In[◦]:=
```
Get@FileNameJoin@{NotebookDirectory[], "SupplementalMaterials.m"};
```

```
------------------------------------------------------------
Package xAct`xPerm`  version 1.2.3, {2015, 8, 23}
CopyRight (C) 2003-2020, Jose M. Martin-Garcia, under the General Public License.
Connecting to external linux executable...
Connection established.
------------------------------------------------------------
Package xAct`xTensor`  version 1.2.0, {2021, 10, 17}
CopyRight (C) 2002-2021, Jose M. Martin-Garcia, under the General Public License.
------------------------------------------------------------
Package xAct`xPlain` version 1.0.0-developer, {2023, 5, 24}
CopyRight © 2023, Will E. V. Barker and Sebastian Zell, under the General Public License.
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------------------------------------------------------------
```

Supplemental materials: field equations

Introduction

How to use this document

These calculations are designed to accompany our manuscript in the form of supplemental materials, for the sake of reproducibility. Throughout, commentary by the authors takes the form of green text. Citations, where needed, will be managed by direct reference to arXiv numbers, and all such references are already provided in full within the body of our manuscript. One exception is the source referred to throughout as `Blagojević'; this pertains to the book `Gravitation and Gauge Symmetries', which is also referenced within the manuscript.

Concrete relation to manuscript: In boxes like this, we will make specific connections between a result which is obtained in the supplemental material and a claim which is made in the manuscript. These points of contact are not always numbered equations, they could be textual.

Note that a programmatical session in the Wolfram language does not really correspond to the clean flow of thoughts in a LaTeX document: there are differences that can't (and shouldn't) be ignored. Thus, whilst this document should be at least readable in standalone format, the reader is encouraged to follow it in tandem with the Wolfram language source codes, so as to avoid ambiguities. In this way, the specific operations and manipulations of quantities will become absolutely clear.

Loading HiGGS and GeoHiGGS

For these calculations, we will use the HiGGS and GeoHiGGS packages. Note that GeoHiGGS was not developed for public release, and so is not documented. The versions of HiGGS and GeoHiGGS used for the computations here are both developer-only, and so we include copies of the sources with these supplemental materials.

```
------------------------------------------------------------

Package xAct`xPert`  version 1.0.6, {2018, 2, 28}

CopyRight (C) 2005-2020, David Brizuela, Jose M. Martin-Garcia
  and Guillermo A. Mena Marugan, under the General Public License.

** Variable $PrePrint assigned value ScreenDollarIndices

** Variable $CovDFormat changed from Prefix to Postfix

** Option AllowUpperDerivatives of ContractMetric changed from False to True

** Option MetricOn of MakeRule changed from None to All

** Option ContractMetrics of MakeRule changed from False to True

------------------------------------------------------------

Package xAct`Invar`  version 2.0.5, {2013, 7, 1}

CopyRight (C) 2006-2020, J. M. Martin-Garcia,
  D. Yllanes and R. Portugal, under the General Public License.

** DefConstantSymbol: Defining constant symbol sigma.

** DefConstantSymbol: Defining constant symbol dim.

** Option CurvatureRelations of DefCovD changed from True to False

** Variable $CommuteCovDsOnScalars changed from True to False

------------------------------------------------------------

Package xAct`xCoba`  version 0.8.6, {2021, 2, 28}

CopyRight (C) 2005-2021, David Yllanes and
  Jose M. Martin-Garcia, under the General Public License.

------------------------------------------------------------

Package xAct`SymManipulator`  version 0.9.5, {2021, 9, 14}

CopyRight (C) 2011-2021, Thomas Bäckdahl, under the General Public License.

------------------------------------------------------------

Package xAct`xTras`  version 1.4.2, {2014, 10, 30}

CopyRight (C) 2012-2014, Teake Nutma, under the General Public License.

** Variable $CovDFormat changed from Postfix to Prefix

** Option CurvatureRelations of DefCovD changed from False to True

------------------------------------------------------------

Package xAct`HiGGS`  version 2.0.0-developer, {2023, 5, 24}

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------------------------------------------------------------

HiGGS incorporates code by Cyril Pitrou.
```

Get: Cannot open xAct`FieldsX`.

Needs: Context xAct`FieldsX` was not created when Needs was evaluated.

```
------------------------------------------------------------
Package xAct`GeoHiGGS`  version 1.0.0-developer, {2023, 5, 24}
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------------------------------------------------------------
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  Disclaimer[]. This is free software, and you are welcome to redistribute
  it under certain conditions. See the General Public License for details.
------------------------------------------------------------
```

All the requisite packages have now been loaded, so we can proceed with the calculations.

Deriving the field equations

HiGGS is designed to study the full ten-parameter Poincaré gauge theory, including nine extra parameters which activate various Lagrange multipliers as defined in arXiv:2205.13534. As a first step, we define the most general case of the set of theories we are interested in by constructing a rule which constrains the Lagrangian couplings.

```
** DefConstantSymbol: Defining constant symbol MPl.
** DefConstantSymbol: Defining constant symbol AlpM.
** DefConstantSymbol: Defining constant symbol Mu2.
** DefConstantSymbol: Defining constant symbol Mu3.
```

$$\left\{\hat{\alpha}_{1}\to 0,\ \hat{\alpha}_{2}\to 0,\ \hat{\alpha}_{3}\to 0,\ \hat{\alpha}_{4}\to 0,\ \hat{\alpha}_{6}\to 0,\ \mathcal{M}_{\text{Pl}}{}^{2}\,\hat{\beta}_{1}\to 0,\ \bar{\alpha}_{1}\to 0,\ \bar{\alpha}_{2}\to 0,\ \bar{\alpha}_{3}\to 0,\ \bar{\alpha}_{4}\to 0,\ \bar{\alpha}_{6}\to 0,\ \mathcal{M}_{\text{Pl}}{}^{2}\,\bar{\beta}_{3}\to 0\right\} \tag{1}$$

We can see that the rules in Eq. (1) are used to disable most of the couplings in the general theory. The couplings which are not suppressed are those which appear in the various Lagrangia in the manuscript. Specifically the $\hat{\alpha}_{5}$ coupling, which mediates the quadratic Riemann-Cartan-Maxwell invariant ${}^{5}\mathcal{R}_{ij}\ {}^{5}\mathcal{R}^{ij}$, the $\mathcal{M}_{\text{Pl}}{}^{2}\,\bar{\beta}_{1}$ and $\mathcal{M}_{\text{Pl}}{}^{2}\,\bar{\beta}_{2}$ couplings, which mediate the torsion multipliers, and the $\mathcal{M}_{\text{Pl}}{}^{2}\,\hat{\beta}_{2}$ and $\mathcal{M}_{\text{Pl}}{}^{2}\,\hat{\beta}_{3}$ couplings which contribute to the torsion mass. These remaining couplings will appear in the equations below.

The generalised momenta

Having done this, we define the generalised momenta associated with this subset of multiplier-constrained Poincaré gauge theory. These quantities are defined on p. 50 of Blagojević.

```
** DefTensor: Defining tensor BGPi[-i, k, l].
** DefTensor: Defining tensor AGPi[-i, -j, k, l].
```

The translational generalised momenta $\pi_{b_{i}}{}^{\mathfrak{k}\mathfrak{l}}$.

$$\pi_{b_{i}}{}^{\mathfrak{k}\mathfrak{l}} \tag{2}$$

$$\begin{aligned}
&-\frac{4}{3}\mathcal{M}_{\mathrm{Pl}}{}^{2}\hat{\beta}_{3}\mathcal{J}\mathcal{N}\mathcal{T}_{i}{}^{\mathfrak{k}\mathfrak{l}}+\frac{4}{3}\mathcal{M}_{\mathrm{Pl}}{}^{2}\hat{\beta}_{3}\mathcal{J}\mathcal{N}\mathcal{T}^{\mathfrak{k}}{}_{i}{}^{\mathfrak{l}}-\frac{4}{3}\mathcal{M}_{\mathrm{Pl}}{}^{2}\hat{\beta}_{3}\mathcal{J}\mathcal{N}\mathcal{T}^{\mathfrak{l}}{}_{i}{}^{\mathfrak{k}}-\frac{4}{3}\mathcal{M}_{\mathrm{Pl}}{}^{2}\hat{\beta}_{2}\,\delta_{i}{}^{\mathfrak{l}}\,\mathcal{J}\mathcal{N}\mathcal{T}^{\mathfrak{y}\mathfrak{k}}{}_{\mathfrak{y}}+\\
&\frac{4}{3}\mathcal{M}_{\mathrm{Pl}}{}^{2}\hat{\beta}_{2}\,\delta_{i}{}^{\mathfrak{k}}\,\mathcal{J}\mathcal{N}\mathcal{T}^{\mathfrak{y}\mathfrak{l}}{}_{\mathfrak{y}}-\frac{4}{3}\mathcal{M}_{\mathrm{Pl}}{}^{2}\bar{\beta}_{1}\mathcal{J}\mathcal{N}\,\lambda_{\mathcal{T}}{}_{i}{}^{\mathfrak{k}\mathfrak{l}}-\frac{2}{3}\mathcal{M}_{\mathrm{Pl}}{}^{2}\bar{\beta}_{1}\mathcal{J}\mathcal{N}\,\lambda_{\mathcal{T}}{}^{\mathfrak{k}}{}_{i}{}^{\mathfrak{l}}+\frac{2}{3}\mathcal{M}_{\mathrm{Pl}}{}^{2}\bar{\beta}_{1}\mathcal{J}\mathcal{N}\,\lambda_{\mathcal{T}}{}^{\mathfrak{l}}{}_{i}{}^{\mathfrak{k}}+\\
&\frac{2}{3}\left(\mathcal{M}_{\mathrm{Pl}}{}^{2}\bar{\beta}_{1}-\mathcal{M}_{\mathrm{Pl}}{}^{2}\bar{\beta}_{2}\right)\delta_{i}{}^{\mathfrak{l}}\,\mathcal{J}\mathcal{N}\,\lambda_{\mathcal{T}}{}^{\mathfrak{y}\mathfrak{k}}{}_{\mathfrak{y}}-\frac{2}{3}\left(\mathcal{M}_{\mathrm{Pl}}{}^{2}\bar{\beta}_{1}-\mathcal{M}_{\mathrm{Pl}}{}^{2}\bar{\beta}_{2}\right)\delta_{i}{}^{\mathfrak{k}}\,\mathcal{J}\mathcal{N}\,\lambda_{\mathcal{T}}{}^{\mathfrak{y}\mathfrak{l}}{}_{\mathfrak{y}}
\end{aligned} \tag{3}$$

The rotational generalised momenta $\pi_{\mathcal{A}_{ij}}{}^{\mathfrak{k}\mathfrak{l}}$.

$$\pi_{\mathcal{A}_{ij}}{}^{\mathfrak{k}\mathfrak{l}} \tag{4}$$

$$\begin{aligned}
&-\mathcal{M}_{\mathrm{Pl}}{}^{2}\hat{\alpha}_{0}\,\delta_{i}{}^{\mathfrak{l}}\,\delta_{j}{}^{\mathfrak{k}}\,\mathcal{J}\mathcal{N}+\mathcal{M}_{\mathrm{Pl}}{}^{2}\hat{\alpha}_{0}\,\delta_{i}{}^{\mathfrak{k}}\,\delta_{j}{}^{\mathfrak{l}}\,\mathcal{J}\mathcal{N}-2\hat{\alpha}_{5}\,\delta_{j}{}^{\mathfrak{l}}\,\mathcal{J}\mathcal{N}\,\mathcal{R}_{i}{}^{\mathfrak{w}\mathfrak{k}}{}_{\mathfrak{w}}+\\
&2\hat{\alpha}_{5}\,\delta_{j}{}^{\mathfrak{k}}\,\mathcal{J}\mathcal{N}\,\mathcal{R}_{i}{}^{\mathfrak{w}\mathfrak{l}}{}_{\mathfrak{w}}+2\hat{\alpha}_{5}\,\delta_{i}{}^{\mathfrak{l}}\,\mathcal{J}\mathcal{N}\,\mathcal{R}_{j}{}^{\mathfrak{w}\mathfrak{k}}{}_{\mathfrak{w}}-2\hat{\alpha}_{5}\,\delta_{i}{}^{\mathfrak{k}}\,\mathcal{J}\mathcal{N}\,\mathcal{R}_{j}{}^{\mathfrak{w}\mathfrak{l}}{}_{\mathfrak{w}}+\\
&2\hat{\alpha}_{5}\,\delta_{j}{}^{\mathfrak{l}}\,\mathcal{J}\mathcal{N}\,\mathcal{R}^{\mathfrak{k}\mathfrak{w}}{}_{i\mathfrak{w}}-2\hat{\alpha}_{5}\,\delta_{i}{}^{\mathfrak{l}}\,\mathcal{J}\mathcal{N}\,\mathcal{R}^{\mathfrak{k}\mathfrak{w}}{}_{j\mathfrak{w}}-2\hat{\alpha}_{5}\,\delta_{j}{}^{\mathfrak{k}}\,\mathcal{J}\mathcal{N}\,\mathcal{R}^{\mathfrak{l}\mathfrak{w}}{}_{i\mathfrak{w}}+2\hat{\alpha}_{5}\,\delta_{i}{}^{\mathfrak{k}}\,\mathcal{J}\mathcal{N}\,\mathcal{R}^{\mathfrak{l}\mathfrak{w}}{}_{j\mathfrak{w}}-\\
&\bar{\alpha}_{5}\,\delta_{j}{}^{\mathfrak{l}}\,\mathcal{J}\mathcal{N}\,\lambda_{\mathcal{R}}{}_{i}{}^{\mathfrak{w}\mathfrak{k}}{}_{\mathfrak{w}}+\bar{\alpha}_{5}\,\delta_{j}{}^{\mathfrak{k}}\,\mathcal{J}\mathcal{N}\,\lambda_{\mathcal{R}}{}_{i}{}^{\mathfrak{w}\mathfrak{l}}{}_{\mathfrak{w}}+\bar{\alpha}_{5}\,\delta_{i}{}^{\mathfrak{l}}\,\mathcal{J}\mathcal{N}\,\lambda_{\mathcal{R}}{}_{j}{}^{\mathfrak{w}\mathfrak{k}}{}_{\mathfrak{w}}-\bar{\alpha}_{5}\,\delta_{i}{}^{\mathfrak{k}}\,\mathcal{J}\mathcal{N}\,\lambda_{\mathcal{R}}{}_{j}{}^{\mathfrak{w}\mathfrak{l}}{}_{\mathfrak{w}}+\\
&\bar{\alpha}_{5}\,\delta_{j}{}^{\mathfrak{l}}\,\mathcal{J}\mathcal{N}\,\lambda_{\mathcal{R}}{}^{\mathfrak{k}\mathfrak{w}}{}_{i\mathfrak{w}}-\bar{\alpha}_{5}\,\delta_{i}{}^{\mathfrak{l}}\,\mathcal{J}\mathcal{N}\,\lambda_{\mathcal{R}}{}^{\mathfrak{k}\mathfrak{w}}{}_{j\mathfrak{w}}-\bar{\alpha}_{5}\,\delta_{j}{}^{\mathfrak{k}}\,\mathcal{J}\mathcal{N}\,\lambda_{\mathcal{R}}{}^{\mathfrak{l}\mathfrak{w}}{}_{i\mathfrak{w}}+\bar{\alpha}_{5}\,\delta_{i}{}^{\mathfrak{k}}\,\mathcal{J}\mathcal{N}\,\lambda_{\mathcal{R}}{}^{\mathfrak{l}\mathfrak{w}}{}_{j\mathfrak{w}}
\end{aligned} \tag{5}$$

At this point we clear up some confusion by defining the lapse function $\mathcal{N}$, the induced measure on the spatial foliation $\mathcal{J}$. Frequently in the expressions below you will see the following cumbersome factor.

$$\mathcal{J}\mathcal{N} \tag{6}$$

Remember that all Eq. (6) means is `the measure in four spacetime dimensions'. The reason it is defined this way is because HiGGS is supposed to be doing Hamiltonian, not Lagrangian, analysis.

Now in the case above, these generalised momenta are obtained from the Lagrangian represented in Eq. (4) of arXiv:2205.13534, with most of the coupling constants set to zero as per the above restrictions. Note that this Lagrangian differs from the most general Lagrangian represented in Blagojević by the use of so-called geometric multiplier fields $\lambda_{\mathcal{R}}{}^{xw}{}_{yz}$ and $\lambda_{\mathcal{T}}{}^{x}{}_{yz}$. These are multipliers which can disable all of the Riemann-Cartan $\mathcal{R}^{xw}{}_{yz}$ or torsion $\mathcal{T}^{x}{}_{yz}$ tensors, but which may typically only be used piecemeal to disable select portions of said tensors. It is this latter use-case which we realise in our manuscript.

The tetrad field equation

Now we define the tetrad equation. In terms of the generalised momenta, the Riemann-Cartan curvature, the torsion, and the gauge fields, and also in terms of the projection operators which are used to define the various quadratic invariants, the left hand side of the tetrad equation as expressed in the first line of Eq. (3.24b) on page 50 of Blagojević is as follows.

$$
\begin{aligned}
&\mathcal{A}^{l}{}_{im}\,\pi_{b_l}{}^{xy}\,h_x{}^{n}\,h_y{}^{m}+\frac{1}{2}\,\pi_{\mathcal{A}_{pq}}{}^{\hbar y}\,h_y{}^{n}\,\mathcal{R}^{pq}{}_{\hbar i}+\pi_{b_p}{}^{\hbar y}\,h_y{}^{n}\,\mathcal{T}^{p}{}_{\hbar i}+\\
&\quad h_i{}^{n}\,\mathcal{J}\mathcal{N}\Bigg(-\frac{1}{2}\mathcal{M}_{Pl}^{2}\,\hat{\alpha}_{0}\,\gamma_{jl}\,\gamma_{q\hbar}\,\mathcal{R}^{qj\hbar l}+\mathcal{R}^{qj}{}_{\hbar l}\Bigg(\Big(\hat{\alpha}_{1}\,{}^{1}\hat{\mathcal{P}}_{\mathcal{R}xw}{}^{yz}{}_{qj}{}^{\hbar l}+\hat{\alpha}_{2}\,{}^{2}\hat{\mathcal{P}}_{\mathcal{R}xw}{}^{yz}{}_{qj}{}^{\hbar l}+\\
&\qquad\qquad \hat{\alpha}_{3}\,{}^{3}\hat{\mathcal{P}}_{\mathcal{R}xw}{}^{yz}{}_{qj}{}^{\hbar l}+\hat{\alpha}_{4}\,{}^{4}\hat{\mathcal{P}}_{\mathcal{R}xw}{}^{yz}{}_{qj}{}^{\hbar l}+\hat{\alpha}_{5}\,{}^{5}\hat{\mathcal{P}}_{\mathcal{R}xw}{}^{yz}{}_{qj}{}^{\hbar l}+\hat{\alpha}_{6}\,{}^{6}\hat{\mathcal{P}}_{\mathcal{R}xw}{}^{yz}{}_{qj}{}^{\hbar l}\Big)\,\mathcal{R}^{xw}{}_{yz}+\\
&\qquad\quad \Big(\bar{\alpha}_{1}\,{}^{1}\hat{\mathcal{P}}_{\mathcal{R}xw}{}^{yz}{}_{qj}{}^{\hbar l}+\bar{\alpha}_{2}\,{}^{2}\hat{\mathcal{P}}_{\mathcal{R}xw}{}^{yz}{}_{qj}{}^{\hbar l}+\bar{\alpha}_{3}\,{}^{3}\hat{\mathcal{P}}_{\mathcal{R}xw}{}^{yz}{}_{qj}{}^{\hbar l}+\bar{\alpha}_{4}\,{}^{4}\hat{\mathcal{P}}_{\mathcal{R}xw}{}^{yz}{}_{qj}{}^{\hbar l}+\\
&\qquad\qquad \bar{\alpha}_{5}\,{}^{5}\hat{\mathcal{P}}_{\mathcal{R}xw}{}^{yz}{}_{qj}{}^{\hbar l}+\bar{\alpha}_{6}\,{}^{6}\hat{\mathcal{P}}_{\mathcal{R}xw}{}^{yz}{}_{qj}{}^{\hbar l}\Big)\,\lambda_{\mathcal{R}}{}^{xw}{}_{yz}\Bigg)+\\
&\qquad \mathcal{T}^{q}{}_{\hbar l}\Bigg(\Big(\mathcal{M}_{Pl}^{2}\,\hat{\beta}_{1}\,{}^{1}\hat{\mathcal{P}}_{\mathcal{T}x}{}^{yz}{}_{q}{}^{\hbar l}+\mathcal{M}_{Pl}^{2}\,\hat{\beta}_{2}\,{}^{2}\hat{\mathcal{P}}_{\mathcal{T}x}{}^{yz}{}_{q}{}^{\hbar l}+\mathcal{M}_{Pl}^{2}\,\hat{\beta}_{3}\,{}^{3}\hat{\mathcal{P}}_{\mathcal{T}x}{}^{yz}{}_{q}{}^{\hbar l}\Big)\,\mathcal{T}^{x}{}_{yz}+\\
&\qquad\quad \Big(\mathcal{M}_{Pl}^{2}\,\bar{\beta}_{1}\,{}^{1}\hat{\mathcal{P}}_{\mathcal{T}x}{}^{yz}{}_{q}{}^{\hbar l}+\mathcal{M}_{Pl}^{2}\,\bar{\beta}_{2}\,{}^{2}\hat{\mathcal{P}}_{\mathcal{T}x}{}^{yz}{}_{q}{}^{\hbar l}+\mathcal{M}_{Pl}^{2}\,\bar{\beta}_{3}\,{}^{3}\hat{\mathcal{P}}_{\mathcal{T}x}{}^{yz}{}_{q}{}^{\hbar l}\Big)\,\lambda_{\mathcal{T}}{}^{x}{}_{yz}\Bigg)\Bigg)-\\
&\quad \pi_{b_i}{}^{xy}\left(-h_a{}^{n}\,h_x{}^{b}\,h_y{}^{m}\,\partial_m b^{a}{}_{b}-h_{a'}{}^{m}\,h_x{}^{n}\,h_y{}^{b'}\,\partial_m b^{a'}{}_{b'}\right)-\\
&\quad h_x{}^{n}\\
&\quad\; h_y{}^{m}\\
&\quad\; \partial_m\pi_{b_i}{}^{xy}
\end{aligned}
\tag{7}
$$

We next impose the restriction on the coupling constants from Eq. (1) in order to go over to the most general case studied in our manuscript, and then we expand the projection operators and the generalised momenta. We subtract the right hand side of the first line of Eq. (3.24b) on page 50 of Blagojević, i.e. the (asymmetric) matter stress-energy tensor $\tau^{n}{}_{i}$, to form the tetrad field equation.

$$
\begin{aligned}
&-\mathcal{M}_{Pl}^{2}\,\hat{\alpha}_{0}\,h^{an}\,\mathcal{R}_{a}{}^{a'}{}_{ia'}+\frac{1}{2}\mathcal{M}_{Pl}^{2}\,\hat{\alpha}_{0}\,h_i{}^{n}\,\mathcal{R}^{aa'}{}_{aa'}+2\,\hat{\alpha}_{5}\,h^{an}\,\mathcal{R}_{a}{}^{a'b}{}_{i}\,\mathcal{R}^{b'}{}_{a'bb'}-\hat{\alpha}_{5}\,h_i{}^{n}\,\mathcal{R}^{aa'b}{}_{a}\,\mathcal{R}^{b'}{}_{a'bb'}+\\
&\quad 2\,\hat{\alpha}_{5}\,h^{an}\,\mathcal{R}^{b'}{}_{abb'}\,\mathcal{R}^{a'b}{}_{ia'}-2\,\hat{\alpha}_{5}\,h^{an}\,\mathcal{R}^{a'b}{}_{ia'}\,\mathcal{R}^{b'}{}_{bab'}-2\,\hat{\alpha}_{5}\,h^{an}\,\mathcal{R}_{a}{}^{a'b}{}_{i}\,\mathcal{R}^{b'}{}_{ba'b'}+\hat{\alpha}_{5}\,h_i{}^{n}\,\mathcal{R}^{aa'b}{}_{a}\,\mathcal{R}^{b'}{}_{ba'b'}+\\
&\quad \bar{\alpha}_{5}\,h^{an}\,\mathcal{R}^{a'b}{}_{ia'}\,\lambda_{\mathcal{R}_a}{}^{b'}{}_{bb'}+\bar{\alpha}_{5}\,h^{an}\,\mathcal{R}_{a}{}^{a'b}{}_{i}\,\lambda_{\mathcal{R}_{a'}}{}^{b'}{}_{bb'}-\bar{\alpha}_{5}\,h_i{}^{n}\,\mathcal{R}^{aa'b}{}_{a}\,\lambda_{\mathcal{R}_{a'}}{}^{b'}{}_{bb'}-\bar{\alpha}_{5}\,h^{an}\,\mathcal{R}^{a'b}{}_{ia'}\,\lambda_{\mathcal{R}_b}{}^{b'}{}_{ab'}-\\
&\quad \bar{\alpha}_{5}\,h^{an}\,\mathcal{R}_{a}{}^{a'b}{}_{i}\,\lambda_{\mathcal{R}_b}{}^{b'}{}_{a'b'}+\bar{\alpha}_{5}\,h_i{}^{n}\,\mathcal{R}^{aa'b}{}_{a}\,\lambda_{\mathcal{R}_b}{}^{b'}{}_{a'b'}-\frac{1}{3}\mathcal{M}_{Pl}^{2}\,\hat{\beta}_{3}\,h_i{}^{n}\,\mathcal{T}_{aa'b}\,\mathcal{T}^{aa'b}+\\
&\quad \frac{2}{3}\mathcal{M}_{Pl}^{2}\,\hat{\beta}_{3}\,h_i{}^{n}\,\mathcal{T}^{aa'b}\,\mathcal{T}_{a'ab}+\frac{4}{3}\mathcal{M}_{Pl}^{2}\,\hat{\beta}_{3}\,h^{an}\,\mathcal{T}_{a'ab}\,\mathcal{T}^{a'b}{}_{i}-\frac{4}{3}\mathcal{M}_{Pl}^{2}\,\hat{\beta}_{3}\,h^{an}\,\mathcal{T}^{a'b}{}_{i}\,\mathcal{T}_{baa'}+
\end{aligned}
$$

$$
\begin{aligned}
&\frac{4}{3}\mathcal{M}_{\text{Pl}}^{2}\hat{\beta}_{3}\, h^{an}\, \mathcal{T}_{ai}{}^{a'}\, \mathcal{T}^{b}{}_{a'b} - \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\hat{\beta}_{2}\, h_{i}{}^{n}\, \mathcal{T}^{a}{}_{a}{}^{a'}\, \mathcal{T}^{b}{}_{a'b} - \frac{4}{3}\mathcal{M}_{\text{Pl}}^{2}\hat{\beta}_{3}\, h^{an}\, \mathcal{T}^{a'}{}_{ia}\, \mathcal{T}^{b}{}_{a'b} - \\
&\frac{4}{3}\mathcal{M}_{\text{Pl}}^{2}\hat{\beta}_{3}\, h^{an}\, \mathcal{T}^{b}{}_{a'b}\, \mathcal{T}_{ia}{}^{a'} - \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\hat{\beta}_{3}\, h^{an}\, \mathcal{T}_{aa'b}\, \mathcal{T}_{i}{}^{a'b} + \frac{\tau^{n}{}_{i}}{\mathcal{J}\,\mathcal{N}} - \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h_{i}{}^{n}\, \mathcal{T}^{aa'b}\, \lambda_{\mathcal{T}\,aa'b} + \\
&\frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h^{an}\, \mathcal{T}^{a'}{}_{i}{}^{b}\, \lambda_{\mathcal{T}\,aa'b} + \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h^{an}\, \mathcal{T}^{a'}{}_{a'}{}^{b}\, \lambda_{\mathcal{T}\,aib} - \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h_{i}{}^{n}\, \mathcal{T}^{aa'b}\, \lambda_{\mathcal{T}\,a'ab} + \\
&\frac{4}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h^{an}\, \mathcal{T}^{a'}{}_{i}{}^{b}\, \lambda_{\mathcal{T}\,a'ab} - \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h^{an}\, \mathcal{T}_{a}{}^{a'b}\, \lambda_{\mathcal{T}\,a'ib} + \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h^{an}\, \mathcal{T}^{a'}{}_{i}{}^{b}\, \lambda_{\mathcal{T}\,baa'} - \\
&\frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h^{an}\, \mathcal{T}^{a'}{}_{a'}{}^{b}\, \lambda_{\mathcal{T}\,bia} + \frac{4}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h^{an}\, \mathcal{T}^{a'}{}_{a'}{}^{b}\, \lambda_{\mathcal{T}\,iab} - \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h^{an}\, \mathcal{T}_{a}{}^{a'b}\, \lambda_{\mathcal{T}\,ia'b} - \\
&\frac{4}{3}\mathcal{M}_{\text{Pl}}^{2}\hat{\beta}_{2}\, h^{a'n}\, h_{i}{}^{a}\left(\mathcal{D}_{a}\mathcal{T}^{b}{}_{a'b}\right) + \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h^{a'n}\, h_{i}{}^{a}\left(\mathcal{D}_{a}\lambda_{\mathcal{T}}{}^{b}{}_{a'b}\right) - \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{2}\, h^{a'n}\, h_{i}{}^{a}\left(\mathcal{D}_{a}\lambda_{\mathcal{T}}{}^{b}{}_{a'b}\right) + \\
&\frac{4}{3}\mathcal{M}_{\text{Pl}}^{2}\hat{\beta}_{2}\, h^{aa'}\, h_{i}{}^{n}\left(\mathcal{D}_{a'}\mathcal{T}^{b}{}_{ab}\right) - \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h^{aa'}\, h_{i}{}^{n}\left(\mathcal{D}_{a'}\lambda_{\mathcal{T}}{}^{b}{}_{ab}\right) + \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{2}\, h^{aa'}\, h_{i}{}^{n}\left(\mathcal{D}_{a'}\lambda_{\mathcal{T}}{}^{b}{}_{ab}\right) + \\
&\frac{4}{3}\mathcal{M}_{\text{Pl}}^{2}\hat{\beta}_{3}\, h^{an}\, h^{a'b}\left(\mathcal{D}_{b}\mathcal{T}_{aia'}\right) - \frac{4}{3}\mathcal{M}_{\text{Pl}}^{2}\hat{\beta}_{3}\, h^{an}\, h^{a'b}\left(\mathcal{D}_{b}\mathcal{T}_{a'ia}\right) - \frac{4}{3}\mathcal{M}_{\text{Pl}}^{2}\hat{\beta}_{3}\, h^{an}\, h^{a'b}\left(\mathcal{D}_{b}\mathcal{T}_{iaa'}\right) - \\
&\frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h^{an}\, h^{a'b}\left(\mathcal{D}_{b}\lambda_{\mathcal{T}\,aia'}\right) + \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h^{an}\, h^{a'b}\left(\mathcal{D}_{b}\lambda_{\mathcal{T}\,a'ia}\right) - \frac{4}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h^{an}\, h^{a'b}\left(\mathcal{D}_{b}\lambda_{\mathcal{T}\,iaa'}\right) == 0
\end{aligned}
$$

The new kinds of derivative appearing here are Poincaré gauge covariant derivatives, whose connection acts only on Lorentz indices but whose own (derivative) index is actually coordinate (Greek), these derivatives are defined in Eq. (3.5) on page 44 of Blagojević (albeit with a nabla symbol, which is a slight difference with our notation). The equation Eq. (8) is nearly ready to use, but the (asymmetric) stress-energy tensor of matter first needs some attention. This is because it still depends on the contorsion, and later on in the analysis we will need to refer only to the metric-based part, from which the symmetric Einstein stress-energy tensor is eventually derived. Before moving on to the spin connection equations therefore, we will need to explore this matter.

Splitting matter stress-energy

First we define the contorsion tensor to have strictly two anholonomic indices and a holonomic index, in line with its interpretation as a part of the connection. This is recovered through Eq. (3.32b) on p. 57 of Blagojević, with comparisons with Eq (3.46) on p. 61, and checks against the methods used on p. 67. Great care must be taken when attempting to achieve this index configuration using contractions of the HiGGS anholonomic torsion tensor with the translational gauge field and inverse:

```
** DefTensor: Defining tensor Contorsion[i, j, -m].
```

$$\mathcal{K}^{ij}{}_{m} \tag{9}$$

$$\frac{1}{2}\, b^{a}{}_{m}\, \mathcal{T}_{a}{}^{ij} - \frac{1}{2}\, b^{a}{}_{m}\, \mathcal{T}^{ij}{}_{a} + \frac{1}{2}\, b^{a}{}_{m}\, \mathcal{T}^{ji}{}_{a} \tag{10}$$

Note that in Eq. (9) and Eq. (10) you don't actually see the holonomic and anholonomic indices with different labels (e.g. Greek and Roman). This is because in HiGGS the tangent space of the index is assumed to be determined by the name of the tensor. Thus all instances of the translational gauge field are Roman in their first index and Greek (coordinate) in their second index.

Now in HiGGS we are used to using the following matter stress-energy tensor.

$$\tau^{m}{}_{\hbar} \tag{11}$$

But the point raised above was that this tensor, the (negative) variational derivative of the matter Lagrangian (density) as defined in Eq. (3.21) on p. 48 of Blagojević, still depends on the rotational gauge field. In the second order formalism, this means that it will depend both on the Ricci rotation coefficients and on the contorsion.

We now define the part which depends only on the Ricci rotation coefficients.

```
** DefTensor: Defining tensor TorsionlessTau[-i, -j].
```

$$\tau(\Delta)_{ij} \tag{12}$$

To understand how the contorsion dependency enters in, we look at Eq. (3.75b) on p. 66 of Blagojević. This separated Lagrangian is varied with respect to the tetrad, in such a way that the Ricci rotation coefficients and contorsion are held constant (they must have two Roman and one Greek index, so that they algebraically inherit the role of the rotational gauge field). The variation of the first term will give us the above quantity, which can then be expressed in terms of the Einstein tensor using the methods of p. 67 (we do this later in the script).

The variation of the second term rests entirely on the variation of the spin tensor. This is rather suspicious, since it means that the details of the spin tensor of matter have a say in whether we can recover the second order formalism at all. However to proceed, we look to the Dirac matter spin tensor in Example 2 at the end of p. 49 of Blagojević. We define a (Lorentz) quantity which is truly independent of the gravitational variables, being composed of Grassmann numbers and (indexed) generators of the Clifford algebra.

```
** DefTensor: Defining tensor SigmaRoman[i, -j, -k].
```

$$\chi^{i}{}_{j\hbar} \tag{13}$$

Understand that Eq. (13) will only have anholonomic Lorentz indices, and is formed from the fermion current.

Now we focus on the spin tensor density $\sigma^{i}{}_{j\hbar}$, which is related to the spin tensor in our manuscript (defined later on in this script) by a factor of the measure in accordance with the paragraph following Eq. (3.58) on p. 67 in Blagojević. It has an initial coordinate index, and then two Lorentz indices. It must also be a density.

$$\sigma^{i}{}_{j\hbar} \tag{14}$$

$$h^{ai}\,\mathcal{J}\,\mathcal{N}\,\mathcal{X}_{aj\hbar} \tag{15}$$

These requirements are met by Eq. (15) w.l.o.g.

Bringing together the contorsion in Eq. (9) and the (Dirac) spin tensor in Eq. (15) the whole of the correction to the second order matter Lagrangian density now takes the following form.

$$\frac{1}{2}\,\mathcal{K}^{ij}{}_{m}\;\sigma^{m}{}_{ij} \tag{16}$$

$$\frac{1}{2}\,\mathcal{K}^{ma}{}_{j}\;h^{ij}\,\mathcal{J}\,\mathcal{N}\,\mathcal{X}_{ima} \tag{17}$$

At this point we will remind ourselves about the derivatives with respect to the tetrad of some quantities. This is really a tangent to the calculations, but the expressions may become useful later in the script. Because HiGGS is oriented towards the ADM formulation, you will see frequent appearances of the following symbol.

$$n^{a} \tag{18}$$

Now Eq. (18) is just a unit-timelike vector with Lorentz indices. Okay, let's find the (coordinate) derivative of the lapse function.

$$\mathcal{N} \tag{19}$$

$$h^{aa'}\;\mathcal{N}\,n_{a}\;\;n^{b}\;\partial_{m}b_{ba'} \tag{20}$$

Now the (coordinate) derivative of the spatial measure.

$$\mathcal{J} \tag{21}$$

$$h^{aa'}\;\mathcal{J}\,\partial_{m}b_{aa'} - h^{aa'}\;\mathcal{J}\,n_{a}\;\;n^{b}\;\partial_{m}b_{ba'} \tag{22}$$

Now the (coordinate) derivative of the (inverse) tetrad.

$$h_{i}{}^{n} \tag{23}$$

$$-h^{a'n}\;\;h_{i}{}^{a}\;\;\partial_{m}b_{a'a} \tag{24}$$

Very well. Now with Eq. (20) Eq. (22) Eq. (24) are mostly obtained as a tangent to the computations.

Now that the dependence of the contorsion correction to the second-order Lagrangian on the translational gauge field has been made clear in Eq. (17), we can use the above derivative laws Eq. (20) Eq. (22)

Eq. (24) to reconstruct the variational derivative of the correction with respect to the translational gauge field. This in turn will tell us how the Ricci rotation coefficient part of the matter stress energy tensor is augmented by terms bilinear in the torsion and the spin tensor.

$$-\frac{1}{2}\mathcal{K}^{in}{}_{a}\, h_{\hbar}{}^{a}\, \sigma^{m}{}_{in}+\frac{1}{2}\mathcal{K}^{ain}\, h_{\hbar}{}^{m}\, \sigma_{nai}+\tau(\Delta)^{m}{}_{\hbar} \tag{25}$$

$$-\frac{1}{2}\sigma^{mai}\, \mathcal{T}_{a\hbar i}+\frac{1}{4}b^{ai}\, h_{\hbar}{}^{m}\, \sigma_{i}{}^{na'}\, \mathcal{T}_{ana'}-\frac{1}{4}\sigma^{min}\, \mathcal{T}_{\hbar in}+\frac{1}{2}b^{ai}\, h_{\hbar}{}^{m}\, \sigma_{i}{}^{na'}\, \mathcal{T}_{naa'}+\tau(\Delta)^{m}{}_{\hbar} \tag{26}$$

We are now ready to write the formula which `splits' the usual HiGGS stress-energy tensor into torsion-free and torsionful parts.

$$\tau^{m}{}_{\hbar} \tag{27}$$

$$\frac{1}{4}b^{aa'}\, h_{\hbar}{}^{m}\, \sigma_{a'}{}^{bb'}\, \mathcal{T}_{abb'}-\frac{1}{2}\sigma^{maa'}\, \mathcal{T}_{a\hbar a'}+\frac{1}{2}b^{aa'}\, h_{\hbar}{}^{m}\, \sigma_{a'}{}^{bb'}\, \mathcal{T}_{bab'}-\frac{1}{4}\sigma^{maa'}\, \mathcal{T}_{\hbar aa'}+\tau(\Delta)^{m}{}_{\hbar} \tag{28}$$

And we will also write a rule to invert this.

```
** MakeQuotientRule: canonicalised expression with tensor substituted by rule:
```

0

$$\tau(\Delta)^{m}{}_{\hbar} \tag{29}$$

$$-\frac{1}{4}b^{aa'}\, h_{\hbar}{}^{m}\, \sigma_{a'}{}^{bb'}\, \mathcal{T}_{abb'}+\frac{1}{2}\sigma^{maa'}\, \mathcal{T}_{a\hbar a'}-\frac{1}{2}b^{aa'}\, h_{\hbar}{}^{m}\, \sigma_{a'}{}^{bb'}\, \mathcal{T}_{bab'}+\frac{1}{4}\sigma^{maa'}\, \mathcal{T}_{\hbar aa'}+\tau^{m}{}_{\hbar} \tag{30}$$

Now the second order formalism `splitting' of the stress-energy current is understood, we will implement the change in our previous version of the tetrad equation Eq. (8).

$$\begin{aligned}
&-\mathcal{M}_{Pl}{}^{2}\hat{\alpha}_{0}\, h^{an}\, \mathcal{R}_{a}{}^{a'}{}_{ia'}+\frac{1}{2}\mathcal{M}_{Pl}{}^{2}\hat{\alpha}_{0}\, h_{i}{}^{n}\, \mathcal{R}^{aa'}{}_{aa'}+2\hat{\alpha}_{5}\, h^{an}\, \mathcal{R}_{a}{}^{a'b}{}_{i}\, \mathcal{R}_{a'}{}^{b'}{}_{bb'}-\hat{\alpha}_{5}\, h_{i}{}^{n}\, \mathcal{R}^{aa'b}{}_{a}\, \mathcal{R}_{a'}{}^{b'}{}_{bb'}+\\
&\quad 2\hat{\alpha}_{5}\, h^{an}\, \mathcal{R}_{a}{}^{b'}{}_{bb'}\, \mathcal{R}^{a'b}{}_{ia'}-2\hat{\alpha}_{5}\, h^{an}\, \mathcal{R}^{a'b}{}_{ia'}\, \mathcal{R}_{b}{}^{b'}{}_{ab'}-2\hat{\alpha}_{5}\, h^{an}\, \mathcal{R}_{a}{}^{a'b}{}_{i}\, \mathcal{R}_{b}{}^{b'}{}_{a'b'}+\\
&\quad \hat{\alpha}_{5}\, h_{i}{}^{n}\, \mathcal{R}^{aa'b}{}_{a}\, \mathcal{R}_{b}{}^{b'}{}_{a'b'}+\bar{\alpha}_{5}\, h^{an}\, \mathcal{R}^{a'b}{}_{ia'}\, \lambda_{\mathcal{R}}{}_{a}{}^{b'}{}_{bb'}+\bar{\alpha}_{5}\, h^{an}\, \mathcal{R}_{a}{}^{a'b}{}_{i}\, \lambda_{\mathcal{R}}{}_{a'}{}^{b'}{}_{bb'}-\bar{\alpha}_{5}\, h_{i}{}^{n}\, \mathcal{R}^{aa'b}{}_{a}\, \lambda_{\mathcal{R}}{}_{a'}{}^{b'}{}_{bb'}-\\
&\quad \bar{\alpha}_{5}\, h^{an}\, \mathcal{R}^{a'b}{}_{ia'}\, \lambda_{\mathcal{R}}{}_{b}{}^{b'}{}_{ab'}-\bar{\alpha}_{5}\, h^{an}\, \mathcal{R}_{a}{}^{a'b}{}_{i}\, \lambda_{\mathcal{R}}{}_{b}{}^{b'}{}_{a'b'}+\bar{\alpha}_{5}\, h_{i}{}^{n}\, \mathcal{R}^{aa'b}{}_{a}\, \lambda_{\mathcal{R}}{}_{b}{}^{b'}{}_{a'b'}+\\
&\quad \frac{b^{aa'}\, h_{i}{}^{n}\, \sigma_{a'}{}^{bb'}\, \mathcal{T}_{abb'}}{4\mathcal{J}\,\mathcal{N}}-\frac{\sigma^{naa'}\, \mathcal{T}_{aia'}}{2\mathcal{J}\,\mathcal{N}}-\frac{1}{3}\mathcal{M}_{Pl}{}^{2}\hat{\beta}_{3}\, h_{i}{}^{n}\, \mathcal{T}_{aa'b}\, \mathcal{T}^{aa'b}+\frac{2}{3}\mathcal{M}_{Pl}{}^{2}\hat{\beta}_{3}\, h_{i}{}^{n}\, \mathcal{T}^{aa'b}\, \mathcal{T}_{a'ab}+\\
&\quad \frac{4}{3}\mathcal{M}_{Pl}{}^{2}\hat{\beta}_{3}\, h^{an}\, \mathcal{T}_{a'ab}\, \mathcal{T}^{a'b}{}_{i}-\frac{4}{3}\mathcal{M}_{Pl}{}^{2}\hat{\beta}_{3}\, h^{an}\, \mathcal{T}^{a'b}{}_{i}\, \mathcal{T}_{baa'}+\frac{b^{aa'}\, h_{i}{}^{n}\, \sigma_{a'}{}^{bb'}\, \mathcal{T}_{bab'}}{2\mathcal{J}\,\mathcal{N}}+
\end{aligned}$$

$$
\begin{aligned}
&\frac{4}{3}\mathcal{M}_{Pl}^2\hat{\beta}_{3}\, h^{an}\,\mathcal{T}_{ai}{}^{a'}\,\mathcal{T}^{b}{}_{a'b}-\frac{2}{3}\mathcal{M}_{Pl}^2\hat{\beta}_{2}\, h_{i}{}^{n}\,\mathcal{T}^{a}{}_{a}{}^{a'}\,\mathcal{T}^{b}{}_{a'b}-\frac{4}{3}\mathcal{M}_{Pl}^2\hat{\beta}_{3}\, h^{an}\,\mathcal{T}^{a'}{}_{ia}\,\mathcal{T}^{b}{}_{a'b}-\\
&\frac{\sigma^{naa'}\,\mathcal{T}_{iaa'}}{4\mathcal{J}\mathcal{N}}-\frac{4}{3}\mathcal{M}_{Pl}^2\hat{\beta}_{3}\, h^{an}\,\mathcal{T}^{b}{}_{a'b}\,\mathcal{T}_{ia}{}^{a'}-\frac{2}{3}\mathcal{M}_{Pl}^2\hat{\beta}_{3}\, h^{an}\,\mathcal{T}_{aa'b}\,\mathcal{T}_{i}{}^{a'b}-\\
&\frac{2}{3}\mathcal{M}_{Pl}^2\bar{\beta}_{1}\, h_{i}{}^{n}\,\mathcal{T}^{aa'b}\,\lambda_{\mathcal{T}\,aa'b}+\frac{2}{3}\mathcal{M}_{Pl}^2\bar{\beta}_{1}\, h^{an}\,\mathcal{T}^{a'}{}_{i}{}^{b}\,\lambda_{\mathcal{T}\,aa'b}+\frac{2}{3}\mathcal{M}_{Pl}^2\bar{\beta}_{1}\, h^{an}\,\mathcal{T}^{a'}{}_{a'}{}^{b}\,\lambda_{\mathcal{T}\,aib}-\\
&\frac{2}{3}\mathcal{M}_{Pl}^2\bar{\beta}_{1}\, h_{i}{}^{n}\,\mathcal{T}^{aa'b}\,\lambda_{\mathcal{T}\,a'ab}+\frac{4}{3}\mathcal{M}_{Pl}^2\bar{\beta}_{1}\, h^{an}\,\mathcal{T}^{a'}{}_{i}{}^{b}\,\lambda_{\mathcal{T}\,a'ab}-\frac{2}{3}\mathcal{M}_{Pl}^2\bar{\beta}_{1}\, h^{an}\,\mathcal{T}_{a}{}^{a'b}\,\lambda_{\mathcal{T}\,a'ib}+\\
&\frac{2}{3}\mathcal{M}_{Pl}^2\bar{\beta}_{1}\, h^{an}\,\mathcal{T}^{a'}{}_{i}{}^{b}\,\lambda_{\mathcal{T}\,baa'}-\frac{2}{3}\mathcal{M}_{Pl}^2\bar{\beta}_{1}\, h^{an}\,\mathcal{T}^{a'}{}_{a'}{}^{b}\,\lambda_{\mathcal{T}\,bia}+\frac{4}{3}\mathcal{M}_{Pl}^2\bar{\beta}_{1}\, h^{an}\,\mathcal{T}^{a'}{}_{a'}{}^{b}\,\lambda_{\mathcal{T}\,iab}-\\
&\frac{2}{3}\mathcal{M}_{Pl}^2\bar{\beta}_{1}\, h^{an}\,\mathcal{T}_{a}{}^{a'b}\,\lambda_{\mathcal{T}\,ia'b}+\frac{\tau(\Delta)^{n}{}_{i}}{\mathcal{J}\mathcal{N}}-\frac{4}{3}\mathcal{M}_{Pl}^2\hat{\beta}_{2}\, h^{a'n}\, h_{i}{}^{a}\left(\mathcal{D}_{a}\mathcal{T}^{b}{}_{a'b}\right)+\\
&\frac{2}{3}\mathcal{M}_{Pl}^2\bar{\beta}_{1}\, h^{a'n}\, h_{i}{}^{a}\left(\mathcal{D}_{a}\lambda_{\mathcal{T}}{}^{b}{}_{a'b}\right)-\frac{2}{3}\mathcal{M}_{Pl}^2\bar{\beta}_{2}\, h^{a'n}\, h_{i}{}^{a}\left(\mathcal{D}_{a}\lambda_{\mathcal{T}}{}^{b}{}_{a'b}\right)+\frac{4}{3}\mathcal{M}_{Pl}^2\hat{\beta}_{2}\, h^{aa'}\, h_{i}{}^{n}\left(\mathcal{D}_{a'}\mathcal{T}^{b}{}_{ab}\right)-\\
&\frac{2}{3}\mathcal{M}_{Pl}^2\bar{\beta}_{1}\, h^{aa'}\, h_{i}{}^{n}\left(\mathcal{D}_{a'}\lambda_{\mathcal{T}}{}^{b}{}_{ab}\right)+\frac{2}{3}\mathcal{M}_{Pl}^2\bar{\beta}_{2}\, h^{aa'}\, h_{i}{}^{n}\left(\mathcal{D}_{a'}\lambda_{\mathcal{T}}{}^{b}{}_{ab}\right)+\frac{4}{3}\mathcal{M}_{Pl}^2\hat{\beta}_{3}\, h^{an}\, h^{a'b}\left(\mathcal{D}_{b}\mathcal{T}_{aia'}\right)-\\
&\frac{4}{3}\mathcal{M}_{Pl}^2\hat{\beta}_{3}\, h^{an}\, h^{a'b}\left(\mathcal{D}_{b}\mathcal{T}_{a'ia}\right)-\frac{4}{3}\mathcal{M}_{Pl}^2\hat{\beta}_{3}\, h^{an}\, h^{a'b}\left(\mathcal{D}_{b}\mathcal{T}_{iaa'}\right)-\frac{2}{3}\mathcal{M}_{Pl}^2\bar{\beta}_{1}\, h^{an}\, h^{a'b}\left(\mathcal{D}_{b}\lambda_{\mathcal{T}\,aia'}\right)+\\
&\frac{2}{3}\mathcal{M}_{Pl}^2\bar{\beta}_{1}\, h^{an}\, h^{a'b}\left(\mathcal{D}_{b}\lambda_{\mathcal{T}\,a'ia}\right)-\frac{4}{3}\mathcal{M}_{Pl}^2\bar{\beta}_{1}\, h^{an}\, h^{a'b}\left(\mathcal{D}_{b}\lambda_{\mathcal{T}\,iaa'}\right)==0
\end{aligned}
$$

Then we say that Eq. (31) is the final version of the tetrad equation in the first-order formulation.

Concrete relation to manuscript: This in Eq. (31) is the (asymmetric) tetrad equation for the most general set of Lagrangian parameters used in our manuscript.

```
** MakeQuotientRule: canonicalised expression with tensor substituted by rule:
```

0

The spin connection equations

Having studied the general tetrad equation, we turn to the spin connection equation. The spin tensor, $\sigma^{i}{}_{jh}$, is defined in Eq. (3.21) on p. 48 of Blagojević.

Once again, in terms of the generalised momenta and the gauge fields, the left hand side of the spin connection equation as expressed in the second line of Eq. (3.24b) on page 50 of Blagojević is as follows.

$$
\begin{aligned}
&\mathcal{A}^{l}{}_{jm}\,\pi_{\mathcal{A}\,il}{}^{xy}\,h_{x}{}^{n}\,h_{y}{}^{m}+\mathcal{A}^{l}{}_{im}\,\pi_{\mathcal{A}\,lj}{}^{xy}\,h_{x}{}^{n}\,h_{y}{}^{m}+\pi_{b\,ij}{}^{y}\,h_{y}{}^{n}-\pi_{b\,ji}{}^{y}\,h_{y}{}^{n}-\\
&h_{x}{}^{n}\,h_{y}{}^{m}\,\partial_{m}\pi_{\mathcal{A}\,ij}{}^{xy}-\pi_{\mathcal{A}\,ij}{}^{xy}\left(-h_{a}{}^{n}\,h_{x}{}^{b}\,h_{y}{}^{m}\,\partial_{m}b^{a}{}_{b}-h_{a'}{}^{m}\,h_{x}{}^{n}\,h_{y}{}^{b'}\,\partial_{m}b^{a'}{}_{b'}\right)
\end{aligned}
\qquad (32)
$$

We now expand the generalised momenta using Eq. (3) Eq. (5), impose the restriction on the coupling constants defined in Eq. (1) and expand out the quadratic invariant projection operators. We also subtract the right hand side of the second line of Eq. (3.24b) on page 50 of Blagojević, to give an actual equation.

$$
\begin{aligned}
&\frac{\sigma^{n}{}_{ij}}{\mathcal{J}\mathcal{N}} + 2\hat{\alpha}_{5}\, h^{an}\, \mathcal{R}_{a'}{}^{b}{}_{jb}\, \mathcal{T}_{ai}{}^{a'} - 2\hat{\alpha}_{5}\, h^{an}\, \mathcal{R}_{j}{}^{b}{}_{a'b}\, \mathcal{T}_{ai}{}^{a'} + \bar{\alpha}_{5}\, h^{an}\, \lambda_{\mathcal{R}\,a'}{}^{b}{}_{jb}\, \mathcal{T}_{ai}{}^{a'} - \bar{\alpha}_{5}\, h^{an}\, \lambda_{\mathcal{R}\,j}{}^{b}{}_{a'b}\, \mathcal{T}_{ai}{}^{a'} + \\
&\quad \mathcal{M}_{\text{Pl}}^{2}\hat{\alpha}_{0}\, h^{an}\, \mathcal{T}_{aij} + \frac{8}{3}\mathcal{M}_{\text{Pl}}^{2}\hat{\beta}_{3}\, h^{an}\, \mathcal{T}_{aij} - 2\hat{\alpha}_{5}\, h^{an}\, \mathcal{R}_{a'}{}^{b}{}_{ib}\, \mathcal{T}_{aj}{}^{a'} + 2\hat{\alpha}_{5}\, h^{an}\, \mathcal{R}_{i}{}^{b}{}_{a'b}\, \mathcal{T}_{aj}{}^{a'} - \\
&\quad \bar{\alpha}_{5}\, h^{an}\, \lambda_{\mathcal{R}\,a'}{}^{b}{}_{ib}\, \mathcal{T}_{aj}{}^{a'} + \bar{\alpha}_{5}\, h^{an}\, \lambda_{\mathcal{R}\,i}{}^{b}{}_{a'b}\, \mathcal{T}_{aj}{}^{a'} + 2\hat{\alpha}_{5}\, h_{j}{}^{n}\, \mathcal{R}_{a'}{}^{b}{}_{ib}\, \mathcal{T}^{a}{}_{a}{}^{a'} - 2\hat{\alpha}_{5}\, h_{i}{}^{n}\, \mathcal{R}_{a'}{}^{b}{}_{jb}\, \mathcal{T}^{a}{}_{a}{}^{a'} - \\
&\quad 2\hat{\alpha}_{5}\, h_{j}{}^{n}\, \mathcal{R}_{i}{}^{b}{}_{a'b}\, \mathcal{T}^{a}{}_{a}{}^{a'} + 2\hat{\alpha}_{5}\, h_{i}{}^{n}\, \mathcal{R}_{j}{}^{b}{}_{a'b}\, \mathcal{T}^{a}{}_{a}{}^{a'} + \bar{\alpha}_{5}\, h_{j}{}^{n}\, \lambda_{\mathcal{R}\,a'}{}^{b}{}_{ib}\, \mathcal{T}^{a}{}_{a}{}^{a'} - \bar{\alpha}_{5}\, h_{i}{}^{n}\, \lambda_{\mathcal{R}\,a'}{}^{b}{}_{jb}\, \mathcal{T}^{a}{}_{a}{}^{a'} - \\
&\quad \bar{\alpha}_{5}\, h_{j}{}^{n}\, \lambda_{\mathcal{R}\,i}{}^{b}{}_{a'b}\, \mathcal{T}^{a}{}_{a}{}^{a'} + \bar{\alpha}_{5}\, h_{i}{}^{n}\, \lambda_{\mathcal{R}\,j}{}^{b}{}_{a'b}\, \mathcal{T}^{a}{}_{a}{}^{a'} - \mathcal{M}_{\text{Pl}}^{2}\hat{\alpha}_{0}\, h_{j}{}^{n}\, \mathcal{T}^{a}{}_{ia} - \frac{4}{3}\mathcal{M}_{\text{Pl}}^{2}\hat{\beta}_{2}\, h_{j}{}^{n}\, \mathcal{T}^{a}{}_{ia} + \\
&\quad \mathcal{M}_{\text{Pl}}^{2}\hat{\alpha}_{0}\, h_{i}{}^{n}\, \mathcal{T}^{a}{}_{ja} + \frac{4}{3}\mathcal{M}_{\text{Pl}}^{2}\hat{\beta}_{2}\, h_{i}{}^{n}\, \mathcal{T}^{a}{}_{ja} - 2\hat{\alpha}_{5}\, h^{an}\, \mathcal{R}_{a}{}^{b}{}_{jb}\, \mathcal{T}^{a'}{}_{ia'} + 2\hat{\alpha}_{5}\, h^{an}\, \mathcal{R}_{j}{}^{b}{}_{ab}\, \mathcal{T}^{a'}{}_{ia'} - \\
&\quad \bar{\alpha}_{5}\, h^{an}\, \lambda_{\mathcal{R}\,a}{}^{b}{}_{jb}\, \mathcal{T}^{a'}{}_{ia'} + \bar{\alpha}_{5}\, h^{an}\, \lambda_{\mathcal{R}\,j}{}^{b}{}_{ab}\, \mathcal{T}^{a'}{}_{ia'} + 2\hat{\alpha}_{5}\, h^{an}\, \mathcal{R}_{a}{}^{b}{}_{ib}\, \mathcal{T}^{a'}{}_{ja'} - 2\hat{\alpha}_{5}\, h^{an}\, \mathcal{R}_{i}{}^{b}{}_{ab}\, \mathcal{T}^{a'}{}_{ja'} + \\
&\quad \bar{\alpha}_{5}\, h^{an}\, \lambda_{\mathcal{R}\,a}{}^{b}{}_{ib}\, \mathcal{T}^{a'}{}_{ja'} - \bar{\alpha}_{5}\, h^{an}\, \lambda_{\mathcal{R}\,i}{}^{b}{}_{ab}\, \mathcal{T}^{a'}{}_{ja'} + \frac{8}{3}\mathcal{M}_{\text{Pl}}^{2}\hat{\beta}_{3}\, h^{an}\, \mathcal{T}_{ija} - \frac{8}{3}\mathcal{M}_{\text{Pl}}^{2}\hat{\beta}_{3}\, h^{an}\, \mathcal{T}_{jia} - \\
&\quad \frac{4}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h^{an}\, \lambda_{\mathcal{T}\,aij} + \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h_{j}{}^{n}\, \lambda_{\mathcal{T}}{}^{a}{}_{ia} - \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{2}\, h_{j}{}^{n}\, \lambda_{\mathcal{T}}{}^{a}{}_{ia} - \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h_{i}{}^{n}\, \lambda_{\mathcal{T}}{}^{a}{}_{ja} + \\
&\quad \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{2}\, h_{i}{}^{n}\, \lambda_{\mathcal{T}}{}^{a}{}_{ja} + \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h^{an}\, \lambda_{\mathcal{T}\,ija} - \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\bar{\beta}_{1}\, h^{an}\, \lambda_{\mathcal{T}\,jia} + 2\hat{\alpha}_{5}\, h^{a'n}\, h_{j}{}^{a}\left(\mathcal{D}_{a}\mathcal{R}_{a'}{}^{b}{}_{ib}\right) - \\
&\quad 2\hat{\alpha}_{5}\, h^{a'n}\, h_{i}{}^{a}\left(\mathcal{D}_{a}\mathcal{R}_{a'}{}^{b}{}_{jb}\right) - 2\hat{\alpha}_{5}\, h^{a'n}\, h_{j}{}^{a}\left(\mathcal{D}_{a}\mathcal{R}_{i}{}^{b}{}_{a'b}\right) + 2\hat{\alpha}_{5}\, h^{a'n}\, h_{i}{}^{a}\left(\mathcal{D}_{a}\mathcal{R}_{j}{}^{b}{}_{a'b}\right) + \\
&\quad \bar{\alpha}_{5}\, h^{a'n}\, h_{j}{}^{a}\left(\mathcal{D}_{a}\lambda_{\mathcal{R}\,a'}{}^{b}{}_{ib}\right) - \bar{\alpha}_{5}\, h^{a'n}\, h_{i}{}^{a}\left(\mathcal{D}_{a}\lambda_{\mathcal{R}\,a'}{}^{b}{}_{jb}\right) - \bar{\alpha}_{5}\, h^{a'n}\, h_{j}{}^{a}\left(\mathcal{D}_{a}\lambda_{\mathcal{R}\,i}{}^{b}{}_{a'b}\right) + \\
&\quad \bar{\alpha}_{5}\, h^{a'n}\, h_{i}{}^{a}\left(\mathcal{D}_{a}\lambda_{\mathcal{R}\,j}{}^{b}{}_{a'b}\right) - 2\hat{\alpha}_{5}\, h^{aa'}\, h_{j}{}^{n}\left(\mathcal{D}_{a'}\mathcal{R}_{a}{}^{b}{}_{ib}\right) + 2\hat{\alpha}_{5}\, h^{aa'}\, h_{i}{}^{n}\left(\mathcal{D}_{a'}\mathcal{R}_{a}{}^{b}{}_{jb}\right) + \\
&\quad 2\hat{\alpha}_{5}\, h^{aa'}\, h_{j}{}^{n}\left(\mathcal{D}_{a'}\mathcal{R}_{i}{}^{b}{}_{ab}\right) - 2\hat{\alpha}_{5}\, h^{aa'}\, h_{i}{}^{n}\left(\mathcal{D}_{a'}\mathcal{R}_{j}{}^{b}{}_{ab}\right) - \bar{\alpha}_{5}\, h^{aa'}\, h_{j}{}^{n}\left(\mathcal{D}_{a'}\lambda_{\mathcal{R}\,a}{}^{b}{}_{ib}\right) + \\
&\quad \bar{\alpha}_{5}\, h^{aa'}\, h_{i}{}^{n}\left(\mathcal{D}_{a'}\lambda_{\mathcal{R}\,a}{}^{b}{}_{jb}\right) + \bar{\alpha}_{5}\, h^{aa'}\, h_{j}{}^{n}\left(\mathcal{D}_{a'}\lambda_{\mathcal{R}\,i}{}^{b}{}_{ab}\right) - \bar{\alpha}_{5}\, h^{aa'}\, h_{i}{}^{n}\left(\mathcal{D}_{a'}\lambda_{\mathcal{R}\,j}{}^{b}{}_{ab}\right) == 0
\end{aligned}
\tag{33}
$$

So in Eq. (33) we have the final version of the spin connection equation in the first-order formulation.

Concrete relation to manuscript: This in Eq. (33) is the spin connection equation for the most general set of Lagrangian parameters used in our manuscript.

Stress-energy conservation

We now have both sets of field equations at our disposal, with reference to their respective source currents. It would be helpful at this stage to verify the second conservation law from Eq. (3.23) p. 49 in Blagojević.

$$-\frac{1}{2}\, b^{l}{}_{m}\, b^{q}{}_{n}\, \mathcal{R}^{ij}{}_{lq}\, \sigma^{n}{}_{ij} - b^{l}{}_{m}\, b^{q}{}_{n}\, \mathcal{T}^{\hbar}{}_{lq}\, \tau^{n}{}_{\hbar} - \mathcal{A}^{l}{}_{\hbar n}\, b^{\hbar}{}_{m}\, \tau^{n}{}_{l} + b^{\hbar}{}_{m}\, \partial_{n}\tau^{n}{}_{\hbar} == 0 \tag{34}$$

We have confirmed that substitution of the field equations into this expression, which refers to the sources only, causes it to vanish. The calculation is computationally quite expensive, so we will omit it here.

Spin conservation

Equally, it is important to verify the second conservation law from Eq. (3.23) page 49 in Blagojević.

First term in the putatively vanishing expression.

$$-b_{jn}\, \tau(\Delta)^{n}{}_{i} + b_{in}\, \tau(\Delta)^{n}{}_{j} \tag{35}$$

Second term in the putatively vanishing expression.

$$-\mathcal{A}^{l}{}_{jn}\, \sigma^{n}{}_{il} - \mathcal{A}^{l}{}_{in}\, \sigma^{n}{}_{lj} + \partial_{n}\sigma^{n}{}_{ij} \tag{36}$$

Once again, by substituting for the source currents on the field equation shell, we have shown these terms to cancel, but the calculation itself is omitted for brevity.

Irreducible decomposition of spin connection equation

Having obtained Eq. (33), we find it important to decompose it into the irreducible parts under the actions of the Lorentz group.

The largest and most cumbersome part of the spin connection equation, the tensor part with 16 degrees of freedom.

$$\begin{aligned}
&-\frac{b_{w}{}^{a}\,\sigma_{a}{}^{r}{}_{u}}{3\mathcal{T}\mathcal{N}} + \frac{b_{u}{}^{a}\,\sigma_{a}{}^{r}{}_{w}}{3\mathcal{T}\mathcal{N}} + \frac{2\, b^{ra}\,\sigma_{auw}}{3\mathcal{T}\mathcal{N}} - \frac{b^{aa'}\,\delta^{r}{}_{w}\,\sigma_{a'ua}}{3\mathcal{T}\mathcal{N}} + \frac{b^{aa'}\,\delta^{r}{}_{u}\,\sigma_{a'wa}}{3\mathcal{T}\mathcal{N}} + \frac{4}{3}\hat{\alpha}_{5}\,\delta^{r}{}_{w}\,\mathcal{R}_{a'}{}^{b}{}_{ub}\,\mathcal{T}^{a}{}_{a}{}^{a'} - \\
&\frac{4}{3}\hat{\alpha}_{5}\,\delta^{r}{}_{u}\,\mathcal{R}_{a'}{}^{b}{}_{wb}\,\mathcal{T}^{a}{}_{a}{}^{a'} - \frac{4}{3}\hat{\alpha}_{5}\,\delta^{r}{}_{w}\,\mathcal{R}_{u}{}^{b}{}_{a'b}\,\mathcal{T}^{a}{}_{a}{}^{a'} + \frac{4}{3}\hat{\alpha}_{5}\,\delta^{r}{}_{u}\,\mathcal{R}_{w}{}^{b}{}_{a'b}\,\mathcal{T}^{a}{}_{a}{}^{a'} + \\
&\frac{2}{3}\bar{\alpha}_{5}\,\delta^{r}{}_{w}\,\lambda_{\mathcal{R}}{}_{a'}{}^{b}{}_{ub}\,\mathcal{T}^{a}{}_{a}{}^{a'} - \frac{2}{3}\bar{\alpha}_{5}\,\delta^{r}{}_{u}\,\lambda_{\mathcal{R}}{}_{a'}{}^{b}{}_{wb}\,\mathcal{T}^{a}{}_{a}{}^{a'} - \frac{2}{3}\bar{\alpha}_{5}\,\delta^{r}{}_{w}\,\lambda_{\mathcal{R}}{}_{u}{}^{b}{}_{a'b}\,\mathcal{T}^{a}{}_{a}{}^{a'} + \\
&\frac{2}{3}\bar{\alpha}_{5}\,\delta^{r}{}_{u}\,\lambda_{\mathcal{R}}{}_{w}{}^{b}{}_{a'b}\,\mathcal{T}^{a}{}_{a}{}^{a'} - \frac{4}{3}\hat{\alpha}_{5}\,\mathcal{R}_{u}{}^{a'}{}_{wa'}\,\mathcal{T}^{ar}{}_{a} + \frac{4}{3}\hat{\alpha}_{5}\,\mathcal{R}_{w}{}^{a'}{}_{ua'}\,\mathcal{T}^{ar}{}_{a} - \frac{2}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{u}{}^{a'}{}_{wa'}\,\mathcal{T}^{ar}{}_{a} + \\
&\frac{2}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{w}{}^{a'}{}_{ua'}\,\mathcal{T}^{ar}{}_{a} - \frac{1}{3}\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\hat{\alpha}_{0}\,\delta^{r}{}_{w}\,\mathcal{T}^{a}{}_{ua} - \frac{2}{3}\hat{\alpha}_{5}\,\mathcal{R}^{ra'}{}_{wa'}\,\mathcal{T}^{a}{}_{ua} + \frac{2}{3}\hat{\alpha}_{5}\,\mathcal{R}_{w}{}^{a'r}{}_{a'}\,\mathcal{T}^{a}{}_{ua} - \\
&\frac{1}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}^{ra'}{}_{wa'}\,\mathcal{T}^{a}{}_{ua} + \frac{1}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{w}{}^{a'r}{}_{a'}\,\mathcal{T}^{a}{}_{ua} + \frac{2}{3}\hat{\alpha}_{5}\,\delta^{r}{}_{w}\,\mathcal{R}_{a}{}^{b}{}_{a'b}\,\mathcal{T}^{a}{}_{u}{}^{a'} - \frac{2}{3}\hat{\alpha}_{5}\,\delta^{r}{}_{w}\,\mathcal{R}_{a'}{}^{b}{}_{ab}\,\mathcal{T}^{a}{}_{u}{}^{a'} + \\
&\frac{1}{3}\bar{\alpha}_{5}\,\delta^{r}{}_{w}\,\lambda_{\mathcal{R}}{}_{a}{}^{b}{}_{a'b}\,\mathcal{T}^{a}{}_{u}{}^{a'} - \frac{1}{3}\bar{\alpha}_{5}\,\delta^{r}{}_{w}\,\lambda_{\mathcal{R}}{}_{a'}{}^{b}{}_{ab}\,\mathcal{T}^{a}{}_{u}{}^{a'} + \frac{1}{3}\mathcal{M}_{\mathrm{Pl}}{}^{2}\,\hat{\alpha}_{0}\,\delta^{r}{}_{u}\,\mathcal{T}^{a}{}_{wa} +
\end{aligned}$$

$$
\begin{aligned}
&\tfrac{2}{3}\hat{\alpha}_{5}\,\mathcal{R}^{ra'}{}_{ua'}\,\mathcal{T}^{a}{}_{wa}-\tfrac{2}{3}\hat{\alpha}_{5}\,\mathcal{R}_{u}{}^{a'r}{}_{a'}\,\mathcal{T}^{a}{}_{wa}+\tfrac{1}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}^{ra'}{}_{ua'}\,\mathcal{T}^{a}{}_{wa}-\tfrac{1}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{u}{}^{a'r}{}_{a'}\,\mathcal{T}^{a}{}_{wa}-\\
&\tfrac{2}{3}\hat{\alpha}_{5}\,\delta^{r}{}_{u}\,\mathcal{R}_{a}{}^{b}{}_{a'b}\,\mathcal{T}^{a}{}_{w}{}^{a'}+\tfrac{2}{3}\hat{\alpha}_{5}\,\delta^{r}{}_{u}\,\mathcal{R}_{a'}{}^{b}{}_{ab}\,\mathcal{T}^{a}{}_{w}{}^{a'}-\tfrac{1}{3}\bar{\alpha}_{5}\,\delta^{r}{}_{u}\,\lambda_{\mathcal{R}}{}_{a}{}^{b}{}_{a'b}\,\mathcal{T}^{a}{}_{w}{}^{a'}+\\
&\tfrac{1}{3}\bar{\alpha}_{5}\,\delta^{r}{}_{u}\,\lambda_{\mathcal{R}}{}_{a'}{}^{b}{}_{ab}\,\mathcal{T}^{a}{}_{w}{}^{a'}+\tfrac{4}{3}\hat{\alpha}_{5}\,\mathcal{R}_{a}{}^{a'}{}_{wa'}\,\mathcal{T}^{r}{}_{u}{}^{a}-\tfrac{4}{3}\hat{\alpha}_{5}\,\mathcal{R}_{w}{}^{a'}{}_{aa'}\,\mathcal{T}^{r}{}_{u}{}^{a}+\tfrac{2}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{a}{}^{a'}{}_{wa'}\,\mathcal{T}^{r}{}_{u}{}^{a}-\\
&\tfrac{2}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{w}{}^{a'}{}_{aa'}\,\mathcal{T}^{r}{}_{u}{}^{a}+\tfrac{2}{3}\mathcal{M}_{\text{Pl}}{}^{2}\hat{\alpha}_{0}\,\mathcal{T}^{r}{}_{uw}-\tfrac{4}{3}\hat{\alpha}_{5}\,\mathcal{R}_{a}{}^{a'}{}_{ua'}\,\mathcal{T}^{r}{}_{w}{}^{a}+\tfrac{4}{3}\hat{\alpha}_{5}\,\mathcal{R}_{u}{}^{a'}{}_{aa'}\,\mathcal{T}^{r}{}_{w}{}^{a}-\\
&\tfrac{2}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{a}{}^{a'}{}_{ua'}\,\mathcal{T}^{r}{}_{w}{}^{a}+\tfrac{2}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{u}{}^{a'}{}_{aa'}\,\mathcal{T}^{r}{}_{w}{}^{a}+\tfrac{2}{3}\hat{\alpha}_{5}\,\mathcal{R}_{a}{}^{a'}{}_{wa'}\,\mathcal{T}_{u}{}^{ra}-\tfrac{2}{3}\hat{\alpha}_{5}\,\mathcal{R}_{w}{}^{a'}{}_{aa'}\,\mathcal{T}_{u}{}^{ra}+\\
&\tfrac{1}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{a}{}^{a'}{}_{wa'}\,\mathcal{T}_{u}{}^{ra}-\tfrac{1}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{w}{}^{a'}{}_{aa'}\,\mathcal{T}_{u}{}^{ra}+\tfrac{1}{3}\mathcal{M}_{\text{Pl}}{}^{2}\hat{\alpha}_{0}\,\mathcal{T}_{u}{}^{r}{}_{w}-\tfrac{2}{3}\hat{\alpha}_{5}\,\mathcal{R}_{a}{}^{a'r}{}_{a'}\,\mathcal{T}_{uw}{}^{a}+\\
&\tfrac{2}{3}\hat{\alpha}_{5}\,\mathcal{R}^{ra'}{}_{aa'}\,\mathcal{T}_{uw}{}^{a}-\tfrac{1}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{a}{}^{a'r}{}_{a'}\,\mathcal{T}_{uw}{}^{a}+\tfrac{1}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}^{ra'}{}_{aa'}\,\mathcal{T}_{uw}{}^{a}-\tfrac{2}{3}\hat{\alpha}_{5}\,\mathcal{R}_{a}{}^{a'}{}_{ua'}\,\mathcal{T}_{w}{}^{ra}+\\
&\tfrac{2}{3}\hat{\alpha}_{5}\,\mathcal{R}_{u}{}^{a'}{}_{aa'}\,\mathcal{T}_{w}{}^{ra}-\tfrac{1}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{a}{}^{a'}{}_{ua'}\,\mathcal{T}_{w}{}^{ra}+\tfrac{1}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{u}{}^{a'}{}_{aa'}\,\mathcal{T}_{w}{}^{ra}-\tfrac{1}{3}\mathcal{M}_{\text{Pl}}{}^{2}\hat{\alpha}_{0}\,\mathcal{T}_{w}{}^{r}{}_{u}+\\
&\tfrac{2}{3}\hat{\alpha}_{5}\,\mathcal{R}_{a}{}^{a'r}{}_{a'}\,\mathcal{T}_{wu}{}^{a}-\tfrac{2}{3}\hat{\alpha}_{5}\,\mathcal{R}^{ra'}{}_{aa'}\,\mathcal{T}_{wu}{}^{a}+\tfrac{1}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{a}{}^{a'r}{}_{a'}\,\mathcal{T}_{wu}{}^{a}-\tfrac{1}{3}\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}^{ra'}{}_{aa'}\,\mathcal{T}_{wu}{}^{a}+\\
&\tfrac{2}{3}\mathcal{M}_{\text{Pl}}{}^{2}\bar{\beta}_{1}\,\delta^{r}{}_{w}\,\lambda_{\mathcal{T}}{}^{a}{}_{ua}-\tfrac{2}{3}\mathcal{M}_{\text{Pl}}{}^{2}\bar{\beta}_{1}\,\delta^{r}{}_{u}\,\lambda_{\mathcal{T}}{}^{a}{}_{wa}-\tfrac{4}{3}\mathcal{M}_{\text{Pl}}{}^{2}\bar{\beta}_{1}\,\lambda_{\mathcal{T}}{}^{r}{}_{uw}-\tfrac{2}{3}\mathcal{M}_{\text{Pl}}{}^{2}\bar{\beta}_{1}\,\lambda_{\mathcal{T}}{}_{u}{}^{r}{}_{w}+\\
&\tfrac{2}{3}\mathcal{M}_{\text{Pl}}{}^{2}\bar{\beta}_{1}\,\lambda_{\mathcal{T}}{}_{w}{}^{r}{}_{u}+\tfrac{2}{3}\hat{\alpha}_{5}\,h_{w}{}^{a}\left(\mathcal{D}_{a}\mathcal{R}^{ra'}{}_{ua'}\right)-\tfrac{2}{3}\hat{\alpha}_{5}\,h_{u}{}^{a}\left(\mathcal{D}_{a}\mathcal{R}^{ra'}{}_{wa'}\right)-\tfrac{2}{3}\hat{\alpha}_{5}\,h_{w}{}^{a}\left(\mathcal{D}_{a}\mathcal{R}_{u}{}^{a'r}{}_{a'}\right)-\\
&\tfrac{4}{3}\hat{\alpha}_{5}\,h^{ra}\left(\mathcal{D}_{a}\mathcal{R}_{u}{}^{a'}{}_{wa'}\right)+\tfrac{2}{3}\hat{\alpha}_{5}\,h_{u}{}^{a}\left(\mathcal{D}_{a}\mathcal{R}_{w}{}^{a'r}{}_{a'}\right)+\tfrac{4}{3}\hat{\alpha}_{5}\,h^{ra}\left(\mathcal{D}_{a}\mathcal{R}_{w}{}^{a'}{}_{ua'}\right)+\tfrac{1}{3}\bar{\alpha}_{5}\,h_{w}{}^{a}\left(\mathcal{D}_{a}\lambda_{\mathcal{R}}{}^{ra'}{}_{ua'}\right)-\\
&\tfrac{1}{3}\bar{\alpha}_{5}\,h_{u}{}^{a}\left(\mathcal{D}_{a}\lambda_{\mathcal{R}}{}^{ra'}{}_{wa'}\right)-\tfrac{1}{3}\bar{\alpha}_{5}\,h_{w}{}^{a}\left(\mathcal{D}_{a}\lambda_{\mathcal{R}}{}_{u}{}^{a'r}{}_{a'}\right)-\tfrac{2}{3}\bar{\alpha}_{5}\,h^{ra}\left(\mathcal{D}_{a}\lambda_{\mathcal{R}}{}_{u}{}^{a'}{}_{wa'}\right)+\tfrac{1}{3}\bar{\alpha}_{5}\,h_{u}{}^{a}\left(\mathcal{D}_{a}\lambda_{\mathcal{R}}{}_{w}{}^{a'r}{}_{a'}\right)+\\
&\tfrac{2}{3}\bar{\alpha}_{5}\,h^{ra}\left(\mathcal{D}_{a}\lambda_{\mathcal{R}}{}_{w}{}^{a'}{}_{ua'}\right)-\tfrac{2}{3}\hat{\alpha}_{5}\,\delta^{r}{}_{w}\,h^{aa'}\left(\mathcal{D}_{a'}\mathcal{R}_{a}{}^{b}{}_{ub}\right)+\tfrac{2}{3}\hat{\alpha}_{5}\,\delta^{r}{}_{u}\,h^{aa'}\left(\mathcal{D}_{a'}\mathcal{R}_{a}{}^{b}{}_{wb}\right)+\\
&\tfrac{2}{3}\hat{\alpha}_{5}\,\delta^{r}{}_{w}\,h^{aa'}\left(\mathcal{D}_{a'}\mathcal{R}_{u}{}^{b}{}_{ab}\right)-\tfrac{2}{3}\hat{\alpha}_{5}\,\delta^{r}{}_{u}\,h^{aa'}\left(\mathcal{D}_{a'}\mathcal{R}_{w}{}^{b}{}_{ab}\right)-\tfrac{1}{3}\bar{\alpha}_{5}\,\delta^{r}{}_{w}\,h^{aa'}\left(\mathcal{D}_{a'}\lambda_{\mathcal{R}}{}_{a}{}^{b}{}_{ub}\right)+\\
&\tfrac{1}{3}\bar{\alpha}_{5}\,\delta^{r}{}_{u}\,h^{aa'}\left(\mathcal{D}_{a'}\lambda_{\mathcal{R}}{}_{a}{}^{b}{}_{wb}\right)+\tfrac{1}{3}\bar{\alpha}_{5}\,\delta^{r}{}_{w}\,h^{aa'}\left(\mathcal{D}_{a'}\lambda_{\mathcal{R}}{}_{u}{}^{b}{}_{ab}\right)-\tfrac{1}{3}\bar{\alpha}_{5}\,\delta^{r}{}_{u}\,h^{aa'}\left(\mathcal{D}_{a'}\lambda_{\mathcal{R}}{}_{w}{}^{b}{}_{ab}\right)==0
\end{aligned}
$$

The simplest vector part of the spin equation, with four degrees of freedom.

$$
\begin{aligned}
&-\frac{b^{aa'}\ \sigma_{a'wa}}{\mathcal{J}\,\mathcal{N}}-2\hat{\alpha}_{5}\,\mathcal{R}_{a'}{}^{b}{}_{wb}\,\mathcal{T}^{a}{}_{a}{}^{a'}+2\hat{\alpha}_{5}\,\mathcal{R}_{w}{}^{b}{}_{a'b}\,\mathcal{T}^{a}{}_{a}{}^{a'}-\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{a'}{}^{b}{}_{wb}\,\mathcal{T}^{a}{}_{a}{}^{a'}+\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{w}{}^{b}{}_{a'b}\,\mathcal{T}^{a}{}_{a}{}^{a'}+\\
&2\mathcal{M}_{\text{Pl}}{}^{2}\hat{\alpha}_{0}\,\mathcal{T}^{a}{}_{wa}+4\mathcal{M}_{\text{Pl}}{}^{2}\hat{\beta}_{2}\,\mathcal{T}^{a}{}_{wa}+2\hat{\alpha}_{5}\,\mathcal{R}_{a}{}^{b}{}_{a'b}\,\mathcal{T}^{a}{}_{w}{}^{a'}-2\hat{\alpha}_{5}\,\mathcal{R}_{a'}{}^{b}{}_{ab}\,\mathcal{T}^{a}{}_{w}{}^{a'}+\\
&\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{a}{}^{b}{}_{a'b}\,\mathcal{T}^{a}{}_{w}{}^{a'}-\bar{\alpha}_{5}\,\lambda_{\mathcal{R}}{}_{a'}{}^{b}{}_{ab}\,\mathcal{T}^{a}{}_{w}{}^{a'}+2\mathcal{M}_{\text{Pl}}{}^{2}\bar{\beta}_{2}\,\lambda_{\mathcal{T}}{}^{a}{}_{wa}+4\hat{\alpha}_{5}\,h^{aa'}\left(\mathcal{D}_{a'}\mathcal{R}_{a}{}^{b}{}_{wb}\right)-
\end{aligned}
\qquad (38)
$$

$$4\hat{\alpha}_{5}\, h^{aa'}\left(\mathcal{D}_{a'}\mathcal{R}_{w}{}^{b}{}_{ab}\right)+2\bar{\alpha}_{5}\, h^{aa'}\left(\mathcal{D}_{a'}\lambda_{\mathcal{R}}{}_{a}{}^{b}{}_{wb}\right)-2\bar{\alpha}_{5}\, h^{aa'}\left(\mathcal{D}_{a'}\lambda_{\mathcal{R}}{}_{w}{}^{b}{}_{ab}\right)==0$$

The other simplest axial vector part of the spin equation, also with four degrees of freedom.

$$\begin{aligned}&-\frac{b^{aa'}\ \epsilon\gamma^{c}{}_{aru}\ \sigma_{a'}{}^{ru}}{\mathcal{J}\mathcal{N}}+8\hat{\alpha}_{5}\ \epsilon\gamma^{c}{}_{a'uw}\ \mathcal{R}^{ru}{}_{r}{}^{w}\ \mathcal{T}^{a}{}_{a}{}^{a'}+4\bar{\alpha}_{5}\ \epsilon\gamma^{c}{}_{a'uw}\ \lambda_{\mathcal{R}}{}^{ru}{}_{r}{}^{w}\ \mathcal{T}^{a}{}_{a}{}^{a'}-\\ &\mathcal{M}_{\mathrm{Pl}}{}^{2}\hat{\alpha}_{0}\ \epsilon\gamma^{c}{}_{aa'r}\ \mathcal{T}^{aa'r}-8\mathcal{M}_{\mathrm{Pl}}{}^{2}\hat{\beta}_{3}\ \epsilon\gamma^{c}{}_{aa'r}\ \mathcal{T}^{aa'r}-4\hat{\alpha}_{5}\ \epsilon\gamma^{c}{}_{arw}\ \mathcal{R}_{a'}{}^{u}{}_{u}{}^{w}\ \mathcal{T}^{aa'r}+\\ &4\hat{\alpha}_{5}\ \epsilon\gamma^{c}{}_{arw}\ \mathcal{R}^{uw}{}_{a'u}\ \mathcal{T}^{aa'r}-2\bar{\alpha}_{5}\ \epsilon\gamma^{c}{}_{arw}\ \lambda_{\mathcal{R}}{}_{a'}{}^{u}{}_{u}{}^{w}\ \mathcal{T}^{aa'r}+2\bar{\alpha}_{5}\ \epsilon\gamma^{c}{}_{arw}\ \lambda_{\mathcal{R}}{}^{uw}{}_{a'u}\ \mathcal{T}^{aa'r}-\\ &8\hat{\alpha}_{5}\ \epsilon\gamma^{c}{}_{auw}\ h^{aa'}\left(\mathcal{D}_{a'}\mathcal{R}^{ru}{}_{r}{}^{w}\right)-4\bar{\alpha}_{5}\ \epsilon\gamma^{c}{}_{auw}\ h^{aa'}\left(\mathcal{D}_{a'}\lambda_{\mathcal{R}}{}^{ru}{}_{r}{}^{w}\right)==0\end{aligned}\tag{39}$$

Irrep conventions for spin and torsion

Here are the conventions used for the irreps of the torsion tensor. Note that the totally antisymmetric tensor will be formatted as $\epsilon\gamma^{m}{}_{nsa}$.

$$\mathcal{T}^{m}{}_{ns}\tag{40}$$

$$\frac{2}{3}\,{}^{1}\!\mathcal{T}^{m}{}_{ns}-\frac{2}{3}\,{}^{1}\!\mathcal{T}^{m}{}_{sn}-\frac{1}{3}\,\delta^{m}{}_{s}\ {}^{2}\!\mathcal{T}_{n}+\frac{1}{3}\,\delta^{m}{}_{n}\ {}^{2}\!\mathcal{T}_{s}+\epsilon\gamma^{m}{}_{nsa}\ {}^{3}\!\mathcal{T}^{a}\tag{41}$$

And we also want to see what the teleparallel Lagrangian looks like, this is the double-bar T symbol which must be multiplied by the measure and half the Planck mass, with a positive sign (see Eq. (15) in arXiv:2006.03581).

$$\frac{4}{9}\,{}^{1}\!\mathcal{T}_{ijk}\ {}^{1}\!\mathcal{T}^{ijk}-\frac{4}{9}\,{}^{1}\!\mathcal{T}_{ikj}\ {}^{1}\!\mathcal{T}^{ijk}-\frac{2}{3}\,{}^{2}\!\mathcal{T}_{i}\ {}^{2}\!\mathcal{T}^{i}+\frac{3}{2}\,{}^{3}\!\mathcal{T}_{i}\ {}^{3}\!\mathcal{T}^{i}\tag{42}$$

We will also take a look at our irrep conventions for the spin tensor. Remembering the form we arrived at in Eq. (15), we see that the following combination must have all Roman (Lorentz) indices, and we will understand that it is the version with all Roman indices which is decomposed.

$$h^{k}{}_{m}\ \sigma^{m}{}_{ij}\tag{43}$$

$$\frac{2}{3}\,{}^{1}\!\sigma_{i}{}^{k}{}_{j}-\frac{2}{3}\,{}^{1}\!\sigma_{j}{}^{k}{}_{i}-\frac{1}{3}\,\delta_{j}{}^{k}\ {}^{2}\!\sigma_{i}+\frac{1}{3}\,\delta_{i}{}^{k}\ {}^{2}\!\sigma_{j}-\epsilon\gamma_{ija}{}^{k}\ {}^{3}\!\sigma^{a}\tag{44}$$

So to make double sure, note that in Eq. (43) and Eq. (44) the decomposed parts are defined only for the version of the spin tensor which has all Lorentz indices, so the version of the tensor we decompose has to have an extra factor of the (inverse) tetrad (and an implicit curved-space metric to lower the Greek index on that field) so as to get rid of the natural coordinate index in our definition of the undecomposed spin tensor.

Some care has to be taken when understanding how the vector and axial vector parts of the torsion couple to the respective parts of the spin tensor. The reason for this is that the spin tensor is constructed, as per the discussion above, with two Lorentz indices, and so in the stress-energy field equation there is a possible factor of two that can go missing unless the indices are kept carefully tracked (think about what it means to take variations with respect to the tetrad vs with respect to the metric tensor). When we go over to the GeoHiGGS second-order formulation in the next part of the script, all appearances of the tetrad and its inverse are simply replaced by the Kronecker symbol. This is safe in the gravity sector of the theory, but not in the matter coupling.

To get a good understanding of how factors of the tetrad enter in, we decompose both factors in Eq. (17) but pretend that the torsion carries only Greek (coordinate) indices. This is an abuse of notation for the HiGGS variables, but we only do it at this one step so as to make sure that the matter coupling is correct.

$$-\frac{1}{3} h^{a a'} \; {}^{\dot{2}}\sigma_{a} \; {}^{\dot{2}}\mathcal{T}_{a'} - \frac{3}{2} h^{a a'} \; {}^{\dot{3}}\sigma_{a} \; {}^{\dot{3}}\mathcal{T}_{a'} \tag{45}$$

The form in Eq. (45) has the tensor parts of both fields neglected. We'll use it later on in our script.

First-to-second order

At this point we have developed the field equations as far as we are able within the HiGGS environment; recall that HiGGS is not purpose-built for this Lagrangian analysis, but rather computes the Hamiltonian constraint structure. HiGGS is useful to us only insofar as it provides the conventions associated with the Poincaré gauge theory, and the geometric multipliers defined in arXiv:2205.13534. To proceed further, we wish to use some of the built-in functionality of xAct.

Specifically, HiGGS is based in the first-order or gauge-theoretic formulation of the theory, which is mostly shared by Blagojević. A key aspect of our manuscript is that the second-order formulation appears rather more revealing in the analysis. xAct and xTensor, which were not developed by the authors, already contain some very sophisticated tools for working with the second-order formulation of torsion gravity.

To access these tools, we face another problem. HiGGS is purposely using a flat spacetime, so that the Poincaré gauge theory is seen from a natural Yang-Mills perspective. In contrast the xAct and xTensor setup relies on the geometric interpretation, whereby the Riemann-Cartan curvature and torsion are geometric properties of the manifold.

It turns out to be possible to switch out a flat manifold for a curved/torsionful one with the same name, within one xAct kernel session. To do so requires some very careful treatment of defined symbols across multiple Wolfram language `contexts' (i.e. scopes). This is what GeoHiGGS is defined to do. Given a HiGGS session, having defined a very large number of first-order formulation quantities, an application of the BuildGeoHiGGS command from GeoHiGGS will destroy the underlying Minkowski spacetime and replace it with a curved/torsionful replacement. Natural re-interpretations of various other HiGGS quantities are also made at the same time. The end result is a standard, working xAct/xTensor session with the HiGGS-defined and user-defined quantities still present in the kernel memory.

We run BuildGeoHiGGS now...

```
** UndefTensor: Undefined weight +2 density DetG
** UndefTensor: Undefined symmetric Christoffel tensor ChristoffelCD
** UndefTensor: Undefined tensor ChristoffelCDGaugeCovD
** UndefTensor: Undefined symmetric cosmological Einstein tensor EinsteinCCCD
** UndefTensor: Undefined Einstein tensor EinsteinCD
** UndefTensor: Undefined Kretschmann scalar KretschmannCD
** UndefTensor: Undefined Ricci tensor RicciCD
** UndefTensor: Undefined Ricci scalar RicciScalarCD
** UndefTensor: Undefined Riemann tensor RiemannCD
** UndefTensor: Undefined symmetric cosmological Schouten tensor SchoutenCCCD
** UndefTensor: Undefined symmetric Schouten tensor SchoutenCD
** UndefTensor: Undefined symmetrized Riemann tensor SymRiemannCD
** UndefTensor: Undefined TFRicci tensor TFRicciCD
** UndefTensor: Undefined torsion tensor TorsionCD
** UndefTensor: Undefined Weyl tensor WeylCD
** UndefCovD: Undefined covariant derivative CD
** UndefTensor: Undefined tetrametric TetraG†
** UndefTensor: Undefined tetrametric TetraG
** UndefTensor: Undefined antisymmetric tensor epsilonG
** UndefTensor: Undefined symmetric metric tensor G
** UndefTensor: Undefined tensor PerturbationGP
** UndefInertHead: Undefined projector inert-head ProjectorGP
** UndefTensor: Undefined acceleration vector AccelerationV
** UndefTensor: Undefined extrinsic curvature tensor ExtrinsicKGP
** UndefTensor: Undefined weight +2 density DetGP
** UndefTensor: Undefined symmetric Christoffel tensor ChristoffelCDP
** UndefTensor: Undefined symmetric cosmological Einstein tensor EinsteinCCCDP
** UndefTensor: Undefined symmetric Einstein tensor EinsteinCDP
** UndefTensor: Undefined Kretschmann scalar KretschmannCDP
```

```
** UndefTensor: Undefined symmetric Ricci tensor RicciCDP
** UndefTensor: Undefined Ricci scalar RicciScalarCDP
** UndefTensor: Undefined Riemann tensor RiemannCDP
** UndefTensor: Undefined symmetric cosmological Schouten tensor SchoutenCCCDP
** UndefTensor: Undefined symmetric Schouten tensor SchoutenCDP
** UndefTensor: Undefined symmetrized Riemann tensor SymRiemannCDP
** UndefTensor: Undefined symmetric TFRicci tensor TFRicciCDP
** UndefTensor: Undefined torsion tensor TorsionCDP
** UndefTensor: Undefined Weyl tensor WeylCDP
** UndefCovD: Undefined covariant derivative CDP
** UndefTensor: Undefined tetrametric TetraGP†
** UndefTensor: Undefined tetrametric TetraGP
** UndefTensor: Undefined antisymmetric tensor epsilonGP
** UndefTensor: Undefined symmetric metric tensor GP
** DefTensor: Defining symmetric metric tensor GeoG[-a, -b].
** DefTensor: Defining antisymmetric tensor epsilonGeoG[-a, -a1, -b, -b1].
** DefTensor: Defining tetrametric TetraGeoG[-a, -a1, -b, -b1].
** DefTensor: Defining tetrametric TetraGeoG†[-a, -a1, -b, -b1].
** DefCovD: Defining covariant derivative GeoCovD[-a].
** DefTensor: Defining vanishing torsion tensor TorsionGeoCovD[a, -a1, -b].
** DefTensor: Defining symmetric Christoffel tensor ChristoffelGeoCovD[a, -a1, -b].
** DefTensor: Defining Riemann tensor RiemannGeoCovD[-a, -a1, -b, -b1].
** DefTensor: Defining symmetric Ricci tensor RicciGeoCovD[-a, -a1].
** DefCovD:  Contractions of Riemann automatically replaced by Ricci.
** DefTensor: Defining Ricci scalar RicciScalarGeoCovD[].
** DefCovD:  Contractions of Ricci automatically replaced by RicciScalar.
** DefTensor: Defining symmetric Einstein tensor EinsteinGeoCovD[-a, -a1].
** DefTensor: Defining Weyl tensor WeylGeoCovD[-a, -a1, -b, -b1].
** DefTensor: Defining symmetric TFRicci tensor TFRicciGeoCovD[-a, -a1].
** DefTensor: Defining Kretschmann scalar KretschmannGeoCovD[].
** DefCovD:  Computing RiemannToWeylRules for dim 4
** DefCovD:  Computing RicciToTFRicci for dim 4
```

```
** DefCovD:  Computing RicciToEinsteinRules for dim 4

** DefTensor: Defining symmetrized Riemann tensor SymRiemannGeoCovD[-a, -a1, -b, -b1].

** DefTensor: Defining symmetric Schouten tensor SchoutenGeoCovD[-a, -a1].

** DefTensor: Defining
 symmetric cosmological Schouten tensor SchoutenCCGeoCovD[LI[_], -a, -a1].

** DefTensor: Defining
 symmetric cosmological Einstein tensor EinsteinCCGeoCovD[LI[_], -a, -a1].

** DefCovD: Defining covariant derivative GeoCovD[-a]. to be symmetrizable

** DefTensor: Defining weight +2 density DetGeoG[]. Determinant.

** DefParameter: Defining parameter PerturbationParameterGeoG.

** DefTensor: Defining tensor PerturbationGeoG[LI[order], -a, -a1].

** DefCovD: Defining covariant derivative GeoGaugeCovD[-a].

** DefTensor: Defining torsion tensor TorsionGeoGaugeCovD[a, -a1, -b].

** DefTensor: Defining non-symmetric Christoffel tensor
 ChristoffelGeoGaugeCovD[a, -a1, -b].

** DefTensor: Defining Riemann tensor
 RiemannGeoGaugeCovD[-a, -a1, -b, -b1]. Antisymmetric pairs cannot be exchanged.

** DefTensor: Defining non-symmetric Ricci tensor RicciGeoGaugeCovD[-a, -a1].

** DefCovD:  Contractions of Riemann automatically replaced by Ricci.

** DefTensor: Defining Ricci scalar RicciScalarGeoGaugeCovD[].

** DefCovD:  Contractions of Ricci automatically replaced by RicciScalar.

** DefTensor: Defining non-symmetric Einstein tensor EinsteinGeoGaugeCovD[-a, -a1].

** DefTensor: Defining Weyl tensor
 WeylGeoGaugeCovD[-a, -a1, -b, -b1]. Antisymmetric pairs cannot be exchanged.

** DefTensor: Defining non-symmetric TFRicci tensor TFRicciGeoGaugeCovD[-a, -a1].

** DefTensor: Defining Kretschmann scalar KretschmannGeoGaugeCovD[].

** DefCovD:  Computing RiemannToWeylRules for dim 4

** DefCovD:  Computing RicciToTFRicci for dim 4

** DefCovD:  Computing RicciToEinsteinRules for dim 4

** DefTensor: Defining non-symmetric Schouten tensor SchoutenGeoGaugeCovD[-a, -a1].

** DefTensor: Defining non-symmetric cosmological Schouten tensor
 SchoutenCCGeoGaugeCovD[LI[_], -a, -a1].

** DefTensor: Defining non-symmetric cosmological Einstein tensor
 EinsteinCCGeoGaugeCovD[LI[_], -a, -a1].
```

** BuildGeoHiGGS: Incorporating the binary at /home/williamb/Documents/paper-f/SupplementalMaterials/FieldEquations/GeoHiGGS.mx...

...and so now BuildGeoHiGGS has completed its execution.

The Belinfante-Rosenfeld tensor

Now we define the usual Einstein stress-energy tensor `Theta[-a, -b]`, according to Eq. (3.59) on p. 67 of Blagojević. This is a further development of our `split' stress-energy tensor in Eq. (28). Note, by the way, that all indices are now Greek (and will remain so from this point onwards) since we ran BuildGeoHiGGS.

** DefTensor: Defining tensor Theta[-a, -b].

$$-\theta_{\kappa}{}^{\mu}+\tau(\Delta)^{\mu}{}_{\kappa}-\frac{1}{2}\left(\mathring{\nabla}_{\lambda}\sigma_{\kappa}{}^{\mu\lambda}\right)+\frac{1}{2}\left(\mathring{\nabla}_{\lambda}\sigma^{\lambda}{}_{\kappa}{}^{\mu}\right)-\frac{1}{2}\left(\mathring{\nabla}_{\lambda}\sigma^{\mu}{}_{\kappa}{}^{\lambda}\right)==0 \tag{46}$$

** MakeQuotientRule: canonicalised expression with tensor substituted by rule:

0

Symmetrising the matter stress-energy tensor

The conservation law for the spin tensor as set out in Eq. (3.23) on page 49 of Blagojević, leads to a rule for symmetrising the stress-energy tensor.

** DefTensor: Defining tensor ChristoffelGeoCovDGeoGaugeCovD[a, -a1, -b].

$$\tau_{\iota\theta} \tag{47}$$

$$-\frac{1}{4}\sigma^{\alpha}{}_{\theta}{}^{\alpha'}\mathcal{T}_{\alpha\iota\alpha'}+\frac{1}{4}\sigma^{\alpha}{}_{\iota}{}^{\alpha'}\mathcal{T}_{\alpha\theta\alpha'}-\frac{1}{4}\sigma^{\alpha}{}_{\theta}{}^{\alpha'}\mathcal{T}_{\alpha'\iota\alpha}+\frac{1}{4}\sigma^{\alpha}{}_{\iota}{}^{\alpha'}\mathcal{T}_{\alpha'\theta\alpha}-\frac{1}{4}\sigma^{\alpha}{}_{\theta}{}^{\alpha'}\mathcal{T}_{\iota\alpha\alpha'}+\frac{1}{4}\sigma^{\alpha}{}_{\iota}{}^{\alpha'}\mathcal{T}_{\theta\alpha\alpha'}+\frac{\tau_{\iota\theta}}{2}+\frac{\tau_{\theta\iota}}{2}+\frac{1}{2}\left(\mathring{\nabla}_{\alpha}\sigma^{\alpha}{}_{\iota\theta}\right) \tag{48}$$

The torsion Maxwell tensors

We define a pair of Maxwell tensors naturally according to the vector torsion.

** DefTensor: Defining tensor Maxwell1[-a, -b].

** DefTensor: Defining tensor NonlinearMaxwell1[-a, -b].

** DefTensor: Defining tensor Maxwell2[-a, -b].

** DefTensor: Defining tensor Maxwell3[-a, -b].

We define a rule to replace the third Maxwell tensor with its dual (with somewhat loosely-defined conventions for the dual: we only require an invertible definition since we will eliminate it in all our final expressions).

```
** DefTensor: Defining tensor DualMaxwell3[-a, -b].
```

```
** MakeQuotientRule: canonicalised expression with tensor substituted by rule:
```

0

$$\overset{3}{\mathcal{F}}_{\mu\nu} \tag{49}$$

$$\overset{3}{\star\mathcal{F}}{}^{\alpha\alpha'}\ \epsilon\mathring{g}_{\mu\nu\alpha\alpha'} \tag{50}$$

$$\overset{3}{\mathcal{F}}_{\mu\nu} \tag{51}$$

We know that these Maxwell tensors will also satisfy the Bianchi identities, even in the curved, Riemannian spacetime.

```
Rules {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ≪374≫} have been declared as generic Rules.
```

```
Rules {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ≪374≫} have been declared as generic Rules.
```

We also construct some rules which reverse divergences of gradients of the torsion vector and axial vector.

$$\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\alpha'}\overset{2}{\mathcal{T}}{}^{\alpha'} \tag{52}$$

$$-R\left[\mathring{\nabla}\right]_{\alpha\alpha'}\ \overset{2}{\mathcal{T}}{}^{\alpha'}+\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{2}{\mathcal{T}}{}^{\alpha'} \tag{53}$$

```
** MakeQuotientRule: canonicalised expression with tensor substituted by rule:
```

0

$$\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\alpha'}\overset{3}{\mathcal{T}}{}^{\alpha'} \tag{54}$$

$$-R\left[\mathring{\nabla}\right]_{\alpha\alpha'}\ \overset{3}{\mathcal{T}}{}^{\alpha'}+\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{3}{\mathcal{T}}{}^{\alpha'} \tag{55}$$

```
** MakeQuotientRule: canonicalised expression with tensor substituted by rule:
```

0

Thus, we have rules to move between Eqs. (52), and (53), or Eqs. (54), and (55) (which are identical pairs of expressions). Similarly, some rules are also constructed to express gradient of divergence as divergence of gradient.

$$\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\chi}\overset{2}{\mathcal{T}}{}^{\alpha} \tag{56}$$

$$R\left[\mathring{\nabla}\right]^{\chi}{}_{\alpha}\ \overset{2}{\mathcal{T}}{}^{\alpha}+\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha}\overset{2}{\mathcal{T}}{}^{\alpha} \tag{57}$$

```
** MakeQuotientRule: canonicalised expression with tensor substituted by rule:
0
```

$$\overset{\circ}{\nabla}_{\alpha}\overset{\circ}{\nabla}^{\chi}\,\overset{3}{\mathcal{T}}^{\alpha} \tag{58}$$

$$R[\overset{\circ}{\nabla}]^{\chi}{}_{\alpha}\;\overset{3}{\mathcal{T}}^{\alpha}+\overset{\circ}{\nabla}^{\chi}\overset{\circ}{\nabla}_{\alpha}\overset{3}{\mathcal{T}}^{\alpha} \tag{59}$$

```
** MakeQuotientRule: canonicalised expression with tensor substituted by rule:
0
```

Again, the pairs of expressions above are clearly identical.

The Maxwell Riemann-Cartan Ricci tensor

Now we move on to translate the irreducible parts of the Riemann-Cartan tensor into kinetic terms for the torsion.

First we expand the Maxwell part of the Riemann-Cartan tensor, whilst suppressing the tensor part of the torsion.

$$\overset{5}{\mathcal{R}}_{\alpha\beta} \tag{60}$$

$$\frac{\overset{2}{\mathcal{F}}_{\alpha\beta}}{3}-\frac{1}{4}\,\epsilon\,\overset{\circ}{g}_{\alpha\beta\alpha'\beta'}\;\overset{3}{\mathcal{F}}^{\alpha'\beta'} \tag{61}$$

Now we can make some rules to convert between the torsion Maxwell tensors and the Riemann-Cartan Maxwell tensor.

$$-\frac{1}{3}\,\overset{2}{\mathcal{F}}_{\alpha\beta}+\frac{1}{4}\,\epsilon\,\overset{\circ}{g}_{\alpha\beta\alpha'\beta'}\;\overset{3}{\mathcal{F}}^{\alpha'\beta'}+\overset{5}{\mathcal{R}}_{\alpha\beta}==0 \tag{62}$$

```
** MakeQuotientRule: canonicalised expression with tensor substituted by rule:
0
```

$$-\frac{1}{3}\,\epsilon\,\overset{\circ}{g}_{\iota\theta\alpha\alpha'}\;\overset{2}{\mathcal{F}}^{\alpha\alpha'}-\overset{3}{\mathcal{F}}_{\iota\theta}+\epsilon\,\overset{\circ}{g}_{\iota\theta\alpha\alpha'}\;\overset{5}{\mathcal{R}}^{\alpha\alpha'}==0 \tag{63}$$

```
** MakeQuotientRule: canonicalised expression with tensor substituted by rule:
0
```

Concrete relation to manuscript: We begin to see here why the dynamics of our new theory are viable. With the tensor torsion suppressed, it is straightforward to obtain the kinetic structure of our effective model from the Riemann-Cartan invariant in our fundamental model. There will be a cross-term, but this is a total divergence.

```
** MakeQuotientRule: canonicalised expression with tensor substituted by rule:
0
```

Non-density source currents

Define a spin tensor, to replace the spin tensor density in Eq. (33) in accordance with the paragraph following Eq. (3.58) on p. 67 in Blagojević.

```
** DefTensor: Defining tensor SigmaTensor[i, j, k].
```

Define irreducible parts of this tensor using HiGGS conventions.

```
** DefTensor: Defining tensor SigmaTensor1[-i, -j, -k].
** DefTensor: Defining tensor SigmaTensor2[i].
** DefTensor: Defining tensor SigmaTensor3[i].
   Rules {1, 2, 3, 4} have been declared as DownValues for SigmaTensor1.
   Rules {1, 2} have been declared as DownValues for SigmaTensor1.
```

$$\frac{\sigma_{\iota\theta\kappa}}{\sqrt{-\overset{\approx}{\mathring{g}}}} \tag{64}$$

$$S_{\iota\theta\kappa} \tag{65}$$

$$\frac{2}{3}\,{}^{1}\!S_{\iota\theta\kappa} - \frac{2}{3}\,{}^{1}\!S_{\iota\kappa\theta} - \frac{1}{3}\,\mathring{g}_{\iota\kappa}\,{}^{2}\!S_{\theta} + \frac{1}{3}\,\mathring{g}_{\iota\theta}\,{}^{2}\!S_{\kappa} + \epsilon\,\mathring{g}_{\iota\theta\kappa\alpha}\,{}^{3}\!S^{\alpha} \tag{66}$$

Similarly, define the symmetric (metrical) matter stress-energy tensor `ThetaTensor[-i, -j]`, to replace the tensor density in Eq. (46).

```
** DefTensor: Defining tensor ThetaTensor[-i, -j].
** DefTensor: Defining tensor ThetaTensor1[-i, -j].
** DefTensor: Defining tensor ThetaTensor2[].
   Rules {1, 2} have been declared as DownValues for ThetaTensor1.
```

$$\frac{\theta_{\iota\theta}}{\sqrt{-\overset{\approx}{\mathring{g}}}} \tag{67}$$

$$T_{\iota\theta} \tag{68}$$

$${}^{1}\!T_{\iota\theta} + \frac{1}{4}\,\mathring{g}_{\iota\theta}\,{}^{2}\!T \tag{69}$$

Now we have set up some tools in the geometric, second-order formulation. The rest of this script will be devoted to an analysis of the case with and without multipliers.

Healthy spectrum with multipliers

So far we have shown that the case without the use of multipliers leads to the sick theory in Eq. (XXX). We would now like to transition to the case where we introduce a multiplier which disables the tensor part of the torsion.

Effective theory ansatz

The first task, since this is a case for which we believe we can exactly recover the effective theory, is to construct an ansatz for the effective Lagrangian.

First we define some constants with which to parameterise the ansatz.

$$\{c_1, c_2, c_3, c_4, c_5, c_6\} \tag{70}$$

Next we define the Lagrangian density; note that we omit the matter Lagrangian.

$$-\frac{1}{4}\sqrt{-\overset{\approx}{g}}\Big(c_1\, {}^{2}\mathcal{F}_{\mu\nu}\, {}^{2}\mathcal{F}^{\mu\nu} + c_2\, {}^{3}\mathcal{F}_{\mu\nu}\, {}^{3}\mathcal{F}^{\mu\nu} + 2\Big(\mathcal{M}_{\text{Pl}}{}^2 R[\overset{\circ}{\nabla}] + c_5\, {}^{2}\mathcal{S}^{\mu}\, {}^{2}\mathcal{T}_{\mu} + c_3\, {}^{2}\mathcal{T}_{\mu}\, {}^{2}\mathcal{T}^{\mu} + c_6\, {}^{3}\mathcal{S}^{\mu}\, {}^{3}\mathcal{T}_{\mu} + c_4\, {}^{3}\mathcal{T}_{\mu}\, {}^{3}\mathcal{T}^{\mu}\Big)\Big) \tag{71}$$

So the Ansatz Eq. (71) is now parameterised in terms of the constants Eq. (70).

In particular, here are the Einstein equations according to Eq. (71), and we note that there is no matter stress-energy tensor because we neglected the Matter Lagrangian.

$$\begin{aligned}&\frac{1}{8} c_1\, \overset{\circ}{g}_{\iota\theta}\, {}^{2}\mathcal{F}_{\alpha\alpha'}\, {}^{2}\mathcal{F}^{\alpha\alpha'} - \frac{1}{2} c_1\, {}^{2}\mathcal{F}_{\iota}{}^{\alpha}\, {}^{2}\mathcal{F}_{\theta\alpha} + \frac{1}{8} c_2\, \overset{\circ}{g}_{\iota\theta}\, {}^{3}\mathcal{F}_{\alpha\alpha'}\, {}^{3}\mathcal{F}^{\alpha\alpha'} - \\ &\quad \frac{1}{2} c_2\, {}^{3}\mathcal{F}_{\iota}{}^{\alpha}\, {}^{3}\mathcal{F}_{\theta\alpha} - \frac{1}{2}\mathcal{M}_{\text{Pl}}{}^2 R[\overset{\circ}{\nabla}]_{\iota\theta} + \frac{1}{4}\mathcal{M}_{\text{Pl}}{}^2\, \overset{\circ}{g}_{\iota\theta} R[\overset{\circ}{\nabla}] + \frac{1}{4} c_5\, \overset{\circ}{g}_{\iota\theta}\, {}^{2}\mathcal{S}^{\alpha}\, {}^{2}\mathcal{T}_{\alpha} + \\ &\quad \frac{1}{4} c_3\, \overset{\circ}{g}_{\iota\theta}\, {}^{2}\mathcal{T}_{\alpha}\, {}^{2}\mathcal{T}^{\alpha} - \frac{1}{8} c_5\, {}^{2}\mathcal{S}_{\theta}\, {}^{2}\mathcal{T}_{\iota} - \frac{1}{8} c_5\, {}^{2}\mathcal{S}_{\iota}\, {}^{2}\mathcal{T}_{\theta} - \frac{1}{2} c_3\, {}^{2}\mathcal{T}_{\iota}\, {}^{2}\mathcal{T}_{\theta} + \frac{1}{4} c_6\, \overset{\circ}{g}_{\iota\theta}\, {}^{3}\mathcal{S}^{\alpha}\, {}^{3}\mathcal{T}_{\alpha} + \\ &\quad \frac{1}{4} c_4\, \overset{\circ}{g}_{\iota\theta}\, {}^{3}\mathcal{T}_{\alpha}\, {}^{3}\mathcal{T}^{\alpha} - \frac{1}{8} c_6\, {}^{3}\mathcal{S}_{\theta}\, {}^{3}\mathcal{T}_{\iota} - \frac{1}{8} c_6\, {}^{3}\mathcal{S}_{\iota}\, {}^{3}\mathcal{T}_{\theta} - \frac{1}{2} c_4\, {}^{3}\mathcal{T}_{\iota}\, {}^{3}\mathcal{T}_{\theta} == 0\end{aligned} \tag{72}$$

Now here is the ${}^{2}\mathcal{T}_{\alpha}$ Proca equation from Eq. (71) which should encode the vector part of the spin equation.

$$-\frac{1}{2} c_5\, {}^{2}\mathcal{S}_{\omega} - c_3\, {}^{2}\mathcal{T}_{\omega} - c_1\left(\overset{\circ}{\nabla}_{\alpha}{}^{2}\mathcal{F}_{\omega}{}^{\alpha}\right) == 0 \tag{73}$$

Here is the ${}^{3}\mathcal{T}_{\alpha}$ Proca equation from Eq. (71) which should encode the axial vector part of the spin equation.

$$-\frac{1}{2} c_6\, {}^{3}\mathcal{S}^{\chi} - c_4\, {}^{3}\mathcal{T}^{\chi} - c_2 \left(\mathring{\nabla}_{\alpha}{}^{3}\mathcal{F}^{\chi\alpha}\right) == 0 \tag{74}$$

The new field equations

Now once again we resurrect the field equations and impose our choices of parameters.

The full tetrad equation, the geometric interpretation of Eq. (31).

$$\begin{aligned}
&-\mathcal{M}_{\text{Pl}}^{2}\, R[\mathring{\nabla}]_{\iota}{}^{\nu} + \frac{1}{2}\mathcal{M}_{\text{Pl}}^{2}\, \delta_{\iota}{}^{\nu}\, R[\mathring{\nabla}] + \frac{\delta_{\iota}{}^{\nu}\sqrt{-\overset{\approx}{\mathring{g}}}\;\sigma^{\alpha'}{}_{\alpha\alpha'}\;{}^{2}\mathcal{T}^{\alpha}}{3\,\overset{\approx}{\mathring{g}}} + \frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\sigma^{\nu}{}_{\iota\alpha}\;{}^{2}\mathcal{T}^{\alpha}}{3\,\overset{\approx}{\mathring{g}}} - \\
&\frac{1}{9}\mathcal{M}_{\text{Pl}}^{2}\,\delta_{\iota}{}^{\nu}\;{}^{2}\mathcal{T}_{\alpha}\;{}^{2}\mathcal{T}^{\alpha} - \frac{2}{9}\mathcal{M}_{\text{Pl}}^{2}\,{}^{2}\mu\,\delta_{\iota}{}^{\nu}\;{}^{2}\mathcal{T}_{\alpha}\;{}^{2}\mathcal{T}^{\alpha} - \frac{2}{9}\mathcal{M}_{\text{Pl}}^{2}\;{}^{2}\mathcal{T}_{\iota}\;{}^{2}\mathcal{T}^{\nu} - \\
&\frac{4}{9}\mathcal{M}_{\text{Pl}}^{2}\,{}^{2}\mu\;{}^{2}\mathcal{T}_{\iota}\;{}^{2}\mathcal{T}^{\nu} - \frac{\delta_{\iota}{}^{\nu}\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\alpha\alpha'\beta\beta'}\;\sigma^{\alpha'\beta\beta'}\;{}^{3}\mathcal{T}^{\alpha}}{4\,\overset{\approx}{\mathring{g}}} - \frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\iota\alpha\alpha'\beta}\;\sigma^{\nu\alpha'\beta}\;{}^{3}\mathcal{T}^{\alpha}}{4\,\overset{\approx}{\mathring{g}}} + \\
&\frac{1}{4}\mathcal{M}_{\text{Pl}}^{2}\,\delta_{\iota}{}^{\nu}\;{}^{3}\mathcal{T}_{\alpha}\;{}^{3}\mathcal{T}^{\alpha} + 2\,\mathcal{M}_{\text{Pl}}^{2}\,{}^{3}\mu\,\delta_{\iota}{}^{\nu}\;{}^{3}\mathcal{T}_{\alpha}\;{}^{3}\mathcal{T}^{\alpha} - \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\,{}^{2}\mu\;\epsilon\mathring{g}_{\iota}{}^{\nu}{}_{\alpha\alpha'}\;{}^{2}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}^{\alpha'} + \\
&\frac{8}{3}\mathcal{M}_{\text{Pl}}^{2}\,{}^{3}\mu\;\epsilon\mathring{g}_{\iota}{}^{\nu}{}_{\alpha\alpha'}\;{}^{2}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}^{\alpha'} + \frac{1}{2}\mathcal{M}_{\text{Pl}}^{2}\;{}^{3}\mathcal{T}_{\iota}\;{}^{3}\mathcal{T}^{\nu} + 4\,\mathcal{M}_{\text{Pl}}^{2}\,{}^{3}\mu\;{}^{3}\mathcal{T}_{\iota}\;{}^{3}\mathcal{T}^{\nu} + \frac{4}{9}\;{}^{2}\mathcal{T}^{\alpha}\;{}^{1}\lambda_{\mathcal{T}}{}_{\iota}{}^{\nu}{}_{\alpha} - \\
&\frac{4}{9}\;{}^{2}\mathcal{T}^{\alpha}\;{}^{1}\lambda_{\mathcal{T}}{}^{\nu}{}_{\alpha\iota} - \frac{2}{3}\,\epsilon\mathring{g}_{\iota\alpha\alpha'\beta}\;{}^{3}\mathcal{T}^{\alpha}\;{}^{1}\lambda_{\mathcal{T}}{}^{\nu\alpha'\beta} - \frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\tau(\Delta)^{\nu}{}_{\iota}}{\overset{\approx}{\mathring{g}}} - \frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\,\delta_{\iota}{}^{\nu}\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}^{\alpha}\right) - \\
&\frac{4}{3}\mathcal{M}_{\text{Pl}}^{2}\,{}^{2}\mu\,\delta_{\iota}{}^{\nu}\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}^{\alpha}\right) + \frac{4}{9}\,\alpha\,\epsilon\mathring{g}_{\iota}{}^{\nu}{}_{\alpha'\beta}\;{}^{2}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}^{\beta}\right) + \frac{4}{27}\,\alpha\;{}^{2}\mathcal{T}^{\alpha}\;{}^{2}\mathcal{T}^{\nu}\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}_{\iota}\right) - \\
&\frac{1}{3}\,\alpha\;{}^{3}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}^{\nu}\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}_{\iota}\right) + \frac{4}{27}\,\alpha\;{}^{2}\mathcal{T}^{\alpha}\;{}^{2}\mathcal{T}_{\iota}\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}^{\nu}\right) - \frac{1}{3}\,\alpha\;{}^{3}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}_{\iota}\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}^{\nu}\right) + \\
&\frac{3}{2}\,\alpha\,\epsilon\mathring{g}_{\iota}{}^{\nu}{}_{\alpha'\beta}\;{}^{3}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}_{\alpha}{}^{3}\mathcal{T}^{\beta}\right) - \frac{1}{3}\,\alpha\;{}^{2}\mathcal{T}^{\nu}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{3}\mathcal{T}_{\iota}\right) - \frac{1}{3}\,\alpha\;{}^{2}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}^{\nu}\left(\mathring{\nabla}_{\alpha}{}^{3}\mathcal{T}_{\iota}\right) - \\
&\frac{1}{3}\,\alpha\;{}^{2}\mathcal{T}_{\iota}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{3}\mathcal{T}^{\nu}\right) - \frac{1}{3}\,\alpha\;{}^{2}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}_{\iota}\left(\mathring{\nabla}_{\alpha}{}^{3}\mathcal{T}^{\nu}\right) + \frac{4}{3}\left(\mathring{\nabla}_{\alpha}{}^{1}\lambda_{\mathcal{T}}{}_{\iota}{}^{\alpha\nu}\right) - \frac{4}{3}\left(\mathring{\nabla}_{\alpha}{}^{1}\lambda_{\mathcal{T}}{}_{\iota}{}^{\nu\alpha}\right) - \\
&\frac{4}{9}\,\alpha\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}^{\nu}\right)\left(\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}_{\iota}\right) + \frac{4}{3}\,\alpha\,R[\mathring{\nabla}]_{\iota\alpha}\left(\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}^{\nu}\right) + \alpha\left(\mathring{\nabla}_{\alpha}{}^{3}\mathcal{T}^{\nu}\right)\left(\mathring{\nabla}^{\alpha}\,{}^{3}\mathcal{T}_{\iota}\right) + \frac{4}{3}\,\alpha\,R[\mathring{\nabla}]_{\iota\alpha}{}^{\nu}{}_{\alpha'}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right) - \\
&\frac{4}{3}\,\alpha\,R[\mathring{\nabla}]_{\iota\alpha'}{}^{\nu}{}_{\alpha}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right) + \frac{2}{3}\,\alpha\,\epsilon\mathring{g}_{\iota}{}^{\nu}{}_{\alpha'\beta}\left(\mathring{\nabla}_{\alpha}{}^{3}\mathcal{T}^{\beta}\right)\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right) - \frac{2}{3}\,\alpha\,\epsilon\mathring{g}_{\iota}{}^{\nu}{}_{\alpha\beta}\left(\mathring{\nabla}_{\alpha'}{}^{3}\mathcal{T}^{\beta}\right)\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right) + \\
&\frac{1}{2}\mathcal{M}_{\text{Pl}}^{2}\,\epsilon\mathring{g}_{\iota}{}^{\nu}{}_{\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right) + 4\,\mathcal{M}_{\text{Pl}}^{2}\,{}^{3}\mu\;\epsilon\mathring{g}_{\iota}{}^{\nu}{}_{\alpha\alpha'}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right) - 2\,\alpha\,\epsilon\mathring{g}^{\nu}{}_{\alpha\alpha'\beta}\,R[\mathring{\nabla}]_{\iota}{}^{\beta}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right) -
\end{aligned} \tag{75}$$

$$2\,\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\iota}{}^{\beta\nu\beta'}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)-\frac{4}{9}\,\alpha\,\epsilon\,\mathring{g}_{\iota}{}^{\nu}{}_{\alpha'\beta}\;{}^{2}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}^{\beta}\,{}^{2}\mathcal{T}_{\alpha}\right)-$$

$$\frac{8}{9}\,\alpha\,\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;{}^{2}\mathcal{T}^{\nu}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{2}\mathcal{T}^{\alpha'}\right)-\frac{2}{9}\,\alpha\,\epsilon\,\mathring{g}^{\nu}{}_{\alpha\alpha'\beta}\;{}^{2}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}^{\beta}\,{}^{2}\mathcal{T}_{\iota}\right)-$$

$$\frac{2}{9}\,\alpha\,\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;{}^{2}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}^{\beta}\,{}^{2}\mathcal{T}^{\nu}\right)-\frac{3}{2}\,\alpha\,\epsilon\,\mathring{g}_{\iota}{}^{\nu}{}_{\alpha'\beta}\;{}^{3}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}^{\beta}\,{}^{3}\mathcal{T}_{\alpha}\right)-$$

$$\frac{2}{9}\,\alpha\,\epsilon\,\mathring{g}_{\iota}{}^{\nu}{}_{\alpha'\beta}\;{}^{2}\mathcal{T}_{\alpha}\;{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{3}\mathcal{T}^{\alpha'}\right)-\frac{2}{9}\,\alpha\,\epsilon\,\mathring{g}^{\nu}{}_{\alpha\alpha'\beta}\;{}^{2}\mathcal{T}^{\alpha}\;{}^{2}\mathcal{T}_{\iota}\left(\mathring{\nabla}^{\beta}\,{}^{3}\mathcal{T}^{\alpha'}\right)-$$

$$\frac{2}{9}\,\alpha\,\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;{}^{2}\mathcal{T}^{\alpha}\;{}^{2}\mathcal{T}^{\nu}\left(\mathring{\nabla}^{\beta}\,{}^{3}\mathcal{T}^{\alpha'}\right)+2\,\alpha\,\epsilon\,\mathring{g}_{\iota}{}^{\nu}{}_{\alpha'\beta}\;{}^{3}\mathcal{T}_{\alpha}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{3}\mathcal{T}^{\alpha'}\right)+$$

$$2\,\alpha\,\epsilon\,\mathring{g}^{\nu}{}_{\alpha\alpha'\beta}\;{}^{3}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}_{\iota}\left(\mathring{\nabla}^{\beta}\,{}^{3}\mathcal{T}^{\alpha'}\right)-\alpha\,\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;{}^{3}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}^{\nu}\left(\mathring{\nabla}^{\beta}\,{}^{3}\mathcal{T}^{\alpha'}\right)-$$

$$\frac{2}{3}\,\alpha\,\epsilon\,\mathring{g}_{\iota}{}^{\nu}{}_{\alpha'\beta}\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\beta}\,{}^{3}\mathcal{T}^{\alpha'}\right)+\frac{4}{9}\,\alpha\,\delta_{\iota}{}^{\nu}\;\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\;{}^{2}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}^{\beta'}\,{}^{2}\mathcal{T}^{\beta}\right)+$$

$$\frac{2}{3}\,\alpha\,\delta_{\iota}{}^{\nu}\;\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\beta'}\,{}^{3}\mathcal{T}^{\beta}\right)-\frac{4}{27}\,\alpha\;{}^{2}\mathcal{T}^{\alpha}\;{}^{2}\mathcal{T}^{\nu}\left(\mathring{\nabla}_{\iota}{}^{2}\mathcal{T}_{\alpha}\right)+\frac{1}{3}\,\alpha\;{}^{3}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}^{\nu}\left(\mathring{\nabla}_{\iota}{}^{2}\mathcal{T}_{\alpha}\right)+$$

$$\frac{2}{3}\,\alpha\,\epsilon\,\mathring{g}^{\nu}{}_{\alpha\alpha'\beta}\left(\mathring{\nabla}^{\beta}\,{}^{3}\mathcal{T}^{\alpha'}\right)\left(\mathring{\nabla}_{\iota}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{2}{9}\,\alpha\,\epsilon\,\mathring{g}^{\nu}{}_{\alpha\alpha'\beta}\;{}^{2}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}_{\iota}{}^{2}\mathcal{T}^{\beta}\right)+\frac{2}{3}\,\mathcal{M}_{\text{Pl}}{}^{2}\left(\mathring{\nabla}_{\iota}{}^{2}\mathcal{T}^{\nu}\right)+$$

$$\frac{4}{3}\,\mathcal{M}_{\text{Pl}}{}^{2}\,{}^{2}\mu\left(\mathring{\nabla}_{\iota}{}^{2}\mathcal{T}^{\nu}\right)-\frac{4}{27}\,\alpha\;{}^{2}\mathcal{T}_{\alpha}\;{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\iota}{}^{2}\mathcal{T}^{\nu}\right)+\frac{1}{3}\,\alpha\;{}^{3}\mathcal{T}_{\alpha}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\iota}{}^{2}\mathcal{T}^{\nu}\right)-\frac{4}{9}\,\alpha\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\iota}{}^{2}\mathcal{T}^{\nu}\right)+$$

$$\frac{1}{3}\,\alpha\;{}^{2}\mathcal{T}^{\nu}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\iota}{}^{3}\mathcal{T}_{\alpha}\right)+\frac{1}{3}\,\alpha\;{}^{2}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}^{\nu}\left(\mathring{\nabla}_{\iota}{}^{3}\mathcal{T}_{\alpha}\right)+\frac{4}{3}\,\alpha\,\epsilon\,\mathring{g}^{\nu}{}_{\alpha\alpha'\beta}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\iota}{}^{3}\mathcal{T}^{\beta}\right)+$$

$$\frac{2}{3}\,\alpha\;{}^{2}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}_{\alpha}\left(\mathring{\nabla}_{\iota}{}^{3}\mathcal{T}^{\nu}\right)+\alpha\left(\mathring{\nabla}_{\alpha}{}^{3}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\iota}{}^{3}\mathcal{T}^{\nu}\right)-\frac{4}{27}\,\alpha\;{}^{2}\mathcal{T}^{\alpha}\;{}^{2}\mathcal{T}_{\iota}\left(\mathring{\nabla}^{\nu}\,{}^{2}\mathcal{T}_{\alpha}\right)+\frac{1}{3}\,\alpha\;{}^{3}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}_{\iota}\left(\mathring{\nabla}^{\nu}\,{}^{2}\mathcal{T}_{\alpha}\right)+$$

$$\frac{4}{9}\,\alpha\left(\mathring{\nabla}_{\iota}{}^{2}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\nu}\,{}^{2}\mathcal{T}_{\alpha}\right)-\frac{4}{3}\,\alpha\,R[\mathring{\nabla}]_{\iota\alpha}\left(\mathring{\nabla}^{\nu}\,{}^{2}\mathcal{T}^{\alpha}\right)+\frac{2}{3}\,\alpha\,\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\left(\mathring{\nabla}^{\beta}\,{}^{3}\mathcal{T}^{\alpha'}\right)\left(\mathring{\nabla}^{\nu}\,{}^{2}\mathcal{T}^{\alpha}\right)+$$

$$\frac{2}{9}\,\alpha\,\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;{}^{2}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}^{\nu}\,{}^{2}\mathcal{T}^{\beta}\right)+\frac{4}{27}\,\alpha\;{}^{2}\mathcal{T}_{\alpha}\;{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\nu}\,{}^{2}\mathcal{T}_{\iota}\right)-\frac{1}{3}\,\alpha\;{}^{3}\mathcal{T}_{\alpha}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\nu}\,{}^{2}\mathcal{T}_{\iota}\right)+$$

$$\frac{4}{9}\,\alpha\left(\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\nu}\,{}^{2}\mathcal{T}_{\iota}\right)+\frac{1}{3}\,\alpha\;{}^{2}\mathcal{T}_{\iota}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\nu}\,{}^{3}\mathcal{T}_{\alpha}\right)+\frac{1}{3}\,\alpha\;{}^{2}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}_{\iota}\left(\mathring{\nabla}^{\nu}\,{}^{3}\mathcal{T}_{\alpha}\right)-$$

$$\alpha\left(\mathring{\nabla}_{\iota}{}^{3}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\nu}\,{}^{3}\mathcal{T}_{\alpha}\right)-\frac{2}{3}\,\alpha\;{}^{2}\mathcal{T}^{\alpha}\;{}^{3}\mathcal{T}_{\alpha}\left(\mathring{\nabla}^{\nu}\,{}^{3}\mathcal{T}_{\iota}\right)-\alpha\left(\mathring{\nabla}_{\alpha}{}^{3}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\nu}\,{}^{3}\mathcal{T}_{\iota}\right)==0$$

The full spin connection equation, the geometric interpretation of Eq. (33).

$$-\frac{\sqrt{-\mathring{\tilde{g}}}\;\sigma^{\nu}{}_{\iota\theta}}{\mathring{\tilde{g}}}+\frac{4}{3}\,\alpha\,\delta_{\theta}{}^{\nu}\;R[\mathring{\nabla}]_{\iota\alpha}\;{}^{2}\mathcal{T}^{\alpha}-\frac{4}{3}\,\alpha\,\delta_{\iota}{}^{\nu}\;R[\mathring{\nabla}]_{\theta\alpha}\;{}^{2}\mathcal{T}^{\alpha}+\frac{4}{3}\,\alpha\,R[\mathring{\nabla}]_{\iota\alpha\theta}{}^{\nu}\;{}^{2}\mathcal{T}^{\alpha}+$$

$$\frac{4}{3}\,\alpha\,R[\mathring{\nabla}]_{\iota\theta}{}^{\nu}{}_{\alpha}\;{}^{2}\mathcal{T}^{\alpha}-\frac{4}{3}\,\alpha\,R[\mathring{\nabla}]_{\iota}{}^{\nu}{}_{\theta\alpha}\;{}^{2}\mathcal{T}^{\alpha}+\frac{2}{3}\,\mathcal{M}_{\text{Pl}}{}^{2}\;\delta_{\theta}{}^{\nu}\;{}^{2}\mathcal{T}_{\iota}+\frac{4}{3}\,\mathcal{M}_{\text{Pl}}{}^{2}\,{}^{2}\mu\;\delta_{\theta}{}^{\nu}\;{}^{2}\mathcal{T}_{\iota}-$$

$$\frac{2}{3}\mathcal{M}_{\text{Pl}}^{2}\,\delta_{\iota}{}^{\nu}\;{}^{2}\!\mathcal{T}_{\theta}-\frac{4}{3}\mathcal{M}_{\text{Pl}}^{2}\,{}^{2}\!\mu\;\delta_{\iota}{}^{\nu}\;{}^{2}\!\mathcal{T}_{\theta}+\mathcal{M}_{\text{Pl}}^{2}\,\epsilon\mathring{g}_{\iota\theta}{}^{\nu}{}_{\alpha}\;{}^{3}\!\mathcal{T}^{\alpha}+8\,\mathcal{M}_{\text{Pl}}^{2}\,{}^{3}\!\mu\;\epsilon\mathring{g}_{\iota\theta}{}^{\nu}{}_{\alpha}\;{}^{3}\!\mathcal{T}^{\alpha}+$$

$$\alpha\,\delta_{\theta}{}^{\nu}\;\epsilon\mathring{g}_{\iota\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\alpha}{}^{\alpha'\beta\beta'}\;{}^{3}\!\mathcal{T}^{\alpha}-\alpha\,\delta_{\iota}{}^{\nu}\;\epsilon\mathring{g}_{\theta\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\alpha}{}^{\alpha'\beta\beta'}\;{}^{3}\!\mathcal{T}^{\alpha}-\frac{4}{3}\,{}^{1}\!\lambda_{\mathcal{T}}{}_{\iota}{}^{\nu}{}_{\theta}+$$

$$\frac{4}{3}\,{}^{1}\!\lambda_{\mathcal{T}}{}_{\theta}{}^{\nu}{}_{\iota}+\frac{4}{9}\alpha\,\delta_{\theta}{}^{\nu}\;{}^{2}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{2}\!\mathcal{T}_{\iota}\right)-\frac{4}{9}\alpha\,\delta_{\iota}{}^{\nu}\;{}^{2}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{2}\!\mathcal{T}_{\theta}\right)-\alpha\,\delta_{\theta}{}^{\nu}\;{}^{3}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{3}\!\mathcal{T}_{\iota}\right)+$$

$$\alpha\,\delta_{\iota}{}^{\nu}\;{}^{3}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{3}\!\mathcal{T}_{\theta}\right)-\frac{4}{3}\alpha\,\delta_{\theta}{}^{\nu}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,{}^{2}\!\mathcal{T}_{\iota}\right)+\frac{4}{3}\alpha\,\delta_{\iota}{}^{\nu}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,{}^{2}\!\mathcal{T}_{\theta}\right)-$$

$$\frac{2}{3}\alpha\,\epsilon\mathring{g}_{\theta}{}^{\nu}{}_{\alpha\alpha'}\;{}^{3}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\!\mathcal{T}_{\iota}\right)+\frac{2}{3}\alpha\,\epsilon\mathring{g}_{\iota}{}^{\nu}{}_{\alpha\alpha'}\;{}^{3}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\!\mathcal{T}_{\theta}\right)+\frac{4}{3}\alpha\,\epsilon\mathring{g}_{\iota\theta\alpha\alpha'}\;{}^{3}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\!\mathcal{T}^{\nu}\right)-$$

$$\frac{2}{3}\alpha\,\epsilon\mathring{g}_{\theta}{}^{\nu}{}_{\alpha\alpha'}\;{}^{2}\!\mathcal{T}_{\iota}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\!\mathcal{T}^{\alpha}\right)+\frac{2}{3}\alpha\,\epsilon\mathring{g}_{\iota}{}^{\nu}{}_{\alpha\alpha'}\;{}^{2}\!\mathcal{T}_{\theta}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\!\mathcal{T}^{\alpha}\right)+\frac{4}{3}\alpha\,\epsilon\mathring{g}_{\iota\theta\alpha\alpha'}\;{}^{2}\!\mathcal{T}^{\nu}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\!\mathcal{T}^{\alpha}\right)+$$

$$\frac{4}{3}\alpha\,\delta_{\theta}{}^{\nu}\;\epsilon\mathring{g}_{\iota\alpha\alpha'\beta}\;{}^{3}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{2}\!\mathcal{T}^{\alpha'}\right)-\frac{4}{3}\alpha\,\delta_{\iota}{}^{\nu}\;\epsilon\mathring{g}_{\theta\alpha\alpha'\beta}\;{}^{3}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{2}\!\mathcal{T}^{\alpha'}\right)-$$

$$\frac{2}{3}\alpha\,\delta_{\theta}{}^{\nu}\;\epsilon\mathring{g}_{\iota\alpha\alpha'\beta}\;{}^{2}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{3}\!\mathcal{T}^{\alpha'}\right)+\frac{2}{3}\alpha\,\delta_{\iota}{}^{\nu}\;\epsilon\mathring{g}_{\theta\alpha\alpha'\beta}\;{}^{2}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{3}\!\mathcal{T}^{\alpha'}\right)-\frac{4}{9}\alpha\,\delta_{\theta}{}^{\nu}\;{}^{2}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\iota}{}^{2}\!\mathcal{T}_{\alpha}\right)+$$

$$\frac{2}{3}\alpha\,\epsilon\mathring{g}_{\theta}{}^{\nu}{}_{\alpha\alpha'}\;{}^{3}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\iota}{}^{2}\!\mathcal{T}^{\alpha'}\right)+\frac{8}{9}\alpha\;{}^{2}\!\mathcal{T}^{\nu}\left(\mathring{\nabla}_{\iota}{}^{2}\!\mathcal{T}_{\theta}\right)+\frac{4}{9}\alpha\;{}^{2}\!\mathcal{T}_{\theta}\left(\mathring{\nabla}_{\iota}{}^{2}\!\mathcal{T}^{\nu}\right)+\alpha\,\delta_{\theta}{}^{\nu}\;{}^{3}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\iota}{}^{3}\!\mathcal{T}_{\alpha}\right)-$$

$$2\,\alpha\;{}^{3}\!\mathcal{T}^{\nu}\left(\mathring{\nabla}_{\iota}{}^{3}\!\mathcal{T}_{\theta}\right)-\alpha\;{}^{3}\!\mathcal{T}_{\theta}\left(\mathring{\nabla}_{\iota}{}^{3}\!\mathcal{T}^{\nu}\right)+\frac{4}{3}\alpha\,\delta_{\theta}{}^{\nu}\left(\mathring{\nabla}_{\iota}\mathring{\nabla}_{\alpha}{}^{2}\!\mathcal{T}^{\alpha}\right)+2\,\alpha\,\epsilon\mathring{g}_{\theta}{}^{\nu}{}_{\alpha\alpha'}\left(\mathring{\nabla}_{\iota}\mathring{\nabla}^{\alpha'}\,{}^{3}\!\mathcal{T}^{\alpha}\right)+$$

$$\frac{4}{9}\alpha\,\delta_{\iota}{}^{\nu}\;{}^{2}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}{}^{2}\!\mathcal{T}_{\alpha}\right)-\frac{2}{3}\alpha\,\epsilon\mathring{g}_{\iota}{}^{\nu}{}_{\alpha\alpha'}\;{}^{3}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}{}^{2}\!\mathcal{T}^{\alpha'}\right)-\frac{8}{9}\alpha\;{}^{2}\!\mathcal{T}^{\nu}\left(\mathring{\nabla}_{\theta}{}^{2}\!\mathcal{T}_{\iota}\right)-\frac{4}{9}\alpha\;{}^{2}\!\mathcal{T}_{\iota}\left(\mathring{\nabla}_{\theta}{}^{2}\!\mathcal{T}^{\nu}\right)-$$

$$\alpha\,\delta_{\iota}{}^{\nu}\;{}^{3}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}{}^{3}\!\mathcal{T}_{\alpha}\right)+2\,\alpha\;{}^{3}\!\mathcal{T}^{\nu}\left(\mathring{\nabla}_{\theta}{}^{3}\!\mathcal{T}_{\iota}\right)+\alpha\;{}^{3}\!\mathcal{T}_{\iota}\left(\mathring{\nabla}_{\theta}{}^{3}\!\mathcal{T}^{\nu}\right)-\frac{4}{3}\alpha\,\delta_{\iota}{}^{\nu}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}{}^{2}\!\mathcal{T}^{\alpha}\right)-$$

$$2\,\alpha\,\epsilon\mathring{g}_{\iota}{}^{\nu}{}_{\alpha\alpha'}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\alpha'}\,{}^{3}\!\mathcal{T}^{\alpha}\right)-\frac{4}{3}\alpha\,\epsilon\mathring{g}_{\iota\theta\alpha\alpha'}\;{}^{3}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\nu}\,{}^{2}\!\mathcal{T}^{\alpha'}\right)-\frac{4}{9}\alpha\;{}^{2}\!\mathcal{T}_{\theta}\left(\mathring{\nabla}^{\nu}\,{}^{2}\!\mathcal{T}_{\iota}\right)+$$

$$\frac{4}{9}\alpha\;{}^{2}\!\mathcal{T}_{\iota}\left(\mathring{\nabla}^{\nu}\,{}^{2}\!\mathcal{T}_{\theta}\right)+\alpha\;{}^{3}\!\mathcal{T}_{\theta}\left(\mathring{\nabla}^{\nu}\,{}^{3}\!\mathcal{T}_{\iota}\right)-\alpha\;{}^{3}\!\mathcal{T}_{\iota}\left(\mathring{\nabla}^{\nu}\,{}^{3}\!\mathcal{T}_{\theta}\right)-\frac{4}{3}\alpha\left(\mathring{\nabla}^{\nu}\mathring{\nabla}_{\iota}{}^{2}\!\mathcal{T}_{\theta}\right)+\frac{4}{3}\alpha\left(\mathring{\nabla}^{\nu}\mathring{\nabla}_{\theta}{}^{2}\!\mathcal{T}_{\iota}\right)==0$$

The tensor part of the spin connection equation, the geometric interpretation of Eq. (37).

$$\frac{\delta^{\rho}{}_{\omega}\sqrt{-\overset{\approx}{g}}\;\sigma^{\alpha}{}_{\upsilon\alpha}}{3\,\overset{\approx}{g}}-\frac{\delta^{\rho}{}_{\upsilon}\sqrt{-\overset{\approx}{g}}\;\sigma^{\alpha}{}_{\omega\alpha}}{3\,\overset{\approx}{g}}-\frac{2\sqrt{-\overset{\approx}{g}}\;\sigma^{\rho}{}_{\upsilon\omega}}{3\,\overset{\approx}{g}}-\frac{\sqrt{-\overset{\approx}{g}}\;\sigma_{\upsilon}{}^{\rho}{}_{\omega}}{3\,\overset{\approx}{g}}+\frac{\sqrt{-\overset{\approx}{g}}\;\sigma_{\omega}{}^{\rho}{}_{\upsilon}}{3\,\overset{\approx}{g}}+$$

$$\frac{4}{9}\alpha\,\delta^{\rho}{}_{\omega}\;R[\mathring{\nabla}]_{\upsilon\alpha}\;{}^{2}\!\mathcal{T}^{\alpha}-\frac{4}{9}\alpha\,\delta^{\rho}{}_{\upsilon}\;R[\mathring{\nabla}]_{\omega\alpha}\;{}^{2}\!\mathcal{T}^{\alpha}+\frac{4}{9}\alpha\,R[\mathring{\nabla}]^{\rho}{}_{\alpha\upsilon\omega}\;{}^{2}\!\mathcal{T}^{\alpha}-\frac{8}{9}\alpha\,R[\mathring{\nabla}]^{\rho}{}_{\upsilon\omega\alpha}\;{}^{2}\!\mathcal{T}^{\alpha}+$$

$$\frac{8}{9}\alpha\,R[\mathring{\nabla}]^{\rho}{}_{\omega\upsilon\alpha}\;{}^{2}\!\mathcal{T}^{\alpha}+\frac{1}{3}\alpha\,\delta^{\rho}{}_{\omega}\;\epsilon\mathring{g}_{\upsilon\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\alpha}{}^{\alpha'\beta\beta'}\;{}^{3}\!\mathcal{T}^{\alpha}-\frac{1}{3}\alpha\,\delta^{\rho}{}_{\upsilon}\;\epsilon\mathring{g}_{\omega\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\alpha}{}^{\alpha'\beta\beta'}\;{}^{3}\!\mathcal{T}^{\alpha}-$$

$$\frac{4}{3}\,{}^{1}\!\lambda_{\mathcal{T}}{}^{\rho}{}_{\upsilon\omega}+\frac{4}{3}\,{}^{1}\!\lambda_{\mathcal{T}}{}^{\rho}{}_{\omega\upsilon}+\frac{4}{9}\alpha\,\delta^{\rho}{}_{\omega}\;{}^{2}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{2}\!\mathcal{T}_{\upsilon}\right)-\frac{4}{9}\alpha\,\delta^{\rho}{}_{\upsilon}\;{}^{2}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{2}\!\mathcal{T}_{\omega}\right)-$$

$$
\begin{aligned}
&\alpha\,\delta^{\rho}{}_{\omega}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{3}\mathcal{T}_{\upsilon}\right)+\alpha\,\delta^{\rho}{}_{\upsilon}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{3}\mathcal{T}_{\omega}\right)-\frac{4}{9}\,\alpha\,\delta^{\rho}{}_{\omega}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}_{\upsilon}\right)+\frac{4}{9}\,\alpha\,\delta^{\rho}{}_{\upsilon}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}_{\omega}\right)+\\
&\frac{4}{3}\,\alpha\,\epsilon\mathring{g}_{\upsilon\omega\alpha\alpha'}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}^{\rho}\right)+\frac{2}{3}\,\alpha\,\epsilon\mathring{g}^{\rho}{}_{\omega\alpha\alpha'}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}_{\upsilon}\right)-\frac{2}{3}\,\alpha\,\epsilon\mathring{g}^{\rho}{}_{\upsilon\alpha\alpha'}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\alpha'}\,{}^{2}\mathcal{T}_{\omega}\right)+\\
&\frac{4}{3}\,\alpha\,\epsilon\mathring{g}_{\upsilon\omega\alpha\alpha'}\;{}^{2}\mathcal{T}^{\rho}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)+\frac{2}{3}\,\alpha\,\epsilon\mathring{g}^{\rho}{}_{\omega\alpha\alpha'}\;{}^{2}\mathcal{T}_{\upsilon}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)-\frac{2}{3}\,\alpha\,\epsilon\mathring{g}^{\rho}{}_{\upsilon\alpha\alpha'}\;{}^{2}\mathcal{T}_{\omega}\left(\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)+\\
&\frac{4}{3}\,\alpha\,\delta^{\rho}{}_{\omega}\;\epsilon\mathring{g}_{\upsilon\alpha\alpha'\beta}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{2}\mathcal{T}^{\alpha'}\right)-\frac{4}{3}\,\alpha\,\delta^{\rho}{}_{\upsilon}\;\epsilon\mathring{g}_{\omega\alpha\alpha'\beta}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{2}\mathcal{T}^{\alpha'}\right)-\\
&\frac{2}{3}\,\alpha\,\delta^{\rho}{}_{\omega}\;\epsilon\mathring{g}_{\upsilon\alpha\alpha'\beta}\;{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{3}\mathcal{T}^{\alpha'}\right)+\frac{2}{3}\,\alpha\,\delta^{\rho}{}_{\upsilon}\;\epsilon\mathring{g}_{\omega\alpha\alpha'\beta}\;{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{3}\mathcal{T}^{\alpha'}\right)-\\
&\frac{4}{3}\,\alpha\,\epsilon\mathring{g}_{\upsilon\omega\alpha\alpha'}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\rho}\,{}^{2}\mathcal{T}^{\alpha'}\right)-\frac{4}{9}\,\alpha\;{}^{2}\mathcal{T}_{\omega}\left(\mathring{\nabla}^{\rho}\,{}^{2}\mathcal{T}_{\upsilon}\right)+\frac{4}{9}\,\alpha\;{}^{2}\mathcal{T}_{\upsilon}\left(\mathring{\nabla}^{\rho}\,{}^{2}\mathcal{T}_{\omega}\right)+\alpha\;{}^{3}\mathcal{T}_{\omega}\left(\mathring{\nabla}^{\rho}\,{}^{3}\mathcal{T}_{\upsilon}\right)-\\
&\alpha\;{}^{3}\mathcal{T}_{\upsilon}\left(\mathring{\nabla}^{\rho}\,{}^{3}\mathcal{T}_{\omega}\right)-\frac{4}{3}\,\alpha\,\epsilon\mathring{g}_{\upsilon\omega\alpha\alpha'}\left(\mathring{\nabla}^{\rho}\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)-\frac{4}{9}\,\alpha\,\delta^{\rho}{}_{\omega}\;{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\upsilon}{}^{2}\mathcal{T}_{\alpha}\right)-\\
&\frac{2}{3}\,\alpha\,\epsilon\mathring{g}^{\rho}{}_{\omega\alpha\alpha'}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\upsilon}{}^{2}\mathcal{T}^{\alpha'}\right)+\frac{4}{9}\,\alpha\;{}^{2}\mathcal{T}_{\omega}\left(\mathring{\nabla}_{\upsilon}{}^{2}\mathcal{T}^{\rho}\right)+\frac{8}{9}\,\alpha\;{}^{2}\mathcal{T}^{\rho}\left(\mathring{\nabla}_{\upsilon}{}^{2}\mathcal{T}_{\omega}\right)+\alpha\,\delta^{\rho}{}_{\omega}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\upsilon}{}^{3}\mathcal{T}_{\alpha}\right)-\\
&\alpha\;{}^{3}\mathcal{T}_{\omega}\left(\mathring{\nabla}_{\upsilon}{}^{3}\mathcal{T}^{\rho}\right)-2\,\alpha\;{}^{3}\mathcal{T}^{\rho}\left(\mathring{\nabla}_{\upsilon}{}^{3}\mathcal{T}_{\omega}\right)+\frac{4}{9}\,\alpha\,\delta^{\rho}{}_{\omega}\left(\mathring{\nabla}_{\upsilon}\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}^{\alpha}\right)-\frac{2}{3}\,\alpha\,\epsilon\mathring{g}^{\rho}{}_{\omega\alpha\alpha'}\left(\mathring{\nabla}_{\upsilon}\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)-\\
&\frac{4}{3}\,\alpha\left(\mathring{\nabla}_{\upsilon}\mathring{\nabla}^{\rho}\,{}^{2}\mathcal{T}_{\omega}\right)+\frac{4}{9}\,\alpha\,\delta^{\rho}{}_{\upsilon}\;{}^{2}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\omega}{}^{2}\mathcal{T}_{\alpha}\right)+\frac{2}{3}\,\alpha\,\epsilon\mathring{g}^{\rho}{}_{\upsilon\alpha\alpha'}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\omega}{}^{2}\mathcal{T}^{\alpha'}\right)-\\
&\frac{4}{9}\,\alpha\;{}^{2}\mathcal{T}_{\upsilon}\left(\mathring{\nabla}_{\omega}{}^{2}\mathcal{T}^{\rho}\right)-\frac{8}{9}\,\alpha\;{}^{2}\mathcal{T}^{\rho}\left(\mathring{\nabla}_{\omega}{}^{2}\mathcal{T}_{\upsilon}\right)-\alpha\,\delta^{\rho}{}_{\upsilon}\;{}^{3}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\omega}{}^{3}\mathcal{T}_{\alpha}\right)+\alpha\;{}^{3}\mathcal{T}_{\upsilon}\left(\mathring{\nabla}_{\omega}{}^{3}\mathcal{T}^{\rho}\right)+\\
&2\,\alpha\;{}^{3}\mathcal{T}^{\rho}\left(\mathring{\nabla}_{\omega}{}^{3}\mathcal{T}_{\upsilon}\right)-\frac{4}{9}\,\alpha\,\delta^{\rho}{}_{\upsilon}\left(\mathring{\nabla}_{\omega}\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}^{\alpha}\right)+\frac{2}{3}\,\alpha\,\epsilon\mathring{g}^{\rho}{}_{\upsilon\alpha\alpha'}\left(\mathring{\nabla}_{\omega}\mathring{\nabla}^{\alpha'}\,{}^{3}\mathcal{T}^{\alpha}\right)+\frac{4}{3}\,\alpha\left(\mathring{\nabla}_{\omega}\mathring{\nabla}^{\rho}\,{}^{2}\mathcal{T}_{\upsilon}\right)==0
\end{aligned}
$$

The vector part of the spin connection equation, the geometric interpretation of Eq. (38).

$$
\frac{\sqrt{-\mathring{\tilde{\tilde{g}}}}\;\sigma^{\alpha}{}_{\omega\alpha}}{\mathring{\tilde{\tilde{g}}}}-\frac{8}{3}\,\alpha\,R[\mathring{\nabla}]_{\omega\alpha}\;{}^{2}\mathcal{T}^{\alpha}-2\,\mathcal{M}_{\mathrm{Pl}}{}^{2}\;{}^{2}\mathcal{T}_{\omega}-4\,\mathcal{M}_{\mathrm{Pl}}{}^{2}\;{}^{2}\mu\;{}^{2}\mathcal{T}_{\omega}+\frac{8}{3}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,{}^{2}\mathcal{T}_{\omega}\right)-\frac{8}{3}\,\alpha\left(\mathring{\nabla}_{\omega}\mathring{\nabla}_{\alpha}{}^{2}\mathcal{T}^{\alpha}\right)==0 \qquad (78)
$$

The axial vector part of the spin connection equation, the geometric interpretation of Eq. (39).

$$
\begin{aligned}
&\frac{\sqrt{-\mathring{\tilde{\tilde{g}}}}\;\epsilon\mathring{g}^{\chi}{}_{\alpha\alpha'\beta}\;\sigma^{\alpha\alpha'\beta}}{\mathring{\tilde{\tilde{g}}}}-8\,\alpha\,R[\mathring{\nabla}]^{\chi}{}_{\alpha}\;{}^{3}\mathcal{T}^{\alpha}-\\
&6\,\mathcal{M}_{\mathrm{Pl}}{}^{2}\;{}^{3}\mathcal{T}^{\chi}-48\,\mathcal{M}_{\mathrm{Pl}}{}^{2}\;{}^{3}\mu\;{}^{3}\mathcal{T}^{\chi}+8\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,{}^{3}\mathcal{T}^{\chi}\right)-8\,\alpha\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha}{}^{3}\mathcal{T}^{\alpha}\right)==0
\end{aligned} \qquad (79)
$$

Algebraic elimination of the multiplier

We now use the symmetries of $\lambda_{\mathcal{T}\,\upsilon\psi\omega}$ to rearrange Eq. (77) into a form where we can solve for the multiplier.

$$
\begin{aligned}
&\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\ \mathring{g}_{\psi\omega}\ \sigma^{\alpha}{}_{\upsilon\alpha}}{6\overset{\approx}{\mathring{g}}}+\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\ \mathring{g}_{\upsilon\omega}\ \sigma^{\alpha}{}_{\psi\alpha}}{6\overset{\approx}{\mathring{g}}}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\ \mathring{g}_{\upsilon\psi}\ \sigma^{\alpha}{}_{\omega\alpha}}{3\overset{\approx}{\mathring{g}}}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\ \sigma_{\upsilon\psi\omega}}{2\overset{\approx}{\mathring{g}}}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\ \sigma_{\psi\upsilon\omega}}{2\overset{\approx}{\mathring{g}}}+\\
&\frac{2}{9}\alpha\,\mathring{g}_{\psi\omega}\,R[\mathring{\nabla}]_{\upsilon\alpha}\,{}^{y}\mathcal{T}^{\alpha}+\frac{2}{9}\alpha\,\mathring{g}_{\upsilon\omega}\,R[\mathring{\nabla}]_{\psi\alpha}\,{}^{y}\mathcal{T}^{\alpha}-\frac{4}{9}\alpha\,\mathring{g}_{\upsilon\psi}\,R[\mathring{\nabla}]_{\omega\alpha}\,{}^{y}\mathcal{T}^{\alpha}+\\
&\frac{2}{3}\alpha\,R[\mathring{\nabla}]_{\upsilon\alpha\psi\omega}\,{}^{y}\mathcal{T}^{\alpha}+\frac{2}{3}\alpha\,R[\mathring{\nabla}]_{\upsilon\omega\psi\alpha}\,{}^{y}\mathcal{T}^{\alpha}-\frac{1}{3}\alpha\,\epsilon\mathring{g}_{\omega\alpha'\beta\beta'}\,\mathring{g}_{\upsilon\psi}\,R[\mathring{\nabla}]_{\alpha}{}^{\alpha'\beta\beta'}\,{}^{z}\mathcal{T}^{\alpha}+\\
&\frac{1}{6}\alpha\,\epsilon\mathring{g}_{\psi\alpha'\beta\beta'}\,\mathring{g}_{\upsilon\omega}\,R[\mathring{\nabla}]_{\alpha}{}^{\alpha'\beta\beta'}\,{}^{z}\mathcal{T}^{\alpha}+\frac{1}{6}\alpha\,\epsilon\mathring{g}_{\upsilon\alpha'\beta\beta'}\,\mathring{g}_{\psi\omega}\,R[\mathring{\nabla}]_{\alpha}{}^{\alpha'\beta\beta'}\,{}^{z}\mathcal{T}^{\alpha}-2\,{}^{x}\lambda_{\mathcal{T}\,\upsilon\psi\omega}+\\
&\frac{2}{9}\alpha\,\mathring{g}_{\psi\omega}\,{}^{y}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{y}\mathcal{T}_{\upsilon}\right)+\frac{2}{9}\alpha\,\mathring{g}_{\upsilon\omega}\,{}^{y}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{y}\mathcal{T}_{\psi}\right)-\frac{4}{9}\alpha\,\mathring{g}_{\upsilon\psi}\,{}^{y}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{y}\mathcal{T}_{\omega}\right)-\\
&\frac{1}{2}\alpha\,\mathring{g}_{\psi\omega}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{z}\mathcal{T}_{\upsilon}\right)-\frac{1}{2}\alpha\,\mathring{g}_{\upsilon\omega}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{z}\mathcal{T}_{\psi}\right)+\alpha\,\mathring{g}_{\upsilon\psi}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{z}\mathcal{T}_{\omega}\right)-\\
&\frac{2}{9}\alpha\,\mathring{g}_{\psi\omega}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,{}^{y}\mathcal{T}_{\upsilon}\right)-\frac{2}{9}\alpha\,\mathring{g}_{\upsilon\omega}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,{}^{y}\mathcal{T}_{\psi}\right)+\frac{4}{9}\alpha\,\mathring{g}_{\upsilon\psi}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,{}^{y}\mathcal{T}_{\omega}\right)+\alpha\,\epsilon\mathring{g}_{\psi\omega\alpha\alpha'}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\alpha\text{⅒}}\,{}^{y}\mathcal{T}_{\upsilon}\right)+\\
&\alpha\,\epsilon\mathring{g}_{\upsilon\omega\alpha\alpha'}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\alpha\text{⅒}}\,{}^{y}\mathcal{T}_{\psi}\right)+\alpha\,\epsilon\mathring{g}_{\psi\omega\alpha\alpha'}\,{}^{y}\mathcal{T}_{\upsilon}\left(\mathring{\nabla}^{\alpha\text{⅒}}\,{}^{z}\mathcal{T}^{\alpha}\right)+\alpha\,\epsilon\mathring{g}_{\upsilon\omega\alpha\alpha'}\,{}^{y}\mathcal{T}_{\psi}\left(\mathring{\nabla}^{\alpha\text{⅒}}\,{}^{z}\mathcal{T}^{\alpha}\right)-\\
&\frac{4}{3}\alpha\,\epsilon\mathring{g}_{\omega\alpha\alpha'\beta}\,\mathring{g}_{\upsilon\psi}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{y}\mathcal{T}^{\alpha'}\right)+\frac{2}{3}\alpha\,\epsilon\mathring{g}_{\psi\alpha\alpha'\beta}\,\mathring{g}_{\upsilon\omega}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{y}\mathcal{T}^{\alpha'}\right)+\\
&\frac{2}{3}\alpha\,\epsilon\mathring{g}_{\upsilon\alpha\alpha'\beta}\,\mathring{g}_{\psi\omega}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{y}\mathcal{T}^{\alpha'}\right)+\frac{2}{3}\alpha\,\epsilon\mathring{g}_{\omega\alpha\alpha'\beta}\,\mathring{g}_{\upsilon\psi}\,{}^{y}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{z}\mathcal{T}^{\alpha'}\right)-\\
&\frac{1}{3}\alpha\,\epsilon\mathring{g}_{\psi\alpha\alpha'\beta}\,\mathring{g}_{\upsilon\omega}\,{}^{y}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{z}\mathcal{T}^{\alpha'}\right)-\frac{1}{3}\alpha\,\epsilon\mathring{g}_{\upsilon\alpha\alpha'\beta}\,\mathring{g}_{\psi\omega}\,{}^{y}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{z}\mathcal{T}^{\alpha'}\right)-\frac{2}{9}\alpha\,\mathring{g}_{\psi\omega}\,{}^{y}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\upsilon}{}^{y}\mathcal{T}_{\alpha}\right)-\\
&\alpha\,\epsilon\mathring{g}_{\psi\omega\alpha\alpha'}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\upsilon}{}^{y}\mathcal{T}^{\alpha'}\right)+\frac{2}{3}\alpha\,{}^{y}\mathcal{T}_{\psi}\left(\mathring{\nabla}_{\upsilon}{}^{y}\mathcal{T}_{\omega}\right)+\frac{1}{2}\alpha\,\mathring{g}_{\psi\omega}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\upsilon}{}^{z}\mathcal{T}_{\alpha}\right)-\frac{3}{2}\alpha\,{}^{z}\mathcal{T}_{\psi}\left(\mathring{\nabla}_{\upsilon}{}^{z}\mathcal{T}_{\omega}\right)+\\
&\frac{2}{9}\alpha\,\mathring{g}_{\psi\omega}\left(\mathring{\nabla}_{\upsilon}\mathring{\nabla}_{\alpha}{}^{y}\mathcal{T}^{\alpha}\right)-\alpha\,\epsilon\mathring{g}_{\psi\omega\alpha\alpha'}\left(\mathring{\nabla}_{\upsilon}\mathring{\nabla}^{\alpha\text{⅒}}\,{}^{z}\mathcal{T}^{\alpha}\right)-\frac{2}{3}\alpha\left(\mathring{\nabla}_{\upsilon}\mathring{\nabla}_{\psi}{}^{y}\mathcal{T}_{\omega}\right)-\frac{2}{9}\alpha\,\mathring{g}_{\upsilon\omega}\,{}^{y}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\psi}{}^{y}\mathcal{T}_{\alpha}\right)-\\
&\alpha\,\epsilon\mathring{g}_{\upsilon\omega\alpha\alpha'}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\psi}{}^{y}\mathcal{T}^{\alpha'}\right)+\frac{2}{3}\alpha\,{}^{y}\mathcal{T}_{\upsilon}\left(\mathring{\nabla}_{\psi}{}^{y}\mathcal{T}_{\omega}\right)+\frac{1}{2}\alpha\,\mathring{g}_{\upsilon\omega}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\psi}{}^{z}\mathcal{T}_{\alpha}\right)-\frac{3}{2}\alpha\,{}^{z}\mathcal{T}_{\upsilon}\left(\mathring{\nabla}_{\psi}{}^{z}\mathcal{T}_{\omega}\right)+\\
&\frac{2}{9}\alpha\,\mathring{g}_{\upsilon\omega}\left(\mathring{\nabla}_{\psi}\mathring{\nabla}_{\alpha}{}^{y}\mathcal{T}^{\alpha}\right)-\alpha\,\epsilon\mathring{g}_{\upsilon\omega\alpha\alpha'}\left(\mathring{\nabla}_{\psi}\mathring{\nabla}^{\alpha\text{⅒}}\,{}^{z}\mathcal{T}^{\alpha}\right)-\frac{2}{3}\alpha\left(\mathring{\nabla}_{\psi}\mathring{\nabla}_{\upsilon}{}^{y}\mathcal{T}_{\omega}\right)+\frac{4}{9}\alpha\,\mathring{g}_{\upsilon\psi}\,{}^{y}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\omega}{}^{y}\mathcal{T}_{\alpha}\right)-\\
&\frac{2}{3}\alpha\,{}^{y}\mathcal{T}_{\psi}\left(\mathring{\nabla}_{\omega}{}^{y}\mathcal{T}_{\upsilon}\right)-\frac{2}{3}\alpha\,{}^{y}\mathcal{T}_{\upsilon}\left(\mathring{\nabla}_{\omega}{}^{y}\mathcal{T}_{\psi}\right)-\alpha\,\mathring{g}_{\upsilon\psi}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\omega}{}^{z}\mathcal{T}_{\alpha}\right)+\frac{3}{2}\alpha\,{}^{z}\mathcal{T}_{\psi}\left(\mathring{\nabla}_{\omega}{}^{z}\mathcal{T}_{\upsilon}\right)+\\
&\frac{3}{2}\alpha\,{}^{z}\mathcal{T}_{\upsilon}\left(\mathring{\nabla}_{\omega}{}^{z}\mathcal{T}_{\psi}\right)-\frac{4}{9}\alpha\,\mathring{g}_{\upsilon\psi}\left(\mathring{\nabla}_{\omega}\mathring{\nabla}_{\alpha}{}^{y}\mathcal{T}^{\alpha}\right)+\frac{2}{3}\alpha\left(\mathring{\nabla}_{\omega}\mathring{\nabla}_{\upsilon}{}^{y}\mathcal{T}_{\psi}\right)+\frac{2}{3}\alpha\left(\mathring{\nabla}_{\omega}\mathring{\nabla}_{\psi}{}^{y}\mathcal{T}_{\upsilon}\right)==0
\end{aligned}
\tag{80}
$$

Concrete relation to manuscript: Here is another spin equation: compare Eq. (80) to Eq. (8a) in the manuscript.

** MakeQuotientRule: canonicalised expression with tensor substituted by rule:

0

** MakeQuotientRule: canonicalised expression with tensor substituted by rule:

0

** MakeQuotientRule: canonicalised expression with tensor substituted by rule:

0

Now we examine Eq. (78) and Eq. (79) to construct rules to replace the divergences of the torsion.

$$-2\,\mathcal{M}_{\text{Pl}}^{y}\left(\mathring{\nabla}_{\alpha}\overset{y}{\mathcal{T}}^{\alpha}\right)-4\,\mathcal{M}_{\text{Pl}}^{y}\,\overset{y}{\mu}\left(\mathring{\nabla}_{\alpha}\overset{y}{\mathcal{T}}^{\alpha}\right)-\frac{\sqrt{-\mathring{\tilde{\tilde{g}}}}\left(\mathring{\nabla}_{\omega}\sigma^{\alpha}{}_{\alpha}{}^{\omega}\right)}{\mathring{\tilde{\tilde{g}}}}==0 \tag{81}$$

** MakeQuotientRule: canonicalised expression with tensor substituted by rule:

$$-6\,\mathcal{M}_{\text{Pl}}^{y}\left(\mathring{\nabla}_{\alpha}\overset{z}{\mathcal{T}}^{\alpha}\right)-48\,\mathcal{M}_{\text{Pl}}^{y}\,\overset{z}{\mu}\left(\mathring{\nabla}_{\alpha}\overset{z}{\mathcal{T}}^{\alpha}\right)-\frac{\sqrt{-\mathring{\tilde{\tilde{g}}}}\ \epsilon\mathring{g}_{\alpha\alpha'\beta\chi}\left(\mathring{\nabla}^{\chi}\sigma^{\alpha\alpha'\beta}\right)}{\mathring{\tilde{\tilde{g}}}}==0 \tag{82}$$

** MakeQuotientRule: canonicalised expression with tensor substituted by rule:

Now we examine Eq. (78) and Eq. (79) to construct rules to replace the d'Alembertians of the torsion.

$$\frac{\sqrt{-\mathring{\tilde{\tilde{g}}}}\ \sigma^{\alpha}{}_{\omega\alpha}}{\mathring{\tilde{\tilde{g}}}}-\frac{8}{3}\,\alpha\,R[\mathring{\nabla}]_{\omega\alpha}\ \overset{y}{\mathcal{T}}^{\alpha}-2\,\mathcal{M}_{\text{Pl}}^{y}\ \overset{y}{\mathcal{T}}_{\omega}-4\,\mathcal{M}_{\text{Pl}}^{y}\,\overset{y}{\mu}\ \overset{y}{\mathcal{T}}_{\omega}+\frac{8}{3}\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\ \overset{y}{\mathcal{T}}_{\omega}\right)-\frac{8\,\alpha\sqrt{-\mathring{\tilde{\tilde{g}}}}\left(\mathring{\nabla}_{\omega}\mathring{\nabla}_{\alpha'}\sigma^{\alpha}{}_{\alpha}{}^{\alpha'}\right)}{3\left(-2\,\mathcal{M}_{\text{Pl}}^{y}-4\,\mathcal{M}_{\text{Pl}}^{y}\,\overset{y}{\mu}\right)\mathring{\tilde{\tilde{g}}}}==0 \tag{83}$$

** MakeQuotientRule: canonicalised expression with tensor substituted by rule:

0

$$\frac{\sqrt{-\mathring{\tilde{\tilde{g}}}}\ \epsilon\mathring{g}^{\chi}{}_{\alpha\alpha'\beta}\ \sigma^{\alpha\alpha'\beta}}{\mathring{\tilde{\tilde{g}}}}-8\,\alpha\,R[\mathring{\nabla}]^{\chi}{}_{\alpha}\ \overset{z}{\mathcal{T}}^{\alpha}-6\,\mathcal{M}_{\text{Pl}}^{y}\ \overset{z}{\mathcal{T}}^{\chi}-$$

(84)

$$48\,\mathcal{M}_{\mathrm{Pl}}^{y}\,\overset{z}{:}\mu\;\overset{z}{:}\mathcal{T}^{\chi}+8\,\alpha\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,\overset{z}{:}\mathcal{T}^{\chi}\right)-\frac{8\,\alpha\,\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\left(\mathring{\nabla}^{\chi}\mathring{\nabla}^{\beta'}\sigma^{\alpha\alpha'\beta}\right)}{\left(-6\,\mathcal{M}_{\mathrm{Pl}}^{y}-48\,\mathcal{M}_{\mathrm{Pl}}^{y}\,\overset{z}{:}\mu\right)\overset{\approx}{\mathring{g}}}==0$$

```
** MakeQuotientRule: canonicalised expression with tensor substituted by rule:
```

0

We can now use the definition of the Belinfante-Rosenfeld tensor in Eq. (46) and the expression for the spin in terms of the gravitational fields in Eq. (76), to construct a rule to replace the general stress-energy tensor with the Belinfante-Rosenfeld tensor and gravitational fields.

$$\frac{8}{9}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;R[\mathring{\nabla}]^{\mu}{}_{\alpha}\;\overset{y}{:}\mathcal{T}^{\alpha}\;\overset{y}{:}\mathcal{T}_{\kappa}+\frac{4}{9}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;R[\mathring{\nabla}]_{\kappa\alpha}\;\overset{y}{:}\mathcal{T}^{\alpha}\;\overset{y}{:}\mathcal{T}^{\mu}+$$

$$\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\alpha}{}^{\alpha'\beta\beta'}\;\overset{y}{:}\mathcal{T}_{\kappa}\;\overset{z}{:}\mathcal{T}^{\alpha}+\frac{1}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\kappa\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\alpha}{}^{\alpha'\beta\beta'}\;\overset{y}{:}\mathcal{T}^{\mu}\;\overset{z}{:}\mathcal{T}^{\alpha}-$$

$$\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\kappa}{}^{\mu}{}_{\alpha'\beta}\;R[\mathring{\nabla}]_{\alpha}{}^{\beta}{}_{\alpha}\;\overset{y}{:}\mathcal{T}^{\alpha}\;\overset{z}{:}\mathcal{T}^{\alpha'}-\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\kappa}{}^{\mu}{}_{\beta\beta'}\;R[\mathring{\nabla}]_{\alpha}{}^{\beta}{}_{\alpha'}{}^{\beta'}\;\overset{y}{:}\mathcal{T}^{\alpha}\;\overset{z}{:}\mathcal{T}^{\alpha'}+$$

$$\frac{2}{3}\alpha\,\delta_{\kappa}{}^{\mu}\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\alpha'\beta\beta'\lambda}\;R[\mathring{\nabla}]_{\alpha}{}^{\beta\beta'\lambda}\;\overset{y}{:}\mathcal{T}^{\alpha}\;\overset{z}{:}\mathcal{T}^{\alpha'}-\frac{1}{3}\alpha\,\delta_{\kappa}{}^{\mu}\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\alpha\beta\beta'\lambda}\;R[\mathring{\nabla}]_{\alpha'}{}^{\beta\beta'\lambda}\;\overset{y}{:}\mathcal{T}^{\alpha}\;\overset{z}{:}\mathcal{T}^{\alpha'}-$$

$$\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\kappa\alpha}{}^{\beta\beta'}\;\overset{y}{:}\mathcal{T}^{\alpha}\;\overset{z}{:}\mathcal{T}^{\alpha'}+\frac{4}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\kappa}{}^{\beta}{}_{\alpha}{}^{\beta'}\;\overset{y}{:}\mathcal{T}^{\alpha}\;\overset{z}{:}\mathcal{T}^{\alpha'}-$$

$$\frac{1}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\kappa\alpha'\beta\beta'}\;R[\mathring{\nabla}]^{\mu}{}_{\alpha}{}^{\beta\beta'}\;\overset{y}{:}\mathcal{T}^{\alpha}\;\overset{z}{:}\mathcal{T}^{\alpha'}+\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\kappa\alpha'\beta\beta'}\;R[\mathring{\nabla}]^{\mu\beta}{}_{\alpha}{}^{\beta'}\;\overset{y}{:}\mathcal{T}^{\alpha}\;\overset{z}{:}\mathcal{T}^{\alpha'}-$$

$$2\,\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;R[\mathring{\nabla}]^{\mu}{}_{\alpha}\;\overset{z}{:}\mathcal{T}^{\alpha}\;\overset{z}{:}\mathcal{T}_{\kappa}-\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;R[\mathring{\nabla}]_{\kappa\alpha}\;\overset{z}{:}\mathcal{T}^{\alpha}\;\overset{z}{:}\mathcal{T}^{\mu}-\theta_{\kappa}{}^{\mu}+\tau(\Delta)^{\mu}{}_{\kappa}-$$

$$\frac{2}{3}\mathcal{M}_{\mathrm{Pl}}^{y}\,\delta_{\kappa}{}^{\mu}\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\alpha}\overset{y}{:}\mathcal{T}^{\alpha}\right)-\frac{4}{3}\mathcal{M}_{\mathrm{Pl}}^{y}\,\overset{y}{:}\mu\;\delta_{\kappa}{}^{\mu}\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\alpha}\overset{y}{:}\mathcal{T}^{\alpha}\right)+\frac{4}{3}\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\alpha}\overset{x}{:}\lambda_{\mathcal{T}\kappa}{}^{\alpha\mu}\right)-$$

$$\frac{4}{3}\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\alpha}\overset{x}{:}\lambda_{\mathcal{T}\kappa}{}^{\mu\alpha}\right)-\frac{4}{9}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\overset{y}{:}\mathcal{T}^{\mu}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\overset{y}{:}\mathcal{T}_{\kappa}\right)-\frac{8}{9}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\overset{y}{:}\mathcal{T}_{\kappa}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\overset{y}{:}\mathcal{T}^{\mu}\right)+$$

$$\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\overset{z}{:}\mathcal{T}^{\mu}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\overset{z}{:}\mathcal{T}_{\kappa}\right)+2\,\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\overset{z}{:}\mathcal{T}_{\kappa}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\overset{z}{:}\mathcal{T}^{\mu}\right)-\frac{4}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\alpha}\overset{y}{:}\mathcal{T}^{\mu}\right)\left(\mathring{\nabla}^{\alpha}\overset{y}{:}\mathcal{T}_{\kappa}\right)+$$

$$\frac{4}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;R[\mathring{\nabla}]_{\kappa\alpha}\left(\mathring{\nabla}^{\alpha}\overset{y}{:}\mathcal{T}^{\mu}\right)+3\,\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\alpha}\overset{z}{:}\mathcal{T}^{\mu}\right)\left(\mathring{\nabla}^{\alpha}\overset{z}{:}\mathcal{T}_{\kappa}\right)+\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\overset{y}{:}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}R[\mathring{\nabla}]_{\kappa}{}^{\mu}{}_{\alpha}{}^{\alpha'}\right)-$$

$$\frac{4}{9}\alpha\,\delta_{\kappa}{}^{\mu}\sqrt{-\overset{\approx}{\mathring{g}}}\;\overset{y}{:}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{y}{:}\mathcal{T}^{\alpha'}\right)+\alpha\,\delta_{\kappa}{}^{\mu}\sqrt{-\overset{\approx}{\mathring{g}}}\;\overset{z}{:}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha'}\mathring{\nabla}_{\alpha}\overset{z}{:}\mathcal{T}^{\alpha'}\right)+$$

$$\frac{4}{9}\alpha\,\delta_{\kappa}{}^{\mu}\sqrt{-\overset{\approx}{\mathring{g}}}\;{}^{y}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha\frac{1}{10}}\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{y}_{\bullet}\mathcal{T}_{\alpha}\right)-\alpha\,\delta_{\kappa}{}^{\mu}\sqrt{-\overset{\approx}{\mathring{g}}}\;{}^{z}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha\frac{1}{10}}\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}_{\bullet}\mathcal{T}_{\alpha}\right)+$$

$$\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;R[\mathring{\nabla}]_{\kappa\alpha}{}^{\mu}{}_{\alpha'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{y}_{\bullet}\mathcal{T}^{\alpha}\right)-\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;R[\mathring{\nabla}]_{\kappa\alpha'}{}^{\mu}{}_{\alpha}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{y}_{\bullet}\mathcal{T}^{\alpha}\right)+\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;R[\mathring{\nabla}]_{\kappa}{}^{\mu}{}_{\alpha\alpha'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{y}_{\bullet}\mathcal{T}^{\alpha}\right)-$$

$$\frac{4}{9}\alpha\,\delta_{\kappa}{}^{\mu}\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\alpha}\,{}^{y}_{\bullet}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{y}_{\bullet}\mathcal{T}^{\alpha}\right)+\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\kappa}{}^{\mu}{}_{\alpha'\beta}\left(\mathring{\nabla}_{\alpha}\,{}^{z}_{\bullet}\mathcal{T}^{\beta}\right)\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{y}_{\bullet}\mathcal{T}^{\alpha}\right)+$$

$$\frac{4}{9}\alpha\,\delta_{\kappa}{}^{\mu}\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\alpha\frac{1}{10}}\,{}^{y}_{\bullet}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{y}_{\bullet}\mathcal{T}^{\alpha}\right)-\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\kappa}{}^{\mu}{}_{\alpha\beta}\left(\mathring{\nabla}_{\alpha\frac{1}{10}}\,{}^{z}_{\bullet}\mathcal{T}^{\beta}\right)\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{y}_{\bullet}\mathcal{T}^{\alpha}\right)+$$

$$\frac{1}{2}\mathcal{M}_{\mathrm{Pl}}{}^{y}\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\kappa}{}^{\mu}{}_{\alpha\alpha'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}_{\bullet}\mathcal{T}^{\alpha}\right)+4\,\mathcal{M}_{\mathrm{Pl}}{}^{y}\,{}^{z}_{\bullet}\mu\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\kappa}{}^{\mu}{}_{\alpha\alpha'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}_{\bullet}\mathcal{T}^{\alpha}\right)-$$

$$\alpha\,\delta_{\kappa}{}^{\mu}\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\alpha'\beta\beta'\lambda}\;R[\mathring{\nabla}]_{\alpha}{}^{\beta\beta'\lambda}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}_{\bullet}\mathcal{T}^{\alpha}\right)+\frac{1}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}^{\mu}{}_{\alpha\beta\beta'}\;R[\mathring{\nabla}]_{\kappa\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}_{\bullet}\mathcal{T}^{\alpha}\right)-$$

$$2\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\kappa}{}^{\beta}{}_{\alpha}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}_{\bullet}\mathcal{T}^{\alpha}\right)+\frac{4}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}^{\mu}{}_{\alpha\beta\beta'}\;R[\mathring{\nabla}]_{\kappa}{}^{\beta}{}_{\alpha'}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}_{\bullet}\mathcal{T}^{\alpha}\right)+$$

$$\alpha\,\delta_{\kappa}{}^{\mu}\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\alpha}\,{}^{z}_{\bullet}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}_{\bullet}\mathcal{T}^{\alpha}\right)-\alpha\,\delta_{\kappa}{}^{\mu}\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\alpha\frac{1}{10}}\,{}^{z}_{\bullet}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}_{\bullet}\mathcal{T}^{\alpha}\right)+$$

$$\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\kappa}{}^{\mu}{}_{\alpha\alpha'}\;{}^{z}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\beta}\mathring{\nabla}^{\beta}\,{}^{y}_{\bullet}\mathcal{T}^{\alpha'}\right)-\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\kappa}{}^{\mu}{}_{\alpha'\beta}\left(\mathring{\nabla}_{\alpha}\,{}^{y}_{\bullet}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\beta}\,{}^{z}_{\bullet}\mathcal{T}^{\alpha'}\right)-$$

$$\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\kappa}{}^{\mu}{}_{\alpha'\beta}\;{}^{y}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha}\,{}^{z}_{\bullet}\mathcal{T}^{\alpha'}\right)-\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\kappa}{}^{\mu}{}_{\alpha\beta}\;{}^{z}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\mathring{\nabla}_{\alpha\frac{1}{10}}\,{}^{y}_{\bullet}\mathcal{T}^{\alpha'}\right)-$$

$$\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\;{}^{z}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta\frac{1}{10}}R[\mathring{\nabla}]_{\kappa}{}^{\alpha'}{}_{\alpha}{}^{\beta}\right)+\frac{2}{3}\alpha\,\delta_{\kappa}{}^{\mu}\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{y}_{\bullet}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\beta\frac{1}{10}}\,{}^{z}_{\bullet}\mathcal{T}^{\beta}\right)+$$

$$\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;{}^{y}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\kappa}R[\mathring{\nabla}]^{\mu}{}_{\alpha}\right)+\frac{1}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}^{\mu}{}_{\alpha'\beta\beta'}\;{}^{z}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\kappa}R[\mathring{\nabla}]_{\alpha}{}^{\alpha'\beta\beta'}\right)+$$

$$\frac{8}{9}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}^{\alpha}\,{}^{y}_{\bullet}\mathcal{T}^{\mu}\right)\left(\mathring{\nabla}_{\kappa}\,{}^{y}_{\bullet}\mathcal{T}_{\alpha}\right)+\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}^{\mu}{}_{\alpha\alpha'\beta}\left(\mathring{\nabla}^{\beta}\,{}^{z}_{\bullet}\mathcal{T}^{\alpha'}\right)\left(\mathring{\nabla}_{\kappa}\,{}^{y}_{\bullet}\mathcal{T}^{\alpha}\right)+$$

$$\frac{2}{3}\mathcal{M}_{\mathrm{Pl}}{}^{y}\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\kappa}\,{}^{y}_{\bullet}\mathcal{T}^{\mu}\right)+\frac{4}{3}\mathcal{M}_{\mathrm{Pl}}{}^{y}\,{}^{y}_{\bullet}\mu\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\kappa}\,{}^{y}_{\bullet}\mathcal{T}^{\mu}\right)-\frac{4}{9}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\alpha}\,{}^{y}_{\bullet}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\kappa}\,{}^{y}_{\bullet}\mathcal{T}^{\mu}\right)-$$

$$2\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}^{\alpha}\,{}^{z}_{\bullet}\mathcal{T}^{\mu}\right)\left(\mathring{\nabla}_{\kappa}\,{}^{z}_{\bullet}\mathcal{T}_{\alpha}\right)+\frac{4}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}^{\mu}{}_{\alpha\alpha'\beta}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{y}_{\bullet}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\kappa}\,{}^{z}_{\bullet}\mathcal{T}^{\beta}\right)+$$

$$\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\alpha}\,{}^{z}_{\bullet}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\kappa}\,{}^{z}_{\bullet}\mathcal{T}^{\mu}\right)+\frac{4}{9}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;{}^{y}_{\bullet}\mathcal{T}^{\mu}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\,{}^{y}_{\bullet}\mathcal{T}^{\alpha}\right)-\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;{}^{z}_{\bullet}\mathcal{T}^{\mu}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}_{\alpha}\,{}^{z}_{\bullet}\mathcal{T}^{\alpha}\right)-$$

$$\frac{2}{3}\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}^{\mu}{}_{\alpha\alpha'\beta}\;{}^{y}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\kappa}\mathring{\nabla}^{\beta}\,{}^{z}_{\bullet}\mathcal{T}^{\alpha'}\right)+\alpha\,\delta_{\kappa}{}^{\mu}\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\alpha'\beta\beta'\lambda}\;{}^{z}_{\bullet}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\lambda}R[\mathring{\nabla}]_{\alpha}{}^{\alpha'\beta\beta'}\right)-$$

$$\frac{2}{3}\,\alpha\,\sqrt{-\overset{\approx}{\mathring{g}}}\;{}^{y}_{\cdot}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\mu}R\big[\mathring{\nabla}\big]_{\kappa\alpha}\right)+\frac{8}{9}\,\alpha\,\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}^{\alpha}\,{}^{y}_{\cdot}\mathcal{T}_{\kappa}\right)\left(\mathring{\nabla}^{\mu}\,{}^{y}_{\cdot}\mathcal{T}_{\alpha}\right)-\frac{4}{9}\,\alpha\,\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\kappa}\,{}^{y}_{\cdot}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\mu}\,{}^{y}_{\cdot}\mathcal{T}_{\alpha}\right)-$$

$$\frac{4}{3}\,\alpha\,\sqrt{-\overset{\approx}{\mathring{g}}}\;R\big[\mathring{\nabla}\big]_{\kappa\alpha}\left(\mathring{\nabla}^{\mu}\,{}^{y}_{\cdot}\mathcal{T}^{\alpha}\right)+\frac{2}{3}\,\alpha\,\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\kappa\alpha\alpha'\beta}\left(\mathring{\nabla}^{\beta}\,{}^{z}_{\cdot}\mathcal{T}^{\alpha'}\right)\left(\mathring{\nabla}^{\mu}\,{}^{y}_{\cdot}\mathcal{T}^{\alpha}\right)+$$

$$\frac{4}{9}\,\alpha\,\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\alpha}\,{}^{y}_{\cdot}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\mu}\,{}^{y}_{\cdot}\mathcal{T}_{\kappa}\right)-2\,\alpha\,\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}^{\alpha}\,{}^{z}_{\cdot}\mathcal{T}_{\kappa}\right)\left(\mathring{\nabla}^{\mu}\,{}^{z}_{\cdot}\mathcal{T}_{\alpha}\right)+$$

$$\alpha\,\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\kappa}\,{}^{z}_{\cdot}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\mu}\,{}^{z}_{\cdot}\mathcal{T}_{\alpha}\right)-\alpha\,\sqrt{-\overset{\approx}{\mathring{g}}}\left(\mathring{\nabla}_{\alpha}\,{}^{z}_{\cdot}\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}^{\mu}\,{}^{z}_{\cdot}\mathcal{T}_{\kappa}\right)+\frac{8}{9}\,\alpha\,\sqrt{-\overset{\approx}{\mathring{g}}}\;{}^{y}_{\cdot}\mathcal{T}_{\kappa}\left(\mathring{\nabla}^{\mu}\mathring{\nabla}_{\alpha}\,{}^{y}_{\cdot}\mathcal{T}^{\alpha}\right)+$$

$$\frac{4}{9}\,\alpha\,\sqrt{-\overset{\approx}{\mathring{g}}}\;{}^{y}_{\cdot}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\mu}\mathring{\nabla}_{\alpha}\,{}^{y}_{\cdot}\mathcal{T}_{\kappa}\right)-2\,\alpha\,\sqrt{-\overset{\approx}{\mathring{g}}}\;{}^{z}_{\cdot}\mathcal{T}_{\kappa}\left(\mathring{\nabla}^{\mu}\mathring{\nabla}_{\alpha}\,{}^{z}_{\cdot}\mathcal{T}^{\alpha}\right)-\alpha\,\sqrt{-\overset{\approx}{\mathring{g}}}\;{}^{z}_{\cdot}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\mu}\mathring{\nabla}_{\alpha}\,{}^{z}_{\cdot}\mathcal{T}_{\kappa}\right)-$$

$$\frac{2}{3}\,\alpha\,\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\kappa\alpha\alpha'\beta}\;{}^{z}_{\cdot}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\mu}\mathring{\nabla}^{\beta}\,{}^{y}_{\cdot}\mathcal{T}^{\alpha'}\right)-\frac{4}{9}\,\alpha\,\sqrt{-\overset{\approx}{\mathring{g}}}\;{}^{y}_{\cdot}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\mu}\mathring{\nabla}_{\kappa}\,{}^{y}_{\cdot}\mathcal{T}_{\alpha}\right)+\alpha\,\sqrt{-\overset{\approx}{\mathring{g}}}\;{}^{z}_{\cdot}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\mu}\mathring{\nabla}_{\kappa}\,{}^{z}_{\cdot}\mathcal{T}_{\alpha}\right)==0$$

```
** MakeQuotientRule: canonicalised expression with tensor substituted by rule:
```

0

Symmetric part of tetrad equation

We now jump straight in with the symmetric part of the tetrad equation in Eq. (75).

We eliminate the second-order stress-energy tensor in favour of the Einstein tensor and gravitational variables using the equation Eq. (85).

$$-\mathcal{M}_{\text{Pl}}{}^{y}\;R\big[\mathring{\nabla}\big]_{\iota\theta}+\frac{1}{2}\,\mathcal{M}_{\text{Pl}}{}^{y}\;\mathring{g}_{\iota\theta}\;R\big[\mathring{\nabla}\big]+\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\mathring{g}_{\iota\theta}\;\sigma^{\alpha'}{}_{\alpha\alpha'}\;{}^{y}_{\cdot}\mathcal{T}^{\alpha}}{3\,\overset{\approx}{\mathring{g}}}+\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\sigma_{\iota\theta\alpha}\;{}^{y}_{\cdot}\mathcal{T}^{\alpha}}{6\,\overset{\approx}{\mathring{g}}}+$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\sigma_{\theta\iota\alpha}\;{}^{y}_{\cdot}\mathcal{T}^{\alpha}}{6\,\overset{\approx}{\mathring{g}}}-\frac{1}{9}\,\mathcal{M}_{\text{Pl}}{}^{y}\;\mathring{g}_{\iota\theta}\;{}^{y}_{\cdot}\mathcal{T}_{\alpha}\;{}^{y}_{\cdot}\mathcal{T}^{\alpha}-\frac{2}{9}\,\mathcal{M}_{\text{Pl}}{}^{y}\;{}^{y}_{\cdot}\mu\;\mathring{g}_{\iota\theta}\;{}^{y}_{\cdot}\mathcal{T}_{\alpha}\;{}^{y}_{\cdot}\mathcal{T}^{\alpha}-$$

$$\frac{2}{3}\,\alpha\,R\big[\mathring{\nabla}\big]_{\theta\alpha}\;{}^{y}_{\cdot}\mathcal{T}^{\alpha}\;{}^{y}_{\cdot}\mathcal{T}_{\iota}-\frac{2}{3}\,\alpha\,R\big[\mathring{\nabla}\big]_{\iota\alpha}\;{}^{y}_{\cdot}\mathcal{T}^{\alpha}\;{}^{y}_{\cdot}\mathcal{T}_{\theta}-\frac{2}{9}\,\mathcal{M}_{\text{Pl}}{}^{y}\;{}^{y}_{\cdot}\mathcal{T}_{\iota}\;{}^{y}_{\cdot}\mathcal{T}_{\theta}-\frac{4}{9}\,\mathcal{M}_{\text{Pl}}{}^{y}\;{}^{y}_{\cdot}\mu\;{}^{y}_{\cdot}\mathcal{T}_{\iota}\;{}^{y}_{\cdot}\mathcal{T}_{\theta}-$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\;\mathring{g}_{\iota\theta}\;\sigma^{\alpha'\beta\beta'}\;{}^{z}_{\cdot}\mathcal{T}^{\alpha}}{4\,\overset{\approx}{\mathring{g}}}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;\sigma_{\iota}{}^{\alpha'\beta}\;{}^{z}_{\cdot}\mathcal{T}^{\alpha}}{8\,\overset{\approx}{\mathring{g}}}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;\sigma_{\theta}{}^{\alpha'\beta}\;{}^{z}_{\cdot}\mathcal{T}^{\alpha}}{8\,\overset{\approx}{\mathring{g}}}-$$

$$\frac{1}{2}\,\alpha\,\epsilon\,\mathring{g}_{\theta\alpha'\beta\beta'}\;R\big[\mathring{\nabla}\big]_{\alpha}{}^{\alpha'\beta\beta'}\;{}^{y}_{\cdot}\mathcal{T}_{\iota}\;{}^{z}_{\cdot}\mathcal{T}^{\alpha}-\frac{1}{2}\,\alpha\,\epsilon\,\mathring{g}_{\iota\alpha'\beta\beta'}\;R\big[\mathring{\nabla}\big]_{\alpha}{}^{\alpha'\beta\beta'}\;{}^{y}_{\cdot}\mathcal{T}_{\theta}\;{}^{z}_{\cdot}\mathcal{T}^{\alpha}+$$

$$
\begin{aligned}
&\frac{1}{4}\mathcal{M}_{\text{Pl}}^{y}\,\mathring{g}_{\iota\theta}\,{}^{z}\mathcal{T}_{\alpha}\,{}^{z}\mathcal{T}^{\alpha}+2\mathcal{M}_{\text{Pl}}^{y}\,{}^{z}\mu\,\mathring{g}_{\iota\theta}\,{}^{z}\mathcal{T}_{\alpha}\,{}^{z}\mathcal{T}^{\alpha}-\frac{2}{3}\alpha\,\epsilon\mathring{g}_{\alpha'\beta\beta'\chi}\,\mathring{g}_{\iota\theta}\,R[\mathring{\nabla}]_{\alpha}{}^{\beta\beta'\chi}\,{}^{y}\mathcal{T}^{\alpha}\,{}^{z}\mathcal{T}^{\alpha'}+\\
&\frac{1}{3}\alpha\,\epsilon\mathring{g}_{\alpha\beta\beta'\chi}\,\mathring{g}_{\iota\theta}\,R[\mathring{\nabla}]_{\alpha'}{}^{\beta\beta'\chi}\,{}^{y}\mathcal{T}^{\alpha}\,{}^{z}\mathcal{T}^{\alpha'}+\frac{1}{2}\alpha\,\epsilon\mathring{g}_{\theta\alpha'\beta\beta'}\,R[\mathring{\nabla}]_{\iota\alpha}{}^{\beta\beta'}\,{}^{y}\mathcal{T}^{\alpha}\,{}^{z}\mathcal{T}^{\alpha'}-\\
&\alpha\,\epsilon\mathring{g}_{\theta\alpha'\beta\beta'}\,R[\mathring{\nabla}]_{\iota}{}^{\beta}{}_{\alpha}{}^{\beta'}\,{}^{y}\mathcal{T}^{\alpha}\,{}^{z}\mathcal{T}^{\alpha'}+\frac{1}{2}\alpha\,\epsilon\mathring{g}_{\iota\alpha'\beta\beta'}\,R[\mathring{\nabla}]_{\theta\alpha}{}^{\beta\beta'}\,{}^{y}\mathcal{T}^{\alpha}\,{}^{z}\mathcal{T}^{\alpha'}-\\
&\alpha\,\epsilon\mathring{g}_{\iota\alpha'\beta\beta'}\,R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha}{}^{\beta'}\,{}^{y}\mathcal{T}^{\alpha}\,{}^{z}\mathcal{T}^{\alpha'}+\frac{3}{2}\alpha\,R[\mathring{\nabla}]_{\theta\alpha}\,{}^{z}\mathcal{T}^{\alpha}\,{}^{z}\mathcal{T}_{\iota}+\frac{3}{2}\alpha\,R[\mathring{\nabla}]_{\iota\alpha}\,{}^{z}\mathcal{T}^{\alpha}\,{}^{z}\mathcal{T}_{\theta}+\\
&\frac{1}{2}\mathcal{M}_{\text{Pl}}^{y}\,{}^{z}\mathcal{T}_{\iota}\,{}^{z}\mathcal{T}_{\theta}+4\mathcal{M}_{\text{Pl}}^{y}\,{}^{z}\mu\,{}^{z}\mathcal{T}_{\iota}\,{}^{z}\mathcal{T}_{\theta}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\theta_{\iota\theta}}{\overset{\approx}{\mathring{g}}}-\frac{2}{9}\,{}^{y}\mathcal{T}^{\alpha}\,{}^{x}\lambda_{\mathcal{T}\,\iota\alpha\theta}-\\
&\frac{1}{3}\epsilon\mathring{g}_{\theta\alpha\alpha'\beta}\,{}^{z}\mathcal{T}^{\alpha}\,{}^{x}\lambda_{\mathcal{T}\,\iota}{}^{\alpha'\beta}+\frac{4}{9}\,{}^{y}\mathcal{T}^{\alpha}\,{}^{x}\lambda_{\mathcal{T}\,\iota\theta\alpha}-\frac{2}{9}\,{}^{y}\mathcal{T}^{\alpha}\,{}^{x}\lambda_{\mathcal{T}\,\theta\alpha\iota}-\frac{1}{3}\epsilon\mathring{g}_{\iota\alpha\alpha'\beta}\,{}^{z}\mathcal{T}^{\alpha}\,{}^{x}\lambda_{\mathcal{T}\,\theta}{}^{\alpha'\beta}+\\
&\frac{4}{27}\alpha\,{}^{y}\mathcal{T}^{\alpha}\,{}^{y}\mathcal{T}_{\theta}\left(\mathring{\nabla}_{\alpha}{}^{y}\mathcal{T}_{\iota}\right)-\frac{1}{3}\alpha\,{}^{z}\mathcal{T}^{\alpha}\,{}^{z}\mathcal{T}_{\theta}\left(\mathring{\nabla}_{\alpha}{}^{z}\mathcal{T}_{\iota}\right)+\frac{4}{27}\alpha\,{}^{y}\mathcal{T}^{\alpha}\,{}^{y}\mathcal{T}_{\iota}\left(\mathring{\nabla}_{\alpha}{}^{y}\mathcal{T}_{\theta}\right)-\\
&\frac{1}{3}\alpha\,{}^{z}\mathcal{T}^{\alpha}\,{}^{z}\mathcal{T}_{\iota}\left(\mathring{\nabla}_{\alpha}{}^{y}\mathcal{T}_{\theta}\right)-\frac{1}{3}\alpha\,{}^{y}\mathcal{T}_{\theta}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{z}\mathcal{T}_{\iota}\right)-\frac{1}{3}\alpha\,{}^{y}\mathcal{T}^{\alpha}\,{}^{z}\mathcal{T}_{\theta}\left(\mathring{\nabla}_{\alpha}{}^{z}\mathcal{T}_{\iota}\right)-\\
&\frac{1}{3}\alpha\,{}^{y}\mathcal{T}_{\iota}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha}{}^{z}\mathcal{T}_{\theta}\right)-\frac{1}{3}\alpha\,{}^{y}\mathcal{T}^{\alpha}\,{}^{z}\mathcal{T}_{\iota}\left(\mathring{\nabla}_{\alpha}{}^{z}\mathcal{T}_{\theta}\right)+\frac{2}{3}\alpha\,{}^{y}\mathcal{T}_{\theta}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,{}^{y}\mathcal{T}_{\iota}\right)+\frac{2}{3}\alpha\,{}^{y}\mathcal{T}_{\iota}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,{}^{y}\mathcal{T}_{\theta}\right)-\\
&\frac{3}{2}\alpha\,{}^{z}\mathcal{T}_{\theta}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,{}^{z}\mathcal{T}_{\iota}\right)-\frac{3}{2}\alpha\,{}^{z}\mathcal{T}_{\iota}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\,{}^{z}\mathcal{T}_{\theta}\right)+\frac{8}{9}\alpha\left(\mathring{\nabla}_{\alpha}{}^{y}\mathcal{T}_{\theta}\right)\left(\mathring{\nabla}^{\alpha}\,{}^{y}\mathcal{T}_{\iota}\right)-2\alpha\left(\mathring{\nabla}_{\alpha}{}^{z}\mathcal{T}_{\theta}\right)\left(\mathring{\nabla}^{\alpha}\,{}^{z}\mathcal{T}_{\iota}\right)+\\
&\frac{4}{9}\alpha\,\mathring{g}_{\iota\theta}\,{}^{y}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha\frac{1}{10}}\mathring{\nabla}_{\alpha}{}^{y}\mathcal{T}^{\alpha'}\right)-\alpha\,\mathring{g}_{\iota\theta}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha\frac{1}{10}}\mathring{\nabla}_{\alpha}{}^{z}\mathcal{T}^{\alpha'}\right)-\frac{4}{9}\alpha\,\mathring{g}_{\iota\theta}\,{}^{y}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha\frac{1}{10}}\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{y}\mathcal{T}_{\alpha}\right)+\\
&\alpha\,\mathring{g}_{\iota\theta}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha\frac{1}{10}}\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}\mathcal{T}_{\alpha}\right)+\frac{4}{9}\alpha\,\mathring{g}_{\iota\theta}\left(\mathring{\nabla}_{\alpha}{}^{y}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{y}\mathcal{T}^{\alpha}\right)-\frac{4}{9}\alpha\,\mathring{g}_{\iota\theta}\left(\mathring{\nabla}_{\alpha\frac{1}{10}}{}^{y}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{y}\mathcal{T}^{\alpha}\right)-\\
&\alpha\,\epsilon\mathring{g}_{\theta\alpha\alpha'\beta}\,R[\mathring{\nabla}]_{\iota}{}^{\beta}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}\mathcal{T}^{\alpha}\right)-\alpha\,\epsilon\mathring{g}_{\iota\alpha\alpha'\beta}\,R[\mathring{\nabla}]_{\theta}{}^{\beta}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}\mathcal{T}^{\alpha}\right)+\alpha\,\epsilon\mathring{g}_{\alpha'\beta\beta'\chi}\,\mathring{g}_{\iota\theta}\,R[\mathring{\nabla}]_{\alpha}{}^{\beta\beta'\chi}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}\mathcal{T}^{\alpha}\right)-\\
&\frac{1}{6}\alpha\,\epsilon\mathring{g}_{\theta\alpha\beta\beta'}\,R[\mathring{\nabla}]_{\iota\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}\mathcal{T}^{\alpha}\right)+\alpha\,\epsilon\mathring{g}_{\theta\alpha'\beta\beta'}\,R[\mathring{\nabla}]_{\iota}{}^{\beta}{}_{\alpha}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}\mathcal{T}^{\alpha}\right)-\\
&\frac{2}{3}\alpha\,\epsilon\mathring{g}_{\theta\alpha\beta\beta'}\,R[\mathring{\nabla}]_{\iota}{}^{\beta}{}_{\alpha'}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}\mathcal{T}^{\alpha}\right)-\frac{1}{6}\alpha\,\epsilon\mathring{g}_{\iota\alpha\beta\beta'}\,R[\mathring{\nabla}]_{\theta\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}\mathcal{T}^{\alpha}\right)+\\
&\alpha\,\epsilon\mathring{g}_{\iota\alpha'\beta\beta'}\,R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}\mathcal{T}^{\alpha}\right)-\frac{2}{3}\alpha\,\epsilon\mathring{g}_{\iota\alpha\beta\beta'}\,R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha'}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}\mathcal{T}^{\alpha}\right)-\alpha\,\mathring{g}_{\iota\theta}\left(\mathring{\nabla}_{\alpha}{}^{z}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}\mathcal{T}^{\alpha}\right)+\\
&\alpha\,\mathring{g}_{\iota\theta}\left(\mathring{\nabla}_{\alpha\frac{1}{10}}{}^{z}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\,{}^{z}\mathcal{T}^{\alpha}\right)-\frac{4}{9}\alpha\,\epsilon\mathring{g}_{\theta\alpha\alpha'\beta}\,{}^{y}\mathcal{T}_{\iota}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{y}\mathcal{T}^{\alpha'}\right)-\frac{4}{9}\alpha\,\epsilon\mathring{g}_{\iota\alpha\alpha'\beta}\,{}^{y}\mathcal{T}_{\theta}\,{}^{z}\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta}\,{}^{y}\mathcal{T}^{\alpha'}\right)-\\
&\frac{2}{9}\alpha\,\epsilon\mathring{g}_{\theta\alpha\alpha'\beta}\,{}^{y}\mathcal{T}^{\alpha}\,{}^{z}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}^{\beta}\,{}^{y}\mathcal{T}_{\iota}\right)-\frac{2}{9}\alpha\,\epsilon\mathring{g}_{\iota\alpha\alpha'\beta}\,{}^{y}\mathcal{T}^{\alpha}\,{}^{z}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}^{\beta}\,{}^{y}\mathcal{T}_{\theta}\right)-\\
&\frac{2}{9}\alpha\,\epsilon\mathring{g}_{\theta\alpha\alpha'\beta}\,{}^{y}\mathcal{T}^{\alpha}\,{}^{y}\mathcal{T}_{\iota}\left(\mathring{\nabla}^{\beta}\,{}^{z}\mathcal{T}^{\alpha'}\right)-\frac{2}{9}\alpha\,\epsilon\mathring{g}_{\iota\alpha\alpha'\beta}\,{}^{y}\mathcal{T}^{\alpha}\,{}^{y}\mathcal{T}_{\theta}\left(\mathring{\nabla}^{\beta}\,{}^{z}\mathcal{T}^{\alpha'}\right)+
\end{aligned}
$$

$$\frac{1}{2}\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\,{}^{z}\!\mathcal{T}^{\alpha}\,{}^{z}\!\mathcal{T}_{\iota}\left(\mathring{\nabla}^{\beta}\,{}^{z}\!\mathcal{T}^{\alpha'}\right)+\frac{1}{2}\alpha\,\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\,{}^{z}\!\mathcal{T}^{\alpha}\,{}^{z}\!\mathcal{T}_{\theta}\left(\mathring{\nabla}^{\beta}\,{}^{z}\!\mathcal{T}^{\alpha'}\right)+$$

$$\frac{1}{3}\alpha\,\epsilon\,\mathring{g}_{\theta\alpha'\beta\beta'}\,{}^{z}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta\frac{1}{10}}R[\mathring{\nabla}]_{\iota}{}^{\alpha'}{}_{\alpha}{}^{\beta}\right)+\frac{1}{3}\alpha\,\epsilon\,\mathring{g}_{\iota\alpha'\beta\beta'}\,{}^{z}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\beta\frac{1}{10}}R[\mathring{\nabla}]_{\theta}{}^{\alpha'}{}_{\alpha}{}^{\beta}\right)+$$

$$\frac{4}{9}\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\,\mathring{g}_{\iota\theta}\,{}^{y}\!\mathcal{T}^{\alpha}\,{}^{z}\!\mathcal{T}^{\alpha'}\left(\mathring{\nabla}^{\beta\frac{1}{10}}\,{}^{y}\!\mathcal{T}^{\beta}\right)-\alpha\,\epsilon\,\mathring{g}_{\alpha'\beta\beta'\chi}\,\mathring{g}_{\iota\theta}\,{}^{z}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}^{\chi}R[\mathring{\nabla}]_{\alpha}{}^{\alpha'\beta\beta'}\right)-$$

$$\frac{1}{6}\alpha\,\epsilon\,\mathring{g}_{\theta\alpha'\beta\beta'}\,{}^{z}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\iota}R[\mathring{\nabla}]_{\alpha}{}^{\alpha'\beta\beta'}\right)-\frac{4}{27}\alpha\,{}^{y}\!\mathcal{T}^{\alpha}\,{}^{y}\!\mathcal{T}_{\theta}\left(\mathring{\nabla}_{\iota}{}^{y}\!\mathcal{T}_{\alpha}\right)+\frac{1}{3}\alpha\,{}^{z}\!\mathcal{T}^{\alpha}\,{}^{z}\!\mathcal{T}_{\theta}\left(\mathring{\nabla}_{\iota}{}^{y}\!\mathcal{T}_{\alpha}\right)-$$

$$\frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha}\,{}^{y}\!\mathcal{T}_{\theta}\right)\left(\mathring{\nabla}_{\iota}{}^{y}\!\mathcal{T}_{\alpha}\right)+\frac{2}{9}\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\,{}^{y}\!\mathcal{T}^{\alpha}\,{}^{z}\!\mathcal{T}^{\alpha'}\left(\mathring{\nabla}_{\iota}{}^{y}\!\mathcal{T}^{\beta}\right)+\frac{1}{3}\alpha\,{}^{y}\!\mathcal{T}_{\theta}\,{}^{z}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\iota}{}^{z}\!\mathcal{T}_{\alpha}\right)+$$

$$\frac{1}{3}\alpha\,{}^{y}\!\mathcal{T}^{\alpha}\,{}^{z}\!\mathcal{T}_{\theta}\left(\mathring{\nabla}_{\iota}{}^{z}\!\mathcal{T}_{\alpha}\right)+2\alpha\left(\mathring{\nabla}^{\alpha}\,{}^{z}\!\mathcal{T}_{\theta}\right)\left(\mathring{\nabla}_{\iota}{}^{z}\!\mathcal{T}_{\alpha}\right)-\frac{2}{3}\alpha\,{}^{y}\!\mathcal{T}_{\theta}\left(\mathring{\nabla}_{\iota}\mathring{\nabla}_{\alpha}{}^{y}\!\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\,{}^{y}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\iota}\mathring{\nabla}_{\alpha}{}^{y}\!\mathcal{T}_{\theta}\right)+$$

$$\frac{3}{2}\alpha\,{}^{z}\!\mathcal{T}_{\theta}\left(\mathring{\nabla}_{\iota}\mathring{\nabla}_{\alpha}{}^{z}\!\mathcal{T}^{\alpha}\right)+\frac{1}{2}\alpha\,{}^{z}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\iota}\mathring{\nabla}_{\alpha}{}^{z}\!\mathcal{T}_{\theta}\right)+\frac{1}{3}\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\,{}^{z}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\iota}\mathring{\nabla}^{\beta}\,{}^{y}\!\mathcal{T}^{\alpha'}\right)+$$

$$\frac{1}{3}\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\,{}^{y}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\iota}\mathring{\nabla}^{\beta}\,{}^{z}\!\mathcal{T}^{\alpha'}\right)-\frac{1}{6}\alpha\,\epsilon\,\mathring{g}_{\iota\alpha'\beta\beta'}\,{}^{z}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}R[\mathring{\nabla}]_{\alpha}{}^{\alpha'\beta\beta'}\right)-\frac{4}{27}\alpha\,{}^{y}\!\mathcal{T}^{\alpha}\,{}^{y}\!\mathcal{T}_{\iota}\left(\mathring{\nabla}_{\theta}{}^{y}\!\mathcal{T}_{\alpha}\right)+$$

$$\frac{1}{3}\alpha\,{}^{z}\!\mathcal{T}^{\alpha}\,{}^{z}\!\mathcal{T}_{\iota}\left(\mathring{\nabla}_{\theta}{}^{y}\!\mathcal{T}_{\alpha}\right)-\frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha}\,{}^{y}\!\mathcal{T}_{\iota}\right)\left(\mathring{\nabla}_{\theta}{}^{y}\!\mathcal{T}_{\alpha}\right)+\frac{8}{9}\alpha\left(\mathring{\nabla}_{\iota}{}^{y}\!\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\theta}{}^{y}\!\mathcal{T}_{\alpha}\right)+$$

$$\frac{2}{9}\alpha\,\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\,{}^{y}\!\mathcal{T}^{\alpha}\,{}^{z}\!\mathcal{T}^{\alpha'}\left(\mathring{\nabla}_{\theta}{}^{y}\!\mathcal{T}^{\beta}\right)+\frac{1}{3}\alpha\,{}^{y}\!\mathcal{T}_{\iota}\,{}^{z}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}{}^{z}\!\mathcal{T}_{\alpha}\right)+\frac{1}{3}\alpha\,{}^{y}\!\mathcal{T}^{\alpha}\,{}^{z}\!\mathcal{T}_{\iota}\left(\mathring{\nabla}_{\theta}{}^{z}\!\mathcal{T}_{\alpha}\right)+$$

$$2\alpha\left(\mathring{\nabla}^{\alpha}\,{}^{z}\!\mathcal{T}_{\iota}\right)\left(\mathring{\nabla}_{\theta}{}^{z}\!\mathcal{T}_{\alpha}\right)-2\alpha\left(\mathring{\nabla}_{\iota}{}^{z}\!\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\theta}{}^{z}\!\mathcal{T}_{\alpha}\right)-\frac{2}{3}\alpha\,{}^{y}\!\mathcal{T}_{\iota}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}{}^{y}\!\mathcal{T}^{\alpha}\right)-\frac{2}{9}\alpha\,{}^{y}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}{}^{y}\!\mathcal{T}_{\iota}\right)+$$

$$\frac{3}{2}\alpha\,{}^{z}\!\mathcal{T}_{\iota}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}{}^{z}\!\mathcal{T}^{\alpha}\right)+\frac{1}{2}\alpha\,{}^{z}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}{}^{z}\!\mathcal{T}_{\iota}\right)+\frac{1}{3}\alpha\,\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\,{}^{z}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\beta}\,{}^{y}\!\mathcal{T}^{\alpha'}\right)+$$

$$\frac{1}{3}\alpha\,\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\,{}^{y}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\beta}\,{}^{z}\!\mathcal{T}^{\alpha'}\right)+\frac{4}{9}\alpha\,{}^{y}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\iota}{}^{y}\!\mathcal{T}_{\alpha}\right)-\alpha\,{}^{z}\!\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\iota}{}^{z}\!\mathcal{T}_{\alpha}\right)==0$$

We also eliminate from Eq. (86) the tensor multiplier in favour of the spin and gravitational variables using our Eq. (80).

$$-\mathcal{M}_{\mathrm{Pl}}^{y}\,R[\mathring{\nabla}]_{\iota\theta}+\frac{1}{2}\mathcal{M}_{\mathrm{Pl}}^{y}\,\mathring{g}_{\iota\theta}\,R[\mathring{\nabla}]+\frac{2\sqrt{-\overset{\approx}{\mathring{g}}}\;\mathring{g}_{\iota\theta}\,\sigma^{\alpha'}{}_{\alpha\alpha'}\,{}^{y}\!\mathcal{T}^{\alpha}}{9\,\overset{\approx}{\mathring{g}}}-\frac{1}{9}\mathcal{M}_{\mathrm{Pl}}^{y}\,\mathring{g}_{\iota\theta}\,{}^{y}\!\mathcal{T}_{\alpha}\,{}^{y}\!\mathcal{T}^{\alpha}-$$

$$\frac{2}{9}\mathcal{M}_{\mathrm{Pl}}^{y}\,{}^{y}\!\mu\,\mathring{g}_{\iota\theta}\,{}^{y}\!\mathcal{T}_{\alpha}\,{}^{y}\!\mathcal{T}^{\alpha}+\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\sigma^{\alpha}{}_{\theta\alpha}\,{}^{y}\!\mathcal{T}_{\iota}}{18\,\overset{\approx}{\mathring{g}}}-\frac{16}{27}\alpha\,R[\mathring{\nabla}]_{\theta\alpha}\,{}^{y}\!\mathcal{T}^{\alpha}\,{}^{y}\!\mathcal{T}_{\iota}+$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\sigma^{\alpha}{}_{\iota\alpha}\;\overset{y}{\mathcal{T}}_{\theta}}{18\,\overset{\approx}{\mathring{g}}} - \frac{16}{27}\alpha\, R[\mathring{\nabla}]_{\iota\alpha}\;\overset{y}{\mathcal{T}}^{\alpha}\;\overset{y}{\mathcal{T}}_{\theta} - \frac{2}{9}\mathcal{M}_{\mathrm{Pl}}^{y}\;\overset{y}{\mathcal{T}}_{\iota}\;\overset{y}{\mathcal{T}}_{\theta} - \frac{4}{9}\mathcal{M}_{\mathrm{Pl}}^{y}\,\overset{y}{\mu}\;\overset{y}{\mathcal{T}}_{\iota}\;\overset{y}{\mathcal{T}}_{\theta} -$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\;\mathring{g}_{\iota\theta}\;\sigma^{\alpha'\beta\beta'}\;\overset{z}{\mathcal{T}}^{\alpha}}{4\,\overset{\approx}{\mathring{g}}} + \frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;\sigma^{\alpha'\beta}{}_{\iota}\;\overset{z}{\mathcal{T}}^{\alpha}}{12\,\overset{\approx}{\mathring{g}}} + \frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;\sigma^{\alpha'\beta}{}_{\theta}\;\overset{z}{\mathcal{T}}^{\alpha}}{12\,\overset{\approx}{\mathring{g}}} -$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;\sigma_{\iota}{}^{\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}}{24\,\overset{\approx}{\mathring{g}}} - \frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;\sigma_{\theta}{}^{\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}}{24\,\overset{\approx}{\mathring{g}}} + \frac{1}{4}\mathcal{M}_{\mathrm{Pl}}^{y}\;\mathring{g}_{\iota\theta}\;\overset{z}{\mathcal{T}}_{\alpha}\;\overset{z}{\mathcal{T}}^{\alpha} +$$

$$2\mathcal{M}_{\mathrm{Pl}}^{y}\,\overset{z}{\mu}\;\mathring{g}_{\iota\theta}\;\overset{z}{\mathcal{T}}_{\alpha}\;\overset{z}{\mathcal{T}}^{\alpha} + \frac{4}{3}\alpha\, R[\mathring{\nabla}]_{\theta\alpha}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}_{\iota} + \frac{4}{3}\alpha\, R[\mathring{\nabla}]_{\iota\alpha}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}_{\theta} +$$

$$\frac{1}{2}\mathcal{M}_{\mathrm{Pl}}^{y}\;\overset{z}{\mathcal{T}}_{\iota}\;\overset{z}{\mathcal{T}}_{\theta} + 4\mathcal{M}_{\mathrm{Pl}}^{y}\,\overset{z}{\mu}\;\overset{z}{\mathcal{T}}_{\iota}\;\overset{z}{\mathcal{T}}_{\theta} - \frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\theta_{\iota\theta}}{\overset{\approx}{\mathring{g}}} + \frac{16}{27}\alpha\;\overset{y}{\mathcal{T}}_{\theta}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\;\overset{y}{\mathcal{T}}_{\iota}\right) +$$

$$\frac{16}{27}\alpha\;\overset{y}{\mathcal{T}}_{\iota}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\;\overset{y}{\mathcal{T}}_{\theta}\right) - \frac{4}{3}\alpha\;\overset{z}{\mathcal{T}}_{\theta}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\;\overset{z}{\mathcal{T}}_{\iota}\right) - \frac{4}{3}\alpha\;\overset{z}{\mathcal{T}}_{\iota}\left(\mathring{\nabla}_{\alpha}\mathring{\nabla}^{\alpha}\;\overset{z}{\mathcal{T}}_{\theta}\right) + \frac{8}{9}\alpha\left(\mathring{\nabla}_{\alpha}\overset{y}{\mathcal{T}}_{\theta}\right)\left(\mathring{\nabla}^{\alpha}\;\overset{y}{\mathcal{T}}_{\iota}\right) -$$

$$2\alpha\left(\mathring{\nabla}_{\alpha}\overset{z}{\mathcal{T}}_{\theta}\right)\left(\mathring{\nabla}^{\alpha}\;\overset{z}{\mathcal{T}}_{\iota}\right) + \frac{8}{27}\alpha\;\mathring{g}_{\iota\theta}\;\overset{y}{\mathcal{T}}^{\alpha}\left(\mathring{\nabla}_{\alpha''}\mathring{\nabla}_{\alpha}\overset{y}{\mathcal{T}}^{\alpha'}\right) - \frac{2}{3}\alpha\;\mathring{g}_{\iota\theta}\;\overset{z}{\mathcal{T}}^{\alpha}\left(\mathring{\nabla}_{\alpha''}\mathring{\nabla}_{\alpha}\overset{z}{\mathcal{T}}^{\alpha'}\right) -$$

$$\frac{8}{27}\alpha\;\mathring{g}_{\iota\theta}\;\overset{y}{\mathcal{T}}^{\alpha}\left(\mathring{\nabla}_{\alpha''}\mathring{\nabla}^{\alpha''}\;\overset{y}{\mathcal{T}}_{\alpha}\right) + \frac{2}{3}\alpha\;\mathring{g}_{\iota\theta}\;\overset{z}{\mathcal{T}}^{\alpha}\left(\mathring{\nabla}_{\alpha''}\mathring{\nabla}^{\alpha''}\;\overset{z}{\mathcal{T}}_{\alpha}\right) + \frac{4}{9}\alpha\;\mathring{g}_{\iota\theta}\left(\mathring{\nabla}_{\alpha}\overset{y}{\mathcal{T}}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha''}\;\overset{y}{\mathcal{T}}^{\alpha}\right) -$$

$$\frac{4}{9}\alpha\;\mathring{g}_{\iota\theta}\left(\mathring{\nabla}_{\alpha''}\overset{y}{\mathcal{T}}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha''}\;\overset{y}{\mathcal{T}}^{\alpha}\right) - \alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;R[\mathring{\nabla}]_{\iota}{}^{\beta}\left(\mathring{\nabla}^{\alpha''}\;\overset{z}{\mathcal{T}}^{\alpha}\right) - \alpha\,\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}\left(\mathring{\nabla}^{\alpha''}\;\overset{z}{\mathcal{T}}^{\alpha}\right) +$$

$$\alpha\,\epsilon\,\mathring{g}_{\theta\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\iota}{}^{\beta}{}_{\alpha}{}^{\beta'}\left(\mathring{\nabla}^{\alpha''}\;\overset{z}{\mathcal{T}}^{\alpha}\right) - \alpha\,\epsilon\,\mathring{g}_{\theta\alpha\beta\beta'}\;R[\mathring{\nabla}]_{\iota}{}^{\beta}{}_{\alpha'}{}^{\beta'}\left(\mathring{\nabla}^{\alpha''}\;\overset{z}{\mathcal{T}}^{\alpha}\right) +$$

$$\alpha\,\epsilon\,\mathring{g}_{\iota\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha}{}^{\beta'}\left(\mathring{\nabla}^{\alpha''}\;\overset{z}{\mathcal{T}}^{\alpha}\right) - \alpha\,\epsilon\,\mathring{g}_{\iota\alpha\beta\beta'}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha'}{}^{\beta'}\left(\mathring{\nabla}^{\alpha''}\;\overset{z}{\mathcal{T}}^{\alpha}\right) - \alpha\;\mathring{g}_{\iota\theta}\left(\mathring{\nabla}_{\alpha}\overset{z}{\mathcal{T}}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha''}\;\overset{z}{\mathcal{T}}^{\alpha}\right) +$$

$$\alpha\;\mathring{g}_{\iota\theta}\left(\mathring{\nabla}_{\alpha''}\overset{z}{\mathcal{T}}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha''}\;\overset{z}{\mathcal{T}}^{\alpha}\right) - \frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha}\;\overset{y}{\mathcal{T}}_{\theta}\right)\left(\mathring{\nabla}_{\iota}\overset{y}{\mathcal{T}}_{\alpha}\right) + 2\alpha\left(\mathring{\nabla}^{\alpha}\;\overset{z}{\mathcal{T}}_{\theta}\right)\left(\mathring{\nabla}_{\iota}\overset{z}{\mathcal{T}}_{\alpha}\right) -$$

$$\frac{16}{27}\alpha\;\overset{y}{\mathcal{T}}_{\theta}\left(\mathring{\nabla}_{\iota}\mathring{\nabla}_{\alpha}\overset{y}{\mathcal{T}}^{\alpha}\right) + \frac{4}{3}\alpha\;\overset{z}{\mathcal{T}}_{\theta}\left(\mathring{\nabla}_{\iota}\mathring{\nabla}_{\alpha}\overset{z}{\mathcal{T}}^{\alpha}\right) - \frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha}\;\overset{y}{\mathcal{T}}_{\iota}\right)\left(\mathring{\nabla}_{\theta}\overset{y}{\mathcal{T}}_{\alpha}\right) + \frac{8}{9}\alpha\left(\mathring{\nabla}_{\iota}\overset{y}{\mathcal{T}}^{\alpha}\right)\left(\mathring{\nabla}_{\theta}\overset{y}{\mathcal{T}}_{\alpha}\right) +$$

$$2\alpha\left(\mathring{\nabla}^{\alpha}\;\overset{z}{\mathcal{T}}_{\iota}\right)\left(\mathring{\nabla}_{\theta}\overset{z}{\mathcal{T}}_{\alpha}\right) - 2\alpha\left(\mathring{\nabla}_{\iota}\overset{z}{\mathcal{T}}^{\alpha}\right)\left(\mathring{\nabla}_{\theta}\overset{z}{\mathcal{T}}_{\alpha}\right) - \frac{16}{27}\alpha\;\overset{y}{\mathcal{T}}_{\iota}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\overset{y}{\mathcal{T}}^{\alpha}\right) + \frac{4}{3}\alpha\;\overset{z}{\mathcal{T}}_{\iota}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\overset{z}{\mathcal{T}}^{\alpha}\right) == 0$$

We use dynamical torsion equations Eq. (83) and Eq. (84) to eliminate in Eq. (87) d'Alembertians of torsion in favour of the spin tensor.

$$-\mathcal{M}_{\mathrm{Pl}}^{y}\;R[\mathring{\nabla}]_{\iota\theta} + \frac{1}{2}\mathcal{M}_{\mathrm{Pl}}^{y}\;\mathring{g}_{\iota\theta}\;R[\mathring{\nabla}] + \frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\mathring{g}_{\iota\theta}\;\sigma^{\alpha'}{}_{\alpha\alpha'}\;\overset{y}{\mathcal{T}}^{\alpha}}{3\,\overset{\approx}{\mathring{g}}} - \frac{8\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\mathring{g}_{\iota\theta}\;R[\mathring{\nabla}]_{\alpha}{}^{\alpha'}\;\sigma^{\beta}{}_{\alpha'\beta}\;\overset{y}{\mathcal{T}}^{\alpha}}{27\left(-2\mathcal{M}_{\mathrm{Pl}}^{y} - 4\mathcal{M}_{\mathrm{Pl}}^{y}\,\overset{y}{\mu}\right)\overset{\approx}{\mathring{g}}} -$$

$$\frac{1}{3}\mathcal{M}_{\text{Pl}}^{y}\,\mathring{g}_{\iota\theta}\,\overset{y}{:}\mathcal{T}_{\alpha}\,\overset{y}{:}\mathcal{T}^{\alpha}-\frac{2}{3}\mathcal{M}_{\text{Pl}}^{y}\,\overset{y}{:}\mu\,\mathring{g}_{\iota\theta}\,\overset{y}{:}\mathcal{T}_{\alpha}\,\overset{y}{:}\mathcal{T}^{\alpha}-\frac{8}{27}\alpha\,\mathring{g}_{\iota\theta}\,R[\mathring{\nabla}]_{\alpha\alpha'}\,\overset{y}{:}\mathcal{T}^{\alpha}\,\overset{y}{:}\mathcal{T}^{\alpha'}-$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\sigma^{\alpha}{}_{\theta\alpha}\,\overset{y}{:}\mathcal{T}_{\iota}}{6\,\overset{\approx}{\mathring{g}}}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\sigma^{\alpha}{}_{\iota\alpha}\,\overset{y}{:}\mathcal{T}_{\theta}}{6\,\overset{\approx}{\mathring{g}}}+\frac{2}{3}\mathcal{M}_{\text{Pl}}^{y}\,\overset{y}{:}\mathcal{T}_{\iota}\,\overset{y}{:}\mathcal{T}_{\theta}+\frac{4}{3}\mathcal{M}_{\text{Pl}}^{y}\,\overset{y}{:}\mu\,\overset{y}{:}\mathcal{T}_{\iota}\,\overset{y}{:}\mathcal{T}_{\theta}-$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\alpha\alpha'\beta\beta'}\,\mathring{g}_{\iota\theta}\,\sigma^{\alpha'\beta\beta'}\,\overset{z}{:}\mathcal{T}^{\alpha}}{3\,\overset{\approx}{\mathring{g}}}+\frac{2\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\beta\beta'\chi\chi'}\,\mathring{g}_{\iota\theta}\,R[\mathring{\nabla}]_{\alpha}{}^{\chi}{}_{\alpha'}{}^{\chi'}\,\sigma^{\alpha'\beta\beta'}\,\overset{z}{:}\mathcal{T}^{\alpha}}{3\left(-6\mathcal{M}_{\text{Pl}}^{y}-48\mathcal{M}_{\text{Pl}}^{y}\,\overset{z}{:}\mu\right)\overset{\approx}{\mathring{g}}}-$$

$$\frac{4\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\alpha'\beta'\chi\chi'}\,\mathring{g}_{\iota\theta}\,R[\mathring{\nabla}]_{\alpha}{}^{\chi}{}_{\beta}{}^{\chi'}\,\sigma^{\alpha'\beta\beta'}\,\overset{z}{:}\mathcal{T}^{\alpha}}{3\left(-6\mathcal{M}_{\text{Pl}}^{y}-48\mathcal{M}_{\text{Pl}}^{y}\,\overset{z}{:}\mu\right)\overset{\approx}{\mathring{g}}}+\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\theta\alpha\alpha'\beta}\,\sigma^{\alpha'}{}_{\iota}{}^{\beta}\,\overset{z}{:}\mathcal{T}^{\alpha}}{12\,\overset{\approx}{\mathring{g}}}+$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\iota\alpha\alpha'\beta}\,\sigma^{\alpha'}{}_{\theta}{}^{\beta}\,\overset{z}{:}\mathcal{T}^{\alpha}}{12\,\overset{\approx}{\mathring{g}}}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\theta\alpha\alpha'\beta}\,\sigma_{\iota}{}^{\alpha'\beta}\,\overset{z}{:}\mathcal{T}^{\alpha}}{24\,\overset{\approx}{\mathring{g}}}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\iota\alpha\alpha'\beta}\,\sigma_{\theta}{}^{\alpha'\beta}\,\overset{z}{:}\mathcal{T}^{\alpha}}{24\,\overset{\approx}{\mathring{g}}}+$$

$$\frac{3}{4}\mathcal{M}_{\text{Pl}}^{y}\,\mathring{g}_{\iota\theta}\,\overset{z}{:}\mathcal{T}_{\alpha}\,\overset{z}{:}\mathcal{T}^{\alpha}+6\mathcal{M}_{\text{Pl}}^{y}\,\overset{z}{:}\mu\,\mathring{g}_{\iota\theta}\,\overset{z}{:}\mathcal{T}_{\alpha}\,\overset{z}{:}\mathcal{T}^{\alpha}+\frac{2}{3}\alpha\,\mathring{g}_{\iota\theta}\,R[\mathring{\nabla}]_{\alpha\alpha'}\,\overset{z}{:}\mathcal{T}^{\alpha}\,\overset{z}{:}\mathcal{T}^{\alpha'}+$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\theta\alpha\alpha'\beta}\,\sigma^{\alpha\alpha'\beta}\,\overset{z}{:}\mathcal{T}_{\iota}}{6\,\overset{\approx}{\mathring{g}}}+\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\iota\alpha\alpha'\beta}\,\sigma^{\alpha\alpha'\beta}\,\overset{z}{:}\mathcal{T}_{\theta}}{6\,\overset{\approx}{\mathring{g}}}-\frac{3}{2}\mathcal{M}_{\text{Pl}}^{y}\,\overset{z}{:}\mathcal{T}_{\iota}\,\overset{z}{:}\mathcal{T}_{\theta}-12\mathcal{M}_{\text{Pl}}^{y}\,\overset{z}{:}\mu\,\overset{z}{:}\mathcal{T}_{\iota}\,\overset{z}{:}\mathcal{T}_{\theta}-$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\theta_{\iota\theta}}{\overset{\approx}{\mathring{g}}}+\frac{8}{9}\alpha\left(\mathring{\nabla}_{\alpha}\overset{y}{:}\mathcal{T}_{\theta}\right)\left(\mathring{\nabla}^{\alpha}\overset{y}{:}\mathcal{T}_{\iota}\right)-2\alpha\left(\mathring{\nabla}_{\alpha}\overset{z}{:}\mathcal{T}_{\theta}\right)\left(\mathring{\nabla}^{\alpha}\overset{z}{:}\mathcal{T}_{\iota}\right)+\frac{8}{27}\alpha\,\mathring{g}_{\iota\theta}\,\overset{y}{:}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha\frac{1}{10}}\mathring{\nabla}_{\alpha}\overset{y}{:}\mathcal{T}^{\alpha'}\right)-$$

$$\frac{2}{3}\alpha\,\mathring{g}_{\iota\theta}\,\overset{z}{:}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha\frac{1}{10}}\mathring{\nabla}_{\alpha}\overset{z}{:}\mathcal{T}^{\alpha'}\right)+\frac{4}{9}\alpha\,\mathring{g}_{\iota\theta}\left(\mathring{\nabla}_{\alpha}\overset{y}{:}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{y}{:}\mathcal{T}^{\alpha}\right)-\frac{4}{9}\alpha\,\mathring{g}_{\iota\theta}\left(\mathring{\nabla}_{\alpha\frac{1}{10}}\overset{y}{:}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{y}{:}\mathcal{T}^{\alpha}\right)-$$

$$\alpha\,\epsilon\mathring{g}_{\theta\alpha\alpha'\beta}\,R[\mathring{\nabla}]_{\iota}{}^{\beta}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{:}\mathcal{T}^{\alpha}\right)-\alpha\,\epsilon\mathring{g}_{\iota\alpha\alpha'\beta}\,R[\mathring{\nabla}]_{\theta}{}^{\beta}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{:}\mathcal{T}^{\alpha}\right)+\alpha\,\epsilon\mathring{g}_{\theta\alpha'\beta\beta'}\,R[\mathring{\nabla}]_{\iota}{}^{\beta}{}_{\alpha}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{:}\mathcal{T}^{\alpha}\right)-$$

$$\alpha\,\epsilon\mathring{g}_{\theta\alpha\beta\beta'}\,R[\mathring{\nabla}]_{\iota}{}^{\beta}{}_{\alpha'}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{:}\mathcal{T}^{\alpha}\right)+\alpha\,\epsilon\mathring{g}_{\iota\alpha'\beta\beta'}\,R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{:}\mathcal{T}^{\alpha}\right)-$$

$$\alpha\,\epsilon\mathring{g}_{\iota\alpha\beta\beta'}\,R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha'}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{:}\mathcal{T}^{\alpha}\right)-\alpha\,\mathring{g}_{\iota\theta}\left(\mathring{\nabla}_{\alpha}\overset{z}{:}\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{:}\mathcal{T}^{\alpha}\right)+\alpha\,\mathring{g}_{\iota\theta}\left(\mathring{\nabla}_{\alpha\frac{1}{10}}\overset{z}{:}\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{:}\mathcal{T}^{\alpha}\right)-$$

$$\frac{8\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\mathring{g}_{\iota\theta}\;\overset{y}{\mathcal{T}}{}^{\alpha}\left(\mathring{\nabla}_{\beta}\mathring{\nabla}_{\alpha}\sigma^{\alpha'}{}_{\alpha'}{}^{\beta}\right)}{27\left(-2\mathcal{M}_{\rm Pl}^{y}-4\mathcal{M}_{\rm Pl}^{y}\overset{y}{\mu}\right)\overset{\approx}{\mathring{g}}}+\frac{2\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\alpha'\beta\beta'\chi}\;\mathring{g}_{\iota\theta}\;\overset{z}{\mathcal{T}}{}^{\alpha}\left(\mathring{\nabla}^{\chi}\mathring{\nabla}_{\alpha}\sigma^{\alpha'\beta\beta'}\right)}{3\left(-6\mathcal{M}_{\rm Pl}^{y}-48\mathcal{M}_{\rm Pl}^{y}\overset{z}{\mu}\right)\overset{\approx}{\mathring{g}}}-$$

$$\frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha}\overset{y}{\mathcal{T}}{}_{\theta}\right)\left(\mathring{\nabla}_{\iota}\overset{y}{\mathcal{T}}{}_{\alpha}\right)+2\alpha\left(\mathring{\nabla}^{\alpha}\overset{z}{\mathcal{T}}{}_{\theta}\right)\left(\mathring{\nabla}_{\iota}\overset{z}{\mathcal{T}}{}_{\alpha}\right)-\frac{16}{27}\alpha\,\overset{y}{\mathcal{T}}{}_{\theta}\left(\mathring{\nabla}_{\iota}\mathring{\nabla}_{\alpha}\overset{y}{\mathcal{T}}{}^{\alpha}\right)+\frac{4}{3}\alpha\,\overset{z}{\mathcal{T}}{}_{\theta}\left(\mathring{\nabla}_{\iota}\mathring{\nabla}_{\alpha}\overset{z}{\mathcal{T}}{}^{\alpha}\right)+$$

$$\frac{16\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\overset{y}{\mathcal{T}}{}_{\theta}\left(\mathring{\nabla}_{\iota}\mathring{\nabla}_{\alpha\frac{1}{10}}\sigma^{\alpha}{}_{\alpha}{}^{\alpha'}\right)}{27\left(-2\mathcal{M}_{\rm Pl}^{y}-4\mathcal{M}_{\rm Pl}^{y}\overset{y}{\mu}\right)\overset{\approx}{\mathring{g}}}-\frac{4\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\alpha\alpha'\beta\beta'}\;\overset{z}{\mathcal{T}}{}_{\theta}\left(\mathring{\nabla}_{\iota}\mathring{\nabla}^{\beta\frac{1}{10}}\sigma^{\alpha\alpha'\beta}\right)}{3\left(-6\mathcal{M}_{\rm Pl}^{y}-48\mathcal{M}_{\rm Pl}^{y}\overset{z}{\mu}\right)\overset{\approx}{\mathring{g}}}-\frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha}\overset{y}{\mathcal{T}}{}_{\iota}\right)\left(\mathring{\nabla}_{\theta}\overset{y}{\mathcal{T}}{}_{\alpha}\right)+$$

$$\frac{8}{9}\alpha\left(\mathring{\nabla}_{\iota}\overset{y}{\mathcal{T}}{}^{\alpha}\right)\left(\mathring{\nabla}_{\theta}\overset{y}{\mathcal{T}}{}_{\alpha}\right)+2\alpha\left(\mathring{\nabla}^{\alpha}\overset{z}{\mathcal{T}}{}_{\iota}\right)\left(\mathring{\nabla}_{\theta}\overset{z}{\mathcal{T}}{}_{\alpha}\right)-2\alpha\left(\mathring{\nabla}_{\iota}\overset{z}{\mathcal{T}}{}^{\alpha}\right)\left(\mathring{\nabla}_{\theta}\overset{z}{\mathcal{T}}{}_{\alpha}\right)-\frac{16}{27}\alpha\,\overset{y}{\mathcal{T}}{}_{\iota}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\overset{y}{\mathcal{T}}{}^{\alpha}\right)+$$

$$\frac{4}{3}\alpha\,\overset{z}{\mathcal{T}}{}_{\iota}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha}\overset{z}{\mathcal{T}}{}^{\alpha}\right)+\frac{16\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\overset{y}{\mathcal{T}}{}_{\iota}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}_{\alpha\frac{1}{10}}\sigma^{\alpha}{}_{\alpha}{}^{\alpha'}\right)}{27\left(-2\mathcal{M}_{\rm Pl}^{y}-4\mathcal{M}_{\rm Pl}^{y}\overset{y}{\mu}\right)\overset{\approx}{\mathring{g}}}-\frac{4\alpha\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\alpha\alpha'\beta\beta'}\;\overset{z}{\mathcal{T}}{}_{\iota}\left(\mathring{\nabla}_{\theta}\mathring{\nabla}^{\beta\frac{1}{10}}\sigma^{\alpha\alpha'\beta}\right)}{3\left(-6\mathcal{M}_{\rm Pl}^{y}-48\mathcal{M}_{\rm Pl}^{y}\overset{z}{\mu}\right)\overset{\approx}{\mathring{g}}}==0$$

We use dynamical torsion equations Eq. (81) and Eq. (82) to eliminate in Eq. (88) divergences of torsion in favour of the spin tensor.

$$-\mathcal{M}_{\rm Pl}^{y}\,R[\mathring{\nabla}]_{\iota\theta}+\frac{1}{2}\mathcal{M}_{\rm Pl}^{y}\,\mathring{g}_{\iota\theta}\,R[\mathring{\nabla}]+\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\mathring{g}_{\iota\theta}\;\sigma^{\alpha'}{}_{\alpha\alpha'}\;\overset{y}{\mathcal{T}}{}^{\alpha}}{3\,\overset{\approx}{\mathring{g}}}-\frac{1}{3}\mathcal{M}_{\rm Pl}^{y}\,\mathring{g}_{\iota\theta}\;\overset{y}{\mathcal{T}}{}_{\alpha}\;\overset{y}{\mathcal{T}}{}^{\alpha}-$$

$$\frac{2}{3}\mathcal{M}_{\rm Pl}^{y}\overset{y}{\mu}\;\mathring{g}_{\iota\theta}\;\overset{y}{\mathcal{T}}{}_{\alpha}\;\overset{y}{\mathcal{T}}{}^{\alpha}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\sigma^{\alpha}{}_{\theta\alpha}\;\overset{y}{\mathcal{T}}{}_{\iota}}{6\,\overset{\approx}{\mathring{g}}}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\sigma^{\alpha}{}_{\iota\alpha}\;\overset{y}{\mathcal{T}}{}_{\theta}}{6\,\overset{\approx}{\mathring{g}}}+\frac{2}{3}\mathcal{M}_{\rm Pl}^{y}\;\overset{y}{\mathcal{T}}{}_{\iota}\;\overset{y}{\mathcal{T}}{}_{\theta}+$$

$$\frac{4}{3}\mathcal{M}_{\rm Pl}^{y}\overset{y}{\mu}\;\overset{y}{\mathcal{T}}{}_{\iota}\;\overset{y}{\mathcal{T}}{}_{\theta}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\alpha\alpha'\beta\beta'}\;\mathring{g}_{\iota\theta}\;\sigma^{\alpha'\beta\beta'}\;\overset{z}{\mathcal{T}}{}^{\alpha}}{3\,\overset{\approx}{\mathring{g}}}+\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\theta\alpha\alpha'\beta}\;\sigma^{\alpha'}{}_{\iota}{}^{\beta}\;\overset{z}{\mathcal{T}}{}^{\alpha}}{12\,\overset{\approx}{\mathring{g}}}+$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\iota\alpha\alpha'\beta}\;\sigma^{\alpha'}{}_{\theta}{}^{\beta}\;\overset{z}{\mathcal{T}}{}^{\alpha}}{12\,\overset{\approx}{\mathring{g}}}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\theta\alpha\alpha'\beta}\;\sigma_{\iota}{}^{\alpha'\beta}\;\overset{z}{\mathcal{T}}{}^{\alpha}}{24\,\overset{\approx}{\mathring{g}}}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\mathring{g}_{\iota\alpha\alpha'\beta}\;\sigma_{\theta}{}^{\alpha'\beta}\;\overset{z}{\mathcal{T}}{}^{\alpha}}{24\,\overset{\approx}{\mathring{g}}}+$$

$$\frac{3}{4}\mathcal{M}_{\text{Pl}}{}^{y}\,\mathring{g}_{\iota\theta}\;{}^{z}\!\mathcal{T}_{\alpha}\;{}^{z}\!\mathcal{T}^{\alpha}+6\,\mathcal{M}_{\text{Pl}}{}^{y}\,{}^{z}\!\mu\;\mathring{g}_{\iota\theta}\;{}^{z}\!\mathcal{T}_{\alpha}\;{}^{z}\!\mathcal{T}^{\alpha}+\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;\sigma^{\alpha\alpha'\beta}\;{}^{z}\!\mathcal{T}_{\iota}}{6\,\overset{\approx}{\mathring{g}}}+$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;\sigma^{\alpha\alpha'\beta}\;{}^{z}\!\mathcal{T}_{\theta}}{6\,\overset{\approx}{\mathring{g}}}-\frac{3}{2}\mathcal{M}_{\text{Pl}}{}^{y}\;{}^{z}\!\mathcal{T}_{\iota}\;{}^{z}\!\mathcal{T}_{\theta}-12\,\mathcal{M}_{\text{Pl}}{}^{y}\,{}^{z}\!\mu\;{}^{z}\!\mathcal{T}_{\iota}\;{}^{z}\!\mathcal{T}_{\theta}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\theta_{\iota\theta}}{\overset{\approx}{\mathring{g}}}+$$

$$\frac{8}{9}\alpha\left(\mathring{\nabla}_{\alpha}{}^{y}\!\mathcal{T}_{\theta}\right)\left(\mathring{\nabla}^{\alpha}\,{}^{y}\!\mathcal{T}_{\iota}\right)-2\,\alpha\left(\mathring{\nabla}_{\alpha}{}^{z}\!\mathcal{T}_{\theta}\right)\left(\mathring{\nabla}^{\alpha}\,{}^{z}\!\mathcal{T}_{\iota}\right)+\frac{4}{9}\alpha\;\mathring{g}_{\iota\theta}\left(\mathring{\nabla}_{\alpha}{}^{y}\!\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha⅒}\,{}^{y}\!\mathcal{T}^{\alpha}\right)-$$

$$\frac{4}{9}\alpha\;\mathring{g}_{\iota\theta}\left(\mathring{\nabla}_{\alpha⅒}{}^{y}\!\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha⅒}\,{}^{y}\!\mathcal{T}^{\alpha}\right)-\alpha\;\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;R[\mathring{\nabla}]_{\iota}{}^{\beta}\left(\mathring{\nabla}^{\alpha⅒}\,{}^{z}\!\mathcal{T}^{\alpha}\right)-\alpha\;\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}\left(\mathring{\nabla}^{\alpha⅒}\,{}^{z}\!\mathcal{T}^{\alpha}\right)+$$

$$\alpha\;\epsilon\,\mathring{g}_{\theta\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\iota}{}^{\beta}{}_{\alpha}{}^{\beta'}\left(\mathring{\nabla}^{\alpha⅒}\,{}^{z}\!\mathcal{T}^{\alpha}\right)-\alpha\;\epsilon\,\mathring{g}_{\theta\alpha\beta\beta'}\;R[\mathring{\nabla}]_{\iota}{}^{\beta}{}_{\alpha'}{}^{\beta'}\left(\mathring{\nabla}^{\alpha⅒}\,{}^{z}\!\mathcal{T}^{\alpha}\right)+$$

$$\alpha\;\epsilon\,\mathring{g}_{\iota\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha}{}^{\beta'}\left(\mathring{\nabla}^{\alpha⅒}\,{}^{z}\!\mathcal{T}^{\alpha}\right)-\alpha\;\epsilon\,\mathring{g}_{\iota\alpha\beta\beta'}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha'}{}^{\beta'}\left(\mathring{\nabla}^{\alpha⅒}\,{}^{z}\!\mathcal{T}^{\alpha}\right)-\alpha\;\mathring{g}_{\iota\theta}\left(\mathring{\nabla}_{\alpha}{}^{z}\!\mathcal{T}_{\alpha'}\right)\left(\mathring{\nabla}^{\alpha⅒}\,{}^{z}\!\mathcal{T}^{\alpha}\right)+$$

$$\alpha\;\mathring{g}_{\iota\theta}\left(\mathring{\nabla}_{\alpha⅒}{}^{z}\!\mathcal{T}_{\alpha}\right)\left(\mathring{\nabla}^{\alpha⅒}\,{}^{z}\!\mathcal{T}^{\alpha}\right)-\frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha}\,{}^{y}\!\mathcal{T}_{\theta}\right)\left(\mathring{\nabla}_{\iota}{}^{y}\!\mathcal{T}_{\alpha}\right)+2\,\alpha\left(\mathring{\nabla}^{\alpha}\,{}^{z}\!\mathcal{T}_{\theta}\right)\left(\mathring{\nabla}_{\iota}{}^{z}\!\mathcal{T}_{\alpha}\right)-$$

$$\frac{8}{9}\alpha\left(\mathring{\nabla}^{\alpha}\,{}^{y}\!\mathcal{T}_{\iota}\right)\left(\mathring{\nabla}_{\theta}{}^{y}\!\mathcal{T}_{\alpha}\right)+\frac{8}{9}\alpha\left(\mathring{\nabla}_{\iota}{}^{y}\!\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\theta}{}^{y}\!\mathcal{T}_{\alpha}\right)+2\,\alpha\left(\mathring{\nabla}^{\alpha}\,{}^{z}\!\mathcal{T}_{\iota}\right)\left(\mathring{\nabla}_{\theta}{}^{z}\!\mathcal{T}_{\alpha}\right)-2\,\alpha\left(\mathring{\nabla}_{\iota}{}^{z}\!\mathcal{T}^{\alpha}\right)\left(\mathring{\nabla}_{\theta}{}^{z}\!\mathcal{T}_{\alpha}\right)==0$$

Convert remaining torsion derivatives to Maxwell tensors in Eq. (89).

$$-\frac{2}{9}\alpha\;\mathring{g}_{\iota\theta}\;{}^{y}\!\mathcal{F}_{\alpha\alpha'}\;{}^{y}\!\mathcal{F}^{\alpha\alpha'}+\frac{8}{9}\alpha\;{}^{y}\!\mathcal{F}_{\iota}{}^{\alpha}\;{}^{y}\!\mathcal{F}_{\theta\alpha}+\frac{1}{2}\alpha\;\mathring{g}_{\iota\theta}\;{}^{z}\!\mathcal{F}_{\alpha\alpha'}\;{}^{z}\!\mathcal{F}^{\alpha\alpha'}-2\,\alpha\;{}^{z}\!\mathcal{F}_{\iota}{}^{\alpha}\;{}^{z}\!\mathcal{F}_{\theta\alpha}+$$

$$\frac{1}{2}\alpha\;\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;{}^{z}\!\mathcal{F}^{\alpha\alpha'}\;R[\mathring{\nabla}]_{\iota}{}^{\beta}-\mathcal{M}_{\text{Pl}}{}^{y}\;R[\mathring{\nabla}]_{\iota\theta}+\frac{1}{2}\alpha\;\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;{}^{z}\!\mathcal{F}^{\alpha\alpha'}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}+$$

$$\frac{1}{2}\mathcal{M}_{\text{Pl}}{}^{y}\;\mathring{g}_{\iota\theta}\;R[\mathring{\nabla}]-\alpha\;\epsilon\,\mathring{g}_{\theta\alpha'\beta\beta'}\;{}^{z}\!\mathcal{F}^{\alpha\alpha'}\;R[\mathring{\nabla}]_{\iota}{}^{\beta}{}_{\alpha}{}^{\beta'}-\alpha\;\epsilon\,\mathring{g}_{\iota\alpha'\beta\beta'}\;{}^{z}\!\mathcal{F}^{\alpha\alpha'}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha}{}^{\beta'}+$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\mathring{g}_{\iota\theta}\;\sigma^{\alpha'}{}_{\alpha\alpha'}\;{}^{y}\!\mathcal{T}^{\alpha}}{3\,\overset{\approx}{\mathring{g}}}-\frac{1}{3}\mathcal{M}_{\text{Pl}}{}^{y}\;\mathring{g}_{\iota\theta}\;{}^{y}\!\mathcal{T}_{\alpha}\;{}^{y}\!\mathcal{T}^{\alpha}-\frac{2}{3}\mathcal{M}_{\text{Pl}}{}^{y}\,{}^{y}\!\mu\;\mathring{g}_{\iota\theta}\;{}^{y}\!\mathcal{T}_{\alpha}\;{}^{y}\!\mathcal{T}^{\alpha}-$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\sigma^{\alpha}{}_{\theta\alpha}\;{}^{y}\!\mathcal{T}_{\iota}}{6\,\overset{\approx}{\mathring{g}}}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\sigma^{\alpha}{}_{\iota\alpha}\;{}^{y}\!\mathcal{T}_{\theta}}{6\,\overset{\approx}{\mathring{g}}}+\frac{2}{3}\mathcal{M}_{\text{Pl}}{}^{y}\;{}^{y}\!\mathcal{T}_{\iota}\;{}^{y}\!\mathcal{T}_{\theta}+\frac{4}{3}\mathcal{M}_{\text{Pl}}{}^{y}\,{}^{y}\!\mu\;{}^{y}\!\mathcal{T}_{\iota}\;{}^{y}\!\mathcal{T}_{\theta}-$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\;\mathring{g}_{\iota\theta}\;\sigma^{\alpha'\beta\beta'}\;{}^{z}_{:}\mathcal{T}^{\alpha}}{3\,\overset{\approx}{\mathring{g}}}+\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;\sigma^{\alpha'\,\beta}{}_{\iota}\;{}^{z}_{:}\mathcal{T}^{\alpha}}{12\,\overset{\approx}{\mathring{g}}}+$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;\sigma^{\alpha'\,\beta}{}_{\theta}\;{}^{z}_{:}\mathcal{T}^{\alpha}}{12\,\overset{\approx}{\mathring{g}}}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;\sigma_{\iota}{}^{\alpha'\beta}\;{}^{z}_{:}\mathcal{T}^{\alpha}}{24\,\overset{\approx}{\mathring{g}}}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;\sigma_{\theta}{}^{\alpha'\beta}\;{}^{z}_{:}\mathcal{T}^{\alpha}}{24\,\overset{\approx}{\mathring{g}}}+$$

$$\frac{3}{4}\mathcal{M}_{\mathrm{Pl}}^{\,y}\;\mathring{g}_{\iota\theta}\;{}^{z}_{:}\mathcal{T}_{\alpha}\;{}^{z}_{:}\mathcal{T}^{\alpha}+6\,\mathcal{M}_{\mathrm{Pl}}^{\,y}\;{}^{z}_{:}\mu\;\mathring{g}_{\iota\theta}\;{}^{z}_{:}\mathcal{T}_{\alpha}\;{}^{z}_{:}\mathcal{T}^{\alpha}+\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;\sigma^{\alpha\alpha'\beta}\;{}^{z}_{:}\mathcal{T}_{\iota}}{6\,\overset{\approx}{\mathring{g}}}+$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;\sigma^{\alpha\alpha'\beta}\;{}^{z}_{:}\mathcal{T}_{\theta}}{6\,\overset{\approx}{\mathring{g}}}-\frac{3}{2}\mathcal{M}_{\mathrm{Pl}}^{\,y}\;{}^{z}_{:}\mathcal{T}_{\iota}\;{}^{z}_{:}\mathcal{T}_{\theta}-12\,\mathcal{M}_{\mathrm{Pl}}^{\,y}\;{}^{z}_{:}\mu\;{}^{z}_{:}\mathcal{T}_{\iota}\;{}^{z}_{:}\mathcal{T}_{\theta}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\theta_{\iota\theta}}{\overset{\approx}{\mathring{g}}}==0$$

Some curvature cross terms persist in Eq. (90), so we notice that the third Maxwell tensor can be expressed in terms of its dual (and back again) using Eq. (50), and the intermediate Levi-Civita kills off those curvature couplings due to the Bianchi identity obeyed by the Maxwell tensors.

$$-\frac{2}{9}\alpha\;\mathring{g}_{\iota\theta}\;{}^{y}_{:}\mathcal{F}_{\alpha\alpha'}\;{}^{y}_{:}\mathcal{F}^{\alpha\alpha'}+\frac{8}{9}\alpha\;{}^{y}_{:}\mathcal{F}_{\iota}{}^{\alpha}\;{}^{y}_{:}\mathcal{F}_{\theta\alpha}+\frac{1}{2}\alpha\;\mathring{g}_{\iota\theta}\;{}^{z}_{:}\mathcal{F}_{\alpha\alpha'}\;{}^{z}_{:}\mathcal{F}^{\alpha\alpha'}-2\,\alpha\;{}^{z}_{:}\mathcal{F}_{\iota}{}^{\alpha}\;{}^{z}_{:}\mathcal{F}_{\theta\alpha}-$$

$$\mathcal{M}_{\mathrm{Pl}}^{\,y}\;R[\mathring{\nabla}]_{\iota\theta}+\frac{1}{2}\mathcal{M}_{\mathrm{Pl}}^{\,y}\;\mathring{g}_{\iota\theta}\;R[\mathring{\nabla}]+\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\mathring{g}_{\iota\theta}\;\sigma^{\alpha'}{}_{\alpha\alpha'}\;{}^{y}_{:}\mathcal{T}^{\alpha}}{3\,\overset{\approx}{\mathring{g}}}-\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}^{\,y}\;\mathring{g}_{\iota\theta}\;{}^{y}_{:}\mathcal{T}_{\alpha}\;{}^{y}_{:}\mathcal{T}^{\alpha}-$$

$$\frac{2}{3}\mathcal{M}_{\mathrm{Pl}}^{\,y}\;{}^{y}_{:}\mu\;\mathring{g}_{\iota\theta}\;{}^{y}_{:}\mathcal{T}_{\alpha}\;{}^{y}_{:}\mathcal{T}^{\alpha}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\sigma^{\alpha}{}_{\theta\alpha}\;{}^{y}_{:}\mathcal{T}_{\iota}}{6\,\overset{\approx}{\mathring{g}}}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\sigma^{\alpha}{}_{\iota\alpha}\;{}^{y}_{:}\mathcal{T}_{\theta}}{6\,\overset{\approx}{\mathring{g}}}+\frac{2}{3}\mathcal{M}_{\mathrm{Pl}}^{\,y}\;{}^{y}_{:}\mathcal{T}_{\iota}\;{}^{y}_{:}\mathcal{T}_{\theta}+$$

$$\frac{4}{3}\mathcal{M}_{\mathrm{Pl}}^{\,y}\;{}^{y}_{:}\mu\;{}^{y}_{:}\mathcal{T}_{\iota}\;{}^{y}_{:}\mathcal{T}_{\theta}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\;\mathring{g}_{\iota\theta}\;\sigma^{\alpha'\beta\beta'}\;{}^{z}_{:}\mathcal{T}^{\alpha}}{3\,\overset{\approx}{\mathring{g}}}+\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;\sigma^{\alpha'\,\beta}{}_{\iota}\;{}^{z}_{:}\mathcal{T}^{\alpha}}{12\,\overset{\approx}{\mathring{g}}}+ \quad (91)$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\,\sigma^{\alpha'}{}_{\theta}{}^{\beta}\,{}^{z}\mathcal{T}^{\alpha}}{12\,\overset{\approx}{\mathring{g}}}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\,\sigma_{\iota}{}^{\alpha'\beta}\,{}^{z}\mathcal{T}^{\alpha}}{24\,\overset{\approx}{\mathring{g}}}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\,\sigma_{\theta}{}^{\alpha'\beta}\,{}^{z}\mathcal{T}^{\alpha}}{24\,\overset{\approx}{\mathring{g}}}+$$

$$\frac{3}{4}\mathcal{M}_{\mathrm{Pl}}^{y}\,\mathring{g}_{\iota\theta}\,{}^{z}\mathcal{T}_{\alpha}\,{}^{z}\mathcal{T}^{\alpha}+6\,\mathcal{M}_{\mathrm{Pl}}^{y}\,{}^{z}\mu\,\mathring{g}_{\iota\theta}\,{}^{z}\mathcal{T}_{\alpha}\,{}^{z}\mathcal{T}^{\alpha}+\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\,\sigma^{\alpha\alpha'\beta}\,{}^{z}\mathcal{T}_{\iota}}{6\,\overset{\approx}{\mathring{g}}}+$$

$$\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\,\sigma^{\alpha\alpha'\beta}\,{}^{z}\mathcal{T}_{\theta}}{6\,\overset{\approx}{\mathring{g}}}-\frac{3}{2}\mathcal{M}_{\mathrm{Pl}}^{y}\,{}^{z}\mathcal{T}_{\iota}\,{}^{z}\mathcal{T}_{\theta}-12\,\mathcal{M}_{\mathrm{Pl}}^{y}\,{}^{z}\mu\,{}^{z}\mathcal{T}_{\iota}\,{}^{z}\mathcal{T}_{\theta}-\frac{\sqrt{-\overset{\approx}{\mathring{g}}}\;\theta_{\iota\theta}}{\overset{\approx}{\mathring{g}}}==0$$

Now we replace in Eq. (90) the source current densities by non-density counterparts, defined in Eq. (65) and Eq. (68).

$$\begin{aligned}&-\frac{2}{9}\alpha\,\mathring{g}_{\iota\theta}\,{}^{y}\mathcal{F}_{\alpha\alpha'}\,{}^{y}\mathcal{F}^{\alpha\alpha'}+\frac{8}{9}\alpha\,{}^{y}\mathcal{F}_{\iota}{}^{\alpha}\,{}^{y}\mathcal{F}_{\theta\alpha}+\frac{1}{2}\alpha\,\mathring{g}_{\iota\theta}\,{}^{z}\mathcal{F}_{\alpha\alpha'}\,{}^{z}\mathcal{F}^{\alpha\alpha'}-2\,\alpha\,{}^{z}\mathcal{F}_{\iota}{}^{\alpha}\,{}^{z}\mathcal{F}_{\theta\alpha}-\\&\quad\mathcal{M}_{\mathrm{Pl}}^{y}\,R[\mathring{\nabla}]_{\iota\theta}+\frac{1}{2}\mathcal{M}_{\mathrm{Pl}}^{y}\,\mathring{g}_{\iota\theta}\,R[\mathring{\nabla}]+\frac{1}{3}\mathring{g}_{\iota\theta}\,{}^{y}S^{\alpha}\,{}^{y}\mathcal{T}_{\alpha}-\frac{1}{3}\mathcal{M}_{\mathrm{Pl}}^{y}\,\mathring{g}_{\iota\theta}\,{}^{y}\mathcal{T}_{\alpha}\,{}^{y}\mathcal{T}^{\alpha}-\\&\quad\frac{2}{3}\mathcal{M}_{\mathrm{Pl}}^{y}\,{}^{y}\mu\,\mathring{g}_{\iota\theta}\,{}^{y}\mathcal{T}_{\alpha}\,{}^{y}\mathcal{T}^{\alpha}-\frac{1}{6}{}^{y}S_{\theta}\,{}^{y}\mathcal{T}_{\iota}-\frac{1}{6}{}^{y}S_{\iota}\,{}^{y}\mathcal{T}_{\theta}+\frac{2}{3}\mathcal{M}_{\mathrm{Pl}}^{y}\,{}^{y}\mathcal{T}_{\iota}\,{}^{y}\mathcal{T}_{\theta}+\\&\quad\frac{4}{3}\mathcal{M}_{\mathrm{Pl}}^{y}\,{}^{y}\mu\,{}^{y}\mathcal{T}_{\iota}\,{}^{y}\mathcal{T}_{\theta}+\frac{3}{2}\mathring{g}_{\iota\theta}\,{}^{z}S^{\alpha}\,{}^{z}\mathcal{T}_{\alpha}+\frac{3}{4}\mathcal{M}_{\mathrm{Pl}}^{y}\,\mathring{g}_{\iota\theta}\,{}^{z}\mathcal{T}_{\alpha}\,{}^{z}\mathcal{T}^{\alpha}+6\,\mathcal{M}_{\mathrm{Pl}}^{y}\,{}^{z}\mu\,\mathring{g}_{\iota\theta}\,{}^{z}\mathcal{T}_{\alpha}\,{}^{z}\mathcal{T}^{\alpha}-\\&\quad\frac{3}{4}{}^{z}S_{\theta}\,{}^{z}\mathcal{T}_{\iota}-\frac{3}{4}{}^{z}S_{\iota}\,{}^{z}\mathcal{T}_{\theta}-\frac{3}{2}\mathcal{M}_{\mathrm{Pl}}^{y}\,{}^{z}\mathcal{T}_{\iota}\,{}^{z}\mathcal{T}_{\theta}-12\,\mathcal{M}_{\mathrm{Pl}}^{y}\,{}^{z}\mu\,{}^{z}\mathcal{T}_{\iota}\,{}^{z}\mathcal{T}_{\theta}+T_{\iota\theta}==0\end{aligned}\tag{92}$$

And at last here in Eq. (92) is the symmetric part of the tetrad equation.

Concrete relation to manuscript: Here in Eq. (92) is the symmetric tetrad equation of the effective theory.

Antisymmetric part of tetrad equation

Now we want to understand what the antisymmetric part of the tetrad equation is telling us in Eq. (75).

The first step this time is to make use of the conservation law for the spin tensor as written in Eq. (48), since it allows us to fully replace precisely the antisymmetric part of the matter tetrad tensor.

$$\begin{aligned}&2\,\alpha\,\epsilon\,\mathring{g}_{\iota\theta\alpha'\beta}\,{}^{z}\mathcal{T}^{\alpha}\,{}^{z}\mathcal{T}^{\alpha'}\left(\mathring{\nabla}_{\alpha}{}^{z}\mathcal{T}^{\beta}\right)-\frac{2}{3}\alpha\,{}^{y}\mathcal{T}^{\alpha}\left(\mathring{\nabla}_{\alpha⅒}R[\mathring{\nabla}]_{\iota\theta\alpha}{}^{\alpha'}\right)+\frac{2}{3}\alpha\,R[\mathring{\nabla}]_{\iota\alpha\theta\alpha'}\left(\mathring{\nabla}^{\alpha⅒}\,{}^{y}\mathcal{T}^{\alpha}\right)-\\&\quad\frac{2}{3}\alpha\,R[\mathring{\nabla}]_{\iota\alpha'\theta\alpha}\left(\mathring{\nabla}^{\alpha⅒}\,{}^{y}\mathcal{T}^{\alpha}\right)-\frac{2}{3}\alpha\,R[\mathring{\nabla}]_{\iota\theta\alpha\alpha'}\left(\mathring{\nabla}^{\alpha⅒}\,{}^{y}\mathcal{T}^{\alpha}\right)-\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\,R[\mathring{\nabla}]_{\iota}{}^{\beta}\left(\mathring{\nabla}^{\alpha⅒}\,{}^{z}\mathcal{T}^{\alpha}\right)+\end{aligned}$$

$$
\begin{aligned}
&\alpha\,\epsilon\,\mathring{g}_{\imath\alpha\alpha'\beta}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)-\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\beta\beta'}\;R[\mathring{\nabla}]_{\imath\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)+\\
&\alpha\,\epsilon\,\mathring{g}_{\theta\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\imath}{}^{\beta}{}_{\alpha}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)-\frac{2}{3}\,\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\beta\beta'}\;R[\mathring{\nabla}]_{\imath}{}^{\beta}{}_{\alpha'}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)-\\
&2\,\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\imath}{}^{\beta}{}_{\theta}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)+\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}_{\imath\alpha\beta\beta'}\;R[\mathring{\nabla}]_{\theta\alpha'}{}^{\beta\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)-\\
&\alpha\,\epsilon\,\mathring{g}_{\imath\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)+\frac{2}{3}\,\alpha\,\epsilon\,\mathring{g}_{\imath\alpha\beta\beta'}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha'}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)-\\
&2\,\alpha\,\epsilon\,\mathring{g}_{\imath\theta\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}^{\alpha'}\left(\mathring{\nabla}^{\beta}\overset{z}{\mathcal{T}}_{\alpha}\right)+2\,\alpha\,\epsilon\,\mathring{g}_{\imath\theta\alpha'\beta}\;\overset{z}{\mathcal{T}}_{\alpha}\;\overset{z}{\mathcal{T}}^{\alpha}\left(\mathring{\nabla}^{\beta}\overset{z}{\mathcal{T}}^{\alpha'}\right)+\\
&2\,\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}_{\imath}\left(\mathring{\nabla}^{\beta}\overset{z}{\mathcal{T}}^{\alpha'}\right)-2\,\alpha\,\epsilon\,\mathring{g}_{\imath\alpha\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}_{\theta}\left(\mathring{\nabla}^{\beta}\overset{z}{\mathcal{T}}^{\alpha'}\right)+\\
&\frac{1}{3}\,\alpha\,\epsilon\,\mathring{g}_{\theta\alpha'\beta\beta'}\;\overset{z}{\mathcal{T}}^{\alpha}\left(\mathring{\nabla}^{\beta\frac{1}{10}}R[\mathring{\nabla}]_{\imath}{}^{\alpha'}{}_{\alpha}{}^{\beta}\right)-\frac{1}{3}\,\alpha\,\epsilon\,\mathring{g}_{\imath\alpha'\beta\beta'}\;\overset{z}{\mathcal{T}}^{\alpha}\left(\mathring{\nabla}^{\beta\frac{1}{10}}R[\mathring{\nabla}]_{\theta}{}^{\alpha'}{}_{\alpha}{}^{\beta}\right)-\frac{2}{3}\,\alpha\,\overset{y}{\mathcal{T}}^{\alpha}\left(\mathring{\nabla}_{\imath}R[\mathring{\nabla}]_{\theta\alpha}\right)-\\
&\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}_{\theta\alpha'\beta\beta'}\;\overset{z}{\mathcal{T}}^{\alpha}\left(\mathring{\nabla}_{\imath}R[\mathring{\nabla}]_{\alpha}{}^{\alpha'\beta\beta'}\right)+\frac{2}{3}\,\alpha\,\overset{y}{\mathcal{T}}^{\alpha}\left(\mathring{\nabla}_{\theta}R[\mathring{\nabla}]_{\imath\alpha}\right)+\frac{1}{6}\,\alpha\,\epsilon\,\mathring{g}_{\imath\alpha'\beta\beta'}\;\overset{z}{\mathcal{T}}^{\alpha}\left(\mathring{\nabla}_{\theta}R[\mathring{\nabla}]_{\alpha}{}^{\alpha'\beta\beta'}\right)==0
\end{aligned}
$$

Now we take Eq. (93) and we run through exactly the same steps as we did to get from Eq. (86) to Eq. (92).

Now again eliminate the multiplier.

$$
\begin{aligned}
&2\,\alpha\,\epsilon\,\mathring{g}_{\imath\theta\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}^{\alpha'}\left(\mathring{\nabla}_{\alpha}\overset{z}{\mathcal{T}}^{\beta}\right)+\frac{2}{3}\,\alpha\,R[\mathring{\nabla}]_{\imath\alpha\theta\alpha'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{y}{\mathcal{T}}^{\alpha}\right)-\\
&\frac{2}{3}\,\alpha\,R[\mathring{\nabla}]_{\imath\alpha'\theta\alpha}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{y}{\mathcal{T}}^{\alpha}\right)-\frac{2}{3}\,\alpha\,R[\mathring{\nabla}]_{\imath\theta\alpha\alpha'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{y}{\mathcal{T}}^{\alpha}\right)-\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;R[\mathring{\nabla}]_{\imath}{}^{\beta}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)+\\
&\alpha\,\epsilon\,\mathring{g}_{\imath\alpha\alpha'\beta}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)+\alpha\,\epsilon\,\mathring{g}_{\theta\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\imath}{}^{\beta}{}_{\alpha}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)-\\
&\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\beta\beta'}\;R[\mathring{\nabla}]_{\imath}{}^{\beta}{}_{\alpha'}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)-2\,\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\imath}{}^{\beta}{}_{\theta}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)-\\
&\alpha\,\epsilon\,\mathring{g}_{\imath\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)+\alpha\,\epsilon\,\mathring{g}_{\imath\alpha\beta\beta'}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha'}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)-\\
&2\,\alpha\,\epsilon\,\mathring{g}_{\imath\theta\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}^{\alpha'}\left(\mathring{\nabla}^{\beta}\overset{z}{\mathcal{T}}_{\alpha}\right)+2\,\alpha\,\epsilon\,\mathring{g}_{\imath\theta\alpha'\beta}\;\overset{z}{\mathcal{T}}_{\alpha}\;\overset{z}{\mathcal{T}}^{\alpha}\left(\mathring{\nabla}^{\beta}\overset{z}{\mathcal{T}}^{\alpha'}\right)+\\
&2\,\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}_{\imath}\left(\mathring{\nabla}^{\beta}\overset{z}{\mathcal{T}}^{\alpha'}\right)-2\,\alpha\,\epsilon\,\mathring{g}_{\imath\alpha\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}_{\theta}\left(\mathring{\nabla}^{\beta}\overset{z}{\mathcal{T}}^{\alpha'}\right)==0
\end{aligned}
\tag{94}
$$

Again eliminate d'Alembertians.

$$
\begin{aligned}
&2\,\alpha\,\epsilon\,\mathring{g}_{\imath\theta\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}^{\alpha'}\left(\mathring{\nabla}_{\alpha}\overset{z}{\mathcal{T}}^{\beta}\right)+\frac{2}{3}\,\alpha\,R[\mathring{\nabla}]_{\imath\alpha\theta\alpha'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{y}{\mathcal{T}}^{\alpha}\right)-\\
&\frac{2}{3}\,\alpha\,R[\mathring{\nabla}]_{\imath\alpha'\theta\alpha}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{y}{\mathcal{T}}^{\alpha}\right)-\frac{2}{3}\,\alpha\,R[\mathring{\nabla}]_{\imath\theta\alpha\alpha'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{y}{\mathcal{T}}^{\alpha}\right)-\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;R[\mathring{\nabla}]_{\imath}{}^{\beta}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)+\\
&\alpha\,\epsilon\,\mathring{g}_{\imath\alpha\alpha'\beta}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)+\alpha\,\epsilon\,\mathring{g}_{\theta\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\imath}{}^{\beta}{}_{\alpha}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)-
\end{aligned}
$$

$$\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\beta\beta'}\;R[\mathring{\nabla}]_{\iota}{}^{\beta}{}_{\alpha'}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)-2\,\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\iota}{}^{\beta}{}_{\theta}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)-$$
$$\alpha\,\epsilon\,\mathring{g}_{\iota\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)+\alpha\,\epsilon\,\mathring{g}_{\iota\alpha\beta\beta'}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha'}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)-$$
$$2\,\alpha\,\epsilon\,\mathring{g}_{\iota\theta\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}^{\alpha'}\left(\mathring{\nabla}^{\beta}\overset{z}{\mathcal{T}}_{\alpha}\right)+2\,\alpha\,\epsilon\,\mathring{g}_{\iota\theta\alpha'\beta}\;\overset{z}{\mathcal{T}}_{\alpha}\;\overset{z}{\mathcal{T}}^{\alpha}\left(\mathring{\nabla}^{\beta}\overset{z}{\mathcal{T}}^{\alpha'}\right)+$$
$$2\,\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}_{\iota}\left(\mathring{\nabla}^{\beta}\overset{z}{\mathcal{T}}^{\alpha'}\right)-2\,\alpha\,\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}_{\theta}\left(\mathring{\nabla}^{\beta}\overset{z}{\mathcal{T}}^{\alpha'}\right)==0$$

Again eliminate divergences.

$$2\,\alpha\,\epsilon\,\mathring{g}_{\iota\theta\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}^{\alpha'}\left(\mathring{\nabla}_{\alpha}\overset{z}{\mathcal{T}}^{\beta}\right)+\frac{2}{3}\,\alpha\,R[\mathring{\nabla}]_{\iota\alpha\theta\alpha'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{y}{\mathcal{T}}^{\alpha}\right)-$$
$$\frac{2}{3}\,\alpha\,R[\mathring{\nabla}]_{\iota\alpha'\theta\alpha}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{y}{\mathcal{T}}^{\alpha}\right)-\frac{2}{3}\,\alpha\,R[\mathring{\nabla}]_{\iota\theta\alpha\alpha'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{y}{\mathcal{T}}^{\alpha}\right)-\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;R[\mathring{\nabla}]_{\iota}{}^{\beta}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)+$$
$$\alpha\,\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)+\alpha\,\epsilon\,\mathring{g}_{\theta\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\iota}{}^{\beta}{}_{\alpha}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)-$$
$$\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\beta\beta'}\;R[\mathring{\nabla}]_{\iota}{}^{\beta}{}_{\alpha'}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)-2\,\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\iota}{}^{\beta}{}_{\theta}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)-\quad(96)$$
$$\alpha\,\epsilon\,\mathring{g}_{\iota\alpha'\beta\beta'}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)+\alpha\,\epsilon\,\mathring{g}_{\iota\alpha\beta\beta'}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha'}{}^{\beta'}\left(\mathring{\nabla}^{\alpha\frac{1}{10}}\overset{z}{\mathcal{T}}^{\alpha}\right)-$$
$$2\,\alpha\,\epsilon\,\mathring{g}_{\iota\theta\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}^{\alpha'}\left(\mathring{\nabla}^{\beta}\overset{z}{\mathcal{T}}_{\alpha}\right)+2\,\alpha\,\epsilon\,\mathring{g}_{\iota\theta\alpha'\beta}\;\overset{z}{\mathcal{T}}_{\alpha}\;\overset{z}{\mathcal{T}}^{\alpha}\left(\mathring{\nabla}^{\beta}\overset{z}{\mathcal{T}}^{\alpha'}\right)+$$
$$2\,\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}_{\iota}\left(\mathring{\nabla}^{\beta}\overset{z}{\mathcal{T}}^{\alpha'}\right)-2\,\alpha\,\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}_{\theta}\left(\mathring{\nabla}^{\beta}\overset{z}{\mathcal{T}}^{\alpha'}\right)==0$$

Again convert to Maxwell tensors.

$$\frac{1}{2}\,\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;\overset{z}{\mathcal{F}}^{\alpha\alpha'}\;R[\mathring{\nabla}]_{\iota}{}^{\beta}-\frac{1}{2}\,\alpha\,\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;\overset{z}{\mathcal{F}}^{\alpha\alpha'}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}-\alpha\,\epsilon\,\mathring{g}_{\theta\alpha'\beta\beta'}\;\overset{z}{\mathcal{F}}^{\alpha\alpha'}\;R[\mathring{\nabla}]_{\iota}{}^{\beta}{}_{\alpha}{}^{\beta'}+$$
$$\alpha\,\epsilon\,\mathring{g}_{\alpha\alpha'\beta\beta'}\;\overset{z}{\mathcal{F}}^{\alpha\alpha'}\;R[\mathring{\nabla}]_{\iota}{}^{\beta}{}_{\theta}{}^{\beta'}+\alpha\,\epsilon\,\mathring{g}_{\iota\alpha'\beta\beta'}\;\overset{z}{\mathcal{F}}^{\alpha\alpha'}\;R[\mathring{\nabla}]_{\theta}{}^{\beta}{}_{\alpha}{}^{\beta'}-\alpha\,\epsilon\,\mathring{g}_{\iota\theta\alpha'\beta}\;\overset{z}{\mathcal{F}}^{\alpha'\beta}\;\overset{z}{\mathcal{T}}_{\alpha}\;\overset{z}{\mathcal{T}}^{\alpha}+\quad(97)$$
$$2\,\alpha\,\epsilon\,\mathring{g}_{\iota\theta\alpha'\beta}\;\overset{z}{\mathcal{F}}_{\alpha}{}^{\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}^{\alpha'}-\alpha\,\epsilon\,\mathring{g}_{\theta\alpha\alpha'\beta}\;\overset{z}{\mathcal{F}}^{\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}_{\iota}+\alpha\,\epsilon\,\mathring{g}_{\iota\alpha\alpha'\beta}\;\overset{z}{\mathcal{F}}^{\alpha'\beta}\;\overset{z}{\mathcal{T}}^{\alpha}\;\overset{z}{\mathcal{T}}_{\theta}==0$$

Again employ the dual/Bianchi trick.

True (98)

Now at last here is the antisymmetric part of the tetrad equation.

True (99)

So, we quickly see that in Eq. (99) the antisymmetric part of the tetrad equation just boils away to an identity.

Concrete relation to manuscript: The antisymmetric part of the tetrad equa-

tion just boils away to an identity.

Spin connection equations

Finally, we would like to write the remaining spin connection equations in their most compact form.

The vector part of the spin connection equation in Eq. (78).

$$ {}^{y}\!S_{\omega} - 2\,\mathcal{M}_{\text{Pl}}^{y}\;{}^{y}\!\mathcal{T}_{\omega} - 4\,\mathcal{M}_{\text{Pl}}^{y}\;{}^{y}\!\mu\;{}^{y}\!\mathcal{T}_{\omega} - \frac{8}{3}\,\alpha\left(\mathring{\nabla}_{\alpha}\,{}^{y}\!\mathcal{F}_{\omega}{}^{\alpha}\right) == 0 \tag{100}$$

Concrete relation to manuscript: Here is another of the field equations: compare Eq. (100) to Eq. (8b) in the manuscript.

The axial vector part of the spin connection equation in Eq. (79).

$$ -6\;{}^{z}\!S^{\chi} - 6\,\mathcal{M}_{\text{Pl}}^{y}\;{}^{z}\!\mathcal{T}^{\chi} - 48\,\mathcal{M}_{\text{Pl}}^{y}\;{}^{z}\!\mu\;{}^{z}\!\mathcal{T}^{\chi} - 8\,\alpha\left(\mathring{\nabla}_{\alpha}\,{}^{z}\!\mathcal{F}^{\chi\alpha}\right) == 0 \tag{101}$$

Concrete relation to manuscript: Here is another of the field equations: compare Eq. (101) to Eq. (8c) in the manuscript.

Reconstruction of the effective action

Now finally we can compare the field equations Eq. (92), Eq. (100) and Eq. (101) with those obtained from our ansatz above in Eq. (72), Eq. (73) and Eq. (74). Note that for the Einstein equations, we must remove the stress-energy tensor of matter to make the comparison. The solutions for the ansatz parameters are as follows.

$$\left\{\left\{c_{x} \to -\frac{8\,\alpha}{9},\, c_{y} \to 2\,\alpha,\, c_{z} \to -\frac{2\,\mathcal{M}_{\text{Pl}}^{y}}{3} - \frac{4\,\mathcal{M}_{\text{Pl}}^{y}\;{}^{y}\!\mu}{3},\, c_{è} \to \frac{3\,\mathcal{M}_{\text{Pl}}^{y}}{2} + 12\,\mathcal{M}_{\text{Pl}}^{y}\;{}^{z}\!\mu,\, c_{é} \to \frac{2}{3},\, c_{ę} \to 3\right\}\right\} \tag{102}$$

Substituting Eq. (102) into our ansatz in Eq. (71) we obtain the reconstructed Lagrangian density.

$$\begin{aligned}&\frac{2}{9}\,\alpha\sqrt{-\mathring{\tilde{\tilde{g}}}}\;{}^{y}\!\mathcal{F}_{\mu\nu}\;{}^{y}\!\mathcal{F}^{\mu\nu} - \frac{1}{2}\,\alpha\sqrt{-\mathring{\tilde{\tilde{g}}}}\;{}^{z}\!\mathcal{F}_{\mu\nu}\;{}^{z}\!\mathcal{F}^{\mu\nu} - \frac{1}{2}\,\mathcal{M}_{\text{Pl}}^{y}\sqrt{-\mathring{\tilde{\tilde{g}}}}\;R[\mathring{\nabla}] - \frac{1}{3}\sqrt{-\mathring{\tilde{\tilde{g}}}}\;{}^{y}\!S^{\mu}\;{}^{y}\!\mathcal{T}_{\mu} + \\ &\frac{1}{3}\,\mathcal{M}_{\text{Pl}}^{y}\left(1 + 2\,{}^{y}\!\mu\right)\sqrt{-\mathring{\tilde{\tilde{g}}}}\;{}^{y}\!\mathcal{T}_{\mu}\;{}^{y}\!\mathcal{T}^{\mu} - \frac{3}{2}\sqrt{-\mathring{\tilde{\tilde{g}}}}\;{}^{z}\!S^{\mu}\;{}^{z}\!\mathcal{T}_{\mu} - \frac{3}{4}\,\mathcal{M}_{\text{Pl}}^{y}\left(1 + 8\,{}^{z}\!\mu\right)\sqrt{-\mathring{\tilde{\tilde{g}}}}\;{}^{z}\!\mathcal{T}_{\mu}\;{}^{z}\!\mathcal{T}^{\mu}\end{aligned} \tag{103}$$

Concrete relation to manuscript: We obtain our effective Lagrangian: compare Eq. (103) with Eq. (9) in the manuscript.

This concludes the supplemental materials to our manuscript.