Part IB Physics A: Lent 2022

QUANTUM PHYSICS EXAMPLES I
MODEL ANSWERS

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1. The deflection d of the electrometer tends to zero when the photocurrent vanishes as the stopping voltage is reached. Therefore, one can use the data in the table to find the value of V_0 for each λ. One can then use the equation from the lecture notes $eV_0 = h\nu - W_{\mathrm{Na}}$ and perform a linear fit of V_0 vs. $1/\lambda$ to obtain $\hbar$ and the work function W_{Na} for Sodium.

2. Bookwork using $p = h/\lambda$, with $p = mv$ for (a), $p = E/c$ for (b), and $p = \sqrt{2mE}$ for (c).

3. The wave function is an entangled superposition of the photon being at each of the two detectors. However, the wavefunction collapses upon measurement. and the photon is detected at any given time only at *one* of the two detectors, stochastically with equal probability.

4. The time taken by the dart to reach the ground is $t = \sqrt{2H/g}$, H being the height from which it starts falling. The accuracy in the transverse direction Δx evolves in time from some initial value Δx_0 to a final value

$$\Delta x(t) = \sqrt{\Delta x_0^2 + \left(\frac{\Delta p_{x,0}}{m} t\right)^2} \, ,$$

where $\Delta p_{x,0}$ is the initial accuracy in the transverse momentum. The uncertainty principle tells us that $\Delta x_0 \, \Delta p_{x,0} \geq \hbar/2$, and therefore

$$\Delta x(t) \geq \sqrt{\Delta x_0^2 + \left(\frac{\hbar}{2\Delta x_0 m} t\right)^2} \, .$$

The right hand side can then be minimised to find $\Delta x_0 = \sqrt{\hbar t/(2m)}$ and

$$\Delta x(t) \geq \sqrt{\frac{\hbar t}{m}} \simeq \sqrt{\frac{\hbar}{m}} \left(\frac{2H}{g}\right)^{1/4} \, .$$

Notice that there are two independent transverse directions (x and y) to the vertical (z) direction along which the dart falls. Therefore, one ought to add them in quadrature to find the total transverse uncertainty ($r = \sqrt{x^2 + y^2}$):

$$\Delta r(t) = \sqrt{[\Delta x(t)]^2 +]\Delta y(t)]^2} = \sqrt{2} \, \Delta x(t) \simeq 1 \text{ mm} \, .$$

5. For a minimal uncertainty packet, $\Delta x \Delta p = \hbar/2$. This enforces a lower bound on the energy of the system, given a spatial accuracy Δx, of

$$E_0 = \frac{\hbar^2}{8m\Delta x^2} + \frac{1}{2}\kappa \Delta x^2 \,.$$

Minimising the right hand side, one finds

$$\Delta x = \left(\frac{\hbar^2}{4m\kappa}\right)^{1/4}$$

and therefore

$$E_0 = \frac{1}{2}\hbar\sqrt{\frac{\kappa}{m}} \,.$$

6. By straightforward integration: $\int_0^a |\psi(x)|^2 \, dx = 1$, $\int_0^a \int_0^b \int_0^c |\psi(x,y,z)|^2 \, dxdydz = 1$, and $4\pi \int_0^\infty |\psi(r)|^2 r^2 \, dr = 1$.

7. The wave function can be computed by Gaussian integration following the same steps as in the lecture notes (end of Chapter II). Note the difference $a^2 \to 2a^2$ in notation between the notes and the question. The integration gives

$$\psi(x,t) \propto \frac{1}{\sqrt{a^2 + i\hbar t/2m}} \exp\left[-\frac{(x - \hbar k_0 t/m)^2}{4(a^2 + i\hbar t/2m)}\right] e^{i(k_0 x - \omega_0 t)} \,.$$

In order to find the width of the wave packet, we ought to compute $|\psi(x,t)|^2$. Focusing on the exponential terms,

$$\exp\left[-\frac{x^2}{2a^2\left(1 + \frac{\hbar^2 t^2}{4m^2 a^4}\right)}\right] ,$$

we see that the width is proportional to $\sqrt{a^2(1 + \hbar^2 t^2/4m^2 a^4)}$ which is a at $t = 0$ and doubles when

$$1 + \frac{\hbar^2 t^2}{4m^2 a^4} = 4 \qquad \to \qquad t = \frac{2\sqrt{3} m a^2}{\hbar} \,.$$

Simple algebra gives the values requested in (a) and (b), see end of question sheet.

8. Given $\psi(x) = Ax\exp(-\alpha x^2)$

(a) we can calculate the normalising constant A by Gaussian integration, for example using the trick

$$\int |\psi(x)|^2 dx = |A|^2 \int x^2 e^{-2\alpha x^2} dx = -\frac{|A|^2}{2}\frac{d}{d\alpha}\int e^{-2\alpha x^2} dx = -\frac{|A|^2}{2}\frac{d}{d\alpha}\sqrt{\frac{\pi}{2\alpha}} = \frac{|A|^2\sqrt{\pi}}{4\sqrt{2\alpha^3}} = 1.$$

Therefore, $|A|^2 = 4\sqrt{2\alpha^3}/\sqrt{\pi}$.

(b) we can similarly calculate $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{3/(4\alpha)}$ (by symmetry, $\langle x \rangle = 0$). To compute $\Delta p_x = \hbar \Delta k = \hbar\sqrt{\langle k^2 \rangle - \langle k \rangle^2}$, we need to first obtain the wave function in reciprocal space by Fourier Transform:

$$\psi(k) = i\left(\frac{1}{2\pi\alpha^3}\right)^{1/4} k \exp\left(-\frac{k^2}{4\alpha}\right) ,$$

and then calculate in a similar way $\langle k \rangle = 0$ and $\langle k^2 \rangle = 3\alpha$, leading to $\Delta p_x = \hbar\sqrt{3\alpha}$.
(c) it is then trivial to show that $\Delta x \Delta p_x = 3\hbar/2$.

9. This problem is illustrated in Fig.29 in the lecture notes and discussed in the corresponding section. Let us call regions I and III the ones before and after the well ($x < 0$ and $x > a$), where particles have energy E, and region II in the well ($0 < x < a$), where particles have energy $E + V$. Correspondingly,

$$k_{\mathrm{I}} = k_{\mathrm{III}} = k = \sqrt{\frac{2mE}{\hbar^2}} \qquad k_{\mathrm{II}} = \tilde{k} = \sqrt{\frac{2m(E+V)}{\hbar^2}}\,.$$

Using plane wave solutions as in the lecture notes, in region I we have $\psi \sim e^{ikx} + re^{-ikx}$, in region II we have $\psi \sim t'e^{i\tilde{k}x} + r'e^{-i\tilde{k}x}$, and in region III we have $\psi \sim t''e^{ikx}$. Imposing b.c. at $x = 0$ and $x = a$ (continuity of the wave function and of its first derivative), after a few lines of algebra we obtain

$$\begin{cases} 1 + r = t' + r' \\ 1 - r = \frac{\tilde{k}}{k}(t' - r') \\ t'e^{i\tilde{k}a} + r'e^{-i\tilde{k}a} = t''e^{ika} \\ t'e^{i\tilde{k}a} - r'e^{-i\tilde{k}a} = \frac{k}{\tilde{k}}t''e^{ika}\,. \end{cases}$$

If we demand no reflection, $r = 0$, the first two equations give $t' = (\tilde{k} + k)/(2\tilde{k})$ and $r' = (\tilde{k} - k)/(2\tilde{k})$. The remaining two equations can be linearly combined to obtain the identity

$$t'e^{i\tilde{k}a} + r'e^{-i\tilde{k}a} - \frac{\tilde{k}}{k}\left(t'e^{i\tilde{k}a} - r'e^{-i\tilde{k}a}\right) = 0\,.$$

Substituting the values of t' and r' that we just found, this identity reduces to $e^{i2\tilde{k}a} = 1$, which is satisfied for any $\tilde{k} = n\pi/a$, for $n \in \mathbb{N}$; finally, after substituting for $\tilde{k}$, we find $E_n = \hbar^2 n^2 \pi^2/(2ma^2) - V$.
The transmission coefficient is given by $T = J^+_{\mathrm{III}}/J^+_{\mathrm{I}} = |t''|^2$. The full solution is given and sketched in the lecture notes. Incidentally, as expected, $T = 1$ when $r = 0$.
A minimum in the scattering appears at maximal transmission. If the first one occurs at $E = 0.5$ eV, then substituting this value into

$$\sqrt{\frac{2m(E + U_{\mathrm{Kr}})a^2}{\hbar^2}} = \pi$$

gives an estimate of the effective potential $U_{\mathrm{Kr}} \simeq -1.85$ eV inside the Kr atoms.

10. Following the derivation in the lecture notes, we find the energy eigenstates

$$\psi_n(x) = \begin{cases} A_n \sin[k_n(x - a/2)] & n \text{ even} \\ B_n \cos[(k_n(x - a/2)] & n \text{ odd}\,, \end{cases}$$

where $k_n = n\pi/a$, $n = 1, 2, 3, \ldots$ and $A_n = B_n = \sqrt{2/a}$. Thence

$$\begin{aligned} \langle x \rangle &= \begin{cases} \int_0^a xA_n^2 \sin^2[k_n(x - a/2)]\,dx = a/2 & n \text{ even} \\ \int_0^a xB_n^2 \cos^2[k_n(x - a/2)]\,dx = a/2 & n \text{ odd} \end{cases} \\ \langle x^2 \rangle &= \begin{cases} \int_0^a x^2A_n^2 \sin^2[k_n(x - a/2)]\,dx = \frac{a^2}{6}\left(2 - \frac{3}{n^2\pi^2}\right) & n \text{ even} \\ \int_0^a x^2B_n^2 \cos^2[k_n(x - a/2)]\,dx = \frac{a^2}{6}\left(2 - \frac{3}{n^2\pi^2}\right) & n \text{ odd}\,. \end{cases} \end{aligned} \tag{1}$$

The uncertainty is then given by

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a\sqrt{\frac{1}{12} - \frac{3}{n^2\pi^2}} \quad \xrightarrow{n\to\infty} \quad \frac{a}{\sqrt{12}}\,.$$

In the limit of $n \to \infty$, these values are consistent with the results from classical mechanics, where the probability of finding a particle between 0 and a is uniform, $P(x) = 1/a$, and it vanishes outside ($x < 0$ and $x > a$). One can straightforwardly obtain $\int xP(x)\,dx = a/2$, $\int x^2 P(x)\,dx = a^2/3$ and $(\Delta x)_{\text{classical}} = a/\sqrt{12}$.

11. Quantum mechanically, let us define region 1 ($0 < x < a$) and region 2 ($a < x < 2a$). The wave function has wave vectors $k_1 = \sqrt{2mE/\hbar^2} = 5\pi/(2a)$ and $k_2 = \sqrt{2m(E-V_0)/\hbar^2} = 3\pi/(2a)$, respectively. Requiring that the boundary conditions at $x = 0$ and $x = 2a$ are satisfied ($\psi(0) = \psi(2a) = 0$), we obtain: $\psi_1(x) = A\sin(k_1 x)$ and $\psi_2(x) = B\sin[k_2(2a - x)] = B\sin(k_2 x)$. We must finally impose the boundary conditions at $x = a$:

$$\begin{aligned} &A\sin(5\pi/2) = B\sin(3\pi/2) \\ &Ak_1\cos(5\pi/2) = -Bk_2\cos(3\pi/2)\,. \end{aligned}$$

The latter is automatically satisfied ($\cos(5\pi/2) = \cos(3\pi/2) = 0$) whereas the former requires $A = -B$ ($\sin(5\pi/2) = -\sin(3\pi/2) = 1$). Since

$$\int_0^a \sin^2(k_1 x)\,dx = \int_a^{2a} \sin^2(k_2 x)]\,dx = a/2\,, \qquad (*)$$

the correct normalisation choice is $A = 1/\sqrt{a}$ and

$$\begin{aligned} \psi(x) \quad &= \quad \begin{cases} \frac{1}{\sqrt{a}}\sin(k_1 x) & 0 < x < a \\ -\frac{1}{\sqrt{a}}\sin(k_2 x) & a < x < 2a \end{cases} \\ P_{\text{quantum}} \quad &= \quad \frac{1}{a}\int_0^a \sin^2(k_1 x)\,dx = \frac{1}{2} \end{aligned}$$

(which could have actually been argued directly from ($*$) without further calculations).

Classically, the probability $P_{\text{classical}}$ of finding the particle in region 1 ($0 < x < a$) is given by the ratio of the time t_1 spent in region 1 divided by the total time ($t_1 + t_2$, where t_2 is the time spent in region 2). The time spent in a region is given by the size of the region divided by the velocity of the particle in that region, $t_{1,2} = a/v_{1,2}$. And the velocity can be obtained from the kinetic energy, $v_{1,2} = \sqrt{2mE_{1,2}}$. Recalling that $E_1 = E \propto 25/8$ and $e_2 = E - V_0 \propto 9/8$, and noticing that all dimensional factors cancel out, after a few lines of algebra we obtain

$$P_{\text{classical}} = \frac{t_1}{t_1 + t_2} = \frac{v_2}{v_1 + v_2} = \frac{\sqrt{E - V_0}}{\sqrt{E} = \sqrt{E - V_0}} = \frac{3/(2\sqrt{2})}{5/(2\sqrt{2}) + 3/(2\sqrt{2})} = \frac{3}{8}\,.$$

12. A one-dimensional rectangular potential well of depth V_0 has width $2a$. Show that there is one and only one bound state for a particle of mass m if

$$\frac{2ma^2V_0}{\hbar^2} < \frac{\pi^2}{4}$$

Let us divide space into regions 1 ($x < -a$), 2 ($-a < x < a$), and 3 ($x > a$). A bound state exists if at least one of the energy eigenstates is negative, $E < 0$. The corresponding behaviour of the wavefunction is

$$\psi(x) = \begin{cases} A_1 \exp(\kappa x) & \text{region 1} \\ A_2 \cos(kx) & \text{region 2} \\ A_1 \exp(-\kappa x) & \text{region 1}\,, \end{cases} \qquad (**)$$

where $\kappa = \sqrt{-2mE/\hbar^2}$ and $k = \sqrt{2m(E+V_0)/\hbar^2}$ and we made use of two key observations: (i) the problem is symmetric in $x \leftrightarrow -x$, and therefore $|\psi(x)|^2 = |\psi(-x)|^2$ (in particular, this means that the term in region 2 must be proportional to either a $\sin(kx)$ or $\cos(kx)$, but not a linear combination thereof; also, $|A_3|^2 = |A_1|^2$); and (ii) the lowest energy state ought to be nodeless (which means we can focus only on the $\cos(kx)$ solution in region 2; and thence $A_3 = A_1$).
We then need to impose the correct boundary conditions (which are now identical at $x = -a$ and $x = a$):

$$\begin{aligned} A_1 \exp(-\kappa a) &= A_2 \cos(ka) \\ A_1 \kappa \exp(-\kappa a) &= A_2 k \sin(ka)\,. \end{aligned}$$

Taking the ratio of the two equations, $\kappa = k\tan(ka)$, and expressing it in terms of the dimensionless variable $x = E/V_0 \in (-1, 0)$, we obtain

$$\sqrt{\frac{-x}{1+x}} = \tan\left[\sqrt{\frac{2ma^2V_0}{\hbar^2}}\sqrt{1+x}\sqrt{\frac{2ma^2V_0}{\hbar^2}}\sqrt{1+x}\right].$$

The l.h.s. diverges for $x \to -1$ and decreases monotonically down to 0 at $x = 0$. The r.h.s. vanishes at $x = -1$ and increases monotonically up to the well-known singular discontinuity when the argument equals $\pi/2$. Therefore a solution (intersection point) always exists.
It is unique if the first excited state is unbound. In order to check this, we ought to replace cos with sin in Eq. (**), as we expect the first excited state to be odd rather than even. One could repeat the procedure above to derive the desired result. Alternatively, one can observe that the continuity of the derivative of the wave function at the boundary requires the sinusoidal function to be increasing / decreasing at $x = -a$ / $+a$, respectively, and therefore $ka > \pi/2$. The energy of the first excited state is therefore bounded from below by

$$\frac{\hbar^2\pi^2}{8ma^2} - V_0\,,$$

which is negative (i.e., the first excited state is a bound state) only if $2ma^2V_0/\hbar^2 > \pi^2/4$.

13. In region 1 ($0 < x < a$), $k = \sqrt{3mV/(2\hbar^2)}$; in region 2 ($x > a$), $\kappa = \sqrt{mV/(2\hbar^2)}$. And the wave function vanishes identically for $x < 0$. Imposing boundary conditions at $x = 0$, the wave function can be written as $\psi_1(x) = A\sin(kx)$ and $\psi_2(x) = B\exp(-\kappa x)$. We then need to impose boundary conditions at $x = a$:

$$A\sin(ka) = Be^{-\kappa a} \qquad \text{and} \qquad Ak\cos(ka) = -B\kappa e^{-\kappa a}\,.$$

Taking the ratio of the two equations, we obtain $\tan(ka) = -k/\kappa = -\sqrt{3}$; therefore, $ka = -\pi/3$ and $\sin(ka) = -\sqrt{3}/2$, $\cos(ka) = 1/2$. We can then use, say, the first of the two equations to solve for $B = A(\sqrt{3}/2)\exp(\kappa a)$ and obtain

$$P(x > a) = \frac{\int_a^\infty A^2(3/4)e^{2\kappa a}e^{-2\kappa x}\,dx}{\int_0^a A^2\sin^2(kx)\,dx + \int_a^\infty A^2(3/4)e^{2\kappa a}e^{-2\kappa x}\,dx} = \frac{3/(8\kappa)}{a/2 + \sqrt{3}/(8k) + 3/(8\kappa)}\,,$$

which gives the desired result upon multiplying numerator and denominator by $24k$ and a few algebraic steps.

14. For a one-dimensional harmonic oscillator with amplitude a, we know that $x(t) = a\sin\omega t$ and $v(t) = a\omega\cos\omega t$. The classical probability is then given by:

$$P_{\rm cl}(x)dx = \frac{dx/v}{\pi/\omega} = \frac{dx}{\pi a\cos\omega t} = \frac{dx}{\pi\sqrt{a^2 - x^2}}\,.$$

(Note that π/ω, the half period, is the time the oscillator takes to travel from $-a$ to a.)

Normalisation can be checked for example by change of variable $x = a\cos\theta$.

The quantum probability distribution corresponding to the state given in the question is:

$$P_{\rm q}(x)dx = \frac{2}{\sqrt{\pi}}\frac{x^2}{x_0^3}\,\mathrm{e}^{-x^2/x_0^2}dx\,,$$

which is remarkably different from the classical case (as a simple sketch illustrates). The correspondence principle applies only for high energy states.

15. The wave functions of the problem in question ought to be the same as a simple harmonic oscillator for $x > 0$ and vanish identically for $x \le 0$. Therefore, we can only retain the eigenstates of a conventional simple harmonic oscillator that vanish at the origin, i.e., the odd states:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \qquad\qquad n = 1, 3, 5, \ldots$$

or equivalently $n = 2n' + 1$ and $E_{n'} = (2n' + 3/2)\hbar\omega$, for $n' \in \mathbb{N}$.

16. Notes about the correspondence principle should mention that: (i) quantum mechanics ought to agree with classical mechanics wherever the latter is expected to hold; (ii) in general, this is expected to hold in the high energy / high-temperature limit, or equivalently in the limit of large quantum numbers; (iii) formally, the scale at which quantum mechanical effects become relevant is set by Planck's constant h, and classical behaviour must be recovered in the limit $h \to 0$.