Part IB Physics A : Lent 2022

QUANTUM PHYSICS EXAMPLES IV

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1. Denote the eigenfunctions of $\widehat{L}^2$ and $\widehat{L}_z$ with eigenvalues $l = 1$ and $m_l = -1, 0, 1$ by $|\phi_{-1}\rangle, |\phi_0\rangle, |\phi_1\rangle$. Use the ladder operators $\widehat{L}_+$ and $\widehat{L}_-$ to find the eigenfunctions of $\widehat{L}_x$ in terms of those of $\widehat{L}_z$.

A beam of atoms with zero spin and in the state $l = 1$ is traveling along the y-axis and passes through an x-Stern-Gerlach apparatus. The emerging beam with $m_l = 1$ is passed through a z-Stern-Gerlach apparatus. Into how many beams is this beam further split and what are the relative numbers of atoms in them?

What happens if the other two beams from the first Stern-Gerlach apparatus are treated in the same way?

2. For a system involving two particles of spin $s_1 = \frac{1}{2}$ and $s_2 = \frac{1}{2}$, find the eigenvalues of $\widehat{\boldsymbol{s}}_1 \cdot \widehat{\boldsymbol{s}}_2$ for the "anti-parallel" spin singlet state and the "parallel" spin triplet states, and comment on your results.

3. Derive the eigenfunctions of the operators $\widehat{J}^2$ and $\widehat{J}_z$ in terms of the eigenfunctions of $\widehat{L}^2$, $\widehat{S}^2$, $\widehat{L}_z$ and $\widehat{S}_z$ for the case $l = 1, s = \frac{1}{2}$.

4. A spin-$\frac{1}{2}$ particle is in state $|\chi\rangle$, having its spin aligned (as far as possible) along a unit vector $\boldsymbol{n}$ in the (θ, ϕ) direction in spherical polar coordinates. This state $|\chi\rangle$ will be such that

$$\boldsymbol{n}.\widehat{\boldsymbol{S}}\,|\chi\rangle = (n_x\widehat{S}_x + n_y\widehat{S}_y + n_z\widehat{S}_z)\,|\chi\rangle = +\frac{1}{2}\hbar\,|\chi\rangle\,.$$

Express $|\chi\rangle$ in terms of the spin eigenstates $|\chi_\uparrow\rangle$ and $|\chi_\downarrow\rangle$ corresponding to the z-axis, and hence find the relative intensities of the two beams produced when a beam of particles in the state $|\chi\rangle$ is passed through a z-Stern-Gerlach apparatus.

5. Given that neutrons, protons and electrons are all fermions, why is ^{4}He a boson? What is ^{3}He?

6. Three non-interacting identical spin-$\frac{1}{2}$ fermions are confined in a rectangular box with edges a, a and d. Find, for the ground state of the system, how: (i) its degeneracy; (ii) its energy; and (iii) its parity with respect to the centre of the box; behave as d varies in the range $0 < d < 2a$.

7. Two identical particles are in an isotropic 3D simple harmonic potential. Show that, if the particles do not interact and there are no spin-orbit forces, the degeneracies of the three lowest energy values are 1, 12, 39 if the particles have spin $\frac{1}{2}$, and 6, 27, 99 if the particles have spin 1.

8. Write brief notes on 'indistinguishability' in quantum mechanics. Comment on its consequences, and list a number of ways in which it reveals itself in experiments.

Some review questions:

9. Write short notes on the following topics:

(a) The position of a particle is measured, and it is found to lie within a region having width Δx. The momentum is then measured, immediately afterwards, and it is found to lie within the range Δp. If the order of the measurements is changed, so that momentum is measured first and then position, do the results have to be the same?

(b) Suppose now that the position of a particle is measured, and it is found to lie within a region having width Δx, but then its position is measured again. What does quantum mechanics say about the positional uncertainty on the second measurement? For a free particle, find a lower bound estimate of the positional uncertainty as a function of time after the first measurement.

10. Observable A has eigenfunctions ψ_1 and ψ_2 with eigenvalues a_1 and a_2. Observable B has eigenfunctions χ_1 and χ_2 with eigenvalues b_1 and b_2, which can be expressed as

$$\chi_1 = (2\psi_1 + 3\psi_2)/\sqrt{13} \qquad\qquad \chi_2 = (3\psi_1 - 2\psi_2)/\sqrt{13}\,.$$

B is measured, and value b_1 is obtained. What would be the probabilities of getting a_1 and a_2 in a measurement of A immediately afterwards? After this measurement of A, B is again measured; what is the probability of getting b_1 again?

11. Consider the creation and annihilation operators of a simple harmonic oscillator, $\hat{a}$ and $\hat{a}^\dagger$, and define the new operator $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, where α is a complex number and α^* is its complex conjugate. Use the Baker–Campbell–Hausdorff formula,

$$e^{\hat{A}}\, e^{\hat{B}} = e^{\hat{A}+\hat{B}+[\hat{A},\hat{B}]/2}$$

where $\hat{A}$ and $\hat{B}$ are two generic operators whose commutator is a complex number, $[\hat{A},\hat{B}] \in \mathbb{C}$, to show that

$$\hat{D}(\alpha) = e^{-|\alpha|^2/2}\, e^{\alpha\hat{a}^\dagger}\, e^{-\alpha^*\hat{a}}$$

$$\hat{D}(\alpha+\beta) = \hat{D}(\alpha)\,\hat{D}(\beta)\, e^{i\,\mathrm{Im}(\alpha^*\beta)}\,,$$

where β is also a complex number.

12. A *coherent state* of the simple harmonic oscillator is defined as $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$, where $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ – see the earlier problem about this operator – and $|0\rangle$ is the ground state. Show that

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,,$$

where $|n\rangle$ are the eigenstates of the SHO. Demonstrate also that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, and show that the coherent state satisfies the minimum uncertainty relation.

13. [*This question is more challenging and optional.*] Define the operator $\hat{S}(r) = \exp\{r\,[(\hat{a}^\dagger)^2 - \hat{a}^2]/2\}$, where $\hat{a}$ and $\hat{a}^\dagger$ are the creation and annihilation operators of a simple harmonic oscillator, and r is a real number. Use the relationship

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \dots \,,$$

together with the fact that $\hat{S}^\dagger(r)\hat{S}(r) = \hat{\mathbb{I}}$, to how that the so called *squeezed state*, $|r\rangle = \hat{S}(r)|0\rangle$, satisfies the uncertainty principle with minimum uncertainty.

14. Consider the Hamiltonian of a spinless particle of mass m in a quartic potential, $\hat{H} = \hat{p}^2/(2m) + \alpha(\hat{x}^4 + \hat{y}^4 + \hat{z}^4)$, where $\alpha > 0$. Demonstrate that the energy is conserved, and also that the Hamiltonian is invariant under rotation about the x-axis by an angle $\pi/2$.
[Hint: the desired rotation can be expressed in operator form as $\hat{D}_x(\pi/2) = \exp[-i\hat{L}_x(\pi/2)/\hbar]$, where $\hat{L}_x$ is the x-component of the angular momentum operator. The Hamiltonian is then invariant if $\hat{D}_x^\dagger \hat{H}\hat{D}_x = \hat{H}$]

ANSWERS:

1. $\frac{1}{\sqrt{2}}(|\phi_1\rangle - |\phi_{-1}\rangle)$; $\frac{1}{2}(|\phi_1\rangle + |\phi_{-1}\rangle \pm \sqrt{2}\,|\phi_0\rangle)$. Split into 3 beams of relative intensities 1:2:1.

2. Singlet: $-\frac{3}{4}\hbar^2$. Triplet: $\frac{1}{4}\hbar^2$.

2. With the notation $\left|\Psi_{l,s,j,m_j}\right\rangle$, $|\phi_{l,m_l}\rangle$, $|\chi_{s,m_s}\rangle$:

$$\left|\Psi_{1,\frac{1}{2},\frac{3}{2},\frac{3}{2}}\right\rangle = |\phi_{1,1}\rangle \left|\chi_{\frac{1}{2},\frac{1}{2}}\right\rangle$$
$$\left|\Psi_{1,\frac{1}{2},\frac{3}{2},\frac{1}{2}}\right\rangle = \frac{1}{\sqrt{3}}\left[\sqrt{2}\,|\phi_{1,0}\rangle \left|\chi_{\frac{1}{2},\frac{1}{2}}\right\rangle + |\phi_{1,1}\rangle \left|\chi_{\frac{1}{2},-\frac{1}{2}}\right\rangle\right]$$
$$\left|\Psi_{1,\frac{1}{2},\frac{3}{2},-\frac{1}{2}}\right\rangle = \frac{1}{\sqrt{3}}\left[|\phi_{1,-1}\rangle \left|\chi_{\frac{1}{2},\frac{1}{2}}\right\rangle + \sqrt{2}\,|\phi_{1,0}\rangle \left|\chi_{\frac{1}{2},-\frac{1}{2}}\right\rangle\right]$$
$$\left|\Psi_{1,\frac{1}{2},\frac{3}{2},-\frac{3}{2}}\right\rangle = |\phi_{1,-1}\rangle \left|\chi_{\frac{1}{2},-\frac{1}{2}}\right\rangle$$
$$\left|\Psi_{1,\frac{1}{2},\frac{1}{2},\frac{1}{2}}\right\rangle = \frac{1}{\sqrt{3}}\left[\sqrt{2}\,|\phi_{1,1}\rangle \left|\chi_{\frac{1}{2},-\frac{1}{2}}\right\rangle - |\phi_{1,0}\rangle \left|\chi_{\frac{1}{2},\frac{1}{2}}\right\rangle\right]$$
$$\left|\Psi_{1,\frac{1}{2},\frac{1}{2},-\frac{1}{2}}\right\rangle = \frac{1}{\sqrt{3}}\left[|\phi_{1,0}\rangle \left|\chi_{\frac{1}{2},-\frac{1}{2}}\right\rangle - \sqrt{2}\,|\phi_{1,-1}\rangle \left|\chi_{\frac{1}{2},\frac{1}{2}}\right\rangle\right].$$

4. $|\chi\rangle = \cos(\theta/2)\, \mathrm{e}^{-i\phi/2}\, |\chi_\uparrow\rangle \;+\; \sin(\theta/2)\, \mathrm{e}^{i\phi/2}\, |\chi_\downarrow\rangle$; intensities $\cos^2(\theta/2)$, $\sin^2(\theta/2)$. [Note that this is not inconsistent with earlier statements about the wavefunction being unchanged by a rotation $\phi \to \phi + 2\pi$ since this referred to the particle's co-ordinates; here ϕ refers to the co-ordinates of the SG experimental set-up.]

6. For $0 < d < a$: $E = (\hbar^2\pi^2/2m)(9/a^2 + 3/d^2)$, fourfold degenerate, parity odd.

For $a < d < 2a$: $E = (\hbar^2\pi^2/2m)(6/a^2 + 6/d^2)$, twofold degenerate, parity odd.

For $d = a$: $E = (\hbar^2\pi^2/2m)(12/a^2)$, sixfold degenerate, parity odd.

10. 4/13; 9/13; 97/169.