Relativistic Astrophysics and Cosmology — Examples 3 — 2018

The cosmological constant should be taken as zero in the cosmology questions below, except where models with Λ are specifically mentioned.

1. A cluster has total galaxy luminosity equal to 500 times that of our Milky Way (which is 10^{11} solar luminosities). The mass to light ratio of the galaxies $M/L = 2$ in solar units. The line of sight velocity dispersion of the galaxies is 1000 km/s and the virial radius is 2 Mpc. What is the total M/L for the cluster? What is the mass fraction in galaxies?

2. The intracluster medium in the inner 100 kpc of a cluster has density $n = 3\times10^3 r^{-1}$ m^{-3}, and temperature $T = 10^8 r^{0.5}$ K, where the radius is in units of 100 kpc. Bremsstrahlung emissivity is $\sim 10^{-40} n^2 T^{1/2}$ W m^{-3}. Estimate where the radiative cooling time of the gas is 5×10^9 yr. What is the total thermal energy inside this radius and what is the minimum mass of the black hole if heat has been supplied to exactly balance cooling of this gas (assume an accretion efficiency of 0.1)?

3. Show that for inverse Compton scattering the final photon energy $\bar{\epsilon}_f = \frac{4}{3}\gamma^2\epsilon_0$, where ϵ_0 is the initial energy of the isotropic photon field and γ is the Lorentz factor of the electron (assumed large).

4. Show that the Einstein radius, where a distant source appears as a perfect ring around a distant point mass, is given by $R_E = \sqrt{2R_s D}$, where $R_s = 2GM/c^2$ and $D = (D_s - D_l)D_l/D_s$. (D_s is the distance to the source and D_l the distance to the lens).

   Consider the situation where the distant lens and source are slightly misaligned by distance r at the lens and the light ray passes at distance R. Show that $R^2 + rR - R_E^2 = 0$, and thus that there are two solutions (images). Show that the total magnification of the two images
   $$A = \left|\frac{R}{r}\frac{dR}{dr}\right| = \frac{u^2+2}{u\sqrt{u^2+4}},$$
   where $u = r/R_E$. Estimate the angular size of the Einstein ring produced by a Solar mass star (a) near the centre of our Galaxy; (b) in a distant galaxy. (See `http://arxiv.org/abs/astro-ph/9604011` if help required.)

5. Find expressions for the scale factor, Hubble parameter and deceleration parameter all as a function of time in (a) a matter dominated Einstein de Sitter universe and (b) a radiation dominated Einstein de Sitter Universe.

6. What is meant by a particle horizon? If our universe evolves according to the Einstein-de-Sitter model show that the total mass contained within our particle horizon is $6c^3tG^{-1}$, where t is the time since the big bang.

   Use this result to find the mass contained within the particle horizon when t is so small that the horizon lies within the radius of a proton. Comment on your answer.

7. The energy density in radiation from quasar light is $\sim 10^{-16}$ Wm^{-3} now. What is the mean mass density in dead black holes? Assume that most quasars occurred at a redshift of 2. What is the mean mass per galaxy of dead black holes if the density of galaxies is 10^{-2} Mpc^{-3}?

8. If galaxies of a certain type now have a space density of $2 \times 10^{-3}\,\mathrm{Mpc}^{-3}$, and each of them went through a very luminous phase lasting 10^8 years after they formed at $z = 3$, what is the observed areal density on the sky (measured per square degree) of galaxies in that phase? [Assume an Einstein-de-Sitter universe and a reasonable value for H_0.]

9. Galaxies having a local number density N_0 are surrounded by gaseous haloes, each of which has a cross section σ. If the numbers and sizes of these galaxies do not change with time show that a line of sight to a quasar at redshift z will intersect, on average, n haloes where
$$n = \int_0^z (N_0\,\sigma\,c\,/H_0)\,(1+z)\,(1+\Omega_{m0}z)^{-1/2}\,dz.$$

10. Starting from the field equations for a flat matter-dominated universe with Λ, prove the result
$$H(z) = H_0\left((1-\Omega_{\Lambda 0})(1+z)^3 + \Omega_{\Lambda 0}\right)^{1/2}$$
where $\Omega_{\Lambda 0}$ is the value of $\Omega_\Lambda = \Lambda/(3H^2)$ today.

11. Show that in a de Sitter universe the angular diameter distance is given by
$$d_\theta = \frac{cz}{1+z}\sqrt{\frac{3}{\Lambda}}.$$
What does this result imply for the angular diameter subtended by an object of fixed proper diameter D as a function of redshift for both low and high redshift?

12. Suppose that in a de Sitter universe a photon is emitted from the origin of coordinates ($\chi = 0$), at time $t = t_0$ (now) towards an object which is currently seen at redshift z. Show that that the photon is received at the object at a time t_{recep} given by
$$t_{\mathrm{recep}} = t_0 - \sqrt{\frac{3}{\Lambda}}\ln(1-z),$$
provided $z < 1$, and that it is never received if $z \geq 1$.

13. Show that for all reasonable values of Ω_{m0} and H_0, the spatial curvature (i.e. the term kc^2/R^2 in the second field equation) is negligible during the whole of the radiation dominated phase of the universe. Use this to calculate how long the universe is radiation dominated if the epoch of equality (i.e. the time at which the energy density in radiation equals that in matter) is given by $z_{eq} = 9600\,\Omega_{m0}h^2$ (where $h = H_0/50\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$). Obtain another estimate of the same quantity by assuming the present universe is matter dominated Einstein-de-Sitter, and finding the time corresponding to z_{eq}. [Answers: $10,410\,(\Omega_{m0}h^2)^{-2}$ and $13,880\,h^{-4}$ years respectively.]

14. If radiation and matter were in thermal equilibrium prior to recombination, and if no condensation into galaxies had subsequently occurred, what would be the present temperature of the cosmic gas? If residual gas at this temperature still exists how would you attempt to detect it? (Assume that after recombination the matter cools according to the adiabatic law $TV^{\gamma-1}$ =constant, where $\gamma = 5/3$.)

15. By assuming that the universe is radiation dominated Einstein-de-Sitter before recombination, and matter dominated Einstein-de-Sitter afterwards, show (a) that the proper

distance across the particle horizon at recombination is $2ct_{rec}$, and (b) that this distance subtends an angle $\approx \frac{2}{3}(1+z_{rec})^{-1/2}$ radians today. Comment on the implications for the microwave background.

16. For the dynamics of inflation as discussed in the lectures, derive the scalar field equivalent of the (A) equation (the 'acceleration equation') by differentiating the scalar field (B) ('energy') equation w.r.t. time and employing the scalar field equivalent of the continuity equation to eliminate $\ddot{\phi}$.

17. (From Astrophysics and Cosmology Major Option examination paper, January 2007.)

    (a) The cosmological field equations can be written as

$$\frac{\ddot{R}}{R} + \frac{4\pi G\rho}{3}(1+\epsilon) - \frac{\Lambda}{3} = 0,$$

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G\rho}{3} - \frac{\Lambda}{3} = -\frac{kc^2}{R^2}.$$

    Explain briefly the meanings of the quantities appearing in these equations.

    For a flat universe with a cosmological constant of zero, obtain a single equation involving just R and its derivatives in time by eliminating ρ between the two equations, Then, substituting a trial solution of the form $R(t) \propto t^{\beta}$, find what values of β are compatible with this equation.

    By this means, determine the age of a universe which follows a matter-dominated Einstein de Sitter model with a current value for the Hubble constant of $H_0 = 72\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. [Note $1\,\mathrm{pc} = 3.0857 \times 10^{16}\,\mathrm{m}$.]

    Discuss briefly what is known about the minimum ages of objects in the actual universe, and compare these with the result you have just obtained.

    (b) Define what is meant by the term *particle horizon*, and derive an expression for the proper distance across the particle horizon at time t involving the scale factor $R(t)$ and the speed of light.

    In a simplified model of the very early universe, the universe starts with a period of inflation at time $t = 0$, which lasts until a time t_{inf}. During the period $0 \le t \le t_{\mathrm{inf}}$ the Hubble parameter is constant, and has a large value H_{inf}, such that $H_{\mathrm{inf}} t_{\mathrm{inf}} \gg 1$.

    Within this model, calculate the proper distance d_{inf} across the particle horizon at time t_{inf}.

    If, instead, inflation did not occur, and the universe started with a big bang at $t = 0$ and was flat and radiation-dominated from $t = 0$ to $t = t_{\mathrm{inf}}$, what would be the proper distance d_{rad} across the particle horizon at $t = t_{\mathrm{inf}}$ in this case? Show that the ratio of the answers with and without inflation is given by

$$\frac{d_{\mathrm{inf}}}{d_{\mathrm{rad}}} \approx \frac{\exp(H_{\mathrm{inf}} t_{\mathrm{inf}})}{2 H_{\mathrm{inf}} t_{\mathrm{inf}}}.$$

    Discuss the relevance of the size of the particle horizon at this early time relevant to observations of the cosmic microwave background radiation.