Natural Sciences Tripos Part 1A
Lent Term 2018
Course A Mathematics Mock Exam
Time Limit: 2 hours
Supervisor: Nora Martin

Name (Print): ____________________

Please attempt all questions in Part A and 3 questions in Part B.

Good luck!

Problem	Points	Score
1	2	
2	4	
3	5	
4	6	
5	8	
6	12	
7	12	
8	12	
9	12	
10	12	
Total:	85	

Section A

1. (2 points) Given $\vec{OC} = (1, 2, 3)$ and $\vec{AC} = (2, 1, 5)$, find an expression for the cosine of the angle between vectors $\vec{OA}$ and $\vec{OC}$.

2. (4 points) Evaluate $\ln(\frac{1}{\sqrt{2}}(1 + i))$ and show the result on the complex plane.

3. (a) (2 points) Calculate the second derivative with respect to x of the real function $y = \sin(\exp x)$.

   (b) (3 points) Sketch the function $y = \sin(\exp x)$ in the range $-\infty < x < \infty$.

4. (6 points) Find the zeros, the stationary points and the inflection points of $y = x^3 - 3x^2 + 4$ and state whether the stationary points are maxima or minima. Indicate these points on a graph of the function.

5. (a) (3 points) State Taylor's theorem for the expansion about $x = x_0$ of a function that is differentiable n times and give an expression for the remainder term R_n after n terms.

   (b) (5 points) Give the first three terms of the Taylor Series of $f(x) = \exp(-2x)$ around $x = 0$. What is the n^{th} term of this series?

Section B

6. Two planes are defined by the equations:
   Plane A: $x + y - 2z = 1$

   Plane B: $x = 2$

   (a) (4 points) A third plane, plane C, is defined by two lines. These lines are:

$$\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + r \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} \tag{1}$$

$$\vec{x} = \begin{pmatrix} 5 \\ -4 \\ -3 \end{pmatrix} + s \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix} \tag{2}$$

   Find a coordinate equation of plane C.

   (b) (4 points) Calculate the volume of the parallelepiped of side lengths 2, 3 and 5, which exactly occupies one of the corners of the point of intersection of planes A, B and C.

   (c) (4 points) Find the distance of plane A from the origin. Also find the coordinates of the point at the foot of the perpendicular from the origin to this plane.

7. Integration:

   (a) (2 points) Evaluate from first principles, by considering elementary areas, the integral:

$$\int_a^b x \; dx \tag{3}$$

(b) (4 points) Evaluate the integral:

$$\int_0^\infty e^{-x}\ sin(2x)\ dx \tag{4}$$

(c) (6 points) Find a recurrence relation between I_n and I_{n-1} for

$$I_n = \int_0^\infty x^n e^{-x}\ dx \tag{5}$$

and use it to calculate I_n for the case $n = 5$.

8. Complex Numbers I *(adapted from Tripos paper 1, 2002)*:

   (a) (5 points) Let $z = 2\exp(i\phi)$ where $0 < \phi < \pi/2$. Express z^* and $-z$ in the form $r\exp(i\theta)$ where $r > 0$ and $0 < \theta < 2\pi$. For $\phi = \pi/4$, sketch the location of z, z^* and $-z$ in the complex plane.

   (b) (3 points) If $z = 2\exp(i\phi)$ where $0 < \phi < \pi$, calculate the real and imaginary parts of $w = (z - 2)/(z + 2)$.

   (c) (4 points) Write down formulae for $\cos\theta$ and $\sin\theta$ in terms of complex exponentials. Use these to derive formulae for $\cos 2\theta$ and sin $\sin 2\theta$ in terms of $\cos\theta$ and $\sin\theta$.

9. Complex Numbers II *(adapted from Tripos paper 1, 2008)*:

   (a) (4 points) Let $z = x + i\ y$ with x and y real. Find the real and imaginary parts of the following in terms of x and y: $z\sin(z)$

   (b) (3 points) Find all the roots of the equation $z^4 - z^2 - 2 = 0$ and plot them in the Argand diagram.

   (c) (5 points) Let the complex numbers z_1, z_2, z_3 and z_4 represent the vertices of a plane quadrilateral ABCD in the complex plane. Show that ABCD is a parallelogram if $z_1 - z_2 + z_3 - z_4 = 0$.

10. Probabilities:

    (a) (5 points) Find the probability that in a group of k people at least two have the same birthday (ignoring 29th February).

    (b) (7 points) The random variable X can take any non-negative value and its **cumulative** density function is $P(0 \le X \le x) = k\ (1 - \exp(-x))$, where k is a constant. Find the probability **density** function for X, the value of k, and the mean and variance of X.