VIII. SCALE-INVARIANT VARIABLES: CODE IMPLEMENTATION

First, we note that $\mathcal{T}^\dagger$ is equivalent in form when A is replaced by $A^\dagger$ (equation 20 of Lin's draft):

$$\mathcal{T}^{\dagger A}{}_{BC} = \mathcal{T}^{\natural A}{}_{BC} + \frac{1}{3}(\delta^A_B \mathcal{T}^\natural_C - \delta^A_C \mathcal{T}^\natural_B), \tag{52}$$

where $\mathcal{T}^\natural$ is given by (draft eqn. 18)

$$\mathcal{T}^{\natural A}{}_{\mu\nu} \equiv 2(\partial_{[\mu} b^A{}_{\nu]} + A^{\dagger A}{}_{E[\mu} b^E{}_{\nu]}). \tag{53}$$

It is clear that for the $\mathcal{R}^\dagger$ and $\mathcal{T}^\dagger$ terms in the eWGT Lagrangian, using $A^\dagger$ instead as the variable does not lead to change in form of the Lagrangian.
Second, there is a degeneracy in the Weyl vector in eWGT (equation 19 of draft)

$$\mathcal{T}^\natural_B \equiv \mathcal{T}^{\natural A}{}_{BA} = \mathcal{T}_B - 3B_B, \tag{54}$$

with the quantity $\mathcal{T}_B - 3B_B$ appearing in the expressions for $\mathcal{H}^\dagger_{\mu\nu}$ and $\mathcal{D}^\dagger_A$ (draft eqns. 10, 13):

$$\mathcal{H}^\dagger_{\mu\nu} = \partial_\mu(B_\nu - \frac{1}{3}\mathcal{T}_\nu) - \partial_\nu(B_\mu - \frac{1}{3}\mathcal{T}_\mu), \tag{55}$$

$$\mathcal{D}^\dagger_A = \partial_A\phi - (B_A - \frac{1}{3}\mathcal{T}_A)\phi. \tag{56}$$

We can choose to make a variable redefinition $B_A - \frac{1}{3}\mathcal{T}_A \to B'_A$, preserving Weyl vector B' as a variable (WGT style). Alternatively, we can choose to eliminate B as a variable (after choosing $A^\dagger$) via (54), i.e. $B_A - \frac{1}{3}\mathcal{T}_A \to -\frac{1}{3}\mathcal{T}^\natural_A$ (PGT style). (It seems to me the ability to choose the PGT/WGT style Lagrangians to reflect a gauge fixing of the 'TVG' symmetry?)
File `.../WeylGaugeTheoryExtended/LagrangianWGTEScaleInvariantRescaling.m`, section "Setting the rescaling", implements the scale-invariant variables (removing compensator ϕ) of Lasenby and Hobson [9, eqns. 199a-d] for the WGT-style Lagrangian:

$$\hat{h}_A{}^\mu \equiv \left(\frac{\phi}{\phi_0}\right)^{-1} h_A{}^\mu, \quad \hat{b}^A{}_\mu \equiv \left(\frac{\phi}{\phi_0}\right) b^A{}_\mu, \quad \hat{h}^{-1} = \left(\frac{\phi}{\phi_0}\right)^4 h^{-1}, \tag{57}$$

$$\hat{A}^{\dagger A}{}_{\mu\nu} \equiv A^{\dagger A}{}_{\mu\nu}, \quad \hat{B}_\mu \equiv B_\mu - \frac{1}{3}\mathcal{T}_\mu - \partial_\mu \ln\frac{\phi}{\phi_0}. \tag{58}$$

The general eWGT Lagrangian expressed with these variables, in the WGT-style with $\hat{B}$, is kept in `NonlinearLagrangianWGTEScaleInvariantRescaling`. Section "WGT Lagrangian: removing $\hat{B}$ to get a PGT Lagrangian" implements the PGT-style Lagrangian by setting $\hat{B}_\mu = -\frac{1}{3}\hat{\mathcal{T}}^\natural_\mu$ [9, sec. III.M], storing it in `NonlinearLagrangianWGTEScaleInvariantRescalingPGT`. (It seems to me the scale-invariant variables account to a Weyl-symmetry gauge choice, but not too sure about ϕ itself.)

A. Overview of cases analysed

See Table I for an overview of the cases analysed. The cases are contained in the following files and are called up by `.../WeylGaugeTheoryExtended.m` in the following order:

1. `.../WeylGaugeTheoryExtended/WGTESimpleTestCases.m`
2. `.../WeylGaugeTheoryExtended/WGTEGeneralCase.m`
3. `.../WeylGaugeTheoryExtended/WGTETestCasesScaleInvariantRescaling.m`
4. `.../WeylGaugeTheoryExtended/WGTEGeneralCaseScaleInvariantRescaling.m`

Case (indices suppressed)	File 1	File 2	File 3	File 4
Test 1: $\phi^2\mathcal{R}^\dagger$	EP	-	-	-
Test 2: $\phi^2\mathcal{R}^\dagger + (\mathcal{D}^\dagger\phi)^2$	EP	-	-	-
Test 3: $\phi^2\mathcal{R}^\dagger + (\mathcal{D}^\dagger\phi)^2 + \mathcal{H}^\dagger\mathcal{H}^\dagger + \mathcal{R}^\dagger\mathcal{H}^\dagger$	EP	-	WGT	-
Test 4: $(\mathcal{D}^\dagger\phi)^2 + \mathcal{H}^\dagger\mathcal{H}^\dagger$	EP	-	PGT	-
General eWGT, draft eqn. 13	-	EP	-	WGT, PGT

Table I: Overview of the cases analysed with respect to how the compensator ϕ is handled; the file numbers refer to the order they are called on for analysis, see section VIII A. Dashes - case not analysed; EP - perturbations around Einstein gauge $\phi \to \phi_0(1+\phi)$; WGT(PGT) - scale-invariant variables keeping(removing) $\hat{B}$.

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