Part IB Physics A

Oscillations Waves and Optics Examples (2021/2022)

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Questions are tentatively graded A (mostly straightforward), B (standard level), and C (may involve deeper insight or mathematical sophistication). AB and BC questions are intermediate between these levels. If I have got these levels wrong, please let me know!

Q 1 (B) A damped oscillator consists of a mass m suspended from a light spring, and obeys the equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

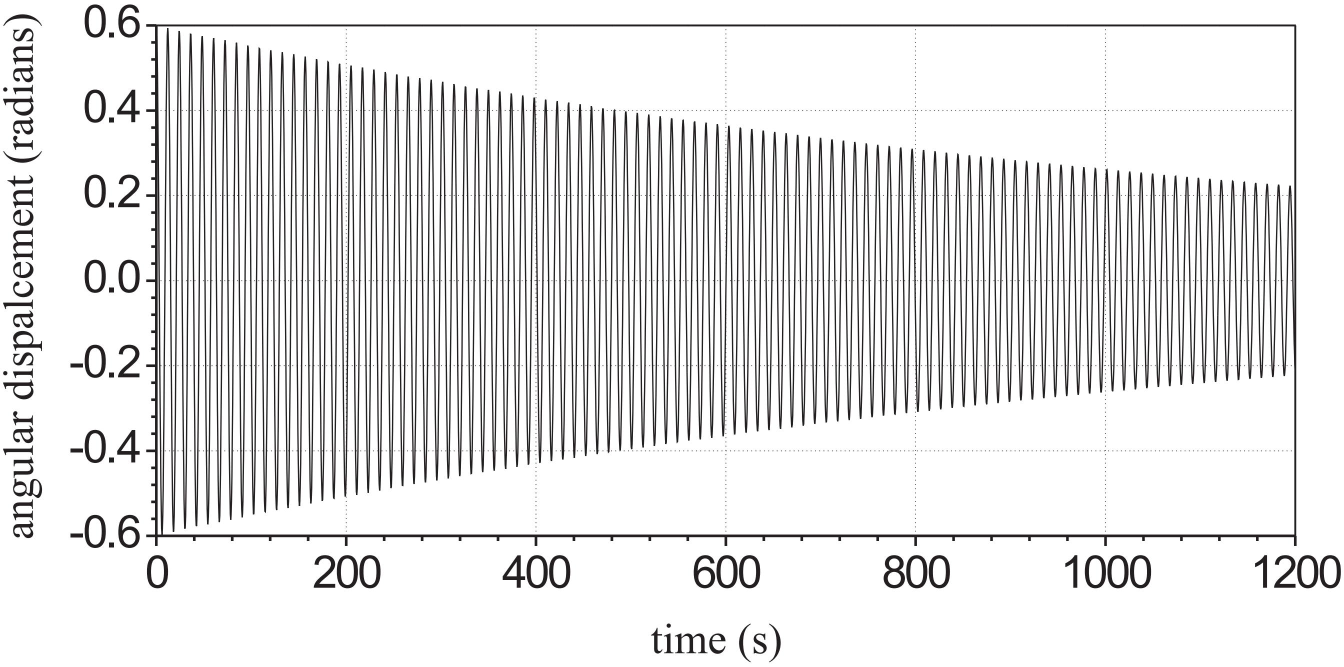
The damping coefficient b is related to the spring constant k by $k = 4b^2/(25m)$. At time $t = 0$, with the mass stationary and in its equilibrium position, the upper end of the spring is suddenly moved upwards by a distance X.

a. Sketch the subsequent motion.

b. Show that, for large times t, the motion can be approximated by the single exponential term $\frac{4}{3}X\mathrm{e}^{-bt/(5m)}$.

c. Show that the time required for the mass to move to within $X/100$ of its new equilibrium position is approximately $24.5m/b$.

Q 2 (B) A torsional oscillator consists of a bob, with moment of inertia $I = 5.90 \times 10^{-5}\,\mathrm{kg\,m^2}$, suspended from a thin wire which provides a restoring torque when the bob is rotated away from its equilibrium position. The graph shows the motion of the bob as a function of time.



Estimate, with errors:

a. the angular frequency ω_f of the oscillation;

b. the quality factor Q of the oscillator;

c. the free oscillation frequency ω_0 if there were no damping forces;

d. the restoring torque per unit angular displacement (torsional spring constant c) of the wire.

Q 3 (A) Estimate the Q-factor of

a. The pendulum of a grandfather clock;
b. Big Ben striking: there are some data available at `https://www.youtube.com/watch?v=E9wWBjnaEck` or `http://www.parliament.uk/audio/images/bigben-images/bigbenstrikes.mp3`. Be as quantitative as possible.
c. A car suspension;
d. A piece of cheese.

Q 4 (A) A damped oscillator of mass m, being driven by a force $F = \Re\{F_0 e^{i\omega t}\}$, obeys the equation:

$$m\ddot{x} + b\dot{x} + kx = F$$

The force has a fixed magnitude ($F_0 = 2\,\mathrm{N}$), but ω can be varied. Assume that $m = 0.2\,\mathrm{kg}$, $b = 1\,\mathrm{N\,m^{-1}\,s}$ and $k = 80\,\mathrm{N\,m^{-1}}$.

a. Calculate the values of ω_0 and Q for this system and deduce the fractional difference between ω_0 and the angular frequency at which the system has its peak amplitude response.
b. Sketch the oscillator's amplitude, velocity and acceleration as a function of angular frequency.
c. What are the values of A and δ for the steady-state response described by $x = A\cos(\omega t - \delta)$ when (i) $\omega = 30\,\mathrm{rad\,s^{-1}}$, (ii) $\omega = 20\,\mathrm{rad\,s^{-1}}$ and (iii) $\omega = 0\,\mathrm{rad\,s^{-1}}$?
d. What is the mean power input when (i) $\omega = 30\,\mathrm{rad\,s^{-1}}$ and (ii) $\omega = 20\,\mathrm{rad\,s^{-1}}$? Comment on these results.

Q 5 (B) A driving force $F(t) = F_1(t) + F_2(t)$ is applied to a damped oscillator, where $F_1(t) = \Re[A_1 \exp(i\omega_1 t)]$ and $F_2(t) = \Re[A_2 \exp(i\omega_2 t)]$.
Show that, if $\omega_1 \neq \omega_2$, the mean power absorbed by the system is given by $\langle P\rangle = \langle P_1\rangle + \langle P_2\rangle$ where $\langle P_1\rangle$ and $\langle P_2\rangle$ are the mean powers absorbed if only F_1 or only F_2 were applied respectively. What happens if $\omega_1 = \omega_2$?
Consider now an external force having the form of a square wave, which may be approximated as

$$F(t) = A_0\left(\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t\right).$$

Sketch the power absorption as a function of the angular frequency ω of the fundamental, if the oscillator has $\omega_0 = 15\,\mathrm{rad\,s^{-1}}$ and $Q = 30$.

Q 6 (B) A simple pendulum whose damping is negligible oscillates freely with angular velocity ω_0.
A small horizontal force $F\sin\omega t$ is applied to the bob which has mass m.

a. Show that the displacement x of the bob follows

$$\ddot{x} + \omega_0^2 x = \frac{F}{m}\sin\omega t$$

b. By considering the steady state response (the particular integral) and the free response (the complementary function), obtain the general solution of the above equation.
c. If, at $t = 0$, the bob is at rest in its equilibrium position, show that

$$x = \frac{F}{m(\omega_0^2 - \omega^2)}\left(\sin\omega t - \frac{\omega}{\omega_0}\sin\omega_0 t\right)$$

Sketch a graph of this response as a function of time for the case $\omega = 2\omega_0$.
d. Without detailed calculation, discuss and sketch a graph of what happens if the damping is small but not negligible.

Q 7 (B) [A computational problem] A simple pendulum consists of a small bob of mass m hanging from a massless string of length l in a gravitational field g. Its equation of motion is

$$\ddot{\theta} = -(g/l)\sin\theta$$

The pendulum is displaced by an angle θ_0 and released from rest. Write a program in `MATLAB` (or another language, perhaps python) to plot the displacement θ as a function of time for a few periods, for initial values $\theta_0 = 0.01, 0.03, 0.1, 1.0, 2.0, 3.0$ radians and comment on the results. (Assume for simplicity that $g = 10\,\mathrm{m\,s^{-2}}$ and $l = 10\,\mathrm{m}$. You may find it helpful to plot the normalised displacements, θ/θ_0, against time.).
[Do *not* assume that the displacement is small. Using `MATLAB` to solve the differential equation, you will need to rewrite it as two coupled first order equations in new variables $y_0 \equiv \theta$, $y_1 \equiv \dot{\theta}$. Then use the routine `ode45` to integrate the system. You may need to ensure the solution is sufficiently accurate by setting the tolerance parameters appropriately e.g. something like `odeset('RelTol', 1e-6);`]

Q 8 (B) A uniform inextensible string of length l and mass m is suspended vertically. It is tapped at the top end so that a transverse pulse runs down it. At the same instant, a small body is released from rest at the top of the string, so that it falls freely parallel to the string.

a. How far from the top of the string does the falling body catch up with the impulse?
b. What mass should be hung from the string so that the pulse and the falling body reach the bottom simultaneously?

Q 9 (A)

a. A steel piano string has a diameter of $0.5\,\mathrm{mm}$ and a length of $700\,\mathrm{mm}$. The steel has a density of $\rho = 7800\,\mathrm{kg\,m^{-3}}$ and a Young's modulus of $Y = 200\,\mathrm{GPa}$. What tension does the string have to be put under to tune the string to middle C (261.6 Hz)? High-strength steel has a yield strength of about $3 \times 10^9\,\mathrm{N\,m^{-2}}$. Comment on the dangers of piano tuning.
b. What is the frequency of the fundamental longitudinal oscillation of this string?
c. With a few exceptions, most solids break if they are stretched by more than 1-2% of their unstretched length. Use this fact to show that transverse waves on a stretched string are almost always slower than compression waves on the same string.

Q 10 (A) Three transverse waves propagating in the z direction have the following displacements in the x direction:

(1) A wave which is linearly polarized at an angle θ to the x axis with

$$\Psi_{x,1} = a\cos\theta\cos(kz - \omega t + \phi_0).$$

(2) A right-circularly polarized wave (optical convention — looking into the source) with

$$\Psi_{x,2} = b\cos(kz - \omega t).$$

(3) A right-elliptically polarized wave with

$$\Psi_{x,3} = c\cos(kz - \omega t)$$

(the major axis of the ellipse is twice the length of the minor axis and lies along the x axis).

a. Write down the corresponding displacements in the y direction for each of these waves.
b. An elliptically polarized wave can be produced by the superposition of the circularly polarized wave and the linearly polarized wave. Show that this is true for the waves given above for appropriate values of θ, ϕ_0, a and b; find these values (you may assume $0 \le \theta \le \pi/2$,

$0 \le \phi_0 \le \pi/2$). Draw vector diagrams in the x-y plane showing the relationship between the displacements produced by the three waves at (i) $z = 0$, $t = 0$, (ii) $z = 0$, $\omega t = \pi/4$, (iii) $z = 0$, $\omega t = \pi/2$ and (iv) $kz = \pi/4$, $t = 0$.

Q 11 (B) A sinusoidal wave given by $\psi = \Re\{C \exp(\mathrm{i}[\omega t - kz])$ is travelling along a string of impedance Z, where C is complex and Z is real. Given that the instantaneous power transmitted by a travelling wave is given by $P(t) = Z(\frac{\partial\psi}{\partial t})^2$, show that the average power passing a given point in the string is given by $\langle P \rangle = \frac{1}{2} Z\omega^2 |C|^2$.
How would you interpret a complex value for Z of $Z(\omega)$ at a given angular frequency ω, and what would be the average power transmitted in this case?

Q 12 (BC) A string has mass per unit length μ and tension T. One end is free to move transversely but is attached to a massless vane which moves in a viscous fluid and therefore experiences a drag force of αu_v where u_v is its transverse velocity. Show that if α is very small or very large, the wave is totally reflected. Is a node or an anti-node produced at the free end? What value of α leads to perfect absorption?
A harmonic wave travels along the string towards the end attached to the vane. For a general value of α, superposition of the incident and reflected waves gives rise to a combination of a standing wave and a travelling wave, no point on the string having zero amplitude. Show that for $\alpha < \sqrt{T\mu}$ the standing wave ratio, i.e. the ratio of the maximum amplitude to the minimum amplitude, is given by $\sqrt{T\mu}/\alpha$. Sketch the displacement of the string when the free end has (i) its maximum displacement and (ii) zero displacement, indicating the positions of the points of maximum and minimum amplitude. (Measuring the standing wave ratio is a convenient way of finding the impedance of a termination.)

Q 13 (B)

a. By considering the appropriate continuity conditions, derive from first principles the *pressure amplitude reflection coefficient* for a sound wave passing from a liquid of impedance Z_1 into a fluid of impedance Z_2. (You may assume normal incidence on a plane boundary.) What is the corresponding *amplitude* reflection coefficient?

b. Figure 1a shows apparatus in which a pulse of high-frequency sound is launched upwards through water by a transducer. In Figure 1b the feature labelled A shows the initial excitation of the transducer and B shows the arrival of the sound pulse back at the transducer after it has been reflected from the water-air interface. Calculate the velocity of sound, the acoustic impedance and the bulk modulus for water.

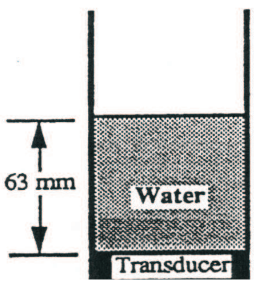
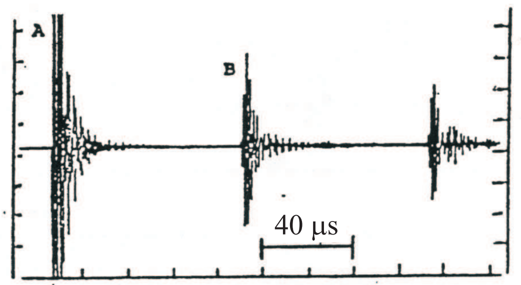


Figure 1a



Transducer output versus time

Figure 1b

c. Figure 2a shows a similar experiment in which the water has the same depth but a thick layer of oil has been floated onto the water surface. Identify the pulses marked C and D in figure 2b. Calculate the velocity of sound and hence the acoustic impedance Z_2 for the oil.

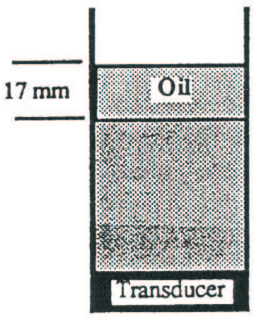


Figure 2a

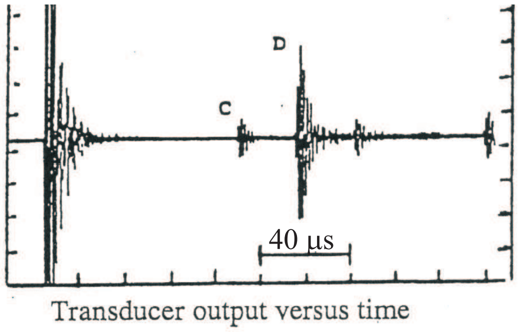


Figure 2b

d. Explain why pulse C in Figure 2b is smaller than pulse B in Figure 1b. Use the relative sizes of these pulses to test the formula derived in (a) for the reflection of sound at the water-oil interface. Does this result need correction for divergence or absorption of the sound beam?

[Densities: water $1000\,\mathrm{kg\,m^{-3}}$; the oil $790\,\mathrm{kg\,m^{-3}}$. Acoustic impedance of air = $400\,\mathrm{kg\,m^{-2}\,s^{-1}}$]

Q 14 (B)

a. A plane sinusoidal sound wave of displacement amplitude $1.2 \times 10^{-3}\,\mathrm{mm}$ and frequency $680\,\mathrm{Hz}$ is propagated in a gas of density $1.3\,\mathrm{kg\,m^{-3}}$ and pressure $1.0 \times 10^{5}\,\mathrm{N\,m^{-2}}$. The ratio of principal specific heat capacities (γ) is 1.4. Find the pressure amplitude of the wave.

b. What frequency is heard if a thin steel bar of length $1.0\,\mathrm{m}$ is tapped on the end so as to launch a longitudinal wave pulse? What fraction of the pulse energy is lost into air each time the pulse reaches the end of the bar? Assuming this is the only mechanism of energy loss, calculate a characteristic decay time for the ringing of the bar. [The Young's modulus of steel is $210\,\mathrm{GPa}$ and the density $7800\,\mathrm{kg\,m^{-3}}$. The density of air is $1.2\,\mathrm{kg\,m^{-3}}$ and the speed of sound in air is $340\,\mathrm{m\,s^{-1}}$.]

Q 15 (C) [A computational question] A plane wave of unit amplitude, $\exp(\mathrm{i}\omega t - \mathrm{i}k_1 x)$ propagates in the $+x$ direction of a region of impedance Z_1 which fills the region $x < 0$. It is incident normally on a layer of material of thickness l and impedance Z_2 which fills the region $0 < x < l$. Some of it is in general reflected, so that the total disturbance in $x < 0$ is

$$\psi_1 = \exp(\mathrm{i}\omega t - \mathrm{i}k_1 x) + r\exp(\mathrm{i}\omega t + \mathrm{i}k_1 x),$$

where r is the complex amplitude of the reflected wave. In steady state, region 2 in general contains forward and backward travelling waves of complex amplitudes a and b:

$$\psi_2 = a\exp(\mathrm{i}\omega t - \mathrm{i}k_2 x) + b\exp(\mathrm{i}\omega t + \mathrm{i}k_2 x)$$

The layer lies on an infinite substrate of impedance Z_3, which fills the region $x > l$. In this region, the transmitted wave can be written

$$\psi_3 = \tau\exp\left[\mathrm{i}\omega t - \mathrm{i}k_3(x - l)\right]$$

for some complex transmitted amplitude τ. Note that k_1, k_2, k_3 are the wavenumbers in the three regions.

Show that the boundary conditions at $x = 0$ and $x = l$ can be written in the form

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & Z_1/Z_2 & -Z_1/Z_2 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}k_2 l} & \mathrm{e}^{\mathrm{i}k_2 l} & -1 \\ 0 & \mathrm{e}^{-\mathrm{i}k_2 l} & -\mathrm{e}^{\mathrm{i}k_2 l} & -Z_2/Z_3 \end{pmatrix} \begin{pmatrix} r \\ a \\ b \\ \tau \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Using `MATLAB` (or another language, perhaps Python), write a function which solves this matrix equation, taking as arguments the values Z_1, Z_2, Z_3, k_2 and l, and returning the vector of complex amplitudes (r, a, b, τ). Hence plot the amplitude and phase of the reflection (r) and transmission (t) coefficients as a function of thickness l in the range 0 to $1\,\mathrm{m}$ for the cases

a. $k_2 = 2\pi\mathrm{m}^{-1}$, $Z_1 = 1\,\Omega$, $Z_2 = 2\,\Omega$, $Z_3 = 4\,\Omega$.

b. $k_2 = 2\pi\mathrm{m}^{-1}$, $Z_1 = 1\,\Omega$, $Z_2 = 3\,\Omega$, $Z_3 = 4\,\Omega$

To get the correct form of the equations, assume that for this wave system, for a wave of amplitude ψ, the quantities ψ and ψ/Z are continuous (Z is the impedance). Recall that the impedance for a wave traveling in the negative direction is $-Z$. The continuity equations at $x = 0$ are then $(1 + r) = (a + b)$ and $(1 - r)/Z_1 = (a - b)/Z_2$

Q 16 (C) A long string (extending from $x = -\infty$ to $x = \infty$) with mass per unit length $\mu = 0.2\,\mathrm{kg\,m^{-1}}$ lies on a smooth horizontal table, and is put under uniform tension. A point mass $m = 0.005\,\mathrm{kg}$ is attached to the string at $x = 0$. A wave of wavelength $\lambda = 0.1\,\mathrm{m}$ propagates along the string, in the plane of the table, from $x = -\infty$ and strikes the mass.

a. What fraction of energy is reflected by the mass?

b. (more challenging) Where should another identical mass be attached so as to eliminate the reflection?

Q 17 (B)

a. Show that the group velocity v_g can be written as

$$v_g = v_p - \lambda \frac{\mathrm{d}v_p}{\mathrm{d}\lambda}$$

where v_p is the phase velocity and λ the wavelength of the disturbance.
The phase velocity for surface waves in a liquid of depth $\gg \lambda$ is given by

$$v_p = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi\sigma}{\rho\lambda}}$$

where σ is the surface tension and ρ is the density. Find an expression for the group velocity. Hence show that in the long wavelength limit, when gravity dominates,

$$v_g = \frac{v_p}{2}.$$

A disturbance of frequency $1\,\mathrm{Hz}$ occurs on the surface of a deep lake. How long will it take the resulting wave group to reach a point $1\,\mathrm{km}$ distant?

b. Storms in the Pacific ocean with sustained strong winds cause swells (groups of gravity-driven waves) which can create great surfing conditions when they arrive on California's beaches. A surfer in California notes that the group of waves she is surfing has a typical wave period of $18\,\mathrm{s}$. Twenty-four hours later, she goes surfing again and notices the period is now $17\,\mathrm{s}$. Explain this phenomenon, and estimate the distance to the storm, and the windspeed in the storm.

Q 18 (B) Discuss what is meant by group velocity and phase velocity, and distinguish between dispersive and non-dispersive media.

For electromagnetic waves of frequency f in a dispersive medium of refractive index n, the frequency-dependent phase velocity is given by c/n, where c is the speed of light in a vacuum. Denoting the group velocity u_g, show that

$$\frac{c}{u_g} = n + f\left(\frac{\mathrm{d}n}{\mathrm{d}f}\right)$$

A communication system uses square pulses of infrared radiation transmitted along optical fibres. The infrared laser has a frequency of $f_0 = 2.0 \times 10^{14}\,\mathrm{Hz}$, and each pulse has a duration $T_P = 1.0 \times 10^{-10}\,\mathrm{s}$. The refractive index of the fibres in this frequency range is given by

$$n(f) = n_0 + \tau(f - f_0)$$

where $\tau = 1.0 \times 10^{-16}$ s and $n_0 = 1.5$. The time interval between successive pulses is $T_R = 1.0 \times 10^{-9}$ s.

a. Sketch the power spectrum of the signal, and estimate the range of frequencies present in the signal.

b. Find the group velocity at the centre frequency ν_0 and estimate the range of group velocities present in each pulse.

c. Explain why the pulses spread out as they travel along the fibre. Estimate the maximum length of fibre that can be used without the pulses running into each other (neglecting waveguide dispersion).

Q 19 (B)

a. A particular wavepacket can represented by a sum of travelling waves with different amplitudes, all with wavevectors close in magnitude to the central wavevector k_0:

$$\psi(x,t) = \sum_{m=-m_0}^{m_0} \exp\left[\frac{-(k_m - k_0)^2}{2\sigma_k^2}\right] \cos\left[k_m x - \omega(k_m)t\right].$$

The dispersion relation $\omega(k)$ for this wavesystem is

$$\omega(k) = ck + dk^3$$

and the wavevector of the m'th component is $k_m = k_0 + m\Delta k$.

Using the values $k_0 = 1\,\mathrm{m}^{-1}$ (i.e. a central wavelength of 2πm), $\sigma_k = k_0/10$, $\Delta k = \sigma_k/10$, $c = 1\,\mathrm{m\,s^{-1}}$, $d = 0$ (i.e. no dispersion), $m_0 = 30$, use MATLAB (or another language) to calculate and plot the disturbance at times t =0 s, 100 s, 200 s and 300 s.

Repeat the plots with differing amounts of dispersion (e.g. $d = 0.1\,\mathrm{m^3\,s^{-1}}$, $d = -0.1\,\mathrm{m^3\,s^{-1}}$, $d = 0.2\,\mathrm{m^3\,s^{-1}}$, $d = -0.2\,\mathrm{m^3\,s^{-1}}$) and describe the results qualitatively, and where possible quantitatively. In particular, discuss the speed at the which the packet propagates, and the rate at which is spreads.

b. *Only attempt this if you enjoyed part (a)!* A travelling square wave can be modelled as a Fourier series:

$$\psi(x,t) = \cos\left[k_0 x - \omega(k_0)t\right] - \frac{1}{3}\cos\left[3k_0 x - \omega(3k_0)t\right] + \frac{1}{5}\cos\left[5k_0 x - \omega(5k_0)t\right] + \ldots,$$

or, truncating after $m_0 + 1$ terms, we have:

$$\psi(x,t) \approx \sum_{m=0}^{m_0} \frac{(-1)^m}{2m+1} \cos\left[k_m x - \omega(k_m)t\right]$$

The dispersion relation $\omega(k)$ for this system is taken to be $\omega = ck + dk^3$, and the wavevector of the m'th component wave is $k_m = (2m+1)k_0$.

Using the values $k_0 = 1\,\mathrm{m}^{-1}$, $c = 1\,\mathrm{m\,s^{-1}}$, $d = 0$ (i.e. no dispersion), $m_0 = 10$, use MATLAB to calculate and plot the disturbance at times $t = 0$, $t = \pi$s, $t = 2\pi$s.

Repeat the plots with differing amounts of dispersion (d =0.001 $\mathrm{m^3\,s^{-1}}$, $-0.001\,\mathrm{m^3\,s^{-1}}$, $0.02\,\mathrm{m^3\,s^{-1}}$ and $0.05\,\mathrm{m^3\,s^{-1}}$) and describe the results qualitatively, and where possible quantitatively.

Explain how these results relate to the design of digital communication systems.

Q 20 (B) A transmission line, such as a coaxial cable, consists of two conductors, and is used to carry electrical signals. The voltage $V(x,t)$ along such a cable lying on the x-axis obeys the equation

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + RC\frac{\partial V}{\partial t},$$

where R, C, L are the resistance, capacitance and inductance *per unit length of the cable* respectively. A sinusoidal voltage of angular frequency ω is applied at the end $x = 0$ of a very long cable, resulting in a travelling wave $V(x,t) = V_0 \exp(ikx - i\omega t)$. Assuming that the resistance per unit length R is small ($R \ll \omega L$), find expressions for (i) the wavevector k, and (ii) the phase speed of the signal. Show that the signal is attenuated as it propagates, and find an expression for the attenuation length (the length over which the signal amplitude falls by a factor e).

Q 21 (AB) Two plane waves are represented by

$$\Psi_1 = \mathrm{e}^{-\mathrm{i}(k_x x + k_y y)} \quad \Psi_2 = \mathrm{e}^{-\mathrm{i}(k_x x - k_y y)}$$

(Note that the term $\mathrm{e}^{\mathrm{i}\omega t}$ has been omitted from the representations).

a. Show that the result of superposing these disturbances is a wave travelling in the positive x direction with amplitude $A = 2\cos k_y y$.

b. Show that l, the spacing of the nodal lines in the y direction, is given by $l = \pi/k_y$.

c. By substituting into the wave equation show that $k_x^2 + k_y^2 = \omega^2/u^2$, where u is the speed of the waves.

d. A rubber sheet is fastened to rigid supports along two lines parallel to the x axis so as to make a waveguide. The width of the guide is $0.1\,\mathrm{m}$. In the absence of the supports the speed of waves on the sheet is $2\,\mathrm{m\,s^{-1}}$. For waves of frequency $12\,\mathrm{Hz}$, calculate the phase velocity and group velocity for propagation along the guide. What happens to waves of frequency $8\,\mathrm{Hz}$? Sketch the instantaneous displacement of the membrane in a section of the guide at (i) $12\,\mathrm{Hz}$ and (ii) $8\,\mathrm{Hz}$.

Q 22 (B) An aperture consists of two long slits each of width a, with their centres separated by a distance b. The aperture is illuminated by a plane wave at normal incidence.

a. Show that the diffracted amplitude in direction θ for monochromatic illumination by light of wavelength λ is

$$\psi(\theta) \propto 2\psi_0 a \cos\left(\frac{kb\sin\theta}{2}\right) \operatorname{sinc}\left(\frac{ka\sin\theta}{2}\right).$$

where $k = 2\pi/\lambda$.

b. Show how your result can be derived from the Fourier transforms of a delta-function and a top-hat function. Describe with the aid of sketches how the Fourier transform changes with a and b.

c. Show that the ratio of the width of the central peak of the envelope function to that of the central interference fringe is $2(b/a)$. (Note that this result is independent of λ.)

d. Derive the diffraction pattern above by treating the aperture as a superposition of a slit of width $b + a$ and a second slit of width $b - a$, the latter covered by a sheet of glass which inverts the phase of the light passing through it.

Q 23 (B) A digital camera has a lens with a focal length f which can varied between $14\,\mathrm{mm}$ and $150\,\mathrm{mm}$. Light enters the lens via a circular iris diaphragm of variable diameter. The *f-number* of the lens, N, can be varied from $N = 4$ to $N = 32$ (The *f-number* is the ratio of the lens focal length to diameter of the circular aperture; an f-number of 4 is usually written $f/4$ etc). The image is focused onto a CCD detector of size $17.3\,\mathrm{mm} \times 13.0\,\mathrm{mm}$, which contains 16 million detectors (pixels), which can be assumed to be square with no spaces between them. Evaluate, as a function of N, whether the image quality is limited by diffraction or by the size of the detectors. (Assume a typical wavelength of $\lambda = 600\,\mathrm{nm}$.)

Q 24 (C) Calculate and sketch the Fraunhofer diffraction patterns of the following apertures, using the conventional Fourier space coordinates $(p,q) = (k\sin\theta, k\sin\xi)$):

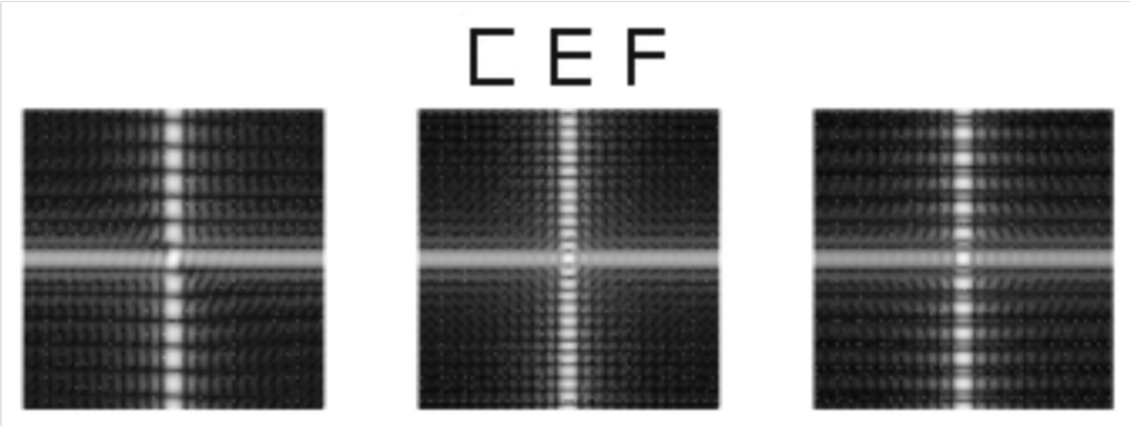
a. an infinite chequerboard consisting of transparent and opaque squares, each of side a;

b. a transparent square aperture (with sides of length $2a$) whose lower half ($|x| < a$, $y < 0$) is covered by a sheet of transparent film which introduces a phase delay of $\pi/2$;

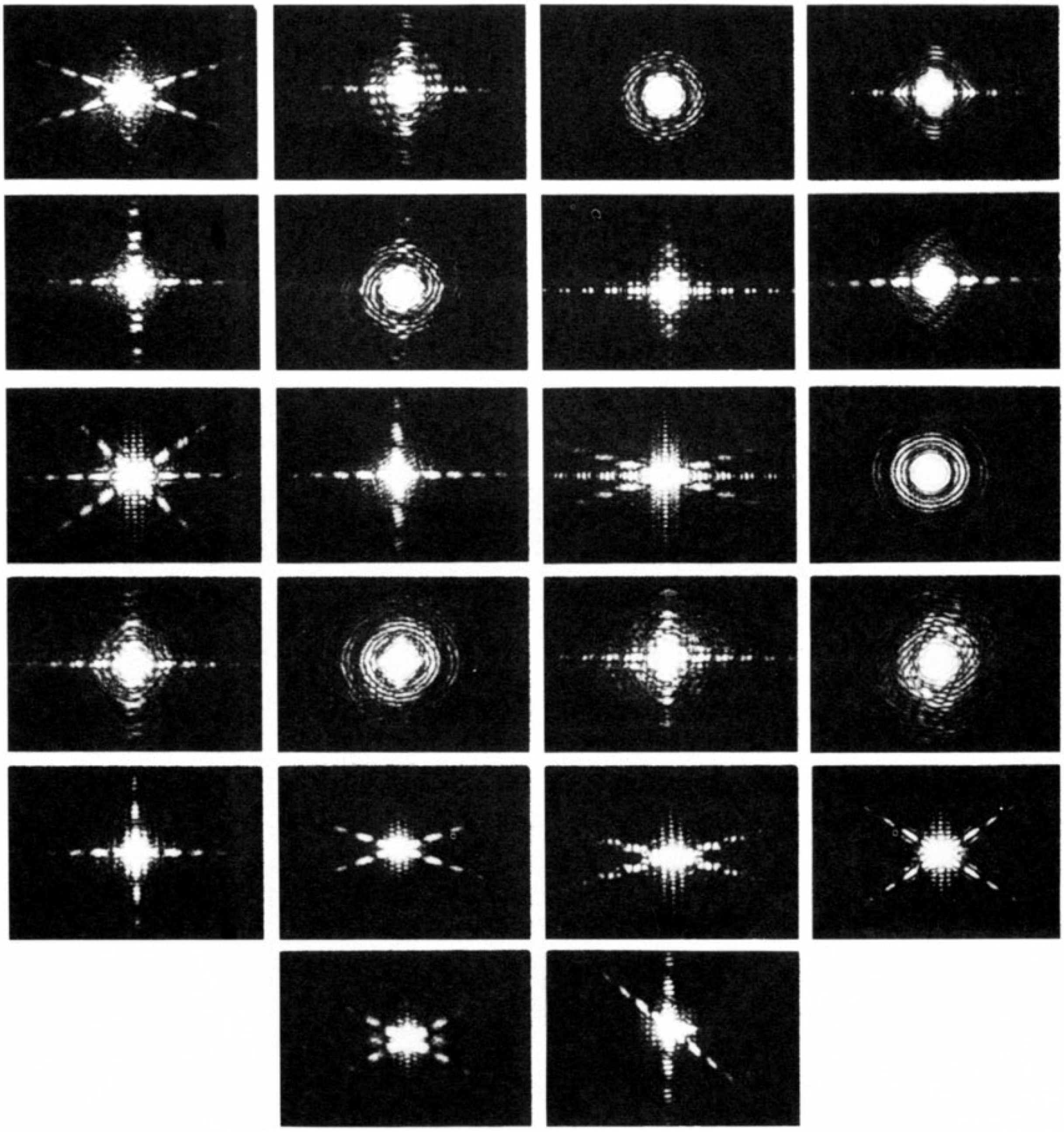
c. a transparent elliptical aperture with major axis $2a$ and minor axis $2b$, the major axis being aligned with x axis (hint: scaling);

d. an array of 6 very small, identical circles arranged in a regular hexagon whose sides are of length a, with 2 of the circles located at $(x_i, y_i) = (0, \pm 2a)$.

e. The figure shows the Fraunhofer diffraction patterns of the letters C, E, and F, in unknown order: which is which?



Q 25 (B) The Fraunhofer diffraction patterns of the letters of the alphabet are presented below, in alphabetical order, with 4 letters missing. Which ones?



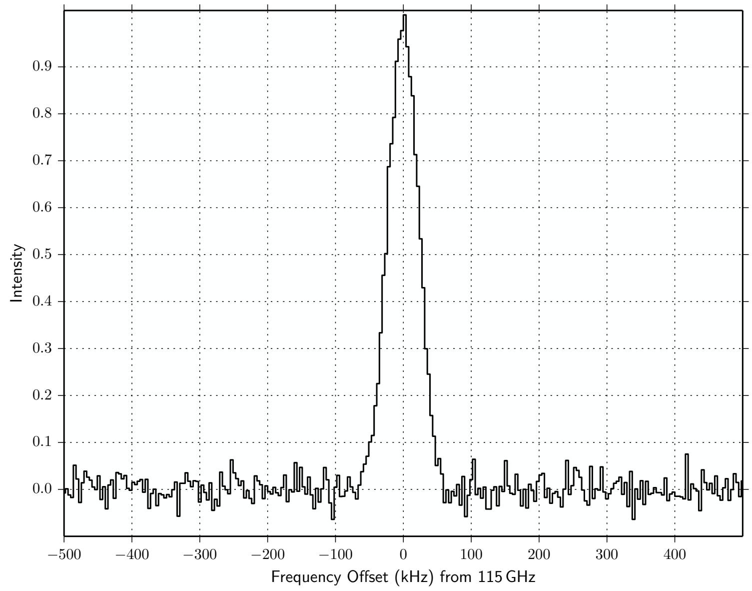
Q 26 (B) A diffraction grating has period D, and each element of the grating consists of a pair of long narrow slits separated by D/N where N is an integer.

a. Sketch the diffraction pattern when the grating is normally illuminated.

b. Calculate the relative intensities I_n of the various diffraction peaks of order n for the case $N = 4$. Which peaks are "forbidden"?

c. For the case $N = 2$ show that the solution reduces to that of a simple grating of period $D/2$.

Q 27 (B) (a) This plot shows the spectrum of the rotational emission line from carbon monoxide molecules in an interstellar gas cloud. The rest frequency of the transition is 115 GHz. Estimate the temperature of the cloud, assuming the linewidth is due to Doppler broadening.



(b) Estimate the contribution to the linewidth caused by pressure broadening in such a cloud if the density of the gas (assumed for simplicity to be pure carbon monoxide) is $1 \times 10^9\,\mathrm{m}^{-3}$ and that the cross section for collisions between molecules is $1 \times 10^{-19}\,\mathrm{m}^2$.

Q 28 (B) Describe how the Fraunhofer diffraction pattern of a grating depends on:

a. the spacing between the grating lines;
b. the overall extent of the grating;
c. the amplitude distribution over an individual line;
d. the phase distribution over an individual line.

A grating with 2500 lines per metre, each line $100\,\mu\mathrm{m}$ wide, is illuminated at normal incidence with light of $600\,\mathrm{nm}$ wavelength. Each line is divided lengthwise into halves, one of which introduces an additional half wavelength of path. Sketch the form of the Fraunhofer diffraction pattern. Show that it is symmetrical about the centre and derive the intensities I_n of the diffracted beams for order $n = 0,\ \pm1,\ \pm2,\ \pm3$, expressed relative to I_1. How might the phase distribution of each line be further modified in order to increase preferentially the intensity of beams on one side of the pattern?

Q 29 (B) What is meant by the resolution and order of the spectrum of a diffraction grating? It is planned to resolve a pair of spectral lines of mean wavelength $656.3\,\mathrm{nm}$ and separation $0.015\,\mathrm{nm}$, using a standard spectrometer configuration of spectral slit and collimator lens ($100\,\mathrm{mm}$ focal length) to produce a parallel beam of light which falls normally onto a plane diffraction grating (600 lines per mm); the diffracted beams are viewed with a telescope. What requirements does the required spectral resolution place on:

a. the size of the diffraction grating;
b. the diameters of the collimator and telescope objective lenses?

Q 30 (B) Signals from a transmitter of $0.1\,\mathrm{m}$ wavelength situated at a height of $2\,\mathrm{m}$ above the ground are picked up by a receiver in a radio telescope $22\,\mathrm{m}$ above ground at a distance of $1\,\mathrm{km}$. It is desired to screen the receiver from these signals by erecting an absorbing screen $50\,\mathrm{m}$ from the transmitter. How high must it be if the received power is to be reduced by a factor of 100? (Neglect reflections from the ground.)

Q 31 (B) [An optional `MATLAB` exercise] An aperture lies in the xy plane at $z = 0$, and consists of a long slit of width d. The slit is transparent in the region $|y| < d/2$; elsewhere it blocks incoming light.

It is illuminated at normal incidence by a plane wave of wavelength λ travelling in the z direction. The diffraction pattern is observed on a screen at $z = D$ parallel to the aperture.
The complex diffraction pattern observed on the screen at coordinate $(0, Y, D)$ can be written in the approximate form

$$\psi(Y) \propto \int_{-d/2}^{d/2} \exp(ikr)K(\theta)\,\mathrm{d}y$$

where $k = 2\pi/\lambda$, $r = \sqrt{(Y-y)^2 + D^2}$, $\tan\theta = (Y-y)/D$ and $K(\theta)$ is the obliquity factor. Explain the origin of this equation in terms of Huygens' construction.
Use MATLAB (or another language) to compute the complex diffraction pattern $\psi(Y)$ by integration. Use the values $\lambda = 500\,\mathrm{nm}$, $d = 1\,\mathrm{mm}$, and assume the obliquity factor is unity: $K(\theta) = 1$. Plot the diffracted intensity $|\psi(Y)|^2$ for D =10 mm, 100 mm, 326 mm, 640 mm and 4000 mm and comment on the patterns as D varies (recall the Fresnel distance is d^2/λ).
[You will need to use a MATLAB integration routine such as `quad` or `quadgk`. You can check some of your patterns against Fig. 10.70 in Hecht's Optics book.]

Q 32 (BC) While an observer on Earth is observing a distant star, the Moon moves across the observer's line of sight, blocking out the star: this is a lunar occultation. The speed of the Moon relative to the Earth's surface is $500\,\mathrm{m\,s^{-1}}$ and the apparent velocity vector lies perpendicular to the edge of the moon that blocks the star. The star is being observed through a filter which transmits only a small range of wavelengths near $500\,\mathrm{nm}$ and the Moon is $4 \times 10^5\,\mathrm{km}$ from the observer.

a. Sketch the apparent brightness of the star as a function of time (the 'lightcurve') during the occultation, assuming the star is a point source of light. (You may assume that the limb of the moon acts as a straight edge).
b. What is the time between the last two maxima in brightness before the star disappears?
c. Explain how the observed light curve would change if the star's angular size was not negligible.

Q 33 (B) Plane waves of monochromatic ($\lambda = 0.5\,\mu\mathrm{m}$) light are incident normally on an aperture. A detector is situated on axis at a distance of $D = 20\,\mathrm{mm}$ from the aperture plane.

a. What is the value of ρ_1, the radius of the first Fresnel half-period zone on the aperture, as observed from the detector?
b. If the aperture is a circular hole of radius $r = 1\,\mathrm{mm}$, centred on axis, how many half-period zones does it contain? Sketch the intensity received by the detector if it were moved along the axis such that D takes on values in the range $(20 \pm 1)\,\mathrm{mm}$. If the detector is moved away from the aperture along the optical axis, what is the furthest distance from the aperture that zero intensity will be detected?
c. If the aperture is a zone plate with odd numbered zones blocked out, and with the radius of the first zone equal to ρ_1 as in (a), determine the first three focal lengths of the zone plate.
d. If the detector is moved along the axis away from the aperture, what is the distance D at which the final clear minimum is observed?

Q 34 (B) In an experiment to observe Poisson's spot, a black circular disc is placed in front of, and orthogonal to, a plane wave light source of wavelength $\lambda = 0.5\,\mu\mathrm{m}$. The light falls onto a screen orthogonal to the incoming rays at a distance $D = 2.0\,\mathrm{m}$. The disc has a nominal radius of $r = 4.0\,\mathrm{mm}$, but manufacturing errors cause its radius to vary irregularly, with standard deviation $\pm\sigma$ from its nominal radius.
For Poisson's spot to be observed, how accurately circular must the disc be, i.e. what value of σ can be tolerated? Give your answer in terms of the Fresnel half-period zones, and calculate its size in this example. Estimate the size of the bright spot in this example if the disc is perfectly circular.
How is the argument for Poisson's spot affected if the incident light is: (a) white; (b) not perfectly collimated (i.e. from an extended, incoherent source)?

Q 35 (B) A Fresnel zone plate has overall radius a and contains N half-period zones. The odd-numbered zones are open but the even-numbered zones are covered by glass so as to invert the phase of light passing through them. The zone plate is illuminated with monochromatic, collimated light of wavelength λ and intensity I_0.

a. Show that the principal focal length f is given by

$$f = \frac{a^2}{N\lambda}.$$

b. Show that the intensity in the focal spot is

$$I = \frac{4I_0 a^4}{f^2\lambda^2}.$$

c. By equating the total power in the focal spot with the total power incident on the zone plate, show that the radius of the focal spot is roughly equal to the width of the outermost half-period zone.

d. Compare the performance of the zone plate with that of a perfect glass lens of radius a and focal length f.

e. Show that along the optic axis the focal spot has a length (between two points of zero intensity) of $4f/N$. (HINT:- the intensity on axis will be zero if the phase change across the first open zone is $\pi(1 + 2/N)$.)

Q 36 (BC) The sodium D-lines are a pair of narrow, closely spaced, approximately equal intensity spectral lines with a mean wavelength of approximately $589\,\mathrm{nm}$. A Michelson interferometer is set up to study the D-lines from a sodium lamp. High contrast fringes are seen for zero pathlength difference between the two arms of the interferometer. The fringes disappear when the pathlength difference is increased to $0.29\,\mathrm{mm}$.

a. What is the wavelength difference between the lines?

b. What would you expect to see if the pathlength difference were increased to $0.58\,\mathrm{mm}$, assuming the spectral lines are very narrow?

c. If the spectral lines have approximately Gaussian shapes, with a width of $50\,\mathrm{pm}$ (taken between the points of the line shape where the intensity falls to $\mathrm{e}^{-1/2}$ of the peak intensity), what is the maximum fringe contrast (visibility) seen for a pathlength difference of around $4\,\mathrm{mm}$?

Numerical answers

Q2 (a) $(0.521 \pm 0.002)\,\mathrm{rad\,s^{-1}}$; (b) 310 ± 7. (c) $(1.60 \pm 0.01) \times 10^{-5}\,\mathrm{N\,m\,rad^{-1}}$;

Q4 (a) $20\,\mathrm{rad\,s^{-1}}$, 4, 0.016; (b) (i) $19\,\mathrm{mm}$, $-163°$; (ii) $100\,\mathrm{mm}$, $90°$; (iii) $25\,\mathrm{mm}$, $0°$; (c) (i) $0.17\,\mathrm{W}$; (ii) $2.0\,\mathrm{W}$

Q8 $8l/9$; $m/8$

Q9 $204\,\mathrm{N}$

Q10 (a) $a\sin\theta\cos(kz - \omega t + \phi_0)$, $b\sin(kz - \omega t)$, $(c/2)\sin(kz - \omega t)$; (b) either $a = b = c/2$, $\theta = 0$, $\phi_0 = 0$, or $a = c/2$, $b = c$, $\theta = \pi/2$, $\phi_0 = \pi/2$.

Q13 (a) $(Z_2 - Z_1)/(Z_2 + Z_1)$;
(b) $1.5 \times 10^3\,\mathrm{m\,s^{-1}}$, $1.5 \times 10^6\,\mathrm{kg\,m^{-2}\,s^{-1}}$, $2.3 \times 10^9\,\mathrm{Pa}$;
(c) $1.3 \times 10^3\,\mathrm{m\,s^{-1}}$, $1.0 \times 10^6\,\mathrm{kg\,m^{-2}\,s^{-1}}$

Q14 (a) $2.2\,\mathrm{Pa}$ (b) $2.6\,\mathrm{kHz}$, $4.8\,\mathrm{s}$

Q16 38%; $14.4\,\mathrm{mm}$

Q17 (a) $1300\,\mathrm{s}$ (b) Approximately $2.1 \times 10^4\,\mathrm{km}$; approximately $27\,\mathrm{m\,s^{-1}}$

Q18 (a) $2 \times 10^{14}\,\mathrm{Hz} \pm 5 \times 10^9\,\mathrm{Hz}$; (b) $1.97 \times 10^8\,\mathrm{m\,s^{-1}}$, $\pm 130\,\mathrm{m\,s^{-1}}$; (c) $1.5 \times 10^5\,\mathrm{m}$

Q20 $\frac{2}{R}\sqrt{\frac{L}{C}}$

Q25 E or F, I, N, U

Q26 (b) $I_n/I_0 = 1$ for $n = 0, \pm 4, \pm 8, \ldots$; $I_n/I_0 = 0.5$ for $n = \pm 1, \pm 3, \pm 5, \ldots$; $I_n/I_0 = 0$ for $n = \pm 2, \pm 6, \ldots$

Q27 (a) $12\,\mathrm{K}$ (b) $1.0 \times 10^{-8}\,\mathrm{Hz}$
Q28 $I_0/I_1 = 0;\ I_{\pm1}/I_1 = 1;\ I_{\pm2}/I_1 = 2.9.;\ I_{\pm3}/I_1 = 3.77$
Q30 $3.6\,\mathrm{m}$ above the line joining the transmitter and receiver, i.e. $6.6\,\mathrm{m}$ above the ground.
Q32 $23\,\mathrm{ms}$
Q33 (a) $100\,\mathrm{\mu m}$ (b) At locations on axis $(20 \pm 0.2\mathrm{i})\,\mathrm{mm}$ there are maxima for even j, and zero intensity for odd j. (c)$20\,\mathrm{mm}$, $6.7\,\mathrm{mm}$, $4\,\mathrm{mm}$
Q36 (a)0.6 nm (c)0.0014