Seminar 4

Special relativity

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What is special relativity?

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  - **classical mechanics**$\rightarrow$ **relativistic mechanics**

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What is special relativity?

- Most importanty the **regime** $v \sim c$
- This regime tends to **break** classical laws of phyics:
  - **quantum mechanics**$\rightarrow$ **quantum field theory**
  - **classical mechanics**$\rightarrow$ **relativistic mechanics**
- Three space dimensions and one time dimension are **aspects** of a connected whole known as **spacetime**

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Special vs general relativity

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Special vs general relativity

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    - **Special relativity** $\rightarrow$ **flat spacetime**

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- What does that mean about the **shape** of spacetime?
  - **Special relativity** $\rightarrow$ **flat spacetime**
  - **General relativity** $\rightarrow$ **curved spacetime**

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Special vs general relativity

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- The **three** dimensional space we all live in is known as **Euclidean** space
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  - We can **rotate** the space
  - We can define **distances** which don't change when we **rotate** the space

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We want everything simple...

- We will only deal with **two** dimensions at a time, otherwise everything will become very complicated!

Euclidean rotations

- Start with **two** ordinary space dimensions and coordinates x and y, what is a **vector** which describes a **position** in that space?

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Euclidean rotations

- Start with **two** ordinary space dimensions and coordinates x and y, what is a **vector** which describes a **position** in that space?
- Should have:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

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Euclidean rotations

- Now we need to introduce **matrices**!

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Euclidean rotations

- Now we need to introduce **matrices**!
- Just a **table** of numbers:

$$\mathbf{R} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

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Euclidean rotations

- Now we **multiply** a **vector** by a **matrix**!

Euclidean rotations

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- Here is the **rule** for doing this:

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- **WRITE THIS DOWN, YOU WILL NEED IT**

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Euclidean rotations

- **Over to you**: find x' and y' if:

$$\mathbf{R} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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- Emre: $\theta = 10°$, Mason: $\theta = 360°$, Ali Goktug: $\theta = 45°$, Claudia: $\theta = 70°$, Federico: $\theta = 180°$, Beltran: $\theta = 270°$

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Euclidean rotations

- **Still over to you**: Now find $\sqrt{x'^2+y'^2}$ for these new vectors!

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Euclidean rotations

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  - The matrix $\mathbf{R}$ just **rotates** the position in space through the **angle**

Euclidean rotations

- We should have found the following **exciting** things:
  - The matrix $\mathbf{R}$ just **rotates** the position in space through the **angle**
  - This leaves the **distance** from the origin **unchanged**

Euclidean rotations

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Euclidean rotations

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  - **Rotations** are done with **trigonometric** functions $\sin(\theta)$ and $\cos(\theta)$

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Euclidean rotations

- So we can say this about **Euclidean** space:
  - **Rotations** are done with **trigonometric** functions $\sin(\theta)$ and $\cos(\theta)$
  - **Distances** are done like this: $\sqrt{x^2 + y^2 + z^2}$

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Galilean rotations

- This was rather **boring** actually, it gets more interesting if we have one **space** dimension and one **time** dimension

Galilean rotations

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- Let's say **space** is just given by x, what would be a good coordinate for time?

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- This was rather **boring** actually, it gets more interesting if we have one **space** dimension and one **time** dimension
- It would be nice if we could set up this **two** dimensional space with coordinates which have the same **units**
- Let's say **space** is just given by x, what would be a good coordinate for time?
- So our new **position vector** is:

$$\mathbf{x} = \begin{bmatrix} ct \\ x \end{bmatrix}$$

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Galilean rotations

- We've just started to put together the idea of **spacetime**!

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Galilean rotations

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- Some important points:
  - In **space** positions are usually known as **points**
  - In **spacetime** the positions are known as **events**

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Galilean rotations

- We've just started to put together the idea of **spacetime**!
- Some important points:
  - In **space** positions are usually known as **points**
  - In **spacetime** the positions are known as **events**
- Why?

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Galilean rotations

- Now things begin getting complicated...

Galilean rotations

- Now things begin getting complicated. . .
- Let's say we start moving along the x direction at **constant** velocity v (couldn't think of a simpler motion really. . .)

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- What is our x at general t?

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- At $t = 0$, we were at $x = 0$
- What is our x at general t?
- What is our t at general t (trick question!)?

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Galilean rotations

- With this in mind, how will we measure t' and x' of some **event**:

$$\mathbf{x} = \begin{bmatrix} ct \\ x \end{bmatrix}$$

Galilean rotations

- With this in mind, how will we measure t' and x' of some **event**:

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- Just using common sense, we should end up with these very simple formulae:

$$t' = t,$$
$$x' = x - vt$$

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Galilean rotations

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- You all already **knew these**: they just say that if you **move**, the **time** you observe an event is **the same**, but the position of the event **moves toward you** (or away if $v \to -v$)

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- Does anyone **disagree**?
- These formulae are **completely and utterly wrong**, but nobody noticed until 1905!

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Galilean rotations

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- That is because the idea of **rotating space** makes **perfect sense**, but **rotating space and time** sounds like **nonsense**...
- Let's do it anyway!

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Lorentz rotations

- **Over to you again**: find t' and x' if:

$$\mathbf{R} = \begin{bmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & \frac{-v/c}{\sqrt{1-v^2/c^2}} \\ \frac{-v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} ct \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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- Emre: $v = c$, Mason: $v = 2c$, Ali Goktug: $v = 0$, Claudia: $v = 0.5c$, Federico: $v = 0.2c$, Beltran: $v = 0.1c$

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Lorentz rotations

- **Next task**: find $\sqrt{c^2 t'^2 - x'^2}$ for your vectors!

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Lorentz rotations

- **Now find this**:

$$\left(\frac{1}{\sqrt{1-v^2/c^2}}\right)^2-\left(\frac{-v/c}{\sqrt{1-v^2/c^2}}\right)^2$$

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- **Finally who can remember what this is** for any ψ:

$$\cosh(\psi)^2-\sinh(\psi)^2$$

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$$\cosh(\psi)^2 - \sinh(\psi)^2$$

- Claudia: ψ is another Greek letter pronounced 'psi' ;)

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Lorentz rotations

- So it turns out we can just write that horrible matrix in a simple form:

$$\mathbf{R} = \begin{bmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & \frac{-v/c}{\sqrt{1-v^2/c^2}} \\ \frac{-v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} \end{bmatrix} = \begin{bmatrix} \cosh(\psi) & \sinh(\psi) \\ \sinh(\psi) & \cosh(\psi) \end{bmatrix} = \mathbf{\Lambda}$$

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- Claudia: Λ is uppercase Greek letter 'lambda', lowercase is λ ;)

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Lorentz rotations

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Lorentz rotations

- So unlike last time, the things we get now should actually be interesting:
    - When we move, space and time **rotate**, but instead of **trigonometric** rotations with $\sin(\theta)$ and $\cos(\theta)$ for some angle θ we have **hyperbolic** rotations with **hyperbolic** functions $\sinh(\psi)$ and $\cosh(\psi)$ – don't bother with what ψ actually is, there is a formula for it in terms of v/c (Mason, find $\psi(v/c)$)

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    - The **distance** in spacetime is just $\sqrt{c^2t^2 - x^2 - y^2 - z^2}$

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Lorentz rotations

- Some long words:

Lorentz rotations

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  - **Distance** in spacetime is known as the **interval** – in the same sense that the **point** is known as an **event**

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  - **Minkowskian spacetime** is flat, but has a **negative metric signature** – tomorrow (Seminar 5) we will look at general relativity, in which the metric signature is the same, but because there is **gravity** the spacetime is **curved**

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  - **Minkowskian spacetime** is flat, but has a **negative metric signature** – tomorrow (Seminar 5) we will look at general relativity, in which the metric signature is the same, but because there is **gravity** the spacetime is **curved**
  - You might want to look up some of these terms, but we don't have time (or the methematical development) to go into them

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Lorentz rotations

- Note to self: find some Lorentz transformations on Youtube

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Lorentz rotations

- But I didn't go into why any of this **hyperbolic/Lorentz** rotation stuff is true...

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Lorentz rotations

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- In fact, we were happy with the simpler Galilean transformations before weren't we?

Lorentz rotations

- But I didn't go into why any of this **hyperbolic/Lorentz** rotation stuff is true...
- In fact, we were happy with the simpler Galilean transformations before weren't we?
- Go back to t and x (i.e. **two dimensional case**) – what if the **event** was a photon being **emitted** in the past at $t < 0$ and some position x, and detected by us ($v = 0$) at the **origin** of spacetime ($t_0 = 0$ and $x_0 = 0$) **Find the interval for the photon**:

$$\sqrt{c^2t^2 - x^2}$$

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Lorentz rotations

- We should find:

$$\sqrt{c^2t^2 - x^2} = 0$$

Lorentz rotations

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- And even if we start moving at v we just spent ages proving that the **interval** of the photon's motion remains **unchanged**:

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- This relies on the photon moving at c, right?
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$$\sqrt{c^2t'^2 - x'^2} = 0$$

- So when we are moving at v, **what is the speed of the photon**?
- So all this **hyperbolic** rotation stuff just ensures that **the speed of light is the same, no matter how fast you are moving**

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Lorentz rotations

- Finally, we **Lorentz** rotations:

$$ct' = \frac{ct}{\sqrt{1 - v^2/c^2}} - \frac{vx/c}{\sqrt{1 - v^2/c^2}},$$
$$x' = \frac{x}{\sqrt{1 - v^2/c^2}} - \frac{vt}{\sqrt{1 - v^2/c^2}},$$

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- And **Galilean** transformations:

$$t' = t,$$
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Lorentz rotations

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$$
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ct' &= \frac{ct}{\sqrt{1-v^2/c^2}} - \frac{vx/c}{\sqrt{1-v^2/c^2}}, \\
x' &= \frac{x}{\sqrt{1-v^2/c^2}} - \frac{vt}{\sqrt{1-v^2/c^2}},
\end{aligned}
$$

- And **Galilean** transformations:

$$
\begin{aligned}
t' &= t, \\
x' &= x - vt,
\end{aligned}
$$

- **Explain why Galilean transformations are okay so long as** $v \ll c$

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Lorentz rotations

- We also have **relativistic** momentum:

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

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Lorentz rotations

- We also have **relativistic** momentum:

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- And **non-relativistic** momentum:

$$p = mv$$

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Lorentz rotations

- We also have **relativistic** momentum:

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- And **non-relativistic** momentum:

$$p = mv$$

- **Explain why they agree at** $v \ll c$

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Lorentz rotations

- Finally we have **relativistic** energy:

$$\mathcal{E} = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

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Lorentz rotations

- Finally we have **relativistic** energy:

$$\mathcal{E} = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

- **What happens at $v \ll c$?**
- Should have **non-relativistic** energy:

$$\mathcal{E} = mc^2 + \frac{1}{2}mv^2$$

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Lorentz rotations

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- Should have **non-relativistic** energy:

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- What happens if the particle is standing still?

$$\mathcal{E} = mc^2$$

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- Should have **non-relativistic** energy:

$$\mathcal{E} = mc^2 + \frac{1}{2}mv^2$$

- What happens if the particle is standing still?

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- **The end.**

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Astrophysical phenomena

"""einstein".jpg

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