Part IB Physics : Lent 2022

QUANTUM PHYSICS EXAMPLES II

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1. A particle of mass m is confined by the potential:

$$\begin{aligned} V(x) &= 0 && 0 < x < a \\ V(x) &= V_0 && a < x < 2a \\ V(x) &= \infty && \text{elsewhere} \end{aligned}$$

where $V_0 = 2\hbar^2\pi^2/ma^2$. If the particle's energy is $25\hbar^2\pi^2/8ma^2$, what is the probability of finding the particle in the interval $0 < x < a$: (a) quantum mechanically; (b) classically?

2. A one-dimensional rectangular potential well of depth V_0 has width $2a$. Show that there is one and only one bound state for a particle of mass m if

$$\frac{2ma^2V_0}{\hbar^2} < \frac{\pi^2}{4}$$

3. A particle is bound in a one-dimensional potential well:

$$\begin{aligned} V(x) &= \infty && x < 0 \\ V(x) &= -V < 0 && 0 < x < a \\ V(x) &= 0 && x > a \end{aligned}$$

in the lowest energy state with total energy $-V/4$.

Show that the probability that the particle is outside the attractive part of the well is

$$\frac{9\sqrt{3}}{8\pi + 12\sqrt{3}}$$

4. For a one-dimensional harmonic oscillator oscillating with amplitude a, show that the probability of finding the particle in the interval x to $x + \mathrm{d}x$ is, according to classical mechanics,

$$\begin{aligned} P_{\mathrm{cl}}(x)\,\mathrm{d}x &= \frac{1}{\pi\sqrt{a^2 - x^2}}\,\mathrm{d}x; && |x| < a \\ &= 0 && |x| > a\,. \end{aligned}$$

With the aid of sketches compare this probability with the quantum mechanical one for the $n = 1$ eigenstate with normalised eigenfunction

$$\psi_1(x) = \frac{\sqrt{2}}{\pi^{1/4}}\frac{x}{x_0^{3/2}}\,\mathrm{e}^{-x^2/2x_0^2}$$

where $x_0 = \sqrt{\hbar/m\omega}$.

(Check the normalization of the classical distribution.)

5. Find, by inspecting the wave functions of a quantum simple harmonic oscillator, the energy eigenvalues of a particle of mass m moving in the potential:

$$\begin{aligned} V(x) &= \infty & \qquad x &\leq 0 \\ V(x) &= m\omega^2 x^2/2 & \qquad x &> 0 \ . \end{aligned}$$

6. Write a few brief notes on the *correspondence principle*, and discuss these with your supervisor.

7. Consider the following operations, which act on $f(x)$ as described below, where c is a constant:

(a) $cf(x)$ – vertical scaling;
(b) $f(x) + c$ – vertical displacement;
(c) $f^2(x)$ – squaring;
(d) $\mathrm{d}f/\mathrm{d}x$ – differentiation;
(e) $g(x)f(x)$ – multiplication by a function;
(f) $f(\mathrm{d}f/\mathrm{d}x)$;
(g) $\mathrm{d}^2 f/\mathrm{d}x^2$ – double differentiation;
(h) $f(cx)$ – horizontal scaling;
(i) $\sin f(x)$;
(j) $f(-x)$ – inversion.

Which of these operations are linear?

What are the eigenfunctions of the operations that are linear? (Note: some may not be normalizable.)

8. Which of the following operators are Hermitian, given that $\widehat{A}$ and $\widehat{B}$ are Hermitian?

$$\widehat{A} + \widehat{B} \qquad c\widehat{A} \qquad \widehat{A}\widehat{B} \qquad \widehat{A}\widehat{B} + \widehat{B}\widehat{A}$$

Show that in one dimension, for functions that tend to zero as $x \to \pm\infty$, the operator $\mathrm{d}/\mathrm{d}x$ is not Hermitian, but the operator $-i\hbar\mathrm{d}/\mathrm{d}x$ is Hermitian. Is the operator $\mathrm{d}^2/\mathrm{d}x^2$ Hermitian?

9. Show that any non-Hermitian operator $\widehat{A}$ can be written as a linear combination of two Hermitian operators.

10. Show that, in one dimension, the state functions e^{-x^2}, $x\mathrm{e}^{-x^2}$ and $(4x^2 - 1)\mathrm{e}^{-x^2}$ are mutually orthogonal.

11. ϕ_1 and ϕ_2 are normalised eigenfunctions of observable A which are degenerate, and hence not necessarily orthogonal. If $\langle \phi_1 | \phi_2 \rangle = c$ and c is real, find linear combinations of ϕ_1 and ϕ_2 which are normalised and orthogonal to: (a) ϕ_1; (b) $\phi_1 + \phi_2$.

12. A space-domain wave function $\psi(x)$ is shifted by x_0 to give a new wave function $\psi(x - x_0)$. Calculate the corresponding momentum-domain operator. Show that the momentum-domain wave function remains normalised even after the operator has been applied.

13. For a certain system, the observable A has eigenvalues ± 1, with corresponding eigenfunctions u_+ and u_-. Another observable B also has eigenvalues ± 1, but the corresponding eigenfunctions are:

$$v_+ = (u_+ + u_-)/\sqrt{2} \qquad\qquad v_- = (u_+ - u_-)/\sqrt{2}$$

Show that $C \equiv A + B$ is an observable and find the possible results of a measurement of C.
Find the probability of obtaining each result when a measurement of C is performed on an atom in the state u_+, and express the corresponding eigenstates $w_\pm$ of the system immediately after the measurement in terms of u_+ and u_-.

14. By writing $\widehat{x}$ and $\widehat{p}$ in terms of the raising and lowering operators $\widehat{a}^\dagger$ and $\widehat{a}$, prove that, for the n^{th} excited state of a one-dimensional harmonic oscillator, $\Delta x \Delta p = (n + \frac{1}{2})\hbar$.

ANSWERS:

1. (a) 1/2; (b) 3/8.

5. $E_n = \hbar\omega(2n + \frac{3}{2})$, $n = 0, 1, 2, 3, \dots$.

7. (a) any $f(x)$; (d) $\mathrm{e}^{\alpha x}$; (e) $\delta(x - x_0)$; (g) $\mathrm{e}^{\alpha x}$ or $\cos(kx + \phi)$; (h) constant or x^b; (j) $f(x) = \pm f(-x)$.

8. The following are Hermitian: $\widehat{A} + \widehat{B}$; $c\widehat{A}$ if c is real; $\widehat{A}\widehat{B}$ if $[\widehat{A}, \widehat{B}] = 0$; $\widehat{A}\widehat{B} + \widehat{B}\widehat{A}$; $\mathrm{d}^2/\mathrm{d}x^2$.

11. (a) $\dfrac{c\phi_1 - \phi_2}{\sqrt{1 - c^2}}$; (b) $\dfrac{\phi_1 - \phi_2}{\sqrt{2(1 - c)}}$.

15. $C = \pm\sqrt{2}$, with probabilities $\dfrac{(2 \pm \sqrt{2})}{4}$. And $w_\pm = \sqrt{\frac{1}{2}\left(1 \pm \frac{1}{\sqrt{2}}\right)}u_+ \pm \sqrt{\frac{1}{2}\left(1 \mp \frac{1}{\sqrt{2}}\right)}u_-$.