_{\theta\alpha\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha}\right)+\frac{2}{9}\alpha\,\overset{1}{\mathcal{T}}_{\kappa\alpha\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha}\right)+\frac{2}{9}\alpha\,\overset{2}{\mathcal{T}}{}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\theta}\overset{1}{\mathcal{T}}_{\kappa\alpha}{}^{\alpha'}\right)-\\
&\frac{2}{9}\alpha\,\overset{2}{\mathcal{T}}{}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\kappa}\overset{1}{\mathcal{T}}_{\theta\alpha}{}^{\alpha'}\right)-\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\overset{1}{\mathcal{T}}_{\theta\alpha'\kappa}\right)\left(\mathring{\nabla}^{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha}\right)+\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\overset{1}{\mathcal{T}}_{\kappa\alpha'\theta}\right)\left(\mathring{\nabla}^{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha}\right)-\\
&\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha'}\overset{1}{\mathcal{T}}_{\theta\alpha\kappa}\right)\left(\mathring{\nabla}^{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha}\right)+\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha'}\overset{1}{\mathcal{T}}_{\kappa\alpha\theta}\right)\left(\mathring{\nabla}^{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha}\right)-\frac{1}{9}\alpha\,\overset{1}{\mathcal{T}}_{\alpha\alpha'\kappa}\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}^{\alpha}\overset{2}{\mathcal{T}}_{\theta}\right)+\\
&\frac{1}{9}\alpha\,\overset{1}{\mathcal{T}}_{\kappa\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}^{\alpha}\overset{2}{\mathcal{T}}_{\theta}\right)+\frac{1}{9}\alpha\,\overset{1}{\mathcal{T}}_{\alpha\alpha'\theta}\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}^{\alpha}\overset{2}{\mathcal{T}}_{\kappa}\right)-\frac{1}{9}\alpha\,\overset{1}{\mathcal{T}}_{\theta\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}^{\alpha}\overset{2}{\mathcal{T}}_{\kappa}\right)-\\
&\frac{1}{9}\alpha\,\overset{1}{\mathcal{T}}_{\alpha\alpha'\kappa}\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}_{\theta}\overset{2}{\mathcal{T}}{}^{\alpha}\right)+\frac{1}{9}\alpha\,\overset{1}{\mathcal{T}}_{\kappa\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}_{\theta}\overset{2}{\mathcal{T}}{}^{\alpha}\right)+\frac{1}{9}\alpha\,\overset{1}{\mathcal{T}}_{\alpha\alpha'\theta}\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}_{\kappa}\overset{2}{\mathcal{T}}{}^{\alpha}\right)-\\
&\frac{1}{9}\alpha\,\overset{1}{\mathcal{T}}_{\theta\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}_{\kappa}\overset{2}{\mathcal{T}}{}^{\alpha}\right)+\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\,\overset{2}{\mathcal{T}}{}^{\alpha}\left(\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa\alpha}\right)+\frac{1}{9}\alpha\,\overset{1}{\mathcal{T}}_{\kappa}{}^{\alpha\alpha'}\left(\mathring{\nabla}_{\theta}\overset{2}{\mathcal{F}}_{\alpha\alpha'}\right)-\\
&\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{2}{9}\alpha\,\overset{2}{\mathcal{T}}{}^{\alpha}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha'}\overset{1}{\mathcal{T}}_{\kappa\alpha}{}^{\alpha'}\right)+\frac{2}{9}\alpha\,\overset{1}{\mathcal{T}}_{\alpha\alpha'\kappa}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha}\right)-\\
&\frac{1}{9}\alpha\,\overset{1}{\mathcal{T}}_{\kappa\alpha\alpha'}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha}\right)-\frac{1}{9}\alpha\,\overset{1}{\mathcal{T}}_{\kappa\alpha'\alpha}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha}\right)-\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\,\overset{2}{\mathcal{T}}{}^{\alpha}\left(\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta\alpha}\right)-\\
&\frac{1}{9}\alpha\,\overset{1}{\mathcal{T}}_{\theta}{}^{\alpha\alpha'}\left(\mathring{\nabla}_{\kappa}\overset{2}{\mathcal{F}}_{\alpha\alpha'}\right)+\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)+\frac{2}{9}\alpha\,\overset{2}{\mathcal{T}}{}^{\alpha}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha'}\overset{1}{\mathcal{T}}_{\theta\alpha}{}^{\alpha'}\right)-\\
&\frac{2}{9}\alpha\,\overset{1}{\mathcal{T}}_{\alpha\alpha'\theta}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha}\right)+\frac{1}{9}\alpha\,\overset{1}{\mathcal{T}}_{\theta\alpha\alpha'}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha}\right)+\frac{1}{9}\alpha\,\overset{1}{\mathcal{T}}_{\theta\alpha'\alpha}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha}\right)==0
\end{aligned}
$$

Next we will take Eq. (68) and expand the vector and axial Maxwell tensors back into derivatives, and substitute for the solutions in terms of the 2-form field that we obtained from Eqs. (62), and (63). We apply the suggestive perturbative solution approach for torsion in terms of the two-form field, finally ending up with the following nonlinear field equation (i.e. from the cubic action).

$$
\begin{aligned}
&-\frac{1}{3}\mathcal{M}_{\text{Pl}}{}^{3}\,\mathcal{B}_{\theta\kappa}-\frac{4}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)+\frac{16}{27}\alpha^{2}\left(\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)\left(\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\alpha'}\right)-\frac{16}{27}\alpha^{2}\,\mathcal{B}_{\theta\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)-\\
&\frac{8}{27}\alpha^{2}\,\mathcal{B}_{\kappa}{}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha'}\right)+\frac{8}{27}\alpha^{2}\,\mathcal{B}_{\theta}{}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha'}\right)+\frac{128\,\alpha^{3}\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\beta}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^{2}}+\\
&\frac{64\,\alpha^{3}\left(\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)\left(\mathring{\nabla}_{\beta}\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\alpha'}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^{2}}-\frac{64\,\alpha^{3}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\beta}\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\theta\kappa}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^{2}}+\frac{8}{27}\alpha^{2}\left(\mathring{\nabla}^{\alpha'}\mathcal{B}_{\kappa}{}^{\alpha}\right)\left(\mathring{\nabla}_{\theta}\mathcal{B}_{\alpha\alpha'}\right)-\\
&\frac{8}{27}\alpha^{2}\left(\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\alpha'}\right)\left(\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{32\,\alpha^{3}\left(\mathring{\nabla}_{\beta}\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\alpha'}\right)\left(\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^{2}}-\frac{4}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)-
\end{aligned}
$$

$$\frac{8}{27}\alpha^2\,\mathcal{B}_{\kappa}{}^{\alpha}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\alpha'}\right)+\frac{8}{27}\alpha^2\,\mathcal{B}^{\alpha\alpha'}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\kappa\alpha}\right)+\frac{64\,\alpha^3\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\alpha'}{}^{\beta}\right)\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha'}\mathcal{B}_{\kappa}{}^{\alpha}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}-$$

$$\frac{32\,\alpha^3\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\beta}\right)\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha'}\mathcal{B}_{\kappa}{}^{\alpha}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}+\frac{32\,\alpha^3\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\beta}\mathcal{B}_{\alpha'}{}^{\beta}\right)}{27\,\mathcal{M}_{\text{Pl}}{}^2}+$$

$$\frac{64\,\alpha^3\left(\mathring{\nabla}^{\alpha'}\mathcal{B}_{\kappa}{}^{\alpha}\right)\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\alpha'}{}^{\beta}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}+\frac{32\,\alpha^3\left(\mathring{\nabla}^{\alpha'}\mathcal{B}_{\kappa}{}^{\alpha}\right)\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\beta}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}-$$

$$\frac{32\,\alpha^3\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\kappa}{}^{\beta}\right)}{27\,\mathcal{M}_{\text{Pl}}{}^2}+\frac{32\,\alpha^3\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\beta}\mathring{\nabla}^{\beta}\mathcal{B}_{\kappa\alpha'}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}-$$

$$\frac{8}{27}\alpha^2\left(\mathring{\nabla}^{\alpha'}\mathcal{B}_{\theta}{}^{\alpha}\right)\left(\mathring{\nabla}_{\kappa}\mathcal{B}_{\alpha\alpha'}\right)+\frac{32\,\alpha^3\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\beta}\right)\left(\mathring{\nabla}_{\kappa}\mathcal{B}^{\alpha\alpha'}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}+\frac{8}{27}\alpha^2\left(\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\alpha'}\right)\left(\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right)+$$

$$\frac{32\,\alpha^3\left(\mathring{\nabla}_{\beta}\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\alpha'}\right)\left(\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}+\frac{4}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)+\frac{8}{27}\alpha^2\,\mathcal{B}_{\theta}{}^{\alpha}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\alpha'}\right)-$$

$$\frac{8}{27}\alpha^2\,\mathcal{B}^{\alpha\alpha'}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\theta\alpha}\right)-\frac{64\,\alpha^3\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\alpha'}{}^{\beta}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha'}\mathcal{B}_{\theta}{}^{\alpha}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}+\frac{32\,\alpha^3\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\beta}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha'}\mathcal{B}_{\theta}{}^{\alpha}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}-$$

$$\frac{32\,\alpha^3\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\beta}\mathcal{B}_{\alpha'}{}^{\beta}\right)}{27\,\mathcal{M}_{\text{Pl}}{}^2}-\frac{64\,\alpha^3\left(\mathring{\nabla}^{\alpha'}\mathcal{B}_{\theta}{}^{\alpha}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\alpha'}{}^{\beta}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}-$$

$$\frac{32\,\alpha^3\left(\mathring{\nabla}^{\alpha'}\mathcal{B}_{\theta}{}^{\alpha}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\beta}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}-\frac{32\,\alpha^3\left(\mathring{\nabla}_{\theta}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\beta}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}+$$

$$\frac{32\,\alpha^3\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\theta}{}^{\beta}\right)}{27\,\mathcal{M}_{\text{Pl}}{}^2}-\frac{32\,\alpha^3\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\beta}\mathring{\nabla}^{\beta}\mathcal{B}_{\theta\alpha'}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}==0$$

So now the equation in Eq. (69) contains all the dynamical information about the linear spectrum of the theory Eq. (53), since Eqs. (62), (63), and (65) serve only to determine the torsion in terms of the 2-form. So the key question is how much of the 2-form does Eq. (69) propagate?

For the linearised version of Eq. (69), we found that taking the divergence led to a constraint (cf. Eq. (51)). Let's take the divergence of the full Eq. (69).

$$\frac{1}{3}\mathcal{M}_{\text{Pl}}{}^3\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{8}{9}\alpha^2\left(\mathring{\nabla}^{\alpha'}\mathcal{B}_{\kappa}{}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\alpha'}{}^{\beta}\right)-\frac{16}{27}\alpha^2\left(\mathring{\nabla}^{\alpha'}\mathcal{B}_{\kappa}{}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\beta}\right)+$$

$$
\begin{aligned}
&\frac{8}{9}\alpha^2\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\kappa}{}^{\beta}\right)+\frac{8}{9}\alpha^2\,\mathcal{B}_{\kappa}{}^{\alpha}\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha'\beta}\right)-\frac{8}{27}\alpha^2\,\mathcal{B}^{\alpha\alpha'}\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\beta}\right)+\\
&\frac{8}{27}\alpha^2\left(\mathring{\nabla}^{\alpha'}\mathcal{B}_{\kappa}{}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}\mathring{\nabla}^{\beta}\mathcal{B}_{\alpha\alpha'}\right)+\frac{8}{27}\alpha^2\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\beta}\mathring{\nabla}^{\beta}\mathcal{B}_{\kappa\alpha'}\right)-\frac{8}{27}\alpha^2\,\mathcal{B}_{\kappa}{}^{\alpha}\left(\mathring{\nabla}_{\beta}\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\alpha'}\right)+\\
&\frac{8}{27}\alpha^2\,\mathcal{B}^{\alpha\alpha'}\left(\mathring{\nabla}_{\beta}\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\kappa\alpha}\right)+\frac{16}{27}\alpha^2\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\kappa\alpha}\right)\left(\mathring{\nabla}^{\beta}\mathcal{B}^{\alpha\alpha'}\right)-\frac{64\,\alpha^3\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\kappa}{}^{\theta}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}-\\
&\frac{32\,\alpha^3\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha'}\mathcal{B}_{\kappa}{}^{\alpha}\right)\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\theta}\mathring{\nabla}_{\beta}\mathcal{B}_{\alpha}{}^{\beta}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}+\frac{32\,\alpha^3\left(\mathring{\nabla}^{\alpha'}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\theta}\mathring{\nabla}_{\beta}\mathcal{B}_{\alpha'}{}^{\beta}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}+\\
&\frac{64\,\alpha^3\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\theta}\mathring{\nabla}_{\beta}\mathcal{B}_{\kappa\alpha'}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}-\frac{32\,\alpha^3\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\theta}\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\kappa}{}^{\beta}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}+\\
&\frac{32\,\alpha^3\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\theta}\mathring{\nabla}_{\beta}\mathring{\nabla}^{\beta}\mathcal{B}_{\kappa\alpha'}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}-\frac{32\,\alpha^3\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}^{\beta\theta}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)}{27\,\mathcal{M}_{\text{Pl}}{}^2}+\\
&\frac{32\,\alpha^3\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\theta}\mathring{\nabla}_{\beta}\mathcal{B}_{\alpha'}{}^{\beta}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}+\frac{16}{27}\alpha^2\left(\mathring{\nabla}^{\beta}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha\beta}\right)+\\
&\frac{16}{27}\alpha^2\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\beta}\mathcal{B}_{\alpha'}{}^{\beta}\right)+\frac{64\,\alpha^3\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\mathcal{B}_{\beta}{}^{\theta}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\beta}\mathcal{B}^{\alpha\alpha'}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}-\\
&\frac{32\,\alpha^3\left(\mathring{\nabla}_{\beta}\mathring{\nabla}^{\beta}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\alpha}{}^{\theta}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}+\frac{64\,\alpha^3\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha'}\mathcal{B}_{\beta}{}^{\theta}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}+\\
&\frac{64\,\alpha^3\left(\mathring{\nabla}^{\beta}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\mathcal{B}_{\beta}{}^{\theta}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}-\frac{32\,\alpha^3\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha'}\mathcal{B}^{\beta\theta}\right)}{27\,\mathcal{M}_{\text{Pl}}{}^2}+\\
&\frac{32\,\alpha^3\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\mathring{\nabla}^{\theta}\mathcal{B}_{\alpha'\beta}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}+\frac{32\,\alpha^3\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\alpha\alpha'}\right)\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\mathring{\nabla}^{\theta}\mathring{\nabla}_{\beta}\mathcal{B}_{\alpha'}{}^{\beta}\right)}{81\,\mathcal{M}_{\text{Pl}}{}^2}==0
\end{aligned}
$$

So Eq. (70) is no longer a constraint, and it no longer knocks out 3 d.o.f from the 2-form.

Concrete relation to manuscript: We regain 3 d.o.f from the relaxed constraint in the cubic extension to the theory, we conclude that all the 2-form propagates, i.e. 6 propagating d.o.f in the nonlinear spectrum. Note that we may directly integrate Eq. (69) to produce (in flat space) the first and second lines in the effective theory Eq. (6) in our manuscript.

Column 3 of Eq. (16): Vector mass, but no axial multiplier

Setting up the Lagrangian

We define the Lagrangian. It contains nonvanishing vector mass parameter $\overset{2}{\cdot}\mu$ but it does not contain a multiplier field $\overset{3}{\cdot}\lambda_{\mathcal{T}}{}^{\mu}$, which would ordinarily disable the axial vector torsion.

$$\alpha\ \mathrm{R}[\nabla]_{\alpha\beta}\ \mathrm{R}[\nabla]^{\alpha\beta} - \alpha\ \mathrm{R}[\nabla]^{\alpha\beta}\ \mathrm{R}[\nabla]_{\beta\alpha} - \frac{\mathcal{M}_{\mathrm{Pl}}{}^{2}\ \mathrm{R}[\nabla]}{2} + \mathcal{M}_{\mathrm{Pl}}{}^{2}\ \overset{2}{\cdot}\mu\ \overset{2}{\cdot}\mathcal{T}_{\alpha}\ \overset{2}{\cdot}\mathcal{T}^{\alpha} \tag{71}$$

Now we would like to have the post-Riemannian decomposition of the Lagrangian Eq. (71).

$$\begin{aligned}
&-\frac{\mathcal{M}_{\mathrm{Pl}}{}^{2}\ \mathrm{R}[\mathring{\nabla}]}{2} - \frac{1}{8}\mathcal{M}_{\mathrm{Pl}}{}^{2}\ \mathcal{T}_{\alpha\alpha'\beta}\ \mathcal{T}^{\alpha\alpha'\beta} - \frac{1}{4}\mathcal{M}_{\mathrm{Pl}}{}^{2}\ \mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\alpha\beta} - \\
&\frac{1}{2}\mathcal{M}_{\mathrm{Pl}}{}^{2}\ \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\ \mathcal{T}^{\beta}{}_{\alpha'\beta} - \mathcal{M}_{\mathrm{Pl}}{}^{2}\ \overset{2}{\cdot}\mu\ \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\ \mathcal{T}^{\beta}{}_{\alpha'\beta} + \frac{1}{4}\alpha\ \mathcal{T}_{\alpha}{}^{\beta'\chi}\ \mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\beta}{}^{\prime}\ \mathcal{T}_{\beta'\chi\prime} + \\
&\frac{1}{4}\alpha\ \mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\beta}{}^{\beta'}\ \mathcal{T}_{\beta'}{}^{\chi\prime}\ \mathcal{T}_{\chi\alpha\prime} - \alpha\ \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\ \mathcal{T}_{\alpha'}{}^{\beta\beta'}\ \mathcal{T}_{\beta}{}^{\chi\prime}\ \mathcal{T}_{\chi\beta'\prime} - \\
&\frac{1}{2}\alpha\ \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\ \mathcal{T}_{\alpha'}{}^{\beta\beta'}\ \mathcal{T}^{\chi}{}_{\beta\beta'}\ \mathcal{T}^{\prime}{}_{\chi\prime} + \alpha\ \mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\beta}{}^{\beta'}\ \left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\chi}{}_{\beta'\chi}\right) + \\
&\mathcal{M}_{\mathrm{Pl}}{}^{2}\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right) + \alpha\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\beta'}{}_{\beta\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right) - \alpha\left(\mathring{\nabla}_{\beta}\mathcal{T}^{\beta'}{}_{\alpha'\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right) + \\
&\frac{1}{2}\alpha\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right) + 2\,\alpha\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right) - \alpha\ \mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\beta}{}^{\beta'}\ \left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\chi}{}_{\alpha\chi}\right) - \\
&2\,\alpha\ \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\ \mathcal{T}_{\alpha'}{}^{\beta\beta'}\ \left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\chi}{}_{\beta\chi}\right) + \alpha\ \mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\beta}{}^{\beta'}\ \left(\mathring{\nabla}_{\chi}\mathcal{T}^{\chi}{}_{\alpha\beta'}\right) + \alpha\ \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\ \mathcal{T}_{\alpha'}{}^{\beta\beta'}\ \left(\mathring{\nabla}_{\chi}\mathcal{T}^{\chi}{}_{\beta\beta'}\right)
\end{aligned} \tag{72}$$

We want to study the theory when it is linearised. As an intermediate step in order to do this, we just keep in Eq. (72) the second-order terms in torsion and no higher. Also from this point onwards we completely neglect factors of the curvature which may arise in the field equations by commuting covariant derivatives.

$$\begin{aligned}
&-\frac{\mathcal{M}_{\mathrm{Pl}}{}^{2}\ \mathrm{R}[\mathring{\nabla}]}{2} - \frac{1}{8}\mathcal{M}_{\mathrm{Pl}}{}^{2}\ \mathcal{T}_{\alpha\alpha'\beta}\ \mathcal{T}^{\alpha\alpha'\beta} - \frac{1}{4}\mathcal{M}_{\mathrm{Pl}}{}^{2}\ \mathcal{T}^{\alpha\alpha'\beta}\ \mathcal{T}_{\alpha'\alpha\beta} - \frac{1}{2}\mathcal{M}_{\mathrm{Pl}}{}^{2}\ \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\ \mathcal{T}^{\beta}{}_{\alpha'\beta} - \\
&\mathcal{M}_{\mathrm{Pl}}{}^{2}\ \overset{2}{\cdot}\mu\ \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\ \mathcal{T}^{\beta}{}_{\alpha'\beta} + \mathcal{M}_{\mathrm{Pl}}{}^{2}\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right) + \alpha\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\beta'}{}_{\beta\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right) - \\
&\alpha\left(\mathring{\nabla}_{\beta}\mathcal{T}^{\beta'}{}_{\alpha'\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right) + \frac{1}{2}\alpha\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right) + 2\,\alpha\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right)
\end{aligned} \tag{73}$$

Now we decompose the torsion in Eq. (73) into the Lorentz irreps.

$$-\frac{\mathcal{M}_{\mathrm{Pl}}{}^{2}\ \mathrm{R}[\mathring{\nabla}]}{2} - \frac{2}{9}\mathcal{M}_{\mathrm{Pl}}{}^{2}\ \overset{1}{\cdot}\mathcal{T}_{\alpha\alpha'\beta}\ \overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta} + \frac{2}{9}\mathcal{M}_{\mathrm{Pl}}{}^{2}\ \overset{1}{\cdot}\mathcal{T}_{\alpha\beta\alpha'}\ \overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta} +$$

$$\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\ \overset{2}{\cdot}\mathcal{T}_{\alpha}\ \ \overset{2}{\cdot}\mathcal{T}^{\alpha}+\mathcal{M}_{\text{Pl}}^{2}\ \overset{2}{\cdot}\mu\ \overset{2}{\cdot}\mathcal{T}_{\alpha}\ \ \overset{2}{\cdot}\mathcal{T}^{\alpha}-\frac{3}{4}\mathcal{M}_{\text{Pl}}^{2}\ \overset{3}{\cdot}\mathcal{T}_{\alpha}\ \ \overset{3}{\cdot}\mathcal{T}^{\alpha}+\mathcal{M}_{\text{Pl}}^{2}\left(\mathring{\nabla}_{\alpha}\overset{2}{\cdot}\mathcal{T}^{\alpha}\right)-$$
$$\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\overset{2}{\cdot}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha'}\ \overset{2}{\cdot}\mathcal{T}^{\alpha}\right)+\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha'}\overset{2}{\cdot}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha'}\ \overset{2}{\cdot}\mathcal{T}^{\alpha}\right)+\alpha\left(\mathring{\nabla}_{\alpha}\overset{3}{\cdot}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha'}\ \overset{3}{\cdot}\mathcal{T}^{\alpha}\right)-$$
$$\alpha\left(\mathring{\nabla}_{\alpha'}\overset{3}{\cdot}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha'}\ \overset{3}{\cdot}\mathcal{T}^{\alpha}\right)+\frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha'}\ \overset{2}{\cdot}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}\overset{1}{\cdot}\mathcal{T}_{\alpha}{}^{\beta}{}_{\alpha'}\right)-\frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha'}\ \overset{2}{\cdot}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}\overset{1}{\cdot}\mathcal{T}_{\alpha'}{}^{\beta}{}_{\alpha}\right)-$$
$$\frac{4}{3}\alpha\ \epsilon\,\mathring{g}_{\alpha\alpha'\beta'\chi}\left(\mathring{\nabla}^{\alpha'}\ \overset{3}{\cdot}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}\overset{1}{\cdot}\mathcal{T}^{\beta\beta'\chi}\right)+\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}_{\alpha'}{}^{\beta'}{}_{\beta}\right)-$$
$$\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}_{\beta}{}^{\beta'}{}_{\alpha'}\right)-\frac{4}{3}\alpha\ \epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\left(\mathring{\nabla}^{\alpha'}\ \overset{2}{\cdot}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\beta'}\ \overset{3}{\cdot}\mathcal{T}^{\beta}\right)$$

Manipulating the field equations

Here is the tensor field equation.

$$\begin{aligned}-&\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\ \overset{1}{\cdot}\mathcal{T}'{}_{\theta\kappa}+\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\ \overset{1}{\cdot}\mathcal{T}'{}_{\kappa\theta}-\frac{2}{9}\alpha\ \delta'{}_{\kappa}\left(\mathring{\nabla}_{\alpha}\overset{2}{\cdot}\mathcal{F}_{\theta}{}^{\alpha}\right)+\\
&\frac{2}{9}\alpha\ \delta'{}_{\theta}\left(\mathring{\nabla}_{\alpha}\overset{2}{\cdot}\mathcal{F}_{\kappa}{}^{\alpha}\right)-\frac{2}{9}\alpha\ \delta'{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'}{}_{\theta}\right)+\frac{2}{9}\alpha\ \delta'{}_{\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'}{}_{\kappa}\right)+\\
&\frac{2}{9}\alpha\ \delta'{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\theta}{}^{\alpha\alpha'}\right)-\frac{2}{9}\alpha\ \delta'{}_{\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\kappa}{}^{\alpha\alpha'}\right)-\frac{1}{3}\alpha\ \epsilon\,\mathring{g}_{\theta\kappa\alpha\alpha'}\left(\mathring{\nabla}'\,\overset{3}{\cdot}\mathcal{F}^{\alpha\alpha'}\right)-\\
&\frac{4}{9}\alpha\left(\mathring{\nabla}'\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\right)+\frac{4}{9}\alpha\left(\mathring{\nabla}'\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\right)+\frac{1}{3}\alpha\left(\mathring{\nabla}_{\theta}\overset{2}{\cdot}\mathcal{F}'{}_{\kappa}\right)-\frac{1}{6}\alpha\ \epsilon\,\mathring{g}'{}_{\kappa\alpha\alpha'}\left(\mathring{\nabla}_{\theta}\overset{3}{\cdot}\mathcal{F}^{\alpha\alpha'}\right)-\\
&\frac{2}{9}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}'{}^{\alpha}{}_{\kappa}\right)+\frac{2}{9}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\kappa}{}^{\alpha\prime}\right)+\frac{1}{3}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}'\,\overset{2}{\cdot}\mathcal{T}_{\kappa}\right)-\frac{1}{3}\alpha\left(\mathring{\nabla}_{\kappa}\overset{2}{\cdot}\mathcal{F}'{}_{\theta}\right)+\\
&\frac{1}{6}\alpha\ \epsilon\,\mathring{g}'{}_{\theta\alpha\alpha'}\left(\mathring{\nabla}_{\kappa}\overset{3}{\cdot}\mathcal{F}^{\alpha\alpha'}\right)+\frac{2}{9}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}'{}^{\alpha}{}_{\theta}\right)-\frac{2}{9}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\theta}{}^{\alpha\prime}\right)-\frac{1}{3}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}'\,\overset{2}{\cdot}\mathcal{T}_{\theta}\right)==0\end{aligned}\tag{75}$$

There is some nuance to how we obtain Eq. (75), in that we can't just vary with respect to the tensor irrep. If we do that, then the resulting equation can have traces which are not true on-shell. It is safest to actually vary with respect to the whole torsion tensor and then project out the (traceless by construction) tensor irrep from that equation.

Now for the vector equation.

$$\frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\ \overset{2}{\cdot}\mathcal{T}^{\mu}+2\,\mathcal{M}_{\text{Pl}}^{2}\ \overset{2}{\cdot}\mu\ \overset{2}{\cdot}\mathcal{T}^{\mu}+\frac{8}{9}\alpha\left(\mathring{\nabla}_{\alpha}\overset{2}{\cdot}\mathcal{F}^{\mu\alpha}\right)+\frac{8}{9}\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\mu}\right)-\frac{8}{9}\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\mu\alpha\alpha'}\right)==0\tag{76}$$

And for the axial vector equation.

$$-\frac{3}{2}\mathcal{M}_{\text{Pl}}^{2}\ \overset{3}{\cdot}\mathcal{T}^{\mu}-2\,\alpha\left(\mathring{\nabla}_{\alpha}\overset{3}{\cdot}\mathcal{F}^{\mu\alpha}\right)+\frac{4}{3}\alpha\ \epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\left(\mathring{\nabla}^{\beta'}\ \mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\right)==0\tag{77}$$

Now we can also take the dual of Eq. (77), and so we write this out for completeness.

$$\frac{3}{2}\mathcal{M}_{\text{Pl}}^{2}\,\epsilon\mathring{g}^{\alpha\beta\chi}{}_{\alpha'}\;{}^{3}\mathcal{T}^{\alpha'}-2\,\alpha\,\epsilon\mathring{g}^{\alpha\beta\chi}{}_{\mu}\left(\mathring{\nabla}_{\alpha'}{}^{3}\mathcal{F}^{\alpha'\mu}\right)-$$
$$\frac{4}{3}\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha}\,{}^{1}\mathcal{T}^{\beta\alpha'\chi}\right)+\frac{4}{3}\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha}\,{}^{1}\mathcal{T}^{\chi\alpha'\beta}\right)+\frac{4}{3}\alpha\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}^{\alpha\alpha'\chi}\right)-$$
$$\frac{4}{3}\alpha\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}^{\chi\alpha'\alpha}\right)-\frac{4}{3}\alpha\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}^{\alpha\alpha'\beta}\right)+\frac{4}{3}\alpha\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}^{\beta\alpha'\alpha}\right)==0$$

With the effective 2-form field in Eq. (14), we are ready to simplify the equations of motion.

Here is the vector equation Eq. (76).

$$\frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\left(1+3\,{}^{2}\mu\right){}^{2}\mathcal{T}^{\mu}-\frac{8}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\mu\alpha}\right)==0 \tag{79}$$

Here is the axial vector equation Eq. (77).

$$-\frac{3}{2}\mathcal{M}_{\text{Pl}}^{2}\;{}^{3}\mathcal{T}^{\mu}+\frac{2}{3}\alpha\,\mathcal{M}_{\text{Pl}}\,\epsilon\mathring{g}^{\mu}{}_{\alpha\alpha'\beta}\left(\mathring{\nabla}^{\beta}\mathcal{B}^{\alpha\alpha'}\right)==0 \tag{80}$$

Again here is the dual part of Eq. (80)

$$\frac{3}{2}\mathcal{M}_{\text{Pl}}^{2}\,\epsilon\mathring{g}_{\iota\theta\kappa\alpha}\;{}^{3}\mathcal{T}^{\alpha}-\frac{4}{3}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\iota}\mathcal{B}_{\theta\kappa}\right)+\frac{4}{3}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathcal{B}_{\iota\kappa}\right)-\frac{4}{3}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathcal{B}_{\iota\theta}\right)==0 \tag{81}$$

Note that Eqs. (79), and (80) allow us to solve for ${}^{3}\mathcal{T}_{\mu}$ and ${}^{2}\mathcal{T}^{\mu}$ purely in terms of the 2-form.

And here is the tensor equation Eq. (75).

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\;{}^{1}\mathcal{T}^{\iota}{}_{\theta\kappa}+\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\;{}^{1}\mathcal{T}^{\iota}{}_{\kappa\theta}+\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)-\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)-$$
$$\frac{4}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}^{\iota}\mathcal{B}_{\theta\kappa}\right)-\frac{4}{9}\alpha\left(\mathring{\nabla}^{\iota}\,{}^{2}\mathcal{F}_{\theta\kappa}\right)-\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathcal{B}^{\iota}{}_{\kappa}\right)+\frac{5}{18}\alpha\left(\mathring{\nabla}_{\theta}\,{}^{2}\mathcal{F}^{\iota}{}_{\kappa}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\iota}\,{}^{2}\mathcal{T}_{\kappa}\right)+$$
$$\frac{1}{6}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}\,{}^{2}\mathcal{T}^{\iota}\right)+\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathcal{B}^{\iota}{}_{\theta}\right)-\frac{5}{18}\alpha\left(\mathring{\nabla}_{\kappa}\,{}^{2}\mathcal{F}^{\iota}{}_{\theta}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\iota}\,{}^{2}\mathcal{T}_{\theta}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\,{}^{2}\mathcal{T}^{\iota}\right)==0 \tag{82}$$

Now the next thing we do is to take the divergence of Eq. (82).

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\left(\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\right)+\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\left(\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\right)-\frac{4}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)-$$
$$\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{F}_{\theta\kappa}\right)+\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\,{}^{2}\mathcal{F}_{\kappa}{}^{\alpha}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}_{\kappa}\right)+$$
$$\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}\,{}^{2}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right)+\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\,{}^{2}\mathcal{F}_{\theta}{}^{\alpha}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}_{\theta}\right)-$$
$$\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\,{}^{2}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)==0 \tag{83}$$

Next we substitute into Eq. (83) for the 2-form field in Eq. (14) again.

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{3}\,\mathcal{B}_{\theta\kappa}-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\,\overset{2}{:}\mathcal{F}_{\theta\kappa}+\frac{1}{4}\mathcal{M}_{\text{Pl}}^{2}\,\epsilon\,\mathring{g}_{\theta\kappa\alpha\alpha'}\,\overset{3}{:}\mathcal{F}^{\alpha\alpha'}-\frac{4}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)-$$
$$\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,\overset{2}{:}\mathcal{F}_{\theta\kappa}\right)+\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\overset{2}{:}\mathcal{F}_{\kappa}{}^{\alpha}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha}\,\overset{2}{:}\mathcal{T}_{\kappa}\right)+$$
$$\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}\overset{2}{:}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right)+\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\overset{2}{:}\mathcal{F}_{\theta}{}^{\alpha}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha}\,\overset{2}{:}\mathcal{T}_{\theta}\right)-$$
$$\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\overset{2}{:}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{2}{9}\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)==0$$

Next we will take Eq. (84) and expand the vector and axial Maxwell tensors back into derivatives, and substitute for the solutions in terms of the 2-form field that we obtained from Eqs. (79), and (80).

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{3}\,\mathcal{B}_{\theta\kappa}+\frac{4\,\alpha\,\mathcal{M}_{\text{Pl}}\,\overset{2}{:}\mu\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)}{3+9\,\overset{2}{:}\mu}-\frac{4\,\alpha\,\mathcal{M}_{\text{Pl}}\,\overset{2}{:}\mu\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)}{3+9\,\overset{2}{:}\mu}==0 \quad (85)$$

So now the equation in Eq. (85) contains all the dynamical information about the linear spectrum of the theory Eq. (71), since Eqs. (79), (80), and (82) serve only to determine the torsion in terms of the 2-form. So the key question is how much of the 2-form does Eq. (85) propagate?

Let's first take the divergence of Eq. (85).

$$\frac{1}{3}\mathcal{M}_{\text{Pl}}^{3}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{4\,\alpha\,\mathcal{M}_{\text{Pl}}\,\overset{2}{:}\mu\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)}{3+9\,\overset{2}{:}\mu}==0 \quad (86)$$

So Eq. (86) still looks like a propagating equation.

Let's now take the dual of the gradient of Eq. (85).

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{3}\,\epsilon\,\mathring{g}^{\psi}{}_{\alpha\theta\kappa}\left(\mathring{\nabla}^{\kappa}\mathcal{B}^{\alpha\theta}\right)==0 \quad (87)$$

So Eq. (87) is just a constraint, which knocks out 3 d.o.f from the 2-form.

Concrete relation to manuscript: Since we lose 3 d.o.f to a constraint, we conclude that half the 2-form propagates, i.e. 3 propagating d.o.f in the linear spectrum.

Concrete relation to manuscript: We won't compute the last column in Eq. (16) yet: instead we'll transition to the equivalent table for which the axial and vector torsion are switched, and for which we are validating all our nonlinear spectral claims with full Hamiltonian analyses, and study it in that more thorough context.

Summary of findings: when we want the axial

vector to propagate

These final few sections are focussed on the following class of torsion theory, in which the Einstein-Hilbert term is augmented by the square of the antisymmetric Ricci tensor, and by an axial vector torsion mass term, and by a multiplier which switches off the vector torsion.

$$\alpha\ R[\nabla]_{\alpha\beta}\ R[\nabla]^{\alpha\beta} - \alpha\ R[\nabla]^{\alpha\beta}\ R[\nabla]_{\beta\alpha} - \frac{\mathcal{M}_{\text{Pl}}{}^{2}\ R[\nabla]}{2} + \mathcal{M}_{\text{Pl}}{}^{2}\ \overset{3}{\vdots}\mu\ \overset{3}{\vdots}\mathcal{T}_{\alpha}\ \overset{3}{\vdots}\mathcal{T}^{\alpha} + \overset{2}{\vdots}\mathcal{T}^{\alpha}\ \overset{2}{\vdots}\lambda_{\mathcal{T}\,\alpha} \tag{88}$$

By setting to zero either or both of the mass and multiplier terms in Eq. (88), we find some interesting effects in the linear spectra. These are summarised in the following table. Note that there are always an extra +2 d.o.f from Einstein's graviton.

	(No conditions)	$\overset{3}{\vdots}\mu == 0$	$\overset{2}{\vdots}\lambda_{\mathcal{T}}{}^{\mu} == 0$	$\overset{3}{\vdots}\mu == \overset{2}{\vdots}\lambda_{\mathcal{T}}{}^{\mu} == 0$
Linear d.o.f	6 + 2	3 + 2	3 + 2	0 + 2
Nonlinear d.o.f	6 + 2	6 + 2	6 + 2	6 + 2

(89)

Concrete relation to manuscript: Of course, as we argue in our manuscript, Eqs. (15), and (88) are equivalent theories, the only difference being in the sign of α for which we might be led to believe (naively) that each is viable. Once again, the table is surprising, and we will study each column. Unlike in the case for Eq. (15), we will not study any of the theories at cubic or higher order using Lagrangian methods. Instead, we use the field equations (Lagrangian methods) to study the linear spectra, and then use the Hamiltonian analysis to study the full nonlinear spectra, including the linear limit.

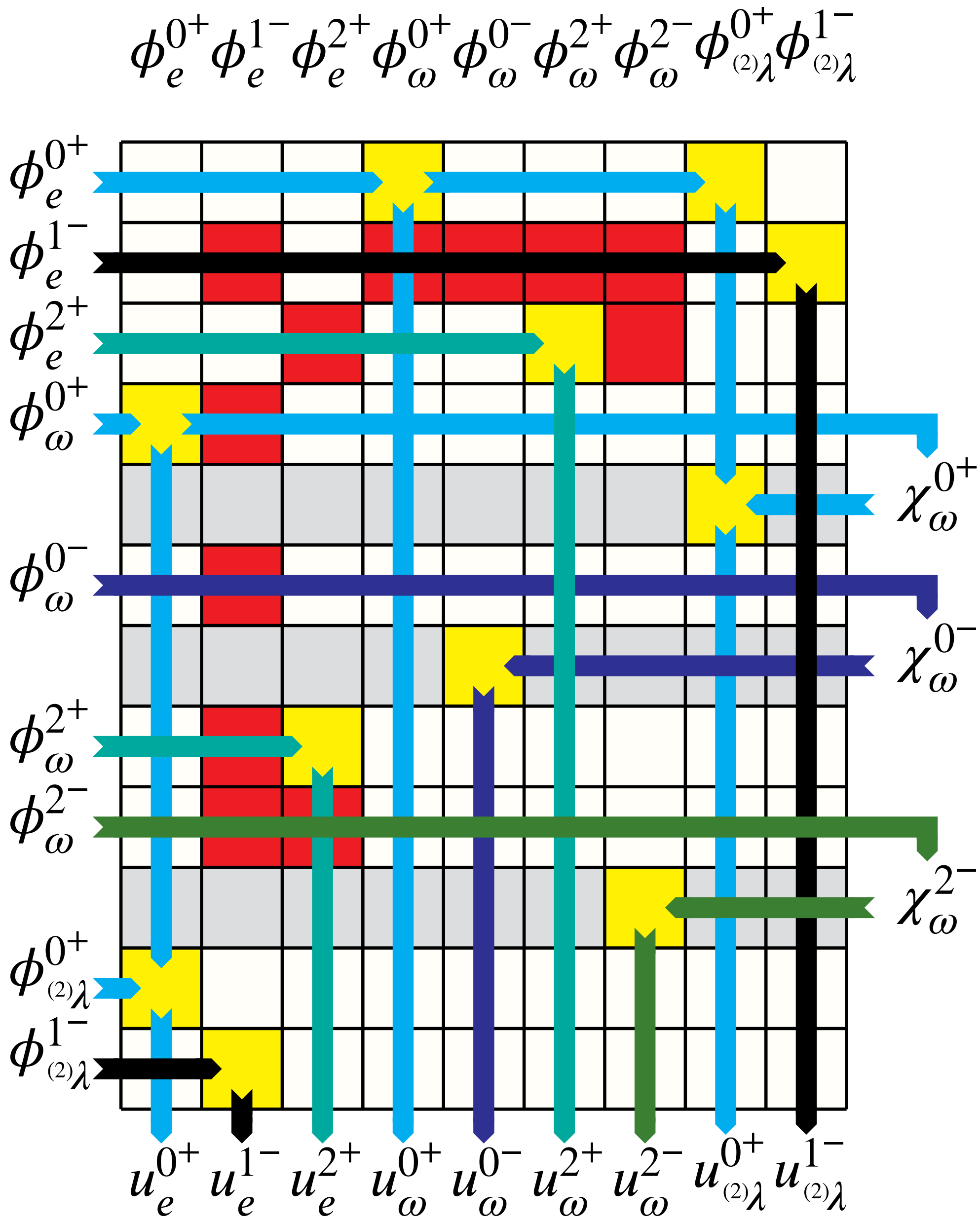
Column 1 of Eq. (89): Axial mass and vector multiplier

Setting up the Lagrangian

We define the Lagrangian. It contains both the nonvanishing axial mass parameter $\overset{3}{\vdots}\mu$ and a multiplier field $\overset{2}{\vdots}\lambda_{\mathcal{T}}{}^{\mu}$ to disable the vector torsion.

$$\alpha\ R[\nabla]_{\alpha\beta}\ R[\nabla]^{\alpha\beta} - \alpha\ R[\nabla]^{\alpha\beta}\ R[\nabla]_{\beta\alpha} - \frac{\mathcal{M}_{\text{Pl}}{}^{2}\ R[\nabla]}{2} + \mathcal{M}_{\text{Pl}}{}^{2}\ \overset{3}{\vdots}\mu\ \overset{3}{\vdots}\mathcal{T}_{\alpha}\ \overset{3}{\vdots}\mathcal{T}^{\alpha} + \overset{2}{\vdots}\mathcal{T}^{\alpha}\ \overset{2}{\vdots}\lambda_{\mathcal{T}\,\alpha} \tag{90}$$

Let's now look at the full nonlinear Hamiltonian analysis of Eq. (90).



According to this analysis we will have the following d.o.f in the linear spectrum. We compute this by adding the naive canonical d.o.f from the tetrad and the spin connection, and any multiplier fields, then subtracting the Poincaré and kinematic constraints, then subtracting the primaries and then the secondaries, and the result is then halved.

$$\frac{1}{2}(2\times 16+2\times 24+2\times 4-2\times 10-2\times 10-(1+3+5+1+1+5+5+1+3)-(1+1+5)) == 6+2 \tag{91}$$

Concrete relation to manuscript: So we expect 6 d.o.f in the linear spectrum, apart from the two polarisations of Einstein's graviton, from the Hamiltonian analysis. The nonlinear analysis yields the same number of d.o.f, so there is no strong coupling.

Now we would like to have the post-Riemannian decomposition of the Lagrangian Eq. (90).

$$
\begin{aligned}
&-\frac{\mathcal{M}_{\text{Pl}}{}^{2}\,\mathrm{R}[\overset{\circ}{\nabla}]}{2}-\frac{1}{8}\mathcal{M}_{\text{Pl}}{}^{2}\,\mathcal{T}_{\alpha\beta\chi}\,\mathcal{T}^{\alpha\beta\chi}-\frac{1}{18}\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{3}{\cdot}\mu\,\mathcal{T}_{\alpha\beta\chi}\,\mathcal{T}^{\alpha\beta\chi}-\\
&\frac{1}{4}\mathcal{M}_{\text{Pl}}{}^{2}\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\alpha\chi}+\frac{1}{9}\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{3}{\cdot}\mu\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\alpha\chi}-\frac{1}{2}\mathcal{M}_{\text{Pl}}{}^{2}\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\,\mathcal{T}^{\chi}{}_{\beta\chi}+\\
&\frac{1}{4}\alpha\,\mathcal{T}_{\alpha}{}^{\iota\theta}\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\chi}{}^{\kappa}\,\mathcal{T}_{\iota\theta\kappa}+\frac{1}{4}\alpha\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\chi}{}^{\iota}\,\mathcal{T}_{\iota}{}^{\theta\kappa}\,\mathcal{T}_{\theta\alpha\kappa}-\\
&\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\,\mathcal{T}_{\beta}{}^{\chi\iota}\,\mathcal{T}_{\chi}{}^{\theta\kappa}\,\mathcal{T}_{\theta\iota\kappa}-\frac{1}{2}\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\,\mathcal{T}_{\beta}{}^{\chi\iota}\,\mathcal{T}^{\theta}{}_{\chi\iota}\,\mathcal{T}^{\kappa}{}_{\theta\kappa}-\mathcal{T}^{\beta}{}_{\alpha\beta}\,\overset{2}{\cdot}\lambda_{\mathcal{T}}{}^{\alpha}+\\
&\alpha\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\chi}{}^{\iota}\left(\overset{\circ}{\nabla}_{\alpha}\mathcal{T}^{\theta}{}_{\iota\theta}\right)+\mathcal{M}_{\text{Pl}}{}^{2}\left(\overset{\circ}{\nabla}_{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\right)+\alpha\left(\overset{\circ}{\nabla}_{\beta}\mathcal{T}^{\iota}{}_{\chi\iota}\right)\left(\overset{\circ}{\nabla}{}^{\chi}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\right)-\alpha\left(\overset{\circ}{\nabla}_{\chi}\mathcal{T}^{\iota}{}_{\beta\iota}\right)\left(\overset{\circ}{\nabla}{}^{\chi}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\right)+\\
&\frac{1}{2}\alpha\left(\overset{\circ}{\nabla}_{\alpha}\mathcal{T}^{\alpha\beta\chi}\right)\left(\overset{\circ}{\nabla}_{\iota}\mathcal{T}^{\iota}{}_{\beta\chi}\right)+2\,\alpha\left(\overset{\circ}{\nabla}{}^{\chi}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\right)\left(\overset{\circ}{\nabla}_{\iota}\mathcal{T}^{\iota}{}_{\beta\chi}\right)-\alpha\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\chi}{}^{\iota}\left(\overset{\circ}{\nabla}_{\iota}\mathcal{T}^{\theta}{}_{\alpha\theta}\right)-\\
&2\,\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\,\mathcal{T}_{\beta}{}^{\chi\iota}\left(\overset{\circ}{\nabla}_{\iota}\mathcal{T}^{\theta}{}_{\chi\theta}\right)+\alpha\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\chi}{}^{\iota}\left(\overset{\circ}{\nabla}_{\theta}\mathcal{T}^{\theta}{}_{\alpha\iota}\right)+\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\,\mathcal{T}_{\beta}{}^{\chi\iota}\left(\overset{\circ}{\nabla}_{\theta}\mathcal{T}^{\theta}{}_{\chi\iota}\right)
\end{aligned}
\tag{92}
$$

We want to study the theory when it is linearised. As an intermediate step in order to do this, we just keep in Eq. (92) the second-order terms in torsion and no higher. Also from this point onwards we completely neglect factors of the curvature which may arise in the field equations by commuting covariant derivatives.

$$
\begin{aligned}
&-\frac{\mathcal{M}_{\text{Pl}}{}^{2}\,\mathrm{R}[\overset{\circ}{\nabla}]}{2}-\frac{1}{8}\mathcal{M}_{\text{Pl}}{}^{2}\,\mathcal{T}_{\alpha\beta\chi}\,\mathcal{T}^{\alpha\beta\chi}-\frac{1}{18}\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{3}{\cdot}\mu\,\mathcal{T}_{\alpha\beta\chi}\,\mathcal{T}^{\alpha\beta\chi}-\\
&\frac{1}{4}\mathcal{M}_{\text{Pl}}{}^{2}\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\alpha\chi}+\frac{1}{9}\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{3}{\cdot}\mu\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\alpha\chi}-\frac{1}{2}\mathcal{M}_{\text{Pl}}{}^{2}\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\,\mathcal{T}^{\chi}{}_{\beta\chi}-\\
&\mathcal{T}^{\beta}{}_{\alpha\beta}\,\overset{2}{\cdot}\lambda_{\mathcal{T}}{}^{\alpha}+\mathcal{M}_{\text{Pl}}{}^{2}\left(\overset{\circ}{\nabla}_{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\right)+\alpha\left(\overset{\circ}{\nabla}_{\beta}\mathcal{T}^{\iota}{}_{\chi\iota}\right)\left(\overset{\circ}{\nabla}{}^{\chi}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\right)-\\
&\alpha\left(\overset{\circ}{\nabla}_{\chi}\mathcal{T}^{\iota}{}_{\beta\iota}\right)\left(\overset{\circ}{\nabla}{}^{\chi}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\right)+\frac{1}{2}\alpha\left(\overset{\circ}{\nabla}_{\alpha}\mathcal{T}^{\alpha\beta\chi}\right)\left(\overset{\circ}{\nabla}_{\iota}\mathcal{T}^{\iota}{}_{\beta\chi}\right)+2\,\alpha\left(\overset{\circ}{\nabla}{}^{\chi}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\right)\left(\overset{\circ}{\nabla}_{\iota}\mathcal{T}^{\iota}{}_{\beta\chi}\right)
\end{aligned}
\tag{93}
$$

Now we decompose the torsion in Eq. (93) into the Lorentz irreps.

$$
\begin{aligned}
&-\frac{\mathcal{M}_{\text{Pl}}{}^{2}\,\mathrm{R}[\overset{\circ}{\nabla}]}{2}-\frac{2}{9}\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{1}{\cdot}\mathcal{T}_{\alpha\alpha'\beta}\,\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}+\frac{2}{9}\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{1}{\cdot}\mathcal{T}_{\alpha\beta\alpha'}\,\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}+\frac{1}{3}\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{2}{\cdot}\mathcal{T}_{\alpha}\,\overset{2}{\cdot}\mathcal{T}^{\alpha}-\\
&\frac{3}{4}\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{3}{\cdot}\mathcal{T}_{\alpha}\,\overset{3}{\cdot}\mathcal{T}^{\alpha}+\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{3}{\cdot}\mu\,\overset{3}{\cdot}\mathcal{T}_{\alpha}\,\overset{3}{\cdot}\mathcal{T}^{\alpha}+\overset{2}{\cdot}\mathcal{T}^{\alpha}\,\overset{2}{\cdot}\lambda_{\mathcal{T}\alpha}+\mathcal{M}_{\text{Pl}}{}^{2}\left(\overset{\circ}{\nabla}_{\alpha}\overset{2}{\cdot}\mathcal{T}^{\alpha}\right)-\\
&\frac{4}{9}\alpha\left(\overset{\circ}{\nabla}_{\alpha}\overset{2}{\cdot}\mathcal{T}_{\alpha'}\right)\left(\overset{\circ}{\nabla}{}^{\alpha'}\overset{2}{\cdot}\mathcal{T}^{\alpha}\right)+\frac{4}{9}\alpha\left(\overset{\circ}{\nabla}_{\alpha'}\overset{2}{\cdot}\mathcal{T}_{\alpha}\right)\left(\overset{\circ}{\nabla}{}^{\alpha'}\overset{2}{\cdot}\mathcal{T}^{\alpha}\right)+\alpha\left(\overset{\circ}{\nabla}_{\alpha}\overset{3}{\cdot}\mathcal{T}_{\alpha'}\right)\left(\overset{\circ}{\nabla}{}^{\alpha'}\overset{3}{\cdot}\mathcal{T}^{\alpha}\right)-\\
&\alpha\left(\overset{\circ}{\nabla}_{\alpha'}\overset{3}{\cdot}\mathcal{T}_{\alpha}\right)\left(\overset{\circ}{\nabla}{}^{\alpha'}\overset{3}{\cdot}\mathcal{T}^{\alpha}\right)+\frac{8}{9}\alpha\left(\overset{\circ}{\nabla}{}^{\alpha'}\overset{2}{\cdot}\mathcal{T}^{\alpha}\right)\left(\overset{\circ}{\nabla}_{\beta}\overset{1}{\cdot}\mathcal{T}_{\alpha}{}^{\beta}{}_{\alpha'}\right)-\frac{8}{9}\alpha\left(\overset{\circ}{\nabla}{}^{\alpha'}\overset{2}{\cdot}\mathcal{T}^{\alpha}\right)\left(\overset{\circ}{\nabla}_{\beta}\overset{1}{\cdot}\mathcal{T}_{\alpha'}{}^{\beta}{}_{\alpha}\right)-
\end{aligned}
\tag{94}
$$

$$\frac{4}{3}\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\chi\iota}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}{}^{1}\mathcal{T}^{\beta\chi\iota}\right)+\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\chi}{}^{1}\mathcal{T}_{\alpha'}{}^{\chi}{}_{\beta}\right)-\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\chi}{}^{1}\mathcal{T}_{\beta}{}^{\chi}{}_{\alpha'}\right)-\frac{4}{3}\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\chi}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\chi}\,{}^{3}\mathcal{T}^{\beta}\right)$$

Manipulating the field equations

Here is the tensor field equation.

$$\begin{aligned}&-\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}^{2}\,{}^{1}\mathcal{T}^{\iota}{}_{\theta\kappa}+\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}^{2}\,{}^{1}\mathcal{T}^{\iota}{}_{\kappa\theta}-\frac{2}{9}\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{F}_{\theta}{}^{\alpha}\right)+\\&\frac{2}{9}\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{F}_{\kappa}{}^{\alpha}\right)-\frac{2}{9}\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}^{\alpha\alpha'}{}_{\theta}\right)+\frac{2}{9}\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}^{\alpha\alpha'}{}_{\kappa}\right)+\\&\frac{2}{9}\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}_{\theta}{}^{\alpha\alpha'}\right)-\frac{2}{9}\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}_{\kappa}{}^{\alpha\alpha'}\right)-\frac{1}{3}\alpha\,\epsilon\,\mathring{g}_{\theta\kappa\alpha\alpha'}\left(\mathring{\nabla}^{\iota}\,{}^{3}\mathcal{F}^{\alpha\alpha'}\right)-\\&\frac{4}{9}\alpha\left(\mathring{\nabla}^{\iota}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\right)+\frac{4}{9}\alpha\left(\mathring{\nabla}^{\iota}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\right)+\frac{1}{3}\alpha\left(\mathring{\nabla}_{\theta}{}^{2}\mathcal{F}^{\iota}{}_{\kappa}\right)-\frac{1}{6}\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\kappa\alpha\alpha'}\left(\mathring{\nabla}_{\theta}{}^{3}\mathcal{F}^{\alpha\alpha'}\right)-\\&\frac{2}{9}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}^{\iota\alpha}{}_{\kappa}\right)+\frac{2}{9}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}_{\kappa}{}^{\alpha\iota}\right)+\frac{1}{3}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\iota}\,{}^{2}\mathcal{T}_{\kappa}\right)-\frac{1}{3}\alpha\left(\mathring{\nabla}_{\kappa}{}^{2}\mathcal{F}^{\iota}{}_{\theta}\right)+\\&\frac{1}{6}\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\theta\alpha\alpha'}\left(\mathring{\nabla}_{\kappa}{}^{3}\mathcal{F}^{\alpha\alpha'}\right)+\frac{2}{9}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}^{\iota\alpha}{}_{\theta}\right)-\frac{2}{9}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}_{\theta}{}^{\alpha\iota}\right)-\frac{1}{3}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\iota}\,{}^{2}\mathcal{T}_{\theta}\right)==0\end{aligned}\tag{95}$$

There is some nuance to how we obtain Eq. (95), in that we can't just vary with respect to the tensor irrep. If we do that, then the resulting equation can have traces which are not true on-shell. It is safest to actually vary with respect to the whole torsion tensor and then project out the (traceless by construction) tensor irrep from that equation.

Now for the vector equation.

$$\frac{2}{3}\mathcal{M}_{\mathrm{Pl}}^{2}\,{}^{2}\mathcal{T}^{\mu}+{}^{2}\lambda_{\mathcal{T}}{}^{\mu}+\frac{8}{9}\alpha\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{F}^{\mu\alpha}\right)+\frac{8}{9}\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}^{\alpha\alpha'\mu}\right)-\frac{8}{9}\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}^{\mu\alpha\alpha'}\right)==0\tag{96}$$

And for the axial vector equation.

$$-\frac{3}{2}\mathcal{M}_{\mathrm{Pl}}^{2}\,{}^{3}\mathcal{T}^{\mu}+2\,\mathcal{M}_{\mathrm{Pl}}^{2}\,{}^{3}\mu\,{}^{3}\mathcal{T}^{\mu}-2\,\alpha\left(\mathring{\nabla}_{\alpha}{}^{3}\mathcal{F}^{\mu\alpha}\right)+\frac{4}{3}\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\left(\mathring{\nabla}^{\beta'}\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}^{\alpha\alpha'\beta}\right)==0\tag{97}$$

Now we can also take the dual of Eq. (97), and so we write this out for completeness.

$$\begin{aligned}&\frac{3}{2}\mathcal{M}_{\mathrm{Pl}}^{2}\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\alpha'}\,{}^{3}\mathcal{T}^{\alpha'}-2\,\mathcal{M}_{\mathrm{Pl}}^{2}\,{}^{3}\mu\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\alpha'}\,{}^{3}\mathcal{T}^{\alpha'}-2\,\alpha\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\mu}\left(\mathring{\nabla}_{\alpha'}{}^{3}\mathcal{F}^{\alpha'\mu}\right)-\\&\frac{4}{3}\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha}\,{}^{1}\mathcal{T}^{\beta\alpha'\chi}\right)+\frac{4}{3}\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha}\,{}^{1}\mathcal{T}^{\chi\alpha'\beta}\right)+\frac{4}{3}\alpha\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}^{\alpha\alpha'\chi}\right)-\\&\frac{4}{3}\alpha\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}^{\chi\alpha'\alpha}\right)-\frac{4}{3}\alpha\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}^{\alpha\alpha'\beta}\right)+\frac{4}{3}\alpha\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha'}{}^{1}\mathcal{T}^{\beta\alpha'\alpha}\right)==0\end{aligned}\tag{98}$$

Finally we have the equation for the multiplier itself.

$$\overset{2}{\cdot}\mathcal{T}_{\mu} == 0 \tag{99}$$

With the effective 2-form field in Eq. (14), we are ready to simplify the equations of motion.

Here is the vector equation Eq. (96).

$$\overset{2}{\cdot}\lambda_{\mathcal{T}}{}^{\mu} - \frac{8}{9}\alpha \mathcal{M}_{\text{Pl}} \left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\mu\alpha}\right) == 0 \tag{100}$$

Here is the axial vector equation Eq. (97).

$$\frac{1}{2}\mathcal{M}_{\text{Pl}}{}^{2}\left(-3+4\,\overset{3}{\cdot}\mu\right)\,\overset{3}{\cdot}\mathcal{T}^{\mu} + \frac{2}{3}\alpha\mathcal{M}_{\text{Pl}}\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha\alpha'\beta}\left(\mathring{\nabla}^{\beta}\mathcal{B}^{\alpha\alpha'}\right) == 0 \tag{101}$$

Again here is the dual part of Eq. (101)

$$\frac{1}{2}\mathcal{M}_{\text{Pl}}{}^{2}\left(3-4\,\overset{3}{\cdot}\mu\right)\,\epsilon\,\mathring{g}_{\iota\theta\kappa\alpha}\,\overset{3}{\cdot}\mathcal{T}^{\alpha} - \frac{4}{3}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\iota}\mathcal{B}_{\theta\kappa}\right) + \frac{4}{3}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathcal{B}_{\iota\kappa}\right) - \frac{4}{3}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathcal{B}_{\iota\theta}\right) == 0 \tag{102}$$

Note that Eqs. (100), and (101) allow us to solve for $\overset{3}{\cdot}\mathcal{T}_{\mu}$ and $\overset{2}{\cdot}\lambda_{\mathcal{T}}{}^{\mu}$ purely in terms of the 2-form.

And here is the tensor equation Eq. (95).

$$\begin{aligned}&-\frac{1}{3}\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{1}{\cdot}\mathcal{T}'{}_{\theta\kappa} + \frac{1}{3}\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{1}{\cdot}\mathcal{T}'{}_{\kappa\theta} + \frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\,\delta'{}_{\kappa}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right) - \\ &\quad \frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\,\delta'{}_{\theta}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right) - \frac{4}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}'\mathcal{B}_{\theta\kappa}\right) - \frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathcal{B}'{}_{\kappa}\right) + \frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathcal{B}'{}_{\theta}\right) == 0\end{aligned} \tag{103}$$

Now the next thing we do is to take the divergence of Eq. (103).

$$\begin{aligned}&-\frac{1}{3}\mathcal{M}_{\text{Pl}}{}^{2}\left(\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\right) + \frac{1}{3}\mathcal{M}_{\text{Pl}}{}^{2}\left(\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\right) - \frac{4}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right) + \frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right) - \\ &\quad \frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right) - \frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right) + \frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right) == 0\end{aligned} \tag{104}$$

Next we substitute into Eq. (104) for the 2-form field in Eq. (14) again.

$$\begin{aligned}&-\frac{1}{3}\mathcal{M}_{\text{Pl}}{}^{3}\,\mathcal{B}_{\theta\kappa} - \frac{1}{3}\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{2}{\cdot}\mathcal{F}_{\theta\kappa} + \frac{1}{4}\mathcal{M}_{\text{Pl}}{}^{2}\,\epsilon\,\mathring{g}_{\theta\kappa\alpha\alpha'}\,\overset{3}{\cdot}\mathcal{F}^{\alpha\alpha'} - \frac{4}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right) + \\ &\quad \frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right) - \frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right) - \frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right) + \frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right) == 0\end{aligned} \tag{105}$$

Next we will take Eq. (105) and expand the vector and axial Maxwell tensors back into derivatives, and substitute for the solutions in terms of the 2-form field that we obtained from Eqs. (100), and (101).

$$-\frac{1}{3}\mathcal{M}_{Pl}{}^{3}\ \mathcal{B}_{\theta\kappa}+\frac{16\,\alpha\,\mathcal{M}_{Pl}\ \overset{3}{\cdot}\mu\left(\overset{\circ}{\nabla}_{\alpha}\overset{\circ}{\nabla}{}^{\alpha}\mathcal{B}_{\theta\kappa}\right)}{27-36\ \overset{3}{\cdot}\mu}+\frac{4\,\alpha\,\mathcal{M}_{Pl}\left(\overset{\circ}{\nabla}_{\theta}\overset{\circ}{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)}{9-12\ \overset{3}{\cdot}\mu}+\frac{4\,\alpha\,\mathcal{M}_{Pl}\left(\overset{\circ}{\nabla}_{\kappa}\overset{\circ}{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)}{-9+12\ \overset{3}{\cdot}\mu}==0 \tag{106}$$

So now the equation in Eq. (106) contains all the dynamical information about the linear spectrum of the theory Eq. (90), since Eqs. (100), (101), and (103) serve only to determine the torsion in terms of the 2-form. So the key question is how much of the 2-form does Eq. (106) propagate?

Let's first take the divergence of Eq. (106).

$$\frac{1}{3}\mathcal{M}_{Pl}{}^{3}\left(\overset{\circ}{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{4}{9}\,\alpha\,\mathcal{M}_{Pl}\left(\overset{\circ}{\nabla}_{\theta}\overset{\circ}{\nabla}{}^{\theta}\overset{\circ}{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)==0 \tag{107}$$

So Eq. (107) still looks like a propagating equation.

Let's now take the dual of the gradient of Eq. (106).

$$-\frac{1}{3}\mathcal{M}_{Pl}{}^{3}\ \epsilon\overset{\circ}{g}{}^{\psi}{}_{\alpha\theta\kappa}\left(\overset{\circ}{\nabla}{}^{\kappa}\mathcal{B}^{\alpha\theta}\right)+\frac{16\,\alpha\,\mathcal{M}_{Pl}\ \overset{3}{\cdot}\mu\ \epsilon\overset{\circ}{g}{}^{\psi}{}_{\alpha\theta\omega}\left(\overset{\circ}{\nabla}{}^{\omega}\overset{\circ}{\nabla}_{\kappa}\overset{\circ}{\nabla}{}^{\kappa}\mathcal{B}^{\alpha\theta}\right)}{27-36\ \overset{3}{\cdot}\mu}==0 \tag{108}$$

So Eq. (108) still looks like a propagating equation.

Concrete relation to manuscript: Since we are unable to find non-propagating parts in the field equation, we conclude that the whole 2-form propagates, i.e. 6 propagating d.o.f in the linear spectrum. This agrees with Eq. (91).

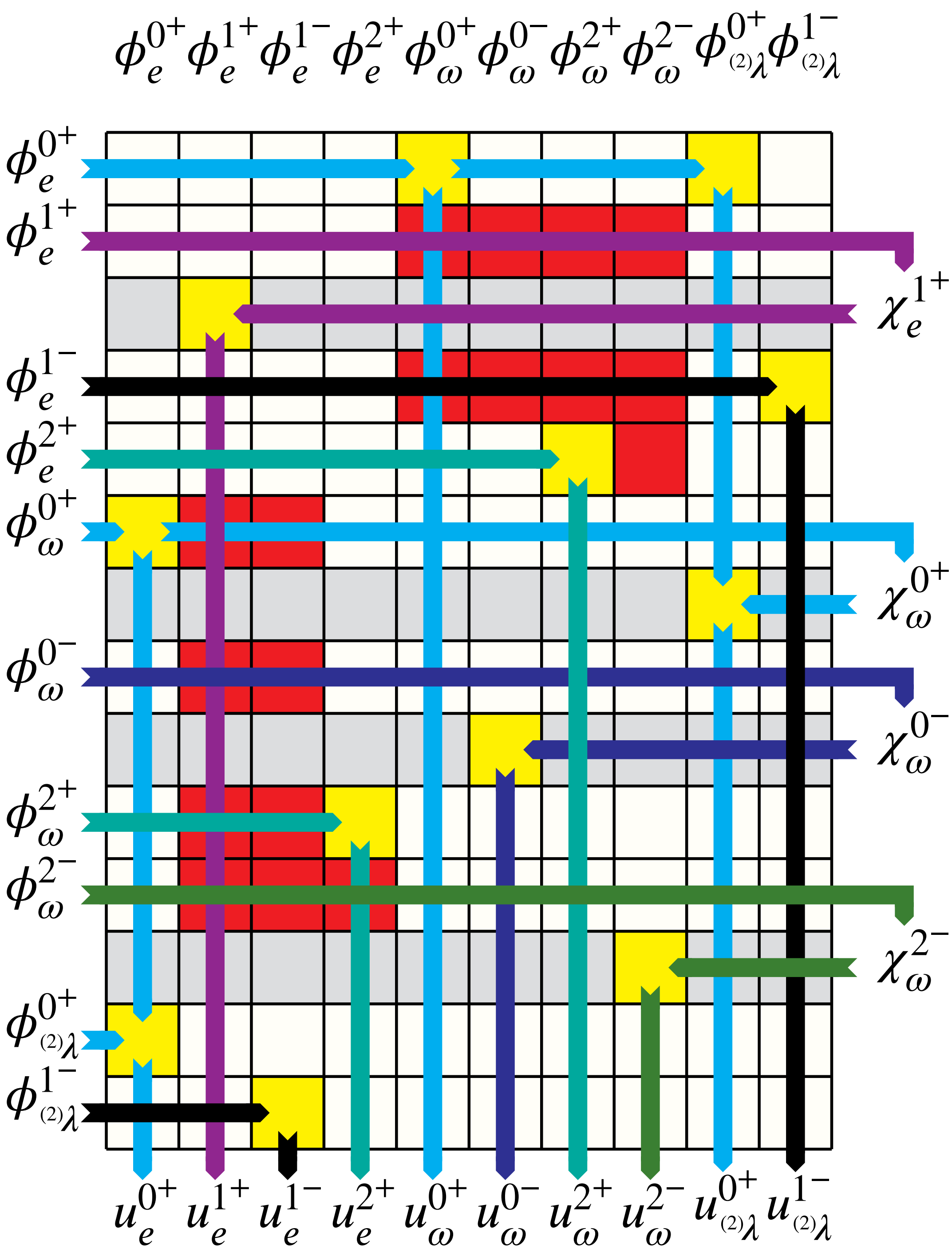
Column 2 of Eq. (89): Vector multiplier, but no axial mass

Setting up the Lagrangian

We define the Lagrangian. It contains vanishing axial mass parameter $\overset{3}{\cdot}\mu$ and a multiplier field $\overset{2}{\cdot}\lambda_{\mathcal{T}}{}^{\mu}$ to disable the vector torsion.

$$\alpha\ R[\nabla]_{\alpha\beta}\ R[\nabla]^{\alpha\beta}-\alpha\ R[\nabla]^{\alpha\beta}\ R[\nabla]_{\beta\alpha}-\frac{\mathcal{M}_{Pl}{}^{2}\ R[\nabla]}{2}+\overset{2}{\cdot}\mathcal{T}^{\alpha}\ \overset{2}{\cdot}\lambda_{\mathcal{T}\alpha} \tag{109}$$

Let's now look at the full nonlinear Hamiltonian analysis of Eq. (109).



According to this analysis we will have the following d.o.f in the linear spectrum. We compute this by adding the naive canonical d.o.f from the tetrad and the spin connection, and any multiplier fields, then subtracting the Poincaré and kinematic constraints, then subtracting the primaries and then the secondaries, and the result is then halved.

$$\frac{1}{2}\,(2\times 16+2\times 24+2\times 4-2\times 10-2\times 10-(1+3+3+5+1+1+5+5+1+3)-(3+1+1+5)) == 3+2 \tag{110}$$

Concrete relation to manuscript: So we expect 3 d.o.f in the linear spectrum, apart from the two polarisations of Einstein's graviton, from the Hamiltonian analysis. Non-linear analysis shows that 6 d.o.f exist, and so there are 3 strongly-coupled d.o.f.

Now we would like to have the post-Riemannian decomposition of the Lagrangian Eq. (109).

$$\begin{aligned}
&-\frac{\mathcal{M}_{Pl}^2\, R[\mathring{\nabla}]}{2}-\frac{1}{8}\mathcal{M}_{Pl}^2\,\mathcal{T}_{\alpha\alpha'\beta}\,\mathcal{T}^{\alpha\alpha'\beta}-\frac{1}{4}\mathcal{M}_{Pl}^2\,\mathcal{T}^{\alpha\alpha'\beta}\,\mathcal{T}_{\alpha'\alpha\beta}-\\
&\frac{1}{2}\mathcal{M}_{Pl}^2\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\,\mathcal{T}^{\beta}{}_{\alpha'\beta}+\frac{1}{4}\alpha\,\mathcal{T}_{\alpha}{}^{\beta'\chi}\,\mathcal{T}^{\alpha\alpha'\beta}\,\mathcal{T}_{\alpha'\beta}{}^{\chi'}\,\mathcal{T}_{\beta'\chi\chi'}+\\
&\frac{1}{4}\alpha\,\mathcal{T}^{\alpha\alpha'\beta}\,\mathcal{T}_{\alpha'\beta}{}^{\beta'}\,\mathcal{T}_{\beta'}{}^{\chi\chi'}\,\mathcal{T}_{\chi\alpha\chi'}-\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\,\mathcal{T}_{\alpha'}{}^{\beta\beta'}\,\mathcal{T}_{\beta}{}^{\chi\chi'}\,\mathcal{T}_{\chi\beta'\chi'}-\\
&\frac{1}{2}\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\,\mathcal{T}_{\alpha'}{}^{\beta\beta'}\,\mathcal{T}^{\chi}{}_{\beta\beta'}\,\mathcal{T}^{\chi'}{}_{\chi\chi'}-\mathcal{T}^{\alpha'}{}_{\alpha\alpha'}\;{}^{2}\lambda_{\mathcal{T}}{}^{\alpha}+\alpha\,\mathcal{T}^{\alpha\alpha'\beta}\,\mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\chi}{}_{\beta'\chi}\right)+\\
&\mathcal{M}_{Pl}^2\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)+\alpha\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\beta'}{}_{\beta\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)-\alpha\left(\mathring{\nabla}_{\beta}\mathcal{T}^{\beta'}{}_{\alpha'\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)+\\
&\frac{1}{2}\alpha\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right)+2\,\alpha\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right)-\alpha\,\mathcal{T}^{\alpha\alpha'\beta}\,\mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\chi}{}_{\alpha\chi}\right)-\\
&2\,\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\,\mathcal{T}_{\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\chi}{}_{\beta\chi}\right)+\alpha\,\mathcal{T}^{\alpha\alpha'\beta}\,\mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\mathring{\nabla}_{\chi}\mathcal{T}^{\chi}{}_{\alpha\beta'}\right)+\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\,\mathcal{T}_{\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}_{\chi}\mathcal{T}^{\chi}{}_{\beta\beta'}\right)
\end{aligned} \tag{111}$$

We want to study the theory when it is linearised. As an intermediate step in order to do this, we just keep in Eq. (111) the second-order terms in torsion and no higher. Also from this point onwards we completely neglect factors of the curvature which may arise in the field equations by commuting covariant derivatives.

$$\begin{aligned}
&-\frac{\mathcal{M}_{Pl}^2\, R[\mathring{\nabla}]}{2}-\frac{1}{8}\mathcal{M}_{Pl}^2\,\mathcal{T}_{\alpha\alpha'\beta}\,\mathcal{T}^{\alpha\alpha'\beta}-\frac{1}{4}\mathcal{M}_{Pl}^2\,\mathcal{T}^{\alpha\alpha'\beta}\,\mathcal{T}_{\alpha'\alpha\beta}-\\
&\frac{1}{2}\mathcal{M}_{Pl}^2\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\,\mathcal{T}^{\beta}{}_{\alpha'\beta}-\mathcal{T}^{\alpha'}{}_{\alpha\alpha'}\;{}^{2}\lambda_{\mathcal{T}}{}^{\alpha}+\mathcal{M}_{Pl}^2\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)+\alpha\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\beta'}{}_{\beta\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)-\\
&\alpha\left(\mathring{\nabla}_{\beta}\mathcal{T}^{\beta'}{}_{\alpha'\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)+\frac{1}{2}\alpha\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right)+2\,\alpha\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right)
\end{aligned} \tag{112}$$

Now we decompose the torsion in Eq. (112) into the Lorentz irreps.

$$-\frac{\mathcal{M}_{Pl}^2\, R[\mathring{\nabla}]}{2}-\frac{2}{9}\mathcal{M}_{Pl}^2\;{}^{1}\mathcal{T}_{\alpha\alpha'\beta}\;{}^{1}\mathcal{T}^{\alpha\alpha'\beta}+\frac{2}{9}\mathcal{M}_{Pl}^2\;{}^{1}\mathcal{T}_{\alpha\beta\alpha'}\;{}^{1}\mathcal{T}^{\alpha\alpha'\beta}+$$

$$\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\ \overset{2}{\cdot}\mathcal{T}_{\alpha}\ \overset{2}{\cdot}\mathcal{T}^{\alpha}-\frac{3}{4}\mathcal{M}_{\text{Pl}}^{2}\ \overset{3}{\cdot}\mathcal{T}_{\alpha}\ \overset{3}{\cdot}\mathcal{T}^{\alpha}+\overset{2}{\cdot}\mathcal{T}^{\alpha}\ \overset{2}{\cdot}\lambda_{\mathcal{T}\alpha}+\mathcal{M}_{\text{Pl}}^{2}\left(\mathring{\nabla}_{\alpha}\overset{2}{\cdot}\mathcal{T}^{\alpha}\right)-$$
$$\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\overset{2}{\cdot}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha'}\ \overset{2}{\cdot}\mathcal{T}^{\alpha}\right)+\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha'}\overset{2}{\cdot}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha'}\ \overset{2}{\cdot}\mathcal{T}^{\alpha}\right)+\alpha\left(\mathring{\nabla}_{\alpha}\overset{3}{\cdot}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha'}\ \overset{3}{\cdot}\mathcal{T}^{\alpha}\right)-$$
$$\alpha\left(\mathring{\nabla}_{\alpha'}\overset{3}{\cdot}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha'}\ \overset{3}{\cdot}\mathcal{T}^{\alpha}\right)+\frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha'}\ \overset{2}{\cdot}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}\overset{1}{\cdot}\mathcal{T}_{\alpha}{}^{\beta}{}_{\alpha'}\right)-\frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha'}\ \overset{2}{\cdot}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}\overset{1}{\cdot}\mathcal{T}_{\alpha'}{}^{\beta}{}_{\alpha}\right)-$$
$$\frac{4}{3}\alpha\ \epsilon\mathring{g}_{\alpha\alpha'\beta'\chi}\left(\mathring{\nabla}^{\alpha'}\ \overset{3}{\cdot}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}\overset{1}{\cdot}\mathcal{T}^{\beta\beta'\chi}\right)+\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}_{\alpha'}{}^{\beta'}{}_{\beta}\right)-$$
$$\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}\overset{1}{\cdot}\mathcal{T}_{\beta}{}^{\beta'}{}_{\alpha'}\right)-\frac{4}{3}\alpha\ \epsilon\mathring{g}_{\alpha\alpha'\beta\beta'}\left(\mathring{\nabla}^{\alpha'}\ \overset{2}{\cdot}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\beta'}\ \overset{3}{\cdot}\mathcal{T}^{\beta}\right)$$

Manipulating the field equations

Here is the tensor field equation.

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\ \overset{1}{\cdot}\mathcal{T}^{\iota}{}_{\theta\kappa}+\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\ \overset{1}{\cdot}\mathcal{T}^{\iota}{}_{\kappa\theta}-\frac{2}{9}\alpha\ \delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha}\overset{2}{\cdot}\mathcal{F}_{\theta}{}^{\alpha}\right)+$$
$$\frac{2}{9}\alpha\ \delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha}\overset{2}{\cdot}\mathcal{F}_{\kappa}{}^{\alpha}\right)-\frac{2}{9}\alpha\ \delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'}{}_{\theta}\right)+\frac{2}{9}\alpha\ \delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'}{}_{\kappa}\right)+$$
$$\frac{2}{9}\alpha\ \delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\theta}{}^{\alpha\alpha'}\right)-\frac{2}{9}\alpha\ \delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\kappa}{}^{\alpha\alpha'}\right)-\frac{1}{3}\alpha\ \epsilon\mathring{g}_{\theta\kappa\alpha\alpha'}\left(\mathring{\nabla}^{\iota}\overset{3}{\cdot}\mathcal{F}^{\alpha\alpha'}\right)-$$
$$\frac{4}{9}\alpha\left(\mathring{\nabla}^{\iota}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\right)+\frac{4}{9}\alpha\left(\mathring{\nabla}^{\iota}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\right)+\frac{1}{3}\alpha\left(\mathring{\nabla}_{\theta}\overset{2}{\cdot}\mathcal{F}^{\iota}{}_{\kappa}\right)-\frac{1}{6}\alpha\ \epsilon\mathring{g}^{\iota}{}_{\kappa\alpha\alpha'}\left(\mathring{\nabla}_{\theta}\overset{3}{\cdot}\mathcal{F}^{\alpha\alpha'}\right)-$$
$$\frac{2}{9}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\iota\alpha}{}_{\kappa}\right)+\frac{2}{9}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\kappa}{}^{\alpha\iota}\right)+\frac{1}{3}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\iota}\overset{2}{\cdot}\mathcal{T}_{\kappa}\right)-\frac{1}{3}\alpha\left(\mathring{\nabla}_{\kappa}\overset{2}{\cdot}\mathcal{F}^{\iota}{}_{\theta}\right)+$$
$$\frac{1}{6}\alpha\ \epsilon\mathring{g}^{\iota}{}_{\theta\alpha\alpha'}\left(\mathring{\nabla}_{\kappa}\overset{3}{\cdot}\mathcal{F}^{\alpha\alpha'}\right)+\frac{2}{9}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\iota\alpha}{}_{\theta}\right)-\frac{2}{9}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\theta}{}^{\alpha\iota}\right)-\frac{1}{3}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\iota}\overset{2}{\cdot}\mathcal{T}_{\theta}\right)==0 \tag{114}$$

There is some nuance to how we obtain Eq. (114), in that we can't just vary with respect to the tensor irrep. If we do that, then the resulting equation can have traces which are not true on-shell. It is safest to actually vary with respect to the whole torsion tensor and then project out the (traceless by construction) tensor irrep from that equation.

Now for the vector equation.

$$\frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\ \overset{2}{\cdot}\mathcal{T}^{\mu}+\overset{2}{\cdot}\lambda_{\mathcal{T}}{}^{\mu}+\frac{8}{9}\alpha\left(\mathring{\nabla}_{\alpha}\overset{2}{\cdot}\mathcal{F}^{\mu\alpha}\right)+\frac{8}{9}\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\mu}\right)-\frac{8}{9}\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\mu\alpha\alpha'}\right)==0 \tag{115}$$

And for the axial vector equation.

$$-\frac{3}{2}\mathcal{M}_{\text{Pl}}^{2}\ \overset{3}{\cdot}\mathcal{T}^{\mu}-2\alpha\left(\mathring{\nabla}_{\alpha}\overset{3}{\cdot}\mathcal{F}^{\mu\alpha}\right)+\frac{4}{3}\alpha\ \epsilon\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\left(\mathring{\nabla}^{\beta'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\right)==0 \tag{116}$$

Now we can also take the dual of Eq. (116), and so we write this out for completeness.

$$\frac{3}{2}\mathcal{M}_{\text{Pl}}^{2}\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\alpha'}\,{}^{3}_{\bullet}\mathcal{T}^{\alpha'}-2\,\alpha\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\mu}\left(\mathring{\nabla}_{\alpha'}\,{}^{3}_{\bullet}\mathcal{F}^{\alpha'\mu}\right)-\frac{4}{3}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha}\,{}^{1}_{\bullet}\mathcal{T}^{\beta\alpha'\chi}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha}\,{}^{1}_{\bullet}\mathcal{T}^{\chi\alpha'\beta}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\,{}^{1}_{\bullet}\mathcal{T}^{\alpha\alpha'\chi}\right)-\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\,{}^{1}_{\bullet}\mathcal{T}^{\chi\alpha'\alpha}\right)-\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha'}\,{}^{1}_{\bullet}\mathcal{T}^{\alpha\alpha'\beta}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha'}\,{}^{1}_{\bullet}\mathcal{T}^{\beta\alpha'\alpha}\right)==0$$

Finally we have the equation for the multiplier itself.

$${}^{2}_{\bullet}\mathcal{T}_{\mu}==0 \tag{118}$$

With the effective 2-form field in Eq. (14), we are ready to simplify the equations of motion.

Here is the vector equation Eq. (115).

$${}^{2}_{\bullet}\lambda_{\mathcal{T}}{}^{\mu}-\frac{8}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\mu\alpha}\right)==0 \tag{119}$$

Here is the axial vector equation Eq. (116).

$$-\frac{3}{2}\mathcal{M}_{\text{Pl}}^{2}\,{}^{3}_{\bullet}\mathcal{T}^{\mu}+\frac{2}{3}\,\alpha\,\mathcal{M}_{\text{Pl}}\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha\alpha'\beta}\left(\mathring{\nabla}^{\beta}\mathcal{B}^{\alpha\alpha'}\right)==0 \tag{120}$$

Again here is the dual part of Eq. (120)

$$\frac{3}{2}\mathcal{M}_{\text{Pl}}^{2}\,\epsilon\,\mathring{g}_{\iota\theta\kappa\alpha}\,{}^{3}_{\bullet}\mathcal{T}^{\alpha}-\frac{4}{3}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\iota}\mathcal{B}_{\theta\kappa}\right)+\frac{4}{3}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathcal{B}_{\iota\kappa}\right)-\frac{4}{3}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathcal{B}_{\iota\theta}\right)==0 \tag{121}$$

Note that Eqs. (119), and (120) allow us to solve for ${}^{3}_{\bullet}\mathcal{T}_{\mu}$ and ${}^{2}_{\bullet}\lambda_{\mathcal{T}}{}^{\mu}$ purely in terms of the 2-form.

And here is the tensor equation Eq. (114).

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\,{}^{1}_{\bullet}\mathcal{T}^{\iota}{}_{\theta\kappa}+\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\,{}^{1}_{\bullet}\mathcal{T}^{\iota}{}_{\kappa\theta}+\frac{2}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)-\frac{2}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{4}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}^{\iota}\mathcal{B}_{\theta\kappa}\right)-\frac{2}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathcal{B}^{\iota}{}_{\kappa}\right)+\frac{2}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathcal{B}^{\iota}{}_{\theta}\right)==0 \tag{122}$$

Now the next thing we do is to take the divergence of Eq. (122).

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\left(\mathring{\nabla}_{\alpha}\,{}^{1}_{\bullet}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\right)+\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\left(\mathring{\nabla}_{\alpha}\,{}^{1}_{\bullet}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\right)-\frac{4}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)+\frac{2}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{2}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right)-\frac{2}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{2}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)==0 \tag{123}$$

Next we substitute into Eq. (123) for the 2-form field in Eq. (14) again.

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{3}\,\mathcal{B}_{\theta\kappa}-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\,\overset{2}{\vdots}\mathcal{F}_{\theta\kappa}+\frac{1}{4}\mathcal{M}_{\text{Pl}}^{2}\,\epsilon\,\mathring{g}_{\theta\kappa\alpha\alpha'}\;\overset{3}{\vdots}\mathcal{F}^{\alpha\alpha'}-\frac{4}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,\mathcal{B}_{\theta\kappa}\right)+\frac{2}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{2}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right)-\frac{2}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{2}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)==0 \quad (124)$$

Next we will take Eq. (124) and expand the vector and axial Maxwell tensors back into derivatives, and substitute for the solutions in terms of the 2-form field that we obtained from Eqs. (119), and (120).

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{3}\,\mathcal{B}_{\theta\kappa}+\frac{4}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{4}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)==0 \quad (125)$$

So now the equation in Eq. (125) contains all the dynamical information about the linear spectrum of the theory Eq. (109), since Eqs. (119), (120), and (122) serve only to determine the torsion in terms of the 2-form. So the key question is how much of the 2-form does Eq. (125) propagate?

Let's first take the divergence of Eq. (125).

$$\frac{1}{3}\mathcal{M}_{\text{Pl}}^{3}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{4}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)==0 \quad (126)$$

So Eq. (126) still looks like a propagating equation.

Let's now take the dual of the gradient of Eq. (125).

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{3}\,\epsilon\,\mathring{g}^{\psi}{}_{\alpha\theta\kappa}\left(\mathring{\nabla}^{\kappa}\mathcal{B}^{\alpha\theta}\right)==0 \quad (127)$$

So Eq. (127) is just a constraint, which knocks out 3 d.o.f from the 2-form.

Concrete relation to manuscript: Since we lose 3 d.o.f to a constraint, we conclude that half the 2-form propagates, i.e. 3 propagating d.o.f in the linear spectrum. This agrees with Eq. (110).

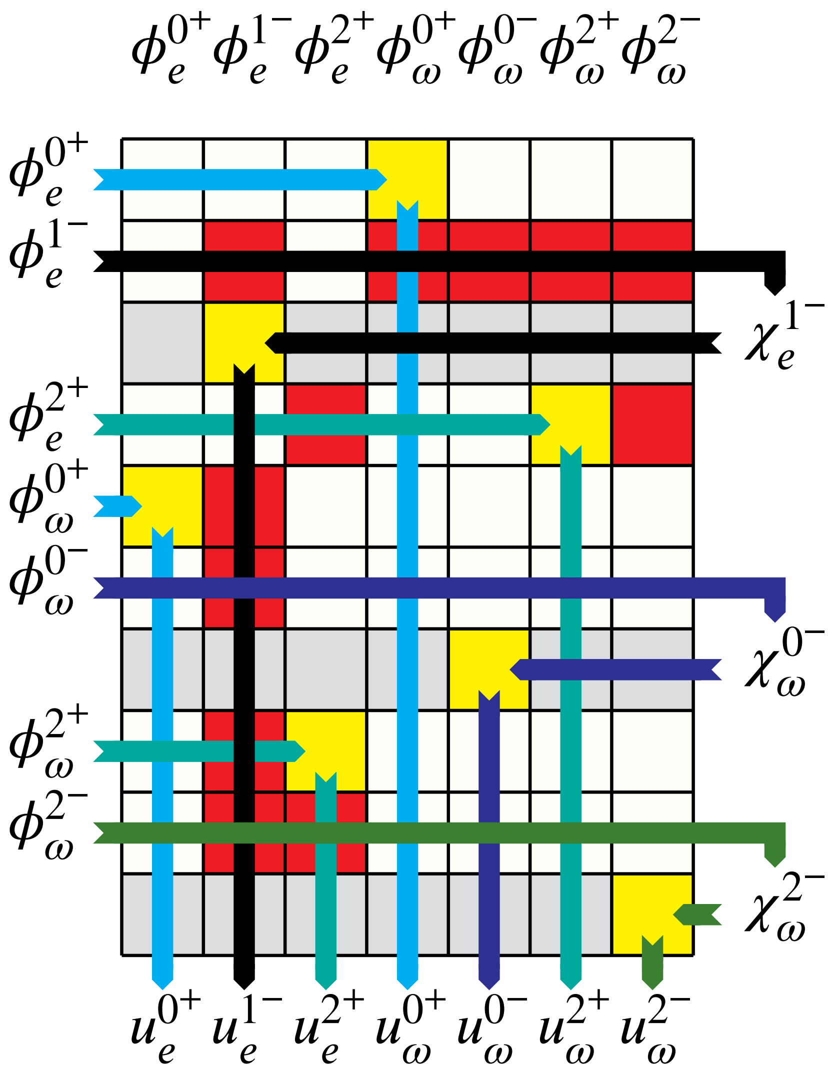
Column 3 of Eq. (89): Axial mass, but no vector multiplier

Setting up the Lagrangian

We define the Lagrangian. It contains nonvanishing axial mass parameter $\overset{3}{\vdots}\mu$ but it does not contain a multiplier field $\overset{2}{\vdots}\lambda_{\mathcal{T}}{}^{\mu}$, which would ordinarily disable the vector torsion.

$$\alpha\;\mathrm{R}[\nabla]_{\alpha\beta}\;\;\mathrm{R}[\nabla]^{\alpha\beta}-\alpha\;\mathrm{R}[\nabla]^{\alpha\beta}\;\;\mathrm{R}[\nabla]_{\beta\alpha}-\frac{\mathcal{M}_{\text{Pl}}^{2}\,\mathrm{R}[\nabla]}{2}+\mathcal{M}_{\text{Pl}}^{2}\;\overset{3}{\vdots}\mu\;\;\overset{3}{\vdots}\mathcal{T}_{\alpha}\;\;\overset{3}{\vdots}\mathcal{T}^{\alpha} \quad (128)$$

Let's now look at the full nonlinear Hamiltonian analysis of Eq. (128).



According to this analysis we will have the following d.o.f in the linear spectrum. We compute this by adding the naive canonical d.o.f from the tetrad and the spin connection, and any multiplier fields, then subtracting the Poincaré and kinematic constraints, then subtracting the primaries and then the secondaries, and the result is then halved.

$$\frac{1}{2}(2\times 16+2\times 24-2\times 10-2\times 10-(1+3+5+1+1+5+5)-(3+1+5)) == 3+2 \quad (129)$$

Concrete relation to manuscript: So we expect 3 d.o.f in the linear spectrum, apart from the two polarisations of Einstein's graviton, from the Hamiltonian analysis. Non-linear analysis shows that 6 d.o.f exist, and so there are 3 strongly-coupled d.o.f.

Now we would like to have the post-Riemannian decomposition of the Lagrangian Eq. (128).

$$-\frac{\mathcal{M}_{\text{Pl}}{}^2\, \mathrm{R}[\mathring{\nabla}]}{2}-\frac{1}{8}\mathcal{M}_{\text{Pl}}{}^2\,\mathcal{T}_{\alpha\beta\chi}\,\mathcal{T}^{\alpha\beta\chi}-\frac{1}{18}\mathcal{M}_{\text{Pl}}{}^2\,\overset{3}{\vdots}\mu\,\mathcal{T}_{\alpha\beta\chi}\,\mathcal{T}^{\alpha\beta\chi}-$$

$$\frac{1}{4}\mathcal{M}_{\text{Pl}}^{2}\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\alpha\chi}+\frac{1}{9}\mathcal{M}_{\text{Pl}}^{2}\,{}^{3}_{\bullet}\mu\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\alpha\chi}-\frac{1}{2}\mathcal{M}_{\text{Pl}}^{2}\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\,\mathcal{T}^{\chi}{}_{\beta\chi}+$$
$$\frac{1}{4}\alpha\,\mathcal{T}_{\alpha}{}^{\iota\theta}\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\chi}{}^{\kappa}\,\mathcal{T}_{\iota\theta\kappa}+\frac{1}{4}\alpha\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\chi}{}^{\iota}\,\mathcal{T}_{\iota}{}^{\theta\kappa}\,\mathcal{T}_{\theta\alpha\kappa}-$$
$$\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\,\mathcal{T}_{\beta}{}^{\chi\iota}\,\mathcal{T}_{\chi}{}^{\theta\kappa}\,\mathcal{T}_{\theta\iota\kappa}-\frac{1}{2}\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\,\mathcal{T}_{\beta}{}^{\chi\iota}\,\mathcal{T}^{\theta}{}_{\chi\iota}\,\mathcal{T}^{\kappa}{}_{\theta\kappa}+\alpha\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\chi}{}^{\iota}\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\theta}{}_{\iota\theta}\right)+$$
$$\mathcal{M}_{\text{Pl}}^{2}\left(\mathring{\nabla}_{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\right)+\alpha\left(\mathring{\nabla}_{\beta}\mathcal{T}^{\iota}{}_{\chi\iota}\right)\left(\mathring{\nabla}^{\chi}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\right)-\alpha\left(\mathring{\nabla}_{\chi}\mathcal{T}^{\iota}{}_{\beta\iota}\right)\left(\mathring{\nabla}^{\chi}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\right)+$$
$$\frac{1}{2}\alpha\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\alpha\beta\chi}\right)\left(\mathring{\nabla}_{\iota}\mathcal{T}^{\iota}{}_{\beta\chi}\right)+2\alpha\left(\mathring{\nabla}^{\chi}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\right)\left(\mathring{\nabla}_{\iota}\mathcal{T}^{\iota}{}_{\beta\chi}\right)-\alpha\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\chi}{}^{\iota}\left(\mathring{\nabla}_{\iota}\mathcal{T}^{\theta}{}_{\alpha\theta}\right)-$$
$$2\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\,\mathcal{T}_{\beta}{}^{\chi\iota}\left(\mathring{\nabla}_{\iota}\mathcal{T}^{\theta}{}_{\chi\theta}\right)+\alpha\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\chi}{}^{\iota}\left(\mathring{\nabla}_{\theta}\mathcal{T}^{\theta}{}_{\alpha\iota}\right)+\alpha\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\,\mathcal{T}_{\beta}{}^{\chi\iota}\left(\mathring{\nabla}_{\theta}\mathcal{T}^{\theta}{}_{\chi\iota}\right)$$

We want to study the theory when it is linearised. As an intermediate step in order to do this, we just keep in Eq. (130) the second-order terms in torsion and no higher. Also from this point onwards we completely neglect factors of the curvature which may arise in the field equations by commuting covariant derivatives.

$$-\frac{\mathcal{M}_{\text{Pl}}^{2}\,R[\mathring{\nabla}]}{2}-\frac{1}{8}\mathcal{M}_{\text{Pl}}^{2}\,\mathcal{T}_{\alpha\beta\chi}\,\mathcal{T}^{\alpha\beta\chi}-\frac{1}{18}\mathcal{M}_{\text{Pl}}^{2}\,{}^{3}_{\bullet}\mu\,\mathcal{T}_{\alpha\beta\chi}\,\mathcal{T}^{\alpha\beta\chi}-\frac{1}{4}\mathcal{M}_{\text{Pl}}^{2}\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\alpha\chi}+$$
$$\frac{1}{9}\mathcal{M}_{\text{Pl}}^{2}\,{}^{3}_{\bullet}\mu\,\mathcal{T}^{\alpha\beta\chi}\,\mathcal{T}_{\beta\alpha\chi}-\frac{1}{2}\mathcal{M}_{\text{Pl}}^{2}\,\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\,\mathcal{T}^{\chi}{}_{\beta\chi}+\mathcal{M}_{\text{Pl}}^{2}\left(\mathring{\nabla}_{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\right)+\alpha\left(\mathring{\nabla}_{\beta}\mathcal{T}^{\iota}{}_{\chi\iota}\right)\left(\mathring{\nabla}^{\chi}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\right)- \tag{131}$$
$$\alpha\left(\mathring{\nabla}_{\chi}\mathcal{T}^{\iota}{}_{\beta\iota}\right)\left(\mathring{\nabla}^{\chi}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\right)+\frac{1}{2}\alpha\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\alpha\beta\chi}\right)\left(\mathring{\nabla}_{\iota}\mathcal{T}^{\iota}{}_{\beta\chi}\right)+2\alpha\left(\mathring{\nabla}^{\chi}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\right)\left(\mathring{\nabla}_{\iota}\mathcal{T}^{\iota}{}_{\beta\chi}\right)$$

Now we decompose the torsion in Eq. (131) into the Lorentz irreps.

$$-\frac{\mathcal{M}_{\text{Pl}}^{2}\,R[\mathring{\nabla}]}{2}-\frac{2}{9}\mathcal{M}_{\text{Pl}}^{2}\,{}^{1}_{\bullet}\mathcal{T}_{\alpha\alpha'\beta}\,{}^{1}_{\bullet}\mathcal{T}^{\alpha\alpha'\beta}+\frac{2}{9}\mathcal{M}_{\text{Pl}}^{2}\,{}^{1}_{\bullet}\mathcal{T}_{\alpha\beta\alpha'}\,{}^{1}_{\bullet}\mathcal{T}^{\alpha\alpha'\beta}+$$
$$\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\,{}^{2}_{\bullet}\mathcal{T}_{\alpha}\,{}^{2}_{\bullet}\mathcal{T}^{\alpha}-\frac{3}{4}\mathcal{M}_{\text{Pl}}^{2}\,{}^{3}_{\bullet}\mathcal{T}_{\alpha}\,{}^{3}_{\bullet}\mathcal{T}^{\alpha}+\mathcal{M}_{\text{Pl}}^{2}\,{}^{3}_{\bullet}\mu\,{}^{3}_{\bullet}\mathcal{T}_{\alpha}\,{}^{3}_{\bullet}\mathcal{T}^{\alpha}+\mathcal{M}_{\text{Pl}}^{2}\left(\mathring{\nabla}_{\alpha}{}^{2}_{\bullet}\mathcal{T}^{\alpha}\right)-$$
$$\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}{}^{2}_{\bullet}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha'}{}^{2}_{\bullet}\mathcal{T}^{\alpha}\right)+\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha'}{}^{2}_{\bullet}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha'}{}^{2}_{\bullet}\mathcal{T}^{\alpha}\right)+\alpha\left(\mathring{\nabla}_{\alpha}{}^{3}_{\bullet}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha'}{}^{3}_{\bullet}\mathcal{T}^{\alpha}\right)-$$
$$\alpha\left(\mathring{\nabla}_{\alpha'}{}^{3}_{\bullet}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha'}{}^{3}_{\bullet}\mathcal{T}^{\alpha}\right)+\frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha'}{}^{2}_{\bullet}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}{}^{1}_{\bullet}\mathcal{T}_{\alpha}{}^{\beta}{}_{\alpha'}\right)-\frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha'}{}^{2}_{\bullet}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}{}^{1}_{\bullet}\mathcal{T}_{\alpha'}{}^{\beta}{}_{\alpha}\right)- \tag{132}$$
$$\frac{4}{3}\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\chi\iota}\left(\mathring{\nabla}^{\alpha'}{}^{3}_{\bullet}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}{}^{1}_{\bullet}\mathcal{T}^{\beta\chi\iota}\right)+\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}{}^{1}_{\bullet}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\chi}{}^{1}_{\bullet}\mathcal{T}_{\alpha'}{}^{\chi}{}_{\beta}\right)-$$
$$\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}{}^{1}_{\bullet}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\chi}{}^{1}_{\bullet}\mathcal{T}_{\beta}{}^{\chi}{}_{\alpha'}\right)-\frac{4}{3}\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\chi}\left(\mathring{\nabla}^{\alpha'}{}^{2}_{\bullet}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\chi}{}^{3}_{\bullet}\mathcal{T}^{\beta}\right)$$

Manipulating the field equations

Here is the tensor field equation.

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\,{}^{1}_{\bullet}\mathcal{T}^{\iota}{}_{\theta\kappa}+\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\,{}^{1}_{\bullet}\mathcal{T}^{\iota}{}_{\kappa\theta}-\frac{2}{9}\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha}{}^{2}_{\bullet}\mathcal{F}_{\theta}{}^{\alpha}\right)+$$

$$\begin{aligned}
&\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha}\overset{2}{\cdot}\mathcal{F}_{\kappa}{}^{\alpha}\right)-\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'}{}_{\theta}\right)+\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'}{}_{\kappa}\right)+\\
&\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\theta}{}^{\alpha\alpha'}\right)-\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\kappa}{}^{\alpha\alpha'}\right)-\frac{1}{3}\,\alpha\,\epsilon\,\mathring{g}_{\theta\kappa\alpha\alpha'}\left(\mathring{\nabla}^{\iota}\overset{3}{\cdot}\mathcal{F}^{\alpha\alpha'}\right)-\\
&\frac{4}{9}\,\alpha\left(\mathring{\nabla}^{\iota}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\right)+\frac{4}{9}\,\alpha\left(\mathring{\nabla}^{\iota}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\right)+\frac{1}{3}\,\alpha\left(\mathring{\nabla}_{\theta}\overset{2}{\cdot}\mathcal{F}^{\iota}{}_{\kappa}\right)-\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\kappa\alpha\alpha'}\left(\mathring{\nabla}_{\theta}\overset{3}{\cdot}\mathcal{F}^{\alpha\alpha'}\right)-\\
&\frac{2}{9}\,\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\iota\alpha}{}_{\kappa}\right)+\frac{2}{9}\,\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\kappa}{}^{\alpha\iota}\right)+\frac{1}{3}\,\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\iota}\overset{2}{\cdot}\mathcal{T}_{\kappa}\right)-\frac{1}{3}\,\alpha\left(\mathring{\nabla}_{\kappa}\overset{2}{\cdot}\mathcal{F}^{\iota}{}_{\theta}\right)+\\
&\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\theta\alpha\alpha'}\left(\mathring{\nabla}_{\kappa}\overset{3}{\cdot}\mathcal{F}^{\alpha\alpha'}\right)+\frac{2}{9}\,\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\iota\alpha}{}_{\theta}\right)-\frac{2}{9}\,\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}_{\theta}{}^{\alpha\iota}\right)-\frac{1}{3}\,\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\iota}\overset{2}{\cdot}\mathcal{T}_{\theta}\right)==0
\end{aligned}$$

There is some nuance to how we obtain Eq. (133), in that we can't just vary with respect to the tensor irrep. If we do that, then the resulting equation can have traces which are not true on-shell. It is safest to actually vary with respect to the whole torsion tensor and then project out the (traceless by construction) tensor irrep from that equation.

Now for the vector equation.

$$\frac{2}{3}\,\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{2}{\cdot}\mathcal{T}^{\mu}+\frac{8}{9}\,\alpha\left(\mathring{\nabla}_{\alpha}\overset{2}{\cdot}\mathcal{F}^{\mu\alpha}\right)+\frac{8}{9}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\mu}\right)-\frac{8}{9}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\mu\alpha\alpha'}\right)==0 \tag{134}$$

And for the axial vector equation.

$$-\frac{3}{2}\,\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{3}{\cdot}\mathcal{T}^{\mu}+2\,\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{3}{\cdot}\mu\,\overset{3}{\cdot}\mathcal{T}^{\mu}-2\,\alpha\left(\mathring{\nabla}_{\alpha}\overset{3}{\cdot}\mathcal{F}^{\mu\alpha}\right)+\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\left(\mathring{\nabla}^{\beta'}\mathring{\nabla}_{\alpha}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\right)==0 \tag{135}$$

Now we can also take the dual of Eq. (135), and so we write this out for completeness.

$$\begin{aligned}
&\frac{3}{2}\,\mathcal{M}_{\text{Pl}}{}^{2}\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\alpha'}\,\overset{3}{\cdot}\mathcal{T}^{\alpha'}-2\,\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{3}{\cdot}\mu\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\alpha'}\,\overset{3}{\cdot}\mathcal{T}^{\alpha'}-2\,\alpha\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\mu}\left(\mathring{\nabla}_{\alpha'}\overset{3}{\cdot}\mathcal{F}^{\alpha'\mu}\right)-\\
&\quad\frac{4}{3}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha}\overset{1}{\cdot}\mathcal{T}^{\beta\alpha'\chi}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha}\overset{1}{\cdot}\mathcal{T}^{\chi\alpha'\beta}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\chi}\right)-\\
&\quad\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\overset{1}{\cdot}\mathcal{T}^{\chi\alpha'\alpha}\right)-\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha'}\overset{1}{\cdot}\mathcal{T}^{\alpha\alpha'\beta}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha'}\overset{1}{\cdot}\mathcal{T}^{\beta\alpha'\alpha}\right)==0
\end{aligned} \tag{136}$$

With the effective 2-form field in Eq. (14), we are ready to simplify the equations of motion.

Here is the vector equation Eq. (134).

$$\frac{2}{3}\,\mathcal{M}_{\text{Pl}}{}^{2}\,\overset{2}{\cdot}\mathcal{T}^{\mu}-\frac{8}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\mu\alpha}\right)==0 \tag{137}$$

Here is the axial vector equation Eq. (135).

$$\frac{1}{2}\,\mathcal{M}_{\text{Pl}}{}^{2}\left(-3+4\,\overset{3}{\cdot}\mu\right)\overset{3}{\cdot}\mathcal{T}^{\mu}+\frac{2}{3}\,\alpha\,\mathcal{M}_{\text{Pl}}\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha\alpha'\beta}\left(\mathring{\nabla}^{\beta}\mathcal{B}^{\alpha\alpha'}\right)==0 \tag{138}$$

Again here is the dual part of Eq. (138)

$$\frac{1}{2}\mathcal{M}_{\text{Pl}}^{2}\left(3-4\,\overset{3}{:}\mu\right)\epsilon\,\mathring{g}_{\iota\theta\kappa\alpha}\;\overset{3}{:}\mathcal{T}^{\alpha}-\frac{4}{3}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\iota}\mathcal{B}_{\theta\kappa}\right)+\frac{4}{3}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathcal{B}_{\iota\kappa}\right)-\frac{4}{3}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathcal{B}_{\iota\theta}\right)==0 \qquad (139)$$

Note that Eqs. (137), and (138) allow us to solve for $\overset{3}{:}\mathcal{T}_{\mu}$ and $\overset{2}{:}\mathcal{T}^{\mu}$ purely in terms of the 2-form.

And here is the tensor equation Eq. (133).

$$\begin{aligned}&-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\;\overset{1}{:}\mathcal{T}^{\iota}{}_{\theta\kappa}+\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\;\overset{1}{:}\mathcal{T}^{\iota}{}_{\kappa\theta}+\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)-\\&\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{4}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}^{\iota}\mathcal{B}_{\theta\kappa}\right)-\frac{4}{9}\alpha\left(\mathring{\nabla}^{\iota}\,\overset{2}{:}\mathcal{F}_{\theta\kappa}\right)-\\&\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathcal{B}^{\iota}{}_{\kappa}\right)+\frac{5}{18}\alpha\left(\mathring{\nabla}_{\theta}\overset{2}{:}\mathcal{F}^{\iota}{}_{\kappa}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\iota}\,\overset{2}{:}\mathcal{T}_{\kappa}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}\overset{2}{:}\mathcal{T}^{\iota}\right)+\\&\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathcal{B}^{\iota}{}_{\theta}\right)-\frac{5}{18}\alpha\left(\mathring{\nabla}_{\kappa}\overset{2}{:}\mathcal{F}^{\iota}{}_{\theta}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\iota}\,\overset{2}{:}\mathcal{T}_{\theta}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\overset{2}{:}\mathcal{T}^{\iota}\right)==0\end{aligned} \qquad (140)$$

Now the next thing we do is to take the divergence of Eq. (140).

$$\begin{aligned}&-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\left(\mathring{\nabla}_{\alpha}\overset{1}{:}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\right)+\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\left(\mathring{\nabla}_{\alpha}\overset{1}{:}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\right)-\frac{4}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)-\\&\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,\overset{2}{:}\mathcal{F}_{\theta\kappa}\right)+\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\overset{2}{:}\mathcal{F}_{\kappa}{}^{\alpha}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha}\,\overset{2}{:}\mathcal{T}_{\kappa}\right)+\\&\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}\overset{2}{:}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right)+\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\overset{2}{:}\mathcal{F}_{\theta}{}^{\alpha}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha}\,\overset{2}{:}\mathcal{T}_{\theta}\right)-\\&\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\overset{2}{:}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)==0\end{aligned} \qquad (141)$$

Next we substitute into Eq. (141) for the 2-form field in Eq. (14) again.

$$\begin{aligned}&-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{3}\,\mathcal{B}_{\theta\kappa}-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\;\overset{2}{:}\mathcal{F}_{\theta\kappa}+\frac{1}{4}\mathcal{M}_{\text{Pl}}^{2}\,\epsilon\,\mathring{g}_{\theta\kappa\alpha\alpha'}\;\overset{3}{:}\mathcal{F}^{\alpha\alpha'}-\frac{4}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)-\\&\frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,\overset{2}{:}\mathcal{F}_{\theta\kappa}\right)+\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\overset{2}{:}\mathcal{F}_{\kappa}{}^{\alpha}\right)+\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha}\,\overset{2}{:}\mathcal{T}_{\kappa}\right)+\\&\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}\overset{2}{:}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right)+\frac{5}{18}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\overset{2}{:}\mathcal{F}_{\theta}{}^{\alpha}\right)-\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha}\,\overset{2}{:}\mathcal{T}_{\theta}\right)-\\&\frac{1}{6}\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\overset{2}{:}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{2}{9}\alpha\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)==0\end{aligned} \qquad (142)$$

Next we will take Eq. (142) and expand the vector and axial Maxwell tensors back into derivatives, and substitute for the solutions in terms of the 2-form field that we obtained from Eqs. (137), and (138).

$$-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{3}\,\mathcal{B}_{\theta\kappa}+\frac{16\,\alpha\mathcal{M}_{\text{Pl}}\,\overset{3}{:}\mu\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)}{27-36\,\overset{3}{:}\mu}+ \qquad (143)$$

$$\frac{16\,\alpha\,\mathcal{M}_{\text{Pl}}\,\overset{3}{:}\mu\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)}{27-36\,\overset{3}{:}\mu}+\frac{16\,\alpha\,\mathcal{M}_{\text{Pl}}\,\overset{3}{:}\mu\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)}{9\left(-3+4\,\overset{3}{:}\mu\right)}==0$$

So now the equation in Eq. (143) contains all the dynamical information about the linear spectrum of the theory Eq. (128), since Eqs. (137), (138), and (140) serve only to determine the torsion in terms of the 2-form. So the key question is how much of the 2-form does Eq. (143) propagate?

Let's first take the divergence of Eq. (143).

$$\frac{1}{3}\,\mathcal{M}_{\text{Pl}}{}^{3}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)==0 \tag{144}$$

So Eq. (144) is just a constraint, which knocks out 3 d.o.f from the 2-form.

Let's now take the dual of the gradient of Eq. (143).

$$-\frac{1}{3}\,\mathcal{M}_{\text{Pl}}{}^{3}\;\epsilon\,\mathring{g}^{\psi}{}_{\alpha\theta\kappa}\left(\mathring{\nabla}^{\kappa}\mathcal{B}^{\alpha\theta}\right)+\frac{16\,\alpha\,\mathcal{M}_{\text{Pl}}\,\overset{3}{:}\mu\;\epsilon\,\mathring{g}^{\psi}{}_{\alpha\theta\omega}\left(\mathring{\nabla}^{\omega}\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\kappa}\mathcal{B}^{\alpha\theta}\right)}{27-36\,\overset{3}{:}\mu}==0 \tag{145}$$

So Eq. (145) still looks like a propagating equation.

Concrete relation to manuscript: Since we lose 3 d.o.f to a constraint, we conclude that half the 2-form propagates, i.e. 3 propagating d.o.f in the linear spectrum. This agrees with Eq. (129).

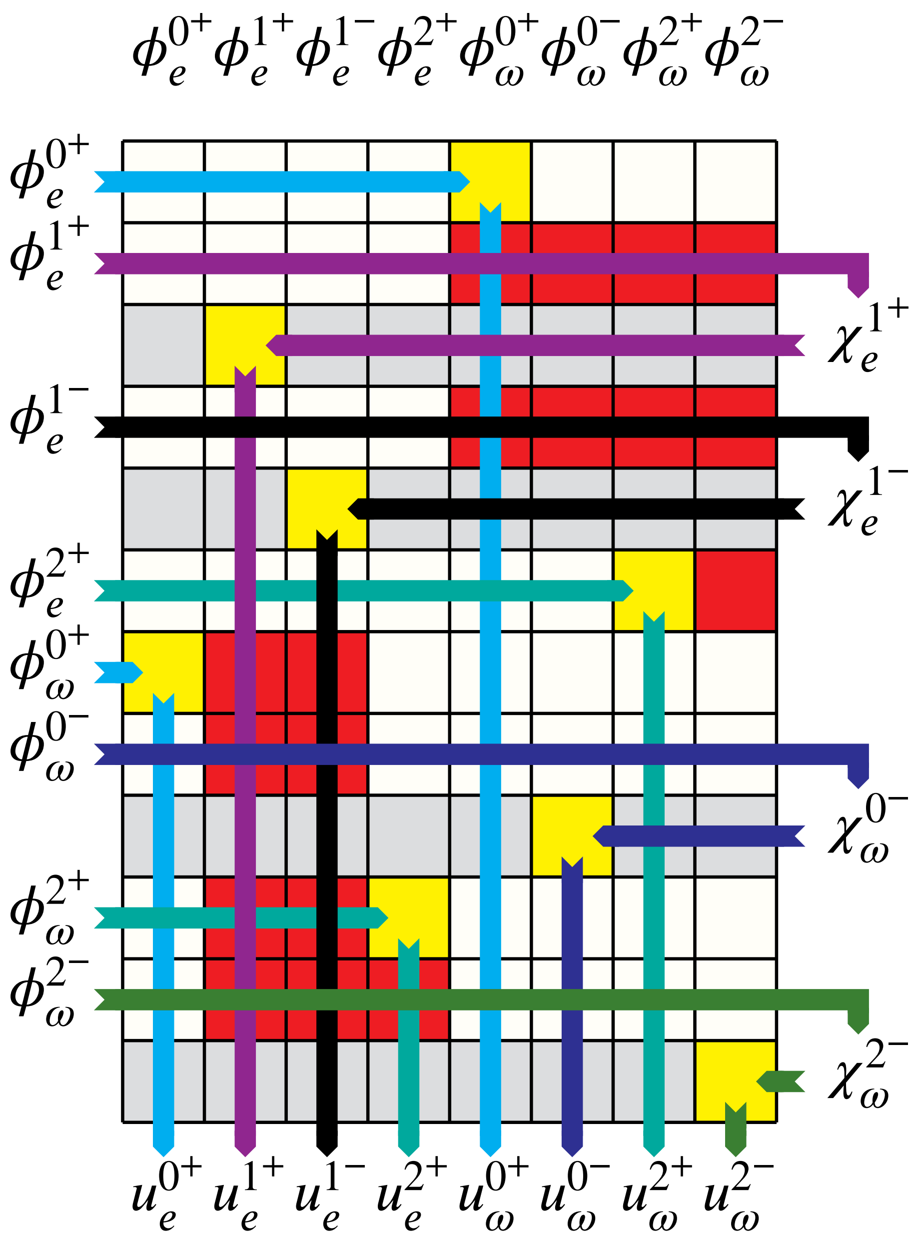
Column 4 of Eq. (89): No axial mass, no vector multiplier

Setting up the Lagrangian

We define the Lagrangian. It contains vanishing axial mass parameter $\overset{3}{:}\mu$ and it does not contain a multiplier field $\overset{2}{:}\lambda_{\mathcal{T}}{}^{\mu}$, which would ordinarily disable the vector torsion.

$$\alpha\;\text{R}[\nabla]_{\alpha\beta}\;\;\text{R}[\nabla]^{\alpha\beta}-\alpha\;\text{R}[\nabla]^{\alpha\beta}\;\;\text{R}[\nabla]_{\beta\alpha}-\frac{\mathcal{M}_{\text{Pl}}{}^{2}\;\text{R}[\nabla]}{2} \tag{146}$$

Let's now look at the full nonlinear Hamiltonian analysis of Eq. (146).



According to this analysis we will have the following d.o.f in the linear spectrum. We compute this by adding the naive canonical d.o.f from the tetrad and the spin connection, and any multiplier fields, then subtracting the Poincaré and kinematic constraints, then subtracting the primaries and then the secondaries, and the result is then halved.

$$\frac{1}{2}(2\times 16+2\times 24-2\times 10-2\times 10-(1+3+3+5+1+1+5+5)-(3+3+1+5)) == 2 \quad (147)$$

Concrete relation to manuscript: So we expect 0 d.o.f in the linear spectrum, apart from the two polarisations of Einstein's graviton, from the Hamiltonian analysis. The non-linear analysis confirms the existence of all 6 d.o.f, and so all 6 of them are strongly coupled.

Now we would like to have the post-Riemannian decomposition of the Lagrangian Eq. (146).

$$-\frac{\mathcal{M}_{\mathrm{Pl}}^2\, \mathrm{R}[\mathring{\nabla}]}{2} - \frac{1}{8}\mathcal{M}_{\mathrm{Pl}}^2\, \mathcal{T}_{\alpha\alpha'\beta}\, \mathcal{T}^{\alpha\alpha'\beta} - \frac{1}{4}\mathcal{M}_{\mathrm{Pl}}^2\, \mathcal{T}^{\alpha\alpha'\beta}\, \mathcal{T}_{\alpha'\alpha\beta} - \frac{1}{2}\mathcal{M}_{\mathrm{Pl}}^2\, \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\, \mathcal{T}^{\beta}{}_{\alpha'\beta} + \frac{1}{4}\alpha\, \mathcal{T}_{\alpha}{}^{\beta'\chi}\, \mathcal{T}^{\alpha\alpha'\beta}\, \mathcal{T}_{\alpha'\beta}{}^{\chi'}\, \mathcal{T}_{\beta'\chi\chi'} + \frac{1}{4}\alpha\, \mathcal{T}^{\alpha\alpha'\beta}\, \mathcal{T}_{\alpha'\beta}{}^{\beta'}\, \mathcal{T}_{\beta'}{}^{\chi\chi'}\, \mathcal{T}_{\chi\alpha\chi'} - \alpha\, \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\, \mathcal{T}_{\alpha'}{}^{\beta\beta'}\, \mathcal{T}_{\beta}{}^{\chi\chi'}\, \mathcal{T}_{\chi\beta'\chi'} - \frac{1}{2}\alpha\, \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\, \mathcal{T}_{\alpha'}{}^{\beta\beta'}\, \mathcal{T}^{\chi}{}_{\beta\beta'}\, \mathcal{T}^{\chi'}{}_{\chi\chi'} + \alpha\, \mathcal{T}^{\alpha\alpha'\beta}\, \mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\chi}{}_{\beta'\chi}\right) + \mathcal{M}_{\mathrm{Pl}}^2\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right) + \alpha\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\beta'}{}_{\beta\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right) - \alpha\left(\mathring{\nabla}_{\beta}\mathcal{T}^{\beta'}{}_{\alpha'\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right) + \frac{1}{2}\alpha\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right) + 2\,\alpha\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right) - \alpha\, \mathcal{T}^{\alpha\alpha'\beta}\, \mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\chi}{}_{\alpha\chi}\right) - 2\,\alpha\, \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\, \mathcal{T}_{\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\chi}{}_{\beta\chi}\right) + \alpha\, \mathcal{T}^{\alpha\alpha'\beta}\, \mathcal{T}_{\alpha'\beta}{}^{\beta'}\left(\mathring{\nabla}_{\chi}\mathcal{T}^{\chi}{}_{\alpha\beta'}\right) + \alpha\, \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\, \mathcal{T}_{\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}_{\chi}\mathcal{T}^{\chi}{}_{\beta\beta'}\right) \tag{148}$$

We want to study the theory when it is linearised. As an intermediate step in order to do this, we just keep in Eq. (148) the second-order terms in torsion and no higher. Also from this point onwards we completely neglect factors of the curvature which may arise in the field equations by commuting covariant derivatives.

$$-\frac{\mathcal{M}_{\mathrm{Pl}}^2\, \mathrm{R}[\mathring{\nabla}]}{2} - \frac{1}{8}\mathcal{M}_{\mathrm{Pl}}^2\, \mathcal{T}_{\alpha\alpha'\beta}\, \mathcal{T}^{\alpha\alpha'\beta} - \frac{1}{4}\mathcal{M}_{\mathrm{Pl}}^2\, \mathcal{T}^{\alpha\alpha'\beta}\, \mathcal{T}_{\alpha'\alpha\beta} - \frac{1}{2}\mathcal{M}_{\mathrm{Pl}}^2\, \mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\, \mathcal{T}^{\beta}{}_{\alpha'\beta} + \mathcal{M}_{\mathrm{Pl}}^2\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right) + \alpha\left(\mathring{\nabla}_{\alpha'}\mathcal{T}^{\beta'}{}_{\beta\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right) - \alpha\left(\mathring{\nabla}_{\beta}\mathcal{T}^{\beta'}{}_{\alpha'\beta'}\right)\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right) + \frac{1}{2}\alpha\left(\mathring{\nabla}_{\alpha}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right) + 2\,\alpha\left(\mathring{\nabla}^{\beta}\mathcal{T}^{\alpha}{}_{\alpha}{}^{\alpha'}\right)\left(\mathring{\nabla}_{\beta'}\mathcal{T}^{\beta'}{}_{\alpha'\beta}\right) \tag{149}$$

Now we decompose the torsion in Eq. (149) into the Lorentz irreps.

$$-\frac{\mathcal{M}_{\mathrm{Pl}}^2\, \mathrm{R}[\mathring{\nabla}]}{2} - \frac{2}{9}\mathcal{M}_{\mathrm{Pl}}^2\, {}^{1}\mathcal{T}_{\alpha\alpha'\beta}\, {}^{1}\mathcal{T}^{\alpha\alpha'\beta} + \frac{2}{9}\mathcal{M}_{\mathrm{Pl}}^2\, {}^{1}\mathcal{T}_{\alpha\beta\alpha'}\, {}^{1}\mathcal{T}^{\alpha\alpha'\beta} + \frac{1}{3}\mathcal{M}_{\mathrm{Pl}}^2\, {}^{2}\mathcal{T}_{\alpha}\, {}^{2}\mathcal{T}^{\alpha} - \frac{3}{4}\mathcal{M}_{\mathrm{Pl}}^2\, {}^{3}\mathcal{T}_{\alpha}\, {}^{3}\mathcal{T}^{\alpha} + \mathcal{M}_{\mathrm{Pl}}^2\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}^{\alpha}\right) - \frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right) + \frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha'}{}^{2}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right) + \alpha\left(\mathring{\nabla}_{\alpha}{}^{3}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right) - \alpha\left(\mathring{\nabla}_{\alpha'}{}^{3}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right) + \frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}{}^{1}\mathcal{T}_{\alpha}{}^{\beta}{}_{\alpha'}\right) - \frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}{}^{1}\mathcal{T}_{\alpha'}{}^{\beta}{}_{\alpha}\right) - \frac{4}{3}\alpha\, \epsilon\, \mathring{g}_{\alpha\alpha'\beta'\chi}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\beta}{}^{1}\mathcal{T}^{\beta\beta'\chi}\right) + \frac{4}{9}\alpha\left(\mathring{\nabla}_{\alpha}{}^{1}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}{}^{1}\mathcal{T}_{\alpha'}{}^{\beta'}{}_{\beta}\right) - \tag{150}$$

$$\frac{4}{9}\,\alpha\left(\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}^{\alpha\alpha'\beta}\right)\left(\mathring{\nabla}_{\beta'}\,{}^{1}\mathcal{T}_{\beta}{}^{\beta'}{}_{\alpha'}\right)-\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\beta'}\,{}^{3}\mathcal{T}^{\beta}\right)$$

Manipulating the field equations

Here is the tensor field equation.

$$\begin{aligned}
&-\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\,{}^{1}\mathcal{T}^{\iota}{}_{\theta\kappa}+\frac{1}{3}\mathcal{M}_{\text{Pl}}^{2}\,{}^{1}\mathcal{T}^{\iota}{}_{\kappa\theta}-\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha}\,{}^{2}\mathcal{F}_{\theta}{}^{\alpha}\right)+\\
&\quad\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha}\,{}^{2}\mathcal{F}_{\kappa}{}^{\alpha}\right)-\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}^{\alpha\alpha'}{}_{\theta}\right)+\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}^{\alpha\alpha'}{}_{\kappa}\right)+\\
&\quad\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha\alpha'}\right)-\frac{2}{9}\,\alpha\,\delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha\alpha'}\right)-\frac{1}{3}\,\alpha\,\epsilon\,\mathring{g}_{\theta\kappa\alpha\alpha'}\left(\mathring{\nabla}^{\iota}\,{}^{3}\mathcal{F}^{\alpha\alpha'}\right)-\\
&\quad\frac{4}{9}\,\alpha\left(\mathring{\nabla}^{\iota}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\right)+\frac{4}{9}\,\alpha\left(\mathring{\nabla}^{\iota}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\right)+\frac{1}{3}\,\alpha\left(\mathring{\nabla}_{\theta}\,{}^{2}\mathcal{F}^{\iota}{}_{\kappa}\right)-\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\kappa\alpha\alpha'}\left(\mathring{\nabla}_{\theta}\,{}^{3}\mathcal{F}^{\alpha\alpha'}\right)-\\
&\quad\frac{2}{9}\,\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}^{\iota\alpha}{}_{\kappa}\right)+\frac{2}{9}\,\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}_{\kappa}{}^{\alpha\iota}\right)+\frac{1}{3}\,\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\iota}\,{}^{2}\mathcal{T}_{\kappa}\right)-\frac{1}{3}\,\alpha\left(\mathring{\nabla}_{\kappa}\,{}^{2}\mathcal{F}^{\iota}{}_{\theta}\right)+\\
&\quad\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}^{\iota}{}_{\theta\alpha\alpha'}\left(\mathring{\nabla}_{\kappa}\,{}^{3}\mathcal{F}^{\alpha\alpha'}\right)+\frac{2}{9}\,\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}^{\iota\alpha}{}_{\theta}\right)-\frac{2}{9}\,\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}_{\theta}{}^{\alpha\iota}\right)-\frac{1}{3}\,\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\iota}\,{}^{2}\mathcal{T}_{\theta}\right)==0
\end{aligned}\tag{151}$$

There is some nuance to how we obtain Eq. (151), in that we can't just vary with respect to the tensor irrep. If we do that, then the resulting equation can have traces which are not true on-shell. It is safest to actually vary with respect to the whole torsion tensor and then project out the (traceless by construction) tensor irrep from that equation.

Now for the vector equation.

$$\frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\,{}^{2}\mathcal{T}^{\mu}+\frac{8}{9}\,\alpha\left(\mathring{\nabla}_{\alpha}\,{}^{2}\mathcal{F}^{\mu\alpha}\right)+\frac{8}{9}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}^{\alpha\alpha'\mu}\right)-\frac{8}{9}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}^{\mu\alpha\alpha'}\right)==0\tag{152}$$

And for the axial vector equation.

$$-\frac{3}{2}\mathcal{M}_{\text{Pl}}^{2}\,{}^{3}\mathcal{T}^{\mu}-2\,\alpha\left(\mathring{\nabla}_{\alpha}\,{}^{3}\mathcal{F}^{\mu\alpha}\right)+\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\left(\mathring{\nabla}^{\beta'}\mathring{\nabla}_{\alpha}\,{}^{1}\mathcal{T}^{\alpha\alpha'\beta}\right)==0\tag{153}$$

Now we can also take the dual of Eq. (153), and so we write this out for completeness.

$$\begin{aligned}
&\frac{3}{2}\mathcal{M}_{\text{Pl}}^{2}\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\alpha'}\,{}^{3}\mathcal{T}^{\alpha'}-2\,\alpha\,\epsilon\,\mathring{g}^{\alpha\beta\chi}{}_{\mu}\left(\mathring{\nabla}_{\alpha'}\,{}^{3}\mathcal{F}^{\alpha'\mu}\right)-\\
&\quad\frac{4}{3}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha}\,{}^{1}\mathcal{T}^{\beta\alpha'\chi}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}^{\alpha}\,{}^{1}\mathcal{T}^{\chi\alpha'\beta}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}^{\alpha\alpha'\chi}\right)-\\
&\quad\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}^{\chi\alpha'\alpha}\right)-\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}^{\alpha\alpha'\beta}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha'}\,{}^{1}\mathcal{T}^{\beta\alpha'\alpha}\right)==0
\end{aligned}\tag{154}$$

With the effective 2-form field in Eq. (14), we are ready to simplify the equations of motion.

Here is the vector equation Eq. (152).

$$\frac{2}{3}\mathcal{M}_{Pl}{}^{2}\ \overset{2}{\cdot}\mathcal{T}^{\mu}-\frac{8}{9}\,\alpha\,\mathcal{M}_{Pl}\left(\mathring{\nabla}_{\alpha}\mathcal{B}^{\mu\alpha}\right)==0 \tag{155}$$

Here is the axial vector equation Eq. (153).

$$-\frac{3}{2}\mathcal{M}_{Pl}{}^{2}\ \overset{3}{\cdot}\mathcal{T}^{\mu}+\frac{2}{3}\,\alpha\,\mathcal{M}_{Pl}\ \epsilon\,\mathring{g}^{\mu}{}_{\alpha\alpha'\beta}\left(\mathring{\nabla}^{\beta}\mathcal{B}^{\alpha\alpha'}\right)==0 \tag{156}$$

Again here is the dual part of Eq. (156)

$$\frac{3}{2}\mathcal{M}_{Pl}{}^{2}\ \epsilon\,\mathring{g}_{\iota\theta\kappa\alpha}\ \overset{3}{\cdot}\mathcal{T}^{\alpha}-\frac{4}{3}\,\alpha\,\mathcal{M}_{Pl}\left(\mathring{\nabla}_{\iota}\mathcal{B}_{\theta\kappa}\right)+\frac{4}{3}\,\alpha\,\mathcal{M}_{Pl}\left(\mathring{\nabla}_{\theta}\mathcal{B}_{\iota\kappa}\right)-\frac{4}{3}\,\alpha\,\mathcal{M}_{Pl}\left(\mathring{\nabla}_{\kappa}\mathcal{B}_{\iota\theta}\right)==0 \tag{157}$$

Note that Eqs. (155), and (156) allow us to solve for $\overset{3}{\cdot}\mathcal{T}_{\mu}$ and $\overset{2}{\cdot}\mathcal{T}^{\mu}$ purely in terms of the 2-form.

And here is the tensor equation Eq. (151).

$$\begin{aligned}
&-\frac{1}{3}\mathcal{M}_{Pl}{}^{2}\ \overset{1}{\cdot}\mathcal{T}^{\iota}{}_{\theta\kappa}+\frac{1}{3}\mathcal{M}_{Pl}{}^{2}\ \overset{1}{\cdot}\mathcal{T}^{\iota}{}_{\kappa\theta}+\frac{2}{9}\,\alpha\,\mathcal{M}_{Pl}\ \delta^{\iota}{}_{\kappa}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)-\\
&\quad\frac{2}{9}\,\alpha\,\mathcal{M}_{Pl}\ \delta^{\iota}{}_{\theta}\left(\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{4}{9}\,\alpha\,\mathcal{M}_{Pl}\left(\mathring{\nabla}^{\iota}\mathcal{B}_{\theta\kappa}\right)-\frac{4}{9}\,\alpha\left(\mathring{\nabla}^{\iota}\,\overset{2}{\cdot}\mathcal{F}_{\theta\kappa}\right)-\\
&\quad\frac{2}{9}\,\alpha\,\mathcal{M}_{Pl}\left(\mathring{\nabla}_{\theta}\mathcal{B}^{\iota}{}_{\kappa}\right)+\frac{5}{18}\,\alpha\left(\mathring{\nabla}_{\theta}\,\overset{2}{\cdot}\mathcal{F}^{\iota}{}_{\kappa}\right)+\frac{1}{6}\,\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\iota}\,\overset{2}{\cdot}\mathcal{T}_{\kappa}\right)+\frac{1}{6}\,\alpha\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}\,\overset{2}{\cdot}\mathcal{T}^{\iota}\right)+\\
&\quad\frac{2}{9}\,\alpha\,\mathcal{M}_{Pl}\left(\mathring{\nabla}_{\kappa}\mathcal{B}^{\iota}{}_{\theta}\right)-\frac{5}{18}\,\alpha\left(\mathring{\nabla}_{\kappa}\,\overset{2}{\cdot}\mathcal{F}^{\iota}{}_{\theta}\right)-\frac{1}{6}\,\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\iota}\,\overset{2}{\cdot}\mathcal{T}_{\theta}\right)-\frac{1}{6}\,\alpha\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\,\overset{2}{\cdot}\mathcal{T}^{\iota}\right)==0
\end{aligned} \tag{158}$$

Now the next thing we do is to take the divergence of Eq. (158).

$$\begin{aligned}
&-\frac{1}{3}\mathcal{M}_{Pl}{}^{2}\left(\mathring{\nabla}_{\alpha}\,\overset{1}{\cdot}\mathcal{T}_{\theta}{}^{\alpha}{}_{\kappa}\right)+\frac{1}{3}\mathcal{M}_{Pl}{}^{2}\left(\mathring{\nabla}_{\alpha}\,\overset{1}{\cdot}\mathcal{T}_{\kappa}{}^{\alpha}{}_{\theta}\right)-\frac{4}{9}\,\alpha\,\mathcal{M}_{Pl}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)-\\
&\quad\frac{4}{9}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,\overset{2}{\cdot}\mathcal{F}_{\theta\kappa}\right)+\frac{2}{9}\,\alpha\,\mathcal{M}_{Pl}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{5}{18}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\,\overset{2}{\cdot}\mathcal{F}_{\kappa}{}^{\alpha}\right)+\frac{1}{6}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha}\,\overset{2}{\cdot}\mathcal{T}_{\kappa}\right)+\\
&\quad\frac{1}{6}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}\,\overset{2}{\cdot}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\,\alpha\,\mathcal{M}_{Pl}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right)+\frac{5}{18}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\,\overset{2}{\cdot}\mathcal{F}_{\theta}{}^{\alpha}\right)-\frac{1}{6}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha}\,\overset{2}{\cdot}\mathcal{T}_{\theta}\right)-\\
&\quad\frac{1}{6}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\,\overset{2}{\cdot}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\,\alpha\,\mathcal{M}_{Pl}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{2}{9}\,\alpha\,\mathcal{M}_{Pl}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)==0
\end{aligned} \tag{159}$$

Next we substitute into Eq. (159) for the 2-form field in Eq. (14) again.

$$\begin{aligned}
&-\frac{1}{3}\mathcal{M}_{Pl}{}^{3}\ \mathcal{B}_{\theta\kappa}-\frac{1}{3}\mathcal{M}_{Pl}{}^{2}\ \overset{2}{\cdot}\mathcal{F}_{\theta\kappa}+\frac{1}{4}\mathcal{M}_{Pl}{}^{2}\ \epsilon\,\mathring{g}_{\theta\kappa\alpha\alpha'}\ \overset{3}{\cdot}\mathcal{F}^{\alpha\alpha'}-\frac{4}{9}\,\alpha\,\mathcal{M}_{Pl}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\mathcal{B}_{\theta\kappa}\right)-\\
&\quad\frac{4}{9}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,\overset{2}{\cdot}\mathcal{F}_{\theta\kappa}\right)+\frac{2}{9}\,\alpha\,\mathcal{M}_{Pl}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathcal{B}_{\kappa}{}^{\alpha}\right)-\frac{5}{18}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\,\overset{2}{\cdot}\mathcal{F}_{\kappa}{}^{\alpha}\right)+\frac{1}{6}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha}\,\overset{2}{\cdot}\mathcal{T}_{\kappa}\right)+\\
&\quad\frac{1}{6}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\theta}\mathring{\nabla}_{\kappa}\,\overset{2}{\cdot}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\,\alpha\,\mathcal{M}_{Pl}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathcal{B}_{\theta}{}^{\alpha}\right)+\frac{5}{18}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\,\overset{2}{\cdot}\mathcal{F}_{\theta}{}^{\alpha}\right)-\frac{1}{6}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\alpha}\,\overset{2}{\cdot}\mathcal{T}_{\theta}\right)-
\end{aligned} \tag{160}$$

$$\frac{1}{6}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\theta}\overset{2}{\cdot}\mathcal{T}^{\alpha}\right)-\frac{2}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\mathcal{B}_{\kappa}{}^{\alpha}\right)+\frac{2}{9}\,\alpha\,\mathcal{M}_{\text{Pl}}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\mathcal{B}_{\theta}{}^{\alpha}\right)==0$$

Next we will take Eq. (160) and expand the vector and axial Maxwell tensors back into derivatives, and substitute for the solutions in terms of the 2-form field that we obtained from Eqs. (155), and (156).

$$-\frac{1}{3}\,\mathcal{M}_{\text{Pl}}{}^{3}\,\mathcal{B}_{\theta\kappa}==0 \qquad (161)$$

So now the equation in Eq. (161) contains all the dynamical information about the linear spectrum of the theory Eq. (146), since Eqs. (155), (156), and (158) serve only to determine the torsion in terms of the 2-form. But clearly this tells us that none of the 2-form in Eq. (161) is propagating!

Concrete relation to manuscript: Since we lose all 6 d.o.f to a constraint, we conclude that there are 0 propagating d.o.f (besides the graviton) in the linear spectrum. This agrees with Eq. (147).

Concrete relation to manuscript: This concludes our analysis.