Particle spectra of metric-affine gauge theories of gravity

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Metric-affine gravity (MAG) possesses an affine connection *independent* of the metric, and this connection contains the rank-3 torsion and non-metricity tensors. In this paper, we present a parallelised computer algebra procedure to compute the particle spectrum of any general MAG formulated as a linearised Lagrangian, quadratic in the metric and connection and preserving parity. This is based on the spin-parity irreducible representations contained within the tensors. The program is able to generate source constraints to remove matrix singularities due to gauge symmetries, so as to calculate a pseudo-inverse and form the saturated propagator. From the propagator, poles and residues can be systematically identified along with no-ghost and no-tachyon conditions. For massless poles, the constrained sources are expressed in terms of null vectors, from which their eigenvalues are calculated. The generalised Einstein–Hilbert connection was studied using this procedure, and we recover the known result that there are only two propagating massless graviton modes. Future work aims to improve performance on both desktops and clsuters, and automate searches over the propagator determinant to identify critical cases of MAG parameters[a].

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I. INTRODUCTION

One common approach to unifying gravity and quantum mechanics is by formulating gravity as a perturbative quantum field theory (QFT). However, it is well known that the result for general relativity (GR) is not renormalisable [1–4]. This motivates the study of modified theories of gravity, such as Poincaré gravity, for renormalisable theories [5, 6]. Modified gravity can be defined as the class of all theories that respect diffeomorphism invariance [7, 8], of which GR is a special case. That is, in modified gravity, coordinate transforms $\partial x'^{\mu}/\partial x^{\nu}$ are invertible and differentiable, and leave the scalar action invariant. As such, modified gravity can be seen as the gauge theory of the local diffeomorphic symmetry [9–11], with the geometric covariant derivative ∇_{μ} precisely playing the role of the gauge-covariant derivative.

Metric-affine gravity (MAG) theories allow the affine connection $\Gamma_{\mu\nu}{}^{\alpha}$ to be independent of the metric $g_{\mu\nu}$ [7, 9], and is hence a gauge-fixing of the general GL(4) formulation of modified gravity. While MAG has been known to produce renormalisable theories after linearisation [12, 13], the higher derivative terms (in the second-order formulation) e.g. $h\partial^4 h$ can lead to *ghost* particles with negative kinetic energies [3, 14, 15]. In such cases the ground state is not well-defined and unitarity is violated. *Tachyons* - propagator poles with imaginary mass - imply that the field is not at the potential minimum and hence perturbations are unstable [3, 16, 17]. As such, viable QFTs will have to avoid ghosts and tachyons before renormabisability is considered.

Computer algebra [12, 18, 19] has made it possible to calculate the saturated propagator, from which ghosts and tachyons can be identified. Power-counting renormalisability can also be inferred from the powers of momentum k^{-l} in the propagator [13, 19, 20]. MAG theories with the torsion connection have been extensively investigated to produce ghost- and tachyon-free QFTs [20–24], particularly in the cases of Poincaré [19, 25–30] and Weyl [5, 6, 31–33] gauge theories. However, the general class of MAG theories have not been well studied in the literature (for attempts in very recent years see [7, 9, 34]) In particular, current research into MAG particle spectra has been restricted to eliminating either the torsion or the non-metricity *a priori* in the Lagrangian [9, 34–37]. It would hence be instructive to check the validity of such kinematic suppression by exploring whether specific cases of the MAG Lagrangian would generate vanishing torsion or non-metricity.

We developed the computer algebra program `PSALTer` to calculate the saturated propagator and particle spectra in terms of the spin-projections of the tensor fields. In spin-projection decomposition, tensor indices are projected parallel or perpendicular to a particle momentum four-vector, into various components [29, 30, 32, 38]. These components represent the irreducible spin-representations of integer-spin particles via Wigner's classification [3, 14, 39], and are hence natural to work with. In order to test the program, we studied the Einstein–Hilbert action with a general affine connection and compared against known results from the literature [9]. At the time of writing, no other non-trivial MAG theories are yet known.

The structure of the paper is as follows: In Section II, we introduce metric-affine gauge gravity and important examples: Poincaré and Weyl gauge theories. In Section III, we present spin-projection decomposition, and detail how the program computes the saturated propagator and searches for ghosts and tachyons. We then examine the particle spectra of the generalised Einstein–Hilbert action in Section IV before concluding in Section V. In this work we will use the mostly-minus signature $(+,-,-,-)$ and natural units $\hbar = c = 1$.

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[a] Except where specific reference is made to the work of others, this work is original and has not been already submitted either wholly or in part to satisfy any degree requirement at this or any other university. The project notebook can be found at [link].

II. METRIC-AFFINE GRAVITY

A. Non-renormalisability of general relativity

We study the quantisation of gravity by first looking at general relativity. Consider some coordinate frame where the metric can be expressed as a small perturbation from a flat background $g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu} \ll 1$. We *define* the matrix inverse $\eta^{\mu\nu} \equiv (\eta_{\mu\nu})^{-1}$ and raise or lower indices of h with η, e.g. $h^{\mu\nu} \equiv \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$. Then the matrix power series $g^{-1} = \eta^{-1}(1 + h\eta^{-1})^{-1}$ allows the contravariant metric to be expanded as

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\rho}h_{\rho}{}^{\nu} - h^{\mu\rho}h_{\rho}{}^{\sigma}h_{\sigma}{}^{\nu} + ... \qquad (1)$$

In this way, the perturbation $h_{\mu\nu}$ can be seen as a tensor field living on a flat spacetime, instead of having it generate a curved spacetime. This allows gravity to be flattened and incorporated into quantum field theories.

GR can be obtained variationally from the Einstein–Hilbert (EH) action $S_{\rm EH}$ [4], which has a perturbation expansion (suppressing indices for clarity) [1]

$$\begin{aligned} S_{\rm EH} &\equiv M_{\rm Pl}^2 \int \mathrm{d}^4x \, |\det(g_{\mu\nu})|^{1/2} R(\tilde{\Gamma})_{\alpha\mu}{}^{\mu\alpha} \\ &= \int \mathrm{d}^4x \, [\tilde{h}\partial^2\tilde{h} + M_{\rm Pl}^{-1}\tilde{h}(\tilde{h}\partial^2\tilde{h}) + M_{\rm Pl}^{-2}\tilde{h}^2(\tilde{h}\partial^2\tilde{h}) + ...] \end{aligned} \qquad (2)$$

The perturbation is canonically rescaled to $\tilde{h} \equiv M_{\rm Pl}h$ so that the Lagrangian has the QFT kinetic term $\tilde{h}\partial^2\tilde{h}$. Renormalisation requires that only a finite number of Feynman diagrams in a QFT are divergent, and power-counting arguments require that the interaction couplings have mass dimension ≥ 0 [2]. Hence GR is not renormalisable as the first interaction term already has $M_{\rm Pl}^{-1}$ coupling. In Section III E we present power-counting renormalisability via restrictions on the propagator.

B. The MAG action

In order to construct the general Lagrangian and action of metric-affine gravity, we first determine the tensorial content of the general metric-affine connection. The general affine connection is [7, 9, 24, 37]

$$\begin{aligned} \Gamma_{\mu\nu}{}^{\alpha} \equiv \tilde{\Gamma}_{\mu\nu}{}^{\alpha} &+ \frac{1}{2}[A_{\mu\nu}{}^{\alpha} + A^{\alpha}{}_{\mu\nu} + A^{\alpha}{}_{\nu\mu}] \\ &+ \frac{1}{2}[-Q_{\mu\nu}{}^{\alpha} + Q^{\alpha}{}_{\mu\nu} + Q^{\alpha}{}_{\nu\mu}]. \end{aligned} \qquad (3)$$

It consists of the non-tensorial Levi–Civita part

$$\tilde{\Gamma}_{\mu\nu}{}^{\alpha} \equiv \frac{1}{2}g^{\alpha\rho}[\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}], \qquad (4)$$

the anti-symmetric torsion tensor

$$A_{\mu\nu}{}^{\alpha} \equiv \Gamma_{\mu\nu}{}^{\alpha} - \Gamma_{\nu\mu}{}^{\alpha}, \qquad (5)$$

and the symmetric non-metricity tensor

$$Q_{\mu\nu\alpha} \equiv -\nabla_\alpha g_{\mu\nu} = -\partial_\alpha g_{\mu\nu} + \Gamma_{\alpha\mu}{}^{\beta}g_{\beta\nu} + \Gamma_{\alpha\nu}{}^{\beta}g_{\beta\mu}. \qquad (6)$$

Both torsion and non-metricity vanish in GR. The general curvature tensor $R(\Gamma)_{\mu\nu\rho}{}^{\alpha}$ is given by

$$R(\Gamma)_{\mu\nu\rho}{}^{\alpha} \equiv \partial_\mu\Gamma_{\nu\rho}{}^{\alpha} - \partial_\nu\Gamma_{\mu\rho}{}^{\alpha} + \Gamma_{\mu\sigma}{}^{\alpha}\Gamma_{\nu\rho}{}^{\sigma} - \Gamma_{\nu\sigma}{}^{\alpha}\Gamma_{\mu\rho}{}^{\sigma}, \qquad (7)$$

with contractions [9]:

$$R^{(34)}_{\mu\nu} \equiv R_{\mu\nu\alpha}{}^{\alpha}, \quad R^{(14)}_{\mu\nu} \equiv R_{\alpha\mu\nu}{}^{\alpha}, \quad R^{(13)}_{\mu\nu} \equiv R^{\rho}{}_{\mu\rho\nu}. \qquad (8)$$

In GR $R(\tilde{\Gamma})^{(34)}_{\mu\nu} = 0$, while $R(\tilde{\Gamma})^{(14)}_{\mu\nu} = -R(\tilde{\Gamma})^{(13)}_{\mu\nu}$ is the Ricci tensor.

The general free-theory MAG action from [9, 31, 34], preserving diffeomorphism symmetry, parity, and neglecting coupling to matter, including the Einstein–Hilbert action via the $a_0 = M_{\rm Pl}^2$ term, is

$$\begin{aligned} &S_{\rm MAG} \\ &\equiv -\frac{1}{2}\int \mathrm{d}^4x |\det(g_{\mu\nu})|^{1/2}\Big[-a_0 R_{\alpha\mu}{}^{\mu\alpha} + R^{\mu\nu\rho\alpha}\left(c_1 R_{\mu\nu\rho\alpha} + c_2 R_{\mu\nu\alpha\rho} + c_3 R_{\alpha\rho\nu\mu} + c_4 R_{\mu\alpha\rho\nu} + c_5 R_{\mu\rho\alpha\nu} + c_6 R_{\mu\rho\nu\alpha}\right) \\ &+ R^{(14)\mu\nu}\left(c_7 R^{(14)}_{\mu\nu} + c_8 R^{(14)}_{\nu\mu}\right) + R^{(13)\mu\nu}\left(c_9 R^{(13)}_{\mu\nu} + c_{10} R^{(13)}_{\nu\mu} + c_{11} R^{(14)}_{\mu\nu} + c_{12} R^{(14)}_{\nu\mu}\right) \\ &+ R^{(34)\mu\nu}\left(c_{13} R^{(34)}_{\mu\nu} + c_{14} R^{(14)}_{\mu\nu} + c_{15} R^{(13)}_{\mu\nu}\right) + c_{16}\left(R_{\alpha\mu}{}^{\mu\alpha}\right)^2 + A^{\mu\nu\alpha}\left(a_1 A_{\mu\nu\alpha} + a_2 A_{\mu\alpha\nu}\right) + a_3 A_{\mu\alpha}{}^{\alpha}A^{\mu\beta}{}_{\beta} \\ &+ Q^{\mu\nu\alpha}\left(a_4 Q_{\mu\nu\alpha} + a_5 Q_{\alpha\mu\nu}\right) + a_6 Q_{\mu}{}^{\mu\alpha}Q^{\nu}{}_{\nu\alpha} + a_7 Q_{\mu\alpha}{}^{\alpha}Q^{\mu\beta}{}_{\beta} + a_8 Q_{\mu}{}^{\mu\lambda}Q_{\lambda\alpha}{}^{\alpha} + a_9 A^{\mu\nu\alpha}Q_{\alpha\nu\mu} - A_{\lambda\alpha}{}^{\alpha}\left(a_{10} Q_{\mu}{}^{\mu\lambda} + a_{11} Q^{\lambda\beta}{}_{\beta}\right)\Big]. \end{aligned} \qquad (9)$$

The MAG action represents a minimal extension to the EH action by only being constructed out of contractions of the curvature tensor and its sub-components. It only contains terms up to $\sim R^2$ in order to generate the kinetic terms[1] of the connection fields $\sim \Gamma\partial^2\Gamma$. The action contains 28 parameters $\{a_0, a_1, ..., a_{11}, c_1, ..., c_{16}\}$, and additional gauge symmetries will impose constraints and reduce the number of free parameters [9].

C. Kinematically-suppressed example of MAG: Weyl gauge gravity

Various important theory classes can be contained and described in the metric-affine framework [41]. These are usually accomplished by pre-defining the form of the torsion or non-metricity tensor before the formulation of the Lagrangian [9, 35, 42]. In the framework of MAG, such theories are considered kinetmatically suppressed, i.e. the suppression arises from pre-determined constraints rather than from the natural evolution of a general Lagrangian [43].

It is natural to consider turning the global Poincaré symmetry of special relativity into a local symmetry, generating Poincaré gauge theory (PGT) [4, 10]. PGT uses tetrads, which are coordinate transforms $e^i{}_\mu, e_i{}^\mu$ gauge-fixed[2] to the Minkowski metric [1, 9]:

$$g_{\mu\nu} = \eta_{ij} e^i{}_\mu e^j{}_\nu, \quad e^i{}_\mu e_j{}^\mu = \delta^i_j, \quad e^i{}_\mu e_i{}^\nu = \delta^\nu_\mu. \tag{10}$$

Roman indices are used to refer to the Minkowski frame while Greek indices are used for the arbitrary 'world' frame. The tetrads allow quantities in tetrad formulation to be recast in MAG formulation (i.e. pure world indices), e.g. $V^\mu = e^\mu_i V^i$.

Weyl gauge theory (WGT) imposes the local Weyl symmetry

$$\begin{aligned} g_{\mu\nu} &\mapsto g'_{\mu\nu} = e^{2\lambda(x)} g_{\mu\nu}, \\ \varphi &\mapsto \varphi' = e^{w\lambda(x)}\varphi, \end{aligned} \tag{11}$$

where the real number w is the Weyl dimension of the arbitrary field φ (e.g. scalar, vector). It contains the Lie algebra generators of the Poincaré group and can be seen as extending Poincaré symmetry to include scale symmetry. Weyl symmetries can be motivated by experiments such as deep inelastic electron-nucleon scattering, where masses can be neglected [10]. Furthermore, mass scales can be recovered via symmetry breaking, such as via coupling to a scalar field [6, 10], or via quantum anomalies [44].

The Weyl covariant derivative is given by [33]

$$\nabla_\mu = \partial_\mu + \frac{1}{2} A^{ij}{}_\mu \Sigma(\varphi)_{ij} + w B_\mu, \tag{12}$$

with $\Sigma(\varphi)_{ij} = -\Sigma(\varphi)_{ji}$ the representation generators of the field φ. WGT inherits the tetrad e^i_μ and torsion $A^{ij}{}_\mu = -A^{ji}{}_\mu$ gauge fields from PGT, and adds the dilation gauge field B_μ. PGTs are an important example of theories with torsion, while WGTs possess non-zero metricity $Q_{\mu\nu\alpha} = 2g_{\mu\nu}B_\alpha$ [9, 10, 33]. Indeed, Weyl theory is an important example of a theory requiring the non-metricity tensor. However, this non-metricity tensor can be easily separated into the B_α and metric $g_{\mu\nu}$ fields [33], and does not require the heavy machinery of the general MAG connection and Lagrangian. WGT hence begets the question of whether the most general form of the non-metricity tensor would contain important and healthy QFTs, thus directly motivating our study of MAG theories.

III. PARTICLE SPECTRA FROM THE PROPAGATOR

A. Spin-parity decomposition

A general tensor field has many degrees of freedom, hence it is imperative to ascertain the particle modes, and hence quantum states, that can be contained within the tensor. However, the Minkowski metric poses problems for quantum unitarity since probabilities can exceed unity under Lorentz transformations [3]. For example, if a polarisation state was given as a four-vector $|\varepsilon\rangle := (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$, the Lorentz-invariant norm could be defined as $\langle\varepsilon|\varepsilon\rangle := |\varepsilon_0|^2 - |\varepsilon_1|^2 - |\varepsilon_2|^2 - |\varepsilon_3|^2$. However, the probability $P = |\langle\varepsilon|\varepsilon\rangle|^2$ could then exceed unity.

The solution is to split vectors into their time and spatial components with respect to a momentum vector $n^\mu \equiv k^\mu/k$

$$V_\mu \equiv n_\mu V^\perp + V^\parallel_{\bar{\mu}}, \tag{13}$$

where a bar over a Greek index [30] represents the spatial condition $n^\mu V^\parallel_{\bar{\mu}} = 0$. Quantum states are hence only allowed to rotate in spatial dimensions (the SO(3) group), allowing unitarity to be preserved[3] [3]. Each index of a tensor can be similarly decomposed using temporal and spatial projection operators [9, 19]

$$\Omega^\mu{}_\nu \equiv n^\mu n_\nu, \quad \Theta^\mu{}_{\bar{\nu}} \equiv \delta^\mu{}_\nu - n^\mu n_\nu. \tag{14}$$

Algebraically combining these operators would allow a tensor to be decomposed into irreducible representations (irreps) of SO(3).

Here, we would like to introduce full- and reduced-rank representations. A tensor is a direct sum of full-rank irreps with the same rank as the original tensor. Using the four-vector as an example again:

$$\begin{aligned} V_\mu &\equiv V(0^+)_\mu + V(1^-)_{\bar{\mu}} \\ &= n_\mu V(0^+) + V(1^-)_{\bar{\mu}}, \end{aligned} \tag{15a}$$

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[1] Higher derivative terms e.g. $\sim\Gamma\partial^4\Gamma$ suffer from ghosts in general, via the Ostrogradsky instability [3, 40].

[2] Hence, the tetrad formalism is also a gauge choice for diffeomorphisms and is gauge-equivalent to the MAG formulation [9].

[3] For massless on-shell particles, the four-momentum $p^\mu = (E, 0, 0, E)$ forces spatial rotations to be limited to the $x - y$ plane. Nonetheless the spatial-temporal decomposition is still valid.

with the full-rank irreps seen in the first line of Eq. (15a). From a full-rank irrep, we obtain the reduced-rank irrep, which is the smallest-rank tensor possible that contains all the degrees of freedom (d.o.f.) in the full-rank SO(3) irrep. Generally, full-rank irreps are formed from algebraic combinations of the metric and unit vector n_μ, e.g. $V(0^+)_\mu = n_\mu V(0^+)$ in Eq. (15a). A reduced-rank irrep can also be expressed in terms of the original tensor, such as

$$V(0^+) \equiv V^\perp, \quad V^{\parallel}_{\acute{\mu}} \equiv V(1^-)_{\acute{\mu}}. \tag{15b}$$

Wigner's classifcation [3, 14] allows spin-parity states J^P to be assigned to irreps and represent particles. Spin J and parity $P = (-1)^r$ are assigned to a reduced-rank-r irrep with $2J + 1$ degrees of freedom[4]. We decompose the general rank-2 tensor and the rank-3 torsion and non-metricity tensors required for MAG in Appendix A [23, 30, 45]. The spin-parity states contained in these tensors are presented in Table I.

	$Q^{(\mathrm{ts})}$	$Q^{(\mathrm{hs})}$	$A^{(\mathrm{ha})}$	$A^{(\mathrm{ta})}$
$\Omega\Omega\Omega$	$0^+_{\perp\mathrm{t}}$	-	-	-
$\Omega\Omega\Theta + \Omega\Theta\Omega + \Theta\Omega\Omega$	$1^-_{\perp\mathrm{t}}$	$1^-_{\perp\mathrm{h}}$	$1^-_{\perp}$	-
$\Omega\Theta\Theta + \Theta\Omega\Theta - (1/2)\Theta\Theta\Omega$	-	$1^+_{\perp}$	$2^+_{\perp}$, $0^+_{\perp}$	-
$(3/2)\Theta\Theta\Omega$	-	$2^+_{\perp}$, $0^+_{\perp\mathrm{h}}$	$1^+_{\perp}$	-
$\Omega\Theta\Theta + \Theta\Omega\Theta + \Theta\Theta\Omega$	$2^+_{\parallel}$, $0^+_{\parallel}$	-	-	$1^+_{\parallel}$
$\Theta\Theta\Theta$	$3^-_{\parallel}$, $1^-_{\parallel\mathrm{t}}$	$2^-_{\parallel}$, $1^-_{\parallel\mathrm{h}}$	$2^-_{\parallel}$, $1^-_{\parallel}$	$0^-_{\parallel}$
	H			ψ
$\Omega\Omega$	$0^+_{\perp}$			-
$\Omega\Theta + \Theta\Omega$	$1^-_{\perp}$			$1^-_{\perp}$
$\Theta\Theta$	$0^+_{\parallel}$, $2^+_{\parallel}$			$1^+_{\parallel}$

TABLE I. Particle content of metric-affine gauge gravity tensor connections [9], labelled by spin-parity states J^P_s. The index s is used to disambiguate between states of the same spin-parity. The non-metricity tensor splits into a totally symmetric part $Q^{(\mathrm{ts})}$ and a remaining $Q^{(\mathrm{hs})}$ hook-symmetric part; these parts remain in their subspaces under coordinate transformations. The torsion tensor similarly decomposes into total- and hook-antisymmetric parts [24]. For completeness, we include the decomposition of the rank-2 symmetric (H) and antisymmetric (ψ) tensors. The operators have fixed indices which are suppressed, following [9, 12, 20], such as $\Omega\Omega\Omega \sim \Omega^\rho{}_\mu \Omega^\sigma{}_\nu \Omega^\beta{}_\alpha$. They represent invariant subspaces which contain the particle irreps [37].

The polarisation state of the spin-parity irrep can be obtained from the full-rank representation using the projection operator identities [19] (no Einstein sum)

$$\begin{aligned} \Omega^\mu{}_\nu &\equiv \varepsilon^\mu(0^+,0)\, \varepsilon^*_\nu(0^+,0), \\ \Theta^\mu{}_\nu &\equiv \sum_m [\varepsilon^\mu(1^-,m)\, \varepsilon^*_\nu(1^-,m)], \end{aligned} \tag{16}$$

with the polarisation basis vectors

$$\begin{aligned} \varepsilon^\mu(0^+,0) &\equiv \frac{1}{k}(E,0,0,p), \\ \varepsilon^\mu(1^-,0) &\equiv \frac{1}{k}(p,0,0,E), \\ \varepsilon^\mu(1^-,\pm1) &\equiv \frac{1}{\sqrt{2}}(0,\pm1,i,0) \end{aligned} \tag{17}$$

for both on- and off-shell particle momentum $k^\mu = (E,0,0,p)$.

B. Spin projection operators and irrep contractions

In order to arrive at a scalar Lagrangian, the tensor spin-irreps would have to be contracted with each other. This is expressed using the spin-projection operator (SPO) formalism [46, 47]. For an arbitrary tensor labelled by (A) and one of its irreps labelled by s, we notate its full-rank representation as $\varphi(J^P_s)^{(A)}_{\acute{\mu}}$, where $\acute{\mu} \equiv (\mu\nu\alpha\ldots)$ is shorthand for its tensor indices [46]. Then the contraction of two irreps is denoted as (no Einstein sum)

$$\begin{aligned} &\varphi(J^P_s)^{(A)}_{\acute{\mu}} \cdot \varphi(J'^{P'}_r)^{(B)}_{\acute{\nu}} \\ &\equiv \varphi(J^P_s)^{(A)}_{\acute{\mu}} P(J^P_{rs})^{(AB)} \delta_{J'J} \delta_{P'P} \varphi(J'^{P'}_r)^{(B)}_{\acute{\nu}}, \end{aligned} \tag{18}$$

with $P(J^P_{rs})^{(AB)}$ the spin-projection operator [9, 34, 46].

PSALTer allows this tensor contraction to be expressed in terms of the reduced-rank irreps

$$\varphi(J^P_s)^{(A)}_{\acute{\mu},\mathrm{full}} \cdot \varphi(J'^{P'}_r)^{(B)}_{\acute{\nu},\mathrm{full}} \propto \varphi(J^P_s)^{(A)}_{\acute{\mu},\mathrm{red}} \cdot \varphi(J'^{P'}_r)^{(B)}_{\acute{\nu},\mathrm{red}} \delta_{J'J}, \tag{19}$$

where they are normalised such that the constant of proportionality is one when $J^P = J'^{P'}, r = s, A = B$. This method allows spin-states of different parities[5] to be contracted, by using unique contractions of the irreps and an additional Levi–Civita symbol. This was not considered in the SPO formalism of previous works [9, 37], since such terms will only arise if the Lagrangian utilises the Levi–Civita symbol. The full set of reduced-rank irrep contractions are given below, with $\odot$ the elementwise product.

[4] There is an exception for the 0^- state. Our reduced-rank representation for this state is actually a *pseudoscalar*; its reduced-rank irrep *in sensu stricto* is instead a rank-3 totally antisymmetric tensor, see Eq. (A8b) in Appendix A 2.

[5] Note that all spin-irrep contractions will be between states of the same spin J. When expanding the full- to reduced-rank irreps, the reduced-rank irreps in different spin states will vanish as there will be contractions with the metric or unit vector.

For the spin-0 sector:

$$\hat{\varphi}(0^{\pm}) \equiv \left[h(0^+_{\perp})\ \ h(0^+_{\parallel})\ \ A(0^+_{\perp})\ \ Q(0^+_{\perp\mathrm{t}})\ \ Q(0^+_{\parallel})\ \ Q(0^+_{\perp\mathrm{h}})\ \ A(0^-_{\parallel})\right]^{\top}, \tag{20a}$$

$$\hat{\varphi}(0^{\pm})^{(A)} \cdot \hat{\varphi}(0^{\pm})^{(B)} \equiv \hat{\varphi}(0^{\pm})\hat{\varphi}(0^{\pm})^{\dagger}. \tag{20b}$$

For the spin-1 sector:

$$\hat{\varphi}(1^{\pm}) \equiv \left[A(1^+_{\parallel})^{\bar{\mu}\bar{\nu}}\ \ A(1^+_{\perp})^{\bar{\mu}\bar{\nu}}\ \ Q(1^+_{\perp})^{\bar{\mu}\bar{\nu}}\ \ h(1^-_{\perp})_{\bar{\alpha}}\ \ A(1^-_{\parallel})_{\bar{\alpha}}\ \ A(1^-_{\perp})_{\bar{\alpha}}\ \ Q(1^-_{\perp\mathrm{t}})_{\bar{\alpha}}\ \ Q(1^-_{\parallel\mathrm{t}})_{\bar{\alpha}}\ \ Q(1^-_{\perp\mathrm{h}})_{\bar{\alpha}}\ \ Q(1^-_{\parallel\mathrm{h}})_{\bar{\alpha}}\right]^{\top}, \tag{21a}$$

$$\hat{\varphi}(1^{\pm})^{(A)} \cdot \hat{\varphi}(1^{\pm})^{(B)} \equiv \hat{\varphi}(1^{\pm})\hat{\varphi}(1^{\pm})^{\dagger} \odot \begin{bmatrix} \mathcal{J}_{3\times3}(1) & \mathcal{J}_{3\times7}(\epsilon^{0\bar{\mu}\bar{\nu}\bar{\alpha}}) \\ \mathcal{J}_{7\times3}(\epsilon_{0\bar{\mu}\bar{\nu}\bar{\alpha}}) & \mathcal{J}_{7\times7}(1) \end{bmatrix}. \tag{21b}$$

For the spin-2 sector:

$$\hat{\varphi}(2^{\pm}) \equiv \left[h(2^+_{\parallel})_{\bar{\mu}}{}^{\bar{\sigma}}\ \ A(2^+_{\perp})_{\bar{\mu}}{}^{\bar{\sigma}}\ \ Q(2^+_{\parallel})_{\bar{\mu}}{}^{\bar{\sigma}}\ \ Q(2^+_{\perp})_{\bar{\mu}}{}^{\bar{\sigma}}\ \ A(2^-_{\parallel})_{\bar{\mu}\bar{\nu}\bar{\alpha}}\ \ Q(2^-_{\parallel})_{\bar{\mu}\bar{\nu}\bar{\alpha}}\right]^{\top}, \tag{22a}$$

$$\hat{\varphi}(2^{\pm})^{(A)} \cdot \hat{\varphi}(2^{\pm})^{(B)} \equiv \hat{\varphi}(2^{\pm})\hat{\varphi}(2^{\pm})^{\dagger} \odot \begin{bmatrix} \mathcal{J}_{4\times4}(1) & \mathcal{J}_{4\times2}(\epsilon^{0\bar{\nu}\bar{\sigma}\bar{\alpha}}) \\ \mathcal{J}_{2\times4}(\epsilon_{0\bar{\nu}\bar{\sigma}\bar{\alpha}}) & \mathcal{J}_{2\times2}(1) \end{bmatrix}. \tag{22b}$$

For the spin-3 sector:

$$\hat{\varphi}(3^{\pm})^{(A)} \cdot \hat{\varphi}(3^{\pm})^{(B)} \equiv Q(3^-_{\parallel})^{*}_{\bar{\mu}\bar{\nu}\bar{\alpha}} Q(3^-_{\parallel})^{\bar{\mu}\bar{\nu}\bar{\alpha}}. \tag{23}$$

Here, the transpose does not alter the tensor index order, but turns lower indices into upper indices and vice-versa. The matrix $\mathcal{J}_{m\times n}(V)$ refers to a $m \times n$ matrix filled with the same element V.

C. The saturated propagator, ghosts and taychons

We now detail the process of constructing the QFT Lagrangian and action from the tensor contractions. We introduce the sources $j = W, T$ with the same form as the fields $\varphi = \Gamma, h$ in order to book-keep gauge symmetries and formulate the path-integral generating functional [3]. We follow the notation and formalism of [46], expressing sources and fields in a vector form[6] as e.g. $\hat{\varphi} = [\varphi(J^P)^{(1)}_{\acute{\mu}}, \varphi(J^P)^{(2)}_{\acute{\mu}}, ...]^{\top}$. Then the generating functional is

$$Z_0[\hat{j}] \equiv \int \mathcal{D}\hat{\varphi} \exp[i\textstyle\int \mathrm{d}^4x(\frac{1}{2}\hat{\varphi}^{\top}\mathcal{O}\hat{\varphi} - \hat{j}^{\top}\hat{\varphi})]. \tag{24}$$

The Lagrangian can be represented as [19, 46] (no Einstein sum)

$$\begin{aligned}&[\frac{1}{2}\hat{\varphi}^{\top}\mathcal{O} - \hat{j}^{\top}]\hat{\varphi} \\ &\equiv \sum_{AB}[\alpha_{AB}\varphi(J^P)^{(A)} - j(J^P)^{(A)}] \cdot \varphi(J^P)^{(B)}\end{aligned} \tag{25}$$

Gauge symmetries means that the matrix of scalars α_{AB} is singular, with some right eigenvectors $\sum_A v^{(i)}_A \alpha_{AB} = 0$. Since we choose the sources to have the same gauge symmetries as the fields, it suffices to consider the singularity of the fields, inducing source constraints [19]

$$\sum_A v^{(i)}_A j(J^P)^{(A)} = 0. \tag{26}$$

Then there are $\dim[\mathrm{Ker}(\alpha_{\mathrm{AB}})]$ fields that can be gauge fixed $\varphi(J^P)^{(A)} = 0$ such that the remaining square matrix β_{AB} is invertible[7]. However, critical cases exist [46] where the Lagrangian parameters are kinematically restrained in certain combinations to cause $\det(\beta_{AB}) = 0$. In this case, the coupling constants generate additional gauge freedoms. New source constraints need to be calculated from the α-matrix to reformulate β_{AB}.

[6] We suppress the spin-parity degeneracy index s, r for clarity.

[7] This can be done by zeroing a field only if $\mathrm{Rank}(\alpha_{AB})$ remains unchanged [19].

Defining $\hat{\phi} = \hat{\varphi} - \mathcal{O}^{-1}(k)\hat{j}(k)$ and going to the momentum basis (to turn the differential operator into an algebraic one), we obtain the action as [19]

$$S \equiv \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{2} [\hat{\phi}^\dagger(k)\mathcal{O}(k)\hat{\phi}(k) - \hat{j}^\dagger(k)\mathcal{O}^{-1}(k)\hat{j}(k)]. \quad (27)$$

Finally, the path integral becomes determined by the gauge-invariant saturated propagator

$$\begin{aligned} Z_0[\hat{j}] &\equiv \int \mathcal{D}\hat{\phi} \exp[\frac{i}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}(\hat{\phi}^\dagger \mathcal{O}\hat{\phi} - \hat{j}^\dagger \mathcal{O}^{-1}\hat{j})] \\ &= Z_0(0) \exp[\frac{1}{2i}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\Pi(k)]. \end{aligned} \quad (28)$$

After our gauge-fixing procedure, the saturated propagator is given by $\Pi(k) = \sum_{AB} (\beta^{-1})_{AB} j^*(J^P, k)^{(A)} \cdot j(J^P, k)^{(B)}$. The poles are then the (propagating) tensor particles in our gauge choice with masses $k_{\text{pole}} = m$.

The generating functional is evaluated as [19]

$$Z_0(\hat{j}) = \langle 0|0\rangle_j = \exp \int \frac{\mathrm{d}^3 \vec{k}}{(2\pi)^3} \sum_i \frac{-\mathrm{Res}[\Pi(k^2 = m_i^2)]}{2[|\vec{k}|^2 + m_i^2]^{1/2}}. \quad (29)$$

This is the probability of the tensor particle ground state and should not exceed unity. Hence a necessary criterion for unitarity is that $\mathrm{Re}\{\mathrm{Res}[\Pi(k^2 = m^2)]\} \geq 0$ for all the particles in the propagator. Tachyonic solutions $m^2 < 0$ should also be avoided. These solutions correspond to the theory not being expanded at the true vacuum and hence not perturbative [3], e.g. for the tachyonic φ^4 potential $V = \lambda(\varphi^*\varphi + m^2/2\lambda)^2$.

D. Massless propagator poles

In general, the saturated propagator can contain massless poles[8] $\sim k^{-2l}$, and these require special attention as compared to the massive poles. Poles falling faster than k^{-2} would contain ghost states, for example

$$\frac{1}{k^4} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(\frac{1}{k^2} - \frac{1}{k^2 + \varepsilon}\right) \quad (30)$$

contains a normal and ghost state [3, 46]. Hence, such poles would have to be removed. The solution to Eq. (26) is $\hat{j} = \sum_i \chi_i \hat{n}$ where $\hat{n}$ are null vectors and χ_i are d.o.f. in this null space. The null-vectors can contain singularities k^{-n}, however, these are just due to the normalisation factor of the null-vectors [46]. Hence these singularities should be removed by scaling with $(E - p)^n$ until the singularity in the null vector is removed. Defining the vector $\hat{\chi} = [\chi_1, \chi_2, ...]^\top$ the saturated propagator can be written in matrix form as $\Pi = \hat{\chi}^\dagger M \hat{\chi}$. The k^{-2l} poles can be set to zero from the largest l downwards iteratively by

$$M_{2l} = \lim_{E\to p}[k^{2l}M] \mapsto 0, \quad l > 1, \quad (31)$$

until only k^{-2} poles remain. Finally, the no-ghost criterion is achieved by requiring all eigenvalues of M_2 to be non-negative, with the number of non-zero eignvalues the d.o.f. of the propagating massless particles [9, 19, 46].

E. Propagating power-counting renormalisability

With a healthy ghost-less and tachyon-less quantum theory, we follow the power-counting renormalisability criteria and notation of [19, 46] to determine renormalisability of this QFT candidate. Consider a QFT in $d-$dimensional spacetime, with i propagating fields and their propagators going $\sim k^{-l_i}$, $p \to \infty$. The a interactions have N_{ai} legs of i particles, with coupling constants $\lambda_a k^{\delta_a}$. A Feynman diagram has loops L, I_i, E_i internal and external legs of particles i, v_a vertices of interactions a, such that the scattering amplitudes are $\sim \int \mathrm{d}^{Ld} k \prod_{a,i} k^{v_a \delta_a - l_i I_i}$.

The integrand goes by Λ^D with momentum cutoff Λ and the superficial degree of divergence

$$D = d + \frac{1}{2}\sum_{a,i}[(d - l_i)(v_a N_{ai} - E_i) + 2v_a(\delta_a - d)]. \quad (32)$$

Renormalisable theories have finite numbers of superficially divergent diagrams[9], i.e. putting an upper bound on D. This allows the propagator power to be restricted by[10]

$$\delta_a - d + \frac{1}{2}\sum_i (d - l_i)N_{ai} \leq 0 \quad \forall a. \quad (33)$$

For the general MAG action, the kinetic term for h includes $\sim h\partial^4 h$ terms up to quadratic order exapnsion, as discussed in Section II A. Hence $l_h \geq 4$ for the metric field, while $l_{A,Q} \geq 2$ for the connection fields [12, 19, 20].

The propagator power-counting analysis is carried out for theories with β-matrices (see Section III C) where there is no mixing between the different fields h, A, Q [33, 46]. In these cases, non-propagating fields, i.e. where the propagator $\sim k^0$, can be allowed as they would not couple with the other fields. This relaxed power-counting criterion is known as *propagating power-counting renormalisability* (PPCR)[11].

[8] Odd powers of k can be incorporated without loss of generality, e.g. $k^{-3} = k/k^4$ [46].

[9] For $D \geq 0$ divergence can be worse than Λ^D, $\ln\Lambda$ due to divergent subdiagrams. The divergence can be reduced due to symmetries causing terms to cancel [2], or by utilising other expansions [48]. While there are technically infinite diagrams that diverge in renormalisable theories, they only diverge due to a finite number of subdiagrams (the *one-point irreducible* diagrams) due to the BPHZ theorem [2, 3].

[10] When $l_i = d$, a subtlety arises as there are infinite E_i for fixed D in Eq. (32), although we do not further consider this problem.

[11] Note that the PPCR criterion is gauge-dependent as β is gauge-fixed. However, the overall renormalisability should not be affected as the saturated propagator is gauge independent [5].

Cases of WGT and PGT theories have been found to suit the PPCR condition [5, 33] and this suggests that renormalisable MAG theories are possible, at least at the quadratic free-theory level. This is already a marked improvement from the EH action, where the even lowest-order h-h interaction is non-renormalisable.

IV. MODIFIED EINSTEIN–HILBERT ACTION

We developed and implemented the computer algebra program `PSALTer`, utilising the `xAct` library [18, 49–53] and running on Wolfram `Mathematica`, in order to handle the general MAG Lagrangian with 24 particles in the general theory (see Table I). After expanding the MAG action in Eq. (9) to the required order, the `PSALTer` program generates the saturated propagator and source constraints, and identifies the ghosts and tachyons as detailed in Section III.

As a demonstration and check of the `PSALTer` code, we examine the modified EH action $S_{\text{EH}'}$, where the Levi–Civita connection is replaced with the general connection $\Gamma_{\mu\nu}{}^{\alpha}$ in Eq. (2). The linearised Lagrangian with indices raised and lowered by η, e.g. $\Gamma_{\mu\nu\alpha} = \Gamma_{\mu\nu}{}^{\beta}\eta_{\beta\alpha}$, is

$$\begin{aligned} L_{\text{EH}'} &\equiv -M_{\text{Pl}}^2 |\det(g_{\mu\nu})|^{1/2} R(\Gamma)_{\alpha\mu}{}^{\mu\alpha} \\ &= M_{\text{Pl}}^2 [\Gamma^{\mu\nu\alpha}\Gamma_{\alpha\mu\nu} - \Gamma_{\alpha\lambda}{}^{\alpha}\Gamma_{\mu}{}^{\mu\lambda} + h^{\mu\nu}\partial_\alpha\Gamma_{\mu\nu}{}^{\alpha} \\ &\quad - h^{\mu\nu}\partial_\mu\Gamma_{\alpha\nu}{}^{\alpha} + \frac{1}{2}h^{\mu}{}_{\mu}\partial^{\nu}\Gamma_{\alpha\nu}{}^{\alpha} - \frac{1}{2}h^{\mu}{}_{\mu}\partial_\alpha\Gamma_{\nu}{}^{\nu\alpha}]. \end{aligned} \tag{34}$$

This is invariant under the infinitesimal diffeomorphism [9] $\delta x^\mu = \varepsilon\xi^\mu$,

$$\delta h_{\mu\nu} = \varepsilon[\partial_\mu\xi_\nu + \partial_\nu\xi_\mu], \quad \delta\Gamma_{\mu\nu}{}^{\alpha} = \varepsilon\partial_\mu\partial_\nu\xi^\alpha \tag{35}$$

as the volume element d^4x is unchanged. A non-trivial projective symmetry also exists [9, 41] for the Lagrangian with

$$\delta h_{\mu\nu} = 0, \qquad \delta\Gamma_{\mu\nu}{}^{\alpha} = \varepsilon\lambda_\mu\delta_\nu{}^{\alpha}. \tag{36}$$

As the only kinetic terms are $\sim h\partial^2 h$ from the Levi–Civita part of the general connection, we expect only two propagating graviton modes[12]. That is, a general EH action is restricted to produce the same physics as the standard EH action, up to linear-h QFT [9].

Two propagating massless polarisation states were identified by `PSALTer` with residues $14k^2/M_{\text{Pl}}^2$ and $10k^2/M_{\text{Pl}}^2$, with a trivial unitary bound $M_{\text{Pl}}^2 > 0$, $k^2 > 0$. For the modified EH action, the corresponding β-matrices are given below:

$$\begin{aligned} \beta^{-1}(0^\pm)_{AB} &= M_{\text{Pl}}^{-2}(16+3k^2)^{-1} \\ &\cdot \begin{bmatrix} \frac{18k^2}{16+3k^2} & -2\sqrt{3} & -i\sqrt{6}k & \frac{36ik}{16+3k^2} & \frac{-4ik(19+3k^2)}{16+3k^2} & \frac{2i\sqrt{2}k(10+3k^2)}{16+3k^2} \\ -2\sqrt{3} & \frac{-2(16+3k^2)}{k^2} & \frac{-i\sqrt{2}(16+3k^2)}{k} & \frac{-4i\sqrt{3}}{k} & \frac{4i}{\sqrt{3}k} & \frac{4i\sqrt{2/3}}{k} \\ i\sqrt{6}k & \frac{i\sqrt{2}(16+3k^2)}{k} & \cdot & -2\sqrt{6} & 2\sqrt{2/3} & \frac{4}{\sqrt{3}} \\ \frac{-36ik}{16+3k^2} & \frac{4i\sqrt{3}}{k} & -2\sqrt{6} & \frac{72}{16+3k^2} & \frac{-8(19+3k^2)}{16+3k^2} & \frac{4\sqrt{2}(10+3k^2)}{16+3k^2} \\ \frac{4ik(19+3k^2)}{16+3k^2} & \frac{-4i}{\sqrt{3}k} & 2\sqrt{2/3} & \frac{-8(19+3k^2)}{16+3k^2} & \frac{8(35+6k^2)}{3(16+3k^2)} & \frac{4\sqrt{2}(22+3k^2)}{3(16+3k^2)} \\ \frac{-2i\sqrt{2}k(10+3k^2)}{16+3k^2} & \frac{-4i\sqrt{2/3}}{k} & \frac{4}{\sqrt{3}} & \frac{4\sqrt{2}(10+3k^2)}{16+3k^2} & \frac{4\sqrt{2}(22+3k^2)}{3(16+3k^2)} & \frac{16(13+3k^2)}{3(16+3k^2)} \end{bmatrix} \oplus M_{\text{Pl}}^{-2}\mathbf{1}_1, \end{aligned} \tag{37a}$$

[12] All other spin-parity modes of $h_{\mu\nu}$ will not propagate due to gauge-fixing i.e. from diffeomorphic symmetry [2, 3].

$$\beta^{-1}(1^{\pm})_{AB} = M_{\text{Pl}}^{-2}\begin{bmatrix} \cdot & \sqrt{2} & \cdot \\ \sqrt{2} & -1 & \cdot \\ \cdot & \cdot & -2 \end{bmatrix}$$

$$\oplus\, M_{\text{Pl}}^{-2}(2+k^2)^{-1}\begin{bmatrix} \frac{-k^2}{2+k^2} & -i\sqrt{2}k & \frac{-ik(4+k^2)}{2(2+k^2)} & \frac{ik(6+5k^2)}{2\sqrt{6}(2+k^2)} & \frac{-ik\sqrt{5/6}k}{2} & \frac{ik(3+k^2)}{\sqrt{3}(2+k^2)} & \frac{-ik}{\sqrt{6}} \\ i\sqrt{2}k & \cdot & \frac{-(4+k^2)}{\sqrt{2}} & \frac{k^2}{\sqrt{3}} & \cdot & \frac{-k^2}{\sqrt{6}} & \cdot \\ \frac{ik(4+k^2)}{2(2+k^2)} & \frac{-(4+k^2)}{\sqrt{2}} & \frac{-(4+k^2)^2}{2+k^2} & \frac{k^2(2-k^2)}{4\sqrt{6}(2+k^2)} & \frac{\sqrt{5/6}k^2}{4} & \frac{-k^2(5+2k^2)}{2\sqrt{3}(2+k^2)} & \frac{k^2}{2\sqrt{6}} \\ \frac{-ik(6+5k^2)}{2\sqrt{6}(2+k^2)} & \frac{k^2}{\sqrt{3}} & \frac{k^2(2-k^2)}{4\sqrt{6}(2+k^2)} & \frac{76+52k^2+3k^4}{24(2+k^2)} & \frac{\sqrt{5}(10+3k^2)}{24} & \frac{2-k^2}{6\sqrt{2}(2+k^2)} & \frac{1}{4+16/(2+3k^2)} \\ \frac{ik\sqrt{5/6}k}{2} & \cdot & \frac{\sqrt{5/6}k^2}{4} & \frac{\sqrt{5}(10+3k^2)}{24} & \frac{-1}{24} & \frac{\sqrt{5/2}}{6} & \frac{\sqrt{5}}{12} \\ \frac{-ik(3+k^2)}{\sqrt{3}(2+k^2)} & \frac{-k^2}{\sqrt{6}} & \frac{-k^2(5+2k^2)}{2\sqrt{3}(2+k^2)} & \frac{2-k^2}{6\sqrt{2}(2+k^2)} & \frac{\sqrt{5/2}}{6} & \frac{17+14k^2+3k^4}{-3(2+k^2)} & \frac{7+3k^2}{3\sqrt{2}} \\ \frac{ik}{\sqrt{6}} & \cdot & \frac{k^2}{2\sqrt{6}} & \frac{1}{4+16/(2+3k^2)} & \frac{\sqrt{5}}{12} & \frac{7+3k^2}{3\sqrt{2}} & \frac{-5}{6} \end{bmatrix}, \tag{37b}$$

$$\beta^{-1}(2^{\pm})_{AB} = M_{\text{Pl}}^{-2}\begin{bmatrix} \frac{4}{k^2} & \frac{2i\sqrt{2}}{k} & \frac{-2i}{\sqrt{3}k} & \frac{-2i\sqrt{2/3}}{k} \\ \frac{-2i\sqrt{2}}{k} & \cdot & -\sqrt{2/3} & \frac{-2}{\sqrt{3}} \\ \frac{2i}{\sqrt{3}k} & -\sqrt{2/3} & 4/3 & \sqrt{2}/3 \\ \frac{2i\sqrt{2/3}}{k} & \frac{-2}{\sqrt{3}} & \sqrt{2}/3 & -4/3 \end{bmatrix} \oplus (-2M_{\text{Pl}}^{-2})\mathbf{1}_2, \tag{37c}$$

$$\beta^{-1}(3^{\pm})_{AB} = M_{\text{Pl}}^{-2}. \tag{38}$$

The direct sum $\oplus$ generates a block-diagonal matrix starting with the leftmost matrix at the top left of the resultant direct sum, whereas $\mathbf{1}_n$ is the $n \times n$ identity matrix. The saturated propagator is

$$\Pi(k) = \frac{2}{M_{\text{Pl}}^2 k^2}[-h(0^+_{\parallel})^* h(0^+_{\parallel}) + 2h(2^+_{\parallel})^*_{\mu\nu} h(2^+_{\parallel})^{\mu\nu}] \tag{39}$$

and allows identification of the two propagating modes as gravitons[13], with non-PPCR dependence $\sim k^{-2}$.

The source constraints are calculated to be[14]:

$$k[W_{0\alpha}{}^{\alpha} - W_{000}] - 2iT_{00} = 0, \tag{40a}$$

$$kW_{000} + 2iT_{00} = 0, \tag{40b}$$

$$\begin{aligned} &k[W_{\mu\alpha}{}^{\alpha} - n_{\mu}W_{0\alpha}{}^{\alpha} + 2W_{00\mu} - 2n_{\mu}W_{000}] \\ &\qquad + 4i[T_{\mu 0} - n_{\mu}T_{00}] = 0, \end{aligned} \tag{40c}$$

$$k[W_{00\mu} - n_{\mu}W_{000}] + 2i[T_{\mu 0} - n_{\mu}T_{00}] = 0. \tag{40d}$$

Here a 0 index refers to contraction with the unit vector, e.g.$V_0 \equiv V_{\mu}n^{\mu}$. Eqs. (40a) to (40d) can be combined to give source constraints

$$W_{\mu\alpha}{}^{\alpha} = 0, \quad k[W_{00\mu} - n_{\mu}W_{0\alpha}{}^{\alpha}] + 2iT_{\mu 0} = 0. \tag{41}$$

The first constraint corresponds to the projective transformation, while the second corresponds to the diffeomorphism, reproducing Percacci and Sezgin [9, Eqs. 2.19, 4.10].

V. CONCLUDING REMARKS

In this paper, we detailed the processes used to obtain the particle spectra of MAG theories in our implemented code `PSALTer`. In particular, we have calculated the spin-parity irreps for the non-metricity tensor. The Einstein–Hilbert action

[13] Gauge-fixing of our procedure results in the common $(2^+, \pm 2)$ graviton polarisations not being selected.

[14] Since the sources W, T couple to the fields $\sim W\Gamma + Th$, they have mass dimensions $3, 4$ respectively.

with general connection was shown to reduce to GR, with no propagating torsion or non-metricity modes. This behaviour is expected due to the lack of kinetic torsion or non-metric terms, and we also replicate known source constraints in [9, 34].

Our package will be able to calculate the spectrum of any linearised parity-preserving MAG by specifying the parameters of the MAG Lagrangian in Eq. (9). It is again important to note that we have limited our study to the EH action as this is the only-known non-trival MAG theory at the time of writing. In the case of kinematically suppressed torsion, the literature has been extend by novel cases [9, 12, 37], however, the authors note that the general MAG action is too complex for immediate analysis. Our results show otherwise, and we believe our program allows for the calculation accross the landscape of possible MAG models to be automated.

In future iterations of the program, we intend to generate the polarisation states of propagating particles, and also to automate the process of finding gauge and critical-case constraints. It would also be useful to be able to generate Feynman diagrams from the saturated propagator. Work needs to be done to extend the PPCR criterion to interpret mixing terms of the fields in the β-matrix [46]. While gauge constraints have been historically thought to make propagating spin-3 fields unlikely [3, 14], examples of MAG theories with propagating spin-3 fields have been found [37, 54]. However these theories are only gauge-symmetric in the linear order [9], and it would be instructive to search for MAG theories that might be gauge-symmetric even in the full non-linear regime.

VI. ACKNOWLEDGEMENTS

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[1] A. Zee, *Einstein Gravity in a Nutshell*, In a Nutshell (Princeton University Press, Princeton, 2013).
[2] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, USA, 1995).
[3] M. D. Schwartz, *Quantum Field Theory and the Standard Model*, 1st ed. (Cambridge University Press, 2013).
[4] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).
[5] Y.-C. Lin, M. P. Hobson, and A. N. Lasenby, Physical Review D **101**, 064038 (2020).
[6] A. N. Lasenby and M. P. Hobson, Journal of Mathematical Physics **57**, 092505 (2016).
[7] J. B. Jimenez, L. Heisenberg, and T. S. Koivisto, "The Geometrical Trinity of Gravity," (2019), arxiv:1903.06830 [gr-qc, physics:hep-th].
[8] S. Capozziello, A. Finch, J. L. Said, and A. Magro, The European Physical Journal C **81**, 1141 (2021), arxiv:2108.03075 [astro-ph, physics:gr-qc].
[9] R. Percacci and E. Sezgin, Physical Review D **101**, 084040 (2020).
[10] M. Blagojević, *Gravitation and Gauge Symmetries*, Series in High Energy Physics, Cosmology and Gravitation (Inst. of Physics Publ, Bristol, 2002).
[11] F. Gronwald, International Journal of Modern Physics D (2012), 10.1142/S0218271897000157.
[12] E. Sezgin and P. van Nieuwenhuizen, Physical Review D **21**, 3269 (1980).
[13] K. S. Stelle, Physical Review D **16**, 953 (1977).
[14] S. Weinberg, *The Quantum Theory of Fields: Volume 1: Foundations*, Vol. 1 (Cambridge University Press, Cambridge, 1995).
[15] F. Sbisà, European Journal of Physics **36**, 015009 (2015).
[16] A. Zee, *Quantum Field Theory in a Nutshell* (Princeton University Press, Princeton, N.J, 2003).
[17] G. Feinberg, Physical Review **159**, 1089 (1967).
[18] W. Barker, "Hamiltonian Gauge Gravity Surveyor (HiGGS)," (2022).
[19] Y.-C. Lin, *Ghost and Tachyon Free Gauge Theories of Gravity*, Ph.D. thesis, University of Cambridge (2020).
[20] D. E. Neville, Physical Review D **21**, 867 (1980).
[21] T. W. B. Kibble, Journal of Mathematical Physics **2**, 212 (1961).
[22] K. Hayashi and T. Shirafuji, Progress of Theoretical Physics **64**, 1435 (1980).
[23] K. Hayashi and T. Shirafuji, Progress of Theoretical Physics **Vol. 64** (1980).
[24] E. Alvarez, J. Anero, S. Gonzalez-Martin, and R. Santos-Garcia, The European Physical Journal C **78**, 794 (2018).
[25] M. Blagojević and I. A. Nikolić, Physical Review D **28**, 2455 (1983).
[26] G. K. Karananas, Classical and Quantum Gravity **32**, 055012 (2015).
[27] Y. N. Obukhov, "Poincar\'e gauge gravity primer," (2022), arxiv:2206.05205 [gr-qc, physics:hep-th].
[28] I. A. Nikolić, Physical Review D **30**, 2508 (1984).
[29] H.-j. Yo and J. M. Nester, International Journal of Modern Physics D **08**, 459 (1999), arxiv:gr-qc/9902032.
[30] H. J. Yo and J. M. Nester, International Journal of Modern Physics D **11**, 747 (2002), arxiv:gr-qc/0112030.
[31] F. W. Hehl, J. McCrea, E. W. Mielke, and Y. Ne'eman, Physics Reports **258**, 1 (1995).
[32] P. van Nieuwenhuizen, Nuclear Physics B **60**, 478 (1973).
[33] Y.-C. Lin, M. P. Hobson, and A. N. Lasenby, Physical Review D **104**, 024034 (2021).
[34] C. Marzo, Physical Review D **106**, 024045 (2022).
[35] A. Jiménez-Cano and F. J. M. Torralba, Journal of Cosmology and Astroparticle Physics **2022**, 044 (2022).
[36] M. Piva, Physical Review D **105**, 045006 (2022).
[37] C. Marzo, Physical Review D **105**, 065017 (2022).
[38] K. J. Barnes, Journal of Mathematical Physics **6**, 788 (1965).
[39] M. Maggiore, *A Modern Introduction to Quantum Field Theory*, Oxford Master Series in Physics No. 12 (Oxford University Press, Oxford ; New York, 2005).

[40] H. Motohashi and T. Suyama, Physical Review D **91**, 085009 (2015).
[41] D. Iosifidis and K. Pallikaris, "Describing metric-affine theories anew: Alternative frameworks, examples and solutions," (2023), arxiv:2301.11364 [gr-qc, physics:hep-th].
[42] D. Iosifidis, R. Myrzakulov, L. Ravera, G. Yergaliyeva, and K. Yerzhanov, Physics of the Dark Universe **37**, 101094 (2022).
[43] V. Vitagliano, T. P. Sotiriou, and S. Liberati, Annals of Physics **326**, 1259 (2011).
[44] M. J. Duff, Classical and Quantum Gravity **11**, 1387 (1994).
[45] A. Aurilia and H. Umezawa, Physical Review **182**, 1682 (1969).
[46] Y.-C. Lin, M. P. Hobson, and A. N. Lasenby, Physical Review D **99**, 064001 (2019).
[47] L. Buoninfante, "Ghost and singularity free theories of gravity," (2016), arxiv:1610.08744 [gr-qc].
[48] G. Parisi, Nuclear Physics B **100**, 368 (1975).
[49] J. M. Martin-Garcia, R. Portugal, and L. R. U. Manssur, Computer Physics Communications **177**, 640 (2007), arxiv:0704.1756 [gr-qc, physics:hep-th].
[50] J. M. Martín-García, D. Yllanes, and R. Portugal, Computer Physics Communications **179**, 586 (2008).
[51] J. M. Martín-García, Computer Physics Communications **179**, 597 (2008).
[52] T. Nutma, Computer Physics Communications **185**, 1719 (2014).
[53] W. E. V. Barker, "Supercomputers against strong coupling in gravity with curvature and torsion," (2022), arxiv:2206.00658 [gr-qc, physics:physics].
[54] P. Baekler, N. Boulanger, and F. W. Hehl, Physical Review D **74**, 125009 (2006).

Appendix A: Spin-parity decompositions

1. General rank-2 tensor

The notation and projection operators are adapted from [9, 46]. Spin projection can also be performed on tetrads. Defining the tetrad perturbation as $e^i{}_\mu \equiv \delta^i{}_\mu + f^i{}_\mu$ [26], we can generate the tetrad perturbation tensor $F_{\mu\nu} = f^i{}_\mu \delta^j{}_\nu \eta_{ij}$. The metric perturbation is then given by

$$h_{\mu\nu} = F_{\mu\nu} + F_{\nu\mu} + F_{\mu\rho}F^\rho{}_\nu\,. \tag{A1}$$

Limiting to the linear terms requires that only the symmetric part of F propagates, while taking the quadratic part releases this restriction. The tetrad perturbation tensor can be decomposed into the symmetric and antisymmetric tensor parts [47]:

$$F_{\mu\nu} \equiv \frac{1}{2}H_{\mu\nu} + \frac{1}{2}\psi_{\mu\nu}, \tag{A2a}$$

$$H_{\mu\nu} = F_{\mu\nu} + F_{\nu\mu}, \tag{A2b}$$

$$\psi_{\mu\nu} = F_{\mu\nu} - F_{\nu\mu}. \tag{A2c}$$

The antisymmetric part admits the spin decomposition $\psi_{\mu\nu} = \sum_{J^P_s} \psi(J^P_s)_{\mu\nu}$:

$$\begin{aligned}\psi(1^-_\perp)_{\mu\nu} &\equiv [\Omega^\rho{}_\mu\Theta^\sigma{}_{\bar\nu} + \Theta^\rho{}_{\bar\mu}\Omega^\sigma{}_\nu]\psi_{\rho\sigma}\\ &= \frac{1}{\sqrt{2}}[n_\mu\psi(1^-_\perp)_{\bar\nu} - n_\nu\psi(1^-_\perp)_{\bar\mu}],\end{aligned} \tag{A3a}$$

$$\psi(1^+_\parallel)_{\mu\nu} \equiv \Theta^\rho{}_{\bar\mu}\Theta^\sigma{}_{\bar\nu}\psi_{\rho\sigma} = \psi(1^+_\parallel)_{\bar\mu\bar\nu}, \tag{A3b}$$

where the labelling index $s = \perp, \parallel$ refers to whether the state is from the temporal or spatial splitting of the first index μ of $F_{\mu\nu}$, which contains information of the tetrad $f^i{}_\mu$. The reduced-rank spin-irreps are

$$\psi(1^-_\perp)_{\bar\mu} \equiv \sqrt{2}\Theta^\rho{}_{\bar\mu}\psi_{0\rho} = \sqrt{2}\psi_{0\mu}, \tag{A4a}$$

$$\psi(1^+_\parallel)_{\bar\mu\bar\nu} \equiv \psi_{\mu\nu} - n_\mu\psi_{0\nu} + n_\nu\psi_{0\mu}. \tag{A4b}$$

Following the same notation system, the symmetric part $H_{\mu\nu}$ has the following decomposition:

$$\begin{aligned}H(0^+_\perp)_{\mu\nu} &\equiv \Omega^\rho{}_\mu\Omega^\sigma{}_\nu H_{\rho\sigma}\\ &= n_\mu n_\nu H(0^+_\perp),\end{aligned} \tag{A5a}$$

$$\begin{aligned}H(1^-_\perp)_{\mu\nu} &\equiv [\Omega^\rho{}_\mu\Theta^\sigma{}_{\bar\nu} + \Theta^\rho{}_{\bar\mu}\Omega^\sigma{}_\nu]H_{\rho\sigma}\\ &= \frac{1}{\sqrt{2}}[n_\mu H(1^-_\perp)_{\bar\nu} + n_\nu H(1^-_\perp)_{\bar\mu}],\end{aligned} \tag{A5b}$$

$$\begin{aligned}H(0^+_\parallel)_{\mu\nu} &\equiv \frac{1}{3}\Theta_{\bar\mu\bar\nu}\Theta^{\rho\sigma}H_{\rho\sigma}\\ &= \frac{1}{\sqrt{3}}\Theta_{\bar\mu\bar\nu}H(0^+_\parallel),\end{aligned} \tag{A5c}$$

$$\begin{aligned}H(2^+_\parallel)_{\mu\nu} &\equiv [\Theta^\rho{}_{\bar\mu}\Theta^\sigma{}_{\bar\nu} - \frac{1}{3}\Theta_{\bar\mu\bar\nu}\Theta^{\rho\sigma}]H_{\rho\sigma}\\ &= H(2^+_\parallel)_{\bar\mu\bar\nu}.\end{aligned} \tag{A5d}$$

Note that the metric tensor $h_{\mu\nu}$ has the same decomposition structure since this decomposition is valid for any symmetric rank-2 tensor. The reduced-rank spin-irreps are:

$$H(0^+_\perp) \equiv H_{00}, \tag{A6a}$$

$$H(1^-_\perp)_{\bar\mu} \equiv \sqrt{2}\Theta^\rho{}_{\bar\mu}H_{0\rho} = \sqrt{2}[H_{0\mu} - n_\mu H_{00}], \tag{A6b}$$

$$H(0^+_\parallel) \equiv \frac{1}{\sqrt{3}}\Theta^{\rho\sigma}H_{\rho\sigma} = \frac{1}{\sqrt{3}}[H^\mu{}_\mu - H_{00}], \tag{A6c}$$

the irrep $H(2^+_\parallel)_{\bar\mu\bar\nu}$ is expressed in Eq. (A5d).

2. Antisymmetric rank-3 torsion

The antisymmetric torsion tensor $A_{\mu\nu\alpha} \equiv A^{(\mathrm{ta})}_{\mu\nu\alpha} + A^{(\mathrm{ha})}_{\mu\nu\alpha}$ can be decomposed into a totally-antisymmetric part and a remnant hook-antisymmetric part

$$A^{(\mathrm{ta})}_{\mu\nu\alpha} = \frac{1}{3}[A_{\mu\nu\alpha} + A_{\alpha\mu\nu} - A_{\alpha\nu\mu}], \tag{A7a}$$

$$A^{(\mathrm{ha})}_{\mu\nu\alpha} = \frac{1}{3}[2A_{\mu\nu\alpha} - A_{\alpha\mu\nu} + A_{\alpha\nu\mu}]. \tag{A7b}$$

These parts will remain in their respective subspaces under coordinate transformations. The labelling index s now refers to spatial and temporal splitting of the third index α which contain the 'world' information in the PGT and WGT connection $A_{ij}{}^{\alpha}$. The decomposition of $A_{\mu\nu\alpha}$ into spin irreps is given as such:

$$\begin{aligned} A(1^+_\parallel)_{\mu\nu\alpha} &\equiv [\Theta^\rho{}_{\bar\mu}\Theta^\sigma{}_{\bar\nu}\Omega^\beta{}_\alpha + \Theta^\rho{}_{\bar\mu}\Omega^\sigma{}_\nu\Theta^\beta{}_{\bar\alpha} + \Omega^\rho{}_\mu\Theta^\sigma{}_{\bar\nu}\Theta^\beta{}_{\bar\alpha}]A^{(\mathrm{ta})}_{\rho\sigma\beta} \\ &= \frac{1}{\sqrt{3}}[A(1^+_\parallel)_{\bar\mu\bar\nu}n_\alpha + A(1^+_\parallel)_{\bar\alpha\bar\mu}n_\nu - A(1^+_\parallel)_{\bar\alpha\bar\nu}n_\mu], \end{aligned} \tag{A8a}$$

$$\begin{aligned} A(0^-_\parallel)_{\mu\nu\alpha} &\equiv [\Theta^\rho{}_{\bar\mu}\Theta^\sigma{}_{\bar\nu}\Theta^\beta{}_{\bar\alpha}]A^{(\mathrm{ta})}_{\rho\sigma\beta} \\ &= \frac{1}{\sqrt{6}}\epsilon_{0\bar\mu\bar\nu\bar\alpha}A(0^-_\parallel), \end{aligned} \tag{A8b}$$

$$\begin{aligned} A(1^-_\perp)_{\mu\nu\alpha} &\equiv [\Omega^\rho{}_\mu\Theta^\sigma{}_{\bar\nu}\Omega^\beta{}_\alpha + \Theta^\rho{}_{\bar\mu}\Omega^\sigma{}_\nu\Omega^\beta{}_\alpha]A^{(\mathrm{ha})}_{\rho\sigma\beta} \\ &= \frac{1}{\sqrt{2}}n_\alpha[-n_\mu A(1^-_\perp)_{\bar\nu} + n_\nu A(1^-_\perp)_{\bar\mu}], \end{aligned} \tag{A8c}$$

$$\begin{aligned} A(1^+_\perp)_{\mu\nu\alpha} &\equiv \frac{3}{2}\Theta^\rho{}_{\bar\mu}\Theta^\sigma{}_{\bar\nu}\Omega^\beta{}_\alpha A^{(\mathrm{ha})}_{\rho\sigma\beta} \\ &\quad - \frac{1}{\sqrt{6}}[A(1^+_\perp)_{\bar\mu\bar\nu}n_\alpha + A(1^+_\perp)_{\bar\alpha\bar\mu}n_\nu - A(1^+_\perp)_{\bar\alpha\bar\nu}n_\mu] \\ &= \frac{1}{\sqrt{6}}[2A(1^+_\perp)_{\bar\mu\bar\nu}n_\alpha - A(1^+_\perp)_{\bar\alpha\bar\mu}n_\nu + A(1^+_\perp)_{\bar\alpha\bar\nu}n_\mu], \end{aligned} \tag{A8d}$$

$$\begin{aligned} &A(2^+_\perp)_{\mu\nu\alpha} + A(0^+_\perp)_{\mu\nu\alpha} \\ &\equiv [\Omega^\rho{}_\mu\Theta^\sigma{}_{\bar\nu}\Theta^\beta{}_{\bar\alpha} + \Theta^\rho{}_{\bar\mu}\Omega^\sigma{}_\nu\Theta^\beta{}_{\bar\alpha} - \frac{1}{2}\Theta^\rho{}_{\bar\mu}\Theta^\sigma{}_{\bar\nu}\Omega^\beta{}_\alpha]A^{(\mathrm{ha})}_{\rho\sigma\beta} \\ &\quad + \frac{1}{\sqrt{6}}[A(1^+_\perp)_{\bar\mu\bar\nu}n_\alpha + A(1^+_\perp)_{\bar\alpha\bar\mu}n_\nu - A(1^+_\perp)_{\bar\alpha\bar\nu}n_\mu] \\ &= \frac{1}{\sqrt{2}}[-A(2^+_\perp)_{\bar\alpha\bar\mu}n_\nu + A(2^+_\perp)_{\bar\alpha\bar\nu}n_\mu] \\ &\quad + \frac{1}{\sqrt{6}}[-\Theta_{\bar\alpha\bar\mu}n_\nu + \Theta_{\bar\alpha\bar\nu}n_\mu]A(0^+_\perp), \end{aligned} \tag{A8e}$$

$$\begin{aligned} A(1^-_\parallel)_{\mu\nu\alpha} &\equiv \frac{1}{2}[\Theta_{\bar\alpha\bar\mu}\Theta^\sigma{}_{\bar\nu}\Theta^{\rho\beta} + \Theta_{\bar\alpha\bar\nu}\Theta^\rho{}_{\bar\mu}\Theta^{\sigma\beta}]A^{(\mathrm{ha})}_{\rho\sigma\beta} \\ &= \frac{1}{2}[-\Theta_{\bar\alpha\bar\mu}A(1^-_\parallel)_{\bar\nu} + \Theta_{\bar\alpha\bar\nu}A(1^-_\parallel)_{\bar\mu}], \end{aligned} \tag{A8f}$$

$$\begin{aligned} A(2^-_\parallel)_{\mu\nu\alpha} &\equiv [\Theta^\rho{}_{\bar\mu}\Theta^\sigma{}_{\bar\nu}\Theta^\beta{}_{\bar\alpha} \\ &\quad - \frac{1}{2}(\Theta_{\bar\alpha\bar\mu}\Theta^\sigma{}_{\bar\nu}\Theta^{\rho\beta} + \Theta_{\bar\alpha\bar\nu}\Theta^\rho{}_{\bar\mu}\Theta^{\sigma\beta})]A^{(\mathrm{ha})}_{\rho\sigma\beta} \\ &= A(2^-_\parallel)_{\bar\mu\bar\nu\bar\alpha}. \end{aligned} \tag{A8g}$$

Note that the spin-irrep $A(0^-_\parallel)$ is actually a *pseudotensor* since the Levi-Civita symbol $\epsilon_{0\bar\mu\bar\nu\bar\alpha}$ is a pseudotensor [1]. As such, the tensorial representation of the spin-irrep is $A(0^-_\parallel)_{\mu\nu\alpha}$ hence the spin-irrep has spin-parity 0^-. The reduced-rank spin-irreps are:

$$A(1^+_\parallel)_{\bar\mu\bar\nu} \equiv \sqrt{3}\Theta^\rho{}_{\bar\mu}\Theta^\sigma{}_{\bar\nu}A^{(\mathrm{ta})}_{\rho\sigma 0}, \tag{A9a}$$

$$A(0^-_\parallel) \equiv \frac{1}{\sqrt{6}}\epsilon^{0\rho\sigma\beta}A(0^-_\parallel)_{\rho\sigma\beta}, \tag{A9b}$$

$$A(1^-_\perp)_{\bar\mu} \equiv \sqrt{2}\Theta^\rho{}_{\bar\mu}A_{\rho 00}, \tag{A9c}$$

$$A(1^+_\perp)_{\bar\mu\bar\nu} \equiv \sqrt{\frac{3}{2}}\Theta^\rho{}_{\bar\mu}\Theta^\sigma{}_{\bar\nu}A^{(\mathrm{ha})}_{\rho\sigma 0}, \tag{A9d}$$

$$A(0^+_\perp) \equiv \sqrt{\frac{2}{3}}\Theta^{\sigma\beta}A_{0\sigma\beta}, \tag{A9e}$$

$$\begin{aligned} A(2^+_\perp)_{\bar\alpha\bar\mu} &\equiv \frac{1}{\sqrt{2}}[\Theta^\sigma{}_{\bar\alpha}\Theta^\beta{}_{\bar\mu} + \Theta^\sigma{}_{\bar\mu}\Theta^\beta{}_{\bar\alpha} \\ &\quad - \frac{2}{3}\Theta_{\bar\alpha\bar\mu}\Theta^{\sigma\beta}]A_{0\sigma\beta}, \end{aligned} \tag{A9f}$$

$$A(1^-_\parallel)_{\bar\mu} \equiv \Theta^\rho{}_{\bar\mu}\Theta^{\sigma\beta}A^{(\mathrm{ha})}_{\rho\sigma\beta}. \tag{A9g}$$

3. Symmetric rank-3 non-metricity

The symmetric non-metricity tensor is decomposed into its totally-symmetric and hook-symmetric parts

$$Q^{(\mathrm{ts})}_{\mu\nu\alpha} = \frac{1}{3}[Q_{\mu\nu\alpha} + Q_{\alpha\mu\nu} + Q_{\alpha\nu\mu}], \tag{A10a}$$

$$Q^{(\mathrm{hs})}_{\mu\nu\alpha} = \frac{1}{3}[2Q_{\mu\nu\alpha} - Q_{\alpha\mu\nu} - Q_{\alpha\nu\mu}]. \tag{A10b}$$

The labelling indices s contains an additional disambiguation description h, t indicating whether the particle arose from the hook- or totally-symmetric portion of the non-metricity. Its decomposition is given by:

$$\begin{aligned} Q(0^+_{\perp\mathrm{t}})_{\mu\nu\alpha} &\equiv \Omega^{\rho}{}_{\mu}\Omega^{\sigma}{}_{\nu}\Omega^{\beta}{}_{\alpha}Q^{(\mathrm{ts})}_{\rho\sigma\beta} \\ &= n_{\mu}n_{\nu}n_{\alpha}Q(0^+_{\perp\mathrm{t}}), \end{aligned} \tag{A11a}$$

$$\begin{aligned} &Q(1^-_{\perp\mathrm{t}})_{\mu\nu\alpha} \\ &\equiv [\Omega^{\rho}{}_{\mu}\Omega^{\sigma}{}_{\nu}\Theta^{\beta}{}_{\bar{\alpha}} + \Omega^{\rho}{}_{\mu}\Theta^{\sigma}{}_{\bar{\nu}}\Omega^{\beta}{}_{\alpha} + \Theta^{\rho}{}_{\bar{\mu}}\Omega^{\sigma}{}_{\nu}\Omega^{\beta}{}_{\alpha}]Q^{(\mathrm{ts})}_{\rho\sigma\beta} \\ &= \frac{1}{\sqrt{3}}[n_{\mu}n_{\nu}Q(1^-_{\perp\mathrm{t}})_{\bar{\alpha}} + n_{\alpha}n_{\mu}Q(1^-_{\perp\mathrm{t}})_{\bar{\nu}} + n_{\alpha}n_{\nu}Q(1^-_{\perp\mathrm{t}})_{\bar{\mu}}], \end{aligned} \tag{A11b}$$

$$\begin{aligned} &Q(0^+_{\parallel})_{\mu\nu\alpha} \\ &\equiv \frac{1}{3}[\Theta_{\bar{\mu}\bar{\nu}}\Omega^{\beta}{}_{\alpha}\Theta^{\rho\sigma} + \Theta_{\bar{\alpha}\bar{\mu}}\Omega^{\sigma}{}_{\nu}\Theta^{\rho\beta} + \Theta_{\bar{\alpha}\bar{\nu}}\Omega^{\rho}{}_{\mu}\Theta^{\sigma\beta}]Q^{(\mathrm{ts})}_{\rho\sigma\beta} \\ &= \frac{1}{3}[\Theta_{\bar{\mu}\bar{\nu}}n_{\alpha} + \Theta_{\bar{\alpha}\bar{\mu}}n_{\nu} + \Theta_{\bar{\alpha}\bar{\nu}}n_{\mu}]Q(0^+_{\parallel}), \end{aligned} \tag{A11c}$$

$$\begin{aligned} &Q(2^+_{\parallel})_{\mu\nu\alpha} \\ &\equiv [\Theta^{\rho}{}_{\bar{\mu}}\Theta^{\sigma}{}_{\bar{\nu}}\Omega^{\beta}{}_{\alpha} + \Theta^{\rho}{}_{\bar{\mu}}\Omega^{\sigma}{}_{\nu}\Theta^{\beta}{}_{\bar{\alpha}} + \Omega^{\rho}{}_{\mu}\Theta^{\sigma}{}_{\bar{\nu}}\Theta^{\beta}{}_{\bar{\alpha}} \\ &\quad - \frac{1}{3}(\Theta_{\bar{\mu}\bar{\nu}}\Omega^{\beta}{}_{\alpha}\Theta^{\rho\sigma} + \Theta_{\bar{\alpha}\bar{\mu}}\Omega^{\sigma}{}_{\nu}\Theta^{\rho\beta} + \Theta_{\bar{\alpha}\bar{\nu}}\Omega^{\rho}{}_{\mu}\Theta^{\sigma\beta})]Q^{(\mathrm{ts})}_{\rho\sigma\beta} \\ &= \frac{1}{\sqrt{3}}[Q(2^+_{\parallel})_{\bar{\mu}\bar{\nu}}n_{\alpha} + Q(2^+_{\parallel})_{\bar{\alpha}\bar{\mu}}n_{\nu} + Q(2^+_{\parallel})_{\bar{\alpha}\bar{\nu}}n_{\mu}], \end{aligned} \tag{A11d}$$

$$\begin{aligned} &Q(1^-_{\parallel\mathrm{t}})_{\mu\nu\alpha} \\ &\equiv \frac{1}{5}[\Theta_{\bar{\mu}\bar{\nu}}\Theta^{\beta}{}_{\bar{\alpha}}\Theta^{\rho\sigma} + \Theta_{\bar{\alpha}\bar{\mu}}\Theta^{\sigma}{}_{\bar{\nu}}\Theta^{\rho\beta} + \Theta_{\bar{\alpha}\bar{\nu}}\Theta^{\rho}{}_{\bar{\mu}}\Theta^{\sigma\beta}]Q^{(\mathrm{ts})}_{\rho\sigma\beta} \\ &= \frac{1}{\sqrt{15}}[\Theta_{\bar{\mu}\bar{\nu}}Q(1^-_{\parallel\mathrm{t}})_{\bar{\alpha}} + \Theta_{\bar{\alpha}\bar{\mu}}Q(1^-_{\parallel\mathrm{t}})_{\bar{\nu}} + \Theta_{\bar{\alpha}\bar{\nu}}Q(1^-_{\parallel\mathrm{t}})_{\bar{\mu}}], \end{aligned} \tag{A11e}$$

$$\begin{aligned} Q(3^-_{\parallel})_{\mu\nu\alpha} &\equiv [\Theta^{\rho}{}_{\bar{\mu}}\Theta^{\sigma}{}_{\bar{\nu}}\Theta^{\beta}{}_{\bar{\alpha}} - \frac{1}{5}(\Theta_{\bar{\mu}\bar{\nu}}\Theta^{\beta}{}_{\bar{\alpha}}\Theta^{\rho\sigma} \\ &\quad + \Theta_{\bar{\alpha}\bar{\mu}}\Theta^{\sigma}{}_{\bar{\nu}}\Theta^{\rho\beta} + \Theta_{\bar{\alpha}\bar{\nu}}\Theta^{\rho}{}_{\bar{\mu}}\Theta^{\sigma\beta})]Q^{(\mathrm{ts})}_{\rho\sigma\beta} \\ &= Q(3^-_{\parallel})_{\bar{\mu}\bar{\nu}\bar{\alpha}}, \end{aligned} \tag{A11f}$$

$$\begin{aligned} &Q(1^-_{\perp\mathrm{h}})_{\mu\nu\alpha} \\ &\equiv [\Omega^{\rho}{}_{\mu}\Omega^{\sigma}{}_{\nu}\Theta^{\beta}{}_{\bar{\alpha}} + \Omega^{\rho}{}_{\mu}\Theta^{\sigma}{}_{\bar{\nu}}\Omega^{\beta}{}_{\alpha} + \Theta^{\rho}{}_{\bar{\mu}}\Omega^{\sigma}{}_{\nu}\Omega^{\beta}{}_{\alpha}]Q^{(\mathrm{hs})}_{\rho\sigma\beta} \\ &= \frac{1}{\sqrt{6}}[2n_{\mu}n_{\nu}Q(1^-_{\perp\mathrm{h}})_{\bar{\alpha}} - n_{\alpha}n_{\mu}Q(1^-_{\perp\mathrm{h}})_{\bar{\nu}} - n_{\alpha}n_{\nu}Q(1^-_{\perp\mathrm{h}})_{\bar{\mu}}], \end{aligned} \tag{A11g}$$

$$\begin{aligned} &Q(2^+_{\perp})_{\mu\nu\alpha} + Q(0^+_{\perp\mathrm{h}})_{\mu\nu\alpha} \\ &\equiv \frac{3}{2}\Theta^{\rho}{}_{\bar{\mu}}\Theta^{\sigma}{}_{\bar{\nu}}\Omega^{\beta}{}_{\alpha}Q^{(\mathrm{hs})}_{\rho\sigma\beta} \\ &\quad - \frac{1}{\sqrt{6}}[Q(2\oplus 0)_{\bar{\mu}\bar{\nu}}n_{\alpha} + Q(2\oplus 0)_{\bar{\alpha}\bar{\mu}}n_{\nu} + Q(2\oplus 0)_{\bar{\alpha}\bar{\nu}}n_{\mu}] \\ &= \frac{1}{\sqrt{6}}[2Q(2\oplus 0)_{\bar{\mu}\bar{\nu}}n_{\alpha} - Q(2\oplus 0)_{\bar{\alpha}\bar{\mu}}n_{\nu} - Q(2\oplus 0)_{\bar{\alpha}\bar{\nu}}n_{\mu}], \end{aligned} \tag{A11h}$$

where for succinctness we define the tensor

$$Q(2\oplus 0)_{\bar{\mu}\bar{\nu}} \equiv Q(2^+_{\perp})_{\bar{\mu}\bar{\nu}} + \frac{1}{\sqrt{3}}\Theta_{\bar{\mu}\bar{\nu}}Q(0^+_{\perp\mathrm{h}}), \tag{A11i}$$

$$\begin{aligned} &Q(1^+_{\perp})_{\mu\nu\alpha} \\ &\equiv [\Omega^{\rho}{}_{\mu}\Theta^{\sigma}{}_{\bar{\nu}}\Theta^{\beta}{}_{\bar{\alpha}} + \Theta^{\rho}{}_{\bar{\mu}}\Omega^{\sigma}{}_{\nu}\Theta^{\beta}{}_{\bar{\alpha}} - \frac{1}{2}\Theta^{\rho}{}_{\bar{\mu}}\Theta^{\sigma}{}_{\bar{\nu}}\Omega^{\beta}{}_{\alpha}]Q^{(\mathrm{hs})}_{\rho\sigma\beta} \\ &\quad + \frac{1}{\sqrt{6}}[Q(2\oplus 0)_{\bar{\mu}\bar{\nu}}n_{\alpha} + Q(2\oplus 0)_{\bar{\alpha}\bar{\mu}}n_{\nu} + Q(2\oplus 0)_{\bar{\alpha}\bar{\nu}}n_{\mu}] \\ &= -\frac{1}{\sqrt{2}}[Q(1^+_{\perp})_{\bar{\alpha}\bar{\mu}}n_{\nu} + Q(1^+_{\perp})_{\bar{\alpha}\bar{\nu}}n_{\mu}], \end{aligned} \tag{A11j}$$

$$\begin{aligned} &Q(1^-_{\parallel\mathrm{h}})_{\mu\nu\alpha} \\ &\equiv \frac{1}{2}[\Theta_{\bar{\mu}\bar{\nu}}\Theta^{\beta}{}_{\bar{\alpha}}\Theta^{\rho\sigma} + \Theta_{\bar{\alpha}\bar{\mu}}\Theta^{\sigma}{}_{\bar{\nu}}\Theta^{\rho\beta} + \Theta_{\bar{\alpha}\bar{\nu}}\Theta^{\rho}{}_{\bar{\mu}}\Theta^{\sigma\beta}]Q^{(\mathrm{hs})}_{\rho\sigma\beta} \\ &= \frac{1}{2\sqrt{3}}[2\Theta_{\bar{\mu}\bar{\nu}}Q(1^-_{\parallel\mathrm{h}})_{\bar{\alpha}} - \Theta_{\bar{\alpha}\bar{\mu}}Q(1^-_{\parallel\mathrm{h}})_{\bar{\nu}} - \Theta_{\bar{\alpha}\bar{\nu}}Q(1^-_{\parallel\mathrm{h}})_{\bar{\mu}}], \end{aligned} \tag{A11k}$$

$$\begin{aligned} Q(2^-_{\parallel})_{\mu\nu\alpha} &\equiv [\Theta^{\rho}{}_{\bar{\mu}}\Theta^{\sigma}{}_{\bar{\nu}}\Theta^{\beta}{}_{\bar{\alpha}} - \frac{1}{2}(\Theta_{\bar{\mu}\bar{\nu}}\Theta^{\beta}{}_{\bar{\alpha}}\Theta^{\rho\sigma} \\ &\quad + \Theta_{\bar{\alpha}\bar{\mu}}\Theta^{\sigma}{}_{\bar{\nu}}\Theta^{\rho\beta} + \Theta_{\bar{\alpha}\bar{\nu}}\Theta^{\rho}{}_{\bar{\mu}}\Theta^{\sigma\beta})]Q^{(\mathrm{hs})}_{\rho\sigma\beta} \\ &= Q(2^-_{\parallel})_{\bar{\mu}\bar{\nu}\bar{\alpha}}. \end{aligned} \tag{A11l}$$

The reduced rank spin-irreps are:

$$Q(0^+_{\perp\mathrm{t}}) \equiv Q_{000}, \tag{A12a}$$

$$Q(1^-_{\perp\mathrm{t}})_{\bar{\mu}} \equiv \sqrt{3}\Theta^{\rho}{}_{\bar{\mu}}Q^{(\mathrm{ts})}_{\rho 00}, \tag{A12b}$$

$$Q(0^+_{\parallel}) \equiv \Theta^{\rho\sigma}Q^{(\mathrm{ts})}_{\rho\sigma 0}, \tag{A12c}$$

$$Q(2^+_{\parallel})_{\bar{\mu}\bar{\nu}} \equiv \sqrt{3}[\Theta^{\rho}{}_{\bar{\mu}}\Theta^{\sigma}{}_{\bar{\nu}} - \frac{1}{3}\Theta_{\bar{\mu}\bar{\nu}}\Theta^{\rho\sigma}]Q^{(\mathrm{ts})}_{\rho\sigma 0}, \tag{A12d}$$

$$Q(1^-_{\parallel\mathrm{t}})_{\bar{\mu}} \equiv \sqrt{\frac{3}{5}}\Theta^{\rho}{}_{\bar{\mu}}\Theta^{\sigma\beta}Q^{(\mathrm{ts})}_{\rho\sigma\beta}, \tag{A12e}$$

$$Q(1^-_{\perp\mathrm{h}})_{\bar{\mu}} \equiv \sqrt{\frac{3}{2}}\Theta^{\rho}{}_{\bar{\mu}} Q^{(\mathrm{hs})}_{00\rho}, \tag{A12f}$$

$$Q(0^+_{\perp\mathrm{h}}) \equiv \frac{1}{\sqrt{2}}\Theta^{\rho}{}_{\lambda}\Theta^{\sigma\lambda} Q^{(\mathrm{hs})}_{\rho\sigma 0}, \tag{A12g}$$

$$Q(2^+_{\perp}) \equiv \sqrt{\frac{3}{2}}[\Theta^{\rho}{}_{\bar{\mu}}\Theta^{\sigma}{}_{\bar{\nu}} - \frac{1}{3}\Theta_{\bar{\mu}\bar{\nu}}\Theta^{\rho}{}_{\lambda}\Theta^{\sigma\lambda}] Q^{(\mathrm{hs})}_{\rho\sigma 0}, \tag{A12h}$$

$$Q(1^+_{\perp})_{\bar{\alpha}\bar{\mu}} \equiv \frac{1}{\sqrt{2}}[\Theta^{\sigma}{}_{\bar{\alpha}}\Theta^{\beta}{}_{\bar{\mu}} - \Theta^{\sigma}{}_{\bar{\mu}}\Theta^{\beta}{}_{\bar{\alpha}}] Q_{0\sigma\beta}, \tag{A12i}$$

$$Q(1^-_{\|\mathrm{h}})_{\bar{\mu}} \equiv \sqrt{3}\Theta^{\rho}{}_{\bar{\mu}}\Theta^{\sigma\beta} Q^{(\mathrm{hs})}_{\rho\sigma\beta}. \tag{A12j}$$