Large Scale Structure and Galaxy Formation

Lecture 3

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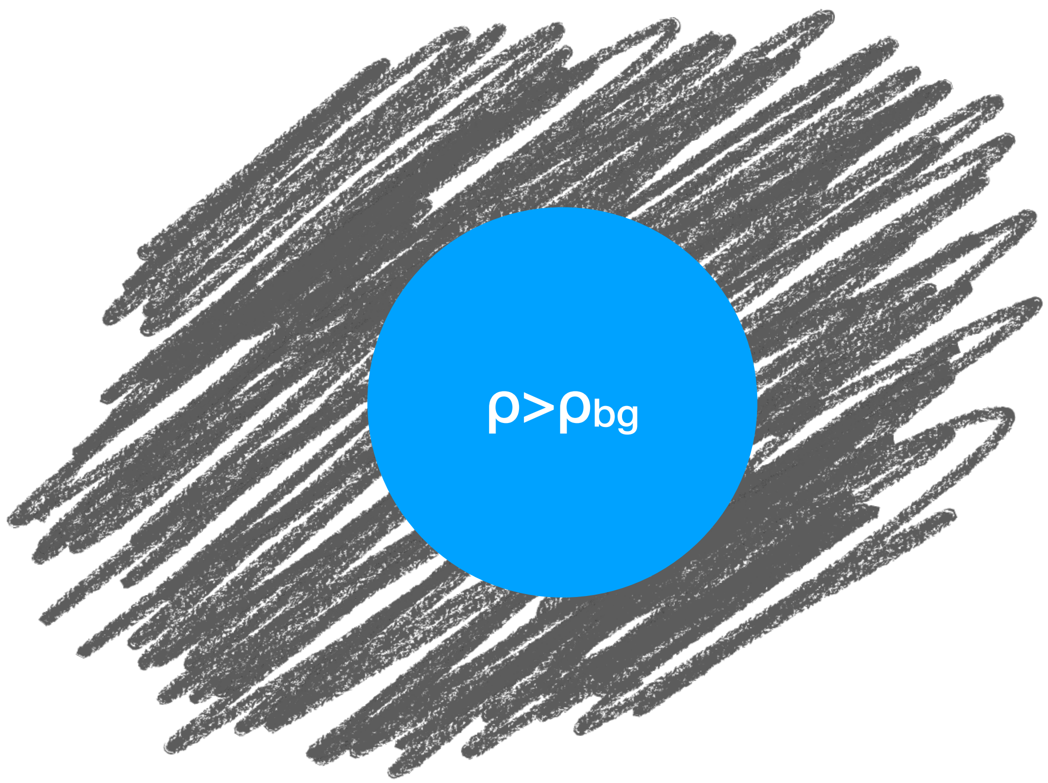
Two topics today:

- Non-linear collapse of a constant-density spherical over-dense region (‘top-hat’ perturbation)
  - turnaround
  - virialisation



- Statistics of large-scale structure:
  - Power spectra and correlation functions

Spherical collapse

$\rho > \rho_{bg}$

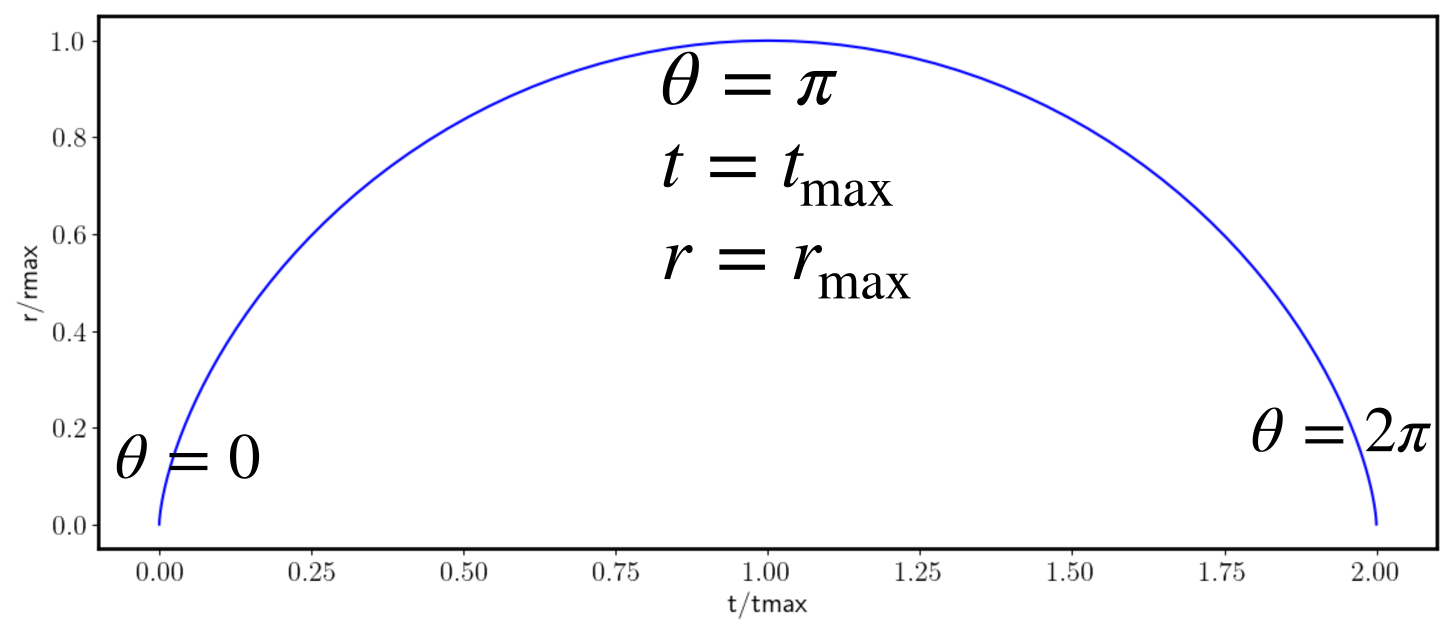
- We can treat an overdense region independent of the surrounding Universe (Birkhoff's theorem). Assume gravity dominates (neglect pressure):

$$\dot{r}^2 = \frac{2GM}{r} + K = 2GM\left(\frac{1}{r} - \frac{1}{r_{\max}}\right)$$

parametric solution: $\begin{cases} r(\theta) = \frac{1}{2} r_{\max}(1 - \cos\theta) \\ t(\theta) = \frac{1}{\pi} t_{\max}(\theta - \sin\theta) \end{cases}$

use $\dot{r} = \dfrac{dr}{d\theta} \Big/ \dfrac{dt}{d\theta} = \dfrac{(r_{\max}/2)\sin\theta}{(t_{\max}/\pi)(1 - \cos\theta)}$

$$\boxed{\Rightarrow \quad r_{\max}^3 = \frac{8GM}{\pi^2} t_{\max}^2}$$



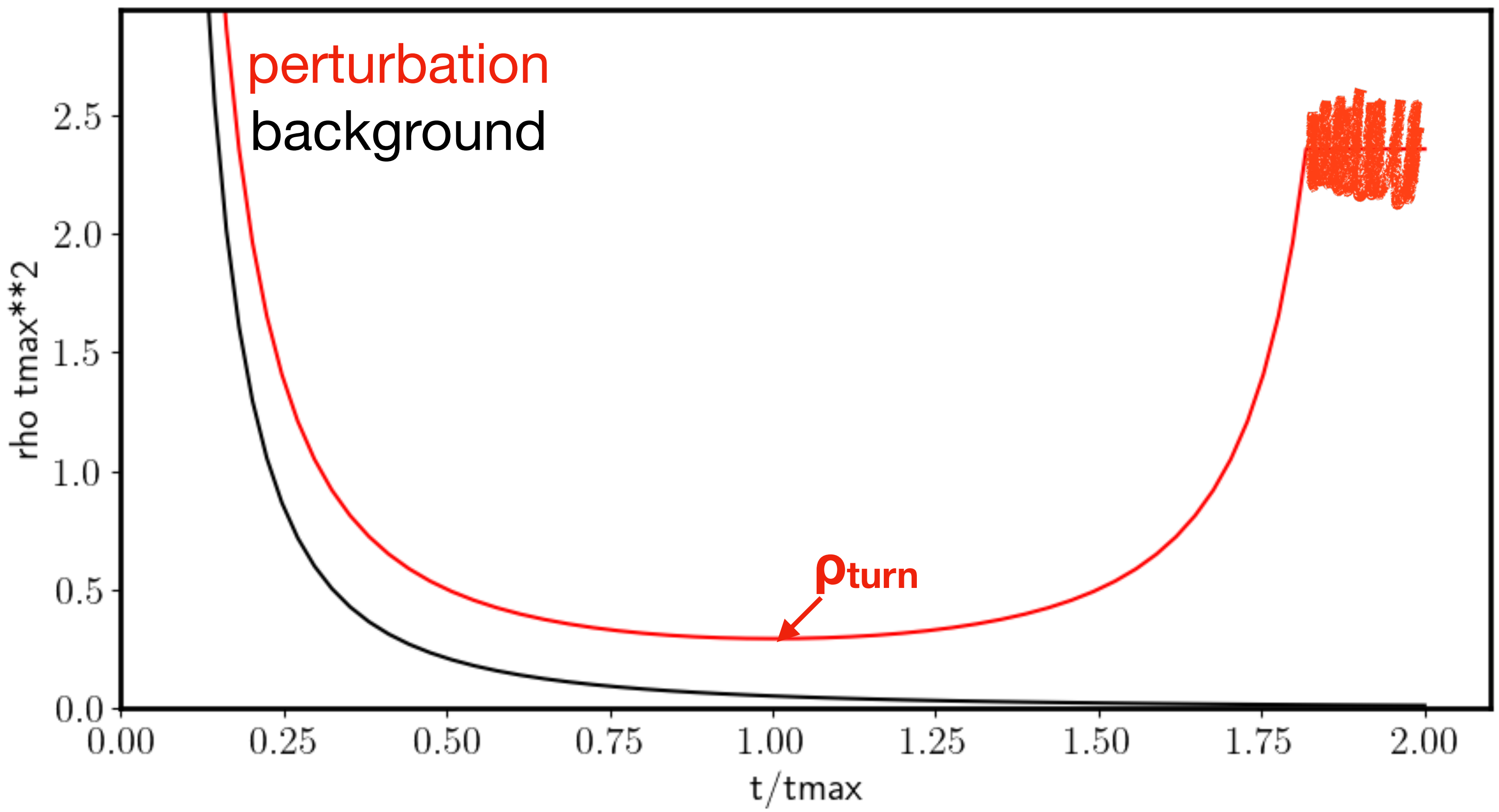
for small θ ($r \ll r_{\max}$):$\Rightarrow$ $\begin{cases} t \simeq \frac{1}{6\pi} t_{\max}\theta^3 \\ r \simeq \frac{1}{4} r_{\max}\theta^2 \simeq \frac{1}{4} r_{\max}(6\pi t/t_{\max})^{2/3} \\ \rho \simeq \frac{M}{\frac{4}{3}\pi r^3} \simeq (6\pi G t^2)^{-1} \end{cases}$

Einstein-de Sitter (Ω_m=1) solution

Spherical collapse

- Now look at the density contrast with respect to EdS background

At $t = t_{\rm max}$, $\rho_{\rm turn} = \dfrac{M}{\frac{4}{3}\pi r_{\rm max}^3} = \dfrac{3\pi}{32 G t_{\rm max}^2}$ and $\rho_{\rm bg} = \dfrac{1}{6\pi G t_{\rm max}^2}$ $\Rightarrow$ $\dfrac{\rho_{\rm turn}}{\rho_{\rm bg}} = \dfrac{9\pi^2}{16} \simeq 5.5$



- This is no longer a small linear density perturbation!
- After turnaround time $t_{\rm max}$ the contrast increases further ↑↓

Spherical collapse

- Relate this to the initial *linear* perturbation (leading-order deviation from background, for small θ):

$$t = \frac{1}{\pi}t_{\max}(\theta - \sin\theta) \simeq \frac{1}{\pi}t_{max}\left(\frac{\theta^3}{6}\right)\left(1 - \frac{\theta^2}{20}\right)$$

$$r = \frac{1}{2}r_{\max}(1 - \cos\theta) \simeq \frac{1}{4}r_{\max}\theta^2\left(1 - \frac{\theta^2}{12}\right)$$

$$\rho(\theta) = \rho_{\rm turn}(r/r_{\max})^{-3} = \frac{8\rho_{\rm turn}}{(1-\cos\theta)^3} \simeq 64\rho_{\rm turn}\theta^{-6}\left(1 + \frac{1}{4}\theta^2\right) \quad \text{with} \quad \rho_{\rm turn} = \frac{3\pi}{32Gt_{\max}^2}$$

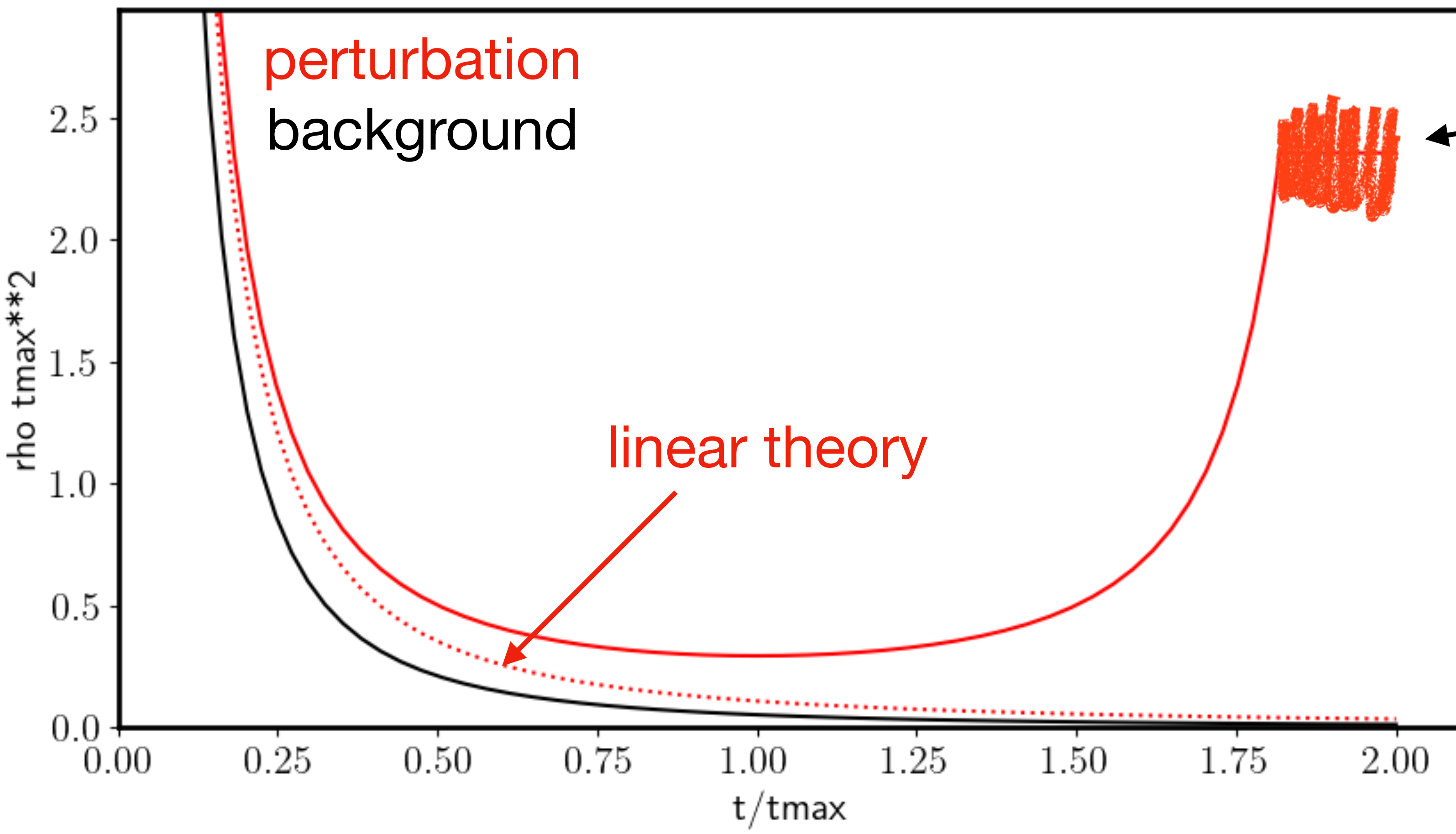
- invert t(θ): $$\theta \simeq \left(\frac{6\pi t}{t_{\max}}\right)^{1/3}\left[1 + \frac{1}{60}\left(\frac{6\pi t}{t_{\max}}\right)^{2/3}\right]$$

- hence $$\rho(t) \simeq \frac{1}{6\pi G t^2}\left[1 - \frac{1}{10}\left(\frac{6\pi t}{t_{\max}}\right)^{2/3}\right]\left[1 + \frac{1}{4}\left(\frac{6\pi t}{t_{\max}}\right)^{2/3}\right] \simeq \rho_{\rm bg}\left[1 + \frac{3}{20}\left(\frac{6\pi t}{t_{\max}}\right)^{2/3}\right]$$

- so to leading order in t, density contrast is $$\delta_{\rm LIN} = \frac{\delta\rho}{\rho_{\rm bg}} \simeq \frac{3}{20}\left(\frac{6\pi t}{t_{\max}}\right)^{2/3}$$

Spherical collapse

- Linear growth formula predicts, at t=t_{max} $\delta_{LIN} = \frac{\delta\rho}{\rho_{bg}} \simeq \frac{3}{20}\left(\frac{6\pi \cancel{t}}{\cancel{t}_{max}}\right)^{2/3} = 1.0624$
  which is NOT ≪1! Linear theory not valid at t_{max} .
- actual overdensity at turn-around is $1 + \delta(t_{max}) = 9\pi^2/16 \simeq 5.5$



collapse at

$t_{coll} = 2t_{max}$:

$\rho_{bg} = \frac{1}{24\pi G t_{max}^2}$

$\delta_{LIN} = 1.69$

$\rho = ??$

- After t_{max}? The region collapses under its own gravity: t_{coll} = 2 t_{max} .
  - ⇒ **Virialisation**

Virialisation

- Because of asymmetries the region will not collapse to a point
- It will reach an *equilibrium* in which motions balance self-gravity

at that point **2x(Kinetic Energy) + (Potential Energy) = 0**

equivalently **2x(Total Energy) = (Potential Energy)**

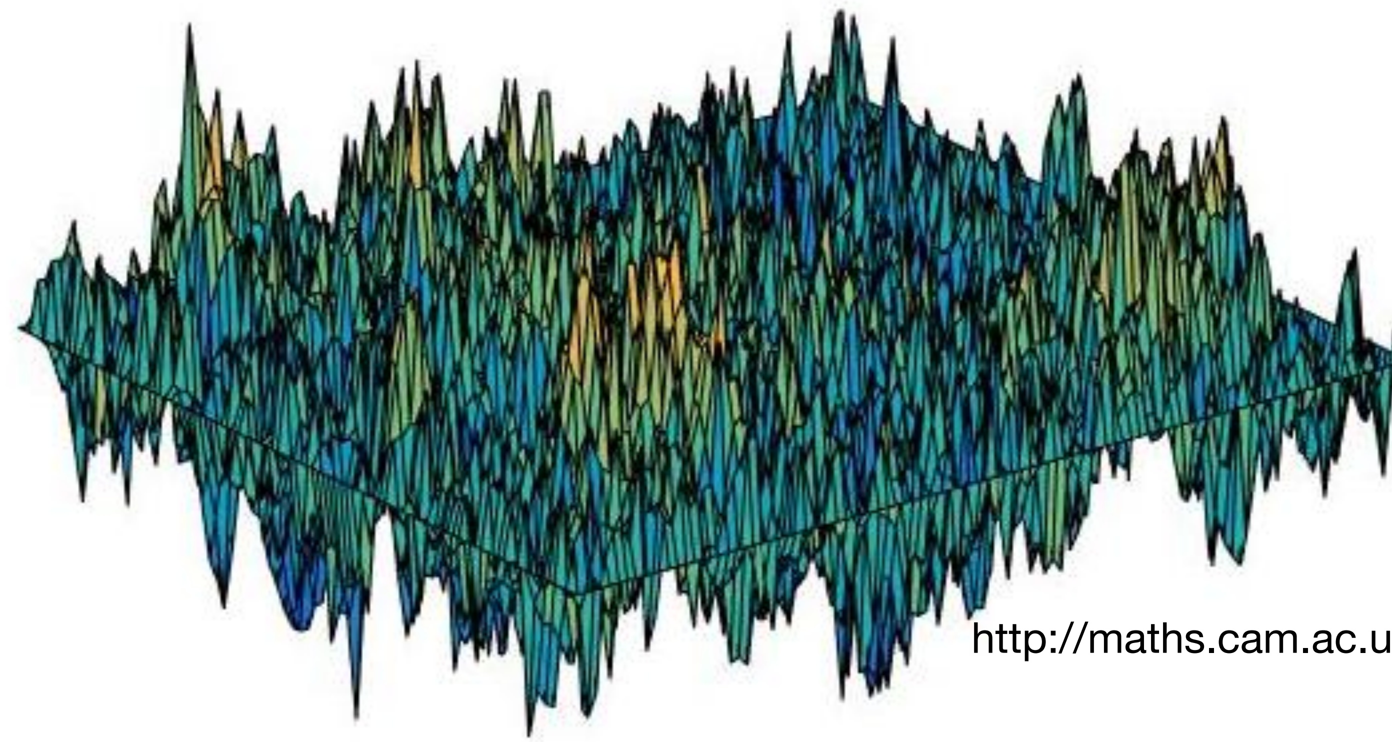
VIRIAL THEOREM

- Total energy = potential energy at turnaround (when kinetic energy=0)
  - so in virial equilibrium $P.E._{vir}=2\ P.E._{turn}$.
  - Since P.E. ~ GM/r : $r_{\rm vir} = r_{\rm max}/2; \qquad \rho_{\rm vir} = 8\rho_{\rm turn} = \dfrac{3\pi}{4Gt_{\rm max}^2}$
  - hence at t_{coll} the density contrast is $\dfrac{\rho_{\rm vir}}{\rho_{\rm bg}} = \dfrac{3\pi}{4Gt_{\rm max}^2} \Big/ \dfrac{1}{6\pi G(2t_{\rm max})^2} = 18\pi^2 \simeq 178$

- A collapsed object has overdensity ~178 at t_{coll}. (today x $(1+z_{coll})^3$)
- It turns around at $t=t_{max}=t_{coll}/2$
- The linear overdensity δ = 1.06 at t_{max}, 1.69 at t_{coll} .

Approximate treatment of halo formation

- Use linear perturbation theory to predict amplitude of fluctuations
- Regions where the linear overdensity reaches 1.06 will turn around
- Regions where the linear overdensity reaches 1.69 have collapsed
  - their density is $178\rho_{bg}(t_{coll})$.



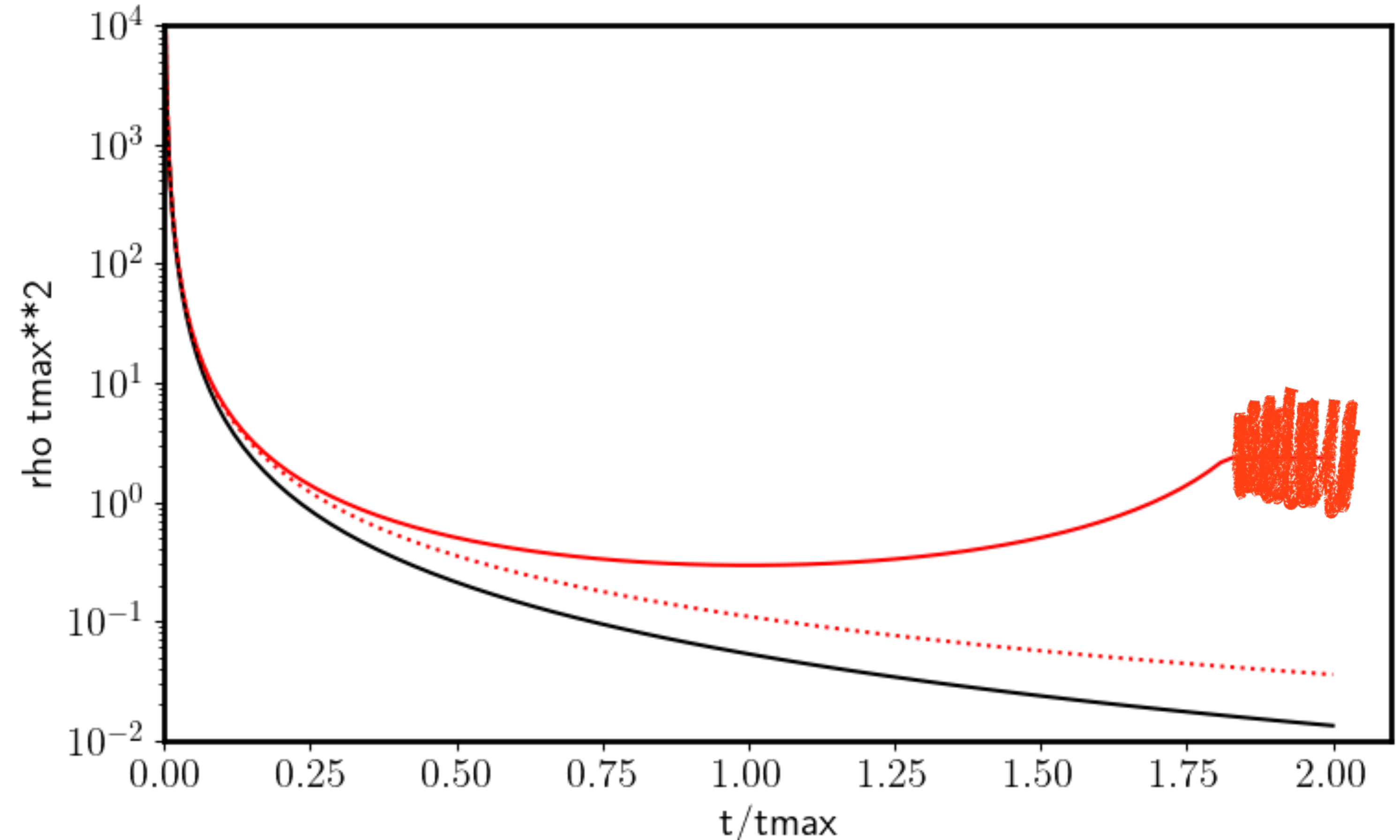
http://maths.cam.ac.uk

- Note that we have assumed an Einstein-de Sitter background
  - but it is common practice to use these numbers in approximate descriptions
- Also common to use overdensity of 200: choose "r_{200}" such that

$$\langle \rho \rangle_{r<r_{200}} = 200\bar{\rho}$$

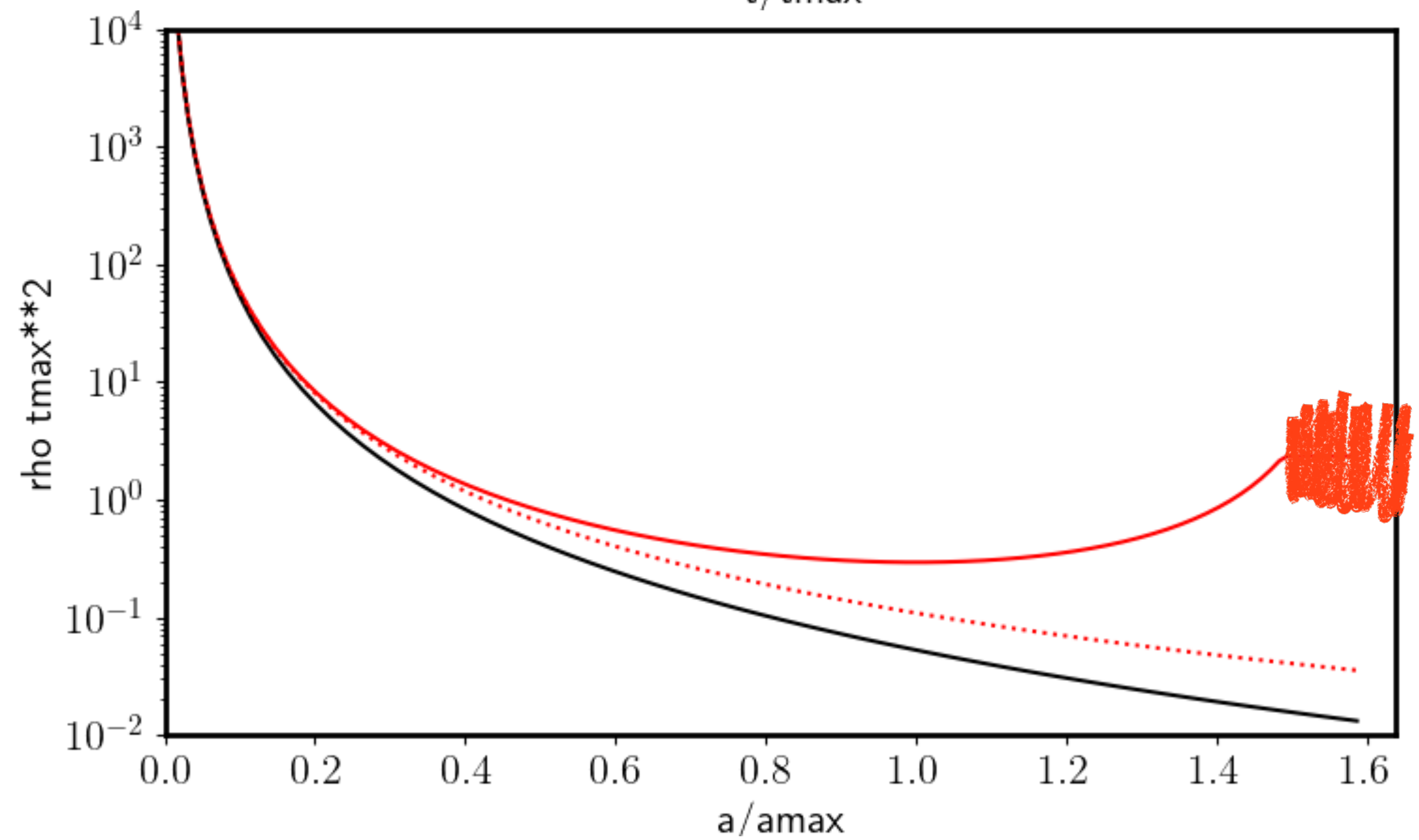
Different versions of the plots

- log(density) vs. t



- log(density) vs. a

(define $a(t_{max})=a_{max}$)

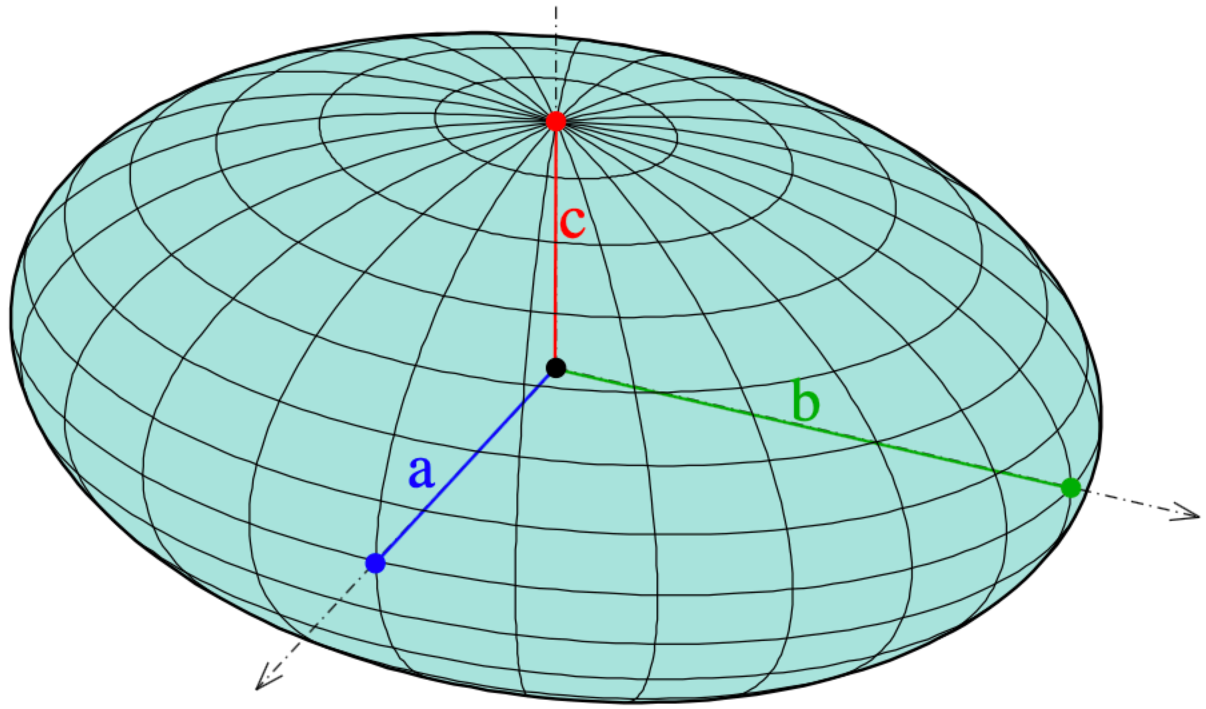


Triaxiality

- Peaks that are slightly elliptical will collapse anisotropically

- Shortest axis will collapse first, increasing the anisotropy

- After virialisation a triaxial object will result



- Typical axis ratios for CDM halos are 1:0.7:0.5 with quite some scatter and weak dependences on mass and redshift

TOPIC 2

The power spectrum of matter density fluctuations

- This tells us how strong fluctuations are as a function of their wavelength (or wavenumber $k = 2\pi/\lambda$)
- Define Fourier Transform of the density perturbation field $\delta(\mathbf{x})$:

$$\hat{\delta}(\mathbf{k}) = \int d^3\mathbf{x}\,\delta(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}} \quad \Leftrightarrow \quad \delta(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\hat{\delta}(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}$$

  - (for a real field we have $\hat{\delta}(-\mathbf{k}) = \hat{\delta}(\mathbf{k})^*$)
- The FT decomposes the field into sine (and cosine) waves
- The power spectrum of any field is defined as $\boxed{P(\mathbf{k}) = |\hat{\delta}(\mathbf{k})|^2}$
- In cosmology we cannot measure P itself (infinite integral) but we can estimate it from the finite volume of space that we can observe.
- So we work with 'expected' $\langle|\hat{\delta}(\mathbf{k})^2|\rangle$ per unit volume

I've cheated…

- If we take the Fourier integral

$$\hat{\delta}(\mathbf{k}) = \int d^3\mathbf{x}\ \delta(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}$$

  over an infinite volume, we get infinity unless $\delta \to 0$ at large $|\mathbf{x}|$ (which conflicts with homogeneity of the universe — cosmol. principle)
- δ averages to zero, but its variance $\langle\delta^2\rangle$ is positive
- So we normalise the integral by 1/√Volume, and take the limit to large Volume

$$\hat{\delta}(\mathbf{k}) = \lim_{V\to\infty} \frac{1}{\sqrt{V}} \int_V d^3\mathbf{x}\delta(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}$$

- This ensures that the values of $|\hat{\delta}_k|^2$ (i.e., the power spectrum) converge.

- Instead of taking the limit to infinite volume we can also average over many random finite-volume realisations of the density field

Statistical Cosmological Principle and Ergodic Theorem

- Cosmological Principle: Universe is homogeneous and isotropic
- This is only true on the largest scales; on smaller scales we have density fluctuations $\delta(\mathbf{x}, t)$.
- Our Universe has one particular random realisation of this field
- SCP says that '*on average*' $\langle \delta(\mathbf{x}, t) \rangle = 0$.
- Theory predicts the probability distribution of $\delta(\mathbf{x}, t)$.
  - and hence of $\hat{\delta}_k(t)$.
  - (isotropic, so only depends on magnitude of $\mathbf{k}$, not direction)

- Ergodic Theorem: ensemble averages over many realisations $\Leftrightarrow$ spatial averages of 1 realisation
  - (in statistical mechanics RHS is usually time average)
  - Applies to Gaussian random fields of interest for LSS.
  - This means all statistical info about δ is available in a single sample over all space

Gaussian Random Fields

- Initial fluctuations were random, Gaussian, isotropic:
    - independent Gaussian distributions of real, imag $\mathscr{R}(\hat{\delta}_k), \mathscr{I}(\hat{\delta}_k)$
      with mean = 0, variance $\frac{1}{2}\left\langle |\hat{\delta}_k|^2 \right\rangle$

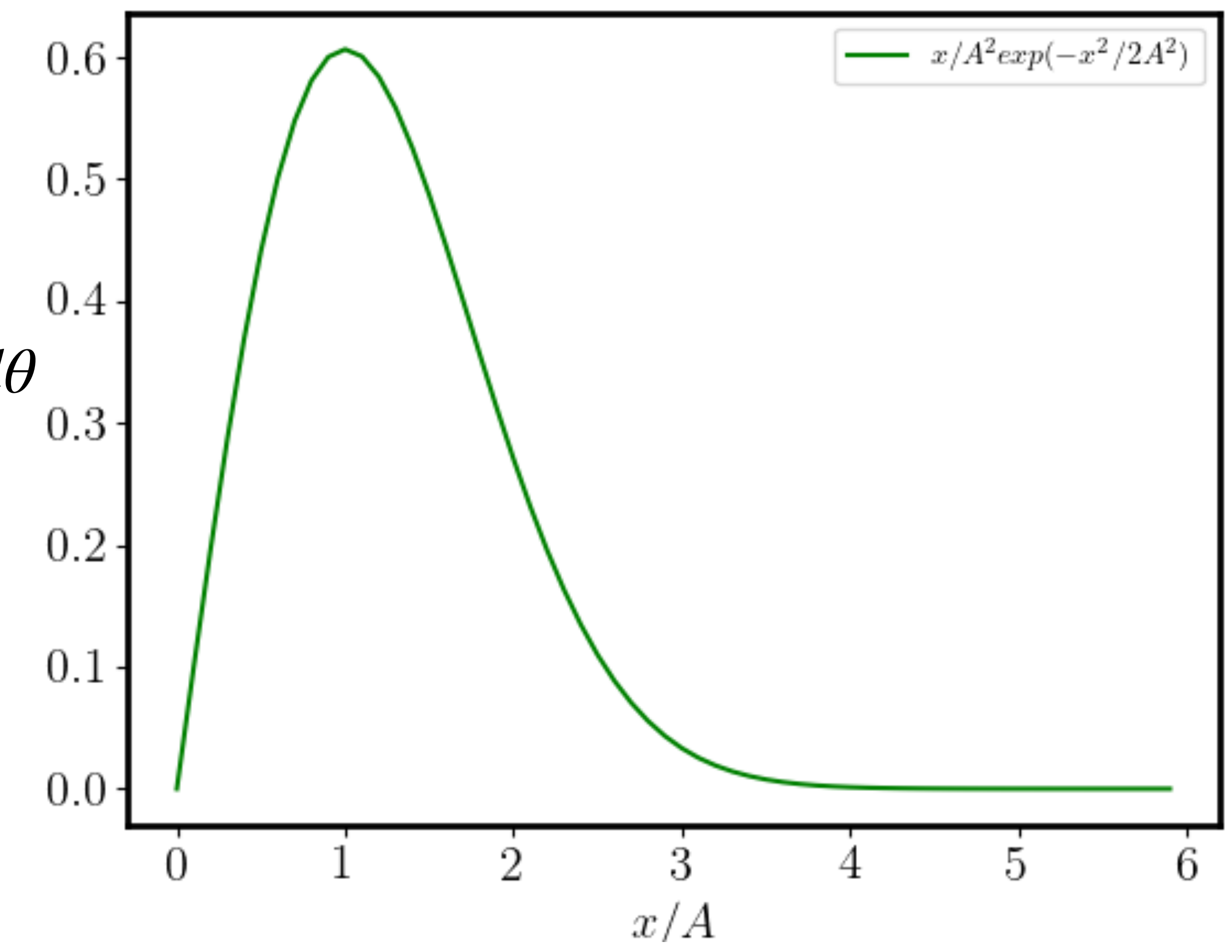
dependence on $k \equiv |\mathbf{k}|$ only

$$\text{Prob}\left(\mathscr{R}(\hat{\delta}_k)\right)\ d\mathscr{R}(\hat{\delta}_k) = \frac{e^{-\mathscr{R}(\hat{\delta}_k)^2/\langle|\hat{\delta}_k|^2\rangle}}{\sqrt{\pi\langle|\hat{\delta}_k|^2\rangle}} d\mathscr{R}(\hat{\delta}_k)$$

similarly for $\mathscr{I}(\hat{\delta}_k)$

- Or in polar coordinates $(|\hat{\delta}_k|, \theta)$

$$\text{Prob}(|\hat{\delta}_k|, \theta)\ d(|\hat{\delta}_k|)d\theta = \frac{e^{-|\hat{\delta}_k|^2/\langle|\hat{\delta}_k|^2\rangle}}{\pi\langle|\hat{\delta}_k|^2\rangle} |\hat{\delta}_k| d(|\hat{\delta}_k|)d\theta$$



- This shows that the phases θ are random, amplitudes follow a Rayleigh distribution

- Statistical distribution fully specified by the power spectrum

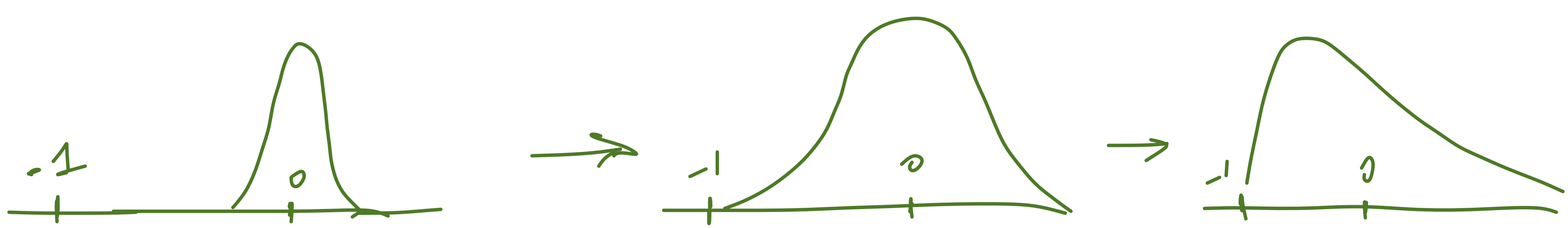
Gaussian density fluctuations

- The density at any point is therefore the sum of many Gaussian random variables:

$$\delta(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \hat{\delta}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

- … and therefore a Gaussian random variable itself

- Remember: linear density fluctuations, $|\delta| = |\rho - \rho_{\rm bg}|/\rho_{\rm bg} \ll 1$
  - By definition $\delta > -1$ so Gaussianity must break down at non-linear density fluctuation amplitudes
  - Gravitational collapse leads to virialisation with $\delta \sim 200$
    - $\Rightarrow$ nonlinear evolution leads to a very skew distribution of δ



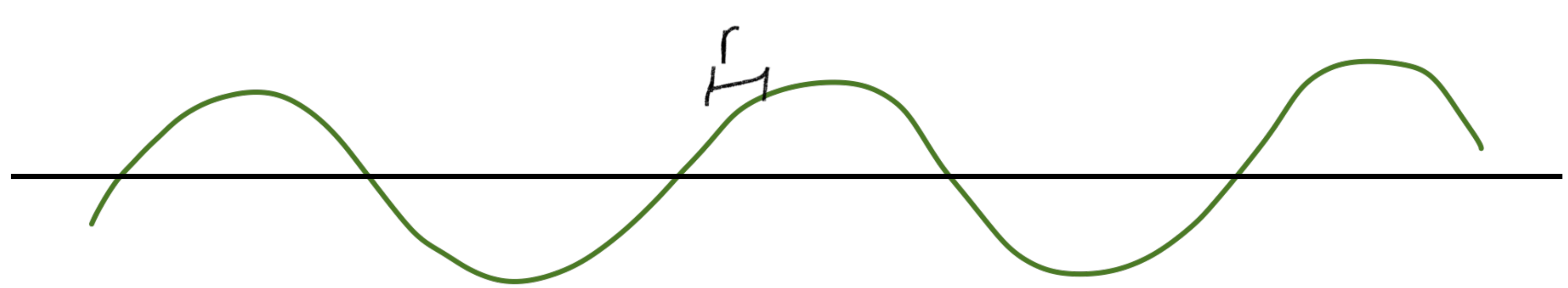
Gaussian density fluctuations

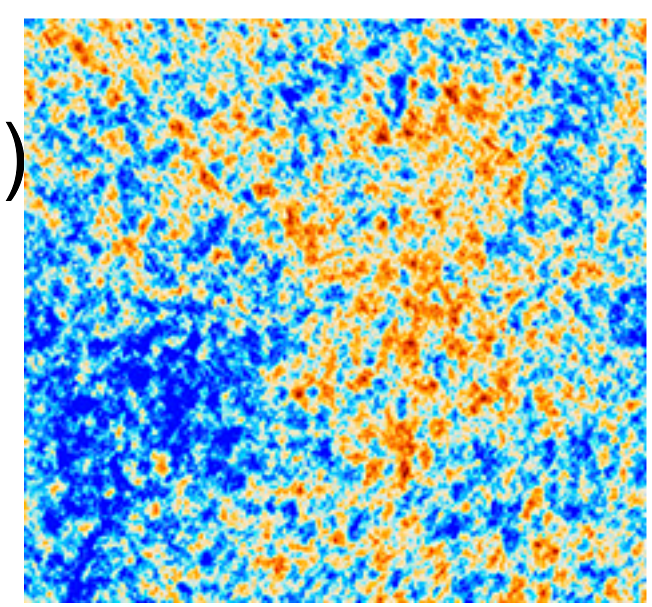
- Important result:

- The statistical properties of the initial Gaussian fluctuations are fully specified by the ***correlation function***
  - *Related to the power spectrum*

Correlation function

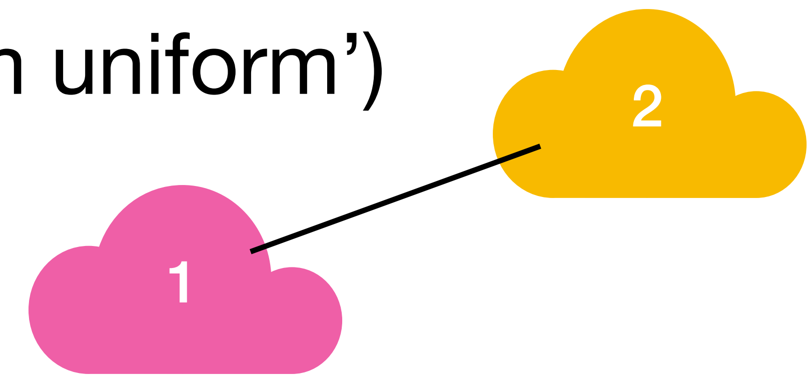
- Defined as
$$\xi(\mathbf{r}) = \langle \delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r}) \rangle_{\mathbf{X}}$$
  - where the average $\langle \rangle$ is defined spatially (i.e. over **x**)
    - (*ergodic theorem:* averaging over many realisations ≡ averaging over many pieces of one realisation).
- ξ identifies whether there is structure on scale r in the δ field
  - If density values at different places are uncorrelated $\xi = 0$
  - But if there are peaks/valleys of characteristic size then neighbouring densities will tend to have the same sign, so that gives a positive ξ on that scale
    - (e.g., calculate $\langle \sin(x)\sin(x+r) \rangle$ for $r \ll 1$)





Correlation function $\xi(\mathbf{r}) = \langle \delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\rangle_{\mathbf{X}}$

- In a density field, the correlation function can be seen as the *excess probability above the Poisson* ('random uniform') expectation for density ρ_{bg}.



- Let $P_{\text{Poisson}}(1,2)$ be the probability of finding galaxies in volumes 1&2, separated by $\mathbf{x}_2 - \mathbf{x}_1$
- Make the volumes small enough that they contain 0 or 1 galaxy
- For a uniform distribution $P_{\text{Poisson}}(1,2) = \rho_{\text{bg}}^2 dV_1 dV_2$.
- In general, for an inhomogeneous density field,

$$
\begin{aligned}
P(1,2) &= \langle \rho(1)\rho(2)\rangle dV_1 dV_2 = \langle \rho_{\text{bg}}(1+\delta(1))\,\rho_{\text{bg}}(1+\delta(2))\rangle dV_1 dV_2 \\
&= \rho_{\text{bg}}^2 \langle 1 + \delta(1) + \delta(2) + \delta(1)\delta(2)\rangle dV_1 dV_2 .
\end{aligned}
$$

- Since $\langle \delta()\rangle = 0$ and $\langle \delta(1)\delta(2)\rangle = \xi(\mathbf{x}_2 - \mathbf{x}_1)$, we have

$$\frac{P(1,2)}{P_{\text{Poisson}}(1,2)} = 1 + \xi(\mathbf{x}_2 - \mathbf{x}_1)$$

Excess Probability

Relation between power spectrum and correlation function

- Recall power spectrum $\boxed{P(\mathbf{k}) = \langle |\hat{\delta}(\mathbf{k})|^2 \rangle}$ for $\hat{\delta}(\mathbf{k}) = \frac{1}{\sqrt{V}} \int_V e^{i\mathbf{k}\cdot\mathbf{x}} \delta(\mathbf{x}) d^3\mathbf{x}$

- Hence…

$$P(\mathbf{k}) = \langle \hat{\delta}(\mathbf{k}) \hat{\delta}(\mathbf{k})^* \rangle = \left\langle \int_V \int_V e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} \delta(\mathbf{x})\delta(\mathbf{y}) \frac{d^3\mathbf{x}}{\sqrt{V}} \frac{d^3\mathbf{y}}{\sqrt{V}} \right\rangle$$

$(\mathbf{x} \to \mathbf{y} + \mathbf{r})$

$$= \left\langle \int_V \int_V e^{i\mathbf{k}\cdot(\cancel{\mathbf{y}}+\mathbf{r})} e^{\cancel{-i\mathbf{k}\cdot\mathbf{y}}} \delta(\mathbf{y})\delta(\mathbf{y}+\mathbf{r}) \frac{d^3\mathbf{x}}{V} d^3\mathbf{r} \right\rangle$$

$$= \left\langle \int_V \int_V e^{i\mathbf{k}\cdot\mathbf{r}} \xi(\mathbf{r}) \boxed{\frac{d^3\mathbf{x}}{V}} d^3\mathbf{r} \right\rangle$$

$$= \hat{\xi}(\mathbf{k})$$

integrates to 1

- … the power spectrum is the Fourier transform of the correlation function (!)
- ⇒ A Gaussian random field is specified by its power spectrum or its correlation function

- Power spectrum $P(\mathbf{k}) = \langle |\hat{\delta}(\mathbf{k})|^2 \rangle$ $\hat{\delta}(\mathbf{k}) = \frac{1}{\sqrt{V}} \int_V e^{i\mathbf{k}\cdot\mathbf{x}} \delta(\mathbf{x}) d^3\mathbf{x}$
- Note the dimensions of *P* are (length)^3.
- We also define the *dimensionless power spectrum* $\Delta^2(k) \equiv k^3 P(k)/(2\pi^2)$ which gives the amount of power per log interval of wavenumber
- From now on we will assume statistical isotropy and only consider the power spectrum to depend on $k = |\mathbf{k}|$, and the correlation function on $r = |\mathbf{x}|$.
- This gives $P(k) = \hat{\xi}(k)$ (still 3-D Fourier transform!)
- In that case we can write the inverse Fourier transform of $P(k)$ as

ˇ means inv.FT

$$\xi(r) = \check{P}(r) = \frac{1}{(2\pi)^3} \underbrace{\int_0^\infty k^2 dk\, P(k) \int_0^\pi \sin\theta d\theta e^{-ikr\cos\theta} \int_0^{2\pi} d\phi} = \frac{1}{(2\pi)^2} \int_0^\infty k^2 dk\, P(k) \underbrace{\left[\frac{e^{-ikr\cos\theta}}{ikr}\right]_0^\pi}_{= 2\sin(kr)/kr}$$

$$= \int_0^\infty \frac{k^2 dk}{2\pi^2} P(k) \frac{\sin(kr)}{kr}$$

(go to spherical polar coordinates in **k**, aligned with **x**)

- The variance of overdensity is

$$\sigma^2 = \langle \delta(\mathbf{x})^2 \rangle = \xi(\mathbf{0}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} P(\mathbf{k}) = \int_0^\infty \frac{4\pi k^2\, dk}{(2\pi)^3} P(k) = \int_{k=0}^\infty \Delta^2(k) d(\ln k)$$

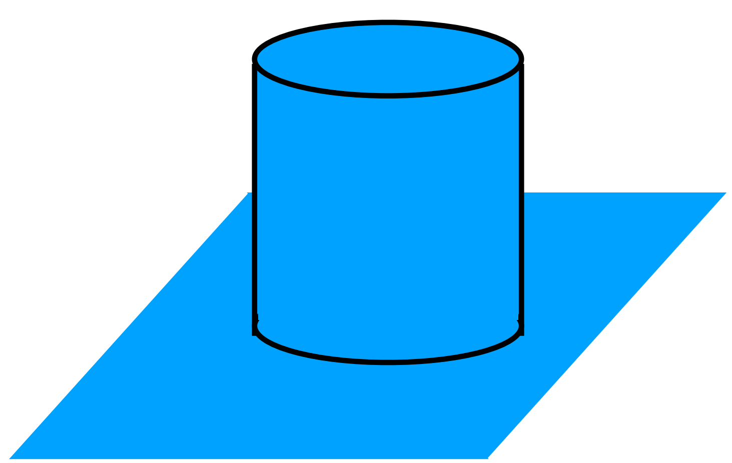
$e^{-i\mathbf{k}\cdot\mathbf{0}} = 1$

Filtered density fields

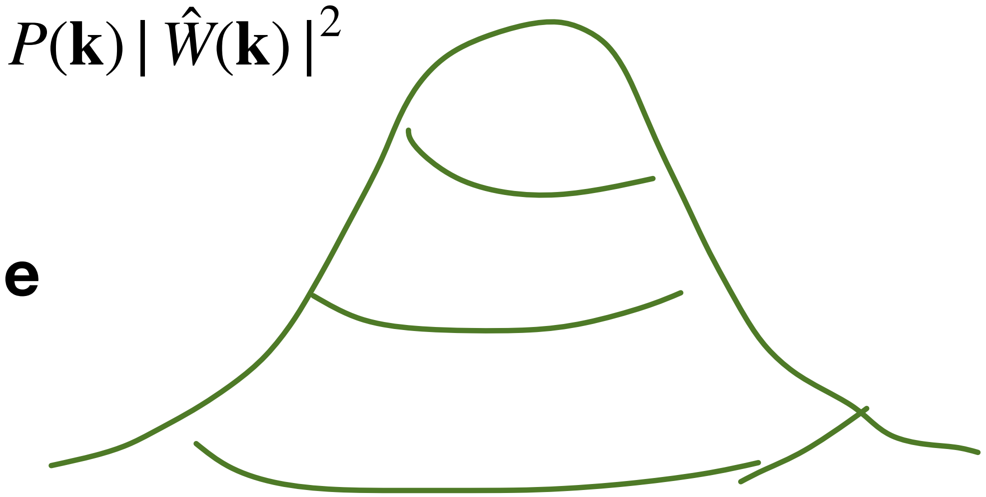
- The density only really makes sense if you specify how you smooth ('filter') it
  - i.e., convolve δ with a filter function $W(\mathbf{y})$ satisfying $\int W(\mathbf{y}) d^3\mathbf{y} = 1$
  - this gives the filtered density $\delta_W(\mathbf{x})$
- In Fourier space, convolution becomes multiplication:

$$\delta_W(\mathbf{x}) = \int \delta(\mathbf{y}) W(\mathbf{x}-\mathbf{y}) d^3\mathbf{y} \quad \Rightarrow \quad \hat{\delta}_W(\mathbf{k}) = \hat{\delta}(\mathbf{k}) \hat{W}(\mathbf{k})$$

$$P_W(\mathbf{k}) = P(\mathbf{k}) \, | \hat{W}(\mathbf{k}) |^2$$



smoothing scale
R_{TH}, R_G



- **"top hat"** filter (average over sphere)

$$W_{\mathrm{TH}}(\mathbf{x}) = \begin{cases} \frac{3}{4\pi R_{\mathrm{TH}}^3} & |\mathbf{x}| < R_{\mathrm{TH}} \\ 0 & |\mathbf{x}| > R_{\mathrm{TH}} \end{cases}$$

Gaussian filter

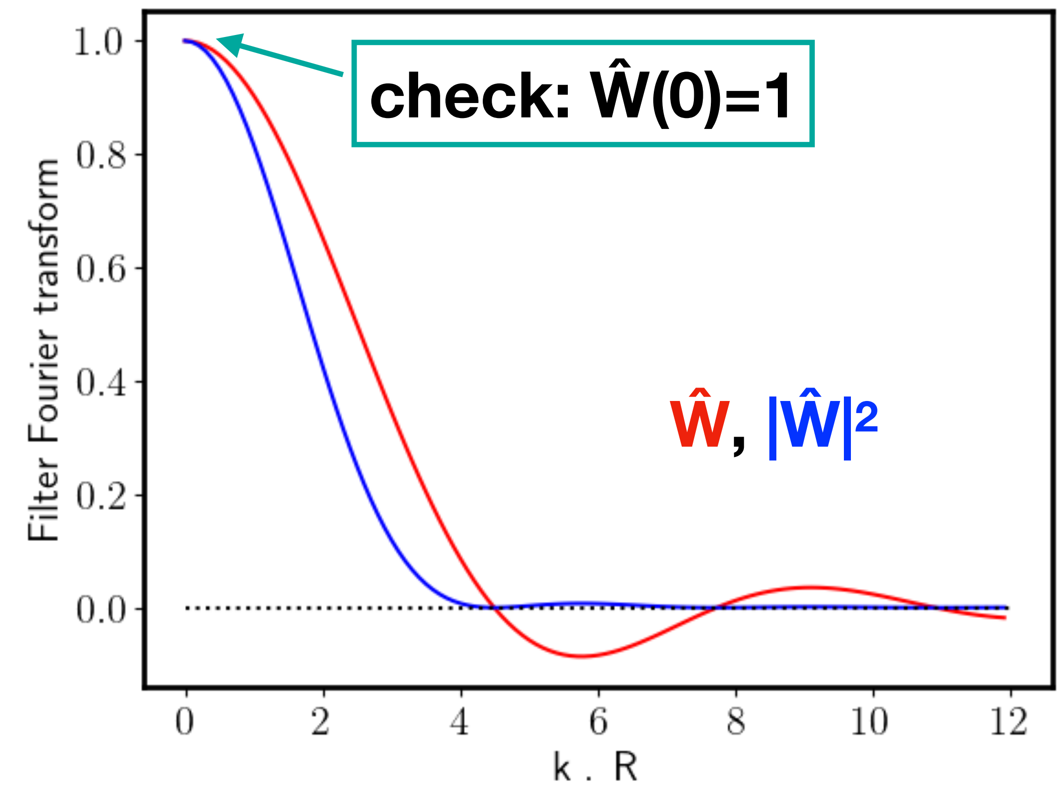
$$W_{\mathrm{G}}(\mathbf{x}) = \frac{e^{-|\mathbf{x}|^2/(2R_{\mathrm{G}}^2)}}{(2\pi)^{3/2} R_{\mathrm{G}}^3}$$

Filters in Fourier space

- Top hat: $$W_{\rm TH}(\mathbf{x}) = \begin{cases} \frac{3}{4\pi R_{\rm TH}^3} & |\mathbf{x}| < R_{\rm TH} \\ 0 & |\mathbf{x}| > R_{\rm TH} \end{cases}$$

$$\hat{W}_{\rm TH}(k) = 4\pi \int_0^\infty r^2 dr\, W(r) \frac{\sin(kr)}{kr} = \frac{3}{k^3 R_{\rm TH}^3}\left[\sin(kR_{\rm TH}) - kR_{\rm TH}\cos(kR_{\rm TH})\right]$$

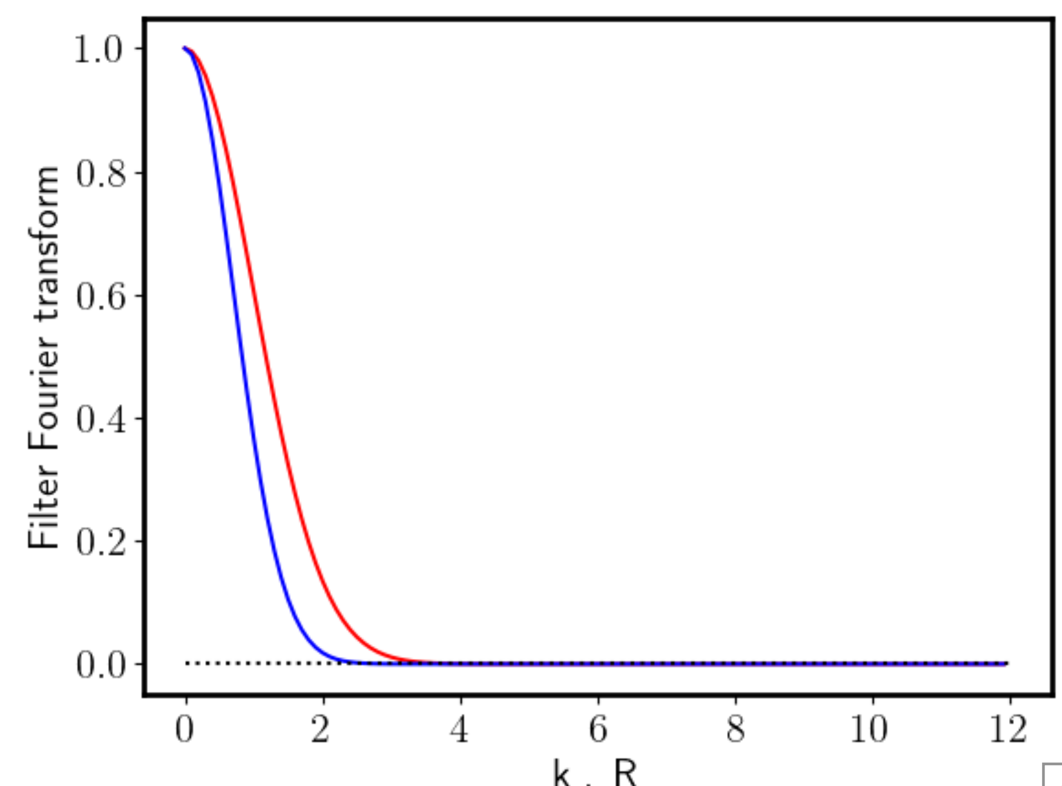
4π for FT,
1/(2π²) for inv.FT



- approx $$\hat{W}_{\rm TH}(k) = \begin{cases} 1 & k < \pi/R_{\rm TH} \\ 0 & k > \pi/R_{\rm TH} \end{cases}$$

- Gaussian $$W_{\rm G}(\mathbf{x}) = \frac{e^{-|\mathbf{x}|^2/(2R_{\rm G}^2)}}{(2\pi)^{3/2} R_{\rm G}^3}$$

$$\hat{W}_G(k) = e^{-k^2 R_G^2/2}$$



- Both examples are *low-pass filters* that suppress small-scale structure (at k>few/R)

Mass fluctuations

- A top-hat filter has an associated mass scale $\overline{M}_{\rm TH} = \dfrac{4\pi}{3} R_{\rm TH}^3 \rho_{\rm bg}$
    - (the average mass within the filter volume)
    - so label the filter with this mass: filter $W_{\rm M}$, filter radius $R_{\rm M}$
- What is the variance of the <u>smoothed</u> density field δ_W? recall

$$\sigma^2 = \langle \delta(\mathbf{x})^2 \rangle = \xi(\mathbf{0}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} P(\mathbf{k}) = \int_0^\infty \frac{4\pi k^2\, dk}{(2\pi)^3} P(k)$$

- so:

$$\sigma^2_{W_M} = \langle \delta_{W_M}(\mathbf{x})^2 \rangle = \xi_{W_M}(\mathbf{0}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} P(\mathbf{k}) \hat{W}_M(\mathbf{k}) \hat{W}_M(\mathbf{k})^* = \int_0^\infty \frac{4\pi k^2\, dk}{(2\pi)^3} P(k) \,|\, \hat{W}_M(k) \,|^2$$

- The integral stops where $\hat{W}_M \simeq 0$, at $k_M \simeq \pi / R_M$
- For a power-law power spectrum $P(k) \propto k^n$, (with $n > -3$)

$$\sigma^2_{W_M} \propto k_M^3 P(k_M) \propto R_{\rm M}^{-(n+3)} \propto \overline{M}^{-(n+3)/3}$$

- Useful approx: $\boxed{\sigma^2_{W_M} \simeq \xi(R_{\rm M})}$ since $\xi(r) = \displaystyle\int_0^\infty \frac{4\pi k^2\, dk}{(2\pi)^3} P(k) \frac{\sin(kr)}{kr}$

Mass fluctuations

- The σ_{W_M} (called σ_M from now on) we have just calculated is the rms scatter in the overdensity $\delta\rho/\rho_{\rm bg} = \delta M/\overline{M}$ of randomly placed spherical regions of radius R_M (that contain mass M on average).

- Observations: $\sigma_M \simeq 1$ for $R_M = 8h^{-1}{\rm Mpc}$ at present.

- It is customary to express the amplitude of the linear power spectrum in terms of σ_8, which is the rms fluctuation of δ filtered with an $8h^{-1}{\rm Mpc}$ top hat, predicted with linear theory for $z = 0$.

Linear evolution of fluctuations

- Time to put everything together!

- Assume an initial *dark matter* density fluctuation power spectrum, at some early time t_i (after inflation) with $P(k) \propto k^n$.
- Fluctuations with co-moving scale k enter horizon when $ct_{\rm ent} \simeq a(t_{\rm ent})/k$
- Recall $\hat{\delta}(k)$ grows as

$$\hat{\delta} \propto \begin{cases} a^2, & t < t_{\rm ent} < t_{\rm eq} \quad \text{or} \quad t < t_{\rm eq} < t_{\rm ent} \\ a^0, & t_{\rm ent} < t < t_{\rm eq} \\ a, & t > t_{\rm eq} \end{cases}$$

- So at late time $t \gg t_{\rm eq}$,

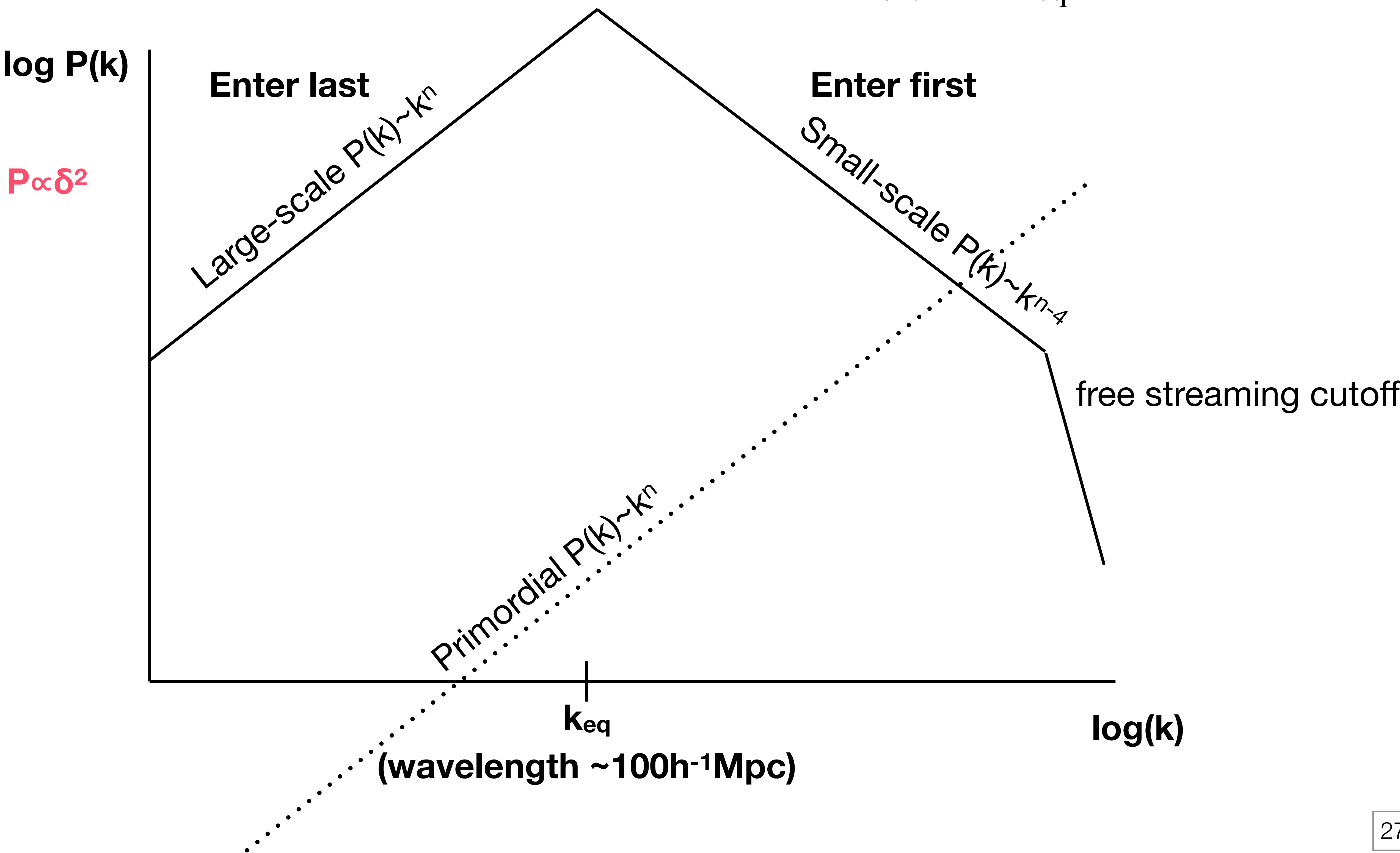
$$\hat{\delta}(k;t) = \hat{\delta}(k;t_i) \times \begin{cases} \left(\frac{a_{\rm eq}}{a_i}\right)^2 \left(\frac{a(t)}{a_{\rm eq}}\right), & t_{\rm ent} > t_{\rm eq} \quad \equiv \quad k < k_{\rm eq} \\ \left(\frac{a_{\rm ent}}{a_i}\right)^2 \left(\frac{a_{\rm eq}}{a_{\rm ent}}\right)^0 \left(\frac{a(t)}{a_{\rm eq}}\right), & t_{\rm ent} < t_{\rm eq} \quad \equiv \quad k > k_{\rm eq} \end{cases}$$

  - (large) scales with $k < k_{\rm eq}$ all grow by the same amount, $\propto a(t)$
  - (small) scales with $k > k_{\rm eq}$ pick up a factor $(a_{\rm ent}/a_{\rm eq})^2 = (k/k_{\rm eq})^{-2}$.

($t < t_{\rm eq}$ so $a \propto t^{1/2}$, $k \propto t_{\rm ent}^{-1/2} \propto a_{\rm ent}^{-1}$.)

Linear evolution of fluctuations

- The shape of the initial power spectrum is modified because the smaller scales grow more slowly between t_{ent} and t_{eq}:



The Harrison-Zel'dovich spectrum

- = Power law power spectrum with **n=1** (CMB tells us n~0.95)
  - Natural outcome of inflation: metric perturbations (~Newtonian potential Ψ) have same amplitude on all scales
    - Recall dimensionless power spectrum $\Delta^2 = \frac{1}{2\pi^2} k^3 P(k)$
      - Δ^2 gives the variance of δ per decade of k
      - F.T. of potential $\hat{\psi} \propto k^{-2}\hat{\delta}_k$, so pow.spec. of ψ is $P_\psi(k) \propto k^{-4}P(k)$
      - with $n = 1$, variance of ψ per decade of k is constant ✔

- If $P(k) = Ak^n$: rms of δ fluctuations / decade of k scales $\propto k^{(n+3)/2}$.
  - Fluctuation of scale k enters the horizon at time $t_{\text{ent}} \propto k^{-2}$ and at that time has grown by factor $\propto a_{\text{ent}}^2 \propto t_{\text{ent}} \propto k^{-2}$
  - Hence (rms δ)/k-decade for fluctuations entering the horizon scales $\propto k^{(n-1)/2}$ — invariant for $n = 1$.

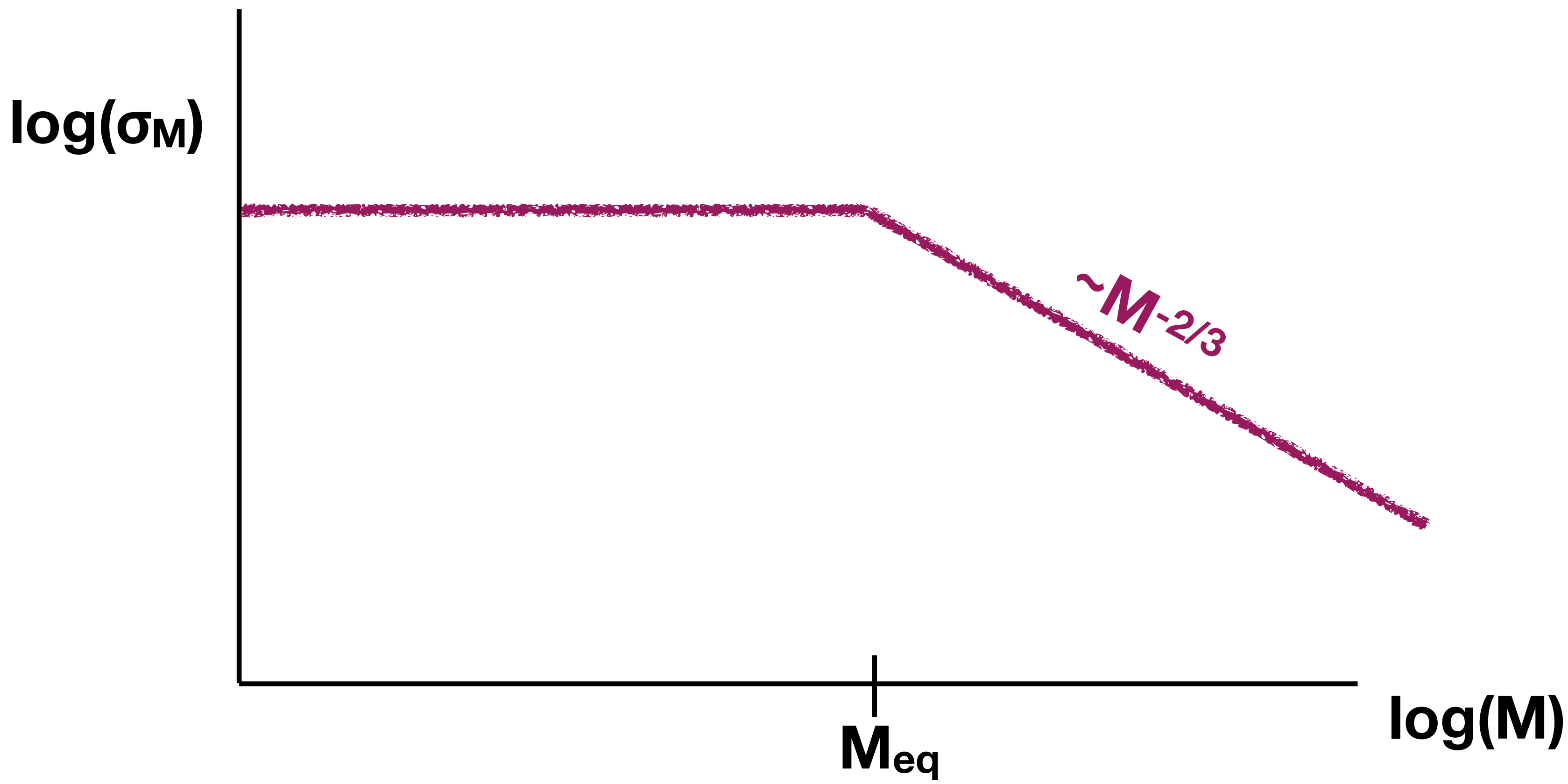
Mass fluctuation spectrum

- From initial H-Z spectrum we obtain, after growth through t_{eq}:

$$\sigma_M^2 \equiv \left\langle \left(\frac{\delta M}{\overline{M}} \right)^2 \right\rangle \propto k_M^3 P(k_M) \propto \begin{cases} M^0, & M_{\rm FS} < M < M_{\rm eq} \\ M^{-4/3}, & M > M_{\rm eq} \end{cases}$$

$\longleftarrow P(k) \sim k^{-3}$

$\longleftarrow P(k) \sim k^{1}$



- All masses below M_{eq} start from same amplitude of fluctuations and so go non-linear and collapse ~ simultaneously(*)
- Larger masses $M>M_{eq}\sim10^{16}M_\odot$ collapse later (>> larger than galaxy cluster scales)

Mass fluctuation spectrum

- (*) actually, smaller masses collapse a little earlier:
  - in any given volume there will be more extreme outliers for smaller mass (more peaks), these will collapse first
  - so a large-M density peak will already be clumpy as it collapses
  - gives a slight tilt to the ‘flat’ part of $\sigma(M)$

- CDM therefore predicts hierarchical structure formation:
  - small scales collapse and cluster to form larger structures
  - wide range of scales present at any time

- In a hot dark matter universe (large free streaming length λ_{FS}) smaller fluctuations ($k>k_{FS}$) are wiped out, and $\sigma_M(k)$=const. for $M<M_{FS}$. In this case structures of mass M_{FS} collapse first, and need to fragment somehow to make smaller structures. “top-down”