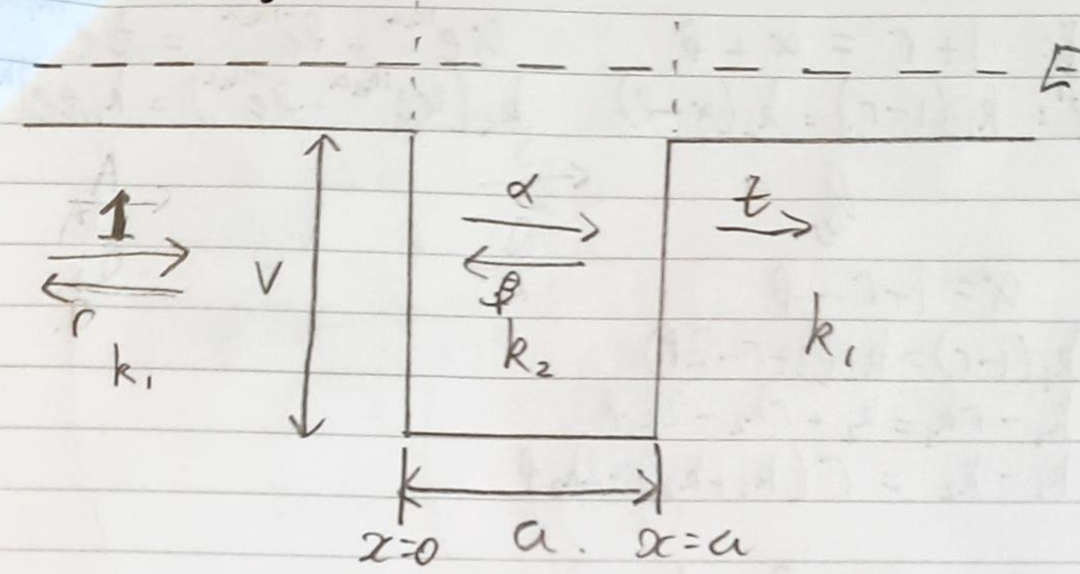
9) QM Questions



Beam of particles can be represented by a plane wave.
Schrödinger:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi.$$

$$E\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi. \qquad \Psi = Ae^{i(kx-\omega t)}$$

$\cancel{i\hbar \cdot E\omega}$

$$(E-V)\Psi = -\frac{\hbar^2}{2m}\nabla_x^2(\Psi)$$

$$(E-V)\cancel{\Psi} = +\frac{\hbar^2}{2m}k^2\cancel{\Psi}$$

$$k = \sqrt{\frac{2m(E-V)}{\hbar^2}}$$

Ψ must be continuous and differentiable.

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}. \qquad k_2 = \sqrt{\frac{2m(E-V)}{\hbar^2}}.$$

$\cancel{\Psi: A + B = C + D}$

$$x=0 \qquad\qquad\qquad x=a$$

$$\psi: \; 1+r = \alpha + \beta \qquad \alpha e^{ik_2a} + \beta e^{-ik_2a} = t e^{ik_1a}$$

$$\psi': \; k_1(1-r) = k_2(\alpha-\beta) \qquad k_2\left(\alpha e^{ik_2a} - \beta e^{-ik_2a}\right) = k_1 t e^{ik_1a}$$

$$\alpha = 1+r-\beta$$

$$k_1(1-r) = k_2(1+r-2\beta)$$

$$k_1 - rk_1 = k_2 + rk_2 - 2k_2\beta$$

$$k_1 - k_2 = r(k_1+k_2) - 2k_2\beta$$

$$r = \alpha + \beta - 1$$

$$k_1(2-\alpha-\beta) = k_2(\alpha-\beta)$$

$$2k_1 - \alpha k_1 - \beta k_1 = \alpha k_2 - \beta k_2$$

$$2k_1 = \alpha(k_1+k_2) + \beta(k_1-k_2)$$

$$2 = \alpha(1+k_2/k_1) + \beta(1-k_2/k_1).$$

$$k_2\left(\alpha e^{ik_1a} - \beta e^{-ika}\right) = k_1\left(\alpha e^{ik_2a} + \beta e^{-ik_2a}\right).$$

$$\alpha\left(k_2 e^{ik_1a} - k_1 e^{ik_2a}\right) = \beta\left(k_1 e^{-ik_2a} + k_2 e^{-ik_1a}\right).$$

$$\alpha = \frac{\beta\left(k_1 e^{-ik_2a} + k_2 e^{-ik_1a}\right)}{k_2 e^{ik_1a} - k_1 e^{ik_2a}}.$$

$$\alpha = \frac{t}{2}\left(1+\frac{k_1}{k_2}\right) e^{-ik_2a} e^{ik_1a}$$

$$\beta = \frac{t}{2}\left(1-\frac{k_1}{k_2}\right) e^{+ik_2a} e^{ik_1a}.$$

$$\Rightarrow t = \frac{2k_1k_2\, e^{-ik_1a}}{2k_1k_2\cos(k_2a) - i(k_1^2+k_2^2)\sin(k_2a)}$$

$$r = \frac{(k_1^2-k_2^2)\sin(ak_2)}{(k_1^2+k_2^2)\sin(k_2a) + 2ik_1k_2\cos(k_2a)}$$

So, when does $R = r^2 = 0$?

Either $k_1 = k_2$ (not true) or $\sin(ak_2) = 0$.

so $ak_2 = n\pi$

$$k_2 = \frac{n\pi}{a}.$$

$$\sqrt{\frac{2m(E-V)}{\hbar^2}} = \frac{n\pi}{a}$$

$$2m(E-V) = \left(\hbar\frac{n\pi}{a}\right)^2$$

$$E = \frac{\left(\frac{\hbar n\pi}{a}\right)^2}{2m} + V.$$

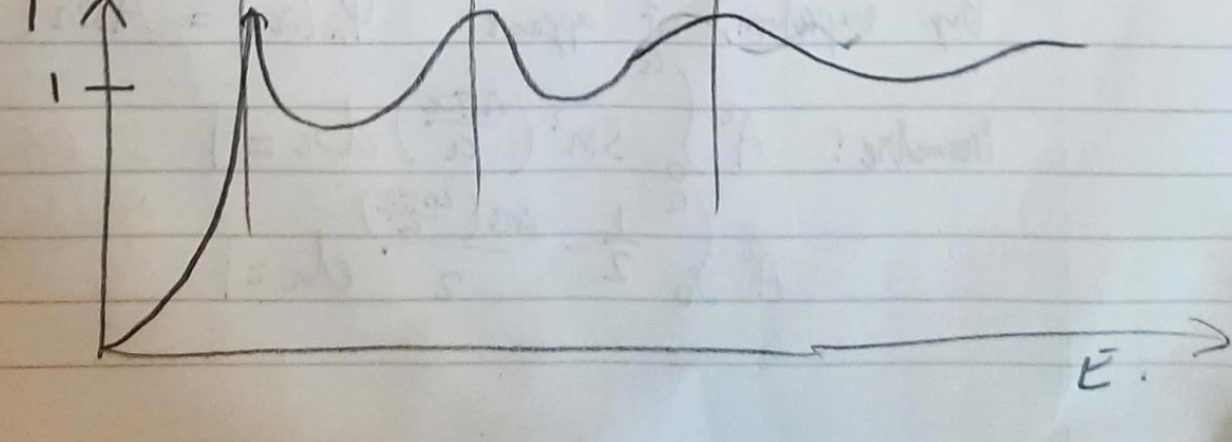
$$T = t^2 = \frac{4k_1^2k_2^2}{4k_1^2k_2^2\cos^2(k_2a) - (k_1^2+k_2^2)^2\sin^2(k_2a) - 4ik_1k_2(k_1^2+k_2^2)\frac{\sin}{\cos}}.$$

$$tt^* = \frac{2k_1k_2e^{-ik_1a}}{2k_1k_2\cos k_2a - i(k_1^2+k_2^2)\sin(k_2a)} \cdot \frac{2k_1k_2e^{ik_1a}}{2k_1k_2\cos k_2a + i(k_1^2+k_2^2)\sin k_2a}$$

$$= \frac{4k_1^2k_2^2}{4k_1^2k_2^2\cos^2(k_2a) + (k_1^2+k_2^2)^2\sin^2(k_2a)}$$

$$\cancel{\left(= 1 / \left(4 \cdot \frac{2mE}{\hbar^2} \cdot \frac{2m(E-V)}{\hbar^2}\right)\right)} \qquad \frac{2m(2E-V)}{\hbar^2}.$$

$$= \cancel{\frac{\frac{4m^2}{\hbar^4}}{\frac{4m^2}{\hbar^4}}} \cdot \frac{4E(E-V)}{4E(E-V)\cos^2\left(a\sqrt{\frac{2m(E-V)}{\hbar^2}}\right) + (2E-V)^2\sin^2(k_2a)}$$



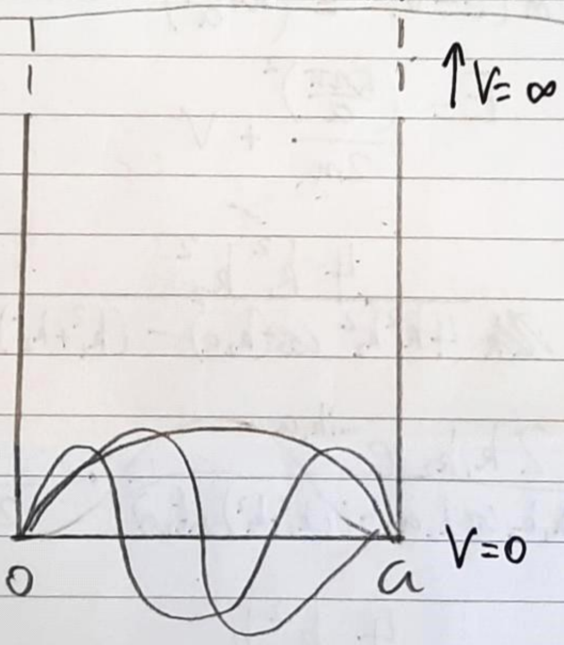
Electrons, $E = 0.5\text{eV}$, $a = 0.4\text{nm}$.
Scattering minimum = no reflection?

$$V = E - \frac{\hbar^2 \pi^2}{2m_e a^2}$$

$$\cancel{0.5(1.6 \cdot 10^{-19})} = V = (0.5e) - \frac{\hbar^2 \pi^2}{2m_e \cdot (0.4 \cdot 10^{-9})^2}$$

$$V = \cancel{k} - 1.853\ \text{eV}$$

10.



Beyond the box $\psi = 0$ as $k \to 0$.
Boundary conditions:

ψ : $\cancel{\psi(0,0)}$ $\psi(0,t) = \psi(a,t) = 0$.

~~ψ' : $\psi'(0,t) = \psi'(a,t) = 0$.~~

V is discontinuous and $\to \infty$, so ψ' need not be continuous

By inspection, it appears $\psi_n(x) = A \sin\left(\frac{n\pi x}{a}\right)$.

Normalise: $A^2 \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx = 1$ $\qquad \cos 2x = 1 - 2\sin^2 x$

$$A^2 \int_0^a \frac{1}{2} - \frac{\cos\left(\frac{2n\pi x}{a}\right)}{2}\, dx = 1 \qquad \frac{-\cos 2x + 1}{2} = \sin^2$$

$$A^2\left[\frac{x}{2}\right]_0^a - A^2\left[\frac{\sin\left(\frac{2n\pi x}{a}\right)}{2}\cdot\frac{a}{2n\pi}\right]_0^a = 1$$

$$\frac{A^2 a}{2} - 0 \text{ for all } n = 1.$$

$$A^2 = \frac{2}{a}, \quad A = \sqrt{\frac{2}{a}}.$$

$$\text{So} \quad \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right).$$

$\int uv' = uv - \int vu'$!

$$\langle x\rangle = \int_0^a x|\psi(x)|^2 dx$$

$u = x \quad v' = \sin^2\frac{n\pi x}{a}$

$u' = 1 \quad v = \text{prev. ans}$

$$= \frac{2}{a}\int_0^a x\sin^2\left(\frac{n\pi x}{a}\right)dx.$$

$$= \frac{2}{a}\left[x\left(\frac{x}{2} - \frac{a\sin\left(\frac{2n\pi x}{a}\right)}{\cancel{2n\pi}}\right)\right]_0^a$$

$$- \frac{2}{a}\int_0^a \frac{x}{2} - \frac{a\sin\left(\frac{2n\pi x}{a}\right)}{2n\pi}dx.$$

$$= \frac{\cancel{a}}{\cancel{a}}a - \frac{2}{a}\left[\frac{x^2}{4}\right]_0^a + \frac{\cancel{2}a}{\cancel{2}n\pi}\left[\cos\left(\frac{2n\pi x}{a}\right)\cdot\frac{a}{2n\pi}\right]_0^a$$

$$= a - \frac{1}{2}\frac{a^2}{a} + \frac{a^2}{2n^2\pi^2}\left[\cos\frac{2n\pi x}{a}\right]_0^a$$

$$= a - \frac{a}{2} \cancel{+ \frac{a^2}{2n^2\pi^2}(1-1)}$$

$$= \frac{a}{2} \text{ for all } n.$$

((ould have done this by eye, thicko!)

or reverse, can't remember.

$$\Delta x = \sqrt{\langle x\rangle^2 - \langle x^2\rangle}$$

$$\langle x^2\rangle = \int_0^a x^2|\psi(x)|^2 dx.$$

$$= \frac{2}{a}\int_0^a x^2\sin^2\left(\frac{n\pi a}{2}\right).$$

$$\frac{2}{a}\int_0^a x^2 \sin^2\left(\frac{n\pi a}{2}\right) dx.$$

$\int uv' = uv - \int u'v$

$u = x^2 \quad v' = \sin^2$

$u' = 2x \quad v =$ you know the drill.

$$= \frac{2}{a}\left[x^2\left(\frac{x}{2} - \cancel{\sin}^{0}\right)\right]_0^a$$

$$- \frac{2}{a}\int_0^a \cancel{2}x\left(\frac{x}{\cancel{2}} - \frac{a\sin\left(\frac{2n\pi x}{a}\right)}{\cancel{2}n\pi}\right) dx$$

$$= a^2 - \frac{2}{a}\left[\frac{x^3}{3}\right]_0^a - \frac{2a}{an\pi}\int_0^a x\sin\left(\frac{2n\pi x}{a}\right) dx$$

$$= a^2 - \frac{2}{3}a^2 - \frac{2}{n\pi}\left(\left[-x\cos\left(\frac{2n\pi x}{a}\right)\frac{a}{2n\pi}\right]_0^a + \int_0^a \frac{a}{2n\pi}\cos\left(\frac{2n\pi x}{a}\right) dx\right)$$

$u = x \qquad v' = \sin\left(\frac{2n\pi x}{a}\right)$

$u' = 1 \qquad v = -\cos\left(\frac{2n\pi x}{a}\right)\cdot\frac{a}{2n\pi}$

$$= \frac{a^2}{3} + \frac{2a}{n^2\pi^2}\left[x\cos\left(\frac{2n\pi x}{a}\right)\right]_0^a - \cancel{\frac{a}{n^2\pi^2}\cdot\frac{a}{2n\pi}\left[\sin\left(\frac{2n\pi x}{a}\right)\right]_0^a}.$$

$$= \frac{a^2}{3} + \frac{2a}{n^2\pi^2}(a) = a^2\left(\frac{1}{3} + \frac{2}{n^2\pi^2}\right).$$

$$\Delta x = \sqrt{\langle x\rangle^2 - \langle x^2\rangle}$$

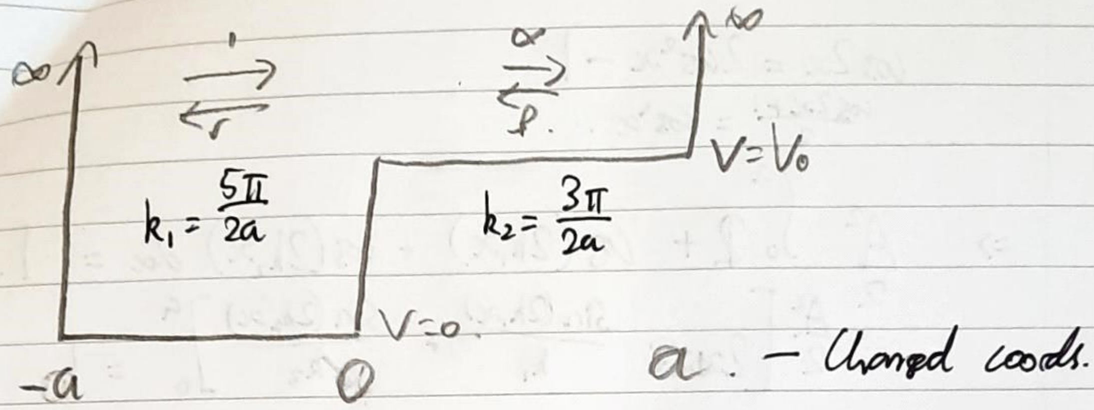
$$\Delta x^2 = \frac{a^2}{4} - \frac{a^2}{3} - \frac{2a^2}{n^2\pi^2}$$

$$\Delta x^2 = \frac{a^2}{12} - \frac{2a^2}{n^2\pi^2}.$$

$$\frac{\Delta x}{a} = \sqrt{\frac{1}{12} - \frac{2}{n^2\pi^2}}.$$

When $n \to \infty$, ~~$\Delta x \to a/\sqrt{12}$~~.

$\Delta x \to a/\sqrt{12}$.



. – Changed coords.

$\psi_1(-a) = 0$. $\psi_2(a) = 0$. $\psi_1(0) = \psi_2(0)$

and $\psi_1'(0) = \psi_2'(0)$

~~Ψ2~~ Since the walls are infinite and impassable, $r = 1$ and $\alpha = \beta$.

$\Rightarrow \psi_1 = e^{ik_1x} + e^{-ik_1x} = (2)\cos(k_1 x)$. Time not considered.

$\psi_2 = \alpha(e^{ik_2x} + e^{-ik_2x}) = (2\alpha)\cos(k_2 x)$.

These cosines must $= 0$ at the far edges.

$\cos(k_1 a) = \cos\left(\frac{5\pi}{2}\right) = 0$ ✓. $\cos(k_2 a) = \cos\left(\frac{3\pi}{2}\right) = 0$ ✓.

The derivatives at 0 are both 0 so this is also satisfied (ψ').

For $\psi_1(0) = \psi_2(0)$, $\alpha = 1$.

Normalisation:

$$\int_{-a}^{0} (A\cos(k_1 x))^2\,dx + \int_0^a (A\cos(k_2 x))^2\,dx = 1$$

$$A^2 \int_0^a (\cos k_1 x)^2 + (\cos k_2 x)^2\,dx = 1$$ (by cos symmetry).

$$\cancel{A^2\left[\frac{\sin k_1 x}{k_1} + \frac{\sin k_2 x}{k_2}\right]_0^a = 1.}$$

$$\cancel{\frac{2a}{\pi} A^2 \left(\frac{\sin\left(\frac{5\pi}{2}\right)}{5} + \frac{\sin\left(\frac{3\pi}{2}\right)}{3} - \frac{\sin 0}{a} - \frac{\sin 0}{0}\right) = 1}$$

forgot to square.

$$\cancel{\frac{2a}{\pi} A^2\left(\frac{1}{5} - \frac{1}{3}\right) = 1.}$$

$\cos 2x = 2\cos^2 x - 1$

$\frac{\cos 2x + 1}{2} = \cos^2 x$.

$$\Rightarrow \frac{A^2}{2} \int_0^a 2 + \cos(2k_1 x) + \cos(2k_2 x)\, dx = 1.$$

$$\frac{A^2}{2}\left[2x + \cancel{\frac{\sin(2k_1 x)}{k_1}} + \cancel{\frac{\sin(2k_2 x)}{k_2}}\right]_0^a = 1.$$

$0 \qquad 0$

$A^2 a = 1$, ~~a~~ $A^2 = 1/a$, $A = \sqrt{1/a}$.

So

$$\Psi(x) = \begin{cases} \sqrt{1/a}\cos\left(\frac{5\pi x}{2a}\right), & -a < x \leq 0 \\ \sqrt{1/a}\cos\left(\frac{3\pi x}{2a}\right), & 0 \leq x < a \end{cases}$$

Odds of finding in the left half:

$$P = \int_{-a}^{0} x\,|\Psi(x)|^2\, dx$$

$$= \frac{1}{a}\int_{-a}^{0} x\cos^2\left(\frac{5\pi x}{2a}\right) dx$$

$U = x \quad V' = \cos\frac{5\pi x}{a}$

$U' = 1 \quad V = \frac{a}{5\pi}\sin\left(\frac{5\pi x}{a}\right)$

$\int uv' = uv - \int vu'$

$$= \frac{1}{2a}\int_{-a}^{0} x + x\cos\left(\frac{5\pi x}{a}\right) dx$$

$$= \frac{1}{2a}\left[\frac{x^2}{2}\right]_{-a}^{0} + \frac{1}{2a}\cancel{\left[\frac{ax}{5\pi}\sin\left(\frac{5\pi x}{a}\right)\right]_{-a}^{0}} \; \frac{1}{2a}\frac{a^2}{25\pi^2}\left[\cos\left(\frac{5\pi x}{a}\right)\right]_{-a}^{0}$$

0

$$= 0 - \frac{a}{4} + \frac{a^2}{25\pi^2}(1 - -1) = \frac{2a^2}{2a \cdot 25\pi^2} - \frac{a}{4}.$$

$$= \frac{a}{25\pi^2} - \frac{a}{4}.$$

X Why do I keep doing this?

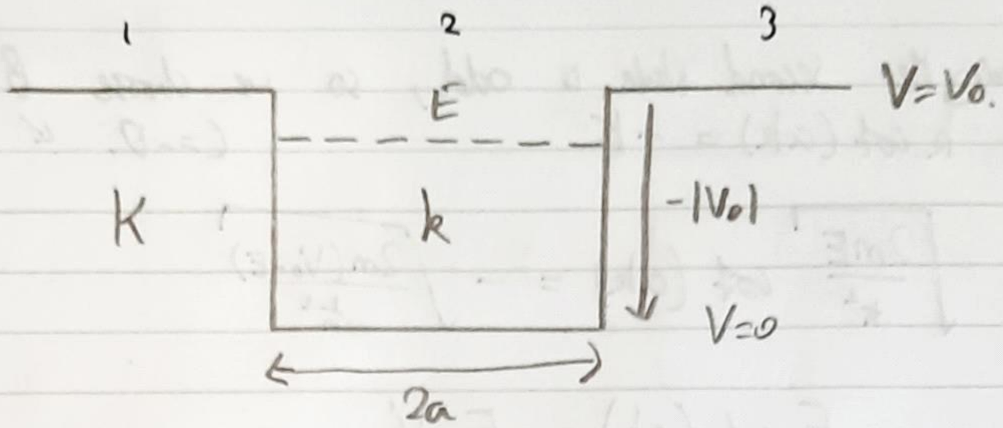
$$P = \int \Psi^2\, dx = \frac{1}{a}\int_{-a}^{0} \cos^2\left(\frac{5\pi}{2a}x\right) dx$$

$$= 1/2.$$

How would you find this classically?

left is ~~4/6~~ 5/8, right 3/8?

12)



If there is one and only one available state, this must be the ground state, with the $n=2$ state being too high energy. Bound states imply $E < V_0$.

$$k = \sqrt{\frac{2m(E - \cancel{V})}{\hbar^2}} = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\Rightarrow \quad k^2\hbar/2m = E, \; < V_0.$$

$$K = \sqrt{\frac{2m\, V_0 - E}{\hbar^2}}.$$

$$\Psi_1 = Ce^{Kx} \qquad \Psi_2 = A \sin kx + B\cos kx.$$
$$\Psi_3 = De^{-Kx}$$

Boundaries: Ψ is continuous and differentiable at a + $-a$.

① $a: \quad De^{-aK} = A\sin ka + B\cos ka$
② $-a: \quad Ce^{-aK} = -A\sin ka + B\cos ka$
③ $a': \quad -KDe^{-Ka} = kA\cos ka - kB\sin ka$
④ $-a': \quad KCe^{-Ka} = kA\cos ka + kB\sin ka.$

$k\cot(ak) = -K$ if $A \neq 0$.

① + ②: $(D+C)e^{-Ka} = 2B\cos(ka).$
① − ②: $(D-C)\,e^{-Ka} = 2A\sin(ka)$
③ − ④: $K(D+C)\,e^{-Ka} = 2kB\sin(ka)$
③ + ④: $-K(D-C)\,e^{-Ka} = 2kA\cos(ka)$

$k\tan(ak) = K.$ if $B \neq 0$

Now the second state is odd, so we choose $B=0$.

$k\cot(ak) = -K$ $\qquad C=-D.$

$$\sqrt{\frac{\cancel{2m}E}{\cancel{\hbar^2}}}\cot(ak) = -\sqrt{\frac{\cancel{2m}(V_0-E)}{\cancel{\hbar^2}}}$$

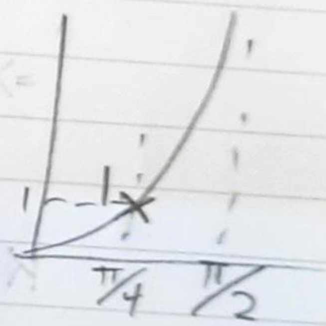
$$E\cot(ak) = E - V_0.$$

$$\cot(ak) = 1 - \frac{V_0}{E}. \qquad \text{note } E<V_0 \text{ so } \underline{\underline{1-\frac{V_0}{E}<0.}}$$

$$\cancel{\cot(ak) = 1 - \frac{2mV_0}{\hbar^2}}$$

$$\cot(ak) < 1.$$

$$\frac{1}{\tan(ak)} < \frac{1}{1}$$

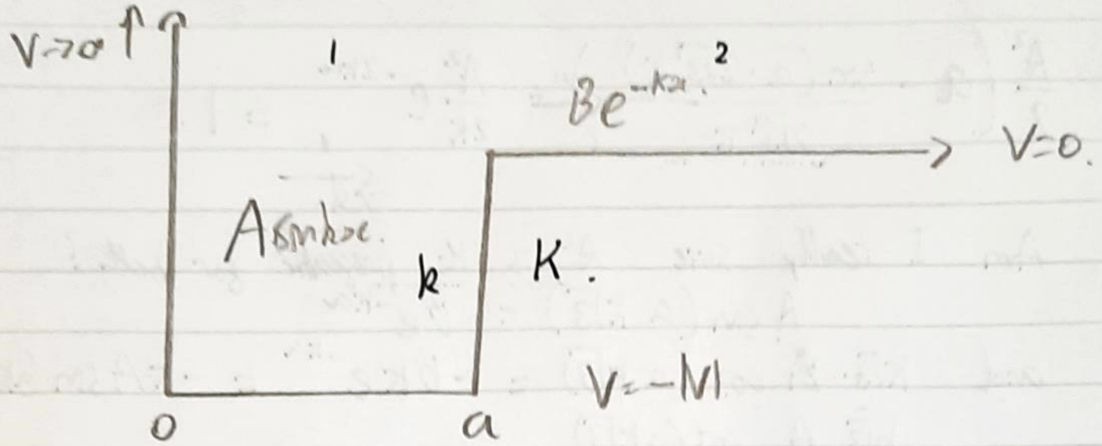


$$a\sqrt{\frac{2mE}{\hbar^2}} < \arctan(1)$$

$$\frac{a^2 \cdot 2mE}{\hbar^2} < \left(\frac{\pi}{4}\right)^2$$

Should have V_0 instead of E so an error somewhere.

13)



$E = -V/4$. $E<0$ so evanescent beyond $x=a$.

$\psi(0) = 0$. $\psi_+(a) = \psi_-(a)$. $\psi_+'(a) = \psi_-'(a)$.

↳ $A = B$, unless a node?

ψ in region 1 must be a sine wave.

ψ in region 2 must be a decaying exponential.

$$K = \sqrt{\frac{2mE}{\hbar^2}} \qquad k = \sqrt{\frac{2m(E-V)}{\hbar^2}}$$

$$= \sqrt{\frac{2mV}{4\hbar^2}} \qquad = \sqrt{\frac{2mV\cdot 3}{4\hbar^2}}$$

$$= \sqrt{\frac{mV}{2\hbar^2}} \qquad = \sqrt{\frac{3mV}{2\hbar^2}} = K\sqrt{3}.$$

$A \sin ka = Be^{-Ka}$

$A \sin(K\sqrt{3}a) = Be^{-Ka}$.

Normalisation:

$\cos 2\theta = 1 - 2\sin^2\theta$

$\frac{1-\cos 2\theta}{2} = \sin^2\theta$

$$\int_{-\infty}^{\infty} |\psi|^2 \, dx = 1.$$

$$\int_0^a A^2 \sin^2(xK\sqrt{3})dx + \int_a^{\infty} A^2 e^{-2Kx} \, dx = 1.$$

$$\frac{A^2}{2}\int_0^a 1 - \cos(2\sqrt{3}Kx)dx + \frac{A^2}{2K}\left[-e^{-2Kx}\right]_a^{\infty} = 1$$

$$\frac{A^2}{2}\left[x - \frac{\sin(2\sqrt{3}Kx)}{2\sqrt{3}K}\right]_0^a + \frac{A^2}{2K}\left(-0 - -e^{-2Ka}\right) = 1$$

$$\underbrace{\frac{A^2}{2}\left(a - \frac{\sin(a \cdot 2\sqrt{3}\,k)}{2\sqrt{3}\,k}\right)}_{\text{inside}} + \underbrace{\frac{A^2}{2k} e^{-2ka}}_{B^2?\ \text{outside}} = 1.$$

Am I really sure A^2 is the prefactor for both?

$$A \sin(a k\sqrt{3}) = B e^{-ka}$$

and $k\sqrt{3} \cdot \cancel{A} \cos(a k\sqrt{3}) = -Bke^{-ka} = -kA \sin(ak\sqrt{3})$.

$$-\frac{\cancel{k}\sqrt{3}}{\cancel{k}} \frac{\cancel{A}}{\cancel{A}} \frac{\cos(ak\sqrt{3})}{\sin(ak\sqrt{3})}^{\cot} = 1$$

$$-\sqrt{3} = \tan(a k\sqrt{3})$$

$$\frac{\pi}{3} = -a k\sqrt{3}$$

$$k = -\frac{\pi}{3\sqrt{3}\,a} \ ?$$

$$\Rightarrow \sin(a \cdot 2\sqrt{3} \cdot k) = \sin\left(-\frac{\pi \cdot \cancel{a} \cdot 2\cancel{\sqrt{3}}}{3\cancel{\sqrt{3}}\,\cancel{a}}\right) = \sin\left(-\frac{2\pi}{3}\right) = -\sqrt{3}/2.$$

$$P(\text{inside}) + P(\text{outside}) = 1.$$

$$\cancel{\frac{A^2}{2}\left(a + \frac{\sqrt{3}/2}{k\, 2\sqrt{3}}\right)} + \frac{A^2}{-2\frac{\pi}{3\sqrt{3}a}} e^{-2a \cdot -\frac{\pi}{3\sqrt{3}a}} = 1$$

$$\frac{A^2}{2}\left(a + \frac{\sqrt{3}/2}{-2\sqrt{3} \cdot \frac{\pi}{3\sqrt{3}a}}\right) + \frac{3\sqrt{3}A^2 a}{-2\pi} e^{\frac{2\pi}{3\sqrt{3}}} = 1$$

$$\frac{A^2}{2}\left(a - \frac{3a\sqrt{3}}{2\pi}\right) - \frac{3\sqrt{3}A^2 a}{2\pi} e^{\frac{2\pi}{3\sqrt{3}}} = 1$$

Attempt at ratio:

$$\frac{\frac{3\sqrt{3}\,aA^2}{2\pi} e^{\cdots}}{\frac{3\sqrt{3}\,aA^2}{2\pi} e^{\cdots} + \frac{A^2}{2}\left(a - \frac{3a\sqrt{3}}{2\pi}\right)}$$

$$\frac{\frac{A^2 \cancel{a}}{\cancel{2\pi}}}{\frac{\cancel{A^2 a}}{\cancel{2\pi}}} \cdot \frac{3\sqrt{3}\, e^{\cdots}}{3\sqrt{3}e^{\cdots} + \frac{1}{\pi} - 3\sqrt{3}} \rightarrow \frac{\sqrt{3}\, e^{\cdots}}{\sqrt{3}\, e^{\cdots} - \sqrt{3} + \frac{1}{3\pi}}.$$

Has $\sqrt{3}$ in at least.