UNIVERSITY OF CAMBRIDGE

Astrophysics out of Triangles: Quantum Gravity with Exotic Geometry

Part III Astrophysics Master's Project
May 2023

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Part III Master's Thesis
Astrophysics out of Triangles: Quantum Gravity with Exotic Geometry

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May 2023

Abstract. We consider the role of geometric algebra in describing the parallel transport of spinors on a two-dimensional triangulated manifold in a three-dimensional embedding. In the development of this model, a new geometric algebra description of spinors was composed for the purpose of introducing the Grassmann-odd nature of the spinor components. This construction builds on the 'minimal ideal' approach, used in Clifford algebra. Ultimately, we were able to build an effective representation of the parallel transport of spinors which culminated in the formation of a geometric algebra equivalent of the Dirac–Wilson Operator. This gives us a solid framework which we could use to develop the first ever geometric algebra description of quantum gravity.

Keywords: Gravity, Quantum Gravity, Dynamical Triangulation, Geometric Algebra, Spinors, Two-dimensional, Grassmann numbers, Lattice Field Theory, Dirac–Wilson operator

1. Introduction

In light of everything that we have been exposed to in our time as physicists, it should now be apparent that our theories are entirely provisional — that is, there is a range (i.e. an energy or distance scale) over which our models work, and *General Relativity* (GR) is no different. This is not to say that Einstein's Gravity [1] ought *not* to be hailed as a triumph. After all, it remains to this day one of the most elegant descriptions of cosmic-scale physics through the curvature of *spacetime*. But, it is ultimately still an *effective field theory* — accurate on astronomical (low energy) scales, but not quantum (high energy) ones.

Of course, we have seen much success in the translation of classical field theories into quantum field theories (QFTs) [2] — indeed, in their 1929 paper on Quantum Electrodynamics (QED) [3], Pauli and Heisenberg anticipated that the process of formulating a quantum theory of gravity would likely bear many similarities to the approach they had just painstakingly outlined. So why is it that a quantum description of gravity continues to elude us? To answer this question, we will need to explore the steps involved in making a QFT, specifically the step which can be said to "make or break" a theory — *renormalization* and *regularization* [4, 5, 6].

Quantum field theories are often plagued with *divergences*, which we can separate into two categories — *infrared* (IR), associated with low momenta, and *ultraviolet* (UV), associated with high momenta. Of the two, UV divergences are considerably more challenging to work around [7], and we have gone to great lengths to find a valid procedure for "taming infinity". This is where renormalization and regularization techniques come into play, allowing us to subtly remove any divergence terms by finely tweaking our Lagrangian with the addition of *counterterms*. Unfortunately, this method does not always work and, in the case of Quantum Gravity (QG), we find that our effective theory (GR) diverges faster than we are able to keep up with it. The reason being that *this* use of renormalization is more of a *perturbative* correction to our effective theory, and there is a massive gulf in energy scales between Einstein's formulation of gravity and quantum gravity. If we wanted to perturb GR to the high energy scales of quantum gravity, we would quickly find ourselves needing to add an infinite number of counterterms to combat the divergences, essentially ruling out perturbative renormalization as an effective means of countering infinities in QG.

So what *can* we do? Interestingly enough, an answer emerges when we examine the other end of the energy spectrum. Just as gravity is *too weak* for perturbation theory to be a useful model at *high energies*, quantum chromodynamics (QCD) is *too strong* when considered at *low energies*. However, thanks to the pioneering work of Kenneth Wilson — who was integral in the development *lattice field theories* (LFTs) [8], high-energy physicists quickly developed a formidable strategy to work with QCD at low energies, essentially procuring a way to work with QFTs *non-perturbatively*. Now, what happens when we try something similar with gravity? There are already a number of lattice models of quantum gravity, with some significant examples being *Dynamical Triangulations* (DT) [9, 10], and its (slightly) more recent counterpart, *Causal Dynamical Triangulations* (CDT) [11]. Each of these models operate by discretizing spacetime [12] and allowing it to vary *dynamically*, although precisely how each model encodes dynamical behaviour is primarily what distinguishes them. We will be concerning ourselves with the ideas associated with DT in which we discretize a spacetime manifold by tessellating it with *simplices*.

Naturally, the process of discretizing spacetime is not free of defects. For instance, we find that by regularizing spacetime, we lose Lorentz invariance (LI). This is now regarded as a minor inconvenience, provided that our theory recovers LI in the *smooth limit*. But the restriction of regaining LI imposed on lattice quantum gravity are worth mentioning. In particular, we require the distribution of spacetime points that form our lattice to be *random*, so as to recover Minkowski spacetime [13]. As such, most DT and CDT models incorporate techniques to procure random manifolds by first generating a set of simplices, and then gluing them together based on some random distribution function. Although our newly proposed model does not yet account for random simplicial manifolds, it was developed with this in mind and we strongly suspect that it will not take much effort to adapt it into a fully functional theory of lattice quantum gravity.

As already stated, this project has concerns itself chiefly with dynamical triangulations — in particular, we have explore the feasibility of constructing a model of DT using *geometric algebra* (GA) [14, 15, 16]. Likewise, we examine how one might couple fermionic matter to our model of quantum gravity — something which has consistently proven to be a hurdle on the road to the quantization of gravity [17]. We adapt the work of Burda [18, 19, 20] into a three-dimensional embedding of two-dimensional Euclidean space. This translates to foliating our manifold with triangles (2-simplices) and devising a method for discrete parallel transport in the language of GA. Furthermore, we establish a new representation of *spinors*, which better reflect the anticommuting nature of fermions through the implementation of *Grassmann-odd* numbers — it should not be understated that this has been a significant objective of GA theorists for a number of years. Finally, we construct the GA equivalent of the Dirac–Wilson operator [8], following on from the formation outlined by Burda [20]. We will, of course, provide definitions for each of these objects as they crop up in the body of the report.

We now provide a brief description of the structure of this report. In Section 2, we provide a pedagogical review of some of the key features of geometric algebra implemented in our research. We then go on to inspect spinors in Section 3, introducing the relevant theory before laying out the development of the new minimal ideal/Grassmann formalism. Section 4 concerns itself with establishing triangulated manifolds within GA, as well as the treatment of frames as we parallel transport them across our discretized manifold. We likewise consider the parallel transport of spinors in Section 5, and use this to construct the Dirac–Wilson operator. Finally, we offer a summary of our work at the end, in Section 6, as well as some supplementary calculations in the Appendix.

2. Geometric Algebra for Astrophysicists

It was the work of Grassmann [16, 21] which began the chain of events that ultimately led us to GA. Indeed, he developed the precursor to linear algebra (sadly inaccessible to the mathematicians of the time) which ultimately inspired Clifford to go on to develop *Clifford algebra* [22]. Clifford algebra has had a renaissance in the past century, regularly making an appearance in QFT in the form of the gamma matrices, and even in basic quantum mechanics through the Pauli matrices. Additionally, David Hestenes introduced a fresh perspective on Clifford algebras, with his introduction of geometric algebra [14, 15] — and his work has gone on to facilitate many interesting theories in physics. Perhaps the most notable addition has been Gauge Theory Gravity (GTG) — first devised in Cambridge by Anthony Lasenby, Chris Doran, and Stephen Gull [23] (see also [24, 25, 26, 27, 28]) — which recasts Einstein–Cartan gravity in the language of GA. For those who may be unfamiliar, Einstein–Cartan gravity is a delightful extension to GR, which introduces spacetime *torsion* as a means of allowing us to couple fermionic matter to gravity. Our ultimate goal is to do something similar, by developing ideas which could later be used to constitute a lattice gauge theory of quantum gravity — this could potentially include introducing torsion at a later stage. But we have allowed ourselves to digress somewhat from our initial point — we will now introduce the reader to GA.

2.1. The Basics

The simplest way to think of geometric algebra is as an extension to standard vector algebra, where we are now able to manipulate more complex mathematical objects, such as oriented planes and volumes, with relative ease. Each of these objects is specified by a *grade* — for instance, we associate scalars with grade-0 objects, vectors with grade-1 objects, and so on — and we are able to sum these graded objects together to form what is referred to as a *multivector*. In addition, we introduce the *geometric product*, which not only serves as a generalization to the standard 'vector products' we are likely familiar with — the inner and outer products — but also allows us to multiply objects of different grades. We will now outline some of the mathematical details — for a complete introduction, we direct the reader to [14, 15, 16, 29, 30].

The geometric product between two vectors a and b can be written as

$$ab = a \cdot b + a \wedge b, \tag{1}$$

where we define $a \cdot b = \frac{1}{2}(ab + ba)$, and $a \wedge b = \frac{1}{2}(ab - ba)$. There are two points worth noting here. Firstly, although not explicitly shown, it ought to be kept in mind that the geometric product is distributive and associative, but generally non-commutative (which comes from the presence of the antisymmetric 'wedge' product). Secondly, by construction, we are adding a grade-2 object ($a \wedge b$, hereafter referred to as a *bivector*) to a grade-0 object ($a \cdot b$, corresponding to a *scalar*) and have thus formed our first multivector. If this feels rather disturbing (after all, the notion of adding a scalar to a vector certainly *feels* unnatural), it may be possible to placate this discomfort by considering the following: we are perfectly happy to add imaginary numbers to real numbers to form complex numbers — why should we treat this any differently? If that simple argument does not sway the malaise, then it should be highlighted that *all* of the objects that exist within GA can be represented as matrices — with the scalar term simply being a multiple of the identity, and this idea is further explored in Section 3.

Before moving on to the next subsection, there is one final comment to be made. The geometric product provides us with the freedom to divide single grade objects — in other words, we now have vector inverses. To see this, we note $aa = a^2 = a \cdot a + a \wedge a = a \cdot a$, where the final equality follows from the antisymmetry of the wedge product. This allows us to give the inverse of a as $a^{-1} = a/|a|^2$, since $aa^{-1} = a\frac{a}{|a|^2} = 1$. With this, we will now discuss the objects that can be spawned from the geometric product in more detail.

2.2. Frames, Blades, and the Generalized Geometric Product

A *frame* in GA, is defined as a set of (not necessarily orthogonal) linearly independent basis vectors $\{e_k\}$. These basis vectors span the n-dimensional space, and we can write any vector in this space as a linear combination of the basis vectors. We supplement the frame via the wedge product, to form 2^n basis *elements* $1,\ \{e_i\},\ \{e_i \wedge e_j\},\ \dots\ ,\ e_1 \wedge e_2 \wedge \cdots \wedge e_n$. And we give the final element the moniker 'pseudoscalar', labelling it as

$$I \equiv e_1 \wedge e_2 \wedge \cdots \wedge e_n. \tag{2}$$

This object plays a rather interesting role in GA — for instance, multiplication by I serves as a means of accessing the dual space of an object. For those unfamiliar with the dual space, it can be thought of as the orthogonal complement. For example, in three dimensional space, the dual space of a plane would be the vector normal to that plane. Of particular importance to this project is the pseudoscalar in two dimensions. This pseudoscalar is bivector valued and, as we shall soon see when we discuss rotations, acts in a similar manner to the imaginary number i, (or equivalently, to the generator of rotations for SO(2)). When we work in a two-dimensional subspace of the three-dimensional Euclidean space, we can regard the pseudoscalar of a planar subspace to be the unit bivector which defines the plane, as we might have expected.

An object formed of purely grade-r components is labelled an r-blade. It can be constructed by taking the outer product of r vectors, since outer products form an object whose grade is the sum of its constituents. If we have a vector a and an r-blade A_r, we can define their geometric product like so

$$aA_r = a \cdot A_r + a \wedge A_r = \frac{1}{2}(aA_r - (-1)^r A_r\, a) + \frac{1}{2}(aA_r + (-1)^r A_r\, a). \tag{3}$$

We identify the inner and outer products as the first and second terms, respectively, of Eq. (3). A full derivation of this result can be found in [16, 30]. The generalization of the geometric product between an r-blade and an s-blade follows from Eq. (3), from which we get

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{r+s-2} + \langle A_r B_s \rangle_{r+s}. \tag{4}$$

Where we have defined the operation of *grade projection* with $\langle\ \rangle_k$. This should be taken to be the isolation of the grade-k part of the multivector. For instance, if we take the grade projection of ab onto the grade-2 component, we would get $\langle ab \rangle_2 = a \wedge b$. Similarly, we could take $\langle ab \rangle_1 = 0$. Note that the standard convention is to drop the subscript on the scalar grade projection ($\langle\ \rangle \equiv \langle\ \rangle_0$). Since this project focuses on two-dimensional surfaces embedded in a three-dimensional Euclidean space, our geometric products will never become too convoluted. That being said, it is worth examining the products of bivectors, as they will be integral to the development of our theory.

In three dimensions with a Euclidean signature, we find that the geometric product between two bivectors A and B is given by $AB = \langle AB \rangle + \langle AB \rangle_2$. We now introduce the *commutator*, characterized by a "$\times$" symbol, to indicate the antisymmetric product between two rotors like so

$$\langle AB \rangle_2 = A \times B = \frac{1}{2}(AB - BA). \tag{5}$$

Note that this is markedly different from the wedge product, which acts to sum the grades of the blades within the product. It is no accident that this product is named the commutator, as it is practically identical to the commutator we find in Lie algebras $[A, B] = AB - BA$. Indeed, commutation relations crop up time and again when we examine some of the deeper aspects of bivector algebras [16]. It can be further demonstrated [14] that the scalar part of the geometric product AB can be given as $\langle AB \rangle = A \cdot B$. We will now cover some more interesting details involving bivectors, specifically with regard to their role in rotations.

2.3. Rotations

With these basic definitions in mind, we can now proceed to discuss *linear functions* which we denote with a capital in Sans-Serif font F, and an underline. As can be surmised from their name, they possess linearity. It can also be seen that these linear functions are *grade preserving*. That is, for vectors a and b, the action of F on $a \wedge b$ is $\underline{\mathsf{F}}(a \wedge b) \equiv \underline{\mathsf{F}}(a) \wedge \underline{\mathsf{F}}(b)$. Rotations are one such example of linear functions, and they lie at heart of this project. In GA, rotation matrices are replaced by objects known as *rotors*. We define the action of a rotor R on a vector a as

$$\underline{\mathsf{R}}(a) \equiv \mathrm{e}^{-B\theta/2} a \mathrm{e}^{B\theta/2}, \tag{6}$$

where B is a constant unit bivector which defines the plane of rotation, and θ is the angle of rotation. This can be extended to an action on a multivector $\mathcal{M}$ via $\underline{\mathsf{R}}(\mathcal{M}) = \mathrm{e}^{-B\theta/2}\mathcal{M}\mathrm{e}^{B\theta/2}$, since the constituent grades of the multivector are conserved. We draw the reader's attention towards the double-sided transformation law for rotors which, as we shall see in Section 3, serves to distinguish every other object that can be formed in GA from spinors. Spinors are unique in the way that they undergo a half-angle rotation as they are acted upon by a single rotor

$$\psi' = \mathrm{e}^{-B\theta/2}\psi. \tag{7}$$

This results in a rotation of a *half-angle* as one would expect. As mentioned in Section 1, we will provide an expansive discussion of spinors in Section 3.

Returning to our broader discussion of rotors, we find that it is useful to introduce the following short-hand notation

$$\underline{\mathsf{R}}(a) = R_\theta\, a\, \tilde{R}_\theta\,, \tag{8}$$

where we have used a tilde "∼" to indicate the *reversion* of the rotor. The reversion of an object in geometric algebra is simply an operation (specifically, we call this an *involution* as performing it twice gives us the identity) which tells us to reverse the order of vectors within a product. To give a brief demonstration, if we were to take the reversion of a bivector B, we would get

$$\tilde{B} = (b_1 \wedge b_2)\tilde{} = (b_2 \wedge b_1) = -B\ , \tag{9}$$

where the last equality holds due to the antisymmetric nature of the wedge product. It should now be apparent that the reversion of a bivector is its negative. We have already drawn an analogy between complex numbers and GA, but this is a point worth expounding upon as we find that the square of a unit bivector is equal to -1 (see Appendix A for a proof of this statement). This simple result allows us to formulate an equivalent expression to Euler's formula ($\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$), which takes the form

$$\mathrm{e}^{B\theta} = \cos\theta + B\sin\theta. \tag{10}$$

We should now see the parallels between reversion and Hermitian conjugation too.

Due to their importance in this project, there is cause to build-up some further intuition for bivectors. Explicitly, we will outline how one might decompose a vector a into its projected and rejected components with regards to a plane B, see Fig. 1, and how a rotation in that plane would affect each of those components. We write $a = a_B + a_{\not B}$, where a_B denotes the part of a which is parallel to the plane B, and $a_{\not B}$ denotes the part of a perpendicular to B. These objects can be expressed as

$$a_B = (a \cdot B)B^{-1}\,, \tag{11a}$$

$$a_{\not B} = (a \wedge B)B^{-1}\,, \tag{11b}$$

where $B^{-1} = \tilde{B}/|B|^2$. If you recall, when we first introduced the geometric product in Eq. (1), we mentioned that the it is generally non-commutative. However, in this instance, the only terms which survive in $a_B B$ or $a_{\not B}B$ are the dot and wedge products respectively. As such, we find

$$a_B B = -Ba_B = \tilde{B}a_B \tag{12a}$$

$$a_{\not B}B = Ba_{\not B}. \tag{12b}$$

Note that, although the inner product between two vectors is symmetric, the inner product between a vector and a bivector is antisymmetric, as per Eq. (3). Finally, we observe that

$$\mathrm{e}^{-B\theta/2}a\mathrm{e}^{B\theta/2} = \mathrm{e}^{-B\theta/2}(a_B + a_{\not B})\mathrm{e}^{B\theta/2}$$

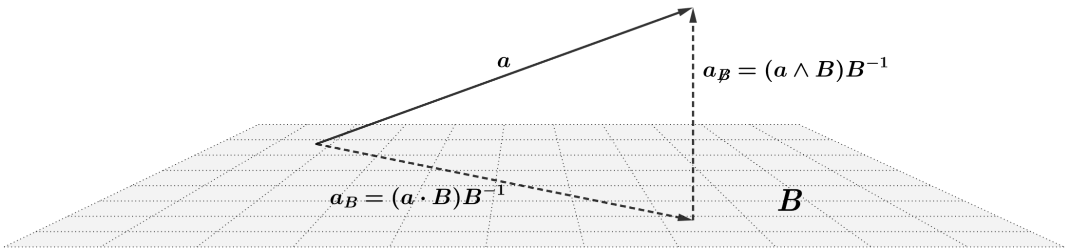


Figure 1: The figure shows a plane, defined by bivector $\boldsymbol{B}$, with a vector $\boldsymbol{a}$. The vector can be decomposed into components which lie in $\boldsymbol{B}$, denoted a_B, and components perpendicular to $\boldsymbol{B}$, denoted as $a_{\not B}$. These are both represented by the dashed vectors. The expressions for each decomposition is also provided.

$$
\begin{aligned}
&= \mathrm{e}^{-B\theta/2} a_B \mathrm{e}^{B\theta/2} + \mathrm{e}^{-B\theta/2} a_{\not B} \mathrm{e}^{B\theta/2} \\
&= a_B \mathrm{e}^{B\theta} + a_{\not B}. \qquad (13)
\end{aligned}
$$

This will be useful later, when we consider the parallel transport of frames on a discretized manifold. With that, we have now concluded our introduction to geometric algebra.

3. Spinors

We know from the Standard Model, that matter falls into two distinct groups — fermions and bosons. Bosons have proven to be relatively forgiving to work with — they obey Bose–Einstein statistics [31], meaning that they are symmetric under exchange (i.e. they commute), and they are able to occupy the same quantum state. On the other hand, fermions do not display this behaviour. They are antisymmetric under exchange (i.e. they anticommute), and they obey Fermi–Dirac statistics [32], so are unable to occupy the same quantum state. Furthermore, fermions have half-integer spins, so they do not transform in the conventional way that one would transform a scalar, vector, or tensor field. These are decidedly more challenging to work with. Unfortunately for us, fermions form the building blocks of matter and so, if we wish to explore quantum gravity, we will inevitably need to generate a means to couple gravity to fermions. As such, we need to find an appropriate way to represent fermions within GA. There are two key properties that we need to encode:

(i) They have half-integer spin — hence they are *spinors*.

(ii) They anticommute and cannot occupy the same quantum state – hence they are *Grassmann-odd*.

But, before we start to build up a description of these objects in GA, it would be helpful to have a suitable understanding of what spinors and Grassmann-odd numbers are.

3.1. Spinors Transform Like Spinors

We briefly discussed the transformation properties of spinors in Eq. (7) but it is important to expound this idea. As with scalars, vectors, and tensors, when we transform a spinor under a rotation, we expect the new object to still be identifiable as a spinor. Furthermore, we know that upon a 2π rotation a spinor does not return to itself as a vector would, but rather it picks up a sign of -1. Upon a further 2π rotation (for a total of 4π) the spinor returns to its initial sign. We therefore conclude that spinors rotate by half-angles and can thus write the transformation law for spinors as $\psi' = \mathrm{e}^{-B\theta/2}\psi$ in GA.

This single-sided transformation law is one of the defining characteristics of spinors in GA.

Of course, as we mentioned above, the reason why spinors are of interest to us is because we find that they provide a representation of Baryonic matter, given that they both share half-integer spins — with the first evidence for this provided by the famous Stern-Gerlach experiment [33, 34]. We notably encounter *Dirac spinors* in the Standard Model, these are eigenfunctions of the "square root" of the *Klein–Gordon* equation in four-dimensional spacetime — namely, the Dirac equation [2, 35], given by $(i\partial\!\!\!/ - m)\psi = 0$, where $\partial\!\!\!/ = \gamma^\mu \partial_\mu$, and γ^μ are the standard *gamma matrices* [2, 35]. The Lagrangian density which gives us the Dirac equation is

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi, \tag{14}$$

where the Dirac adjoint is given by $\overline{\psi} = \psi^\dagger \gamma^0$, and we interpret m as a mass term. Broadly speaking, there are three types of spinors — Dirac, Weyl, and Majorana. Dirac spinors are the standard, complex-valued solution to the Dirac equation. Moreover, Dirac spinors can be decomposed into the sum of two spinors of different chirality (right- and left-handed) — these are known as Weyl spinors. Finally, we have Majorana spinors, which are rather curious entities. For starters, they do not exist in every dimension and, additionally they represent particles which are their own *antiparticle*. This means that they carry a purely real representation, thus having only half the degrees of freedom of a Dirac spinor. There have been many some claims that Majorana spinors hold the key to describing neutrinos [36, 37], stemming from the fact that Majorana spinors are generally massive, as well as how they behave under *charge conjugation*. Charge conjugation is a transformation wherein we swap all particles with their antiparticles within the Lagrangian for the system. We find that Majorana spinors are their own antiparticle, which implies that they are not only neutral, but also highly stable. We similarly observe neutrinos to display a high degree of stability, and we find that there is strong experimental evidence which now suggests that neutrinos are massive [38]. For Majorana spinors, we find [39] the Dirac adjoint is simply $\overline{\psi} = \psi^\intercal C$, where C is the charge conjugation matrix. It was established that the charge conjugation matrix in geometric algebra is simply γ^{12}.

3.2. Grassmann Numbers

As we discussed at the start of this section, we find that two identical fermions are antisymmetric under exchange. This can be explained by the Spin-Statistics theorem, which tells us that fermions cannot occupy the same state, ultimately leading us to the Pauli exclusion principle. Grassmann numbers are exceptionally bizarre objects for those encountering them for the first time. The obey the relation $\theta_i \theta_j = -\theta_j \theta_i \, \forall i, j$. In other words, we have that $\theta_i^2 = 0$ so they are *nilpotent*. But this is *exactly* what we need to represent the anticommuting nature of fermions — we also see, from their nilpotence, that two identical fermions cannot occupy the same state at once! It is important to add that Grassmann-odd numbers will commute with anything that is not Grassmann-odd itself — we refer to objects which commute as Grassmann-even, as they can be formed from the product of an even number of Grassmann-odd numbers. It may have now become apparent that there already exists an operation which plays a similar role in GA — namely the wedge product. In fact, much work has already been done in GA with respect to how one might represent Grassmann numbers [40, 41]. However, it ought to be emphasized that, until this project, nobody has managed to find a working description for spinors with Grassmann-odd characteristics. Having provided a taste of the bizarre nature of Grassmann numbers, as well as how they apply to representations of fermions, we will now discuss the construction of spinors in geometric algebra.

3.3. Spinors as Minimal Ideals

Although minimal ideals are a well-established means of constructing spinors within Clifford algebras [42], the implementation has been all but forgotten within geometric algebra. This is primarily because the GA community favour the even-subalgebra representation of spinors [16, 43, 44, 45, 46]. In effect, this means that spinors are represented via the linear combination of *purely* even-grade basis elements. There is a degree of sense to this approach — for instance this representation is certainly closed under the action of rotors (as per Eq. (4)), and it has been shown by Hestenes [45] that we can construct a working Dirac spinors in $(1+3)$-dimensional spacetime, as well as Pauli spinors in three Euclidean dimensions [43]. However, this is likely due to accidental isomorphisms, since we very quickly find that the even-subalgebra construction does not provide a sufficiently complete description of spinors in other dimensions with different signatures. We can prove this to be the case in a by simply counting the degrees of freedom provided to us by the even-subalgebra representation, and compare it against the degrees of freedom required to describe the various types of spinors that might arise in a particular dimension, with a particular signature. A key innovation of this project, since we choose to work with Majorana spinors, is to use the minimal ideal construction. In particular, this is because we found the even-subalgebra representation to be insufficient for describing Majorana spinors.

A *left ideal* $\mathcal{L}$ of the Geometric algebra $\mathcal{G}$ is a subset where, $\forall a \in \mathcal{G}$ and $\psi \in \mathcal{L}$, we have $a\psi \in \mathcal{L}$. Similarly, for the subset $\mathcal{R}$ containing *right ideals*, we have $\forall a \in \mathcal{G}$ and $\chi \in \mathcal{R}$, $\chi a \in \mathcal{R}$. We can construct left and right minimal ideals through the use of a projection operator (*projector*) which is an *idempotent* (i.e. it squares to the identity or, in the case of GA, a scalar), as one might expect. We will now outline this process in 2-D Euclidean space. For the benefit of visualisation, a matrix representation, alongside the GA formalism, will be provided.

For two-dimensional Euclidean space, we have two (orthonormal) basis vectors $\{\gamma^1, \gamma^2\}$ which form an extended basis $\{1, \gamma^1, \gamma^2, \gamma^1\gamma^2\ (\equiv \gamma^{12})\}$ via the *geometric product*. One can further identify these elements with the Pauli matrices and the identity

$$\begin{aligned} 1 \quad &\longleftrightarrow \quad \mathbb{1} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad & \gamma^1 \quad &\longleftrightarrow \quad \sigma_1 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \gamma^2 \quad &\longleftrightarrow \quad \sigma_3 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad & \gamma^{12} \quad &\longleftrightarrow \quad -i\sigma_2 \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{15}$$

Now, if we define our projector as $P_\pm = \frac{1}{2}(1 \pm \gamma^2)$, we can see that this will have a matrix representation in our chosen basis of

$$P_+ \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_- \;=\; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{16}$$

If we were now to *right* multiply a general multivector $\mathcal{G} \ni v = v_0 + v_1\gamma^1 + v_2\gamma^2 + v_3\gamma^{12}$ by the projector P_+, we would find ourselves with a left ideal

$$\begin{array}{ccc} vP_+ = (v_0 + v_1\gamma^1 + v_2\gamma^2 + v_3\gamma^{12})\frac{1}{2}(1+\gamma^2) & \leftrightarrow & \begin{pmatrix} v_0+v_2 & v_1 - v_3 \\ v_1+v_3 & v_0 - v_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \Downarrow & & \Downarrow \\ \frac{1}{2}(1+\gamma^2)(v_0+v_2) + \frac{1}{2}(\gamma^1+\gamma^{12})(v_1+v_3) & \leftrightarrow & \begin{pmatrix} v_0+v_2 & 0 \\ v_1+v_3 & 0 \end{pmatrix}. \end{array} \tag{17}$$

This is a 2-dimensional Euclidean Majorana spinor. We can see that it is essentially a column spinor. For the sake of notational simplicity, we also introduce following notation $\theta_\pm = \frac{1}{2}(\gamma^1 \pm \gamma^{12})$ so that $\psi = \psi_1 P_+ + \psi_2\theta_+$ and $\psi^\intercal = \psi_1 P_+ + \psi_2\theta_-$ (where we have defined $\psi_1 = v_0 + v_2$ and $\psi_2 = v_1 + v_3$). Once more, visualising geometric algebra with matrices allows us to represent $\theta_\pm$ as

$$\theta_+ \;=\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \theta_- \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{18}$$

The introduction of the $\theta_\pm$ terms will somewhat simplify the calculations, as we will see in the next section. Hereafter, they will be referred to as *nilpotents*, since $\theta_\pm^2 = 0$. Before we go on, it should be noted that one can obtain the equivalent of the complex conjugate transpose of this spinor via the *right* minimal ideal. The calculation is much the same except now we *left* multiply the projector with the reversion of the vector $\tilde{v} = v_0 + v_1\gamma^1 + v_2\gamma^2 - v_3\gamma^{12}$. This will become relevant later when we consider the product $\bar{\psi}\psi$.

3.4. Useful Identities

What follows is a list of products between *projectors* and *nilpotents*

$$\begin{array}{lllllllllll} P_\pm P_\pm &=& P_\pm\,, & P_\pm P_\mp &=& 0\,, & P_\pm\theta_\pm &=& 0\,, & P_\pm\theta_\mp &=& \theta_\mp\,, \\ \theta_\pm\theta_\pm &=& 0\,, & \theta_\pm\theta_\mp &=& P_\mp\,, & \theta_\pm P_\pm &=& \theta_\pm\,, & \theta_\pm P_\mp &=& 0\,. \end{array} \tag{19}$$

We also have

$$P_\pm\gamma^{12} = -\theta_\mp\,, \qquad \theta_\pm\gamma^{12} = P_\mp\,, \tag{20}$$

which we shall use to calculate $\bar{\psi}$.

3.5. A Massive Inconsistency

Given that $\bar{\psi} = \psi^\intercal C$, where C is the charge conjugation matrix, we are now tantalisingly close to being able to perform this simple calculation, provided we can find a suitable representation of C in Geometric algebra. And, as previously stated, this comes in the form of γ^{12} Recalling that $\psi^\intercal = P_+\tilde{v}$, all that remains is the calculation itself

$$\begin{aligned} \bar{\psi}\psi &= \psi^\intercal\hat{\gamma}\psi \\ &= (\psi_0 P_+ + \psi_1\theta_-)\hat{\gamma}(\psi_0 P_+ + \psi_1\theta_+) \\ &= (\psi_0(-\theta_-) + \psi_1 P_+)(\psi_0 P_+ + \psi_1\theta_+) \\ &= -\psi_0^2\ \theta_- P_+ - \psi_0\psi_1\ \theta_-\theta_+ + \psi_0\psi_1\ P_+P_+ + \psi_1^2\ P_+\theta_+) = 0\,, \end{aligned} \tag{21}$$

where we have made use of Eqs. (19) and (20). The result of this calculation would suggest that Majorana spinors are massless, which is decidedly not so. Where did we go wrong? In our haste to form something with spinorial properties, we forgot that the spinor components are Grassmann-odd valued. How might one fix this? We explore this idea in the following subsection.

3.6. Introducing Grassmann Numbers

The previous construction has one glaring issue in that there is no inclusion of Grassmann-odd characteristics in the spinor components. This led to the disastrous result in Eq. (21) that $\bar{\psi}\psi = 0$. But all is not lost! There are a couple of ways in which we could manually insert Grassmann-oddness into our spinor components.

One such way is to introduce a separate 2-dimensional space, containing non-orthonormal basis vectors that we will denote as η_i. We then take the inner product of these vectors with arbitrary vectors a and b to form scalar-valued objects. With these alterations in mind, $\bar{\psi}\psi$ now takes the following form

$$\bar{\psi}\psi = [(\eta_1 \cdot a)P_+ + (\eta_2 \cdot a)\theta_-]\, \gamma^{12}\, [(\eta_1 \cdot b)P_+ + (\eta_2 \cdot b)\theta_+]. \tag{22}$$

It ought to be noted that we have been rather loose with our notation — we are essentially working with two different spinors now since there is generally no way to obtain $\bar{\psi}$ from ψ, unless $a = b$. But, if we ignore this for now, we see that we have essentially the same construction as before, albeit worded in a slightly more complicated manner — we still have scalar spinor components, and the only reason that our result is nonzero is because we have distinguished them through their dot products. So how do we make this work in our favour? Well, we can now *force* the anti-commuting quality by introducing the bivector-valued derivative $\partial_a \wedge \partial_b$. This now gives us the expression

$$\bar{\psi}\psi = \partial_a \wedge \partial_b \big\{[(\eta_1 \cdot a)P_+ + (\eta_2 \cdot a)\theta_-]\, \gamma^{12}\, [(\eta_1 \cdot b)P_+ + (\eta_2 \cdot b)\theta_+]\big\}. \tag{23}$$

Given that we omitted geometric calculus in our prior discussion of geometric algebra in Section 2, we will now explicitly demonstrate how one would evaluate the above expression

$$\begin{aligned}
\bar{\psi}\psi &= \partial_a \wedge \partial_b \big([(\eta_1 \cdot a)P_+ + (\eta_2 \cdot a)\theta_-]\, \gamma^{12}\, [(\eta_1 \cdot b)P_+ + (\eta_2 \cdot b)\theta_+]\big) \\
&= \partial_a \wedge \partial_b \big([-(\eta_1 \cdot a)\theta_- + (\eta_2 \cdot a)P_+][(\eta_1 \cdot b)P_+ + (\eta_2 \cdot b)\theta_+]\big) \\
&= \partial_a \wedge \partial_b \big[-(\eta_1 \cdot a)(\eta_1 \cdot b)\theta_- P_+ - (\eta_1 \cdot a)(\eta_2 \cdot b)\theta_-\theta_+ + (\eta_2 \cdot a)(\eta_1 \cdot b)[P_+]^2 + (\eta_2 \cdot b)(\eta_2 \cdot a)P_+\theta_+\big] \\
&= \partial_a \wedge \partial_b \big[(-(\eta_1 \cdot a)(\eta_2 \cdot b) + (\eta_2 \cdot a)(\eta_1 \cdot b))P_+\big] \\
&= \partial_a \wedge \big[(-(\eta_1 \cdot a)\partial_b(\eta_2 \cdot b) + (\eta_2 \cdot a)\partial_b(\eta_1 \cdot b))P_+\big] \\
&= (-\partial_a(\eta_1 \cdot a) \wedge \eta_2 + \partial_a(\eta_2 \cdot a) \wedge \eta_1)P_+ \\
&= (-\eta_1 \wedge \eta_2 + \eta_2 \wedge \eta_1)P_+ \\
&= 2(\eta_2 \wedge \eta_1)P_+,
\end{aligned} \tag{24}$$

where — as in Eq. (21) — we have made use of Eqs. (19) and (20). Upon evaluation, we see that we end up with the geometric product of a bivector with a projector. Strictly speaking, it would be nicer if we simply ended up with a pure bivector since this would reflect the product of two Grassmann-odd components. But this is a relatively easy fix since we can simply take the grade-projection of Eq. (24) onto a grade-2 subspace. Since P_+ is comprised of a scalar and a vector, the geometric product $2(\eta_2 \wedge \eta_1)P_+$ will contain a vector, a bivector, and a *trivector* (grade-3) term. This is a perfectly valid operation, and one which is frequently carried out in GA literature [16]. Hence we find

$$\langle \bar{\psi}\psi \rangle_2 = \eta_2 \wedge \eta_1, \tag{25}$$

where we lost the factor of two from $P_+ = \frac{1}{2}(1 + \gamma^2)$. And with that, we have now formulated the first ever GA construction of spinors which incorporate Grassmann-odd behaviour into their construction! For this project, we postulate another construction which we outline in Appendix B.

4. Discrete Geometric Algebra

As we have already outlined in Section 1, many of the key theories of QG incorporate lattice field theories into their approach. This process requires the discretization of the geometry of spacetime and, in the case of dynamical triangulations, we choose to work with triangulated manifolds — often referred to as *simplicial complexes*. We thus provide a brief description of how to generate simplices in GA, as well as how one might use these objects to define *frames*. Following on from that, we present the first ever formulation of discrete parallel transport in GA. Finally, we go on to demonstrate that it provides us with a notion of curvature through the *deficit angle*.

4.1. Simplices in Geometric Algebra

For a more in-depth discussion of simplices in GA, we direct the reader to [47]. We will, however, provide a brief overview below.

Given a smooth manifold, one can select a set of ordered points $\{p_i\}$, where i uniquely labels each point. We connect these points with directed lines so as to generate simplices — for a 2D surface, these would be 2-simplices — and these simplices act to provide an approximate curvature of the space (see Section 4.3). Typically, one can select points such that the directed lines each have unit length. As we previously discussed in Section 1, one can produce random manifolds by generating a set of simplices and gluing them together based around a probability distribution. One often employs what is referred to as a *Pachner move* [48] — wherein one deletes an edge of two adjoined simplices and inserts another *different* edge — to systematically transform one triangulation to another. While we considered the option of translating this into GA, given the simplex setup we are about to outline, Pachner moves would remain essentially unchanged and we would simply need to redefine what constitutes our simplices. We will now examine the specific case of two 2-simplices adjoined along an edge, see Fig. 2. For a subset of points from our manifold, say $\{p_i\}_{i=1,2,3,4}$, we can then characterize the vectors which define *directed edges* by the following set of equations

$$\begin{array}{lllllllll} E_1^A & = & p_2 - p_1 \ , & E_2^A & = & p_3 - p_2 \ , & E_3^A & = & p_1 - p_3 \ , \\ E_1^B & = & p_4 - p_2 \ , & E_2^B & = & p_3 - p_4 \ , & E_3^B & = & p_2 - p_3 \ . \end{array} \tag{26}$$

This allows us to construct an area element, which we refer to as the *directed content* ΔX. We observe that the directed content for simplex A is given by [47, 16]

$$\Delta X_A = \frac{1}{2} E_1^A \wedge (-E_3^A) = \frac{1}{2}(p_1 \wedge p_2 + p_2 \wedge p_3 + p_3 \wedge p_1). \tag{27}$$

But, rather than working with these fixed geometric objects like vertices and edges, it would be more useful if we could define *frames* which we will transform with respect to the background simplex. As

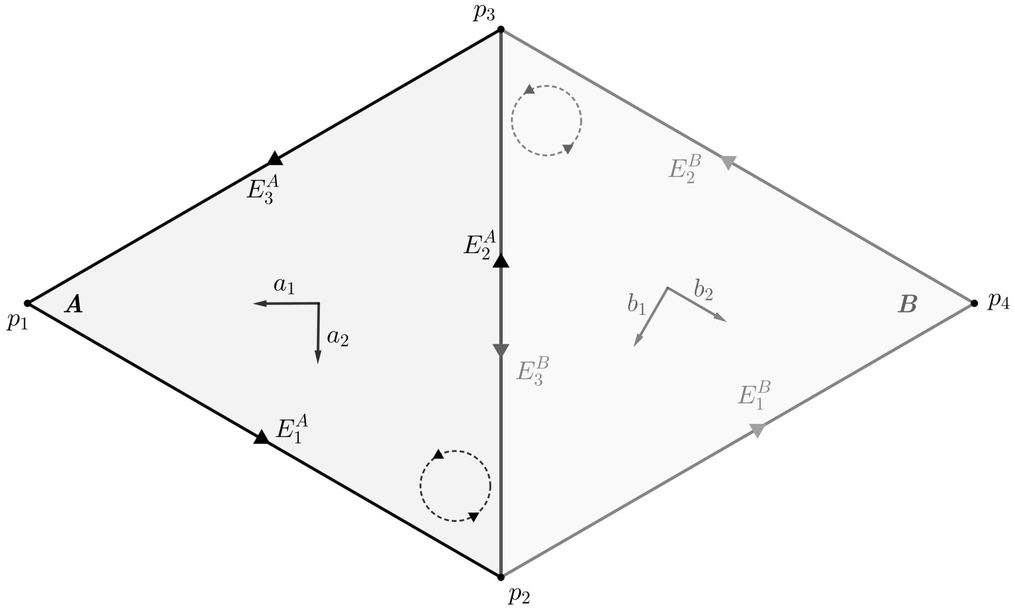


Figure 2: Two 2-simplices, labelled A and B, 'glued' along a single edge. Each 2-simplex is formed from three 1-simplices (edges) — which are labelled E_r^I with I denoting which simplex the directed edge belongs to, and r providing each 1-simplex with a unique arbitrary label — and three 0-simplices (vertices) — these are labelled by $\{p_i\}$ which are a set of ordered points which form a subset of a smooth manifold. Note that the edges specify a anticlockwise *handedness*. As such, for the simplices A and B to have the same orientation, the vectors which define their shared edge— labelled here as E_2^A and E_3^B — will be anti-parallel. We have also included orthonormal frames, labelled as $\{a_i\}$ and $\{b_i\}$, which can be constructed from the elements which define each 2-simplex.

such, we can construct an orthonormal frame $\{a_i\}_{i=1,2}$, see Fig. 2, where

$$a_1 = \tfrac{1}{\sqrt{3}}(E_3^A - E_1^A), \qquad a_2 = -E_2^A. \tag{28}$$

By construction, the wedge of this newly defined frame defines a unit bivector, which we label as A. The relation between A and ΔX_A is therefore given by

$$A \equiv a_1 \wedge a_2 = \frac{1}{\sqrt{3}}(E_2^A \wedge E_3^A + E_1^A \wedge E_2^A) = \frac{2}{\sqrt{3}}(p_1 \wedge p_2 + p_2 \wedge p_3 + p_3 \wedge p_1) = \frac{4}{\sqrt{3}}\Delta X_A\ , \tag{29}$$

where we have used Eqs. (26) and (27). Given that the area of a unit equilateral triangle is $\frac{\sqrt{3}}{4}$, this relation is exactly what one would expect — it likewise holds for ΔX_B and B. Having defined a set of frames for our simplices which we are free to rotate in three-dimensions, we can now discuss parallel transport.

4.2. Parallel Transport of Tetrads

When we discretize a manifold we greatly simplify the geometry, but we must keep in mind that we would like to recover our original *smooth* manifold in the limit as $N \to \infty$, where N is the number of simplices we are using to approximate our manifold structure. We will discover in Section 4.3 that we maintain a notion of the curvature of the manifold, but before encountering that, we should discuss *parallel transport*.

To recover our original manifold in the smooth limit, we need to be careful to include a *discretized* description of the parallel transport of geometric objects about the simplicial complex. This is not as difficult as it may sound at first — indeed, by working with a discretized manifold, we have essentially reduced our spacetime into the union of N local frames. In doing so, we have passively allowed ourselves to define a *local basis* which (on a smooth manifold in $(3+1)$-dimensions) would transform under the Lorentz group. To provide some further intuition, one can imagine a local frame attached to an observer. As that observer moves through spacetime, their local frame will undergo a $(3+1)$-dimensional rotation along that trajectory. In four-dimensions, we call these objects *tetrads* or *vierbeins*‡, and we can now take advantage of our discrete setup to define parallel transport through the rotation of these tetrads§. Of course, we should remember that we are working in a 2-dimensional Euclidean space embedded in 3-dimensions and, as such, we no longer have to contend with the Lorentz group $SO(1,3)$, but rather the group of rotations in 3-D space — $SO(3)$.

So, having said all that, how do we actually parallel transport our tetrads? In standard linear algebra, one approach might be to use rotation matrices to take us from a frame on one simplex to the next. However, this becomes a rather challenging operation once we move into the territory of $SO(3)$. Fortunately, we are not using linear algebra but *geometric algebra* and, as we have already seen in Section 2.3, we can describe the rotation of various geometric objects by making use of *rotors*. We will now outline our devised method for the parallel transport of frames on a discretized manifold using GA. This method takes heavy inspiration from Burda [18, 19, 20] with the added ingredient of working in an embedded picture.

As already outlined by Burda [19, 20], we can break down the parallel transport of tetrads from one simplex to another into a three-step process (see Fig. 3):

(i) Anticlockwise rotation of the initial tetrad frame onto the first intermediary frame.

(ii) Anticlockwise rotation onto adjacent simplex such that the tetrad is aligned with the second intermediary frame.

(iii) Clockwise rotation within the adjacent simplex, from the second intermediary frame to the 'established frame' associated with the adjacent simplex.

As we established earlier, the tetrad frame of the first simplex — in order to retain some consistency, let's call it simplex A — can be derived from the simplex structure itself. We will label the frame as $\{a_i\}$ where $i = 1,2$. In a similar vein, we will label the frame of simplex B as $\{b_i\}$ with $i = 1,2$. Having built up all the intuition we need with regards to the construction of rotors (see Section 2.3 for a reminder), we recall that a rotor is defined by two quantities — the angle we wish to rotate our object by, and the plane of rotation. We therefore anticipate this three-step rotation to be of the form $\underline{\mathsf{R}}\big(\{a_i\}\big) = R_3\, R_2\, R_1\, \{a_i\}\, \tilde{R}_1\, \tilde{R}_2\, \tilde{R}_3$. As described in Section 2.3, we know that we define our planes

‡ It should not go unsaid that the names 'tetrad' and 'vierbein' specifically refer to 4-D space. Since we are working in 2-D subspaces, a better name for these objects might be 'dyad' or 'zweibein'. We, however, will not be using this terminology to avoid potential confusion with readers already familiar with tetrads.

§ It is also worth commenting that, through the introduction of tetrads, we now allow ourselves the possibility of coupling spinors to our spacetime — ordinarily, one would have to contend with the general linear group $GL(4,\mathbb{R})$ when considering the symmetries of our spacetime manifold. By altering our description to that of tetrads, we can now bring in spinors which transform under the *double cover* of the Lorentz group.

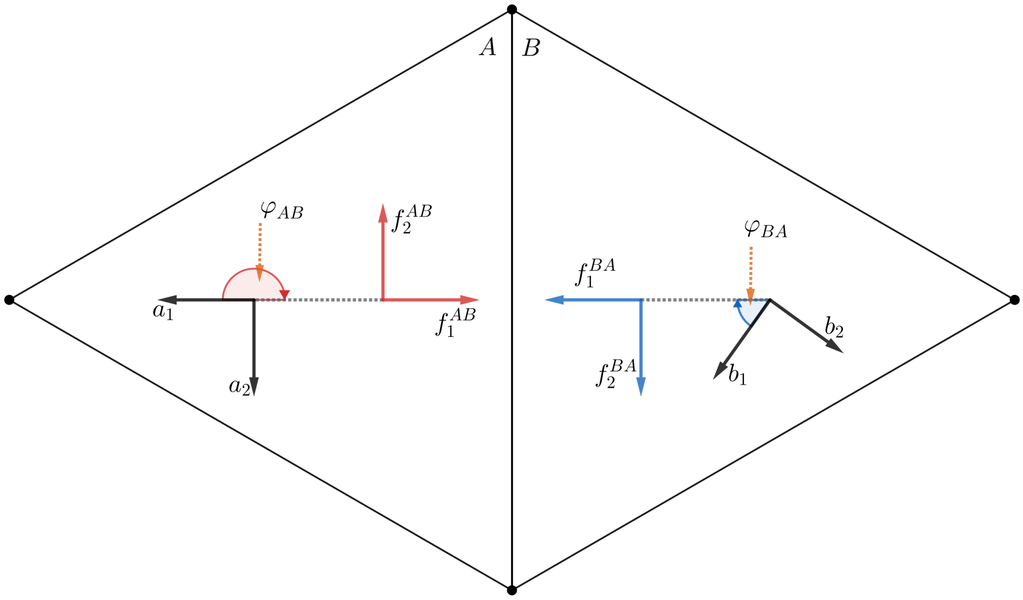


Figure 3: A pictorial demonstration of parallel transport of tetrad frame between adjacent simplices. Transporting the tetrad in A to B, we see that we perform an anticlockwise rotation of the initial frame $\{a_i\}$ by φ_{AB} onto the intermediary frame, labelled $\{f_i^{AB}\}$. We then rotate (anticlockwise) this intermediary frame onto a second intermediary frame $\{f_i^{BA}\}$, before rotating (clockwise) by the angle φ_{BA} onto the frame $\{b_i\}$. Note that each angle labelled in the diagram is defined with respect to the anticlockwise rotation of the "established" frames ($\{a_i\}$ and $\{b_i\}$) of each simplex onto the intermediary frames.

of rotation through bivectors, which we will express like so

$$a_1 a_2 = a_1 \wedge a_2 \equiv \boldsymbol{A}, \qquad b_1 b_2 = b_1 \wedge b_2 \equiv \boldsymbol{B}. \tag{30}$$

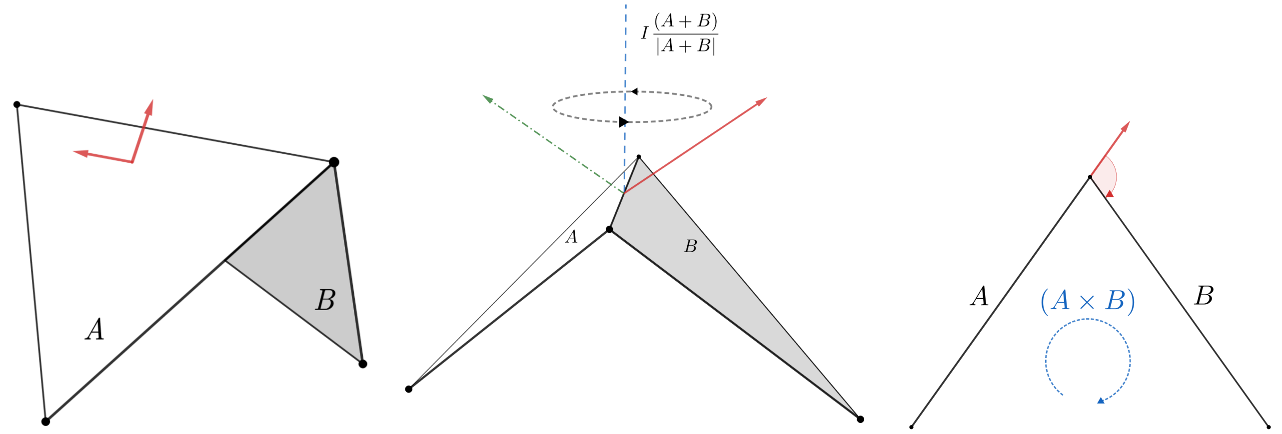
With this, we can now simply write down the first and last rotor as

$$R_1 \equiv \exp\left(-\tfrac{1}{2}\, \boldsymbol{A}\, \varphi_{AB}\right), \qquad R_3 \equiv \exp\left(\tfrac{1}{2}\, \boldsymbol{B}\, \varphi_{BA}\right), \tag{31}$$

where φ_{MN} ($\boldsymbol{M}$ and N can be A or B depending on the context) denotes the angle which rotates the frame from its established position (associated with the simplex) onto the intermediary frame (see Fig. 3). Note that we have different signs in our exponents to indicate anticlockwise or clockwise rotations.

Wonderful — we have managed to make considerable progress with very little work. But how should we define the rotation from simplex A onto simplex $\boldsymbol{B}$ — specifically, what *plane* are we rotating in? There are two obvious candidates for the bivector of our middle rotor: we could either rotate in the plane that is perpendicular to both $\boldsymbol{A}$ and $\boldsymbol{B}$, see Fig. 4c (which is given by the commutator of the two bivectors $\boldsymbol{A} \times \boldsymbol{B}$), or we could rotate by π in an intermediary plane which sits between $\boldsymbol{A}$ and $\boldsymbol{B}$ — given by $(\boldsymbol{A} + \boldsymbol{B})/|\boldsymbol{A} + \boldsymbol{B}|$, see Fig. 4b.

It so happens that rotating in the plane $(\boldsymbol{A} + \boldsymbol{B})/|\boldsymbol{A} + \boldsymbol{B}|$ leads to a far simpler expression to evaluate, as we shall see in Eq. (36). In which case, we obtain an expression for our middle rotor



(a) This diagram demonstrates the state of our frame (given by the orthogonal arrows) just before we rotate it onto simplex B. We have positioned the frame such that it overhangs past the adjoining edge of A and B so as to help with the visualisation of the rotation.

(b) Rotating by π in the plane defined by $\frac{(A+B)}{|A+B|}$ (denoted with a dashed circle, with arrows to indicate the anticlockwise orientation), we can rotate our frame on A (marked by the solid arrow) onto the frame on B (marked by the dash-dotted line). We have also provided the normal to the plane of rotation (dashed straight line), given by its dual $I(A+B)/|A+B|$.

(c) Observing the adjoined 2-simplices along the cross-section containing their dihedral angle, we see that we can rotate the frame of A onto B by a rotation in the plane of $A \times B$ (marked by the dashed circle, with an arrow to indicate its orientation). Explicitly, one would rotate by $\varphi = (\pi - \text{dihedral angle})$.

Figure 4: A collection of diagrams which show the two simplest ways that one might choose to rotate the intermediary frame of A onto the intermediary frame of B.

as $R_2 \equiv \exp(-\frac{1}{2}\frac{(A+B)}{|A+B|}\pi)$, which we can further simplify, using Eq. (10) to give

$$\exp\left(-\frac{1}{2}\frac{(A+B)}{|A+B|}\pi\right) = \cos\pi/2 - \frac{(A+B)}{|A+B|}\sin\pi/2 = -\frac{(A+B)}{|A+B|}. \tag{32}$$

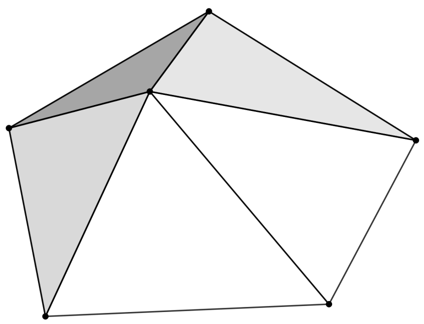
We now have a description of discrete parallel transport within the setting of GA. To assuage any doubts the reader might have about the validity of the rotation within the intermediary plane $(A+B)/|A+B|$, we provide an explicit calculation in Appendix C. This allows us to write our newly-formed discrete parallel transport operator as

$$\underline{\mathsf{R}}_{A\to B}(a) = R_{A\to B}\, a\, \tilde{R}_{A\to B}\,, \quad \text{with} \quad R_{A\to B} = \exp\left(\tfrac{1}{2}\,B\,\varphi_{BA}\right)\left(-\tfrac{A+B}{|A+B|}\right)\exp\left(-\tfrac{1}{2}\,A\,\varphi_{AB}\right). \tag{33}$$

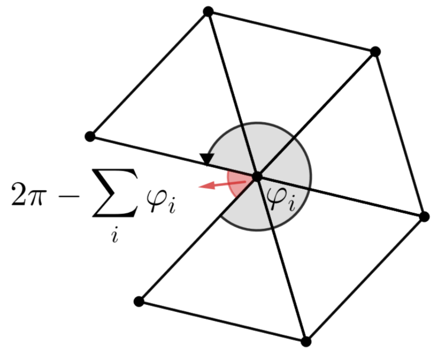
Ordinarily, we would be unable to simplify this expression further as, recalling from Section 2.1, the geometric product is generally non-commutative. However, due to our clever choice of intermediary rotor, we will find in the next section that Eq. (33) *can* indeed be simplified. Let us now see what we can do with this rotor!

4.3. The Deficit Angle

One of the key features of Regge Calculus [49] is that we can describe the local curvature of a discretized manifold through the *deficit angle*. One can quickly develop an intuitive understanding of this concept, and we shall outline it below. As has already been established, one can triangulate a d-dimensional manifold by tessellating it with d-simplices. We 'glue' these simplices along their $(d-1)$-faces, and we claim that any information about the curvature is located at the $(d-2)$-faces [49].



(a) A chain of five 2-simplices sharing a single vertex. The simplices are shown with all relevant edges glued together, so it exists as a two-dimensional surface in a three-dimensional space.



(b) A chain of five 2-simplices, now with a cut along one of the glued edges such that they form a "net" in two-dimensional Euclidean space. The deficit angle is indicated in red, with an arrow pointing to the equation for said deficit angle. Each 2-simplex contributes an angle of φ_i to this expression.

Figure 5: Two diagrams of a chain of five 2-simplices sharing a common vertex. Fig. 5a demonstrates the collective 3-D structure, whereas Fig. 5b provides the same structure if it was 'unfolded' into a 2-D Euclidean plane.

For instance, in the case where $d = 2$, we have 2-simplices (triangles), which we join along their adjacent edges, see Fig. 5a. Any information about the curvature is then contained at the vertices of this simplicial complex. If we were to then cut along one of the edges within a chain of simplices and 'fold out' this chain so that it now forms a 'net' in the 2-dimensional Euclidean plane (Fig. 5b), we would generally find that there is a gap between two of the simplices (i.e. the two simplices that were previous glued together along an adjoined edge), see Fig. 5. This gap corresponds to an angular deficit which we label as the *deficit angle*. Explicitly, the angular deficit is given by $2\pi - \sum_i \varphi_i$, where φ_i is the "edge-vertex-edge" angle, as shown in Fig. 5. It is easy to imagine cases where we have a large, positive deficit angle — corresponding to a large, positive curvature located at that vertex. Similarly, if we had a *negative* deficit angle (i.e. in the case where the simplex chain wraps around itself when we fold it out), this would correspond to a negative curvature located at a vertex.

Before we go on, let us now take a brief pause from all the maths and return to the broader physical picture. When working with LFTs, one often works with *gauge field theories* (GFTs). These are field theories which, due to redundant degrees of freedom relating to how the system has been mathematically described, we find that the Lagrangian is invariant under certain smooth transformations. As such, it is useful to define operators that are similarly *gauge invariant*. One such example of gauge invariant operators are *Wilson Loops* [50] — formulated by, and named after Kenneth Wilson — and they act to provide us with information on how two points in spacetime might differ with regards to this gauge symmetry. One could think of Wilson loops as providing phase factors (and this viewpoint informs the theory of the famous Aharonov-Bohm experiment [51]). But perhaps a more useful analogy, in our case, would be that of parallel transport. In GR, we find that we can relate two vectors living in different tangent spaces through the introduction of a *connection*. The idea is very much the same in GFTs. As a final point, it ought to be highlighted that, by embedding our space in a higher dimension, we have the potential to introduce new *gauge degrees of freedom* into our system. It is also worth noting that we actually already have gauge degrees of freedom present in our theory, based on the fact that we can freely re-orientate our tetrad frames, provided that we keep certain relations between each frame fixed, as we shall soon see. In our specific case, the connection is equivalent to the rotors that we established in the previous section. We likewise find that, in LFTs, when we take the trace of the product of connections which form a loop about a point, we generate a Wilson loop. As such, we will now explore parallel transport about a closed loop.

It has been shown [52] that if we parallel transport a vector about a closed loop which encircles a vertex, we find that the vector has now rotated by the deficit angle associated with that vertex. We will now outline how one can acquire this same result through our discrete parallel transport operator Eq. (33).

We will first consider the case for a closed loop around the shared vertex of three adjoined simplices. We denote the operation as

$$\underline{\mathsf{L}}_3(a) = \underline{\mathsf{R}}_{C\to A}\,\underline{\mathsf{R}}_{B\to C}\,\underline{\mathsf{R}}_{A\to B}(a) = R_{C\to A}\,R_{B\to C}\,R_{A\to B}\;a\;\tilde{R}_{A\to B}\,\tilde{R}_{B\to C}\,\tilde{R}_{C\to A}\,, \tag{34}$$

where a is an arbitrary vector, and we have defined $C \equiv c_1 \wedge c_2$. Likewise, from Eq. (33), we have $R_{M\to N} = \mathrm{e}^{\frac{1}{2}N\varphi_{NM}}\left(-\frac{M+N}{|M+N|}\right)\mathrm{e}^{-\frac{1}{2}M\varphi_{MN}}$, where $M \neq N$ can be $\{A, B, C\}$ in the relevant context. With our earlier discussion on Wilson loops in mind, we see that the linear function $\underline{\mathsf{L}}_3$ is formed of the product of rotors (or connections) which collectively define parallel transport in a closed loop. We can surmise that, if we were to take the trace of Eq. (34), we would find ourselves with a Wilson loop. We will explicitly do this once we have simplified our product of rotors. We can write the composite rotor $R_{C\to A}\,R_{B\to C}\,R_{A\to B}$ as

$$\begin{aligned}
&\mathrm{e}^{\frac{1}{2}A\varphi_{AC}}\left(-\frac{C+A}{|C+A|}\right)\mathrm{e}^{-\frac{1}{2}C\varphi_{CA}}\,\mathrm{e}^{\frac{1}{2}C\varphi_{CB}}\left(-\frac{B+C}{|B+C|}\right)\mathrm{e}^{-\frac{1}{2}B\varphi_{BC}}\,\mathrm{e}^{\frac{1}{2}B\varphi_{BA}}\left(-\frac{A+B}{|A+B|}\right)\mathrm{e}^{-\frac{1}{2}A\varphi_{AB}}\\
&= \mathrm{e}^{\frac{1}{2}A\varphi_{AC}}\left(-\frac{C+A}{|C+A|}\right)\mathrm{e}^{-\frac{1}{2}C(\varphi_{CA}-\varphi_{CB})}\left(-\frac{B+C}{|B+C|}\right)\mathrm{e}^{-\frac{1}{2}B(\varphi_{BC}-\varphi_{BA})}\left(-\frac{A+B}{|A+B|}\right)\mathrm{e}^{-\frac{1}{2}A\varphi_{AB}}.
\end{aligned} \tag{35}$$

Where the last equality comes from the fact that a bivector commutes with itself. It is now that we get to discuss one of the delightful properties of the intermediary bivectors $\frac{M+N}{|M+N|}$. We can show

$$\begin{aligned}
M(M+N) = M^2 + MN = -1 + MN &\iff (M+N)N = MN + N^2 = -1 + MN\\
N(M+N) = NM + N^2 = -1 + NM &\iff (M+N)M = M^2 + NM = -1 + NM.
\end{aligned} \tag{36}$$

This means that we can treat the intermediary bivectors $(M + N)$ as a 'gateway' for the other bivectors wherein we enter from one side with $M(N)$, and leave the other side with $N(M)$. What's more, this process applies to functions of bivectors, provided that they can be expanded in a Taylor series. In light of this, we can make one further simplification to Section 4.3 by passing each exponential through the intermediary bivectors until we are left with the following

$$(-1)^3\left(\frac{C+A}{|C+A|}\right)\left(\frac{B+C}{|B+C|}\right)\left(\frac{A+B}{|A+B|}\right)\mathrm{e}^{-\frac{1}{2}A\left[(\varphi_{AB}-\varphi_{AC})+(\varphi_{BC}-\varphi_{BA})+(\varphi_{CA}-\varphi_{CB})\right]}. \tag{37}$$

This can, of course, be generalised to loops with n simplices about a single vertex. All that is left for us to evaluate is the product of the gateway bivectors. In order to simplify things, we will remove all normalization terms for now, with the assurance that we will re-insert them shortly. The product of the three bivectors on the left is

$$\begin{aligned}(A+C)(C+B)(B+A) &= A(C+B)A + A(C+B)B + C(C+B)A + C(B+C)B \\ &= A(C+B)A - (B+C) + A(C+B)B + (C+B)BA \\ &= 2[A\cdot(C+B)]A + 2[(C+B)\cdot B]A + 2[(C+B)\times B]\cdot A \\ &= 2[(A+B)\cdot(C+B)]A + 2[(C+B)\times B]\cdot A,\end{aligned} \tag{38}$$

where we have made use of Eq. (36) in line 2, and have taken advantage of symmetric and anti-symmetric products to simplify the result in line 3. We can now bring back the normalization term and, given that our result is the sum of a bivector and a scalar as well as the magnitude of the three gateway bivectors being unity, we can now repackage our result from Eq. (38) as an exponential

$$2\mathcal{N}_0\big([(A+B)\cdot(C+B)]A + [(C+B)\times B]\cdot A\big) = \mathrm{e}^{-\frac{1}{2}A\theta}. \tag{39}$$

Putting this back together with Eq. (37), we have the following result

$$R_{C\to A}R_{B\to C}R_{A\to B} \equiv \mathrm{e}^{-\frac{1}{2}A\Delta\phi}, \tag{40}$$

where we take $\Delta\phi$ to be the deficit angle. In other words, if we were to transport an object about a vertex in a closed loop, this would be equivalent to a rotation of that object in the simplex A. One should note that Burda carried out a very similar calculation in [18, 19]. If we were to define our angles φ_{MN} in the same manner as he did, we would not find ourselves with the deficit angle. However, given that we have not defined a rigid position for the 'established frames' for each simplex, we find that we can simply calibrate our parameters in φ_{MN}, to come to the result quoted in Section 4.3. There is fundamentally nothing wrong with this approach as we have sufficient freedom over how the positioning of our 'established frames', as expressed by Burda himself [18, 19, 20].

As we already mentioned above, we can generalize this to a loop comprised of n simplices, and the working is much the same, albeit slightly more involved. To give a taste of this derivation, we write $\mathcal{K} = (B+C)(C+D)...(M+N)$, so that the complete product of intermediary bivectors is given by

$$(A+B)\,\mathcal{K}\,(N+A). \tag{41}$$

We know from Eq. (4) that a product of bivectors will give us a solution which can be decomposed into a scalar and a bivector

$$\mathcal{K} = \langle\mathcal{K}\rangle_0 + \langle\mathcal{K}\rangle_2. \tag{42}$$

But, before we make use of this, we should utilise our findings in Eq. (38) — in fact, it may now be clear *why* we chose to expand our equation in that manner — we can write Eq. (41) in much the same way! Furthermore, the result from Eq. (36) allows us bring the 'end terms' through the chain of gateway bivectors, so that we end up with

$$\begin{aligned}&2\mathcal{N}_0\big[A\,\mathcal{K}\,A - \mathcal{K} + A\,\mathcal{K}\,B + \mathcal{K}\,BA\big] \\ &= 2\mathcal{N}_0\big[\big((A+B)\cdot\langle\mathcal{K}\rangle_2\big)A\big] + 2\mathcal{N}_0\big[\big(\langle\mathcal{K}\rangle_2\times B\big)\cdot A + \big((A\cdot B-1)\langle\mathcal{K}\rangle_0\big].\end{aligned} \tag{43}$$

Once more, we have decomposed our result into the sum of a grade-2 blade and a scalar with a magnitude of 1. We can thus write Eq. (43) as an exponential, with an identical form to that of Eq. (39). Hence, we generally find that $\underline{\mathsf{L}}_n(a) = \mathrm{e}^{-\frac{1}{2}A\Delta\phi}\, a\, \mathrm{e}^{\frac{1}{2}A\Delta\phi}$.

Returning at last to Wilson loops, we find that the trace of a linear function $\underline{\mathsf{F}}$ is given by Tr $\left[\underline{\mathsf{F}}\right] = \partial_v \cdot \underline{\mathsf{F}}(v)$, where v is an arbitrary vector [16, 14]. Moreover, one can show that Tr $\left[\underline{\mathsf{L}}_n\right] = 1 + 2\cos(\Delta\phi)$, thus we have an expression for our Wilson loop. We will not develop this any further in this report, but this object will certainly be important for further research. Furthermore, while Wilson loops are well-established within LFTs, we have not found any mention of it within GA literature.

We will now go on to outline in the next section how one might parallel transport spinors across simplicial complexes, and how we might further use our results to potentially establish the first ever Dirac–Wilson operator in GA.

5. Dirac–Wilson Operator

Previously, we demonstrated how one might recover Grassmann-odd behaviour in spinors (see Section 3) and, in doing so, we implicitly calculated the mass-like part of the Dirac Lagrangian ($\mathcal{L} \supset -m\overline{\psi}\psi$). All we require now is the *kinetic* part and this is what is given by the Dirac–Wilson operator. The Dirac–Wilson operator serves as an extension to the kinetic term in the previously discussed Dirac Lagrangian Eq. (14). Specifically, it allows us to couple fermions to gravity [18, 19, 20]. Now, we will look at how one might transform spinors from one frame to another in a discretized setting. However, in order for us to justify some arguments for spinor transformations, we will have to *quickly* revisit the double-sided parallel transport transformation which we just applied to tetrads. Note that we will be replacing our notation for spinors in Section 3 with that of the previous section, (this is merely a cosmetic choice, and does not reflect a change in the physics). Furthermore, we will initially simplify our calculation by working exclusively in a two-dimensional Euclidean plane, later relaxing this constraint to allow us to consider a ‘curved’ (triangulated) surface embedded in a three-dimensional Euclidean space.

Transformations in Flat Space

5.1. Tetrads

We begin by considering an orthonormal frame $\{a_i\}_{1=1,2}$ where we use (lowercase) "a" to refer to a basis on (uppercase) simplex A. Bear in mind that, on a *globally* flat geometry, we can simply think of this as a label to distinguish different orientations of frames on the same plane. These vectors generate basis elements for the *augmented frame* $\mathcal{G}(A) = \{1, a_1, a_2, A\}$. Now, if we wish to rotate our frame from $\mathcal{G}(A)$ onto $\mathcal{G}(B)$, we can simply do the following

$$b_i = e^{-\frac{1}{2}I\varphi_{AB}}\, a_i\, e^{\frac{1}{2}I\varphi_{AB}}, \tag{44}$$

where we take φ_{AB} as the angle between frame A and frame B, as before, and I is a unit bivector (in this particular geometry, it can be regarded as the pseudoscalar, hence we label it with an "I", as per Eq. (2)). We similarly find, due to the grade-preserving nature of the rotor function $\underline{\mathsf{R}}(a \wedge b) = \underline{\mathsf{R}}(a) \wedge \underline{\mathsf{R}}(b)$, that we can relate $\mathcal{G}(A)$ to $\mathcal{G}(B)$ like so

$$\mathcal{G}(B) = e^{-\frac{1}{2}I\varphi_{AB}}\, \mathcal{G}(A)\, e^{\frac{1}{2}I\varphi_{AB}} = \underline{\mathsf{R}}^{\mathrm{F}}_{AB}(\mathcal{G}(A)), \tag{45}$$

where "F" is used to indicate that we are working in a flat space.

5.2. Spinors

So far, so good it seems. But what if we now attempted to do the same for spinors? We'll first work with spinors *without* the previously established Grassmann-odd characteristics, only inserting them when we formulate the Dirac–Wilson operator. We will now introduce a new notation for our spinors. Recall that we can define our spinors in GA as $\psi = \psi_1 P_+ + \psi_2 \theta_+$ (see Section 3.3 for a reminder). We will now write our idempotents and nilpotents as $P^A_\pm = \frac{1}{2}(1 \pm a_2)$, and $\theta^A_\pm = \frac{1}{2}(a_1 \pm A)$, where the superscripts indicate the simplex associated with the transformation. Presently, we introduce the notation $\psi^I(\mathcal{G}(J))$ to indicate a spinor on simplex I, expressed in terms of the basis elements in the set $\mathcal{G}(J)$. We will now naively consider

$$\begin{aligned} \psi^B\big(\mathcal{G}(B)\big) &= \psi^B\Big(\underline{\mathsf{R}}^{\mathrm{F}}_{AB}\big(\mathcal{G}(A)\big)\Big) \\ &= \psi^B_1\ \underline{\mathsf{R}}^{\mathrm{F}}_{AB}\big(P^A_+\big) + \psi^B_2\ \underline{\mathsf{R}}^{\mathrm{F}}_{AB}\big(\theta^A_+\big) \\ &= \underline{\mathsf{R}}^{\mathrm{F}}_{AB}\big(\psi^B_1 P^A_+ + \psi^B_2 \theta^A_+\big) \\ &= e^{-\frac{1}{2}I\varphi_{AB}}\left[\psi^B_1 P^A_+ + \psi^B_2 \theta^A_+\right] e^{\frac{1}{2}I\varphi_{AB}}. \end{aligned} \tag{46}$$

Alarmingly, by expressing ψ^B in terms of the frame $\mathcal{G}(A)$, we obtain something that demonstrably does *not* transform like a spinor. Put another way, an observer on simplex A will see a spinor on simplex B as an arbitrary multivector. There are no physical grounds for this to be the case, so we will have to amend this transformation slightly. Recall that we can form our spinors by taking the geometric product of an arbitrary multivector — which we'll label as $\mathcal{M}$ to simplify our algebra — with a projector Eq. (17). We can then make the argument that the *projector* undergoes a single-sided transformation giving us

$$\mathcal{M}P^A_+ \longrightarrow \mathcal{M}' \left[P^A_+\right]' = \left[R\,\mathcal{M}\,\tilde{R}\right]\left[R\,P^A_+\right] = R\,\mathcal{M}P^A_+, \tag{47}$$

where R is an arbitrary rotor. Expressed a different way, we find

$$\psi^B\big(\mathcal{G}(B)\big) = e^{-\frac{1}{2}I\varphi_{AB}}\left[\psi^B_1 P^A_+ + \psi^B_2 \theta^A_+\right]. \tag{48}$$

Success! The object before us, without a doubt, can be thought of as a spinor (see Section 3.3). It is worth noting that this modification is not too dissimilar to the transformation of a vector field [44] $\phi(x)$

$$x' = e^{-A\varphi/2}\ x\ e^{A\varphi/2}\,, \tag{49a}$$

$$\phi'(x) = e^{A\varphi/2}\ \phi(x')\ e^{-A\varphi/2}, \tag{49b}$$

except, obviously only having a single-sided rotor transformation in the spinor case. And with that, we are ready to consider curved spaces.

Transformations in Curved Space

We can now simply replace our rotors in a globally flat space, with the ones we developed in Section 4.2. We find that we have an identical calculation to that of Eq. (48), with the only change being that we replace the rotor $e^{-\frac{1}{2}I\varphi_{AB}}$ with $\tilde{R}_{A\to B} = \mathrm{e}^{-\frac{1}{2}B\varphi_{BA}}\left(\frac{A+B}{|A+B|}\right)\mathrm{e}^{\frac{1}{2}A\varphi_{AB}}$

5.3. The Dirac–Wilson Operator

At this stage, let us take stock of what we have achieved thus far: we have produced a discretized model of gravity. Next we devised a process by which we can parallel transport tetrads and spinors across a simplicial complex. Moreover, we completely reformulated how spinors are expressed within GA. We essentially have all the ingredients we could possibly need to form the Dirac–Wilson operator.

Burda writes in his papers on fermions on random lattices in two-dimensions [18, 19, 20] that the Dirac Wilson operator is given by

$$\mathcal{D}_{ij} = -K\mathcal{A}_{ij}\mathcal{H}_{ij} + \delta_{ij}\mathbb{1}, \tag{50}$$

where Burda labels his simplices with i, j. He defines the terms $\mathcal{A}_{ij} = 1$ iff $i$and$j$ are neighbours and $\mathcal{A}_{ij} = 0$ otherwise. And, most importantly, Burda introduces the *hopping operator*, which removes a fermion on one simplex, and inserts it in the adjacent one. Explicitly, Burda gives the hopping operator as

$$\mathcal{H}_{ij} = \frac{1}{2}[\mathcal{B}_j^{(i)}]^{-1}\left[1-\gamma_1\right]\epsilon\, \mathcal{B}_i^{(j)}, \tag{51}$$

where Burda labels his simplices with i, j, and $\mathcal{B}_j^{(i)}$ is used to denote a rotation matrix which rotates a spinor from frame (i) to frame (j). An argument could be made, however, that the expression ought to be

$$\mathcal{H}_{ij} = \frac{1}{2}s_{ij}\left[1-\mathcal{B}_j^{(i)}\gamma^1[\mathcal{B}_j^{(i)}]^{-1}\right][\mathcal{B}_j^{(i)}]^{-1}\epsilon\, \mathcal{B}_i^{(j)}. \tag{52}$$

Unfortunately, we are currently waiting for a response from Professor Burda. Once we have a definitive answer, we will likely be able to form a fully-functional Dirac–Wilson operator for the first time ever in GA.

6. Concluding remarks

As is now apparent, the scope for this project was vast. We therefore will provide a quick reminder of everything that we have done. To summarize:

(i) We have successfully adapted Sobczyk's work on simplices in GA [47], to form a frames which we can freely parallel transport across a simplicial complex.

(ii) We have developed a description in GA for discrete parallel transport, thus discretizing gravity.

(iii) We have shown that parallel transport in a closed loop about a single vertex is equivalent to a rotation within the initial plane by the deficit angle.

(iv) We have formed Wilson loops by taking the trace of the product of rotors in a closed loop.

(v) We have explored the formulation of the Dirac–Wilson operator, as well as the parallel transport of spinors, thus having discretized spinors pending a response from Professor Burda.

(vi) Furthermore, we have managed to form the union of Grassmann-odd numbers with spinors so as to provide an accurate representation of fermionic matter in GA. This has been a long-standing desire within the GA community [40].

Looking back over the sum of work that formed this project, we are reminded of our initial discussion which began with the consideration of GR. General relativity provides a perfectly viable continuum model of gravity and, upon the coupling of the theory to fermions through the introduction of tetrads, we find that the natural next step would be the inclusion of torsion. What we have just described is the underlying theory of the immensely popular Einstein–Cartan gravity [53] (see also [54, 55, 56, 57, 58]) which, as you may recall from Section 2, was re-expressed with geometric algebra in Gauge Theory Gravity[23]. This was a huge success for the Cavendish AP Group, drawing an enormous amount of attention and cementing Cambridge as a pillar of the GA community. GTG is, however, a continuum model and as we outlined in the introduction, can thus take us only so far. What we have accomplished is the construction of the groundwork what we believe to be the first-ever discretized model of gravity in GA. As a reminder for why this is significant, we choose to discretize gravity as a lattice model, because it allows us to probe the possibility of a textitnon-perturbative renormalizable theory of QG, building on the ideas of Wilson [8].

Going forward, it is safe to say that we have many ideas we would like to develop further. Primarily, we would like to reconstitute the torsion already present in GTG and incorporate it into our discrete model. As mentioned earlier in this report, we would like to make use of the extra degrees of freedom provided to us by the embedded picture, feasibly allowing us to introduce another gauge theory into our model. For instance, we hope to relax our constraint on working with equilateral triangles and, in doing so, we could explore the possibility of including information about torsion in the 3N degrees of freedom associated with the positioning of the vertices of each simplices. Alternatively, there the possibility that we could explore torsion as dislocations on a lattice model — in any case, we expect that geometric algebra would likely lend itself exceptionally well to either picture, as it has done here.

Acknowledgments

I would like to extend my sincerest gratitude to Dr William Barker who has provided me with countless hours of his time, as well as many enlightening discussions. Additionally, I would like to thank Professor Zdzisław Burda, and Professor David Tong for steering me in the right direction.

Appendix A. Bivectors and Imaginary Numbers

As mentioned in Section 2.3, there is a strong association to be made between bivectors and imaginary numbers. We used the fact that bivectors square to -1 to argue that the exponential of a bivector could be decomposed into a form similar to that of Euler's formula. We will now demonstrate this to be the case. Without loss of generality, we can say that a unit bivector can be written as the wedge product between two orthonormal vectors. Then, working in $d = 3$ with orthonormal vectors b_1 and b_2, we have

$$\begin{aligned}
B^2 &= (b_1 \wedge b_2)(b_1 \wedge b_2) = \langle (b_1 \wedge b_2)(b_1 \wedge b_2)\rangle + \langle (b_1 \wedge b_2)(b_1 \wedge b_2)\rangle_2 \\
&= (b_1 \wedge b_2) \cdot (b_1 \wedge b_2) \equiv ((b_1 \wedge b_2) \cdot b_1) \cdot b_2 \\
&= -(b_1 \cdot b_1)(b_2 \cdot b_2) + (b_1 \cdot b_2)^2 \\
&= -1,
\end{aligned} \tag{A.1}$$

where we have made use of the standard result in GA $a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b$, as well as the orthonormality of the vectors b_1 and b_2, see [16] for more details.

Appendix B. Another Spinor Formulation

Given that in Section 3.6, we had to expand the dimension of our space to accommodate Grassmann-oddness, perhaps it would be worth examining yet another way of forming the product $\bar{\psi}\psi$. What if, rather than insisting that our spinor components are scalar-valued, we instead relaxed this constraint and allowed them to be vectors? The only difficulty would then be ensuring that these new objects still maintain enough spinor-like characteristics. Let us define our new spinor-esque objects like so

$$\psi = P_+\eta_1 + \theta_+\eta_2, \tag{B.1}$$

where η_i are vectors in this supplementary two-dimensional space, as before. The order here is very important as we find that *left* multiplication by a rotor R results in the immediate transformation of the spinor-like parts. Specifically, we have

$$R\psi = RP_+\eta_1 + R\theta_+\eta_2. \tag{B.2}$$

By a similar reasoning, we would require that $\bar{\psi}$ takes the form

$$\bar{\psi} = \eta_2 P_+ - \eta_1\theta_-. \tag{B.3}$$

Having established our spinor and its Dirac conjugate, let's examine their product

$$\begin{aligned}\bar{\psi}\psi &= [\eta_2 P_+ - \eta_1\theta_-][P_+\eta_1 + \theta_+\eta_2] \\ &= \eta_2 P_+ P_+\eta_1 + \eta_2 P_+\theta_+\eta_2 - \eta_1\theta_- P_+\eta_1 - \eta_1\theta_-\theta_+\eta_2 \\ &= \eta_2 P_+\eta_1 + 0 + 0 - \eta_1 P_+\eta_2 \\ &= \eta_2 P_+\eta_1 - \eta_1 P_+\eta_2\,. \end{aligned} \tag{B.4}$$

Recalling that $P_+ = \frac{1}{2}(1+\gamma^2)$, we obtain

$$\begin{aligned}\bar{\psi}\psi &= \eta_2 P_+\eta_1 - \eta_1 P_+\eta_2 \\ &= \frac{1}{2}[\eta_2(1+\gamma^2)\eta_1 - \eta_1(1+\gamma^2)\eta_2] \\ &= \frac{1}{2}[\eta_2\eta_1 - \eta_1\eta_2] + \frac{1}{2}[\eta_2\gamma^2\eta_1 - \eta_1\gamma^2\eta_2] \\ &= \eta_2 \wedge \eta_1 + \frac{1}{2}[\eta_2\gamma^2\eta_1 - \eta_1\gamma^2\eta_2]\,. \end{aligned} \tag{B.5}$$

We *could* continue evaluating this expression, but we could also save ourselves some time by recognising that the term $\frac{1}{2}[\eta_2\gamma^2\eta_1 - \eta_1\gamma^2\eta_2]$ is comprised of vector and trivector graded objects. As such, we can simply take grade-projection onto the grade-2 subspace mirroring what we did in the previous section. This leaves us with the following

$$\langle\bar{\psi}\psi\rangle_2 = \eta_2 \wedge \eta_1\,. \tag{B.6}$$

If Eq. (B.2) is sufficient to describe the transformation properties of a spinor, then we now have found ourselves with a formulation free from the inclusion of arbitrary vectors, as in Eq. (24). However, we have not yet constructed a method for parallel transport for this object due to the uncertainty in its validity as a representation of a spinor, so we have relegated this suggestion to the appendices. This will likely be the focus of future work.

Appendix C. Intermediary Plane Justification

We will now show that the action of a rotor by π radians in the plane defined by $(A+B)/|A+B|$ on a vector in simplex A results in a vector which exists purely in simplex B. Let a be a general vector which lies in simplex A. We argue that a rotation from A to B is given by

$$R_2\, a\, \tilde{R}_2 = \left(-\frac{A+B}{|A+B|}\right) a \left(\frac{A+B}{|A+B|}\right). \tag{C.1}$$

As we know from Eqs. (11a) and (11b), we can decompose a into components that lie parallel and perpendicular to the plane B. Crucially, both of these components still lie within A, which we will use to our advantage. Explicitly, we have

$$\begin{aligned}
-\frac{A+B}{|A+B|}\, a\, \frac{A+B}{|A+B|} &= -\frac{A+B}{|A+B|}\,(a_B + a_{\not B})\,\frac{A+B}{|A+B|} \\
&= -\frac{A+B}{|A+B|}\, a_B\, \frac{A+B}{|A+B|} - \frac{A+B}{|A+B|}\, a_{\not B}\, \frac{A+B}{|A+B|} \\
&= a_B \left(\frac{A+B}{|A+B|}\right)^2 - a_{\not B}\left(\frac{A-B}{|A+B|}\right)\left(\frac{A+B}{|A+B|}\right) \\
&= -a_B - 2a_{\not B}\,\frac{A\times B}{|A+B|^2},
\end{aligned} \tag{C.2}$$

where we have made use of Eqs. (12a) and (12b) between lines 2 and 3, and that $A^2 = B^2 = \left[(A+B)/|A+B|\right]^2 = -1$ in the final line. We have also made use of Eq. (5) between the penultimate and ultimate lines. It is obvious that a_B still lies in the plane of B, so we can simply disregard it. As such, our question has shifted to ask whether $a_{\not B}\,(A\times B)$ lies in B (ignoring the prefactors for now, since they will only scale our rotated vector).

Now, we *could* expand out the entire geometric product $a_{\not B}\,(A\times B)$ to see if we end up with components that are exclusively in B (this might be a fun exercise to see what runs out first — the ink, the pages, or our sanity). However, it is far less costly to simply observe whether there any components of $a_{\not B}\,(A\times B)$ which lie perpendicular to B. In other words, we ask if $\left(a_{\not B}\,(A\times B)\right)_{\not B} = 0$?

To make this calculation easier, we first observe the following

$$\begin{aligned}
B(A\times B) &= \frac{1}{2}B\,(AB - BA) = \frac{1}{2}\,(BAB + A) \\
(A\times B)B &= \frac{1}{2}\,(AB - BA)\,B = -\frac{1}{2}\,(BAB + A) \\
\Rightarrow\ B(A\times B) &= -(A\times B)B.
\end{aligned} \tag{C.3}$$

Recalling Eqs. (12a) and (12b) once more, we note that if a vector is entirely parallel with a bivector, they anticommute. We likewise know that $a_{\not B}B = Ba_{\not B}$. Coupling this fact with Eq. (C.3), we can now say that $Ba_{\not{B}}(A \times B) = -a_{\not{B}}(A \times B)\, B$. Hence, $a_{\not{B}}(A \times B) \in B$.

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