Part IB Physics A : Lent 2022

QUANTUM PHYSICS EXAMPLES IV
MODEL ANSWERS

Prof. C. Castelnovo

1. From the definitions in the lecture notes, we can write $\hat{L}_x = (\hat{L}_+ + \hat{L}_-)/2$. And from the general action

$$\hat{L}_\pm|\ell, m_\ell\rangle = \hbar|\ell(\ell+1) - m_\ell(m_\ell+1)\,|\ell, m_\ell+1\rangle$$

we find that

$$\begin{array}{rcl}
\hat{L}_+|\phi_{-1}\rangle &=& \sqrt{2}\hbar|\phi_0\rangle \\
\hat{L}_+|\phi_0\rangle &=& \sqrt{2}\hbar|\phi_{+1}\rangle \\
\hat{L}_-|\phi_{+1}\rangle &=& \sqrt{2}\hbar|\phi_0\rangle \\
\hat{L}_-|\phi_0\rangle &=& \sqrt{2}\hbar|\phi_{-1}\rangle\,,
\end{array}$$

and all others vanish.

The matrix representation of $\hat{L}_x$ in the given Z basis $\{|\phi_{-1}\rangle, |\phi_0\rangle, |\phi_{+1}\rangle\}$ is :

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues are then given by the secular equation:

$$-\lambda(\lambda^2 - \hbar^2/2) + \lambda\hbar^2/2 = 0\,,$$

which admits solutions $\lambda = 0$ and $\lambda = \pm\hbar$, as expected.
The eigenvectors $\hat{L}_x|\psi\rangle = \lambda|\psi\rangle$ can be expressed in the given Z basis, $|\psi\rangle = a|\phi_{-1}\rangle + b|\phi_0\rangle + c|\phi_{+1}\rangle$, and obtained for instance by direct calculation:

$$\frac{\hat{L}_+ + \hat{L}_-}{2}|\psi\rangle = \frac{\hbar}{\sqrt{2}}\left(a|\phi_0\rangle + b|\phi_{+1}\rangle + b|\phi_{-1}\rangle + c|\phi_0\rangle\right)\,.$$

For $\lambda = 0$, we must have $b = 0$ and $a = -c$, and upon normalising the state we obtain

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|\phi_{-1}\rangle - |\phi_{+1}\rangle\right)\,.$$

For $\lambda = \pm\hbar$, we must solve the set of equations

$$\left\{\begin{array}{l} \hbar b/\sqrt{2} = \lambda a \\ \hbar(a+c)/\sqrt{2} = \lambda b \\ \hbar b/\sqrt{2} = \lambda c \end{array}\right.\,,$$

to find $c = a$ and $b = \pm\sqrt{2}\,a$. Again, upon normalising the state we obtain

$$|\psi_\pm\rangle = \frac{1}{2}\left(|\phi_{-1}\rangle \pm \sqrt{2}|\phi_0\rangle + |\phi_{+1}\rangle\right) .$$

For a beam of atoms with $\ell = 1$, if after the first (X) Stern-Gerlach experiment the $m_\ell = +1$ beam is selected, then the corresponding state $|\psi_+\rangle$ resolved in the Z basis tells us that there will be 3 beams exiting the second (Z) Stern-Gerlach experiment, of relative intensity $1 : 2 : 1$. The same is true if the $m_\ell = -1$ beam is selected in between the two Stern-Gerlach measurements. On the contrary, if the $m_\ell = 0$ beam is selected, only 2 beams of equal intensity are observed after the second Stern-Gerlach experiment.

2. This problem can be solved in different ways. Pedantically, one can observe that $\hat{\boldsymbol{s}}_1 \cdot \hat{\boldsymbol{s}}_2 = (\hat{s}_1^+\hat{s}_2^- + \hat{s}_1^-\hat{s}_2^+)/2 + \hat{s}_1^z\hat{s}_2^z$, and then can use this expression to compute the expectation value of the operator on the singlet,

$$(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}\,,$$

and triplet,

$$|\uparrow\uparrow\rangle \qquad (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2} \qquad |\downarrow\downarrow\rangle\,,$$

states, as the question asks.
Trivially, the ladder operators vanish when acting on the first and last triplet states and therefore

$$\langle\uparrow\uparrow|\hat{\boldsymbol{s}}_1 \cdot \hat{\boldsymbol{s}}_2|\uparrow\uparrow\rangle = \langle\uparrow\uparrow|\hat{s}_1^z\hat{s}_2^z|\uparrow\uparrow\rangle = \frac{\hbar^2}{4} = \langle\downarrow\downarrow|\hat{\boldsymbol{s}}_1 \cdot \hat{\boldsymbol{s}}_2|\downarrow\downarrow\rangle\,.$$

For the remaining states, one singlet and one triplet, a direct calculation gives the desired result:

$$\begin{aligned}
&\frac{1}{2}\left(\langle\uparrow\downarrow| \pm \langle\downarrow\uparrow|\right)\left[(\hat{s}_1^+\hat{s}_2^- + \hat{s}_1^-\hat{s}_2^+)/2 + \hat{s}_1^z\hat{s}_2^z\right](|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle) \\
&\quad = \frac{\hbar^2}{4}\left(\langle\uparrow\downarrow| \pm \langle\downarrow\uparrow|\right)(\pm|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) - \frac{\hbar^2}{8}\left(\langle\uparrow\downarrow| \pm \langle\downarrow\uparrow|\right)(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle) \\
&\quad = \pm\frac{\hbar^2}{2} - \frac{\hbar^2}{4} = \left\{\begin{array}{l} \hbar^2/4 \\ -3\hbar^2/4 \end{array}\right. ,
\end{aligned}$$

for the triplet (top) and single (bottom) case, respectively.
Notice that, in order to obtain the results above, we used the relationships: $\hat{s}^-|\uparrow\rangle = \hbar|\downarrow\rangle$ and $\hat{s}^+|\downarrow\rangle = \hbar|\uparrow\rangle$.

Alternatively (and setting $\hbar = 1$ for simplicity), one could have arrived at the same answer by observing that $s_1 = s_2 = 1/2$ and the direct sum of these two angular momenta, $\hat{\boldsymbol{s}} = \hat{\boldsymbol{s}}_1 + \hat{\boldsymbol{s}}_2$, has quantum numbers $s = 0, 1$. Since $|\boldsymbol{s}|^2 = |\boldsymbol{s}_1|^2 + 2\boldsymbol{s}_1 \cdot \boldsymbol{s}_2 + |\boldsymbol{s}_2|^2$, then the operator whose eigenvalues we want to obtain can be written as $\boldsymbol{s}_1 \cdot \boldsymbol{s}_2 = (|\boldsymbol{s}|^2 - |\boldsymbol{s}_1|^2 - |\boldsymbol{s}_2|^2)/2$, and therefore

$$\langle \boldsymbol{s}_1 \cdot \boldsymbol{s}_2 \rangle = \frac{1}{2}\left[s(s+1) - 2\frac{3}{4}\right] = \left\{\begin{array}{ll} 1/4 & \text{for } s = 1 \\ -3/4 & \text{for } s = 0 \end{array}\right. .$$

3. Let us label the eigenstates of $\hat{L}^2$ and $\hat{L}_z$ as $|\ell, m_\ell\rangle_L$, for $\ell = 1$ and $m_\ell = -1, 0, 1$, and the eigenstates of $\hat{S}^2$ and $\hat{S}_z$ as $|s, m_s\rangle_S$, for $s = 1/2$ and $m_s = -1/2, 1/2$. From

$\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$, we obtain the corresponding quantum numbers $j = 1/2$, $m_j = \pm 1/2$, and $j = 3/2$, $m_j = \pm 1/2$, $\pm 3/2$; and we label their eigenstates as $|j, m_j\rangle_J$.
We can then straightforwardly write the state with the largest total angular momentum and largest total projection, $|3/2, 3/2\rangle_J = |1, 1\rangle_L \otimes |1/2, 1/2\rangle_S$. Starting from it, we can find the other 3 states in the same set by direct application of the lowering operator $\hat{J}_- = \hat{L}_- + \hat{S}_-$:

$$\begin{aligned}
\hat{J}_-|3/2,3/2\rangle_J &= \hbar\sqrt{15/4 - 3/4}\,|3/2,1/2\rangle_J \\
&= \hbar\sqrt{2}|1,0\rangle_L \otimes |1/2,1/2\rangle_S + \hbar|1,1\rangle_L \otimes |1/2,-1/2\rangle_S \\
\longrightarrow \quad & |3/2,1/2\rangle_J = \sqrt{\frac{2}{3}}|1,0\rangle_L \otimes |1/2,1/2\rangle_S + \frac{1}{\sqrt{3}}|1,1\rangle_L \otimes |1/2,-1/2\rangle_S
\end{aligned}$$

$$\begin{aligned}
\hat{J}_-|3/2,1/2\rangle_J &= \hbar\sqrt{15/4 + 1/4}\,|3/2,-1/2\rangle_J \\
&= \hbar\frac{2}{\sqrt{3}}|1,-1\rangle_L \otimes |1/2,1/2\rangle_S + \hbar\frac{2\sqrt{2}}{\sqrt{3}}|1,0\rangle_L \otimes |1/2,-1/2\rangle_S \\
\longrightarrow \quad & |3/2,-1/2\rangle_J = \frac{1}{\sqrt{3}}|1,-1\rangle_L \otimes |1/2,1/2\rangle_S + \sqrt{\frac{2}{3}}|1,0\rangle_L \otimes |1/2,-1/2\rangle_S
\end{aligned}$$

$$\begin{aligned}
\hat{J}_-|3/2,-1/2\rangle_J &= \hbar\sqrt{15/4 - 3/4}\,|3/2,-3/2\rangle_J \\
&= \hbar\sqrt{3}|1,-1\rangle_L \otimes |1/2,-1/2\rangle_S \\
\longrightarrow \quad & |3/2,-3/2\rangle_J = |1,-1\rangle_L \otimes |1/2,-1/2\rangle_S\,.
\end{aligned}$$

In order to find the remaining two states with quantum number $j = 1/2$, let us consider $|1/2, 1/2\rangle_J$ to begin with. Two states can contribute to it: $|1, 0\rangle_L \otimes |1/2, 1/2\rangle_S$ and $|1, 1\rangle_L \otimes |1/2, -1/2\rangle_S$. The required linear combination of these two states must be orthogonal to all the eigenstates we have already found, which is straightforward apart from :

$${}_J\langle 3/2, 1/2| \left(A|1,0\rangle_L \otimes |1/2,1/2\rangle_S + B|1,1\rangle_L \otimes |1/2,-1/2\rangle_S \right) = 0\,,$$

giving the condition $B = -A\sqrt{2}$. Therefore,

$$|1/2,1/2\rangle_J = \frac{1}{\sqrt{3}}|1,0\rangle_L \otimes |1/2,1/2\rangle_S - \sqrt{\frac{2}{3}}|1,1\rangle_L \otimes |1/2,-1/2\rangle_S\,.$$

Either by similar observations, or direct application of the lowering operator, one can finally obtain the last eigenstate:

$$\begin{aligned}
\hat{J}_-|1/2,1/2\rangle_J &= \hbar\sqrt{3/4 + 1/4}\,|1/2,-1/2\rangle_J \\
&= \hbar\sqrt{\frac{2}{3}}|1,-1\rangle_L \otimes |1/2,1/2\rangle_S - \hbar\frac{1}{\sqrt{3}}|1,0\rangle_L \otimes |1/2,-1/2\rangle_S \\
\longrightarrow \quad & |1/2,-1/2\rangle_J = -\frac{1}{\sqrt{3}}|1,0\rangle_L \otimes |1/2,-1/2\rangle_S + \sqrt{\frac{2}{3}}|1,-1\rangle_L \otimes |1/2,1/2\rangle_S\,.
\end{aligned}$$

4. Using the $\hat{S}_z$ eigenstates $|\uparrow\rangle$, $|\downarrow\rangle$, we can write:

$$|\chi\rangle = a|\uparrow\rangle + b|\downarrow\rangle\,,$$

and (in units where $\hbar = 1$)

$$\begin{aligned}
\hat{S}_x|\chi\rangle &= \frac{\hat{S}_+ + \hat{S}_-}{2}|\chi\rangle = \frac{b}{2}|\uparrow\rangle + \frac{a}{2}|\downarrow\rangle \\
\hat{S}_y|\chi\rangle &= \frac{\hat{S}_+ - \hat{S}_-}{2i}|\chi\rangle = \frac{b}{2i}|\uparrow\rangle - \frac{a}{2i}|\downarrow\rangle \\
\hat{S}_z|\chi\rangle &= \frac{a}{2}|\uparrow\rangle - \frac{b}{2}|\downarrow\rangle \,.
\end{aligned}$$

We then require that the proposed decomposition satisfies the equation in the question, $\boldsymbol{n}.\widehat{\boldsymbol{S}}\,|\chi\rangle = +\frac{1}{2}\,|\chi\rangle$. Here it is convenient to express the components of the unit vector $\boldsymbol{n}$ in terms of the angles: $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, where $\theta \in (0,\pi)$ and $\phi \in (0, 2\pi)$. Finally, we arrive at the equation:

$$[(\sin\theta\cos\phi - i\sin\theta\sin\phi)b + \cos\theta a]|\uparrow\rangle + [(\sin\theta\cos\phi + i\sin\theta\sin\phi)a - \cos\theta b]|\uparrow\rangle = a|\uparrow\rangle + b|\downarrow\rangle \,,$$

which corresponds to the system of equations

$$\begin{cases} \sin\theta e^{-i\phi}b + \cos\theta a = a \\ \sin\theta e^{i\phi}a - \cos\theta b = b \end{cases} .$$

Trigonometric manipulations (using $1 - \cos\theta = 2\sin^2(\theta/2)$, $1 + \cos\theta = 2\cos^2(\theta/2)$, and $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$) show that the two equations actually reduce to a single condition,

$$\cos(\theta/2)\, e^{-i\phi/2}\, b = \sin(\theta/2)\, e^{i\phi/2}\, a \,,$$

and the state can be written as:

$$|\chi\rangle = \cos(\theta/2)\, e^{-i\phi/2}\,|\uparrow\rangle + \sin(\theta/2)\, e^{i\phi/2}\,|\downarrow\rangle \,,$$

which is appropriately normalised.

The intensities of the two beams produced when particles in the state $|\chi\rangle$ pass through a z-Stern-Gerlach apparatus are thus in the ratio $\tan^2(\theta/2)$ (namely, intensities $\cos^2(\theta/2)$ and $\sin^2(\theta/2)$).

5. The wave function of ^{4}He describes an ensemble of 6 spin-1/2 particles and therefore has integer total spin angular momentum: it must satisfy bosonic statistics. The wave function of ^{3}He describes an ensemble of 5 spin-1/2 particles and therefore has half-integer total spin angular momentum: it must satisfy fermionic statistics.

This applies so long as the ensemble is described by a single quantum-coherent wave function with sufficient space-time support to overlap with an identical ensemble, thus becoming indistinguishable in a quantum mechanical sense. When this is no longer the case (e.g., for larger collection of particles, due to dephasing and decoherence effects), exchanging the two ensembles becomes a classical operation and the ensembles can be 'tracked': they are identical in a classical sense, but no longer indistinguishable, and the spin statistics theorem does not apply.

6. Since the electrons are non-interacting, the time-independent Schrödinger equation is separable in the x, y, z coordinates, and solutions take the form seen in lectures for the infinite square well. In the following, we choose x, y to be along the sides of length a, and d to be along the side of length d. It is convenient to exploit the symmetry of the system by choosing the centre of the reference frame to coincide with the centre of the box.

Solutions take the form of products of sine and cosine functions, or argument $k_1 x$, $k_1 y$ and $k_2 z$, where $k_1 = n\pi/a$ and $k_2 = n\pi/d$. The corresponding energies are $E_1 = \hbar^2 n^2 \pi^2/(2ma^2)$ and $E_2 = \hbar^2 n^2 \pi^2/(2md^2)$. Recall also that cosine functions require n to be odd, and sine functions require n to be even.

The lowest energy state has spatial quantum numbers $(n_x, n_y, n_z) = (1, 1, 1)$, with wave-function

$$\psi(x, y, z) \propto \cos(k_1 x)\cos(k_1 y)\cos(k_2 z)\,,$$

and energy

$$E = \frac{\hbar^2\pi^2}{2m}\left(\frac{2}{a^2} + \frac{1}{d^2}\right)\,.$$

It can contain at most 2 electrons, which must form an antisymmetric singlet spin state in order to ensure overall antisymmetry upon exchanging the two electrons (and thus such state is non-degenerate). However, since our system is composed of 3 electrons, this state alone is not an allowed eigenstate and we need to put the third electron in one of the excited states of the spatial wavefunctions of the rectangular box.

In order to find the spatial state occupied by the third electron in the ground state of the 3-electron system, we ought to consider the following two possibilities. Either $(n_x, n_y, n_z) = (2, 1, 1)$, or equivalently $(n_x, n_y, n_z) = (1, 2, 1)$, with wavefunctions

$$\psi(x, y, z) \propto \sin(k_1 x)\cos(k_1 y)\cos(k_2 z) \qquad \text{or} \qquad \psi(x, y, z) \propto \cos(k_1 x)\sin(k_1 y)\cos(k_2 z)\,,$$

and energy

$$E = \frac{\hbar^2\pi^2}{2m}\left(\frac{5}{a^2} + \frac{1}{d^2}\right)\,.$$

Or else $(n_x, n_y, n_z) = (1, 1, 2)$, with wavefunction

$$\psi(x, y, z) \propto \cos(k_1 x)\cos(k_1 y)\sin(k_2 z)\,,$$

and energy

$$E = \frac{\hbar^2\pi^2}{2m}\left(\frac{2}{a^2} + \frac{4}{d^2}\right)\,.$$

Comparing the energies,

$$\frac{5}{a^2} + \frac{1}{d^2} < \frac{2}{a^2} + \frac{4}{d^2} \qquad \to \qquad d < a\,,$$

shows that the former option is favoured for $0 < d < a$ and the latter is favoured for $a < d < 2a$.

We can finally conclude that, for $0 < d < a$ two electrons are in state $(1, 1, 1)$ and one is either in state $(2, 1, 1)$ or in state $(1, 2, 1)$, with total degeneracy $2 \times 2 = 4$ (two spatial and two spin states possible for the third electron); the total energy is

$$E = \frac{\hbar^2\pi^2}{2m}\left(\frac{9}{a^2} + \frac{3}{d^2}\right)$$

and the state is even in y, z and odd in x or even in x, z and odd in y (and therefore odd upon inversion about the centre). Conversely, for $a < d < 2a$ two electrons are in state $(1, 1, 1)$ and one is in state $(1, 1, 2)$, with total degeneracy 2 (two spin states possible for the third electron); the total energy is

$$E = \frac{\hbar^2\pi^2}{2m}\left(\frac{6}{a^2} + \frac{6}{d^2}\right)$$

and the state is even in x, y and odd in z (and therefore odd upon inversion about the centre).

For $d = a$, the two energies are equal and the ground state has degeneracy 6, $E = 6\hbar^2\pi^2/(ma^2)$, and is odd upon inversion.

7. For particles in a 3D simple harmonic potential, in the absence of interactions or spin-orbit forces, one can find spatial eigenfunctions that solve the Schrödinger equation by separation of variables in cartesian coordinates, and thus reducing the problem to a 1D SHO. The eigenstates are therefore labelled by a triplet of integer non-negative quantum numbers, (n_x, n_y, n_z), and they have eigenfunctions

$$\psi_{n_x,n_y,n_z}(x, y, z) = \psi_{n_x}(x)\,\psi_{n_y}(y)\,\psi_{n_z}(z)$$

and eigenenergies

$$E = \hbar\omega\left(n_x + n_y + n_z + \frac{1}{2}\right) = \hbar\omega\left(N + \frac{1}{2}\right),$$

where we introduced for convenience of notation the quantity $N = n_x + n_y + n_z$, which in turn takes on non-negative integer values.

The lowest energy state has $N = 0$ and is unique: $\psi_{0,0,0}(x, y, z) = \psi_0(x)\,\psi_0(y)\,\psi_0(z)$. Two spin-1/2 particles can occupy the same state only if they form an antisymmetric spin singlet, and therefore the total degeneracy is 1. Two spin-1 particles can also occupy the same state, but they must form symmetric spin states, either trivially by having the same value of the spin component (there are 3 such combinations), or forming symmetric superpositions of the form:

$$\psi_0(x)\,\psi_0(y)\,\psi_0(z) \otimes \left[\frac{\chi_{1,-1} + \chi_{-1,1}}{\sqrt{2}}\right]$$

(there are 3 such possible combinations). Giving a total degeneracy of 6.

The next energy level has $N = 1$, which comes in three different forms according to which of the quantum numbers n_x, n_y, n_z takes value 1, while the other two vanish. For the two particles in question, the next energy level corresponds to one particle in the $N = 0$ state and one particle in one of the three $N = 1$ states. For two spin-1/2 particles, the degeneracy is simply $3 \times (2S+1)^2 = 3 \times 2 \times 2 = 12$ (3 options for the spatial wavefunction and 2 options each for the spin states). Similarly, for two spin-1 particles, the degeneracy is simply $3 \times (2S + 1)^2 = 3 \times 3 \times 3 = 27$.
(The formula holds in general, and one can check that it gives the same result if one were to explicitly count the states that have the correct exchange symmetry. For instance,

for two spin-1/2 particles one has to form symmetric / antisymmetric spatial states – $(|0,1a\rangle_N \pm |1a,0\rangle_N)/\sqrt{2}$ – and multiply them respectively by antisymmetric / symmetric spin states – $|s=0,m_s=0\rangle$ and $|s=1,m_s=-1,0,1\rangle$. The label $1a$ spans the three possible $N=1$ states. We thus get 4 states for each value of the label $1a$, for a total of 12 degenerate states.)

Finally, the third energy level can be attained either by placing one particle in the $N=0$ state and the other in one of the $N=2$ states (i.e., $(n_x,n_y,n_z)=(2,0,0)$, or one of its two permutations; or $(n_x,n_y,n_z)=(1,1,0)$ or one of its two permutations). As in the case above, the number of allowed states is $(3+3)\times(2S+1)^2$, giving 24 and 36 for $S=1/2$ and $S=1$, respectively. Alternatively, the same energy can be achieved by placing each of the two particles in an $N=1$ state (i.e., $(n_x,n_y,n_z)=(1,0,0)$, or one of its two permutations). If the states are different, the number of states is again $3\times(2S+1)^2$. If the states are the same (of which there are 3 options), the spatial wave function is necessarily symmetric and one ought to symmetrise accordingly the spin part, leading to the same spin degeneracies as in the case of the lowest energy state. In summary, for two spin-1/2 particles we have $(3+3+3)\times(2S+1)^2=36$ 'general' states, and $3\times 1=3$ states where the spatial part is necessarily symmetric, giving a total degeneracy of 39. For two spin-1 particles we have $(3+3+3)\times(2S+1)^2=81$ 'general' states, and $3\times 6=18$ 'constrained' states, giving a total degeneracy of 99.

8. The notes should include a discussion of identical vs indistinguishable particles and why the former concept does not imply indistinguishability in classical mechanics (because a particle trajectory can in principle be traced with infinite accuracy; this is fundamentally not possible in quantum mechanics due to the uncertainty principle). The notes should also mention the notion of exchange symmetry, the symmetrisation postulate. They can include examples (e.g., singlet and triplet states); and mention the notions of fermionic and bosonic particles; and the spin statistics theorem; multi-particle exchange and Bose-Einstein condensation.

Experimental verifications include the role of the Pauli exclusion principle in determining the structure of the periodic table; and more generally the degeneracy of energy levels in molecules. Electrical and thermal characteristics of solids. Ferromagnetism. The absence of gravitational collapse in White Dwarf stars.

Some review questions:

9. (a) When a system is measured, its wave function changes discontinuously: it is 'projected' onto the corresponding eigenstate(s), and renormalised. When two operators (i.e., observables) do not commute, measuring one of them can substantially alter the measurement of the other. For instance, consider a plane wave of definite momentum. If position is measured within a given interval of width Δx, the uncertainty principle requires that any subsequent measurement of p has an uncertainty Δp greater than $\hbar/(2\Delta x)$. On the contrary, if p had been measured first, one would have obtained $\Delta p = 0$.

(b) Following a position measurement, the particle wave function is projected onto the corresponding eigenstate(s). Therefore, if it was found in a region of width Δx and the position is measured again immediately afterwards, the particle *must* be found in the same region with probability 1. This is strictly true only if the second measurement is

infinitesimally close in time to the first one; time evolution can allow the particle position to change.
For free particles, a lower-bound estimate of the positional uncertainty can be obtained by summing uncertainties in quadrature:

$$\Delta x(t) \simeq \sqrt{\Delta x^2 + \left(\frac{\Delta p\, t}{m}\right)^2} \gtrsim \sqrt{\Delta x^2 + \left(\frac{\hbar t}{2m\Delta x}\right)^2},$$

where we used the uncertainty relation $\Delta x \Delta p \geq \hbar/2$.

10. Following a measurement of B whose outcome is b_1, the system must be in state χ_1. A measurement of A immediately afterwards then gives:

$$\langle\chi_1|\hat{A}|\chi_1\rangle = \frac{4}{13}a_1 + \frac{9}{13}a_2$$

and the probability of outcome $a_{1,2}$ is 4/13, 9/13 (which could have equivalently been estimated by computing $|\langle\psi_{1,2}|\chi_1\rangle|^2$).

Following a measurement of A, the system must be in one of its eigenstates $\psi_{1,2}$. In order to compute the result of a measurement of B immediately afterwards, we ought to express the eigenstates of A in terms of those of B by inverting the equations in the question:

$$\psi_1 = (2\chi_1 + 3\chi_2)/\sqrt{13} \qquad\qquad \psi_2 = (3\chi_1 - 2\chi_2)/\sqrt{13}\,,$$

to find $\langle\psi_1|\hat{B}|\psi_1\rangle = \frac{4}{13}b_1 + \frac{9}{13}b_2$ and $\langle\psi_2|\hat{B}|\psi_2\rangle = \frac{9}{13}b_1 + \frac{4}{13}b_2$. Therefore, for a sequence of measurements, the probability is

$$P(b_1) = P(b_1|a_1)P(a_1) + P(b_1|a_2)P(a_2) = \left(\frac{4}{13}\right)^2 + \left(\frac{9}{13}\right)^2 = \frac{97}{169}\,.$$

(You can similarly check that $P(b_2) = 72/169$ and $P(b_1) + P(b_2) = 1$, as expected.)

11. From the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, one can apply the Baker–Campbell–Hausdorff formula as suggested in the problem to show that

$$e^{c_1\hat{a}}\, e^{c_2\hat{a}^\dagger} = e^{c_1\hat{a}+c_2\hat{a}^\dagger+c_1c_2[\hat{a},\hat{a}^\dagger]/2} = e^{c_1c_2/2}\, e^{c_1\hat{a}+c_2\hat{a}^\dagger}\,,$$

for any $c_1, c_2 \in \mathbb{C}$. From this the result $\hat{D}(\alpha) = e^{-|\alpha|^2/2}\, e^{\alpha\hat{a}^\dagger}\, e^{-\alpha^*\hat{a}}$ follows in a straightforward manner.
On a similar note, we can write

$$\begin{aligned}
\hat{D}(\alpha+\beta) &= \exp[(\alpha+\beta)\hat{a}^\dagger - (\alpha^*+\beta^*)\hat{a}) \\
&= \hat{D}(\alpha)\,\hat{D}(\beta)\, e^{-[\alpha\hat{a}^\dagger-\alpha^*\hat{a},\beta\hat{a}^\dagger-\beta^*\hat{a}]/2} \\
&= \hat{D}(\alpha)\,\hat{D}(\beta)\, e^{\alpha\beta^*[\hat{a}^\dagger,\hat{a}]/2+\alpha^*\beta[\hat{a},\hat{a}^\dagger]/2} = \hat{D}(\alpha)\,\hat{D}(\beta)\, e^{i\,\mathrm{Im}(\alpha^*\beta)}\,,
\end{aligned}$$

where we used the fact that $(\alpha^*\beta - \alpha\beta^*)/2 = \mathrm{Im}(\alpha^*\beta)$.

12. For the first part of this question, we use the power series representation of functions of operators,

$$\begin{aligned}\hat{D}(\alpha) &= \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) = e^{-|\alpha|^2/2}\, e^{\alpha\hat{a}^\dagger}\, e^{-\alpha^*\hat{a}} \\ &= e^{-|\alpha|^2/2}\left[\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}(\hat{a}^\dagger)^n\right]\left[\sum_{n=0}^{\infty}\frac{(-\alpha^*)^n}{n!}\hat{a}^n\right],\end{aligned}$$

and the fact that $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$, $n \geq 0$, and therefore $(\hat{a}^\dagger)^n|0\rangle = \sqrt{n!}\,|n\rangle$, to obtain:

$$\begin{aligned}|\alpha\rangle &= \hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}(\hat{a}^\dagger)^n\sum_{n=0}^{\infty}\frac{(-\alpha^*)^n}{n!}\hat{a}^n\,|0\rangle \\ &= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}(\hat{a}^\dagger)^n\,|0\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\,.\end{aligned}$$

One can easily verify that the state is properly normalised, $\langle\alpha|\alpha\rangle = 1$. For the second part of the question, one can directly compute

$$\hat{a}|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=1}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\sqrt{n}|n-1\rangle = \alpha|\alpha\rangle\,,$$

upon changing the mute summation variable $n \to n-1$; it then follows that $\langle\alpha|\hat{a}|\alpha\rangle = \alpha$, $\langle\alpha|\hat{a}^\dagger|\alpha\rangle = \alpha^*$, and $\langle\alpha|\hat{a}\hat{a}^\dagger|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle + 1 = |\alpha|^2 + 1$. We can then use the definition of the creation and annihilation operators,

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}) \qquad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}^\dagger - \hat{a})\,,$$

to obtain

$$\begin{aligned}\langle\alpha|\hat{x}|\alpha\rangle &= \sqrt{\frac{\hbar}{2m\omega}}(\alpha^* + \alpha) \\ \langle\alpha|\hat{x}^2|\alpha\rangle &= \frac{\hbar}{2m\omega}\left[(\alpha^*)^2 + \alpha^2 + \alpha\alpha^* + \alpha^*\alpha + 1\right] \\ \langle\alpha|\hat{p}|\alpha\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}(\alpha^* - \alpha) \\ \langle\alpha|\hat{p}^2|\alpha\rangle &= -\frac{\hbar m\omega}{2}(\alpha^* - \alpha)\left[(\alpha^*)^2 + \alpha^2 - \alpha\alpha^* - \alpha^*\alpha - 1\right].\end{aligned}$$

Therefore,

$$\Delta x^2 = \langle\alpha|\hat{x}^2|\alpha\rangle - \langle\alpha|\hat{x}|\alpha\rangle^2 = \frac{\hbar}{2m\omega} \qquad \Delta p^2 = \langle\alpha|\hat{p}^2|\alpha\rangle - \langle\alpha|\hat{p}|\alpha\rangle^2 = \frac{\hbar m\omega}{2}$$

and we find the desired result, $\Delta x^2\Delta p^2 = \hbar^2/4$.

13. This question is similar to the former, with the operator $\hat{S}(r) = \exp\{r\,[(\hat{a}^\dagger)^2 - \hat{a}^2]/2\}$, where r is a real number, and with the *squeezed state* $|r\rangle = \hat{S}(r)|0\rangle$. In order to evaluate the uncertainty in position and momentum, we need to evaluate expectation values of the type $\langle r|\hat{a}|r\rangle$ and $\langle r|\hat{a}^2|r\rangle$, and similarly for $\hat{a}^\dagger$. This in turn corresponds to computing vacuum expectation values of $\hat{S}^\dagger(r)\hat{a}\hat{S}(r)$ and of $\hat{S}^\dagger(r)\hat{a}^2\hat{S}(r) = \hat{S}^\dagger(r)\hat{a}\hat{S}(r)\hat{S}^\dagger(r)\hat{a}\hat{S}(r)$. It is therefore useful to obtain an explicit expression for $\hat{S}^\dagger(r)\hat{a}\hat{S}(r)$ and $\hat{S}^\dagger(r)\hat{a}^\dagger\hat{S}(r)$. Following the hint in the question, with $\hat{A} = -r\,[(\hat{a}^\dagger)^2 - \hat{a}^2]/2$ and $\hat{B} = \hat{a}$, we compute

$$\begin{aligned}
[\hat{A},\hat{B}] &= -(r/2)[(\hat{a}^\dagger)^2,\hat{a}] = r\hat{a}^\dagger \\
[\hat{A},[\hat{A},\hat{B}]] &= r[\hat{A},\hat{a}^\dagger] = (r^2/2)[\hat{a}^2,\hat{a}^\dagger] = r^2\hat{a} \\
[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] &= r^2[\hat{A},\hat{a}] = r^3\hat{a}^\dagger \\
\hat{S}^\dagger(r)\hat{a}\hat{S}(r) &= \hat{a} + r\hat{a}^\dagger + \frac{1}{2!}r^2\hat{a} + \frac{1}{3!}r^3\hat{a}^\dagger + \ldots = \cosh(r)\,\hat{a} + \sinh(r)\,\hat{a}^\dagger \\
\hat{S}^\dagger(r)\hat{a}^\dagger\hat{S}(r) &= \cosh(r)\,\hat{a}^\dagger + \sinh(r)\,\hat{a}\,,
\end{aligned}$$

where the last equation can be most conveniently obtained by taking the adjoint of the result in the second to last equation. Therefore,

$$\hat{S}^\dagger(r)\left(\hat{a}^\dagger \pm \hat{a}\right)\hat{S}(r) = [\cosh(r) \pm \sinh(r)]\left(\hat{a}^\dagger \pm \hat{a}\right) = e^{\pm r}\left(\hat{a}^\dagger \pm \hat{a}\right)\,.$$

With these results at hand, we see straightforwardly that $\langle r|\hat{a} \pm \hat{a}^\dagger|r\rangle = 0$, as the expectation values of $\hat{a}$ and $\hat{a}^\dagger$ on the ground state of the simple harmonic oscillator vanish identically. We then need to compute:

$$\begin{aligned}
\Delta x^2 &= \frac{\hbar}{2m\omega}\langle r|(\hat{a}^\dagger + \hat{a})^2|r\rangle = \frac{\hbar}{2m\omega}\langle 0|\left[\hat{S}^\dagger(r)(\hat{a}^\dagger + \hat{a})\hat{S}(r)\right]^2|0\rangle \\
&= \frac{\hbar}{2m\omega}\,e^{2r}\,\langle 0|(\hat{a}^\dagger + \hat{a})^2|0\rangle = \frac{\hbar}{2m\omega}\,e^{2r}\,\langle 0|\hat{a}\hat{a}^\dagger|0\rangle = \frac{\hbar}{2m\omega}\,e^{2r}
\end{aligned}$$

and

$$\Delta p^2 = -\frac{\hbar m\omega}{2}\langle r|(\hat{a}^\dagger - \hat{a})^2|r\rangle = \frac{\hbar m\omega}{2}\,e^{-2r}\,\langle 0|\hat{a}\hat{a}^\dagger|0\rangle = \frac{\hbar m\omega}{2}\,e^{-2r}$$

The desired result, $\Delta x^2\Delta p^2 = \hbar^2/4$, follows then by inspection.

14. The key point of this question is the word '*demonstrate*'. It would be good to see the students produce robust arguments. For example, an operator $\hat{A}$ is conserved in time if $d\hat{A}/dt = 0$; and we know that the time derivative can be computed as $d\hat{A}/dt = (i/\hbar)[\hat{H},\hat{A}] + \partial\hat{A}/\partial t$. Since every operator commutes with itself, a Hamiltonian is conserved if it does not depend directly on time, which is the case for $\hat{H}$ in the question.

With regards to the rotation, one can compute explicitly $\hat{D}_x^\dagger\hat{H}\hat{D}_x$ or observe that the result of the rotation by $\pi/2$ about the x-axis is to map $\hat{y} \to \hat{z}$, $\hat{z} \to -\hat{y}$, and similarly $\hat{p}_y \to \hat{p}_z$, $\hat{p}_z \to -\hat{p}_y$. Such operation leaves $\hat{p}^2$ and $\hat{x}^4 + \hat{y}^4 + \hat{z}^4$ invariant, and as such it leaves $\hat{H}$ invariant.