Part IB Physics A: Lent 2022

QUANTUM PHYSICS EXAMPLES I

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1. In one of Millikan's experiments, a clean sodium surface was irradiated by light having various wavelenths λ, and the emitted electrons subjected to a retarding voltage V. The electron current was determined from the deflection d of an electrometer accumulating charge for 30 s. From the following tabulated results, estimate the stopping potential for each wavelength, and hence determine Planck'c constant $\hbar$, and the work function W_{Na} of sodium.

$\lambda = 546$ (nm)		434		405		365		313	
V (V)	d (mm)	V	d	V	d	V	d	V	d
0.253	28	0.829	44	0.934	82	1.353	67.5	1.929	52
0.305	14	0.881	20	0.986	55	1.405	36	1.981	29
0.358	7	0.934	10	1.039	36	1.458	19	2.034	12
0.410	3	0.986	4	1.091	24	1.510	11	2.086	5
				1.143	3	1.562	4	2.138	2.5

2. Calculate the de Broglie wavelength of:

(a) an 80 kg person walking at 6 km h^{-1};
(b) a photon of energy 20 eV;
(c) an electron of kinetic energy 20 eV;

Comment in each case on the scale of your result in relation to measurable effects.

3. A beam of electromagnetic radiation passes through a 50% beam splitter that is inclined at an angle of 45° with respect to the axis of the beam. Two detectors are used to measure the power that passes through the beam splitter, and the power that is reflected off of the beam splitter. For high intensity beams, the two detectors each read

50% of the power in the beam, as expected. The intensity of the beam is now reduced until photons pass through the system one at a time. If the detectors are sufficiently fast and sensitive, describe how they behave.

4. In an alternative universe, $\hbar$ has the value 10^{-6} J s instead of 10^{-34} J s. A dart with mass 1 kg is dropped from a height of 1 m, the intention being to hit a point target on the ground below. What limitation is imposed by the uncertainty principle on the accuracy that can be achieved?

(Neglect uncertainties in the vertical position and momentum, which produce only second-order effects.)

5. Use the uncertainty principle to estimate the ground state energy E_0 of a particle of mass m moving in a one-dimensional harmonic potential $\frac{1}{2}\kappa x^2$.

6. State the stability condition proposed by Bohr in his early model of the Hydrogen atom, and show how it relates to the quantisation of the angular momentum. Derive the total energy and show that it can be written as

$$E_n = -\frac{hcR}{n^2}\,.$$

Give an expression for the constant R and show that it has the correct dimensions. Thence, obtain expressions for the radius and velocity of the first orbit, and express the latter in terms of the fine structure constant.

7. Determine the normalising constants for the following wave functions:

(a) $\psi(x) = A_1 \sin(\pi x/a), \quad 0 \le x \le a$;
(b) $\psi(x, y, z) = A_2 \sin(\pi x/a)\sin(\pi y/b)\sin(\pi z/c)$, in a rectangular box with sides of length a, b and c;
(c) $\psi(r) = A_3 \exp(-r/a)$, over all space.

8. Show that for a free particle, having a quadratic dispersion relation, of mass m with wavefunction

$$\psi(x,t) \propto \int_{-\infty}^{\infty} \mathrm{e}^{-a^2(k-k_0)^2}\mathrm{e}^{i(kx-\omega t)}\mathrm{d}k,$$

the width of the wave packet is $\sim a$ at $t = 0$, and will double in a time $T = 2\sqrt{3}ma^2/\hbar$.

Calculate T for
(a) a proton localised to within 1 nm;
(b) a 1 kg mass localised to within 0.1 mm.

9. A particle is represented by the wavefunction

$$\psi(x) = Ax\mathrm{e}^{-\alpha x^2}.$$

(a) Calculate the normalising constant A.
(b) Calculate Δx and Δp_x.
(c) Show that $\Delta x \Delta p_x = 3\hbar/2$.

10. A beam of particles travelling in the x-direction with energy E is incident on a one-dimensional potential well with vertical sides of depth V and width a. Show that for certain values of E there is no reflected beam, and sketch the transmission coefficient as a function of E.

The scattering of electrons by Kr atoms shows a (first) minimum when the incident electron kinetic energy is increased from 0 to 0.5 eV. Estimate the effective potential U_{Kr} inside the Kr atoms, assuming that their diameter is 0.4 nm.

11. Take a one-dimensional potential of the form

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & 0 \leq x < a \\ V_1 = V_0 + \frac{3\hbar^2\pi^2}{8ma^2} & x \geq a \end{cases} ,$$

and a beam of particles of mass m incident from the left with energy $E = V_0 + \hbar^2\pi^2/(2ma^2)$. Write a generic expression for the wave function of the beam in the three regions identified by the potential; define the probability current density and write expressions for the relevant probability fluxes in the leftmost and in the rightmost regions. Using the boundary conditions at $x = 0$ and $x = a$, derive the reflection and transmission coefficients for this system and show that $R + T = 1$. Can perfect reflection or perfect transmission occur in this system? Find the corresponding values of V_0 and V_1, if any, and comment on your results.

12. Consider a beam of particles travelling in the x-direction with energy E, incident on a one-dimensional potential barrier of height $V = 2E$ and width a. Obtain the reflection and transmission coefficients and plot them as a function of a (you may quote results from your solution of Q.10 above without re-deriving them).

Assuming a beam of electrons of incident energy 10 eV, compute the barrier width necessary to reduce the beam intensity to 0.01% of its incident value.

13. A particle is confined by the potential:

$$\begin{aligned} V(x) &= 0 \qquad && 0 < x < a \\ V(x) &= \infty \qquad && \text{elsewhere} \end{aligned}$$

Find the expectation value of x and the uncertainty in x when the particle is in its n^{th} energy state. Hence show that, as $n \to \infty$, the average value approach the value obtained from classical mechanics.

ANSWERS:

1. The currently accepted values are $\hbar = h/2\pi = 1.05457266(63) \times 10^{-34}$ J s and $W_{\rm Na} = 2 - 3$ eV.

2. (a) 5×10^{-36} m; (b) 6×10^{-8} m; (c) 3×10^{-10} m.

4. $\simeq 1$ mm.

5. $\langle E_0 \rangle \sim \frac{1}{2}\hbar\sqrt{\kappa/m}$.

6. [bookwork]; $r_1 = 4\pi\epsilon_0\hbar^2/(me^2)$; $v_1 = \hbar/(mr_1) = e^2/2\epsilon_0 h = \alpha c$.

7. (a) $|A_1| = \sqrt{2/a}$; (b) $|A_2| = \sqrt{8/abc}$; (c) $|A_3| = 1/\sqrt{\pi a^3}$.

8. (a) $\sim 6 \times 10^{-11}$ s;(b) $\sim 3 \times 10^{26}$ s $\approx 10^{19}$ years.

9. (a) $|A| = \sqrt{2^{5/2}\alpha^{3/2}/\pi^{1/2}}$; (b) $\Delta x = \sqrt{3/4\alpha}$; (c) $\Delta p_x = \hbar\sqrt{3\alpha}$.

10. No reflected beam for $E = n^2\pi^2\hbar^2/2ma^2 - V$; $U_{\rm Kr} \sim -1.85$ eV.

11. $\boldsymbol{J} = \mathcal{R}\left[\Psi^* \frac{\hbar}{im}\boldsymbol{\nabla}\Psi\right]$; $J_I^+ = |A|^2\hbar k/m$, $J_I^- = |Ar|^2\hbar k/m$, and $J_{III}^+ = |At''|^2\hbar k''/m$, where $t'' = i4ka/(2ka+\pi)$ and $r = (2ka-\pi)/(2ka+\pi)$; $R = J_I^-/J_I^+ = |r|^2 = [(2ka-\pi)/(2ka+\pi)]^2$ and $T = J_{III}^+/J_I^+ = |t''|^2 k''/k = [4ka/(2ka+\pi)]^2\pi/(2ka)$.

12. $T = 1/\cosh^2(\kappa_2 a)$ and $R = \tanh^2(\kappa_2 a)$, with $\kappa_2 = \sqrt{2mE/\hbar^2}$; $a \simeq 3.3$ Å.

13. $\langle x \rangle = a/2$; $\langle x^2 \rangle = (a^2/6)\,(2 - 3/n^2\pi^2)$; $(\Delta x)^2 = (a^2/12)\,(1 - 6/n^2\pi^2)$..