Seminar 3

Classical mechanics

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University of Cambridge

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  - Theory of elasticity

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Quantities of interest

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- What else can we define using **only** those coordinates?
- Any other properties of a point particle?
- In all:

$$t, \quad \mathbf{x}, \quad \mathbf{v}, \quad \mathbf{a}, \quad m$$

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Vectors

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- Acceleration:

$$\mathbf{a} = \left(\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}, \frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2}, \frac{\mathrm{d}^2z(t)}{\mathrm{d}t^2}\right) = (\ddot{x}, \ddot{y}, \ddot{z})$$

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Products of vectors

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- The inner product is also our means of talking about **distances**:

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- Often we will need to take an inner product of a vector with itself:

$$\mathbf{v} \cdot \mathbf{v} = v^2$$

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Kinetic energy

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- As $v \to c$ nothing especially interesting happens, but if we use the **relativistic formula** we find that $T \to \infty$

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- Any other potential fields we can think of?

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Force

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- We are used to **forces** coming into play when we use Newton's laws
- This is dangerous, there is a far better definition which we need to understand:

$$-\mathbf{F}(\mathbf{x}) = \nabla U(\mathbf{x})$$

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Grad

- The upside-down triangle symbol ∇ is known as **grad** or **gradient**

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Grad

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- Grad is a way of getting a vector from a scalar using differentiation:

$$-\mathbf{F}(x,y,z)=\left(\frac{\partial U(x,y,x)}{\partial x},\frac{\partial U(x,y,z)}{\partial y},\frac{\partial U(x,y,z)}{\partial z}\right)$$

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- The Grad vector always points in the direction of **fastest change** of the field

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Energy conservation

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- Much easier to consider a **one-dimensional** example, where we only have the coordinate x

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Energy conservation

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- Now what does the **second** equation tell us about the **first**?
- We should get:

$$\dot{x}\frac{\mathrm{d}U(x)}{\mathrm{d}x} + m\dot{x}\ddot{x} = 0$$

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- We can tidy this up a bit with $\dot{x} = v$, $\ddot{x} = a$ and $\frac{\mathrm{d}U(x)}{\mathrm{d}x} = -F(x)$:

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- But v could be anything, so we can divide it away:

$$F(x) = ma$$

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Newton's laws

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Newton's laws

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- What if we set $F(x) = 0$?

Newton's laws

- This was just the case in **one dimension**, the energy conservation equation for all **three** space dimensions has to be written in textbfvector form:

$$\dot{\mathbf{x}} \cdot \nabla U(\mathbf{x}) + m\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} = 0$$

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- And again we can write these vectors with more sensible names:

$$\mathbf{v} \cdot (-\mathbf{F}(\mathbf{x}) + m\mathbf{a}) = 0$$

- And since $\mathbf{v}$ is completely general:

$$\mathbf{F}(\mathbf{x}) = m\mathbf{a}$$

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  - The **3rd** law guarantees **energy conservation** for **multiple particles** along with **momentum conservation**
  - When we consider all particles and potentials, we are talking about a **system** in which total **energy** and total **momentum** are conserved

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Some deep principles

- There is a vary powerful formalism that describes all of **classical mechanics**, **quantum mechanics** and **general relativity**, known as the **Lagrangian formalism** or **Hamilton's principle**:

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  - **Momentum conservation** happens because the laws of physics are the same at all points in **space**
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- The fancy word for this is **Noether symmetry**

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Some deep principles

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The pendulum

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- What is the potential energy?

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The pendulum

- The pendulum is an example of a **constrained** system, the particle is confined to move only in a circle of radius l, and the coordinate that defines its position is θ

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$$v = l\dot{\theta}$$

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- The **velocity** has the magnitude:

$$v = l\dot{\theta}$$

- The **acceleration** has the magnitude:

$$a = l\ddot{\theta}$$

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The pendulum

- Because the particle is constrained, the $F(\theta)$ that will enter into Newton's 2nd law has to be the **component** of $\mathbf{F}(\mathbf{x})$ that points along the path of the particle

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- Because the particle is constrained, the $F(\theta)$ that will enter into Newton's 2nd law has to be the **component** of $\mathbf{F}(\mathbf{x})$ that points along the path of the particle
- **Find the formula for** $F(\theta)$

The pendulum

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- This means for Newton's 2nd law:

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- Dividing through by m (because $m \neq 0$) gives:

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

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- To make it super easy we use the **small angle approximation** $\theta \ll 1$:

$$\sin(\theta) = \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 - \cdots$$

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The pendulum

- So now we are approximating $\sin(\theta) \approx \theta$, the differential equation becomes:

$$\ddot{\theta} = -\frac{g}{l}\theta$$

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The pendulum

- So now we are approximating $\sin(\theta) \approx \theta$, the differential equation becomes:

$$\ddot{\theta} = -\frac{g}{l}\theta$$

- This looks even easier than before, **solve it** using what we learned in Seminar 1, and assuming $\frac{g}{l} = 1$:

$$f(t) = e^t \implies \dot{f}(t) = e^t, \quad \ddot{f}(t) = e^t$$
$$f(t) = \cosh(t) \implies \dot{f}(t) = \sinh(t), \quad \ddot{f}(t) = \cosh(t)$$
$$f(t) = \sinh(t) \implies \dot{f}(t) = \cosh(t), \quad \ddot{f}(t) = \sinh(t)$$
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Astrophysical phenomena

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