Lecture 3: Control of a system

- Many natural (living and non-living) and man-made systems have/require control (e.g. standing, driving, flying, body and planet T, …).
- In particular many experiments are only possible with control.

- Control achieved by some combination of:

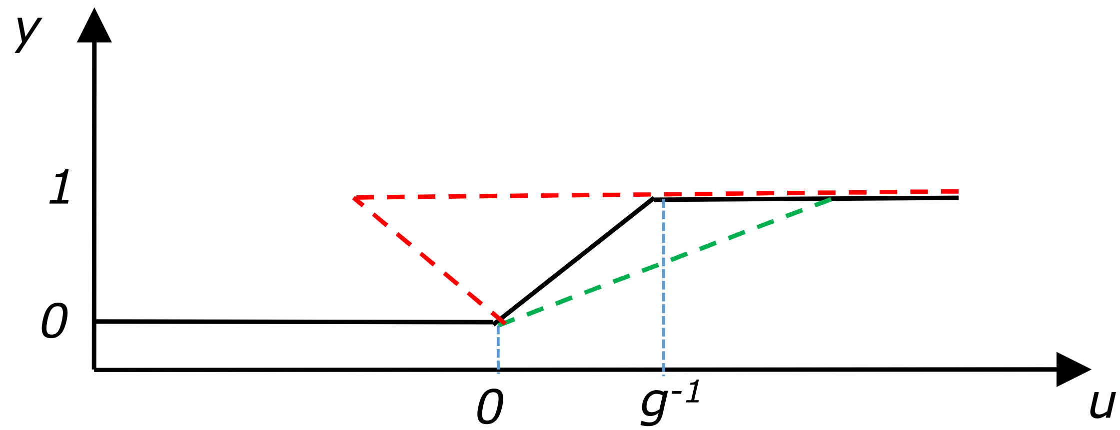
  sensing (measuring), controlling (deciding) and actuating (executing).

- Modern control dates from 1922. Take a “set point” and a “measurement”, their difference is the error (function of time). Key empirical concept is correcting by a feedback that is proportional to (1) the error itself, (2) time integral of error and (3) time derivative of error. → “PID control”.
- Control is essential when the environment can change. PID very commonly used, not the only strategy (idea of “optimal control”).

- Unstable systems can be made stable by control. Many other advantages.

Open and closed loop. +ve and –ve feedback

$$y = \mathrm{sat_g}(u)$$



Sorry, typo on printout

- Taking some fraction of the "output/measurement" and feeding-back as an input to the system constitutes a feedback loop. This enables "closed-loop" control.
- In negative feedback, we subtract the output from the input with a feedback gain *k*.

$$y = \mathrm{sat_g}(u - ky) = \mathrm{sat_{g'}}(u)$$

$$\text{with } g' = g/(1 + kg)$$

So the linear response range has gone from g^{-1} up to $g^{-1} + k$. This is good if we are trying to regulate *y*, since it stays in range over a wider set of inputs.

In positive feedback, we add the output to the input with a feedback gain *k*. Modifies *y* towards a step, and then bistable (memory!)

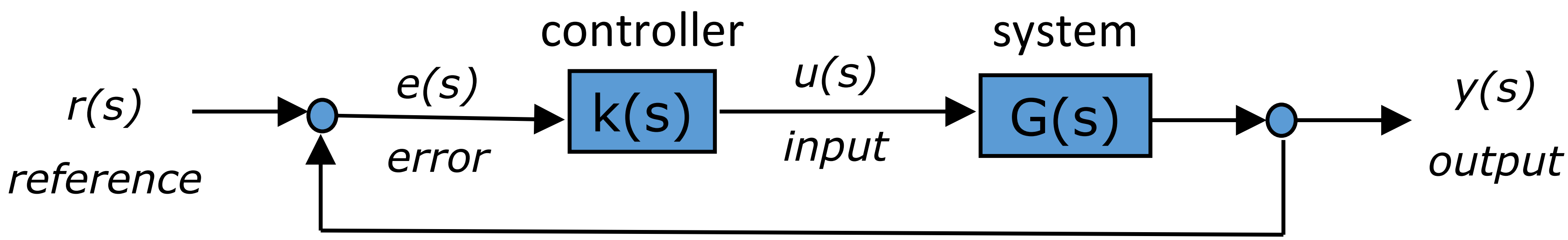
$$g' = g/(1 - kg)$$

In general, systems are complicated!

- We might have frequency dependence on signals and on the gain: $y(s) = G(s)u(s)$

Imagine locking into a target (e.g. staying on lane in a self driving car). We'd like say *y* to track (be equal to) *u*. *G* can be strange and hard to characterise, and perhaps might even change over time.

More generally, we usually have to think in terms of these modules:



With *e=r-y*. Now the closed loop dynamics is: $$y = \frac{GK}{1+G\,K}r$$

Some terminology: L = GK is the *loop transfer function* (product of gains around loop). T = GK/(1+GK) is the *closed loop transfer function*.

S = 1-T = 1/(1+GK) is known as *sensitivity function*.

Proportional control of a first order system

- Consider a physical system that acts as a low-pass filter with gain G_0 and cut-off frequency ω_0:

$$G(s) = \frac{G_0}{\omega_0 + s}$$

The power in the denominator is the "order" of the system

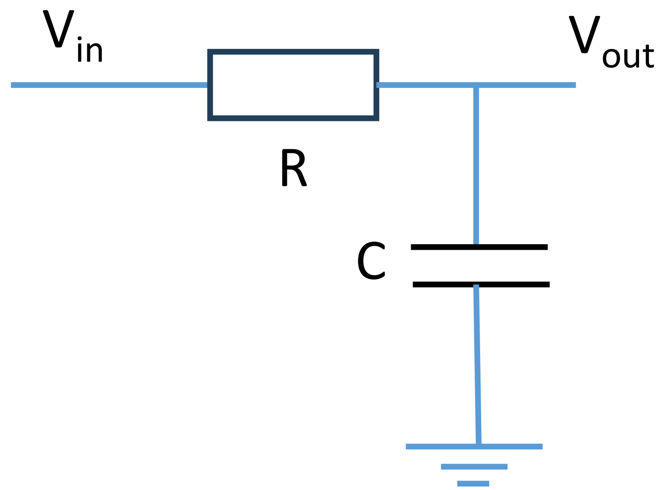
- Let's say the controller is simply a proportional one: $K(s)=K_p$
- So K_p is its gain, and the input is $u(s)= K_p\, e(s)$.
- $e=r-y$, so the feedback is negative, and correction is opposite to the direction of deviation.
- the *closed loop transfer function* T = GK/(1+GK) is in this case

$$T(s) = \frac{GK}{1+G\,K} = \frac{K}{G^{-1}+K} = \frac{K_p}{\frac{\omega_0+s}{G_0}+Kp} = \frac{G_0 K_p}{G_0 K_p+\omega_0+s}$$

- Which is still a first order but different gain and different cut-off.

Let’s use this machinery: The low pass in frequency and time domains

Take the circuit:



We would write the time dynamics here as

$$\dot{V}_{out}(t) = -\frac{1}{RC} V_{out}(t) + \frac{1}{RC} V_{in}(t)$$

Something like this you could integrate in time, sometimes analytically, in general via some numerical method, and get solutions for specific initial conditions.

Can make dimensionless by scaling time with $\tau = RC = 1/\omega_0$:

$$\dot{y}(t) = -y(t) + u(t)$$

LT of the equation gives:

$$sy(s) = -y(s) + u(s)$$

$$G(s) \equiv \frac{y(s)}{u(s)}$$

so

$$G(s) = \frac{1}{1+s}$$

To return to dimensional units:

$$s \to s/\omega_0$$

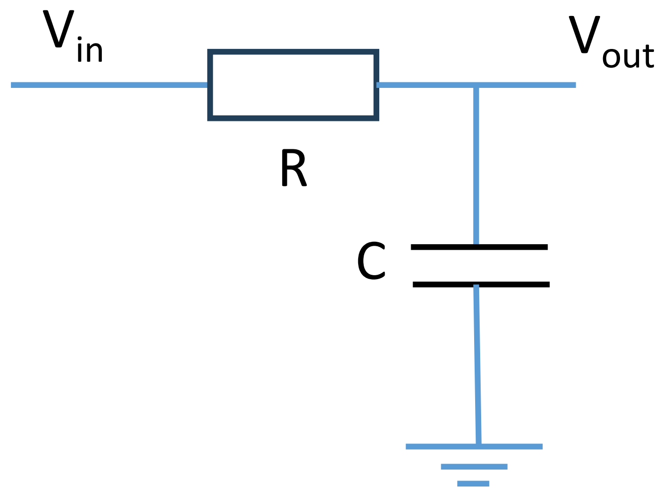
$$\omega \to \omega/\omega_0$$

And the numerator of *G(s)* should be the zero frequency gain.

Memo: The power in the denominator is the “order” of the system

Let's use this machinery: The low pass in frequency and time domains

Take the circuit:



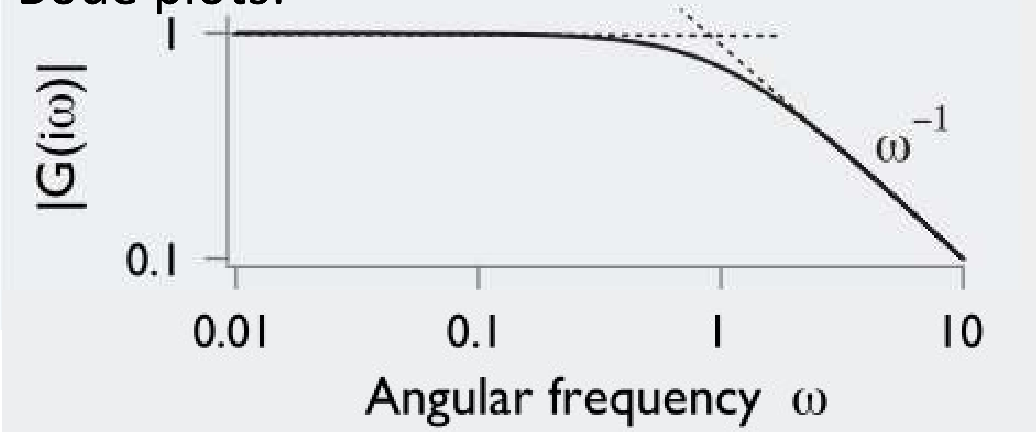
$$G(s) = \frac{1}{1+s}$$

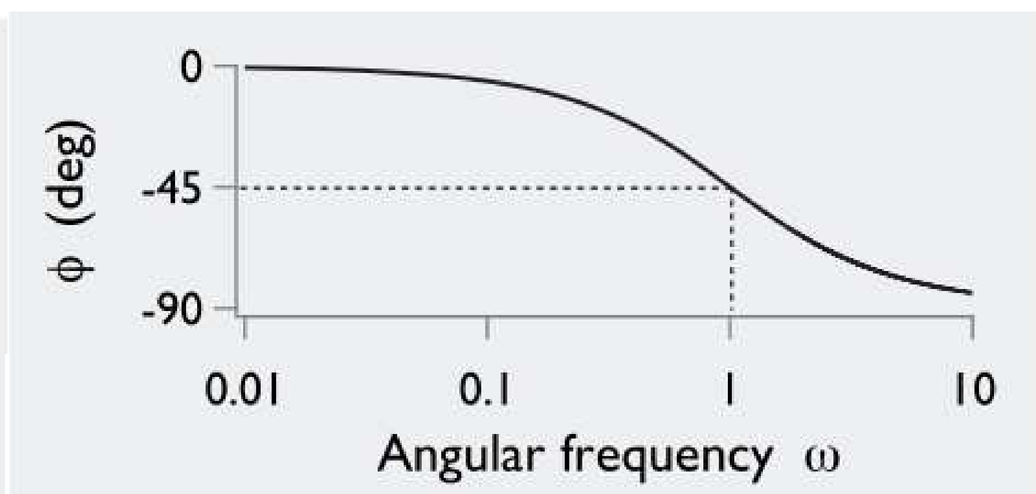
In frequency space:

$$G(i\omega) = \frac{1}{1+i\omega} = |G(i\omega)|e^{i\varphi}$$

$$|G(i\omega)| = \frac{1}{\sqrt{1+\omega^2}} \qquad \phi = \tan^{-1}\frac{\operatorname{Im} G(i\omega)}{\operatorname{Re} G(i\omega)} = -\tan^{-1}(\omega)$$

Bode plots:





In the example we just saw:

$$\dot{y}(t) = -y(t) + u(t)$$

$$G(s) = \frac{1}{1+s}$$

In the proportional control of a first order system we had:

$$G(s) = \frac{G_0}{\omega_0 + s}$$

Corresponds to the time dynamics:

$$\dot{y}(t) = -\omega_0 y(t) + G_0 u(t)$$

and $u = Kp(r - y)$

So: $$\dot{y}(t) = -(\omega_0 + G_0 K_p) y(t) + G_0 K_p r$$

This you know how to solve, for a step $r(t) = \theta(t)$, $y(0) = 0$:

$$y(t) = yss\left[1 - e^{-(\omega_0 + G_0 K_p)t}\right] \qquad y_{ss} = \frac{G_0 K_p}{\omega_0 + G_0 K_p}$$

So from the time solution we see that for large K_p and long times we have

$y(t) \to 1$

Also $T(s) \to 1$

Hence $y = Tr \to r$

i.e. this closed loop has achieved control, and *in this limit* control is even independent of the actual values of ω_0 and G_0 i.e. of the details of the system – this is known as "robust".

We can finally, again under $r(t) = \theta(t)$, look at the control signal looks like

$$u(t) = K_p e(t) = K_p[1 - y(t)]$$

…*exercise, write this out*...

You see that at *t*=0 there is the max of $u(t)$: $u(0) = K_p$ This can be big… so the speedup of the response rate, from

$$\omega_0 \to \omega_0 + G_0 K_p$$

is coming at a cost – the actuator must be able to supply large K_p.

Now instructive to look at what we just did with proportional control, but in frequency domain

$$G(s) = \frac{1}{1+s}$$

$$T(s) = \frac{GK}{1+GK} = \frac{K}{K+G^{-1}} = \frac{K_p}{K_p+1+s} = \frac{K_p}{K_p+1}\left(\frac{1}{1+\frac{s}{K_p+1}}\right)$$

So the closed loop command response is a low pass filter, with an increased cut-off frequency.

Steady state can be obtained from “final-value theorem”

$$y_{ss} = \lim_{s\to 0}\ s\,T(s)r(s)$$

We are considering $r(t) = \theta(t)$, so $r(s) = \int_0^\infty e^{-st}dt = \frac{1}{s}$

$$= \lim_{s\to 0}\ s\,T(s)\frac{1}{s} = \lim_{s\to 0}\ T(s) = \frac{K_p}{K_p+1}$$

Confirming what we had in the time domain.

Note that if we don’t have infinite K_p then $T(s=0) < 1$ which is not ideal. We can fix this, without requiring infinite K_p but by modifying our control strategy.

Proportional + Integral control   Derivative control

now $u(t) = K_p e(t) + K_i \int_0^t dt'\, e(t')$

so $u(s) = Kpe(s) + \frac{K_i}{s} e(t')$

In the system we have been considering $G(s) = \dfrac{1}{1+s}$

Now $K(s) = K_p + \frac{K_i}{s}$ $T(s) = \dfrac{K_p s + Ki}{s^2 + (K_p+1)s + K_i}$ and $T(s=0)$ =1   *yuhoo!*

Achieving control, by just proportional + integral correction, seems excellent… but only works this well on first order systems… most systems are more complex, perhaps approx well by second order, or worse... Then, proportional + integral correction will overshoot, oscillate, etc.

Here comes derivative: $u(t) = Kd\,\frac{de}{dt}$ so $u(s) = Kd\, s\, e(s)$

Note this $u(s)$ diverges at large s, so either the signal really has no high freq noise, or we need to modify our control to tame divergence, can do by

$$K(s) = Kd\, s/(1 + s/\omega_d)$$

Proportional + Integral + Derivative control

this is now very powerful, we can face even a second order system like

$$G(s) = \frac{1}{1 + 2\zeta s + s^2}$$

$$T(s) = \frac{K_p s + Ki + Kds^2}{s^2 + (K_p + 1)s + K_i}$$

And with $K(s) = Kp + \frac{K_i}{s} + K_d\, s$

We get

$$T(s) = \frac{K_p s + Ki + Kds^2}{(K_p s + K_i + Kds^2) + s(1 + 2\zeta s + s^2)}$$

Looks nasty? Actually it's beautiful: with choices of the three parameters you have a completely arbitrary third order denominator, hence can move the zeros to where they matter less to your system.

PID is in some sense using the information from the present (P), past (I) and future (D).

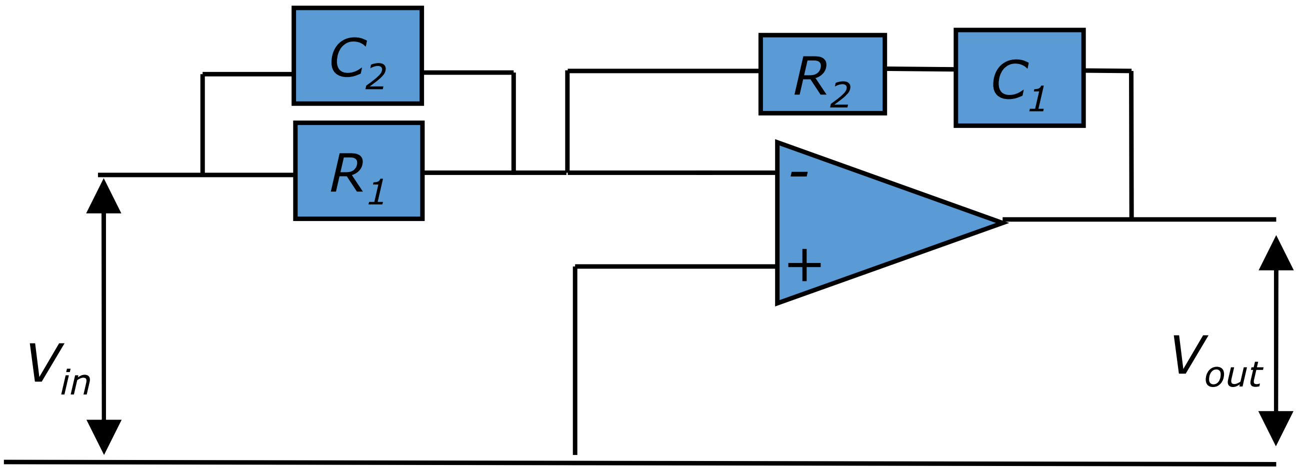
In practice the coefficients are usually going to be set empirically, but the theory we have looked at gives you a sense of why this will work and why it can fail.

PID control

can be achieved digitally (i.e. signals to a microcontroller or computer)

But also analogically, our ever useful op-amp comes into play again!

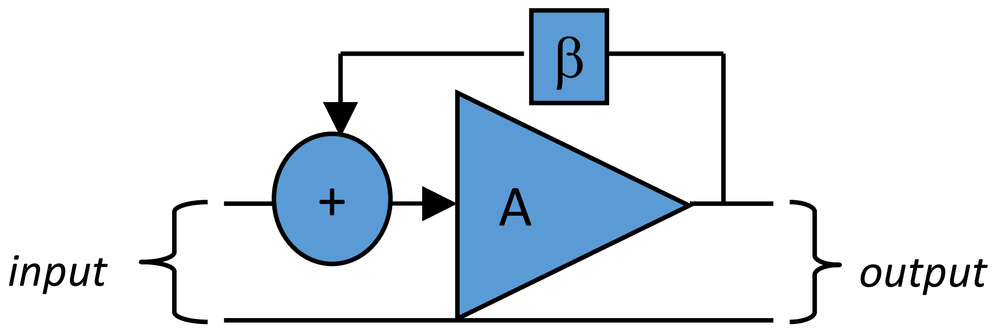
Question sheet exercise:



Makes a controller of the form $u(t) = (K_p + \frac{K_i}{s} + K_d\, s)e(t)$

Op amp Feedback as a case of feedback

- General way of representing a system with feedback:



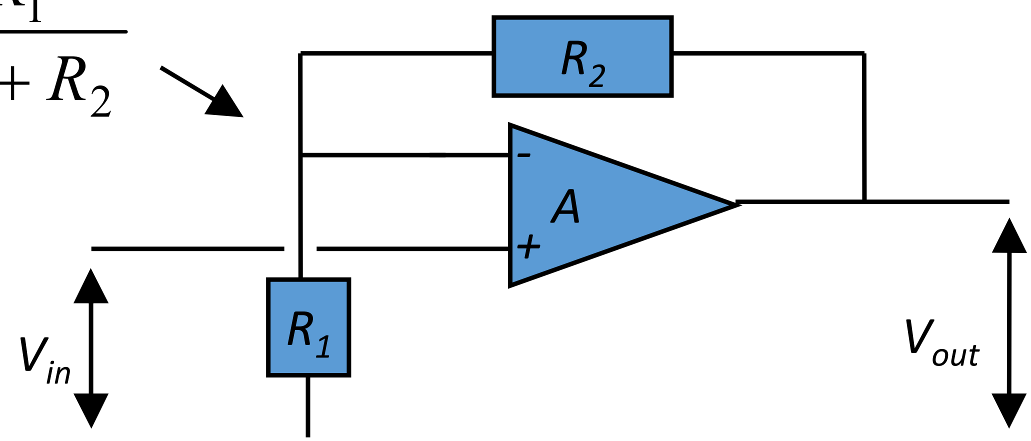
Output = A(input + β × output) [1]

β is the fraction fed back

β is negative for –ve feedback

- For example, for the non-inverting amplifier

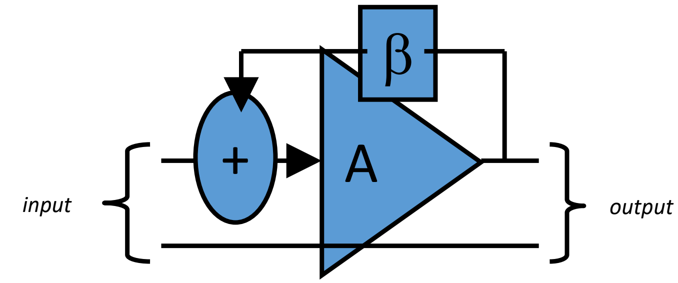
$$\beta = -\frac{R_1}{R_1 + R_2}$$



[The expression for β for the inverting amplifier is more complicated…]

$$\left(\beta = \frac{1}{1 + |-Z_2/Z_1|}\right)$$

We can recast [1] to establish …



$$\text{Closed loop gain} \equiv \frac{Output}{Input} = \frac{A}{1 - A\beta}$$

- Note that if $A\beta = 1$ the output will "explode".

- For β -ve, and, say A ~10^5 and β =-0.1, then |Aβ|>>1 and

$$Output/Input = -(1/\beta)\text{ , independent of } A.$$

- This simple result has profound consequences, e.g., many of our body's regulatory mechanisms rely on this, as do most commercial electronic devices.

Why is this important in an experimental situation?

- It is very likely that in a physical device that amplifies:
  - *A* will change with environmental conditions, e.g., temperature.
  - *A* will fluctuate (from internal disturbance or supply variation).
  - *A* will vary with frequency.
  - *A* will be non-linear, i.e. itself depend on the input strength.

- In the specific case of signal measurement this is bad – but –ve feedback removes these problems provided $|A\beta| >> 1$.

- Quite separately, in the context of voltage sensing circuits, negative feedback desirably reduces Z_{out} and, in the case of the non-inverting amplifier, desirably increases Z_{in}:
  - For example, for the non-inverting amplifier

$$Z_{in} = r_{in} \cdot (1 + |A\beta|) \qquad Z_{out} = R_{out}/(1 + |A\beta|)$$

Feedback in an Op-Amp with frequency dependence

Look back at inverting amplifier, but now let's investigate a gain *G(s)*

$$G(s) = \frac{G_0}{1+s}$$

$$(V_{out} - V_-)\frac{1}{R_2} = (V_- - V_{in})\frac{1}{R_1} \quad \text{and} \quad V_{out} = -GV_- \qquad \text{(see lecture 2)}$$

so

$$\frac{V_{out}}{V_{in}} = -\frac{R_2}{R_1}\frac{G_0}{G_0 + (1+\frac{R_2}{R_1})(1+s)}$$

We recover the gain of the ideal op-amp for low frequencies (s<<1) and high gain $G_0 >> R_2/R_1$.

Above, dimensionless units. In physical units, the condition for ideality is $G_0\,\omega_0 >> (R_2/R_1)\omega$

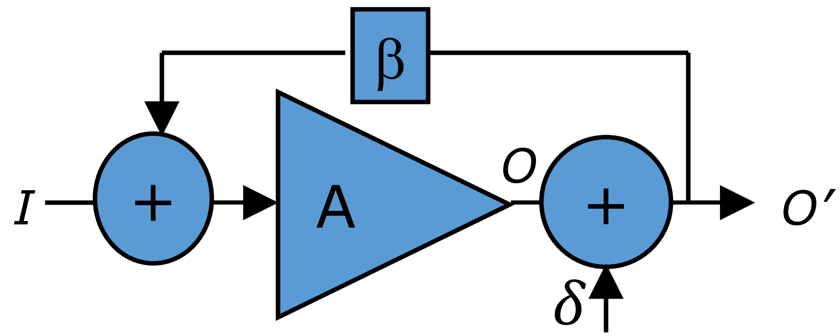
Where $G_0\,\omega_0$ is a property of the op-amp itself, between 1MHz and 1GHz. We can therefore see a trade-off between the gain we can get out of the op-amp circuit and the bandwidth.

How negative feedback helps generally *non-examinable

–ve feedback stabilises systems. How does it do this?

At equilibrium: $O_{eq} = Input \times \frac{A}{1-A\beta} \sim -\frac{1}{\beta} \times Input$

Although this is independent of A, the gain $(-1/\beta)$ is still due to the amplifier.



What if the amplifier response is unstable?

E.g. a noise source causes the instantaneous output to fluctuate by δ, so $O' = O + \delta$

The component fed back to the input is now: $\beta O' = \beta O + \beta\delta$

The input to the Amplifier is therefore: $I + \beta O' = I + \beta O + \beta\delta$

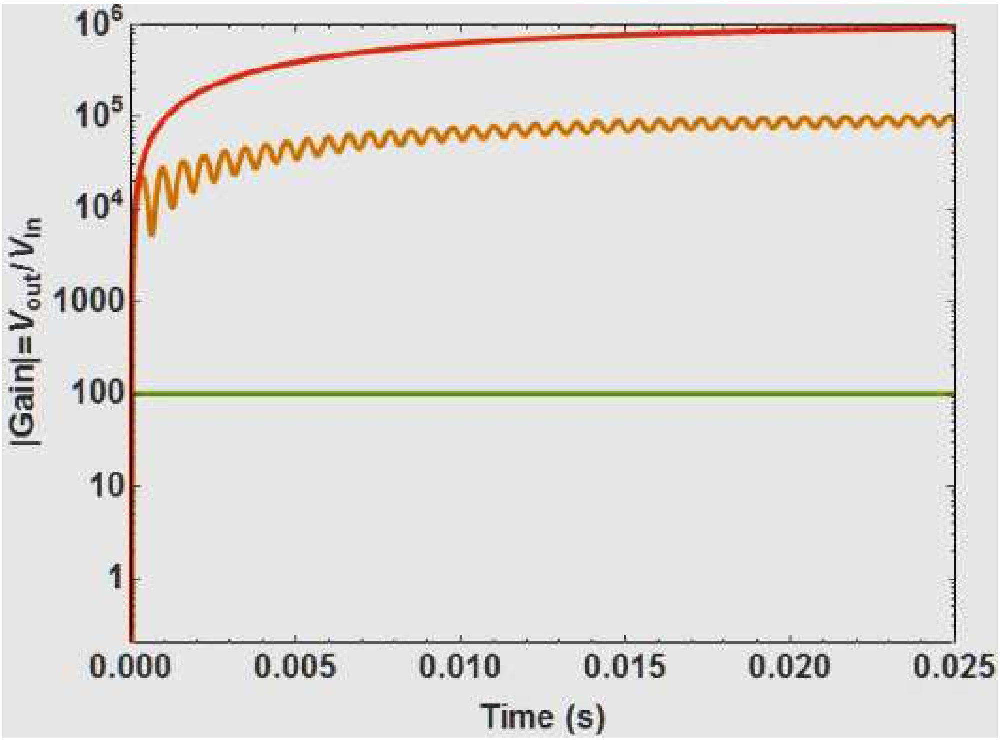
- So for self consistency $O' = A(I + \beta O + \beta\delta) + \delta$ ← sub for O (=O′ - δ) here
- $O' = \frac{AI}{1-A\beta} + \left(\frac{1}{1-A}\right)\delta = O_{eq} + \left(\frac{1}{1-A\beta}\right)\delta$

So δ is largely suppressed

What if A varies (a lot)?

*non-examinable

$$A = A_0 f(t)$$
$$A_0 = 10^5$$



(+ve feedback, ignore)

No feedback

-ve feedback (β = -0.01)

- Most real systems (biological, physical etc.) have very non-linear *and* sensitive mechanism for producing their amplification (A).
- They often operate in the presence of significant external disturbances.

Examples of feedback in physical systems:

- Balancing a horizontal rule:
  - Rest ruler on pair of fingers at ends, bring fingers together.
  - Fingers always meet under center, ruler always balanced.
  - Why?
    - Support that has less friction initially moves toward c-o-m.
    - As this happens, reaction at it increases, and at other reduces.
    - So friction at other reduces, so eventually that support starts moving.

- Stability of stellar burning (life depends on this):
  - Rate of production of E/mass $\propto \rho^a T^b$, with b >4 (can be 20!)
  - Assume T_{core} drops – what happens?
    - Thermal pressure drops.
    - Star contracts, shrinking core.
    - Core heats up due to release of GPE.
    - Restores reaction rate to equilibrium value.

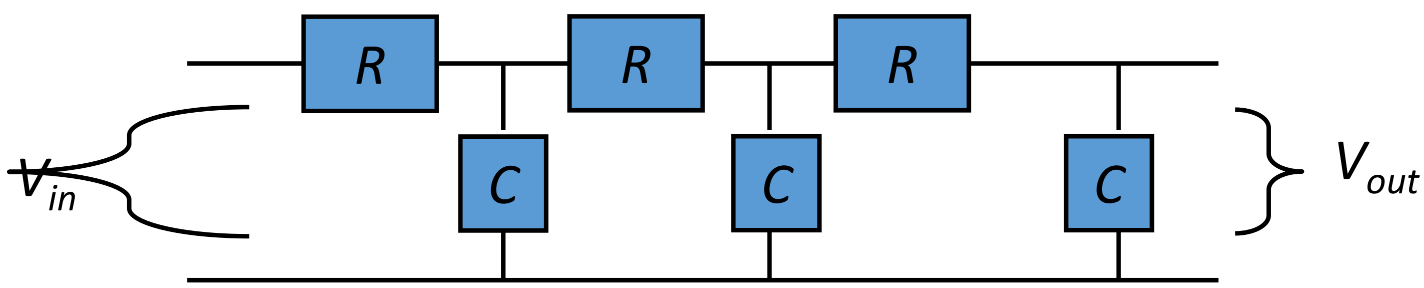
Is +ve feedback ever beneficial?

- For β>0, output will be reinforced by the feedback and the output can become unstable or saturate. Indeed, when $A\beta \to 1$, the gain,

$$Output \;/\; Input = \frac{A}{1-\beta A} \to \infty \;.$$

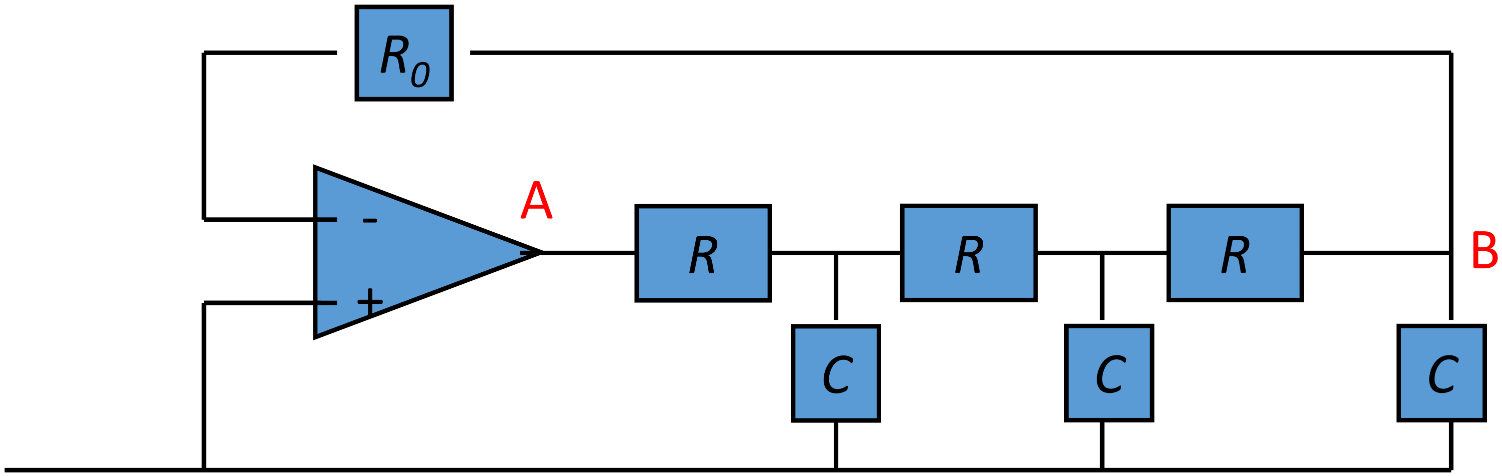
- This is useful if β is a strong function of frequency: if β >0 and $A\beta$ = 1 for only one frequency, systems can oscillate (spontaneously) at that frequency.

- How might we arrange this?



The output is π out of phase with the input at precisely $\omega_0=\sqrt{6}/RC$. So at ω_0, this circuit could be used to make β>0.

How do we exploit this in practice?



- Imagine a small "glitch" at A. The Fourier component at ω_0 has its phase exactly reversed by the RC network, and is fed back to the "-" input, giving in-phase reinforcement at A.
- So, effectively the ladder network acts as a filter for $\omega_{0.}$
- This gives a continuous sine output at B at ω_0: an oscillator.

The details as to how this really works are actually rather subtle. You will try the experiment in the lab…

Recap: positive feedback

- For $\beta > 0$, output will be reinforced by the feedback and output can become unstable. when $A\beta \rightarrow 1$, the gain $\rightarrow \infty$

$$Output \;/\; Input = \frac{A}{1-\beta A} \rightarrow \infty \;.$$

- Useful if β is a strong function of frequency: if $\beta > 0$ and $A\beta = 1$ at a single ω, system oscillates (spontaneously) at that ω.

- We need to recognize that $|A\beta|$ is complex:
  - So $A\beta = 1 \Rightarrow |A\beta| = 1$ and $\mathrm{Arg}\{A\beta\} = 0$
- In practice if $\mathrm{Arg}\{A\beta\} = 0$ and $|A\beta| > 1$, then oscillation occurs
- But we still need some suppression of gain to deliver a "clean" sinusoid

NB: in our analysis we have assumed no "time lags"

If sensing of the output takes time, a more sophisticated time domain analysis is needed

Summary so far

- Elements of Control theory, building up to PID. Time and frequency domains.

- General framework for feedback:
  - $Output = A(input + \beta \times output)$
  - If β -ve and $|A\beta| >> 1$, then the output = $-1/\beta$, finite and independent of A.

- +ve feedback generally bad – latched output, but:
  - If +ve feedback at one frequency $\Rightarrow$ oscillator – very useful.

Try out PID numerically here:

https://colab.research.google.com/drive/1Mv77y7K7hKusgqhra
H4OMTzkqOHHdZHf?usp=sharing