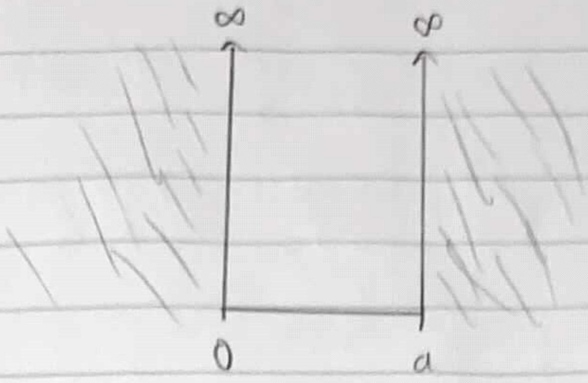
Q10-13 Arjun

10. $V(x) = 0 \qquad 0 < x < a$

$V(x) = \infty \qquad$ elsewhere




$\psi(0) = \psi(a) = 0$ for continuity

$\psi = A\sin kx + B\cos kx$

$B = 0 \qquad \therefore \psi = A\sin kx$

$0 = A\sin ka$

$ka = n\pi$

$k = \frac{n\pi}{a}$

$$\int_0^a \left[A\sin\left(\frac{n\pi x}{a}\right)\right]^2 dx = 1$$

$$A^2 \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx = 1$$

$$A^2 \frac{a^2}{2} = 1$$

$$A = \sqrt{\frac{2}{a}}$$

$$\psi = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$\langle x \rangle = \frac{2}{a}\int_0^a x\sin^2\left(\frac{n\pi x}{a}\right) dx$$

$u = x \qquad v' = \sin^2\left(\frac{n\pi x}{a}\right)$

$u' = 1 \qquad v = \frac{1}{2}\left(x - \frac{2a}{n\pi}\sin\frac{2n\pi x}{a}\right)$

$$= \frac{2}{a}\left[\left[\frac{1}{2}x\left(x - \frac{2a}{n\pi}\sin\frac{2n\pi x}{a}\right)\right]_0^a - \int_0^a \frac{1}{2}\left(x - \frac{2a}{n\pi}\sin\left(\frac{2n\pi x}{a}\right)\right)dx\right]$$

$$= \frac{2}{a}\left(\frac{1}{2}a^2 - \frac{1}{2}\left[\frac{1}{2}x^2 + \frac{a^2}{n^2\pi^2}\cos\left(\frac{2n\pi x}{a}\right)\right]_0^a\right)$$

$$= \frac{2}{a}\left(\frac{1}{2}a^2 - \frac{1}{4}a^2\right)$$

$$= \frac{2}{a}\left(\frac{1}{4}a^2\right) = \frac{1}{2}a$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\langle x^2 \rangle = \frac{2}{a}\int_0^a x^2 \sin^2\left(\frac{n\pi x}{a}\right) dx \qquad u = x^2 \quad v' = \sin^2\left(\frac{n\pi x}{a}\right)$$
$$u' = 2x \quad v = \frac{1}{2}\left(x - \frac{2a}{n\pi}\sin\left(\frac{2n\pi x}{a}\right)\right)$$

$$= \frac{2}{a}\left(\left[\frac{x^2}{2}\left(x - \frac{2a}{n\pi}\sin\left(\frac{2n\pi x}{a}\right)\right)\right]_0^a - \int_0^a x^2 - \frac{2a}{n\pi}x\sin\left(\frac{2n\pi x}{a}\right) dx\right]$$

$$= \frac{2}{a}\left(\frac{a^3}{2} - \left[\frac{1}{3}x^3\right]_0^a + \frac{2a}{n\pi}\int_0^a x\sin\left(\frac{2n\pi x}{a}\right) dx \qquad u = x \quad v' = \sin\frac{2n\pi x}{a}\right.$$
$$u' = 1 \quad v = -\frac{a}{2n\pi}\cos\left(\frac{2n\pi x}{a}\right)$$

$$= \frac{2}{a}\left[\left(\frac{a^3}{2} - \frac{a^3}{3}\right) - \left[\frac{a^2}{4n^2\pi^2}x\cos\left(\frac{2n\pi x}{a}\right)\right]_0^a + \int_0^a \frac{a}{2n\pi}\cos\left(\frac{2n\pi x}{a}\right) dx\right]$$

$$= \frac{2}{a}\left[\frac{a^3}{6} - \frac{a^3}{4n^2\pi^2} + \cancel{\left[\frac{a^2}{4n^2\pi^2}\sin\frac{2n\pi x}{a}\right]_0^a}\right]$$

$$= \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2}$$

$$\Delta x = \sqrt{\frac{a^2}{3} - \frac{a^2}{2n^2\pi^2} - \frac{a^2}{4}}$$

$$= \sqrt{\frac{a^2}{12}\left(1 + \frac{6}{n^2\pi^2}\right)}$$

$$(\Delta x)^2 = \frac{a^2}{12}\left(1 + \frac{6}{n^2\pi^2}\right)$$

When $n \to \infty$ $\Delta x \to \frac{a}{2\sqrt{3}}$

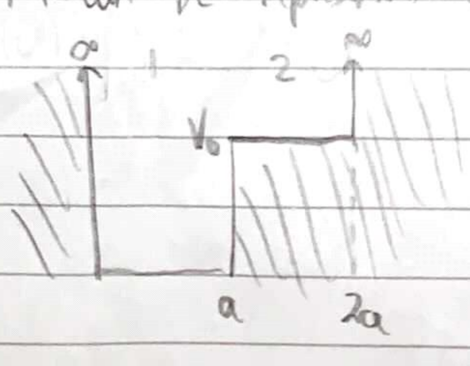
11. $V(x) = 0$ $\quad 0 < x < a$ Region 1

$V(x) = V_0$ $\quad a < x < 2a$ Region 2 $\quad V_0 = \frac{2\hbar^2\pi^2}{ma^2}$

$V(x) = \infty$ $\quad$ elsewhere Region 3 $\quad E = \frac{25\hbar\pi^2}{8ma^2}$

a) $E > V_0$ except where $V = \infty$

$\therefore$ Can be represented as sinusoids throughout



$\psi_1 = Ae^{ik_1x}$ $\quad k_1 = \sqrt{\frac{2mE}{\hbar^2}}$ $\quad \psi_1 = \alpha \sin k_1 x + \beta \cos k_1 x$

$\psi_2 = Be^{ik_2x}$ $\quad k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$ $\quad \psi_2 = \gamma \sin k_2 x + \delta \cos k_2 x$

$\psi_1(0) = 0 \Rightarrow \beta = 0 \quad \therefore \psi_1 = \alpha \sin k_1 x \quad ①$

$\psi_2(2a) = 0 \Rightarrow \gamma \sin 2k_2 a + \delta \cos 2k_2 a = 0 \quad ②$

$\psi_1(a) = \psi_2(a) \Rightarrow \alpha \sin k_1 a = \gamma \sin k_2 a + \delta \cos k_2 a \quad ③$

$\left.\frac{\partial \psi_1}{\partial x}\right|_a = \left.\frac{\partial \psi_2}{\partial x}\right|_a \Rightarrow k_1 \alpha \cos k_1 a = k_2 \gamma \cos k_2 a - k_2 \delta \sin k_2 a \quad ④$

$k_1^2 = \frac{25\pi^2}{4a^2} \quad k_1 = \frac{5\pi}{2a}$

$k_2^2 = \frac{9\pi^2}{4a^2} \quad k_2 = \frac{3\pi}{2a}$

② $\gamma \sin 3\pi + \delta \cos 3\pi = 0 \Rightarrow -\delta = 0$

③ $\alpha \sin \frac{5\pi}{2} = \gamma \sin \frac{3\pi}{2} + \delta \cos \frac{3\pi}{2}$

$\alpha = -\gamma$

④ $\frac{5\pi}{2a} \alpha \cos \frac{5\pi}{2} = -\frac{3\pi}{2a} \alpha \cos \frac{3\pi}{2}$

$\therefore \psi_1 = \alpha \sin k_1 x$

$\psi_2 = -\alpha \sin k_2 x$

$\int_0^a \alpha^2 \sin^2 \frac{5\pi x}{2a} dx + \int_a^{2a} \alpha^2 \sin^2 \frac{3\pi x}{2a} dx = 1$

$1 = \alpha^2 \left[\frac{1}{2}\left(x - \frac{\cancel{a}}{5\pi} \sin\left(\frac{10\pi x}{2a}\right)\right)\right]_0^a + \alpha^2 \left[\frac{1}{2}\left(x - \frac{a}{3\pi} \sin\left(\frac{3\pi x}{a}\right)\right)\right]_a^{2a}$

$$1 = \alpha^2 \left[ \frac{a}{2} + a - \frac{a}{2} \right]$$

$$\alpha^2 = \frac{1}{a}$$

$$\alpha = \frac{1}{\sqrt{a}}$$

$$\therefore \psi_1 = \frac{1}{\sqrt{a}} \sin \frac{5\pi x}{2a}$$

$$\psi_2 = -\frac{1}{\sqrt{a}} \sin \frac{3\pi x}{2a}$$

$$\frac{1}{a} \int_0^a \sin^2 \frac{5\pi x}{2a} dx = \frac{1}{a} \left[ \frac{1}{2} \left( x - \frac{a}{5\pi} \sin \frac{5\pi x}{a} \right) \right]_0^a$$

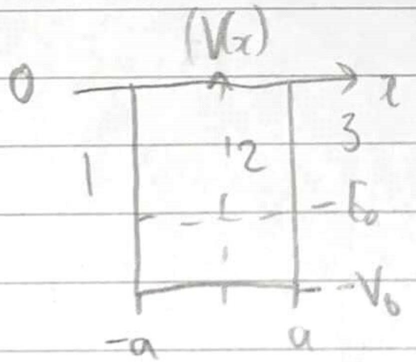
$$= \frac{1}{a} \times \frac{a}{2}$$

$$= \frac{1}{2}$$

b) KE in region 1: $\frac{25\hbar^2\pi^2}{8ma^2}$

KE in region 2: $\frac{9\hbar^2\pi^2}{8ma^2}$

h.



$-V_0 < E < 0$:

let $V = -V_0$

$E = -E_0$

For $|x| < a/2$ $\quad \psi = A\sin kx + B\cos kx$

$$k = \sqrt{\frac{2m(V_0 - E_0)}{\hbar^2}}$$

For $|x| > \frac{a}{2}$ $\quad \psi = Ce^{Kx} + De^{-Kx}$ $\quad K = \sqrt{\frac{2mE_0}{\hbar^2}}$

$\psi \to 0$ as $x \to \pm\infty$

$\therefore \psi_1 = Ce^{Kx}$

$\psi_3 = De^{-Kx}$

BC: $A\sin ka + B\cos ka = De^{-Ka}$

$kA\cos ka + kB\sin ka = -KDe^{-Ka}$

$-A\sin ka + B\cos ka = Ce^{-Ka}$

$kA\cos ka + kB\sin ka = KCe^{-Ka}$

$2A\sin ka = (D-C)e^{-Ka}$

$2B\cos ka = (D+C)e^{-Ka}$

$2kB\sin ka = K(D+C)e^{-Ka}$

$2kA\cos ka = -K(D-C)e^{-Ka}$

$k\cot(ka) = -K$

$k\tan ka = K$

$k^2 = \frac{2m(V_0 - E_0)}{\hbar^2}$

$K^2 = \frac{2mE_0}{\hbar^2}$

$k^2 + K^2 = \frac{2mV_0}{\hbar^2}$

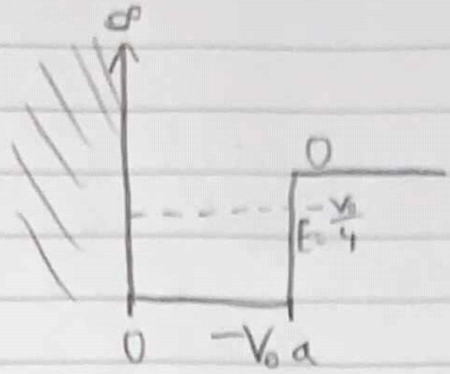
Only one state $\Rightarrow$ No intersection of circle and cot

$k^2a^2 + K^2a^2 = \frac{2ma^2V_0}{\hbar^2}$

$k^2a^2 + K^2a^2 = \frac{\pi^2}{4}$ is where these functions first intersect

ie is $k^2a^2 + K^2a^2 < \frac{\pi^2}{4}$ only one bound state will be possible

13.




$V = -V_0$

$E = -\frac{V_0}{4}$

$0 < x < a$ : $\psi_1 = A\sin kx + B\cos kx$ $\qquad k = \sqrt{\frac{2m(\frac{3}{4}V_0)}{\hbar^2}}$

$x > a$ : $\psi_2 = Ce^{Kx} + De^{-Kx}$ $\qquad K = \sqrt{\frac{2m(\frac{1}{4}V_0)}{\hbar^2}}$

$\psi(0) = 0 \Rightarrow \psi_1 = A\sin kx$ $\qquad k = \sqrt{\frac{3mV}{2\hbar^2}}$

$\qquad K = \sqrt{\frac{mV}{2\hbar^2}}$

$\psi(x) \to 0$ as $x \to \infty$

$\therefore \psi_2 = De^{-Kx}$

$\psi_1(a) = \psi_2(a) \quad \therefore A\sin ka = De^{-Ka}$

$\frac{d\psi_1}{dx}\Big|_a = \frac{\partial \psi_2}{\partial x}\Big|_a \quad \therefore kA\cos ka = -KDe^{-Ka}$

$A = \frac{De^{-Ka}}{\sin ka}$

$A^2\int_0^a \sin^2 kx\,dx + D^2\int_a^\infty e^{-2Kx}\,dx = 1$

$\frac{D^2e^{-2Ka}}{\sin^2 ka}\left[\frac{1}{2}\left(x - \frac{2}{k}\sin 2kx\right)\right]_0^a + D^2\left[-\frac{1}{2K}e^{-2Kx}\right]_a^\infty = 1$

$\frac{D^2e^{-2Ka}}{\sin^2 ka}\left(\frac{1}{2}a - \frac{1}{k}\sin 2ka\right) + \frac{D^2}{2K}e^{-2Ka} = 1$