Part IB Physics A: Lent 2022

QUANTUM PHYSICS EXAMPLES I
MODEL ANSWERS

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1. The deflection d of the electrometer tends to zero when the photocurrent vanishes as the stopping voltage is reached. Therefore, one can use the data in the table to find the value of V_0 for each λ. One can then use the equation from the lecture notes $eV_0 = h\nu - W_{\rm Na}$ and perform a linear fit of V_0 vs. $1/\lambda$ to obtain $\hbar$ and the work function $W_{\rm Na}$ for Sodium.

2. Bookwork using $p = h/\lambda$, with $p = mv$ for (a), $p = E/c$ for (b), and $p = \sqrt{2mE}$ for (c).

3. The wave function is an entangled superposition of the photon being at each of the two detectors. However, the wavefunction collapses upon measurement. and the photon is detected at any given time only at *one* of the two detectors, stochastically with equal probability.

4. The time taken by the dart to reach the ground is $t = \sqrt{2H/g}$, H being the height from which it starts falling. The accuracy in the transverse direction Δx evolves in time from some initial value Δx_0 to a final value

$$\Delta x(t) = \sqrt{\Delta x_0^2 + \left(\frac{\Delta p_{x,0}}{m} t\right)^2} \,,$$

where $\Delta p_{x,0}$ is the initial accuracy in the transverse momentum. The uncertainty principle tells us that $\Delta x_0 \, \Delta p_{x,0} \geq \hbar/2$, and therefore

$$\Delta x(t) \geq \sqrt{\Delta x_0^2 + \left(\frac{\hbar}{2\Delta x_0 m} t\right)^2} \,.$$

The right hand side can then be minimised to find $\Delta x_0 = \sqrt{\hbar t/(2m)}$ and

$$\Delta x(t) \geq \sqrt{\frac{\hbar t}{m}} \simeq \sqrt{\frac{\hbar}{m}} \left(\frac{2H}{g}\right)^{1/4} \,.$$

Notice that there are two independent transverse directions (x and y) to the vertical (z) direction along which the dart falls. Therefore, one ought to add them in quadrature to find the total transverse uncertainty ($r = \sqrt{x^2 + y^2}$):

$$\Delta r(t) = \sqrt{[\Delta x(t)]^2 +]\Delta y(t)]^2} = \sqrt{2} \, \Delta x(t) \simeq 1 \text{ mm} \,.$$

5. For a minimal uncertainty packet, $\Delta x \Delta p = \hbar/2$. This enforces a lower bound on the energy of the system, given a spatial accuracy Δx, of

$$E_0 = \frac{\hbar^2}{8m\Delta x^2} + \frac{1}{2}\kappa \Delta x^2 \,.$$

Minimising the right hand side, one finds

$$\Delta x = \left(\frac{\hbar^2}{4m\kappa}\right)^{1/4}$$

and therefore

$$E_0 = \frac{1}{2}\hbar\sqrt{\frac{\kappa}{m}} \,.$$

6. The condition for a stable circular orbit proposed by Bohr is that the circumference is an integer multiple of the De Broglie wavelength: $2\pi r_n = n\lambda$, where $\lambda = h/p_n$ and $n \in \mathbb{N}$. The angular momentum of such semi-classical orbit is

$$L = m v_n r_n = p_n \frac{n\lambda}{2\pi} = \hbar n \,,$$

demonstrating how Bohr's assumption is indeed equivalent to the quantisation of L in units of $\hbar$.
The kinetic and potential energy of the system are given by the usual expressions:

$$T_n = \frac{1}{2} m v_n^3 \qquad \text{and} \qquad V_n = -\frac{e^2}{4\pi\epsilon_0 r_n} \,.$$

Therefore, in order to find the total energy, $E_n = T_n + V_n$, we need to find expressions for r_n and v_n. In fact, one can take advantage of the virial theorem, as appropriate for potentials that depend on the distance as a power law, to establish that $2T_n = -V_n$ to simplify the task at hand.
For circular motion and Coulomb interaction, we know that

$$\frac{e^2}{4\pi\epsilon_0 r_n^2} = m\frac{v_n^2}{r_n} \,.$$

We can then multiply both sides of the equation by $m r_n^3$ and use the quantisation of the angular momentum, $m v_n r_n = \hbar n$, to obtain:

$$\frac{m e^2 r_n}{4\pi\epsilon_0} = m^2 r_n^2 v_n = \hbar^2 n^2 \qquad \rightarrow \qquad r_n = \frac{4\pi\epsilon_0 \hbar^2 n^2}{m e^2} \,.$$

Substituting into $E_n = V_n/2$ we arrive at the expression

$$E_n = -\frac{m e^4}{8\epsilon_0^2 h^2}\frac{1}{n^2} = -\frac{hcR}{n^2} \,,$$

where $R = m e^4/(8\epsilon_0^2 h^3 c)$. From the dimensions of the known constants involved, one can see that R has units of m^{-1}.

For the first orbit, $n = 1$, we have $r_1 = 4\pi\epsilon_0\hbar^2/(me^2)$ and $v_1 = \hbar/(mr_1) = e^2/2\epsilon_0 h$. If we recall the definition of the fine structure constant, $4\pi\epsilon_0\hbar c\alpha = e^2$, we see that the velocity of Bohr's first orbit can be more simply written as $v_1 = \alpha c$.

7. By straightforward integration: $\int_0^a |\psi(x)|^2\, dx = 1$, $\int_0^a \int_0^b \int_0^c |\psi(x,y,z)|^2\, dxdydz = 1$, and $4\pi \int_0^\infty |\psi(r)|^2 r^2\, dr = 1$.

8. The wave function can be computed by Gaussian integration following the same steps as in the lecture notes (end of Chapter 3). Note the difference $a^2 \to 2a^2$ in notation between the notes and the question. The integration gives

$$\psi(x,t) \propto \frac{1}{\sqrt{a^2 + i\hbar t/2m}} \exp\left[-\frac{(x - \hbar k_0 t/m)^2}{4(a^2 + i\hbar t/2m)}\right] e^{i(k_0 x - \omega_0 t)} .$$

In order to find the width of the wave packet, we ought to compute $|\psi(x,t)|^2$. Focusing on the exponential terms,

$$\exp\left[-\frac{x^2}{2a^2\left(1 + \frac{\hbar^2 t^2}{4m^2 a^4}\right)}\right] ,$$

we see that the width is proportional to $\sqrt{a^2(1 + \hbar^2 t^2/4m^2 a^4)}$ which is a at $t = 0$ and doubles when

$$1 + \frac{\hbar^2 t^2}{4m^2 a^4} = 4 \qquad \to \qquad t = \frac{2\sqrt{3} m a^2}{\hbar} .$$

Simple algebra gives the values requested in (a) and (b), see end of question sheet.

9. Given $\psi(x) = Ax\exp(-\alpha x^2)$

(a) we can calculate the normalising constant A by Gaussian integration, for example using the trick

$$\int |\psi(x)|^2 dx = |A|^2 \int x^2 e^{-2\alpha x^2} dx = -\frac{|A|^2}{2}\frac{d}{d\alpha}\int e^{-2\alpha x^2} dx = -\frac{|A|^2}{2}\frac{d}{d\alpha}\sqrt{\frac{\pi}{2\alpha}} = \frac{|A|^2\sqrt{\pi}}{4\sqrt{2\alpha^3}} = 1.$$

Therefore, $|A|^2 = 4\sqrt{2\alpha^3}/\sqrt{\pi}$.

(b) we can similarly calculate $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{3/(4\alpha)}$ (by symmetry, $\langle x\rangle = 0$). To compute $\Delta p_x = \hbar\Delta k = \hbar\sqrt{\langle k^2\rangle - \langle k\rangle^2}$, we need to first obtain the wave function in reciprocal space by Fourier Transform:

$$\psi(k) = i\left(\frac{1}{2\pi\alpha^3}\right)^{1/4} k\exp\left(-\frac{k^2}{4\alpha}\right) ,$$

and then calculate in a similar way $\langle k\rangle = 0$ and $\langle k^2\rangle = 3\alpha$, leading to $\Delta p_x = \hbar\sqrt{3\alpha}$.

(c) it is then trivial to show that $\Delta x\Delta p_x = 3\hbar/2$.

10. This problem is illustrated in Fig.5.10 in the lecture notes and discussed in the corresponding section. Let us call regions I and III the ones before and after the well

($x < 0$ and $x > a$), where particles have energy E, and region II in the well ($0 < x < a$), where particles have energy $E + V$. Correspondingly,

$$k_{\mathrm{I}} = k_{\mathrm{III}} = k = \sqrt{\frac{2mE}{\hbar^2}} \qquad k_{\mathrm{II}} = \tilde{k} = \sqrt{\frac{2m(E+V)}{\hbar^2}}\,.$$

Using plane wave solutions as in the lecture notes, in region I we have $\psi \sim e^{ikx} + re^{-ikx}$, in region II we have $\psi \sim t'e^{i\tilde{k}x} + r'e^{-i\tilde{k}x}$, and in region III we have $\psi \sim t''e^{ikx}$. Imposing b.c. at $x = 0$ and $x = a$ (continuity of the wave function and of its first derivative), after a few lines of algebra we obtain

$$\begin{cases} 1 + r = t' + r' \\ 1 - r = \frac{\tilde{k}}{k}(t' - r') \\ t'e^{i\tilde{k}a} + r'e^{-i\tilde{k}a} = t''e^{ika} \\ t'e^{i\tilde{k}a} - r'e^{-i\tilde{k}a} = \frac{k}{\tilde{k}}t''e^{ika}\,. \end{cases}$$

If we demand no reflection, $r = 0$, the first two equations give $t' = (\tilde{k} + k)/(2\tilde{k})$ and $r' = (\tilde{k} - k)/(2\tilde{k})$. The remaining two equations can be linearly combined to obtain the identity

$$t'e^{i\tilde{k}a} + r'e^{-i\tilde{k}a} - \frac{\tilde{k}}{k}\left(t'e^{i\tilde{k}a} - r'e^{-i\tilde{k}a}\right) = 0\,.$$

Substituting the values of t' and r' that we just found, this identity reduces to $e^{i2\tilde{k}a} = 1$, which is satisfied for any $\tilde{k} = n\pi/a$, for $n \in \mathbb{N}$; finally, after substituting for $\tilde{k}$, we find $E_n = \hbar^2 n^2 \pi^2/(2ma^2) - V$.
The transmission coefficient is given by $T = J^+_{\mathrm{III}}/J^+_{\mathrm{I}} = |t''|^2$. The full solution is given and sketched in the lecture notes. Incidentally, as expected, $T = 1$ when $r = 0$.
A minimum in the scattering appears at maximal transmission. If the first one occurs at $E = 0.5$ eV, then substituting this value into

$$\sqrt{\frac{2m(E + U_{\mathrm{Kr}})a^2}{\hbar^2}} = \pi$$

gives an estimate of the effective potential $U_{\mathrm{Kr}} \simeq -1.85$ eV inside the Kr atoms.

11. Let us label the three regions from left to right as I, II and III, with respective wave vectors:

$$k = \sqrt{\frac{2mE}{\hbar^2}}\,, \qquad k' = \sqrt{\frac{2m(E - V_0)}{\hbar^2}} = \frac{\pi}{a}\,, \qquad k'' = \sqrt{\frac{2m(E - V_1)}{\hbar^2}} = \frac{\pi}{2a}\,.$$

(Notice that they are all real, as $E > V_1 > V_0$.)
For a generic expression of the wave function in the three regions, we take the incident amplitude to be A and we ought to consider a reflected wave in region I; a transmitted and a reflected wave in region II; and a transmitted way only in region III:

$$\begin{aligned} &A\left[e^{ikx} + re^{-ikx}\right]e^{-i\omega t} && x < 0 \\ &A\left[t'e^{ik'x} + r'e^{-ik'x}\right]e^{-i\omega t} && 0 \le x < a \\ &At''e^{ik''x}e^{-i\omega t} && x \ge a\,. \end{aligned}$$

The probability current density as defined in the lecture notes can be written as

$$\boldsymbol{J} = \mathcal{R}\left[\Psi^* \frac{\hbar}{im}\boldsymbol{\nabla}\Psi\right] ,$$

which for the case of a 1D plane wave reduces to forward (+) and backward (−) fluxes in each of the three regions. Specifically, in region I: $J_I^+ = |A|^2\hbar k/m$ and $J_I^- = |Ar|^2\hbar k/m$; and in region III: $J_{III}^+ = |At''|^2\hbar k''/m$ and $J_{III}^+ = 0$.
In order to progress further, we need to use the boundary conditions at $x = 0$ and $x = a$ (continuity of the wave function and of its spatial derivative) to solve for r and t''. Thanks to the simple values of k' and k'' in this problem, the set of four b.c. equations simplifies to

$$\begin{cases} 1 + r = t' + r' \\ t' + r' = -it'' \\ ka(1 - r) = \pi(t' - r') \\ \pi(t' - r') = -i\frac{\pi}{2}t'' \end{cases} .$$

These can be solved straightforwardly to obtain $t'' = i4ka/(2ka + \pi)$ and $r = (2ka - \pi)/(2ka + \pi)$. The reflection and transmission coefficients are then given by

$$R = \frac{J_I^-}{J_I^+} = |r|^2 = \left[\frac{2ka - \pi}{2ka + \pi}\right]^2$$

and by

$$T = \frac{J_{III}^+}{J_I^+} = |t''|^2\frac{k''}{k} = \left[\frac{4ka}{2ka + \pi}\right]^2 \frac{\pi}{2ka} ,$$

which clearly satisfy $R + T = 1$.
Perfect reflection and perfect transmission occur when $R = 1$ and $T = 1$, respectively. Clearly the former (more conveniently written as $T = 0$) happens only in the non-physical limit of $ka \to \infty$. The latter ($R = 0$) on the other hand is realised when $ka = \pi/2$. A few lines of algebra show that this requires $E = \hbar^2\pi^2/(8ma^2)$, and therefore $V_0 = -3\hbar^2\pi^2/(8ma^2)$ and $v_1 = 0$. This is consistent with what one would expect: (i) perfect transmission can only occur if the initial and final potential tend to the same asymptotic value (i.e., $V_1 = 0$); and (ii) in this problem, $E > V_1 > V_0$ by construction, so perfect transmission can only happen if $V_0 < 0$.

12. The solution to this problem is analogous to Q.10, except for the fact that the wave vector in region II is purely imaginary and has the same absolute value as that in regions II and III:

$$k_{\mathrm{I}} = k_{\mathrm{III}} = k = \sqrt{\frac{2mE}{\hbar^2}} \qquad k_{\mathrm{II}} = \sqrt{\frac{2m(V - E)}{\hbar^2}} = ik .$$

The b.c. at $x = 0$ and $x = a$ (continuity of the wave function and of its first derivative) therefore take the form:

$$\begin{cases} 1 + r = t' + r' \\ 1 - r = i(t' - r') \\ t'e^{-ka} + r'e^{ka} = t''e^{ika} \\ t'e^{-ka} - r'e^{ka} = -it''e^{ika} . \end{cases}$$

After a few lines of algebra, from the first two equations one obtains:

$$r' = \frac{1}{2}(1+r)(1+i)t' = \frac{1-r}{1+i}\,.$$

Substituting into the last two equations and simplifying the factors of $(1+i)$, one arrives at:

$$(1-r)\,e^{-ka} = t''e^{ika}(1+r)\,e^{ka} = t''e^{ika}\,,$$

which can be easily solved to give: $r'' = -\tanh(ka)$ and $t'' = e^{-ika}/\cosh(ka)$. The reflection and transmission coefficients are $R = |r|^2 = \tanh^2(ka)$ and $T = |t|^2 = 1/\cosh^2(ka)$, which satisfy $T + R = 1$, as required. (The plots of these functions are obvious.)

In order to reduce the beam intensity to 0.01% of its incident value, we need $T = 10^{-4}$, that is $ka = \cosh^{-1}(100) \simeq 5.3$. For an electron beam of energy $E = 10$ eV $\simeq 1.6\,10^{-18}$ J, using $m \simeq 9.11\,10^{-31}$ kg and $\hbar \simeq 1.05\,10^{-34}$ J s, one finds $k = \sqrt{2mE/\hbar^2} \simeq 1.6\,10^{10}$ m^{-1}. We therefore need a barrier width $a \simeq 3.3$ Å.

13. Following the derivation in the lecture notes, we find the energy eigenstates

$$\psi_n(x) = \begin{cases} A_n \sin[k_n(x-a/2)] & n \text{ even} \\ B_n \cos[(k_n(x-a/2)] & n \text{ odd}\,, \end{cases}$$

where $k_n = n\pi/a$, $n = 1, 2, 3, \dots$ and $A_n = B_n = \sqrt{2/a}$. Thence

$$\begin{aligned} \langle x \rangle &= \begin{cases} \int_0^a xA_n^2 \sin^2[k_n(x-a/2)]\,dx = a/2 & n \text{ even} \\ \int_0^a xB_n^2 \cos^2[k_n(x-a/2)]\,dx = a/2 & n \text{ odd} \end{cases} \\ \langle x^2 \rangle &= \begin{cases} \int_0^a x^2A_n^2 \sin^2[k_n(x-a/2)]\,dx = \frac{a^2}{6}\left(2 - \frac{3}{n^2\pi^2}\right) & n \text{ even} \\ \int_0^a x^2B_n^2 \cos^2[k_n(x-a/2)]\,dx = \frac{a^2}{6}\left(2 - \frac{3}{n^2\pi^2}\right) & n \text{ odd}\,. \end{cases} \end{aligned}$$

The uncertainty is then given by

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a\sqrt{\frac{1}{12} - \frac{6}{n^2\pi^2}} \quad \xrightarrow{n\to\infty} \quad \frac{a}{\sqrt{12}}\,.$$

In the limit of $n \to \infty$, these values are consistent with the results from classical mechanics, where the probability of finding a particle between 0 and a is uniform, $P(x) = 1/a$, and it vanishes outside ($x < 0$ and $x > a$). One can straightforwardly obtain $\int xP(x)\,dx = a/2$, $\int x^2P(x)\,dx = a^2/3$ and $(\Delta x)_{\text{classical}} = a/\sqrt{12}$.