Part IB Physics : Lent 2020

QUANTUM PHYSICS EXAMPLES II
MODEL ANSWERS

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Operators and States

1. An operation $\hat{P}$ is linear if it preserves the vector space operations of f, namely multiplication by a scalar and addition. That is: $\hat{P}[\alpha f(x)+\beta g(x)] = \alpha\hat{P}[f(x)]+\beta\hat{P}[g(x)]$, for any two functions f and g in the vector space and two scalars α and β. Therefore, (a), (d), (e), (g), (h), (j) are linear operations, whereas (b), (c), (f), (i) are not.
The eigenfunctions are (for any scalars a, b):

(a) any $f(x)$;
(b) none but trivial constant functions;
(c) none but trivial constant functions;
(d) e^{ax};
(e) $\delta(x - x_0)$;
(f) ax;
(g) e^{ax} or $\cos(ax + b)$;
(h) x^a (including $a = 0$, i.e., a constant function);
(i) none but trivial constant functions;
(j) any even or odd f, i.e., s.t. $f(-x) = \pm f(x)$ (functions of definite parity).

2. If $\widehat{A}$ and $\widehat{B}$ are Hermitian, then $\widehat{A} + \widehat{B}$ is trivially Hermitian.

$c\widehat{A}$ is Hermitian if c is real.

$(\widehat{A}\widehat{B})^\dagger = \widehat{B}^\dagger\widehat{A}^\dagger = \widehat{B}\widehat{A}$ which is generally different from $\widehat{A}\widehat{B}$ unless the two operators commute.

$\widehat{A}\widehat{B} + \widehat{B}\widehat{A}$ is Hermitian.

The matrix element of the derivative operator d/dx between two states $|f\rangle$ and $|g\rangle$ is

$$\langle f|\frac{d}{dx}|g\rangle = \int f^*(x)\frac{d}{dx}g(x)\,dx\,.$$

By definition of adjoint, the matrix element of $(d/dx)^\dagger$ is the transpose conjugate of the matrix element of d/dx:

$$\begin{aligned}\langle f|\left(\frac{d}{dx}\right)^\dagger|g\rangle &= \left(\langle g|\frac{d}{dx}|f\rangle\right)^* = \left(\int g^*(x)\frac{d}{dx}f(x)\,dx\right)^* = \int g(x)\frac{d}{dx}f^*(x)\,dx \\ &= \left[f^*(x)g(x)\right]\big|_{-\infty}^{\infty} - \int f^*(x)\frac{d}{dx}g(x)\,dx\,,\end{aligned}$$

where the last line was obtained upon integrating by parts, using the relation $d(f^*g)/dx = f^*dg/dx + gdf^*/dx$. Given that the functions vanish at infinity, as the question states, we finally arrive at the relation:

$$\langle f| \left(\frac{d}{dx}\right)^\dagger |g\rangle = -\langle f|\frac{d}{dx}|g\rangle \,,$$

thus showing that d/dx is not Hermitian; it is in fact anti-Hermitian, and therefore becomes Hermitian if multiplied by the imaginary unit, such as in $-i\hbar \mathrm{d}/\mathrm{d}x$.
A similar calculation, and upon integrating by parts twice, shows that $\mathrm{d}^2/\mathrm{d}x^2$ is instead Hermitian (or more generally, for any anti-Hermitian operator, $\hat{A}^\dagger = -\hat{A}$, it is clear that A^2 is Hermitian, since $(A^2)^\dagger = \hat{A}^\dagger\hat{A}^\dagger = \hat{A}^2$).

3. Trivially, $\hat{a} = (\hat{A}^\dagger + \hat{A}) - i[i(\hat{A}^\dagger - \hat{A})]$, where $\hat{A}^\dagger + \hat{A}$ and $i(\hat{A}^\dagger - i\hat{A})$ are Hermitian.

4. In order to show mutual orthogonality, one needs to demonstrate that $\int \phi_1(x)\phi_2(x)dx = 0$ for any pair of the three functions. Notice that $x\mathrm{e}^{-x^2}$ is odd while the other two are even; therefore it is trivially orthogonal to them. The remaining orthogonality we need to check requires computing the integral:

$$\int_{-\infty}^{\infty} (4x^2 - 1)e^{-2x^2}\, dx = \sqrt{\frac{\pi}{2}} - \sqrt{\frac{\pi}{2}} = 0 \,,$$

as expected.
[
Possibly useful trick, thanks to the absolute convergence of a Gaussian integral:

$$\int_{-\infty}^{\infty} 4x^2 e^{-2x^2} dx = -2\frac{d}{da}\int_{-\infty}^{\infty} e^{-2x^2 a}\, dx\bigg|_{a=1} = -2\frac{d}{da}\sqrt{\frac{\pi}{2a}}\bigg|_{a=1} = -2\sqrt{\frac{\pi}{2}}\frac{(-1)}{2a^{1/4}}\bigg|_{a=1} = \sqrt{\frac{\pi}{2}} \,.$$

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5. For (a), by inspection, $|\psi_a\rangle \propto c|\phi_1\rangle - |\phi_2\rangle$ is orthogonal to ϕ_1. The normalisation constant is obtained by imposing $\langle\psi_a|\psi_a\rangle = 1$:

$$\begin{aligned} (c\langle\phi_1| - \langle\phi_2|)\,(c|\phi_1\rangle - |\phi_2\rangle) &= c^2 - c\langle\phi_1|\phi_2\rangle - c\langle\phi_2|\phi_1\rangle + 1 = 1 - c^2 \\ \rightarrow \qquad |\psi_a\rangle &= \frac{1}{\sqrt{1-c^2}}\,(c|\phi_1\rangle - |\phi_2\rangle) \,. \end{aligned}$$

For (b), one can proceed systematically by solving

$$(\alpha|\phi_1\rangle + \beta|\phi_2\rangle)^\dagger(|\phi_1\rangle + |\phi_2\rangle) = (\alpha+\beta)(1+c) = 0$$

to find $\beta = -\alpha$, and therefore $|\psi_b\rangle \propto |\phi_1\rangle - |\phi_2\rangle$. The normalisation constant can be found in a similar way as above,

$$|\psi_b\rangle = \frac{1}{\sqrt{2(1-c)}}\,(|\phi_1\rangle - |\phi_2\rangle) \,.$$

6. Recall from the lecture notes that

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int \phi(p) e^{ipx/\hbar}\, dp\,,$$

where $\phi(p)$ is the momentum-domain wavefunction. One can then see that the space-domain shift operation $x \to x - x_0$ corresponds to a multiplication by $\exp(-ipx_0/\hbar)$ in momentum-domain:

$$\phi(p) \to \phi(p) e^{-ipx_0/\hbar}\,.$$

This is an operation that trivially preserves the norm of $\phi(p)$.

7. (a) When a system is measured, its wave function changes discontinuously: it is 'projected' onto the corresponding eigenstate(s), and renormalised. When two operators (i.e., observables) do not commute, measuring one of them can substantially alter the measurement of the other. For instance, consider a plane wave of definite momentum. If position is measured within a given interval of width Δx, the uncertainty principle requires that any subsequent measurement of p has an uncertainty Δp greater than $\hbar/(2\Delta x)$. On the contrary, if p had been measured first, one would have obtained $\Delta p = 0$.

(b) Following a position measurement, the particle wave function is projected onto the corresponding eigenstate(s). Therefore, if it was found in a region of width Δx and the position is measured again immediately afterwards, the particle *must* be found in the same region with probability 1. This is strictly true only if the second measurement is infinitesimally close in time to the first one; time evolution can allow the particle position to change.

For free particles, a lower-bound estimate of the positional uncertainty can be obtained by summing uncertainties in quadrature:

$$\Delta x(t) \simeq \sqrt{\Delta x^2 + \left(\frac{\Delta p\, t}{m}\right)^2} \gtrsim \sqrt{\Delta x^2 + \left(\frac{\hbar t}{2m\Delta x}\right)^2}\,,$$

where we used the uncertainty relation $\Delta x \Delta p \geq \hbar/2$.

8. Following a measurement of B whose outcome is b_1, the system must be in state χ_1. A measurement of A immediately afterwards then gives:

$$\langle \chi_1 | \hat{A} | \chi_1 \rangle = \frac{4}{13} a_1 + \frac{9}{13} a_2$$

and the probability of outcome $a_{1,2}$ is $4/13$, $9/13$ (which could have equivalently been estimated by computing $|\langle \psi_{1,2} | \chi_1 \rangle|^2$).

Following a measurement of A, the system must be in one of its eigenstates $\psi_{1,2}$. In order to compute the result of a measurement of B immediately afterwards, we ought to express the eigenstates of A in terms of those of B by inverting the equations in the question:

$$\psi_1 = (2\chi_1 + 3\chi_2)/\sqrt{13} \qquad \psi_2 = (3\chi_1 - 2\chi_2)/\sqrt{13}\,,$$

to find $\langle \psi_1 | \hat{B} | \psi_1 \rangle = \frac{4}{13} b_1 + \frac{9}{13} b_2$ and $\langle \psi_2 | \hat{B} | \psi_2 \rangle = \frac{9}{13} b_1 + \frac{4}{13} b_2$. Therefore, for a sequence of measurements, the probability is

$$P(b_1) = P(b_1|a_1)P(a_1) + P(b_1|a_2)P(a_2) = \left(\frac{4}{13}\right)^2 + \left(\frac{9}{13}\right)^2 = \frac{97}{169}\,.$$

(You can similarly check that $P(b_2) = 72/169$ and $P(b_1) + P(b_2) = 1$, as expected.)

9. Any Hermitian operator is an allowed observable of the system (however cumbersome it may be to measure it); and the sum of two Hermitian operator is Hermitian, therefore C is an observable.

Let us write C in matrix form in the basis of eigenstates of A, and then diagonalise to find its eigenvalues as well as the eigenvectors expressed as linear combinations of $u_\pm$ (which are needed for the last part of the question). We start by inverting the states in the question to find:

$$u_+ = (v_+ + v_-)/\sqrt{2} \qquad\qquad u_- = (v_+ - v_-)/\sqrt{2}\,.$$

Then, computing matrix elements for example as $\langle u_+|\hat{C}|u_-\rangle = \langle u_+|\hat{A}|u_-\rangle + [(\langle v_+| + \langle v_-|)\hat{B}(|v_+\rangle - |v_-\rangle)]/2 = 0 + 2/2 = 1$, we obtain

$$\begin{aligned}\hat{C} &= \langle u_+|\hat{C}|u_+\rangle\,|u_+\rangle\langle u_+| + \langle u_+|\hat{C}|u_-\rangle\,|u_+\rangle\langle u_-| + \langle u_-|\hat{C}|u_+\rangle\,|u_-\rangle\langle u_+| + \langle u_-|\hat{C}|u_-\rangle\,|u_-\rangle\langle u_-| \\ &= |u_+\rangle\langle u_+| + |u_+\rangle\langle u_-| + |u_-\rangle\langle u_+| - |u_-\rangle\langle u_-|\end{aligned}$$

or in matrix form

$$C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The corresponding secular equation $(1-\lambda)(-1-\lambda)-1 = 0$ gives the eigenvalues $\lambda = \pm\sqrt{2}$, and we can solve for the eigenvectors by solving the system of linear equations:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\sqrt{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

to obtain

$$w_\pm = \sqrt{\frac{1}{2}\left(1 \pm \frac{1}{\sqrt{2}}\right)}\,u_+ \pm \sqrt{\frac{1}{2}\left(1 \mp \frac{1}{\sqrt{2}}\right)}\,u_-\,.$$

Once again, we can invert these equations to find

$$u_+ = \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)}\,w_+ + \sqrt{\frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)}\,w_-\,.$$

Therefore, following a measurement of A with outcome $+1$, the probabilities of measuring C and obtaining the values $\pm\sqrt{2}$ are $(2 \pm \sqrt{2})/4$, respectively, and

$$\langle u_+|\hat{C}|u_+\rangle = \sqrt{2}\,\frac{2+\sqrt{2}}{4} - \sqrt{2}\,\frac{2-\sqrt{2}}{4} = 1\,.$$

After C is measured, the system will be in one of its eigenstates, which we already expressed in terms of u_+ and u_- earlier.

10. From the lecture notes, we know that

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right) \qquad \text{and} \qquad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}^\dagger - \hat{a}\right)\,.$$

Also, by symmetry of the system, $\langle \hat{x} \rangle = \langle \hat{p} \rangle = 0$. Therefore, all we need to compute are

$$\begin{aligned}
\langle \phi_n | \hat{x}^2 | \phi_n \rangle &= \frac{\hbar}{2m\omega} \langle \phi_n | \left(\hat{a}^\dagger + \hat{a} \right)^2 | \phi_n \rangle \\
\langle \phi_n | \hat{p}^2 | \phi_n \rangle &= -\frac{m\omega\hbar}{2} \langle \phi_n | \left(\hat{a}^\dagger - \hat{a} \right)^2 | \phi_n \rangle \,.
\end{aligned}$$

Notice that, by the properties of the number operator and of its eigenstates,

$$\begin{aligned}
\langle \phi_n | \left(\hat{a}^\dagger + \hat{a} \right)^2 | \phi_n \rangle &= -\langle \phi_n | \left(\hat{a}^\dagger - \hat{a} \right)^2 | \phi_n \rangle \\
&= \langle \phi_n | \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger | \phi_n \rangle = \langle \phi_n | 2\hat{a}^\dagger \hat{a} + 1 | \phi_n \rangle = 2n + 1 \,,
\end{aligned}$$

where we used the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. Finally,

$$\begin{aligned}
\Delta x &= \sqrt{\langle \phi_n | \hat{x}^2 | \phi_n \rangle} = \sqrt{\frac{\hbar}{m\omega} \left(n + \frac{1}{2} \right)} \\
\Delta p &= \sqrt{\langle \phi_n | \hat{p}^2 | \phi_n \rangle} = \sqrt{m\omega\hbar \left(n + \frac{1}{2} \right)} \qquad (1)
\end{aligned}$$

and one arrives at the desired result $\Delta x \Delta p = \hbar(n + 1/2)$.

11. For a free particle, $\hat{H} = \hat{p}^2/(2m)$. Using Ehrenfest's theorem, we can compute

$$\frac{d\langle \hat{x}^2 \rangle}{dt} = \frac{i}{\hbar} \langle [\hat{H}, \hat{x}^2] \rangle = \frac{1}{2i\hbar m} \langle \hat{x}[\hat{x}, \hat{p}^2] + [\hat{x}, \hat{p}^2]\hat{x} \rangle = \frac{1}{m} \langle \hat{x}\hat{p} + \hat{p}\hat{x} \rangle \,,$$

where we used the commutation relation $[\hat{x}, \hat{p}^2] = 2i\hbar\hat{p}$.
We can then use this result to compute

$$\frac{d^2\langle \hat{x}^2 \rangle}{dt^2} = \frac{1}{m} \frac{d\langle \hat{x}\hat{p} + \hat{p}\hat{x} \rangle}{dt} = \frac{i}{\hbar m} \langle [\hat{H}, \hat{x}\hat{p} + \hat{p}\hat{x}] \rangle = \frac{i}{2\hbar m^2} \langle [\hat{p}^2, \hat{x}]\hat{p} + \hat{p}[\hat{p}^2, \hat{x}] \rangle = \frac{2}{m^2} \langle \hat{p}^2 \rangle \,.$$

The final result follows straightforwardly upon integrating the second order differential equation:

$$\langle \hat{x}^2 \rangle_t = \langle \hat{x}^2 \rangle_{t=0} + \left. \frac{d\langle \hat{x}^2 \rangle}{dt} \right|_{t=0} t + \frac{1}{2} \left. \frac{d^2\langle \hat{x}^2 \rangle}{dt^2} \right|_{t=0} t^2 = \langle \hat{x}^2 \rangle_{t=0} + \langle \hat{p}^2 \rangle_{t=0} \frac{t^2}{m^2} \,.$$

12. Following a measurement of A yielding the eigenvalue a_1, the system is in state ψ_1, which is expressed in terms of energy eigenstates in the question. The state of the system evolves thereafter in time as

$$|\psi_1(t)\rangle = \frac{e^{-iE_1 t/\hbar}}{\sqrt{2}} |u_1\rangle + \frac{e^{-iE_2 t/\hbar}}{\sqrt{2}} |u_2\rangle \,.$$

We can then compute

$$\langle A \rangle(t) = \langle \psi_1(t) | \hat{A} | \psi_1(t) \rangle = \sum_{i,j=1}^{2} \exp\left[i \frac{E_i - E_j}{\hbar} t \right] \langle u_i | \hat{A} | u_j \rangle \,,$$

where, upon inverting the equations given in the question,

$$u_1 = (\psi_1 + \psi_2)/\sqrt{2} \qquad \text{and} \qquad u_2 = (\psi_1 - \psi_2)/\sqrt{2}\,.$$

After some algebra, we finally obtain

$$\begin{aligned}
\langle A\rangle(t) &= \frac{1}{2}\langle u_1|\hat{A}|u_1\rangle + \frac{1}{2}e^{i(E_1-E_2)/\hbar t}\langle u_1|\hat{A}|u_2\rangle + \frac{1}{2}e^{i(E_2-E_1)/\hbar t}\langle u_2|\hat{A}|u_1\rangle + \frac{1}{2}\langle u_2|\hat{A}|u_2\rangle \\
&= \frac{1}{2}\left(a_1 + a_2\right) + \frac{1}{2}\cos\left[\frac{(E_1-E_2)t}{\hbar}\right]\left(a_1 - a_2\right) \\
&= \cos^2\left[\frac{(E_1-E_2)t}{2\hbar}\right] a_1 + \sin^2\left[\frac{(E_1-E_2)t}{2\hbar}\right] a_2\,.
\end{aligned}$$

13. Using the energy eigenstates $|n\rangle$, of eigenvalues ε_n, as a basis to represent the operators, we can write

$$\hat{H} = \sum_n \varepsilon_n |n\rangle\langle n| \qquad \text{and} \qquad \hat{U} = \exp\left(i\hat{H}t\right) = \sum_n e^{i\varepsilon_n t}|n\rangle\langle n|\,.$$

Then one can straightforwardly show that $\hat{U}\exp\left(i\hat{H}t\right) = \exp\left(i\hat{H}t\right)\hat{U}$ by explicit calculation, using the orthonormality of the eigenstates.

Alternatively, one can use the definition of function of an operator,

$$\hat{U} = \exp\left(i\hat{H}t\right) = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left(i\hat{H}t\right)^n\,,$$

to show that the commutator $[\hat{H},\hat{U}]$ reduces to

$$\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\,(it)^n\,[\hat{H},\hat{H}^n] = 0\,,$$

since an operators trivially commutes with itself (and any power thereof).

14. Given that the shift operator in question 6 acted on momentum-space eigenfunctions as multiplication by $\exp\left(-ipx_0/\hbar\right)$, it can be written as

$$\sum_p \exp\left(-ipx_0/\hbar\right)\,|p\rangle\langle p|\,,$$

which defines the operator $\exp\left(-i\hat{p}x_0/\hbar\right)$.

Using, for example, the definition of function of an operator, one can straightforwardly verify the proposed commutation relation:

$$[\exp\left(-i\hat{p}x_{01}/\hbar\right), \exp\left(-i\hat{p}x_{02}/\hbar\right)] = \sum_{n,m=0}^{\infty}\frac{(-ix_{01}/\hbar)^n(-ix_{02}/\hbar)^m}{n!m!}\,[\hat{p}^n,\hat{p}^m] = 0\,.$$