Large Scale Structure and Galaxy Formation

Lecture 2

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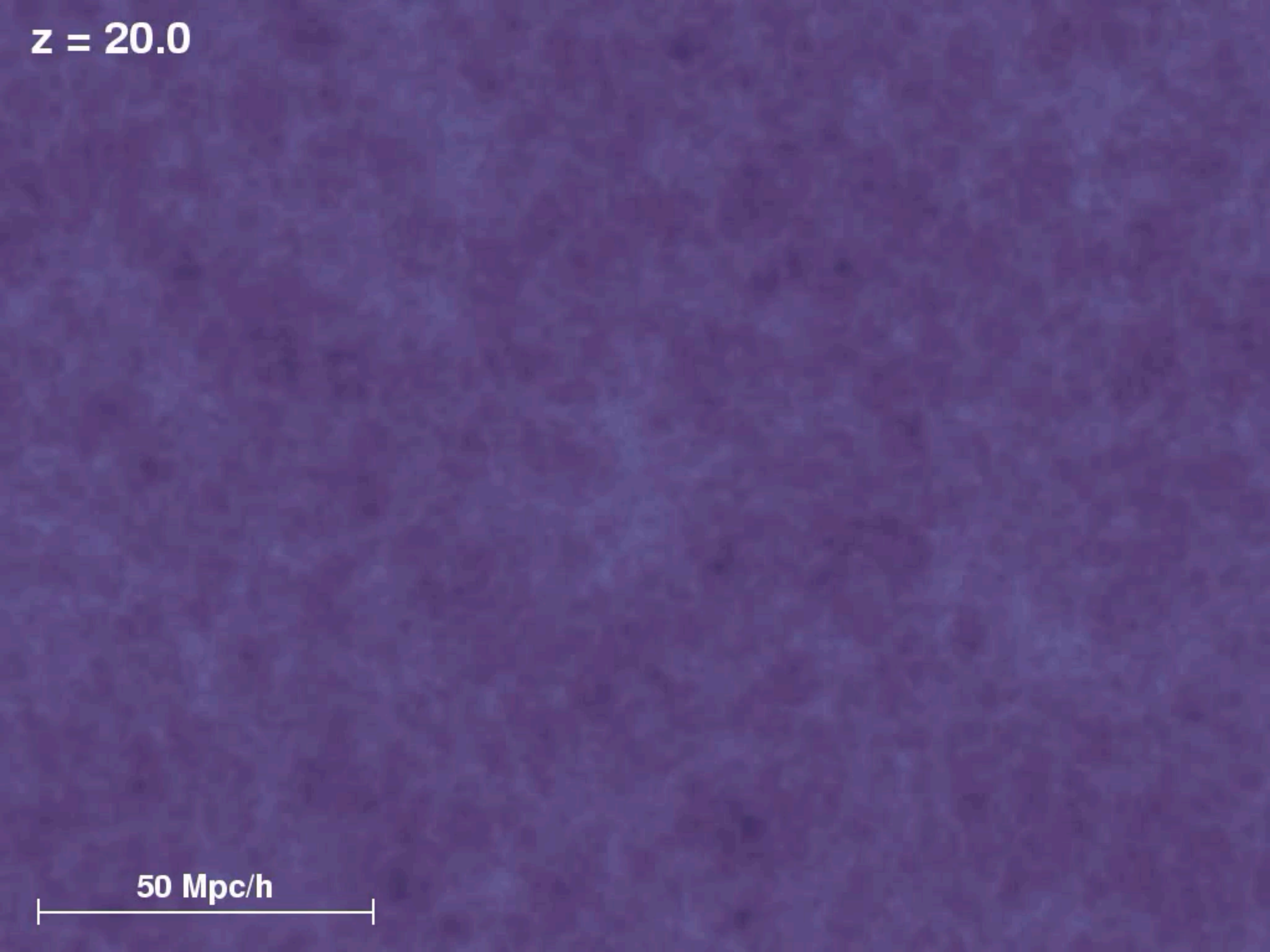
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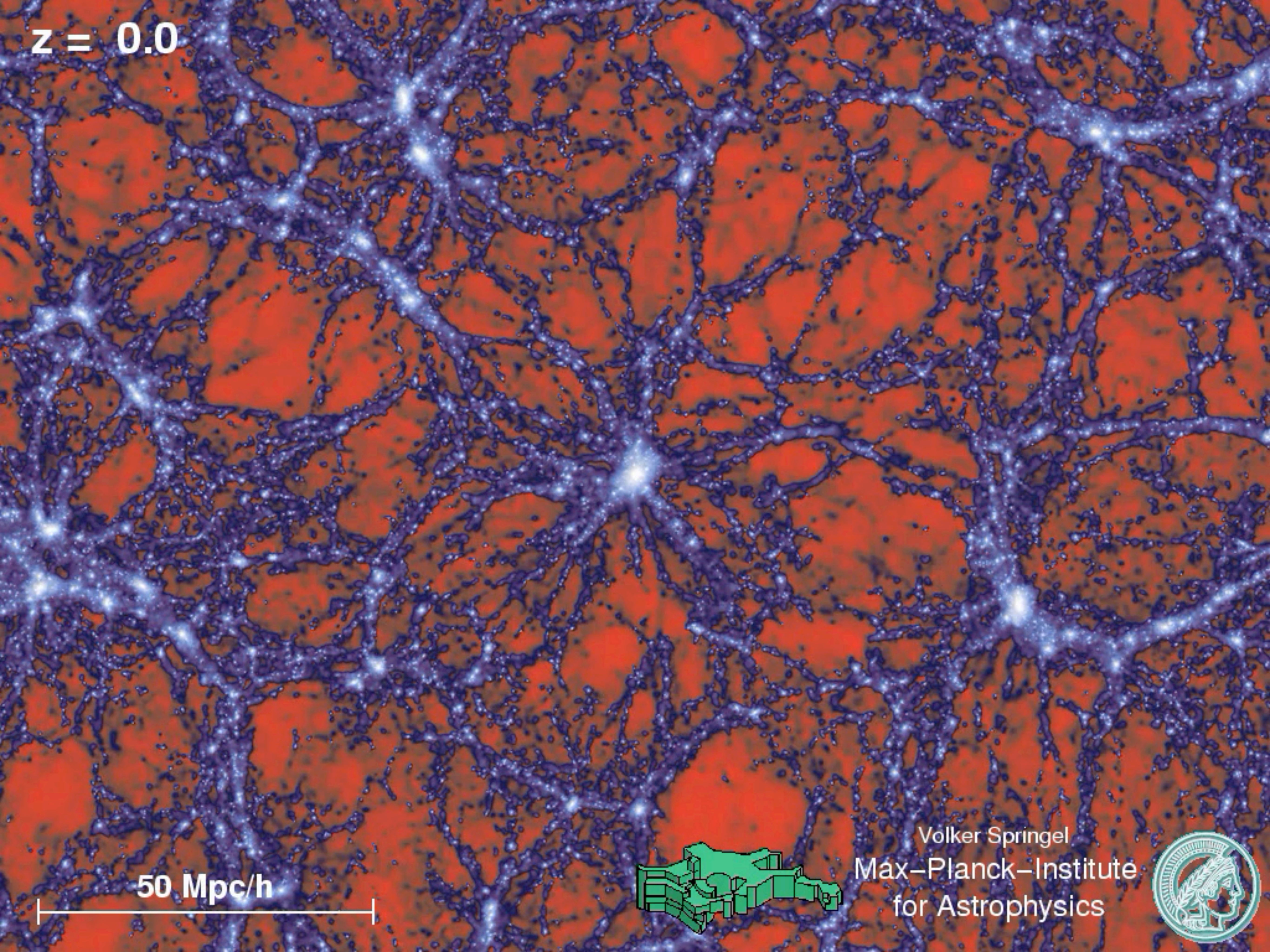
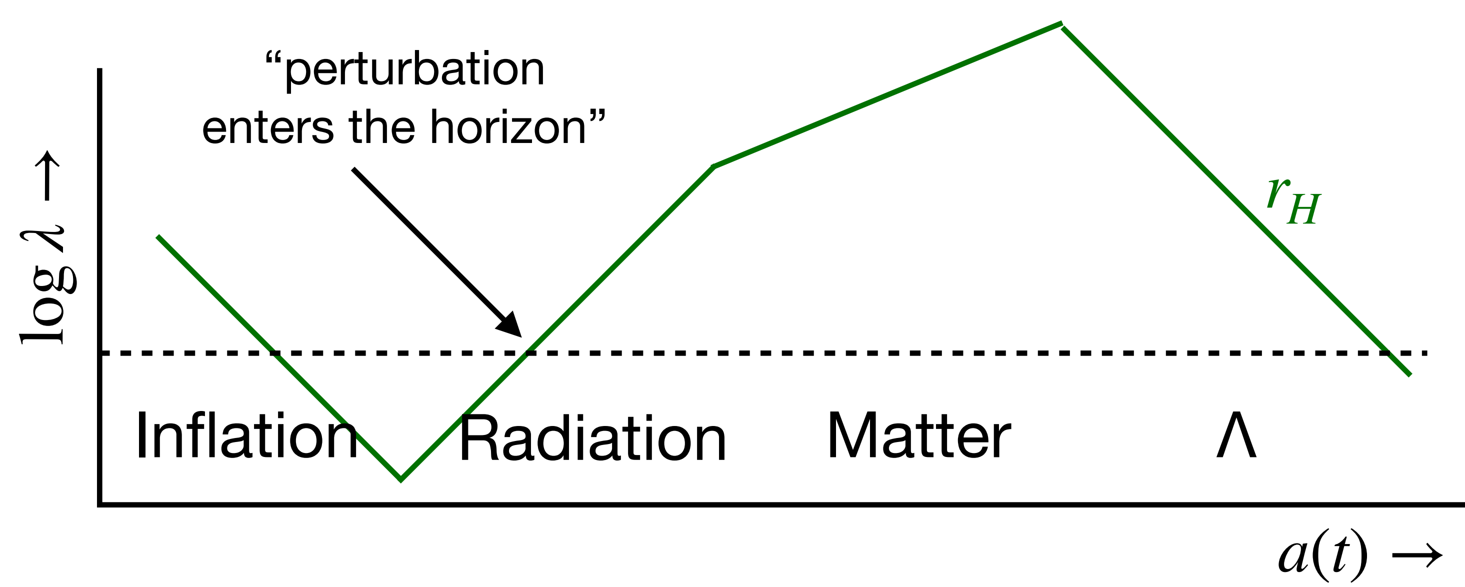
50 Mpc/h


z = 0.0
50 Mpc/h
Volker Springel
Max–Planck–Institute
for Astrophysics

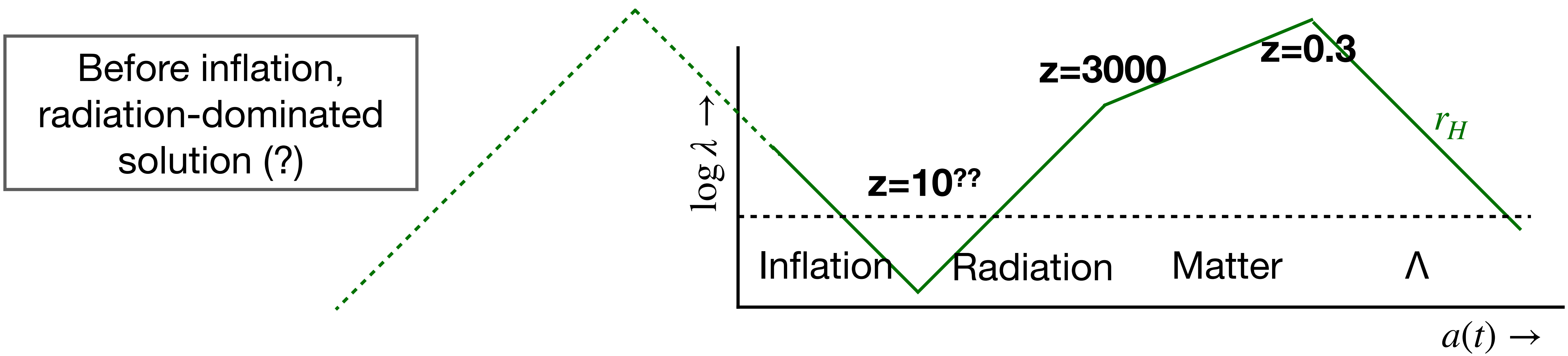
Growth of perturbations

- Inhomogeneities in the Universe can grow through gravitational instability
- The '***Hubble radius***' has a key role in how perturbations grow
- Defined as radius $ca/\dot{a}$ (i.e. distance at which Hubble velocity=c)
  - Perturbations with size larger than the Hubble radius can only evolve very slowly
  - Perturbations smaller than the Hubble radius can grow/damp faster
- Consider <u>co-moving</u> Hubble radius $c/\dot{a}$, remember $a(t) \propto t^{2/(3+3w)}$ so $\dot{a} \propto a/t \propto a \,.\, a^{-(3+3w)/2} = a^{-(1+3w)/2}$
  - then

$$r_H \propto \begin{cases} a^{-1} & - \textbf{ inflation } (w = -1) \\ a & - \textbf{ radiation } (w = 1/3) \\ a^{1/2} & - \textbf{ matter } (w = 0) \\ a^{-1} & - \textbf{ dark energy } (w = -1) \end{cases}$$



Growth of perturbations



- Co-moving horizon size at matter/radiation equality ~ 100 Mpc
  - All <100Mpc structure we see today entered during the radiation era
  - ‘initial’ fluctuations were set during the inflation era

- We will see that perturbations grow at different rates during radiation epoch and matter epoch

Spherical super-horizon perturbations

- “Universe within a Universe”
- Assume background zero curvature Universe, with $a_1(t), \rho_1(t)$ and containing a spherical region with a different (higher) density
  - We can treat this inner region as a different (curved!) solution of the Friedmann eq’s

$$\dot{a}_1^2 = \frac{8\pi G}{3}\rho_1 a_1^2 \quad \text{background}$$

$$(a_1 \dot{+} \delta a)^2 = \frac{8\pi G}{3}(\rho_1 + \delta\rho)(a_1 + \delta a)^2 - \delta K \quad \text{Const.}$$

$$\Rightarrow 2\dot{a}_1\dot{\delta a} = \frac{8\pi G}{3}\left(a_1^2\delta\rho + 2a_1\rho_1\delta a\right) - \delta K \qquad \text{to 1st order}$$

  - Now look for power-law solutions $\delta a(t) \propto t^x, \delta\rho \propto t^y$, given a power-law background solution $a_1(t) \propto t^u, \rho_1 \propto t^\nu$:
  - Find $(u-1) + (x-1) = 0; \qquad 2u + y = 0; \qquad u + v + x = 0$
  - Hence $v = -2; \quad x = 2 - u; \quad y = -2u$
  - So finally $\delta\rho/\rho \propto t^{y-\nu} = t^{-2u+2} = \begin{cases} t & \textbf{if } u = 1/2, \textbf{ radiation domination} \\ t^{2/3} & \textbf{if } u = 2/3, \textbf{ matter domination} \end{cases}$

Sub-horizon fluctuations

- Previous analysis also valid for smaller volumes
- But other processes may oppose growth:
  - Hubble expansion of dominant component
  - Free streaming of matter
  - Effective pressure of collapsed system (virialization)
    - this will be discussed later in the course

- Formulate linearised perturbation equations
  - Use Newtonian dynamics
  - First for static background
  - Then for expanding background

Newtonian density perturbations (static)

ρ: density
v: velocity
p: pressure
ϕ: grav. potential

- Conservation of mass, momentum; Poisson eq.:

$$\begin{aligned} \dot{\rho} + \nabla \cdot \rho \mathbf{v} &= 0 \\ \dot{\mathbf{v}} + \mathbf{v} \cdot \nabla \mathbf{v} + (1/\rho)\nabla p + \nabla \phi &= 0 \\ \nabla^2 \phi - 4\pi G \rho &= 0 \end{aligned}$$

- Static background solution has v=0, ρ=const, const. gravitational field *(NB inconsistent with Poisson eq - correct treatment needs GR)*

- Perturb around this: $\rho = \rho_0 + \delta\rho, \quad \mathbf{v} = \mathbf{0} + \delta\mathbf{v}, \quad \delta p = c_s^2 \delta\rho$

$$\begin{aligned} \dot{\delta\rho} + \rho_0 \nabla \cdot \delta\mathbf{v} &= 0 \\ \dot{\delta\mathbf{v}} + (c_s^2/\rho_0)\nabla\delta\rho + \nabla\delta\phi &= 0 \\ \nabla^2 \delta\phi - 4\pi G \delta\rho &= 0 \end{aligned}$$

- Dispersion relation for wave-like solutions: →

$$\begin{aligned} \delta\rho &= \rho_0 \delta_0 e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)} \\ \delta\mathbf{v} &= \mathbf{V} e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)} \\ \delta\phi &= \Phi e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)} \end{aligned}$$

$$\omega^2 - c_s^2 k^2 + 4\pi G \rho_0 = 0$$

(Aside: dispersion relation)

- Dispersion relation is the condition on frequency and wavenumber that is necessary to allow non-zero amplitude waves:

- Substitute wave-like solutions

$$\begin{aligned} \delta\rho &= \rho_0 \delta_0 e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)} \\ \delta\mathbf{v} &= \mathbf{V} e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)} \\ \delta\phi &= \Phi e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)} \end{aligned}$$

(Note we write overdensity $\delta \equiv \delta\rho/\rho$)

into dynamical eqs.

$$\begin{aligned} \dot{\delta\rho} + \rho_0 \nabla \cdot \delta\mathbf{v} &= 0 \\ \dot{\delta\mathbf{v}} + (c_s^2/\rho_0)\nabla\delta\rho + \nabla\delta\phi &= 0 \\ \nabla^2\delta\phi - 4\pi G\delta\rho &= 0 \end{aligned}$$

with $\partial/\partial t \rightarrow i\omega$ etc.

This results in a matrix equation $\begin{pmatrix} i\omega & i\mathbf{k}\cdot & 0 \\ c_s^2 i\mathbf{k} & i\omega & i\mathbf{k} \\ -4\pi G\rho_0 & 0 & -k^2 \end{pmatrix} \begin{pmatrix} \delta_0 \\ \mathbf{V} \\ \Phi \end{pmatrix} = 0$

which only has non-trivial solutions if the determinant is zero.

Newtonian density perturbations (static)

- Dispersion relation for wave-like solutions:

$$\omega^2 - c_s^2 k^2 + 4\pi G\rho_0 = 0$$

- This gives the relation between frequency ω and wavenumber k for density waves that can propagate in a static nearly uniform self-gravitating medium.

- Simplest case, where self-gravity is unimportant ($G\rho_0 \ll \omega^2$): pressure waves (i.e., sound), for which $\dfrac{\omega}{k} = c_s$, as expected.

Newtonian density perturbations: Jeans mass

$$\omega^2 - c_s^2 k^2 + 4\pi G\rho_0 = 0 \qquad \delta_0,\ \delta\mathbf{v} \propto e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)}$$

- Perturbations are unstable (hence grow) when $\omega^2 < 0$

  i.e., when the wavelength exceeds the *Jeans Length* L_J

$$\lambda = \frac{2\pi}{k} > c_s \left(\frac{\pi}{G\rho_0}\right)^{1/2} \equiv L_J$$

- (Simply said: pressure waves too slow to escape collapse)

- Associated *Jeans Mass* $\quad M_J = \frac{4\pi}{3}\left(\frac{L_J}{2}\right)^3 \rho_0 \sim \frac{c_s^3}{G^{3/2}\rho_0^{1/2}}$

  - M_J is the smallest mass that can collapse in the medium
  - largest scales ($k = 0$) always unstable - “gravity always wins”

Newtonian treatment II: expanding background

- Homogeneously expanding solution with expansion factor $a(t)$, as function of comoving coordinates $\mathbf{r}$:

$$\begin{aligned} \rho_{bg} &= \rho_0 a^{-3} \\ \mathbf{v}_{bg} &= \dot{a}\mathbf{r} \\ \phi_{bg} &= \frac{2}{3}\pi G \rho_{bg} a^2 r^2 \\ p_{bg} &= p(\rho_{bg}) \end{aligned}$$

- Perturb the conservation eqns by setting

$$\begin{aligned} \rho(\mathbf{r},t) &= \rho_{bg} + \delta\rho(\mathbf{r},t) \\ \mathbf{v}(\mathbf{r},t) &= \mathbf{v}_{bg} + \delta\mathbf{v}(\mathbf{r},t) \\ \phi(\mathbf{r},t) &= \phi_{bg} + \delta\phi(\mathbf{r},t) \end{aligned}$$

- and note that in comoving coordinates $\mathbf{r} \equiv \mathbf{x}/a(t)$ the time derivatives and gradients become

$$\left(\frac{\partial}{\partial t}\right)_x = \left(\frac{\partial}{\partial t}\right)_r - \frac{\dot{a}}{a}\mathbf{r}\cdot\nabla \qquad \nabla_x = a\nabla_r$$

- NB: note non-relativistic treatment means background **v** and ϕ are not homogeneous!

Newtonian treatment II: expanding background

- **Mass conservation**: $\dot{\rho} + \nabla \cdot (\rho \mathbf{v}) = 0$ becomes

$$\left(\frac{\partial}{\partial t} - \frac{\dot{a}}{a}\mathbf{r} \cdot \nabla\right)\left[\rho_{bg}(t)(1 + \delta(\mathbf{r}, t))\right] + \frac{\rho_{bg}}{a}\nabla \cdot \left[(1 + \delta(\mathbf{r}, t))(\dot{a}\mathbf{r} + \mathbf{v}(\mathbf{r}, t))\right] = 0$$

- Order 0 terms cancel as $\dot{\rho}_{bg} = -3(\dot{a}/a)\rho_{bg}$
- Remaining terms:

$$\boxed{\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla \cdot \left[(1 + \delta)\mathrm{v}\right] = 0}$$

- **Momentum conservation**: $\dot{\mathbf{v}} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \phi - (1/\rho)\nabla p$ becomes

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} + \frac{\dot{a}}{a}\mathbf{v} + \frac{1}{a}\mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{a}\nabla \delta\phi - \frac{1}{\rho_{bg}}\frac{1}{a}\nabla p}$$

- **Poisson equation**: $\nabla^2 \phi = 4\pi G \rho$ becomes

$$\boxed{\nabla \delta\phi = 4\pi G \rho_{bg} a^2 \delta}$$

Newtonian treatment II: expanding background

- Keeping only first-order terms in δ and $\mathbf{v}$:

$$\dot{\delta} + \frac{1}{a}\nabla \cdot \mathbf{v} = 0 \qquad (1)$$

$$\dot{\mathbf{v}} + \frac{\dot{a}}{a}\mathbf{v} + \frac{1}{a}\nabla\delta\phi + \frac{1}{\rho_{bg}}\frac{1}{a}\nabla p = 0 \qquad (2)$$

- Take $\nabla \cdot (2)$, use $\partial/\partial t(1)$ to eliminate $\nabla \cdot \mathbf{v}$ and Poisson for $\delta\phi$:

$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} - \frac{1}{\rho_{bg}}\frac{1}{a^2}\nabla^2 p - 4\pi G\rho_{bg}\delta = 0$$

- Finally, write pressure as $p(\rho)$ with $dp/d\rho = c_s^2$ (sound speed):

$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} - c_s^2\frac{1}{a^2}\nabla^2\delta - 4\pi G\rho_{bg}\delta = 0$$

- This equation tells us the evolution of small-amplitude density perturbations in expanding background. Note damping effect of $\dot{a}$ term - 'Hubble friction'.

Newtonian treatment II: expanding background

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} - c_s^2 \frac{1}{a^2}\nabla^2 \delta - 4\pi G \rho_{bg} \delta = 0$$

- Now look for wave-like solutions $\delta(\mathbf{r}, t) = \delta_k(t) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}$:
  - note: time-dependent coefficients, so we cannot have $\mathrm{e}^{i\omega t}$!

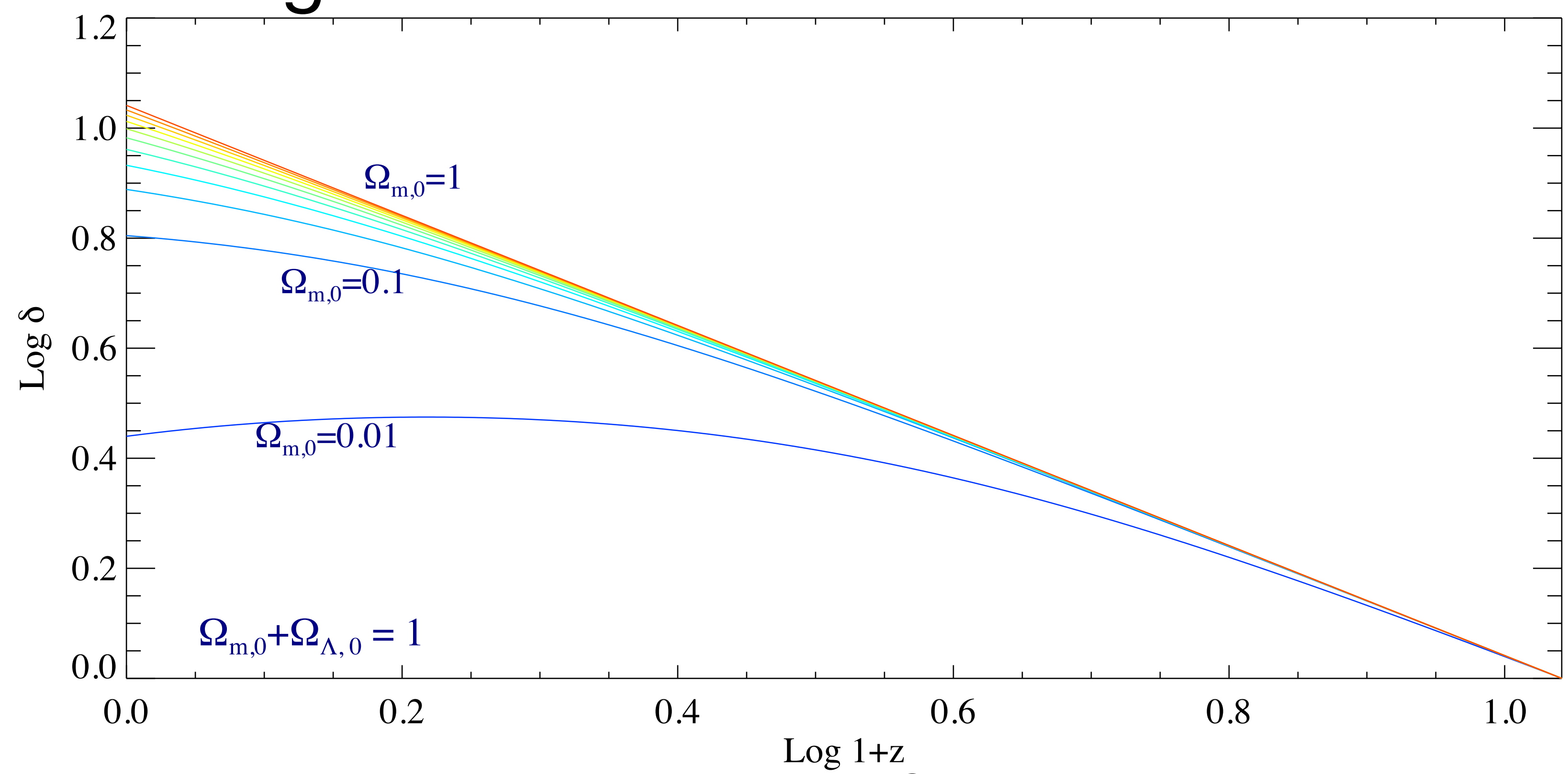
$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k + \left(\frac{k^2 c_s^2}{a^2} - 4\pi G \rho_{bg}\right)\delta_k = 0$$

- Generalization of the static case derived earlier ✔
- Gives evolution of density perturbation waves of wavenumber k as long as $\delta \ll 1$.
- Velocity perturbations: $\mathbf{v}(\mathbf{r}, t) = \mathbf{V}(t)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}$ gives $\dot{\delta}_k + \frac{1}{a} i\mathbf{k}\cdot\mathbf{V} = 0$, so only longitudinal motions along $\mathbf{k}$ affect the density.
- To solve this equation requires prescription of the background: $a(t)$, $\rho_{\rm bg}(t)$, $c_s(t)$.

Linear perturbations

$$\boxed{\ddot{\delta} + 2(\dot{a}/a)\dot{\delta} + (c_s^2 k^2/a^2 - 4\pi G\rho_{\rm bg})\delta = 0}$$

- Pressureless case: $c_s = 0$
  - No k dependence, all modes grow at same rate!
  - Numerical solutions:
    - (for flat universe)



  - Generally requires a numerical integration, but for matter-dominated expansion we had

$$\rho_{\rm bg}(t) = \frac{1}{6\pi G t^2}; \qquad a(t) \propto t^{2/3}$$

- So putting $\delta(t) \propto t^x$ we find $x = -1, \dfrac{2}{3}$: i.e., growing mode $\propto a(t)$
  - Density perturbations grew by factor x3000 since t_{eq} !

Linear perturbations

$$\boxed{\ddot{\delta} + 2(\dot{a}/a)\dot{\delta} + (c_s^2 k^2/a^2 - 4\pi G\rho_{\rm bg})\delta = 0}$$

- ‘Pressure is important’ case: $c_s > 0$, large k (short waves)
  - Evolution of the amplitude depends on k
  - Short waves oscillate (slowly damped, slowly decreasing freq.)
  - Generally requires a numerical integration. Approx., for large k/a:

$$\delta(t) \sim \mathrm{e}^{-H(t)} \sin[c_s k t/a(t)]$$

    - valid as long as self-gravity term remains small, and the frequency $c_s k/a \gg H(t), \dot{H}/H$
    - relativistic gas $c_s^2 \sim a^0$, $\rho_{\rm bg} \sim a^{-4}$ so growth → oscillations
    - cold gas $c_s^2 \sim a^{-2}$, $\rho_{\rm bg} \sim a^{-3}$ so oscillations → growth

  - While oscillating, the perturbations do not grow

Newtonian perturbations?

- Note that we have done a Newtonian treatment. This can only be justified for sub-horizon fluctuations, with wavelengths $< ct$
  - (longer wavelengths would imply causal behaviour beyond horizon scale)

- Another thing to note is that these growing modes are power-law and not exponential. This much slower growth is due to the damping effect ('Hubble friction') of the expansion of space.

Matter-radiation equality

- At early times radiation dominates the energy density of the Universe
- Friedmann eq. $$H(t)^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = H_0^2(\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_\Lambda + \Omega_k a^{-2})$$
- T=2.73K Radiation (+ neutrino) b/g imply $$\rho_r c^2 = \sigma_r T_{\mathrm{CMB}}^4 \times \overbrace{\left[1 + N_\nu \frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right]}^{1.68}$$

Radiation const. 7.566 10^{-16} J/m^3/K^4

- Hence $\Omega_r = 8.5\, 10^{-5} h_{70}^{-2}$
  - At $t = t_{\mathrm{eq}}$, $\Omega_r a^{-4} = \Omega_m a^{-3}$, i.e., $a = a_{\mathrm{eq}} = \Omega_r/\Omega_m$, hence

$$1 + z_{\mathrm{eq}} = \Omega_m/\Omega_r = 3520(\Omega_m/0.3)h_{70}^2$$

- Co-moving horizon radius at t_{eq} :

$$r_{H,\mathrm{eq}} = \int_0^{a_{\mathrm{eq}}} \frac{c\,da}{a^2 H(a)} \simeq \frac{c}{H_0 \Omega_r^{1/2}} a_{\mathrm{eq}} = \frac{c}{H_0 \Omega_m^{1/2}(1 + z_{\mathrm{eq}})^{1/2}}$$

- (We ignored Ω_m term in F.eqn.: exact solution has extra factor $2(\sqrt{2} - 1) \simeq 0.83$)

Matter-radiation equality

- Putting in some numbers:

$$1 + z_{\rm eq} = \Omega_m/\Omega_r = 3520(\Omega_m/0.3)h_{70}^2$$

$$r_{H,\rm eq} = \frac{2(\sqrt{2}-1)c}{H_0\Omega_m^{1/2}(1+z_{\rm eq})^{1/2}} = 0.83\left(\frac{1.68\sigma_r T_{\rm CMB}^4(8\pi G)}{3H_0^2}\right)^{1/2}\frac{1}{H_0\Omega_m} \simeq \frac{109{\rm Mpc}}{(\Omega_m/0.3)h_{70}^2}$$

- A fluctuation with this co-moving wavelength contains a mass of

$$\sim \frac{4\pi}{3}\left(\frac{r_{H,\rm eq}}{2}\right)^3 \Omega_m \rho_{\rm crit} \simeq 2.8\ 10^{16}(\Omega_m/0.3)^{-2}h_{70}^{-4}{\rm M}_\odot$$

- (~10x mass of most massive galaxy clusters)

Linear growth around time t_{eq}

$$\boxed{\ddot{\delta} + 2(\dot{a}/a)\dot{\delta} + (c_s^2 k^2/a^2 - 4\pi G\rho_{\rm bg})\delta = 0}$$

- Consider the behaviour of dark matter (with $c_s = 0$) in a background solution dominated by matter+radiation
  - The relevant ρ_{bg} is the (dark) matter density
  - For $t \ll t_{\rm eq}$ the matter density $\rho_{bg} \propto t^{-3/2}$ so no power-law growth is possible
  - For $t \gg t_{\rm eq}$ the matter density $\rho_{bg} \propto t^{-2}$ so power law solution exists

- [Exercise] $$\delta \propto 1 + \frac{3}{2}\frac{a}{a_{\rm eq}}$$

- This is only true for perturbations <u>**inside the horizon**</u>
- Conclusion: **dark matter perturbations** grow almost exclusively in the matter dominated era, after $t_{\rm eq}$.

Linear growth around time t_{eq}

- Intuitive way to understand this transition to fast growth:
  - Characteristic timescale for the expansion of the universe:

    $$1/H(t)$$

    - Significant change to distances between particles, and density, happens on this timescale
  - Dynamical time for collapse of a fluctuation: $1/\sqrt{G\rho_m}$

    $$(\ddot{R} \simeq GM/R^2 \simeq G\rho R)$$

- <u>The fastest process dominates</u>

- Friedmann eq. reads $H(t) \sim \sqrt{G\rho_{\text{tot}}}$ where ρ_{tot} includes matter & radiation

    - So, as long as $\rho_{\text{tot}} \gg \rho_m$ the expansion will 'win' and fluctuations won't grow
    - (even when matter dominates, the timescales are equal, preventing exponential growth)

Other processes that slow/prevent growth

- Free streaming of particles
    - Perhaps particles move out of fluctuations before they grow?
    - Perhaps radiation pressure moves particles? (Silk damping)

- Effective pressure of a collapsed system
    - Internal motions increase - ‘virialisation’

Free streaming

- Particles do not move precisely with the Hubble flow
  - (thermal motions, perturbations, collisions, …)

- Compare timescale for these **peculiar motions** to dynamical (collapse) time: for typical streaming speed v in region of proper size L

require $\dfrac{L}{v} \ll t_{\rm dyn} \sim \dfrac{1}{\sqrt{G\rho}}$ to erase perturbation, i.e. $L \ll \dfrac{v}{\sqrt{G\rho}} \equiv L_{\rm FS}$

- (if the density is for the dominant component of the Universe then $L_{\rm FS} \sim vt$ since dynamical time is then Hubble time $\sim t$, otherwise it is larger)
- Free streaming most effective for fast-moving particles
- Relativistic particles can stream a distance $ct \sim r_H a$ (proper horizon size)
  - Particles lose momentum $\propto a^{-1}$
  - Once they have lost enough momentum to become non-relativistic their peculiar velocity decays and free streaming becomes less effective at erasing structure

Free streaming

- Take a particle that becomes non-relativistic at $t_{\rm NR}$.
- In a time dt it moves a proper distance $a\,dr = v\,dt$
- We have approximately $v = c \quad (t < t_{\rm NR}); \qquad v = c(a_{\rm NR}/a) \quad (t > t_{\rm NR})$

- So the co-moving distance traveled (free-streamed) after time t is

$$r_{\rm FS} = \int_0^t \dot{r} dt = \int_0^t \frac{v(t)}{a(t)} dt = \int_0^{t_{\rm NR}} \frac{c\,dt}{a(t)} + \int_{t_{\rm NR}}^{t_{\rm eq}} \frac{c\,a_{\rm NR}}{a(t)^2} dt + \int_{t_{\rm eq}}^{t} \frac{c\,a_{\rm NR}}{a(t)^2} dt$$

$$\text{—— } a(t) \propto t^{1/2} \text{ ——} \mid \text{—— } a(t) \propto t^{2/3}$$

  - which shows that the co-moving free streaming length grows as $2c(t_{\rm NR}t)^{1/2}/a_{\rm NR}$ up to $t_{\rm NR}$, logarithmically between $t_{\rm NR}$ and $t_{\rm eq}$, and then saturates in the matter-dominated era (when $a \propto t^{2/3}$).

- Any structure on scales smaller than this will be erased by FS

Free streaming: massive neutrinos

- Example: can neutrinos be the dark matter? They would need to have a rest mass of about 30 eV [check!: $0.3\rho_{crit}/n_{photon}$)]
- Particles of mass m_ν become non-relativistic at the time when their rest mass ~ their temperature, $m_\nu c^2 \simeq 3k_BT$. Neutrino temperature is ~ 2K now and scales as 1/a(t).

$$a_{\rm NR} = \frac{3k_BT_{\nu,0}}{m_\nu c^2} \simeq \frac{5\,10^{-4}}{m_\nu(eV)} \qquad \text{or} \qquad z_{\rm NR} \simeq 2000 m_\nu(eV)$$

- Masses above 2eV put $t_{\rm NR}$ in the radiation epoch, above z~3500.
- Assume Ω_m=0.3, H_0=70 km/s/Mpc. Then

$$z_{\rm eq} = 3520; \quad t_{\rm eq} = 3520^{-3/2}/H_0 \sim 2\,10^{12}\text{s}; \quad t_{\rm NR} = (a_{\rm NR}/a_{\rm eq})^2 t_{\rm eq} \sim \frac{6\,10^{12}}{m_\nu(eV)^2}\text{s}$$

- And the co-moving free streaming length is therefore

$$\frac{2ct_{\rm NR}}{a_{\rm NR}} \simeq \frac{250\text{Mpc}}{m_\nu(eV)}$$

30eV:
L_{FS} ~ 25Mpc

- (more accurate calculation gives length 3x larger)

Free streaming: Silk damping

- Before recombination baryonic matter and radiation were tightly coupled: photons scattered off free electrons (Thomson scattering).
  - Very small *mean free path* $L_{\rm mfp} \equiv (n_e \sigma_T)^{-1}$ where $\sigma_T = 6.65\,10^{-29}\mathrm{m}^2$
  - The electron density decays as time approaches t_{rec}, so m.f.p. increases: just before t_{rec} the electron density was $\rho_{crit}\,\Omega_b/m_p\,(1+z_{rec})^3$, or ~ 2.10^8 / m^3. So mfp ~ 7.10^{19}m, ~2.5kpc (3Mpc in co-moving coordinates).

- Scattering creates a random walk (diffusion), in which average distance travelled L_{Silk} scales as L_{mfp} x $\sqrt{(N/3)}$ after N scatterings.

  - Estimate N as $\dfrac{ct_{\rm rec}}{L_{\rm mfp}(t_{\rm rec})} \simeq 40$ so L_{Silk} ~ 11Mpc (comoving)
  - Allowing for expansion gives other factor of 1/3: **~4Mpc**
- **In absence of dark matter, erases structure < $4(\Omega_b/0.05)^{-1/2}$Mpc**

Perturbations in baryons + dark matter

- We saw that structure can be erased by
  - Free streaming of weakly interacting hot/relativistic particles (e.g., neutrinos, hot dark matter)
  - Silk damping = diffusion of baryons due to radiation scattering
  - Expansion of the universe during radiation dominated era stops growth on sub-horizon scales
- Structure growth comes from **cold, non-interacting dark matter**
  - Slow growth during radiation era, more rapid collapse in matter era

- Treat perturbations in a universe containing multiple fluid components

Perturbations in baryons + dark matter

- For each component X we have

$$\left(\frac{\partial^2}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial}{\partial t}\right)\delta_k^X = 4\pi G\bar{\rho}\delta_k^{\mathrm{tot}} - \frac{c_{s,X}^2 k^2}{a^2}\delta_k^X$$

- e.g., after recombination, the main components, baryons (B) and dark matter (DM) evolve as:

$$\left(\frac{\partial^2}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial}{\partial t}\right)\delta_k^B = 4\pi G\bar{\rho}(\Omega_B\delta_k^B + \Omega_{DM}\delta_k^{DM}) - \frac{c_s^2 k^2}{a^2}\delta_k^B$$

$$\left(\frac{\partial^2}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial}{\partial t}\right)\delta_k^{DM} = 4\pi G\bar{\rho}(\Omega_B\delta_k^B + \Omega_{DM}\delta_k^{DM})$$

- Before recombination, sound speed was c/√3 (relativistic fluid, driven by the photons coupled to matter); after this time it drops very steeply to $\sqrt{(kT/m_H)}$, where m_H is the mass of a Hydrogen atom
    - ~ 10^{-5} c !

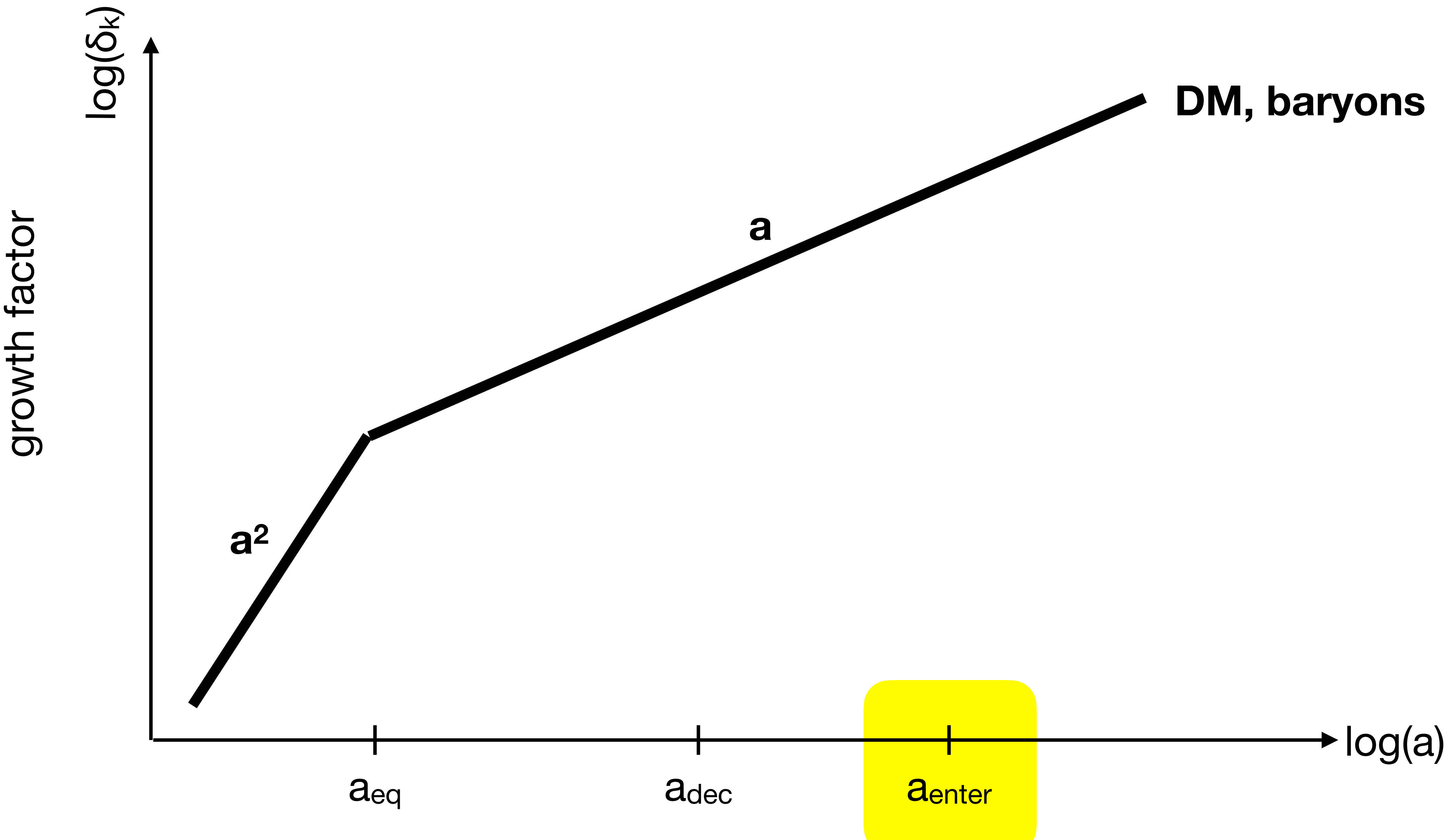
Subhorizon perturbations in baryons + dark matter

- Subhorizon dark matter fluctuations start to grow at t_{eq}.
- Before recombination, matter is a high pressure radiation-dominated plasma
  - Jeans length ~ $\sqrt{(c_s^2/G\rho)}$ ~ $\sqrt{(c^2/G\rho)}$ ~ c/H ~ ct ~ r_H .
  - subhorizon fluctuations in baryons don't grow but oscillate ($\omega^2>0$)
    - pressure (i.e., sound) waves
- After recombination, $c_s \sim 10^{-5}c$ (Jeans mass ~ $10^5 M_\odot$).
  - subhorizon baryon fluctuations: if $\lambda>\lambda_J$ baryons fall into the potential wells created by the dark matter

- Behaviour of the fluctuations depends on when they enter the horizon: before or after t_{eq} (dark matter) and t_{dec} (baryons)
  - i.e. whether $\lambda=2\pi/k$ <> $ct_{eq/dec}$

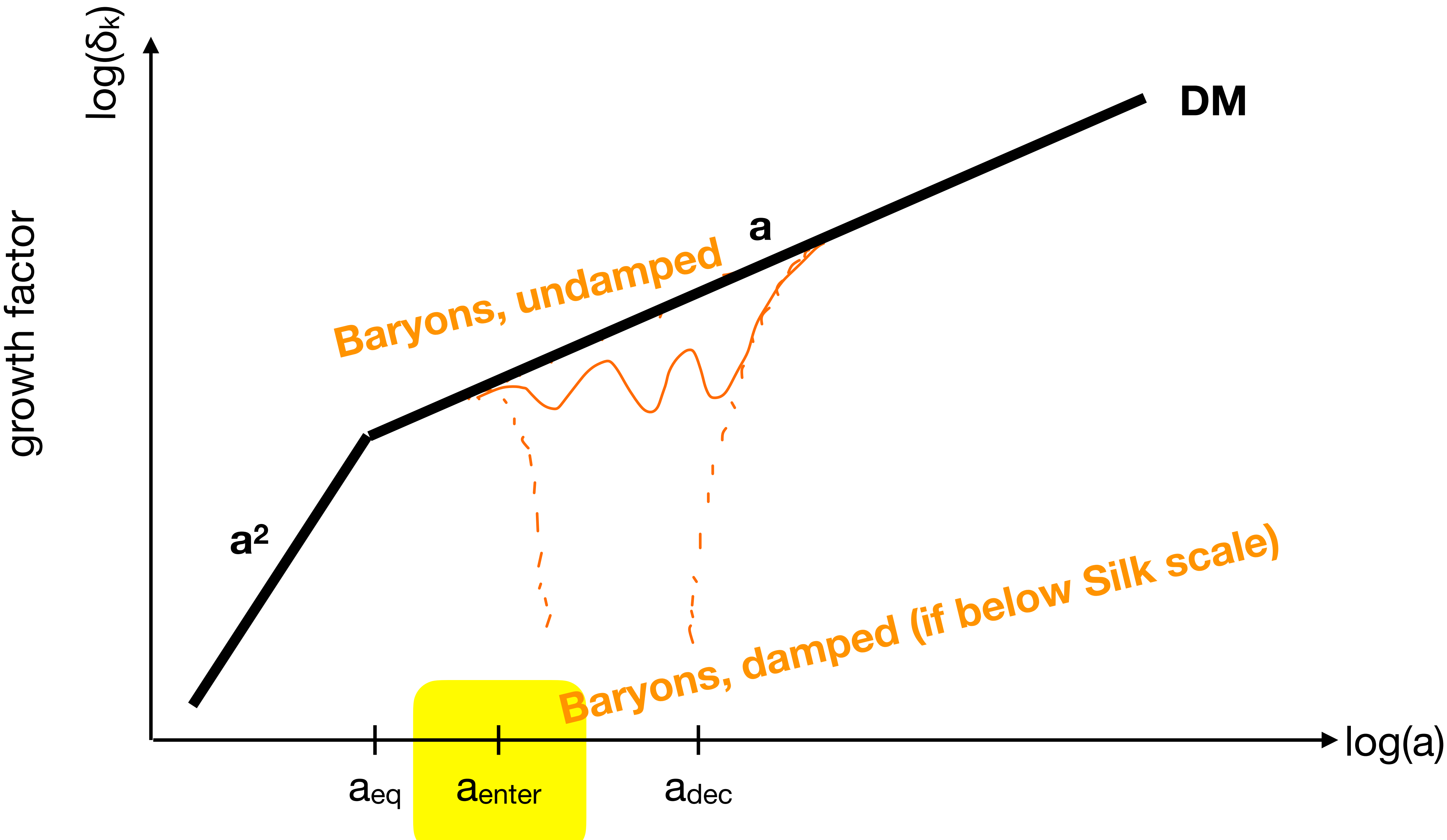
$ct_{dec} < \lambda$

- Largest scales, enter the horizon after decoupling
  - DM and baryons behave as super-horizon fluctuations
    - super-horizon and sub-horizon solutions are identical



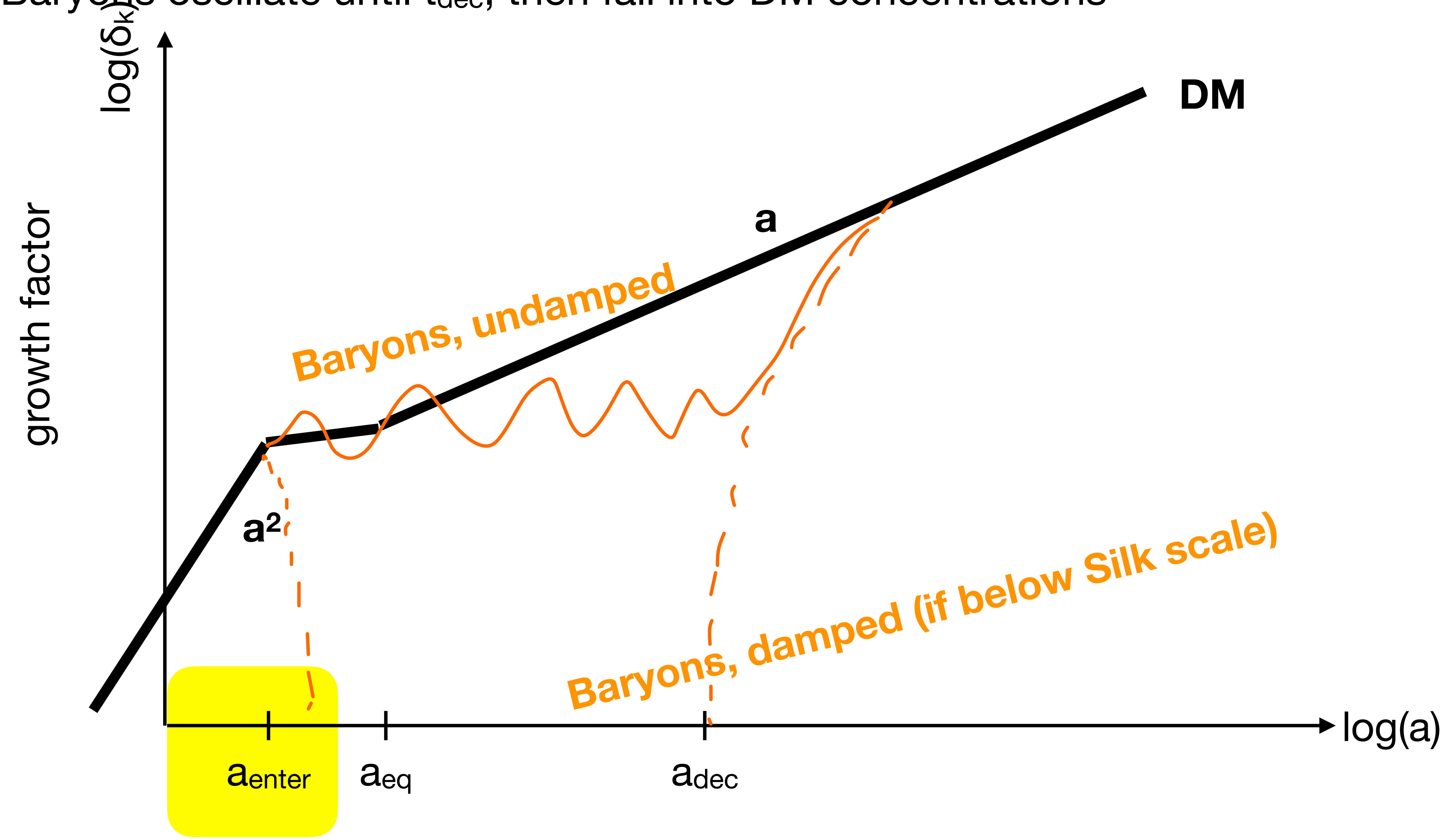
$ct_{eq} < \lambda < ct_{dec}$

- Scales that enter the horizon between decoupling and t_{eq}
  - DM behaves as super-horizon fluctuations
  - Baryons oscillate until t_{dec}, then fall into DM concentrations



$\lambda < ct_{eq} < ct_{dec}$

- Scales that enter the horizon before t_{eq}
  - DM grows as super-horizon fluctuations, until enters horizon
    - but growth pauses between t_{enter} and t_{eq} (follows the $1 + \frac{3}{2}\frac{a}{a_{eq}}$ solution)
  - Baryons oscillate until t_{dec}, then fall into DM concentrations



Interpreting the fluctuations

- Inflation is thought to produce fluctuations that are ~scale-free
  - (the amplitudes of initial density fluctuations vary as power of k)

- The large-scale structure that was ‘released’ at t_{dec}, and is now observed in the Cosmic Microwave Background and in the matter distribution, contains much information:
  - an imprint of t_{eq}, because scales that entered the horizon before this time grew more slowly for a period
  - information about the baryon density, through Silk damping of the smaller fluctuations
  - an imprint of the sound horizon size at t_{dec}.

- REMEMBER that this lecture was about small amplitude fluctuations, $\delta \ll 1$ i.e. $\delta\rho \ll \rho_{bg}$