Question 1.

Refresher on Bayes and maximum Likelihood method.

$$P(\text{data}|\text{model})\cdot P(\text{model}) = P(\text{model}|\text{data})\cdot P(\text{data})$$

Posterior. Likelihood. Prior.

$$\text{Then} \quad P(\text{model}|\text{data}) = \frac{P(\text{data}|\text{model})\cdot P(\text{model})}{P(\text{data})}$$

↑ Marginalisation

Then maximum likelihood is proxy for real goal which is maximum posterior.

Then for chi-squared $\quad p(y_i|\vec{a}) = \frac{1}{\sqrt{2\pi}\,\sigma_i} e^{-\frac{(y_i - f(\vec{a}))^2}{2\sigma_i^2}}$.

Then $L(\vec{y}|\vec{a}) = \prod_i p(y_i|\vec{a})$

and $\ln L(\vec{y}|\vec{a}) = -\frac{1}{2}\underbrace{\sum_i \frac{(y_i - f(\vec{a}))^2}{\sigma_i^2}}_{\chi^2} + \ldots\sigma\ldots$

Key assumption is that all $\sigma_i = \sigma$.

Then solve $\quad \vec{\nabla}_a \chi^2 = 0$.

Then model is $y_i = mx_i + c \Rightarrow \frac{\partial \chi^2}{\partial c} \Rightarrow \sum_i [y_i - (mx_i + c)] = 0$.

$\Rightarrow \langle y\rangle - (m\langle x\rangle + c) = 0$.

$\frac{\partial \chi^2}{\partial m} \Rightarrow \langle xy\rangle - (m\langle x^2\rangle + c\langle x\rangle) = 0$.

Then two equations become $\quad m\langle x\rangle + c - \langle y\rangle = 0$.
$\quad m\langle x^2\rangle + c\langle x\rangle - \langle xy\rangle = 0$.

$$\Rightarrow m = \frac{\langle xy\rangle - \langle x\rangle\langle y\rangle}{\langle x^2\rangle - \langle x\rangle^2} \qquad c = \frac{\langle x^2\rangle\langle y\rangle - \langle x\rangle\langle xy\rangle}{\langle x^2\rangle - \langle x\rangle^2} \quad \text{Nice!}$$

Remember that if model is valid: $\quad \chi^2 = \sum_i \frac{(y_i - f(\vec{a}))^2}{\sigma_i^2} \sim \sum_i \frac{\sigma_i^2}{\sigma_i^2} = \sum_i 1$.

$\chi^2 \gg N_d \Rightarrow$ your model is poor.
$\chi^2 \ll N_d \Rightarrow$ your σ_i are too large. $\quad N_d$ = N of data.

Evaluate.

could say $\quad \sigma_{900} = 10^{-1}$.
$\sigma_i = \sqrt{90}\cdot 10^{-1} = 30\times 10^{-1} = 3\ldots$

24.2
80.4 ⟶ for 8...

Question 2.

Genuine attempt at a solution...

$$\omega^2 - k_x^2 - k_y^2 = 0.$$

But $k_y = \frac{n\pi}{a} \Rightarrow \omega^2 = k^2 + \frac{n^2\pi^2}{a}$.

Compare this to $E^2 = p^2 + m^2$.

Then $\quad v = \frac{d\omega}{dk} = \frac{d}{dk}\sqrt{(k^2+m^2)} = \frac{k}{\sqrt{(k^2+m^2)}}$

$$= \frac{\sqrt{(E^2 - m^2)}}{\sqrt{(E^2 - m^2 + m^2)}} = \sqrt{\left(1 - \frac{m^2}{E^2}\right)}$$

Then $\quad \sqrt{\left(1 - \frac{\pi^2}{E^2a^2}\right)} = 2\sqrt{\left(1 - \frac{4\pi^2}{E^2a^2}\right)}$

$\Rightarrow 1 - 4 = -\frac{16\pi^2}{E^2a^2} + \frac{\pi^2}{E^2a^2}$

$\Rightarrow E^2 = \frac{15\pi^2}{3a^2} = \frac{5\pi^2}{a^2}$

$E = \frac{\sqrt{5}\,\pi}{a}$. $\quad \sqrt{5} \sim 2.2$ i.e. 3rd lightest mode never observed.

Question 3.

Explain conditions under which Fresnel diffraction applies.

$$\phi = \int h(x,y)\frac{e^{ik(r+s)}}{rs}\,dx\,dy$$

Then $\quad ik(r+s) = ik\left(\sqrt{(L^2 + (X-x)^2 + (Y-y)^2)}\right) + \ldots$

$$= ik\left(R\sqrt{\left(1 - \frac{2Xx + 2Yy}{L^2+X^2+Y^2} + \frac{x^2+y^2}{R^2}\right)}\right) + \ldots$$

$$= -ik\underbrace{\frac{Xx+Yy}{R}}_{\text{Fraunhofer term}} + \underbrace{ik\frac{x^2+y^2}{2R}}_{\ll \pi}$$

But for Fresnel $\quad \phi \propto \int h(x,y)\, e^{ik\left(\sqrt{(a^2+x^2+y^2)} + \sqrt{(b^2+x^2+y^2)}\right)}$

$$= \int h(x,y)\frac{e^{ik\left(a+b + x^2+y^2\left(\frac{a}{2a^2} + \frac{b}{2b^2}\right)\right)}}{rs}$$

$$\propto \int h(x,y)\, e^{\frac{ik}{2}\cdot\frac{1}{R}\cdot(x^2+y^2)}.$$

Of course these integrals are separable.

$$\phi \propto \int h(x,y)\cdot e^{\frac{ikx^2}{2R}}\cdot e^{\frac{iky^2}{2R}}.$$

Dimensionless variable $\quad u = x\sqrt{\frac{2}{\lambda R}} \quad v = y\sqrt{\frac{2}{\lambda R}}$

$$\phi \propto \int h(x,y)\, e^{\frac{i\pi u^2}{2}}\, e^{\frac{i\pi v^2}{2}}. \quad \text{Fine}\ldots$$

Then $\quad C(w) = \int_0^w \cos\frac{\pi u^2}{2}\,du \qquad S(w) = \int_0^w \sin\frac{\pi v^2}{2}\,dv$.

Then for annular aperture...

$$\phi \propto \int K h(r)\cdot 2\pi r\, e^{\frac{i\pi r^2}{\lambda z}}\,dr.$$

Then $h(r)$ is top-hat function @ r

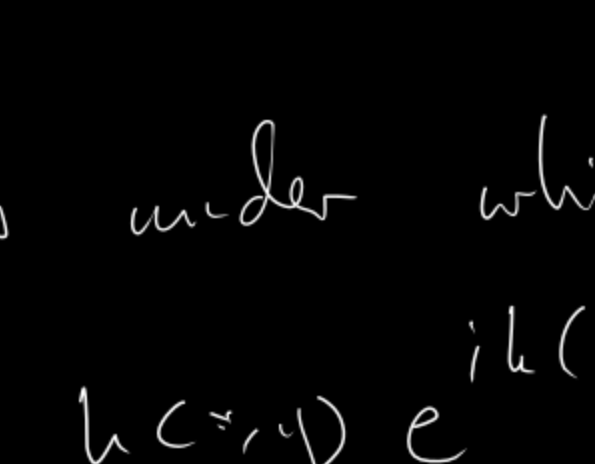
$$\ldots\phi \propto Ke^{\frac{i\pi r^2}{\lambda z}}\cdot 2\pi r\,dr \quad \text{all as required.}$$

The modified Fresnel integral in this case is...

$$\phi \propto \int_0^R 2\pi r K e^{\frac{i\pi r^2}{\lambda z}}\,dr.$$

Now... Change of variable for case of normal incidence.

$$\phi \propto \int \frac{\pi K e^{\frac{i\pi}{\lambda z}}}{\sqrt{(z^2+\rho^2)}}\rho\,d\rho$$

Then $\quad \phi \sim$  etc. $\quad$ where $\frac{d\phi}{d\rho}$ is phase contribution at given annulus...

$\frac{i\pi}{\lambda z}r^2 = in\pi \Rightarrow r^2 = n\lambda z$ all as required. OUTER
So clearly odd $n \rightarrow$ bright spot.

In next part of question we neglect obliquity factor.

$\lambda_1 = 400\,\text{nm}$
$\lambda_2 = 700\,\text{nm}$.

$n = \frac{r^2}{\lambda z}$ $\quad n_1 = 9.025$
$\quad n_2 = 5.15$

$r = \frac{1}{2}D$

$n_a = 7,\ n_b = 9$ are missing

$\lambda = \frac{r^2}{nz} \qquad \lambda_a = 515.7\,\text{nm}$
$\lambda_b = 401.1\,\text{nm}$.