Gravitomagnetism and galaxy rotation curves: a cautionary tale

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We investigate the recent claim by Ludwig [Eur. Phys. J. C 81, 186 (2021)] that gravitomagnetic effects in linearised general relativity can explain flat and rising rotation curves, such as those observed in galaxies, without the need for dark matter. Ludwig's modelling of a galaxy as an axisymmetric, stationary, rotating, non-relativistic and pressureless 'dust' of stars in the gravitoelectromagnetic (GEM) formalism leads to a coupled system of equations that admit a straightforward order of magnitude analysis. We show that gravitomagnetic effects on the circular velocity v of a star are $\mathscr{O}(10^{-6})$ smaller than the standard Newtonian (gravitoelectric) effects and thus any modification of galaxy rotation curves must be negligible, as might be expected. Moreover, we find that gravitomagnetic effects are $\mathscr{O}(10^{-6})$ too small to provide the vertical support necessary to maintain the dynamical equilibrium assumed in Ludwig's model. These issues are obscured when various quantities are eliminated between the system of equations to arrive at the single key equation for v used by Ludwig. We nevertheless solve Ludwig's equation for the case of a galaxy having a Miyamoto–Nagai density profile since this allows for both an exact numerical integration and an accurate analytic approximation. We show that for the derived values of the mass, M, and semi-major and semi-minor axes, a and b, obtained by Ludwig in fitting rotation curve data for NGC 1560, the rotation curve depends only very weakly on the mass M and is independent of M for larger values. We also show that for aspect ratios $a/b > 2$, the resulting rotation curves are concave over their entire range, which does not match observations in any galaxy. Most importantly, we show that, in order to provide the necessary vertical support, the poloidal gravitomagnetic flux ψ must become singular at the origin and have extremely large values near to it. This originates from the unwitting, but forbidden, inclusion of free-space solutions of the Poisson-like equation that determines ψ and also clearly contradicts the linearised treatment implicit in the GEM formalism, hence ruling out the methodology in the form used by Ludwig as a means of explaining flat galaxy rotation curves.

I. INTRODUCTION

It is widely accepted that the modelling of galaxy rotation curves in general relativity (GR) requires the inclusion of a dark matter halo in order to reproduce observations [1–4]. In particular, the modelling of the approximately flat rotation curves observed in the outskirts of large spiral galaxies and, to a lesser extent, the rising rotation curves observed in smaller dwarf galaxies [5–9] is considered to pose a significant challenge to GR without such a component. The absence of any direct experimental evidence for dark matter [10] has thus led to the consideration of various modified gravity theories to attempt to explain the astrophysical data.

There are a number of claims in the literature, however, that such modifications are unnecessary since hitherto neglected effects in GR itself are capable of explaining rotation curves without dark matter. These include gravitoelectric flux confinement arising from graviton self-interaction [11–18], non-linear GR effects arising even in the weak-gravity regime [19] and, most recently, gravitomagnetic effects in linearised GR [20]. An immediate question regarding such claims is how such significant behaviours can have been consistently missed in the long history of numerical relativity [21, 22], or in the well-developed post-Newtonian formalism [23, 24]. Perhaps unsurprisingly therefore, the claims in [11–18] and [19] have been subsequently shown to be non-viable in [25] and [26], respectively. The purpose of this paper is to perform the same function for the claim in [20], by showing that gravitomagnetism in the form used therein cannot explain flat or rising galaxy rotation curves without dark matter.

Our findings concur with the recent results reported in [27], where the gravitoelectromagnetic formulation of linearised GR was used to predict galaxy rotation curves that at all radii differ from those of Newtonian theory at the order of only $v^2/c^2 \approx 10^{-6}$, as one might expect. The main focus of the present paper, however, is to clarify *why* the approach adopted in [20] leads to such different, unexpected and incorrect results, which is not addressed in [27].

The remainder of this paper is arranged as follows. In Section II, we briefly outline linearised GR, focussing on stationary non-relativistic matter sources, and discuss its expression in the GEM formalism in Section III. We then summarise in Section IV the application of the GEM formalism to the modelling of galaxy rotation curves, as proposed in [20]. We lay out the problems with this modelling approach in Section V, before concluding in Section VI.

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II. LINEARISED GENERAL RELATIVITY

In the weak gravitational field limit appropriate for modelling galaxy rotation curves, there exist quasi-Minkowskian coordinate systems $x^\mu = (ct, x^i)$ in which the spacetime metric takes the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $|h_{\mu\nu}| \ll 1$ and the first and higher partial derivatives of $h_{\mu\nu}$ are also small.[1]. One can conveniently reinterpret $h_{\mu\nu}$ simply as a special-relativistic symmetric rank-2 tensor field that represents the weak gravitational field on a Minkowski background spacetime and possesses the gauge freedom $h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$. Imposing the Lorenz gauge condition $\partial_\rho \bar{h}^{\mu\rho} = 0$ on the trace-reverse $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$, where $h = \eta_{\mu\nu} h^{\mu\nu}$, the linearised GR field equations reduce to the simple form

$$\Box^2 \bar{h}^{\mu\nu} = -2\kappa T^{\mu\nu}, \tag{1}$$

where $\Box^2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the d'Alembertian operator, $\kappa = 8\pi G/c^4$ is Einstein's gravitational constant and $T^{\mu\nu}$ is the matter energy-momentum tensor.

For modelling galaxy rotation curves, it is sufficient to a very good approximation to limit one's considerations to stationary, non-relativistic, perfect fluid matter sources. In this case, $\partial_0 T^{\mu\nu} = 0$ and the coordinate 3-speed u of any constituent particle is small enough compared with c that one may neglect terms of order u^2/c^2 and higher in $T^{\mu\nu}$; in particular one may take $\gamma_u = (1 - u^2/c^2)^{-1/2} \approx 1$. Moreover, the fluid pressure p is everywhere much smaller than the energy density and may thus be neglected as a source for the gravitational field. Finally, we note that $|T^{ij}|/|T^{00}| \sim u^2/c^2$ and so one should take $T^{ij} \approx 0$ to the order of our approximation. Thus, for a stationary, non-relativistic source, one approximates its energy-momentum tensor as

$$T^{00} \approx \rho c^2, \qquad T^{i0} \approx c\rho u^i, \qquad T^{ij} \approx 0, \tag{2}$$

where $\rho(\boldsymbol{x})$ is the proper-density distribution of the source and $\boldsymbol{x}$ denotes a spatial 3-vector. As an immediate consequence, the particular integral of (1) yields $\bar{h}^{ij} \approx 0$. Indeed, this is consistent with the Lorenz gauge condition, which implies that $\partial_j \bar{h}^{ij} = -\partial_0 \bar{h}^{i0}$, where the right-hand side vanishes for stationary systems. Thus, only the $\bar{h}^{00}$ and $\bar{h}^{0i} = \bar{h}^{i0}$ components of the gravitational field tensor are non-zero in this approximation.

In linearised GR, there is an inconsistency between the field equations (1) and the equations of motion for matter in a gravitational field. From (1), one quickly finds that $\partial_\mu T^{\mu\nu} = 0$, which should be contrasted with the requirement from the full GR field equations that the covariant divergence should vanish, $\nabla_\mu T^{\mu\nu} = 0$. The latter requirement leads directly to the geodesic equation of motion for the worldline $x^\mu(\tau)$ of a test particle, namely

$$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\sigma}\dot{x}^\nu \dot{x}^\sigma = 0, \tag{3}$$

where the dots denote differentiation with respect to the proper time τ, whereas the former requirement leads to the equation of motion $\ddot{x}^\mu = 0$. This means that the gravitational field has *no effect* on the motion of the particle and so clearly contradicts the geodesic postulate. Despite this inconsistency, one may show that the effect of weak gravitational fields on test particles may still be computed by inserting the linearised connection coefficients into the geodesic equations (3).

III. GRAVITOELECTROMAGNETISM

Gravitoelectromagnetism (GEM) provides a useful and notionally-familiar formalism for linearised GR by drawing a close analogy with classical electromagnetism (EM). Indeed, GEM is ideally suited to modelling galaxy rotation curves, since the assumption of a stationary, non-relativistic matter source leads to GEM field equations and a GEM 'Lorentz' force law (derived below) that are fully consistent and have forms analogous to their counterparts in EM; this is not possible for more general time-dependent scenarios.

The GEM formalism for linear GR with a stationary, non-relativistic source is based on the simple ansatz of relabelling[2] the six independent non-zero components of $\bar{h}^{\mu\nu}$ as $\bar{h}^{00} \equiv 4\Phi/c^2$ and $\bar{h}^{0i} \equiv A^i/c$, where we have defined the gravitational scalar potential Φ and spatial gravitomagnetic vector potential A^i. On lowering indices, the corresponding components of $h_{\mu\nu}$ are $h_{00} = h_{11} = h_{22} = h_{33} = 2\Phi/c^2$ and $h_{0i} = A_i/c$. It should be remembered that raising or lowering a spatial (Roman) index introduces a minus sign with our adopted metric signature. Thus the numerical value of A_i is minus that of A^i, the latter being the ith component of the spatial vector $\boldsymbol{A}$. It is also worth noting that both Φ/c^2 and A_i/c are dimensionless, thereby yielding dimensionless components $h_{\mu\nu}$, which is consistent with our choice of coordinates $x^\mu = (ct, x^i)$ having dimensions of length.

With the above identifications, the linearised field equations (1) with energy-momentum tensor (2) may be written in the scalar/vector form

$$\nabla^2\Phi = 4\pi G\rho, \qquad \nabla^2 \boldsymbol{A} = \frac{16\pi G}{c^2}\boldsymbol{j}, \tag{4}$$

where we have defined the momentum density (or matter current density) $\boldsymbol{j} \equiv \rho\boldsymbol{u}$, and the Lorenz gauge condition $\partial_\rho \bar{h}^{\mu\rho} = 0$ itself becomes $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$. Clearly, the

[1] We adopt the following sign conventions: $(+,-,-,-)$ metric signature, $R^\rho{}_{\sigma\mu\nu} = 2(\partial_{[\mu}\Gamma^\rho{}_{|\sigma|\nu]} + \Gamma^\rho{}_{\lambda[\mu}\Gamma^\lambda{}_{|\sigma|\nu]})$, where the metric (Christoffel) connection $\Gamma^\rho{}_{\lambda\mu} = \frac{1}{2}g^{\rho\sigma}(\partial_\lambda g_{\mu\sigma} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma g_{\lambda\mu})$, and $R^\rho{}_\mu = R^{\rho\sigma}{}_{\mu\sigma}$.

[2] Conventions in the literature vary up to a multiplicative constant for the definition of the gravitomagnetic vector potential A^i. These factors variously modify the analogues of the EM field equations and the Lorentz force law, with no scaling choice allowing all the GEM and EM equations to be perfectly analogous. Here, we follow the convention used in [28].

first equation in (4) recovers the Poisson equation for the gravitational potential, familiar from Newtonian gravity, whereas the second equation determines the gravitomagnetic vector potential that describes the 'extra' (weak) gravitational field predicted in linearised GR, which is produced by the motion of the fluid elements in a stationary, non-relativistic source. Indeed, the general solutions to the equations (4) are given immediately by

$$\begin{aligned} \Phi(\boldsymbol{x}) &= -G\int \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3\boldsymbol{x}', && \text{(5a)}\\ \boldsymbol{A}(\boldsymbol{x}) &= -\frac{4G}{c^2}\int \frac{\boldsymbol{j}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3\boldsymbol{x}'. && \text{(5b)} \end{aligned}$$

One may take the analogy between linearised GR and EM further by defining the gravitoelectric and gravitomagnetic fields $\boldsymbol{E} = -\boldsymbol{\nabla}\Phi$ and $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$, which are easily found to satisfy the gravitational Maxwell equations

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{E} &= -4\pi G\rho, & \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0,\\ \boldsymbol{\nabla}\times\boldsymbol{E} &= \boldsymbol{0}, & \boldsymbol{\nabla}\times\boldsymbol{B} &= -\frac{16\pi G}{c^2}\boldsymbol{j}. \end{aligned} \tag{6}$$

The gravitoelectric field $\boldsymbol{E}$ describes the standard (Newtonian) gravitational field produced by a static matter distribution, whereas the gravitomagnetic field $\boldsymbol{B}$ is the 'extra' gravitational field produced by moving fluid elements in the stationary, non-relativistic source.

The equation of motion for a test particle in the presence of the gravitoelectromagnetic fields is merely the geodesic equation (3) for the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, from which one may determine the trajectories of either massive particles, irrespective of their speed, or massless particles, by considering timelike or null geodesics, respectively. We will assume here, however, that the test particle is massive and slowly-moving, i.e. its coordinate 3-speed v is sufficiently small that we may neglect terms in v^2/c^2 and higher. Hence we may take $\gamma_v = (1 - v^2/c^2)^{-1/2} \approx 1$, so that the 4-velocity of the particle may be written $v^\mu = \gamma_v(c, \boldsymbol{v}) \approx (c, \boldsymbol{v})$. This immediately implies that $\ddot{x}^0 = 0$ and, moreover, that $dt/d\tau = 1$, so one may consider only the spatial components of (3) and replace dots with derivatives with respect to t. Expanding the summation in (3) into terms containing, respectively, two time components, one time and one spatial component, and two spatial components, neglectng the purely spatial terms since their ratio with respect to the purely temporal term is of order v^2/c^2, expanding the connection coefficients to first-order in $h_{\mu\nu}$ and remembering that for a stationary field $\partial_0 h_{\mu\nu} = 0$ and that one inherits a minus sign on raising or lower a spatial (Roman) index, one finally obtains the gravitational Lorentz force law

$$\frac{d\boldsymbol{v}}{dt} = -\boldsymbol{\nabla}\Phi + \boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}. \tag{7}$$

The first term on the right-hand side gives the standard Newtonian result for the motion of a test particle in the field of a static, non-relativistic source, whereas the second term gives the 'extra' force felt by a moving test particle in the presence of the 'extra' field produced by moving fluid elements in the stationary, non-relativistic source.

IV. GRAVITOELECTROMAGNETIC MODELLING OF GALAXY ROTATION CURVES

The GEM formalism is applied to the modelling of galaxy rotation curves in [20], where the galactic density and velocity distribution is assumed to act as a stationary, non-relativistic matter source. Thus, somewhat unusually, the fluid pressure is assumed to vanish and the galaxy is instead modelled as consisting of a 'dust' of stars. This approach therefore uses the field equations (4) and the equation of motion (7), where the velocity distribution $\boldsymbol{u}$ of the galaxy in the former is identified with the velocity $\boldsymbol{v}$ of test particles in the latter, thereby leading to a self-consistent pressureless model.

The central result in [20] can be derived straightforwardly as follows. First, one adopts cylindrical polar coordinates (R, ϕ, z) and assumes azimuthal symmetry, such that $\rho = \rho(R, z)$ and $\boldsymbol{v} = v(R, z)\hat{\boldsymbol{\phi}}$, which from (5) implies that $\Phi = \Phi(R, z)$ and $\boldsymbol{A} = A(R, z)\hat{\boldsymbol{\phi}}$. In this case,

$$\begin{aligned} \boldsymbol{\nabla}\times\boldsymbol{A} &= \frac{1}{R}\left(-\frac{\partial\psi}{\partial z}\hat{\boldsymbol{R}} + \frac{\partial\psi}{\partial R}\hat{\boldsymbol{z}}\right), && \text{(8a)}\\ \boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) &= \frac{v}{R}\left(\frac{\partial\psi}{\partial R}\hat{\boldsymbol{R}} + \frac{\partial\psi}{\partial z}\hat{\boldsymbol{z}}\right), && \text{(8b)} \end{aligned}$$

where we have defined the poloidal gravitomagnetic flux $\psi \equiv RA$. Also, in light of the Lorenz (or Coulomb) gauge condition $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$ (which is easily confirmed by direct calculation), one has

$$\nabla^2\boldsymbol{A} = -\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \left[\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \frac{1}{R}\frac{\partial^2\psi}{\partial z^2}\right]\hat{\boldsymbol{\phi}}. \tag{9}$$

The field equations (4) and the radial and vertical components of the fluid equation of motion (7) may therefore be written as

$$\begin{aligned} \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) + \frac{\partial^2\Phi}{\partial z^2} &= 4\pi G\rho, && \text{(10a)}\\ \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \frac{1}{R}\frac{\partial^2\psi}{\partial z^2} &= \frac{16\pi G}{c^2}\rho v, && \text{(10b)}\\ \frac{\partial\Phi}{\partial R} - \frac{v}{R}\frac{\partial\psi}{\partial R} &= \frac{v^2}{R}, && \text{(10c)}\\ -\frac{\partial\Phi}{\partial z} + \frac{v}{R}\frac{\partial\psi}{\partial z} &= 0. && \text{(10d)} \end{aligned}$$

Using (10c) and (10d) to eliminate $\partial\psi/\partial R$ and $\partial\psi/\partial z$ from (10b), then using (10a) to eliminate the resulting term containing $\partial^2\Phi/\partial z^2$, the field equation (10b) yields

$$\left(v + R\frac{\partial v}{\partial R}\right)\frac{\partial \Phi}{\partial R} + R\frac{\partial v}{\partial z}\frac{\partial \Phi}{\partial z} = \frac{v}{R}\left[v\left(v - R\frac{\partial v}{\partial R}\right) + 4\pi G\rho R^2\left(1 - \frac{4v^2}{c^2}\right)\right]. \quad (11)$$

The non-linear first-order partial differential equation (11) for the galactic velocity field $v(R,z)$ is the key expression in [20][3], and depends only on the galactic density distribution ρ and on the derivatives $\partial\Phi/\partial R$ and $\partial\Phi/\partial z$ of the Newtonian gravitational potential, which are themselves also determined by specifying ρ. Indeed, Φ is given by (5a), which in cylindrical polar coordinates with azimuthal symmetry reads[4]

$$\begin{aligned}\Phi(R,z) &= -G\int_0^\infty dR'\int_0^{2\pi} d\phi'\int_{-\infty}^\infty dz'\,\frac{R'\rho(R',z')}{|\boldsymbol{x}-\boldsymbol{x}'|},\\ &= -2G\int_0^\infty dR'\int_{-\infty}^\infty dz'\,\rho(R',z')R'\sqrt{\frac{m}{RR'}}K(m) \quad (12)\end{aligned}$$

where $K(m)$ is a complete elliptic integral function of the first kind and $m = 4RR'/[(R+R')^2+(z-z')^2]$. Moreover, the derivatives $\partial\Phi/\partial R$ and $\partial\Phi/\partial z$ may also be expressed analytically as

$$\frac{\partial \Phi}{\partial R} = G\int_0^\infty dR'\int_{-\infty}^\infty dz'\,\rho(R',z')\frac{R'}{R}\sqrt{\frac{m}{RR'}}\left[K(m) + \tfrac{1}{2}\left(\frac{R}{R'} - \frac{2-m}{m}\right)\frac{mE(m)}{1-m}\right], \quad (13a)$$

$$\frac{\partial \Phi}{\partial z} = \frac{G}{2}\int_0^\infty dR'\int_{-\infty}^\infty dz'\,\rho(R',z')\left(\frac{z-z'}{R}\right)\sqrt{\frac{m}{RR'}}\frac{mE(m)}{1-m}, \quad (13b)$$

where $E(m)$ denotes a complete elliptic integral of the second kind.

Before considering further the application of equation (11) to modelling galaxy rotation curves, we note that, if one neglects the mass currents on the RHS of (10b) (by letting $c\to\infty$), then one may consistently set $\psi = 0$ (although other solutions to the resulting homogeneous equation (10b) do exist). The radial and vertical components of the fluid equation of motion (10c)–(10d) then immediately yield $\partial\Phi/\partial z = 0$ and thus $v^2(R) = R\,\partial\Phi/\partial R$, where the latter is the usual Newtonian equation assumed in the modelling of galaxy rotation curves.

In applying the full equation (11) to the modelling of galaxy rotation curves, it is noted in [20] that observations of the rotation velocity are typically made along the galactic equatorial plane, so one may take $z = 0$. Assuming further a galactic density distribution that is symmetric about this mid-plane, (11) then reduces to

$$\left(\beta + R\frac{\partial \beta}{\partial R}\right)\frac{\partial \Phi(R,0)}{\partial R} = \frac{c^2\beta}{R}\left[\beta\left(\beta - R\frac{\partial \beta}{\partial R}\right) + \frac{4\pi G}{c^2}\rho(R,0)R^2(1-4\beta^2)\right] \quad (14)$$

where we have defined $\beta(R) \equiv v(R,0)/c$. Equation (14) is applied in [20] to two different models of the galactic density distribution.

The first model considered uses the density and gravitational potential given by the analytical Miyamoto–Nagai (MN) solution to Poisson's equation [29]. In this approach, one begins by assuming the fairly simple potential form

$$\Phi(R,z) = -\frac{GM}{\sqrt{R^2 + (a+\sqrt{b^2+z^2})^2}}, \quad (15)$$

where M is the total galactic mass and a and b are free positive parameters. The density distribution implied by Poisson's equation is then given by

$$\rho(R,z) = \frac{Mb^2}{4\pi}\frac{aR^2 + (a+3\sqrt{b^2+z^2})(a+\sqrt{b^2+z^2})^2}{\left[R^2+(a+\sqrt{b^2+z^2})^2\right]^{5/2}(b^2+z^2)^{3/2}}, \quad (16)$$

which extends to infinity in both R and z. The constant density contours have the form of spheroids of revolution with semi-axes proportional to a and b. It is straightforward to verify that, when integrated over all space, this density distribution yields the total mass M. In [20], this model is fitted to the observed rising rotation curve of NGC 1560 out to 8.3 kpc by varying the parameters M, a and b. The derived parameter values are $M = 7.3\times10^{10}$ $M_\odot$, $a = 0.373$ kpc and $b = 0.300$ kpc, which yield a reasonable fit to the rotation curve, but does not reproduce the luminosity profile of NGC 1560. This occurs because the infinite spheroidal solution does not describe the equilibrium of a finite disk-like object, and thus fails to reproduce its mass distribution and total mass.

Consequently, in the second model, the galaxy is instead considered as an axisymmetric thin disk of finite radius, which is again symmetric about its mid-plane

[3] Equation (11) does, in fact, differ slightly from equation (4.1) in [20], since the latter lacks the factor of 4 muliplying v^2/c^2 in the final term on the RHS. We believe the expression in [20] to be in error as a consequence of the choice of scaling used in the definition therein of the gravitomagnetic vector potential $\boldsymbol{A}$.

[4] Our final expression for Φ differs by a factor of 2 as compared to equation (4.2) in [1]; we believe the latter to be in error.

$z = 0$. The density distribution is assumed to have the functional form

$$\rho(R,z) = \rho(R,0)\exp\left(-\frac{z^2}{2\Delta^2(R)}\right), \tag{17}$$

where $\Delta(R)$ is a characteristic disk width with some assumed radial dependence. For small values of $\Delta(R)$, one can estimate the integral over z' in (13a) analytically using the Laplace approximation, which boils down to setting $z' = 0$ in the integrand and multiplying by the volume $\sqrt{2\pi}\Delta(R)$ of the Gaussian factor in (17); this yields

$$\frac{\partial\Phi(R,0)}{\partial R} \approx 2\sqrt{2\pi}G\int_0^\infty \frac{R'\rho(R',0)\Delta(R')}{R(R+R')}\left[K\left(\frac{4RR'}{(R+R')^2}\right) + \frac{R+R'}{R-R'}E\left(\frac{4RR'}{(R+R')^2}\right)\right] dR'. \tag{18}$$

To evaluate the above integral (numerically), the density distribution $\rho(R,0)$ is taken from the luminosity profile of the galaxy under consideration, which is therefore reproduced *automatically*, but one still requires a model for the radially-dependent characteristic vertical width $\Delta(R)$ of the galaxy. In [1], this is taken to coincide with a given constant density contour of the analytical MN solution (16). In particular, one defines $\Delta(R)$ such that

$$\frac{\rho_{MN}(R,\Delta(R))}{\rho_{MN}(0,0)}\frac{\mathscr{M}(\ell)}{M} = \exp\left(-\frac{\ell^2}{2}\right), \tag{19}$$

where $\rho_{MN}(R,z)$ denotes the right-hand side of (16), ℓ is a pre-defined 'label', which is usually set to $\ell = 3$ so that the chosen contour contains a fraction (approximated by $1 - \exp(-\ell^2/2) = 0.989$) of the total mass M, and $\mathscr{M}(\ell)$ is the resulting approximate mass of the galaxy with the density distributon (17), estimated by using the Laplace approximation to perform the integral over z:

$$\mathscr{M}(\ell) = (2\pi)^{3/2}\int_0^{R_{\max}} \Delta(R)\rho(R,0)R\,dR, \tag{20}$$

where the maximum radius of the galactic disk, $R_{\max}$, is obtained by solving (19) with $\Delta(R) = 0$. This approach requires quite a time-consuming iterative process, but the velocity profile again depends only on the three free parameters M, a and b. When fitted to the same observed rising rotation curve data for NGC 1560 as used above, the derived parameter values for this model are $M = 1.52 \times 10^{10}$ $M_\odot$, $a = 7.19$ kpc and $b = 0.567$ kpc (yielding $R_{\max} = 12.2$ kpc), which again produces a reasonable fit to the rotation curve, but now also reproduces the luminosity profile of NGC 1560 by construction. The model is also used in [20] to reproduce satisfactorily the observed rotation curve data for the spiral galaxy NGC 3198 and the lenticular galaxy NGC 3115.

V. PROBLEMS WITH THE MODEL

Although the approach outlined above appears at first sight to be a reasonable methodology for modelling galaxy rotation curves using the GEM formalism, it does have some unusual features. As mentioned previously, most notably the model assumes that the galaxy consists of a pressureless 'dust' of stars, all of which follow circular orbits. In particular, this means that the vertical support necessary to maintain dynamical equilibrium is assumed all to arise from gravitomagnetic rotational effects, which we will see leads to such effects being massively overestimated. This shortcoming may be addressed by, for example, using a distribution function approach based on the GEM formulation of the Jeans equation [27], since this enables vertical support via a velocity dispersion of the stars, and also allows for individual stars to follow non-circular orbits whilst retaining net currents that are strictly azimuthal.

Whilst the more sophisticated approach of distribution functions makes better physical sense, we will not concern ourselves with such modifications here, since we wish merely to address why the methodology outlined in Section IV can lead to incorrect conclusions regarding the effect of gravitomagnetism on galactic rotation curves. Indeed, we will limit our considerations still further by choosing not to pursue the iterative numerical process for obtaining rotation curves for the thin disk density model (17), since this is rather computationally cumbersome and time consuming. Instead, we will restrict our attention here to the model having the MN density profile (16), which can be treated almost entirely analytically and suffices to demomstrate the shortcomings of the overall approach outlined in Section IV.

A. Order of magnitude analysis

Before considering the single key equation (11) that forms the basis of the approach outlined in Section IV, we begin by making some observations regarding the four separate equations (10) from which (11) is derived. In particular, we first note that in the radial equation of motion (10c) one requires a term in v^2 to obtain sensible results. This occurs because v is $\mathscr{O}(\sqrt{\Phi})$ rather than $\mathscr{O}(\Phi)$. Indeed, one can see from the set of equations (10) that in typical circumstances there will exist a hierarchy of magnitudes for different quantities, and it is worth describing this hierarchy now so as to orient ourselves.

This is most easily achieved by first adopting geometric units $G = c = 1$, which we will assume henceforth. One then requires only a single scale to specify the base of units, which in this application is most conveniently taken to be length. In particular, we take the unit of length to be 1 kpc, which corresponds to typical galactic scales. All other physical quantities can then be expressed in terms of this base unit. For example, a mass M in SI units is given in terms of our units by $GM/c^2 \times (\text{kpc/m})^{-1}$, whereas a density ρ in SI units is given by $G\rho/c^2 \times (\text{kpc/m})^2 \approx 7.07 \times 10^{11}\rho$. Thus, typical galactic densities of $\sim 2 \times 10^{-19}$ kg m^{-3} correspond to $\sim 1.4 \times 10^{-7}$ in our units. For a broad selection of galaxies types, one may therefore take typical densities in our units to lie in the range $\mathscr{O}(10^{-8})$ to $\mathscr{O}(10^{-6})$; we will take the upper of these as indicative, since this maximises the magnitude of gravitomagnetic effects, although in reality they will usually be somewhat smaller.

From the Poisson equation (10a), or its more succinct form in (4), one sees that $|\Phi|$ is also $\mathscr{O}(10^{-6})$, and hence the velocity $v \sim \mathscr{O}(10^{-3})$ (where to convert velocities in SI unit to our units, one needs merely to divide by c). Then, from equation (10b), which one can also write more usefully as $\nabla^2\psi = 16\pi R\rho v - (2/R)\partial\psi/\partial R$, one sees that $|\psi| \sim \mathscr{O}(10^{-9})$, modulo any multiplicative effects from R which are limited to a factor of ~ 10 for a typical galaxy.

Now considering either the radial equation of motion (10c) or its vertical counterpart (10d), one sees any effects arising from ψ, which always appears multiplied by v, must be $\mathscr{O}(10^{-6})$ *smaller* that those arising from Φ. Consequently, any gravitomagnetic effects will have a negligible effect on the circular velocity of a test particle, which will very well approximated simply by the strictly Newtonian expression $\sqrt{R\,\partial\Phi/\partial R}$.

This result is at least allowable (notwithstanding the usual clash with the flat or rising rotation curves observed in many galaxies), if disappointing, but one sees from the vertical equation of motion (10d) that there is a much more serious problem. In this case, one requires the $\mathscr{O}(10^{-6})$ term in Φ to be balanced by the $\mathscr{O}(10^{-12})$ term in ψ; this is simply impossible and indicates that the set of equations (10) has no physically meaningful solution. As mentioned above, this problem arises because one has insisted that all the vertical support force arises from gravitomagnetic effects, which is impossible for ordinary matter.

B. Rotation curves for the MN density profile

In eliminating various quantities between the equations (10) to arrive at the 'master' equation (11) in [20], one can no longer identify the issues discussed above. Indeed, one can go on to find solutions ψ that satisfy (11), although these cannot be physically meaningful, as our analysis above shows. We now illustrate this directly by considering a galaxy having the MN density profile (16) and gravitational potential (15), which was the first model used in [20] to fit the observed rotation curve data of NGC 1560 (although it fails to reproduce its luminosity profile). As discussed above, the resulting derived parameters are $M = 7.3 \times 10^{10}$ M$_\odot$, $a = 0.373$ kpc and $b = 0.300$ kpc, so the fitted MN density profile is moderately oblate. The resulting gravitational potential and density contours are shown in Figure 1.

Inserting the forms for the MN potential (15) and density (16) into the 'master' equation (11) yields a very complicated expression, but one can make progress analytically if one restricts attention to the equatorial plane $z = 0$, as in (14). This is permissible since, although (11) contains the z-derivative of the potential Φ, one can see that for the MN form of the potential this vanishes on the equatorial plane. The resulting equation then reads $A + B = 0$, with

$$A = -R^2M\left[-bR\left(R^2+a^2+2ba+b^2\right)\frac{dv}{dR} + \left(2b^3+5b^2a+\left(4a^2-R^2\right)b+R^2a+a^3\right)v\right] \tag{21a}$$

$$B = 4\left[\tfrac{1}{4}b\left(R^2+a^2+2ba+b^2\right)^{5/2}\left(\frac{dv}{dR}R - v\right) + MR^2v\left(5a^2b+a^3+R^2a+3b^3+7b^2a\right)\right]v^2, \tag{21b}$$

where we have split the LHS into the terms, since it is possible to obtain a simple analytic result for v by just setting $A = 0$. It is not immediately obvious that this is a valid procedure, even as an approximation, since $v \sim \mathscr{O}(10^{-3})$ and ρ, and by extension its volume integral M, are likely $\mathscr{O}(10^{-6})$. Thus, both expression A and the first half the terms in B are likely $\mathscr{O}(10^{-9})$, and hence it is not clear that one can preferentially drop the first half of B. Numerically, however, it transpires that the value of $M = 7.3 \times 10^{10}$ M$_\odot$ derived for NGC 1560 is sufficiently large that one can consider just $A = 0$, and we note that this yields an expression for v that is in fact *independent* of M.

We may illustrate this approach explicitly by comparing the exact and approximate solutions for v in this case. Setting just the first line of (21) to zero and solving for v gives

$$v = \frac{CR^{2+\frac{a}{b}}}{[R^2+(a+b)^2]^{3/2}}, \tag{22}$$

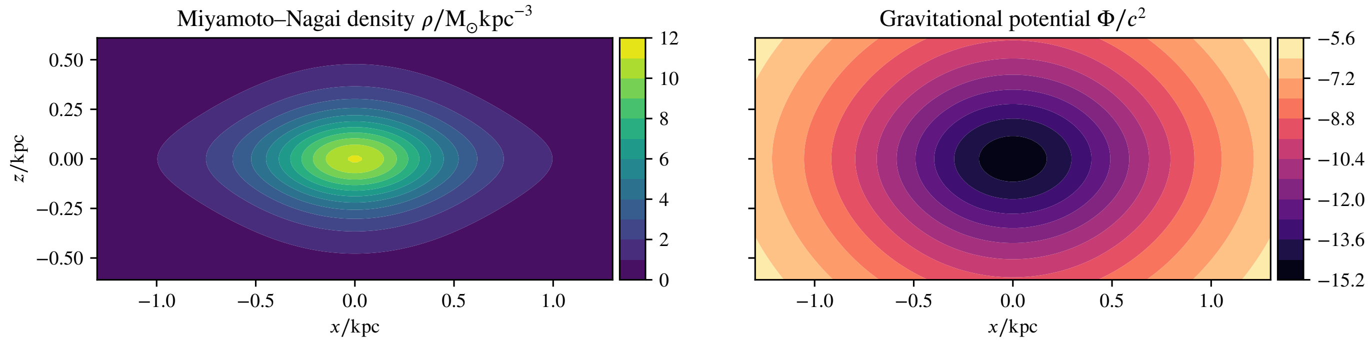

FIG. 1. Density (left) and gravitational potential (right) contours for a MN profile with parameters a and b derived from fitting the rotation curve of NCG 1560 in [20].

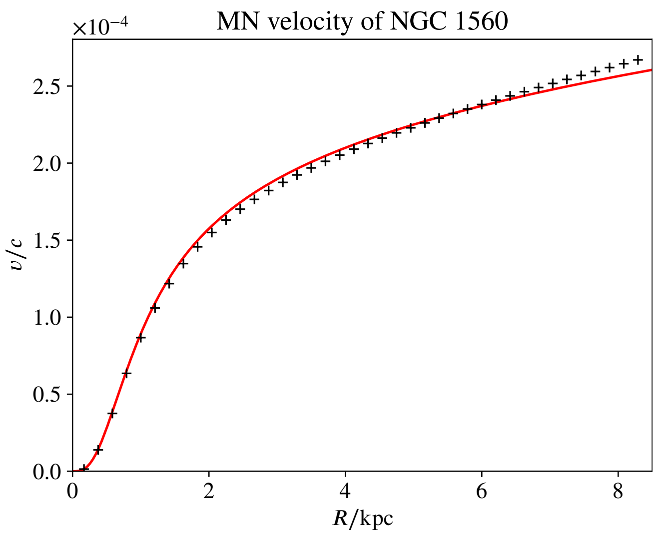

FIG. 2. Rotation velocity v (in units of c) versus R in kpc for a MN profile with parameters derived from NCG 1560. The red curve is obtained using the analytical approximation (22) with $C = 1/6400$ and the black curve is an exact numerical integration using equation (21).

where C is an arbitrary constant. In Figure 2, we show the rotation curve resulting from the analytical approximation (22) as the red curve and an exact numerical integration of the full equation (21) as the black curve. For the analytic approximation, although there is no dependence on mass, one must provide an overall scaling C, and a value of $C = 1/6400$ was used in the plot, which gives reasonably good agreement between the exact result in this case. The latter was calculated by numerical integration starting at the outermost rotation curve data point for NGC 1560, for which $v = 2.67 \times 10^{-4}$ (in units of c) at $R = 8.29$ kpc, and moving inwards towards the origin, in the same way as performed in [20]. Similarly, one could instead fix the scaling C of the analytical result by ensuring that it passes through the outermost data point, which moves the red curve up slightly.

In any case, it is important to note that, while the fit to the NGC 1560 rotation curve data in [20] yields the derived mass $M = 7.3 \times 10^{10}$ $\mathrm{M}_\odot$, the only information about M is in quite small changes in the *shape* of the curve that occur as M drops below this best-fit value. For *larger* values of M, the shape of the curve is invariant, and corresponds to that given in the analytical approximation (22), which does not depend on M. This suggests that there may be a large uncertainty on the mass M derived from the rotation curve data, although no errors on the fitted value are provided in [20].

Nonetheless, let us assume the best-fit value of M to calculate also the rotation curve that one would obtain in the absence gravitomagnetic effects, i.e. $\psi = 0$, and the galaxy is completely static and supported just by usual pressure forces. In this case, the rotational velocity of a test particle is merely $\sqrt{R\partial\Phi/\partial R}$ and one obtains the blue curve in Figure 3, which we plot alongside the exact rotation curve (in black) from Figure 2, which includes gravitomagnetic effects. Figure 3 matches very well with Figure 2 in [20], but is worthy of further comment. First, we note that the conventional rotation curve peaks at velocities around $420\,\mathrm{km\,s^{-1}}$ (readopting SI units for the moment); this is much higher than one would expect for what is meant to be a dwarf galaxy. Second, and more important, we see that the effects of gravitomagnetism here are to *suppress* the rotational velocity of test particles, not *enhance* them. Thus one requires a great deal more matter present in the case with gravitomagnetic effects than that without, in order to explain a given rotation curve level. Gravitomagnetic effects serve here to explain only aspects of the *shape* of rotation curves (here a gradually rising one), but absolutely not whether one requires more matter than appears visible; in other words, it makes the missing matter problem *worse*.

Before moving on to discuss the issue of gravitomagnetic vertical support (or the lack thereof) in the next subsection, it is worth noting some further aspects of the shape of the rotation curves derived above. Although the rotation curves obtained using either (21) or the analytic approximation (22) appear to fit the rotation curve data for NGC 1560 shown in Figure 1 of [20] in a pleasing way,

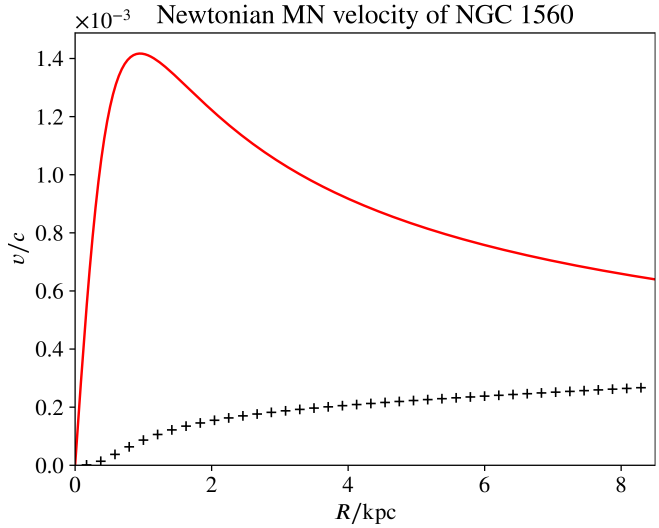

FIG. 3. The conventional Newtonian rotation curve (blue) for NGC 1560 assuming a MN profile with the best-fit values of the parameters a, b and M from [20], together with the exact rotation curve including gravitomagnetic effects (black), already shown in Figure 2.

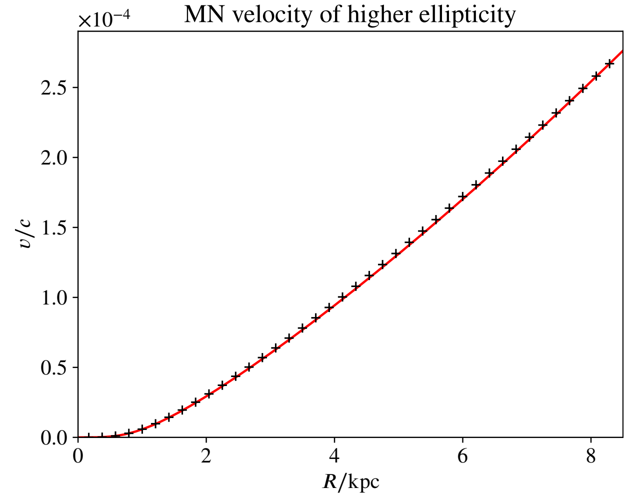

FIG. 4. Same as Figure 2, but for a higher ellipticity case, with $a = 0.7\,\mathrm{kpc}$ and $b = 0.3\,\mathrm{kpc}$.

this disguises the problem that the shape of these rotation curves changes considerably with just small changes in the a and b parameters.

Observations of NGC 1560 in the visible show it to be considerably more 'elliptical' than the ratio $a : b = 0.373 : 0.300$ indicates, with a ratio of $\sim 0.7 : 0.3$ seeming much more appropriate. From the analytical expression (22), however, one can see that this will cause a problem, since the shape of the predicted rotation curve will scale as $v \propto R^{1.33}$ at large R, and so it will be concave rather than convex towards the R axis. Indeed, this will clearly occur for any ratio $a : b > 2 : 1$. No known rotation curves have this shape (concave rather than convex over their whole range), and so this model will be incapable of accommodating galaxies with ellipticities beyond this ratio. That this is not an artefact of our analytical approximation is illustrated in Figure 4, which is the equivalent of the rotation curves plot in Fig. 2, but for a and b values of 0.7 and 0.3 kpc, and using the same mass M. One sees that the red curve (analytical approximation) closely follows the black curve (exact numerical integration), and hence the insights that the analytic approximation (22) provides for what occurs at higher $a : b$ ratios are indeed borne out in the exact integration.

C. Gravitomagnetic vertical support

As our final point we now discuss further the assumption that all vertical support for dynamical equilibrium is provided by gravitomagnetic rotational effects, which in our opinion is the key issue with the modelling approach outlined in Section IV, and applies irrespective of the assumed density profile of the galaxy. As above, however, we will illustrate our findings for the MN profile, since it can again be treated almost entirely analytically.

In particular, we will show that in order to provide the vertical support necessary, ψ has to become infinite at the origin, and have extremely large values near to it. To substantiate this, plus gain some insight into what is happening analytically, we again take a 'dual track' approach in which we carry out exact numerical integrations, as well as develop an analytical approximation. To this end, one can construct an exact ODE in R applicable in the equatorial plane by using radial equation of motion (10c), together with our analytical approximation for circular velocity v in (22). One can then form an approximation to ψ based on the smallness of the coefficient C, which yields the very simple approximate solution

$$\psi = \frac{MbR}{C(b-a)R^{\frac{a}{b}}}. \tag{23}$$

Using the values of the parameters derived for NGC 1560 in [20], this approximation is in fact even better than that for the rotation curve in (22), as we demonstrate in Fig. 5. The curves for the exact numerical integration (black) and the analytic approximation from (23) (red) are virtually indistinguishable. One sees that ψ itself diverges towards the origin, whereas $R\psi$ converges at the origin; this is consistent with the ratio $a/b = 0.373/0.3$ lying between 1 and 2, and hence according to (23) $R\psi$ should go to zero at $R = 0$, whereas ψ diverges.

By comparison, in Fig. 6 we show $R\psi$ for the higher ellipticity case considered above, i.e. $a/b = 0.7/0.3$. We have plotted only $R\psi$ here since even this diverges, as to be expected from (23) with $a/b > 2$. We also note that in all of these plots of ψ the values involved are $\mathcal{O}(1)$ or perhaps $\mathcal{O}(10^{-1})$, which is roughly 10^{8-9} larger than expected to be generated by GEM effects, according to the orders of magnitude analysis given earlier.

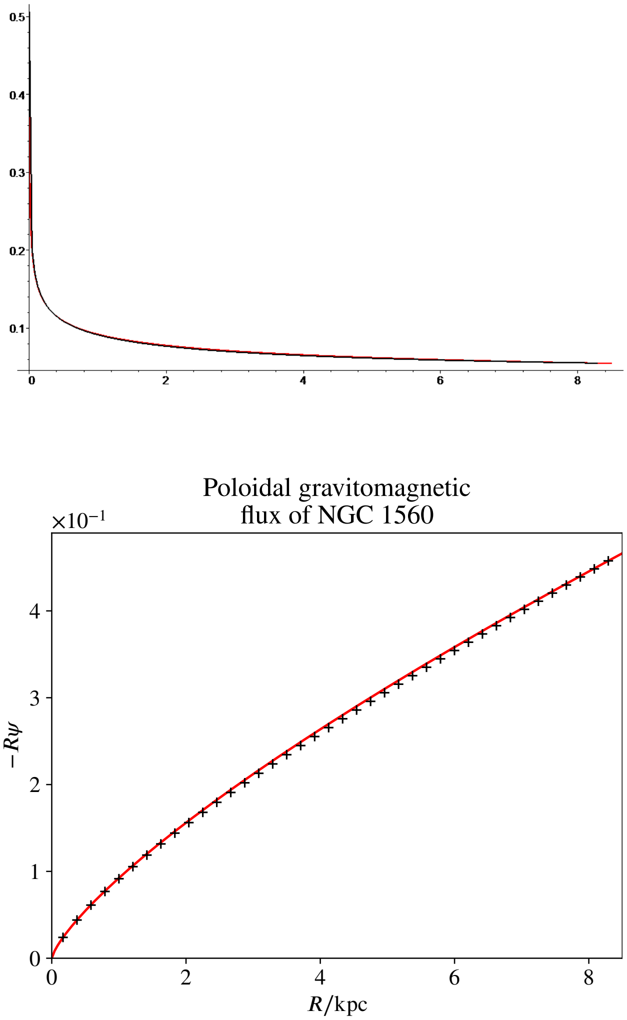

FIG. 5. Top: the function $-\psi$ versus R in kpc using the parameters derived for NGC 1560 in [20]. Bottom: the function $-R\psi$, to indicate better the behaviour near the origin. In each case the black curve is the result of an exact numerical integration, and the red curve shows the analytic approximation (23).

This effect must originate from the unwitting inclusion of *free-space* solutions of the Poisson-like equation (10b) that determines ψ, i.e. solutions for which the source term on the RHS, which would normally generate ψ, are set to zero. One can introduce arbitrary amounts of such homogeneous solutions to any solution of the inhomogeneous equation. However, the penalty is of course that any such solution has to add in singularities at either infinity or the origin. If this were not the case, one would be free to add homogeneous solutions of arbitrary amplitude to, for example, the Poisson equation for the gravitational field around the Sun or Earth, meaning one would lose the ability to predict the force of gravity based on the mass of an object. Such a procedure is forbidden by the need to exclude singularities.

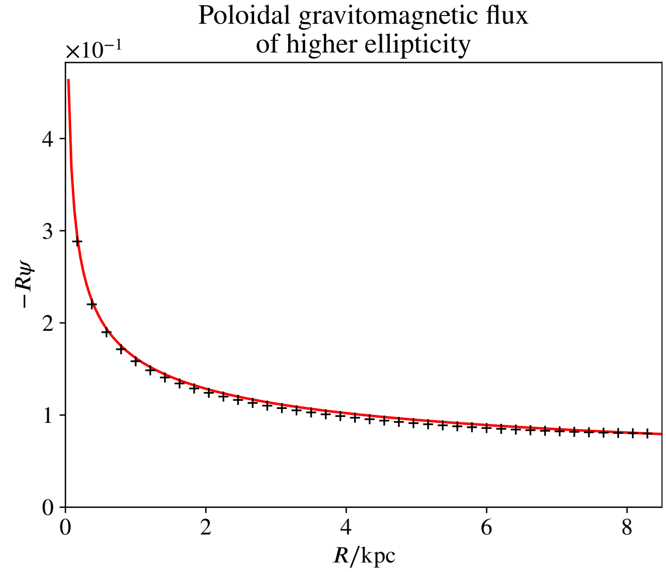

FIG. 6. Same as the Figure 5 (bottom), but for a higher ellipticity case, with $a/b = 0.7/0.3$.

Thus, having demonstrated that a singularity exists (at the origin in this case) with the GEM approach outlined in Section IV, this should definitively rule out the methodology as a means of explaining flat galaxy rotation curves without dark matter. The only 'get-out' might be whether such a singularity (which seems necessary to the GEM modelling) could be related to a black hole being present at the centre of most galaxies, and it may be worth investigating this quantitatively, since one knows the sizes of masses involved. A problem will be, however, that our treatment here is linear, but once ψ becomes large, this is no longer valid. Another problem is that one knows in the Milky Way that stellar orbits very rapidy exhibit precessional and tidal effects that are consistent with a central supermassive black hole as one goes out in radius, so the problem of the radial *scale* over which effects from ψ having very large values manifest themselves, would certainly need to be addressed. Finally, although we have not gone into it here, one finds further that a singularity can exist even if ψ does not diverge, since it turns out that to have the spacetime metric obey 'elementary flatness' [30], one requires not only that ψ is not divergent as R approaches zero, but must behave as $\psi \propto R$ for small R. The ψ functions discussed here are far from having this property, and indeed violate this requirement all the way up the z-axis. This means that one would need to invoke a line-like singularity along the z-axis, not just a point singularity at the origin, and while one might appeal to 'jets' for this, the whole model will look increasingly contrived.

VI. CONCLUSIONS

We have investigated the recent claim by Ludwig in [20] that one need not consider modified gravity theories to explain flat rotation curves, such as those observed in galaxies, without the need for dark matter, since such curves can be explained by gravitomagnetic effects in standard linearised GR. Ludwig adopts the convenient GEM formalism and, somewhat unusually, models a galaxy as an axisymmetric, stationary, rotating, non-relativistic and pressureless 'dust' of stars, all of which follow circular orbits. This approach therefore identifies the bulk velocity distribution of the galaxy with the velocity of stars, thereby aiming to define a self-consistent pressureless model.

The resulting system of GEM field equations for the gravitational (gravitoelectric) potential Φ and the poloidal gravitomagnetic flux ψ, together with the radial and vertical equations of motion, are amenable to an order of magnitude analysis. Indeed, it is straightforward to show that gravitomagnetic effects on the circular velocity v of a star are $\mathcal{O}(10^{-6})$ smaller than the standard Newtonian (gravitoelectric) effects. Thus, as one might have expected, any modification of Newtonian galaxy rotation curves must be negligible. More importantly, we find that the assumption in Ludwig's model that all the vertical support necessary to maintain dynamical equilibrium arises from gravitomagneic effects is impossible to satisfy; if one assumes the presence only of ordinary matter, the gravitomagnetic effects are $\mathcal{O}(10^{-6})$ too small to provide this support.

The above issues are obscured when various quantities are eliminated between the system of equations to arrive at the single key equation for v used by Ludwig. Nevertheless, to understand how Ludwig appears to arrive at a self-consistent pressureless model for a galaxy, we solve this key equation for v in the case of a galaxy having a Miyamoto–Nagai density profile. This allows us to establish an intuition for the results by adopting a 'dual track' approach by performing an exact numerical integration and by developing an accurate anayltic approximation.

Adopting the derived values of the mass, M, and semi-major and semi-minor axes, a and b, obtained by Ludwig in fitting rotation curve data for NGC 1560, we find that the resulting rotation curve depends only very weakly on the mass M. Moreover, we show that for larger values of M, the rotation curve becomes independent of M. In any case, if one compares the rotation curve for the fitted parameters with the corresponding standard Newtonian rotation curve, one finds that the effects of gravitomagnetism are to suppress the rotational velocity of test particles, not enhance them. Thus, although the rotation curve including gravitomagnetic effects has a shape closer to that observed, it requires more matter to be present than in the Newtonian case in order to explain a given rotation curve level, which excerbates the missing matter problem.

Although the predicted rotation curve for the fitted aspect ratio $a/b = 0.373/0.3$ matches the observed one reasonably well, this aspect ratio is somewhat smaller than what would be inferred from observations of NGC 1560 in the visible, which is close to $a/b = 0.7/0.3 \approx 2.33$. We show, however, that for aspect ratios $a/b > 2$, the predicted rotation curves are concave over their entire range, which does not match observations in any galaxy.

The most problematic issue, however, is that in order to provide the necessary vertical support to maintain dynamical equilibrium, the poloidal gravitomagnetic flux ψ must become singular at the origin and have extremely large values near to it. In particular, we show that ψ must be at least $\mathcal{O}(10^{8})$ larger than expected from gravitomagnetic effects. This must occur because free-space solutions of the Poisson-like equation that determines ψ are being unwittingly included, but this is forbidden if one wishes to avoid the presence of singularities. Moreover, the large values of ψ contradict the linearised treatment implicit in the GEM formalism. Consequently, one may rule out the GEM model proposed by Ludwig as a means of explaining flat galaxy rotation curves without the need for dark matter.

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