Mock NST 1A math paper: things done in Michaelmas

14:00-16:00, Wednesday 17$^{\text{th}}$ January 2018

- Try **all** the problems in **Section A** and any **three** problems in **Section B**.
- About 1/4 of the marks are in **Section A**, 3/4 in **Section B**.
- There are no other guidelines for mark distributions.
- Don't bother with a calculator.
- Problems recommended for the B-course folks have a star, (*).

Section A

Try **all** of these in about 1/2 hour.

A1) Use the Cauchy-Schwarz inequality to show that

$$\int_{-\infty}^{\infty} \left[\cosh(\theta)\left(1+\theta^2\right)^{3/4}\right]^{-1} \mathrm{d}\theta \leq 2. \tag{1}$$

A2) Three planes are defined by the equations

$$x + y - 2z = 12, \tag{2}$$
$$2x - 2y + z = 1, \tag{3}$$
$$3x + y + z = 7. \tag{4}$$

Find the volume of the parallelepiped of side lengths 2, 3 and 5, which exactly occupies one of the corners of their point of intersection.

A3) The region Γ_ξ in the $x-y$ plane is bounded by the lines $y = f_\xi(x)$, $x = b_\xi$ and $y = a_\xi$, where $f_\xi(x)$ is a monotonically decreasing function of x for all values of the parameter ξ (in this problem the ξ-subscript indicates implicit dependence on ξ). By considering the ξ-dependence of the double integral of some function $g_\xi(x, y)$ over this region,

$$I_\xi = \int_{\Gamma_\xi} g_\xi(x, y)\mathrm{d}x\mathrm{d}y, \tag{5}$$

show that,

$$\int_{b_\xi}^{f_\xi^{-1}(a_\xi)} g_\xi(u, f_\xi(u)) \frac{\partial}{\partial \xi} f_\xi(u)\mathrm{d}u = \int_{a_\xi}^{f_\xi(b_\xi)} g_\xi(f_\xi^{-1}(u), u) \frac{\partial}{\partial \xi} f_\xi^{-1}(u)\mathrm{d}u. \tag{6}$$

A4) Show that

$$8e^{i\pi/4} \int_0^\infty \theta^5 \left[e^{i\pi/4} \sin\left(i\sqrt[3]{\pi}\theta^4\right) + e^{-i\pi/4}\cos\left(i\sqrt[3]{\pi}\theta^4\right)\right] \mathrm{d}\theta = 1. \tag{7}$$

A5) Show that the number of *disconnected* loci of $\mathbf{x}$ simultaneously satisfying *both* constraints,

$$|\mathbf{x} \times \hat{\mathbf{n}}| = \chi|\mathbf{x}|, \tag{8}$$
$$|\mathbf{x} - \lambda\hat{\mathbf{n}}| = \psi, \tag{9}$$

is two when $|\lambda| < \psi/\chi$, one when $|\lambda| = \psi/\chi$ and zero otherwise, where χ and ψ are constant scalars and $\hat{\mathbf{n}}$ is a constant unit vector‡.

A6) Primed and unprimed coordinates for a three-dimensional Euclidean space are related by

$$\begin{aligned} x_0 &= \cosh(\psi)x_0' - \sinh(\psi)x_1', \\ x_1 &= \cosh(\psi)x_1' - \sinh(\psi)x_0', \\ x_2 &= x_2', \end{aligned} \tag{10}$$

where ψ is a parameter. Find the normalized basis vectors $\{\mathbf{e}_i'\}$ in terms of the Cartesian basis vectors $\{\mathbf{e}_i\}$ and verify that orthogonality has only been lost between the x_0' and x_1' coordinates. Find the reciprocal basis set $\{\mathbf{E}_i'\}$ in terms of the $\{\mathbf{e}_i\}$.

‡ Note that a locus can be a region of any dimension from 0 to 3.

Section B

Try any **three** of these, about 1/2 hour each.

B1) The path P is made up from an infinite series of steps. The nth step involves travelling a distance $l_n = l_0\varepsilon^n/n$ in a straight line and then taking a left turn through an angle θ, so that ε and θ are dimensionless constants and l_0 has dimensions of length.

Sketch the first four steps of the path $\varepsilon = 1/2$ and $\theta = \pi/4$, beginning at $n = 1$.

Show that the series

$$S^N(\varepsilon) = \sum_{n=1}^{N} \frac{\varepsilon^n}{n}, \quad \varepsilon \in \mathbb{R} \tag{11}$$

is convergent for $|\varepsilon| < 1$ but divergent for $|\varepsilon| > 1$, and that convergence for $-1 < \varepsilon < 0$ is absolute.

Show further, with reference to definite integrals of the form $\int_{x_1}^{x_2} \mathrm{d}x/x$, that $S^N(\varepsilon)$ is divergent at $\varepsilon = 1$ and conditionally convergent at $\varepsilon = -1$.

Hence and by means of differentiation also, show that the Taylor series for the real natural logarithm of $x \in \mathbb{R}$ and its range of validity are

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}, \quad -1 < x - 1 \leq 1. \tag{12}$$

Given that in (12) the series expansion (known as the Newton-Mercator series) is also valid for the particular natural logarithm of $x \in \mathbb{C}$ for which $\pi > \Im[\ln(x)] > -\pi$, with the range $|x-1| \leq 1$ and $x \neq 2$, show that a traveller on P with $\varepsilon = 1$ and $\theta = \pi/4$ must travel infinitely far but will eventually find herself a finite distance,

$$l = l_0\sqrt{\left[\frac{1}{2}\ln\left(2-\sqrt{2}\right)\right]^2 + \left[\arctan\left(\frac{1}{1-\sqrt{2}}\right)\right]^2}, \tag{13}$$

from her starting point.

B2)

B3)

B4)

B5)