W6

$\hat{A}|\psi\rangle = \pm|\psi\rangle$

$\hat{A}|u_\pm\rangle = \pm|u_\pm\rangle$

$\hat{B}|v_\pm\rangle = \pm|v_\pm\rangle \longrightarrow |v_\pm\rangle = \frac{1}{\sqrt{2}}\left(|u_+\rangle \pm |u_-\rangle\right)$

Normalisations: $\langle u_\pm|u_\pm\rangle = 1$

$\langle v_\pm|v_\pm\rangle = \frac{1}{2}\left(\langle u_+| \pm \langle u_-|\right)\left(|u_+\rangle \pm |u_-\rangle\right)$

$= \frac{1}{2}\left(1 \pm \langle u_-|u_+\rangle \pm \langle u_+|u_-\rangle \pm 1\right) = 1$

$\langle u_-|u_+\rangle = -\langle u_+|u_-\rangle$

$\hat{C} = \hat{A} + \hat{B}$

$\Downarrow$

$C^\dagger = (\hat{A} + \hat{B})^\dagger$

$= (\hat{A}^\dagger + \hat{B}^\dagger)$ $\} \rightarrow$ $\hat{A}$ & $\hat{B}$ Hermitian.

$= \hat{A} + \hat{B}$

$\therefore$ $\hat{C} = \hat{C}^\dagger$ and is Hermitian therefore is an observable.

$\hat{A} = \cancel{\tfrac{1}{\sqrt{2}}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ assuming ~~$u_+ = \begin{pmatrix}1\\0\end{pmatrix}$ & $u_- = \begin{pmatrix}0\\1\end{pmatrix}$~~ $\}$ hence.

~~$v_+ = \begin{pmatrix}1\\1\end{pmatrix}/\sqrt{2}$ & $v_- = \begin{pmatrix}1\\-1\end{pmatrix}/\sqrt{2}$~~

~~$\hat{B} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$~~

$\hat{A}\begin{pmatrix}1\\0\end{pmatrix} = +1\begin{pmatrix}1\\0\end{pmatrix}$ $\quad \hat{A} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ $\quad$ define $u_+ = \begin{pmatrix}1\\0\end{pmatrix}$ & $u_- = \begin{pmatrix}0\\1\end{pmatrix}$

$\hat{A}\begin{pmatrix}0\\1\end{pmatrix} = -\begin{pmatrix}0\\1\end{pmatrix}$

$$\hat{B}|v_+\rangle = +1|v_+\rangle$$

$$\cancel{\hat{B}\begin{pmatrix} +\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} +\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} \end{pmatrix}}$$

$$\cancel{\hat{B}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = -1\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}}$$

$$\begin{pmatrix} v_+ \\ v_- \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} u_1 + u_2 \\ u_1 - u_2 \end{pmatrix}$$

$$v_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$v_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\frac{1}{-1-1} \times \left(\frac{1}{\sqrt{2}}\right)^{-1}$$

$$\hat{B} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

hence $C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$\left|\begin{pmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix}\right| = -(1-\lambda)(1+\lambda) - 1$$

$$-1 + \lambda^2 - 1 = 0$$

$$\lambda^2 = 2$$

positive results

$$\lambda = \pm\sqrt{2}$$

$$\begin{pmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$\left.\begin{aligned} (1-\sqrt{2})a_1 + a_2 &= 0 \\ a_1 - (1+\sqrt{2})a_2 &= 0 \end{aligned}\right\} \rightarrow \frac{a_1}{a_2} = \frac{1}{\sqrt{2}-1} \checkmark$$

$$\boxed{\cancel{(1-2)a_1 + a_1 = 0} \quad \cancel{a_1 = 0}}$$

Normalise $a_1^2 + a_2^2 =$

$$a_1 = \sqrt{\tfrac{1}{2}\left(1+\tfrac{1}{\sqrt{2}}\right)}$$

$$\therefore\ a_2 = \sqrt{\tfrac{1}{2}\left(1-\tfrac{1}{\sqrt{2}}\right)}$$

$$\begin{pmatrix} 1+\sqrt{2} & 1 \\ 1 & -1+\sqrt{2} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0$$

$$\left.\begin{aligned} (1+\sqrt{2})b_1 + b_2 &= 0 \\ b_1 + (-1+\sqrt{2})b_2 &= 0 \end{aligned}\right\} \quad \frac{b_1}{b_2} = \frac{\sqrt{2}-1}{1} \checkmark$$

$$\boxed{\cancel{b_2 - (1+\sqrt{2})(-1+\sqrt{2})b_2 = 0} \quad \cancel{b_2 + (1-\sqrt{2})b_2 = 0}}$$

normalise

$$b_1 = \sqrt{\tfrac{1}{2}\left(1-\tfrac{1}{\sqrt{2}}\right)}$$

$$b_2 = -\sqrt{\tfrac{1}{2}\left(1+\tfrac{1}{\sqrt{2}}\right)}$$

~~Normalise~~

$$\therefore\ |w_\pm\rangle = \sqrt{\tfrac{1}{2}\left(1\pm\tfrac{1}{\sqrt{2}}\right)}\,|u_+\rangle \pm \sqrt{\tfrac{1}{2}\left(1\mp\tfrac{1}{\sqrt{2}}\right)}\,|u_-\rangle$$

Later part about preparation.

EXAMPLES III

1

$$\langle x^2 \rangle = \langle \psi | x^2 | \psi \rangle$$

Ehrenfest theorem: $\frac{d\langle \hat{A} \rangle}{dt} = \frac{i}{\hbar}\langle [\hat{H}, \hat{A}] \rangle + \langle \frac{d\hat{A}}{dt} \rangle$

$$= \frac{i}{\hbar}\langle \hat{H}\hat{A} - \hat{A}\hat{H} \rangle + \langle \frac{d\hat{A}}{dt} \rangle$$

$$\hat{H} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V$$

$$\hat{x} = -i\hbar \frac{\partial}{\partial x}$$

$$\hat{p} = i\hbar \frac{\partial}{\partial t}$$

2 A has $|\psi_i\rangle$ for $i = 1 \& 2$

(3) testing: $[\hat{H}, e^{i\hat{H}t}] = 0$

$$= \hat{H}e^{i\hat{H}t} - e^{i\hat{H}t}\hat{H}$$

function of operators are taylor expansion with x replaced with $\hat{x}$

$$\cancel{e^{i\hat{H}t} \approx 1 + i\hat{H}t - \frac{\hat{H}^2 t^2}{2} -}$$

$$e^{x} \simeq 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$e^{i\hat{H}t} \simeq 1 + i\hat{H}t - \frac{\hat{H}^2 t^2}{2} - \frac{i\hat{H}^3 t^3}{6} + \dots$$

which means that $e^{i\hat{H}t} = \sum_i e^{+i\underline{H_i}t} |\psi_i\rangle\langle\psi_i|$

$\rightarrow$ eigenvalues.

which has the same eigenvectors of as $\hat{H}$

$$\hat{H} = \sum_i H_i |\psi_i\rangle\langle\psi_i|$$

hence $\hat{H}e^{i\hat{H}t} = \sum_{ij} \underbrace{H_i e^{iH_i t}}_{\text{eigenvalue - constants} \therefore \text{ order doesn't matter.}} \overbrace{|\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|}^{=0 \text{ if orthonormal eigenbasis i.e. } i \neq j}$

$\rightarrow$ $i = j$ for non-zero

therefore order doesn't matter

$$= \sum_i H_i e^{iH_i t} |\psi_i\rangle\langle\psi_i| \; //$$

I think this proves they commute...