Gauge theories of gravity: a dictionary

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Two physically equivalent approaches to constructing gravitational gauge theories are predominant, the passive tensor formalism (PTF) and active multivector formalism (AMF). Whilst the latter offers enormous practical advantages, it requires one to learn geometric algebra, whilst the former is already almost ubiquitous in the community. In preparation for upcoming cosmological results formulated in the AMF, we set out a dictionary between the two, which may be of pedagogical value to future authors.

I. INTRODUCTION

The various incarnations of Poincaré gauge theory (PGT), Weyl gauge theory (WGT) and extended Weyl gauge theory (eWGT) have differed chiefly in several ways:

- Most authors prefer to use tensors and manifolds, but some use multivectors and vector spaces.
- Gauge transformations may be active (alibi) or passive (alias).
- Physical quantities may be totally invariant or form invariant under gauge transformations.
- Gauge fields may or may not be derived from the intrinsic geometry of the fundamental spacetime, which is variously Minkowski, Riemann, teleparallel, Riemann-Cartan, Weyl-Cartan and so forth.

The widely cited book by Blagojević [**?**], advocates the use of tensors on a Minkowskian manifold and total invariance of the fields under passively interpreted gauge transformations. This may be considered a somewhat canonical approach, and was employed to first introduce eWGT to the community [**?**]. Behind the scenes, the conception of eWGT was undertaken using the same formalism as gauge theory gravity (GTG), which uses multivectors in a vector space (whose algebraic structure is Minkowskian), and total invariance under actively interpreted gauge transformations. It is thought that the passive tensor formalism (PTF) and active multivector formalism (AMF) result in the same physical predictions (modulo topological effects), but a document for translating one theory to another appears not to exist. In this work we will attempt to find a passive multivector formalism (PMF) corresponding exactly with the AMF which we can then massage through coordinates and components into a formalism which is manifestly identical to the PTF. In doing this, we will of course lose all the advantages of the AMF and gain those of PTF.

This task is not intended to provide novel physical insights, but should be of pedagogical value and serve to clarify several statements in [**? ? ?**]. Furthermore, it is a necessary step in the conversion of recent cosmological results developed in the AMF to the PTF for upcoming publication.

II. GAUGING TRANSLATIONS

A. The PTF

The tensor formalism is set in the manifold $\mathcal{M}$ with Minkowskian geometry. There is a potentially curvilinear coordinate system x^μ in this spacetime, with coordinates considered to be functions of the points of the manifold, and all fields are written as functions of the coordinates, rather than of points in the manifold. Thus the scalar field $\varphi(x)$ is always shorthand for $\varphi(x^\mu)$. From the x^μ there is defined a basis of tangent vectors $\boldsymbol{e}_\mu$ and cotangent vectors $\boldsymbol{e}^\mu$ in the uaual manner. The metric is then $\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = \gamma_{\mu\nu}$. The first gauge symmetry to consider is that of diffeomorphisms, though these are interpreted as passive general coordinate transformations (GCTs). Particularly, physical quantities should have zero *total* (as supposed to *form*) variations under GCTs. The GCT takes the form of switching to a new set of coordinates x'^μ, so the gauge invariance of the scalar field is expressed as

$$\varphi'(x') = \varphi(x). \tag{1}$$

Furthermore, we can define

$$\boldsymbol{e}'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\boldsymbol{e}^\nu, \quad \boldsymbol{e}'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\boldsymbol{e}_\nu, \quad \partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu. \tag{2}$$

B. The AMF and corresponding PMF

The multivector formalism is set in the vector space $\{x\}$ of the spacetime algebra, we do not think of this as a manifold, but the Minkowskian geometry is encoded in the algebra itself. We use x to denote a particular position vector rather than a set of coordinate functions. These are instead written as $\breve{x} = \{x^\mu\}$, though all multivector-valued fields are expressed as functions of x, for example the scalar field $\varphi(x)$. Diffeomorphism gauge invariance is interpreted more literally in this case, with an active transformation of the fields from their initial value at point x to that at a point $x' = f(x)$,

$$\varphi'(x) = \varphi(x'). \tag{3}$$

Here, $f(x)$ is an arbitrary (i.e. generally nonlinear) function of x. The closest we can get to the diffeomorphism picture is by interpreting f as a map from $\{x\}$ to $\{x'\}$. It is preferred however to stick entirely to $\{x\}$, and consider the active transformation as the 'flipping of a switch', which replaces the $\varphi(x)$ by the new function $\varphi'(x)$ at every x. The statement (3) contains all the information we need about this transformation law, but we can make an equivalent statement that is closer in spirit to (1) by defining $x'' = f^{-1}(x)$, so that

$$\varphi'(x'') = \varphi(x). \tag{4}$$

The final step is to re-cast the active transformation as a passive one. To achieve this, we can define a new set of coordinate functions $\breve{x}' = \{x'^\mu\}$, where $x'^\mu(x) = x^\mu(x'')$. To switch to a coordinate, rather than position based formalism, we adorn a field with a breve accent to refer to the equivalent function whose arguments are the coordinates. Thus for example $\breve{\varphi}(\breve{x}(x)) = \varphi(x)$ and $\breve{\varphi}(\breve{x}'(x)) = \varphi(f^{-1}(x))$. If we now write all instances of the coordinates assuming them to be evaluated at the original point x rather than x' or x'' (unless otherwise stated), we find

$$\breve{\varphi}'(\breve{x}') = \breve{\varphi}(\breve{x}), \tag{5}$$

which has exactly the same meaning as (1). Thus the active transformation of the fields in $\{x\}$ comes with a corresponding passive transformation of the fields expressed as coordinate functions at every point x. In neither picture does the underlying vector space change, rather in the active picture the fields move around across static coordinates, while in the passive the fields are unchanged and the coordinates move. The only difference between the passive picture in the multivector approach and that in the tensor approach is whether we consider points in a manifold or elements of a vector space.

This completes the PMF for the easy case of scalar fields, we must now extend to multivectors. We may again define

$$\mathsf{e}_\mu = \frac{\partial x}{\partial x^\mu}, \quad \mathsf{e}^\mu = \nabla_x x^\mu, \quad \mathsf{e}_\mu \cdot \mathsf{e}_\nu = \gamma_{\mu\nu}, \tag{6}$$

where the usual passive change of coordinate to the system $\breve{x} = \{x'^\mu\}$ invokes precisely the same transformation laws as in (2),

$$\mathsf{e}'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\mathsf{e}^\nu, \quad \mathsf{e}'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\mathsf{e}_\nu, \quad \partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu. \tag{7}$$

To establish equivalent quantities which transform actively, we first define

$$a \cdot \nabla_x f(x) = \underline{\mathsf{f}}(a; x), \tag{8}$$

where $\underline{\mathsf{f}}(a; x)$ is a linear function of a and an arbitrary function of x. Returning again to the notion of mapping vector spaces, we note that f induces the outermorphism $\underline{\mathsf{f}}$, which allows us to map vectors and indeed multivectors of arbitrary grade between $\{x\}$ and $\{x'\}$. It is easiest to begin with vectors such as $\{\mathsf{e}_\mu\}$. Noting that $x^\mu = x^\mu(x)$, we find

$$\begin{aligned} a \cdot \nabla_x x^\mu(x) &= a \cdot \nabla_x x'^\mu(f(x)) \\ &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}(x'^\mu(f(x+\epsilon a)) - x'^\mu(f(x))) \\ &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}(x'^\mu(x' + \epsilon\underline{\mathsf{f}}(a; x)) - x'^\mu(x')) \\ &= \underline{\mathsf{f}}(a; x) \cdot \nabla_{x'} x'^\mu(x') \\ &= a \cdot \bar{\mathsf{f}}(\nabla_{x'} x'^\mu(x'); x). \end{aligned} \tag{9}$$

Next, we define the vector-valued displacement gauge field which transforms actively in the following manner

$$\bar{\mathsf{h}}'(a; x) = \bar{\mathsf{h}}(\bar{\mathsf{f}}^{-1}(a; x); x'). \tag{10}$$

We can now set up four 'gravity frames', $g^\mu(x) = \bar{\mathsf{h}}(\mathsf{e}^\mu(x); x)$, which transform actively to

$$\bar{\mathsf{h}}'(\mathsf{e}^\mu(x); x) = \bar{\mathsf{h}}(\mathsf{e}'^\mu(x'); x'), \tag{11}$$

or as a forward transformation

$$\bar{\mathsf{h}}'(\mathsf{e}^\mu(x''); x'') = \bar{\mathsf{h}}(\mathsf{e}'^\mu(x); x). \tag{12}$$

Admittedly, this is still not especially clear, so we will re-cast everything in terms of the PMF

$$\breve{g}'^\mu(\breve{x}') = \frac{\partial x'^\mu}{\partial x^\nu}\breve{g}^\nu(\breve{x}). \tag{13}$$

This shows that the gravity frames $\{\breve{g}^\mu(\breve{x})\}$ and basis covectors $\{\breve{\mathsf{e}}^\mu(\breve{x})\}$ obey precisely the same passive transformation law under the GCT corresponding to the active displacement. Noting that

$$\frac{\partial f(x)}{\partial x^\mu} = \mathsf{e}_\mu \cdot \nabla_x f(x) = \underline{\mathsf{f}}(\mathsf{e}_\mu; x), \tag{14}$$

we see that if we define an actively transforming gauge field The vector-valued displacement gauge field $\underline{\mathsf{h}}$ is defined to actively transform in the following manner

$$\underline{\mathsf{f}}^{-1}(a) = a \cdot \nabla f^{-1}(x), \tag{15}$$

we can observe the following behaviour among Hestines' gravity frames:

$$\begin{aligned} \breve{g}'^\mu(\breve{x}') &= \breve{\bar{\mathsf{h}}}'(\nabla_{x''} x'^\mu; \breve{x}') = \breve{\bar{\mathsf{h}}}(\nabla_x x'^\mu; \breve{x}) \\ &= \frac{\partial x'^\mu}{\partial x^\nu}\breve{g}^\nu(\breve{x}), \\ \breve{q}'_\mu(\breve{x}') &= \breve{\underline{\mathsf{b}}}'(\partial'_\mu x''; \breve{x}') = \breve{\underline{\mathsf{b}}}(\partial'_\mu x; \breve{x}) \\ &= \frac{\partial x^\nu}{\partial x'^\mu}\breve{q}_\nu(\breve{x}). \end{aligned} \tag{16}$$

Thus we see that for any active displacement of the g^μ and q_μ, there is a corresponding passive GCT of the $\breve{g}^\mu$ and $\breve{q}_\mu$, identical to the transformation of the general coordinate basis in (7).

The rotational gauge transformation in the multivector approach is also interpreted actively

III. GAUGING ROTATIONS

IV. GAUGING DILATIONS