WA

(1) ~~Ehr~~ Ehrenfest's theorem:

$$\frac{d\langle\hat{A}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{A}]\rangle + \langle\frac{d\hat{A}}{dt}\rangle$$

$$\frac{d\langle\hat{A}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{A}]\rangle \qquad \triangleright \langle\frac{d\hat{x}}{dt}\rangle = 0 \;\&\; \langle\frac{d\hat{p}}{dt}\rangle = 0$$

$$\frac{d\langle x^2\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{x}^2]\rangle \xrightarrow[\text{relation}]{\text{commutation}} \frac{i}{\hbar}\hbar 2\hat{x}$$

$$= \frac{i}{\hbar}\langle\hat{H}\hat{x}^2 - \hat{x}^2\hat{H}\rangle$$

$\cancel{\hat{H} = i\hbar\frac{\partial}{\partial t}}$ $\quad \hat{H} = \frac{\hat{p}^2}{2m}$ for a free particle

$\cancel{x = -i\hbar\frac{\partial}{\partial x}}$

$$= \frac{i}{2\hbar m}\langle[\hat{p}^2,\hat{x}^2]\rangle = \frac{i}{2\hbar m}\langle[\hat{p}^2,\hat{x}\hat{x}]\rangle$$

Commutator relations: $[\hat{x},\hat{p}^l] = i\hbar l\hat{p}^{l-1}$ & $[\hat{p},\hat{x}^l] = -i\hbar l\hat{x}^{l-1}$

$$= \cancel{\frac{i}{2\hbar m}(-i\hbar 2}$$

Leibnitz rule: $[\hat{a},\hat{b}\hat{c}] = [\hat{a},\hat{b}]\hat{c} + \hat{b}[\hat{a},\hat{c}]$

$$\frac{i}{2\hbar m}\left([\hat{p}^2,\hat{x}]\hat{x} + \hat{x}[\hat{p}^2,\hat{x}]\right)$$

$$\frac{i}{2\hbar m}\left(-[\hat{x},\hat{p}^2]\hat{x} - \hat{x}[\hat{x},\hat{p}^2]\right)$$

$$\frac{i}{2\hbar m}\left(-i\hbar 2\hat{p}\hat{x} - i\hbar 2\hat{x}\hat{p}\right)$$

$$\frac{1}{m}\left(\hat{p}\hat{x} + \hat{x}\hat{p}\right) \; //$$

$$\frac{d^2\langle x^2\rangle}{dt^2} = \frac{d}{dt}\left(\frac{1}{m}(\hat{x}\hat{p} + \hat{p}\hat{x})\right)$$

$$= \frac{i}{\hbar}\left\langle\left[\frac{\hat{p}^2}{2m}, \frac{1}{m}(\hat{x}\hat{p} + \hat{p}\hat{x})\right]\right\rangle$$

$$= \frac{i}{2m^2\hbar}\left\langle\left[\frac{\hat{p}^2}{\cancel{2m}}, 2\hat{x}\hat{p} + 2\hat{p}\hat{x}\right]\right\rangle$$

$$= \frac{i}{2m\hbar}\left(\left\langle[\hat{p}^2, 2\hat{x}\hat{p}] + [\hat{p}^2, 2\hat{p}\hat{x}]\right\rangle\right)$$

$$= \frac{i}{2m\hbar}\left(\left\langle [\hat{p}^2, \underset{2\hat{x}}{\cancel{\;}}]\hat{p} + \hat{x}[\hat{p}^2, 2\hat{p}] + [\hat{p}^2, 2\hat{p}]\hat{x} + [\hat{p}^2, 2\hat{x}]\hat{p}\right\rangle\right)$$

$$= \frac{i}{2m\hbar}\left\langle -[\cancel{2}\hat{x}, \tfrac{\hat{p}^2}{2}]\hat{p} - \hat{x}[\cancel{2}\hat{p}, \tfrac{\hat{p}^2}{2}] - [\cancel{2}\hat{p}, \tfrac{\hat{p}^2}{2}]\hat{x} - [\cancel{2}\hat{x}, \tfrac{\hat{p}^2}{2}]\hat{p}\right\rangle$$

$$= \frac{i}{2m\hbar}\left\langle -i\hbar\hat{p}^2 - 0 - 0 - i\hbar\hat{p}^2\right\rangle$$

$$= \frac{\cancel{\hbar}\langle\hat{p}^2\rangle}{2\cancel{\hbar}m^2}$$

* How to do another way / $\frac{d^2\langle\hat{x}^2\rangle}{dt^2} = \frac{i}{2\hbar m^2}\langle[\hat{p}^2, \hat{x}]\hat{p} + \hat{p}[\hat{p}^2, \hat{x}]\rangle$ -

$$\cancel{\frac{1}{m}\langle\hat{x}\hat{p} + \hat{p}\hat{x}\rangle} = 0$$

Taylor expansion:

$$\langle x^2\rangle_t \approx \langle x^2\rangle_0 + \overbrace{\left.\frac{d\langle x^2\rangle}{dt}\right|_0}^{=0} + \left.\frac{d^2\langle x^2\rangle}{dt^2}\right|_0 / 2!$$

$$\approx \langle x^2\rangle_0 + \frac{\langle\hat{p}^2\rangle}{m^2}\langle\hat{p}\rangle_0 \;//$$

~~$x + \frac{1}{2}xy + \frac{1}{2}xy + y = 54$~~

~~$x + xy + y = 54$ $\quad x(1 + \frac{1}{2}y) + y(1 + \frac{1}{2}x) = 54$~~

~~$x + y = ?$~~

2 when measured, collapses to state $\psi_1 = \frac{u_1 + u_2}{\sqrt{2}}$ as a_1 found.

Ehrenfest: $\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{A}] + \underbrace{\left\langle \frac{d\hat{A}}{dt} \right\rangle}_{=0}$

$= \frac{i}{\hbar}[\hat{H}, \hat{A}]$

$\hat{H}|u_1\rangle = E_1|u_1\rangle$

$\hat{H}|u_2\rangle = E_2|u_2\rangle$

$|\psi(0)\rangle = |\psi_1\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{\sqrt{2}}|u_2\rangle$

these vary with time

$\hat{A} = a_1|\psi_1\rangle$

energy eigenstates.

$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$

$|\psi(t)\rangle = \sum_n c_n(0) e^{-E_n t/\hbar}|u_n\rangle$

$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-E_1 t/\hbar}|u_1\rangle + \frac{1}{\sqrt{2}} e^{-E_2 t/\hbar}|u_2\rangle$] this is how the state ψ_1 varies with time.

~~$\frac{d\langle\psi|A|\psi\rangle}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{A}]$~~

$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$

$= \frac{1}{2}\left(e^{+\frac{iE_1 t}{\hbar}}\langle u_1| + e^{\frac{iE_2 t}{\hbar}}\langle u_2|\right) A \left(e^{-\frac{iE_1 t}{\hbar}}|u_1\rangle + e^{-\frac{iE_2 t}{\hbar}}|u_2\rangle\right)$

$= \frac{1}{2}\Big(\langle u_1|A|u_1\rangle + e^{i(E_2 - E_1)t/\hbar}\langle u_2|A|u_1\rangle + e^{i(E_1 - E_2)t/\hbar}\langle u_1|A|u_2\rangle$

$+ \langle u_2|A|u_2\rangle\Big)$

$$u_1 = \sqrt{2}\psi_1 - u_2$$

$$\psi_2 = (\sqrt{2}\psi_1 - 2u_2)/\sqrt{2}$$

$$\psi_2 = \psi_1 - \sqrt{2}u_2$$

$$u_2 = \frac{1}{\sqrt{2}}(\psi_1 - \psi_2)$$

$$\therefore u_1 = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$$

$$\hat{A}|\psi_i\rangle = a_i|\psi_i\rangle$$

$$\langle\psi_1|A|\psi_2\rangle = 0 \ ?$$

$$\langle u_1|A|u_1\rangle = \langle \tfrac{1}{\sqrt{2}}(\psi_1+\psi_2)|A|\tfrac{1}{\sqrt{2}}(\psi_1+\psi_2)\rangle$$

$$\boxed{\cancel{\tfrac{1}{2}\left((\langle\psi_1| + \langle\psi_2|)A(|\psi_1\rangle + |\psi_2\rangle)\right)} \quad \cancel{\tfrac{1}{2}(\langle\psi_1|A|\psi_1\rangle +}} \qquad = \frac{1}{2}(a_1 + a_2)$$

B

$$\langle u_2|A|u_1\rangle = \frac{1}{2}\langle\psi_1 - \psi_2|A|\psi_1 + \psi_2\rangle$$

$$= \frac{1}{2}(a_1 - a_2)$$

$$\langle u_1|A|u_2\rangle = \frac{1}{2}(a_1 - a_2)$$

$$\langle u_2|A|u_2\rangle = \frac{1}{2}(a_1 + a_2)$$

hence $\langle\psi(t)|A|\psi(t)\rangle = \frac{1}{4}\left(2(a_1+a_2) + (a_1-a_2)\left(e^{\frac{it}{\hbar}(E_1-E_2)} + e^{\frac{it}{\hbar}(E_2-E_1)}\right)\right)$

$$= \frac{1}{4}\left(2(a_1+a_2) + (a_1-a_2)\times 2\cos\left(\frac{t}{\hbar}(E_1-E_2)\right)\right)$$

$$= \frac{1}{2}\left((a_1+a_2) + (a_1-a_2)\cos\left(\frac{t}{\hbar}(E_1-E_2)\right)\right)$$

$$\frac{1}{2}\left(a_1 + a_2 + (a_1 - a_2)\left(2\cos^2\left(\frac{t}{2\hbar}(E_1 - E_2)\right) - 1\right)\right)$$

$$\frac{1}{2}\left(2a_2 + 2a_1\cos^2\left((E_1 - E_2)\frac{t}{2\hbar}\right) - 2a_2\cos^2\left(\frac{t}{2\hbar}(E_1 - E_2)\right)\right)$$

$\cancel{a_2 + a_1}$

$a_2 = a_2\cos^2(\ldots) + a_2\sin^2(\ldots)$

$$a_2 + a_1\cos^2\left((E_1 - E_2)\frac{t}{2\hbar}\right) - a_2 + a_2\sin^2\left((E_1 - E_2)\frac{t}{2\hbar}\right)$$

$$\therefore \langle A\rangle = a_1\cos^2(\ldots) + a_2\sin^2(\ldots) \quad //$$

3 $\hat{H}|\psi\rangle = E|\psi\rangle$

Did this question last week (would like to check though).

4 Shift operator from Q12?

$$\hat{A}\psi = \psi(x - x_0)$$

~~$\psi = e^{i(kx - \omega t)} \xrightarrow{t=0}$~~

$$\psi(x,t) = Ce^{i(kx - \omega t)}$$
$$\psi(x) = Ce^{ikx}$$

hence $\psi(x - x_0) = Ce^{ik(x - x_0)}$

if $\hat{A}\psi = \psi(x - x_0)$

$$\hat{A}Ce^{ikx} = Ce^{ik(x - x_0)}$$

$$\hat{A} = e^{-ikx_0}$$

~~for~~ $\hat{p} = k\hbar$ $\therefore$ $\hat{A} = e^{-i\frac{\hat{p}}{\hbar}x_0}$ ← this is in terms of momentum representation.

Commutator: $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \underbrace{= 0}_{\text{if operators commute.}}$

$$\psi(x) = \int e^{ikx} \psi(k)\, dk$$ to get position representation from momentum.

Confused as to how to proceed.

5 $[\hat{L}_x, \hat{x}] = L_x\hat{x} - \hat{x}L_x$

$$\hat{L} = \hat{r} \times \hat{p} = \begin{vmatrix} i & j & k \\ \hat{x} & \hat{y} & \hat{z} \\ \hat{p}_x & \hat{p}_y & \hat{p}_z \end{vmatrix}$$

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$$

$= \hat{y}\hat{p}_z\hat{x} - \hat{z}\hat{p}_y\hat{x} - \hat{x}\hat{y}\hat{p}_z + \hat{x}\hat{z}\hat{p}_y$ } all variables are products of independent directions $\therefore$ commute $\therefore$ cancel

$= 0$ //

by analogy, $[\hat{L}_x, \hat{p}_x] = 0$

$$[\hat{L}_x, \hat{y}] = (\hat{y}\hat{p}_z - \hat{z}\hat{p}_y)\hat{y} - \hat{y}(\hat{y}\hat{p}_z - \hat{z}\hat{p}_y)$$

$$= \hat{y}\hat{p}_z\hat{y} - \hat{z}\hat{p}_y\hat{y} - \hat{y}^2\hat{p}_z + \hat{y}\hat{z}\hat{p}_y$$

$$= \cancel{\hat{y}^2\hat{p}_z} - \cancel{\hat{y}^2\hat{p}_z} + \hat{z}i\hbar$$

$$[\hat{L}_x, \hat{p}_y] = \hat{y}\hat{p}_z\hat{p}_y - \cancel{\hat{z}\hat{p}_y\hat{p}_y} - \hat{p}_y\hat{y}\hat{p}_z + \cancel{\hat{p}_y\hat{z}\hat{p}_y}$$

$$= p_z(\hat{y}p_y - p_y\hat{y})$$

$$= \hat{p}_z i\hbar$$

$[L_x, \hat{L}^2] = 0$ (as proved in notes). $[\hat{L}^2, L_x] = 0$ & $[\hat{L}^2, L_x] = -[L_x, \hat{L}^2]$

$$[L_x, \hat{r}^2] = (\hat{y}\hat{p}_z - \hat{z}\hat{p}_y)(\hat{x}^2 + \hat{y}^2 + \hat{z}^2) - (\hat{x}^2 + \hat{y}^2 + \hat{z}^2)(\hat{y}\hat{p}_z - \hat{z}\hat{p}_y)$$

$\hat{r}^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2$

$= [\hat{y}p_z, \hat{x}^2] - [zp_y, x^2] + [yp_z, \hat{y}^2] - [zp_y, \hat{y}^2] + [yp_z, z^2] - [zp_y, \hat{z}^2]$

Leibnitz rule $\times 2$; $[zp_y, x^2] = 0$

$[\hat{y}p_z, \hat{x}\hat{x}] \to [\hat{y}p_z, \hat{x}]\hat{x} + \hat{x}[\hat{y}p_z, \hat{x}] = \hat{y}[p_z, \hat{x}] \ldots = 0$

$\hat{y}[p_z, \hat{y}] = 0$

$[zp_y, y]y - y[zp_y, y]$

$z[p_y, y] + [z, y]p_y = -zi\hbar$, 0

$y - yz$

$+ 2i\hbar y - 2i\hbar y + \dots$ I assume if I work through the terms will all cancel.

6. $\hat{L}_+ \hat{L}_- = (\hat{L}_x + i\hat{L}_y)(L_x - i\hat{L}_y)$

$= \hat{L}_x^2 + i\hat{L}_y \hat{L}_x - i\hat{L}_x \hat{L}_y + \hat{L}_y^2$

$= \hat{L}^2 - \hat{L}_z^2 - i(\hat{L}_x \hat{L}_y - \hat{L}_y \hat{L}_x)$

$-i([\hat{L}_x, \hat{L}_y])$

$= \hat{L}^2 - L_z^2 - i(i\hbar \hat{L}_z)$

$\hat{L}^2 = \hat{L}_+ \hat{L}_- - \hbar \hat{L}_z + L_z^2 \Rightarrow \hat{L}_- \hat{L}_+ = \hat{L}^2 - \hbar L_z - \hat{L}_z^2$

$[\hat{L}_+, \hat{L}_-] = \hat{L}_+ \hat{L}_- - \hat{L}_- \hat{L}_+$

$= \hat{L}^2 - \hat{L}_z^2 + \hbar L_z - \hat{L}^2 + \hat{L}_z^2 + \hbar L_z$

$= 2\hbar L_z$

Ran out of time to finish this ...
(will continue before supo).