Compact Gauge Fields and Quantised Fermions in Lattice Quantum Gravity

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Declarations

1. The following link contains chronologically organised scans of my notes during the course of this work. Most of these notes are handwritten scans, but a small fraction (especially towards the end of the project) consists of Tex files or digitally annotated work.

https://drive.google.com/drive/folders/1dfIO-UG00-H7k193IKE49tR_X3wiNeB-?usp=sharing

2. **Except where specific reference is made to the work of others, this work is original and has not been already submitted either wholly or in part to satisfy any degree requirement at this or any other university.**

Compact Gauge Fields and Quantised Fermions in Lattice Quantum Gravity

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Despite their notable successes, mainstream theories of quantum gravity often require exotic mathematical constructs at a basal level, either to ensure self-consistency or to make contact with the real world. With such constructs translating to multiple unobserved physical predictions, the foundations of these theories are looking increasingly shaky. The dynamical triangulations program is a bold step towards addressing these issues by constructing a *minimalistic*, lattice field theoretic model of quantum gravity. While mathematically well founded, numerical complexities in four dimensions have shifted efforts towards analytic and numerical comprehension of two dimensional toy models. In this project, we pursue a two-pronged approach to studying Quantum Electrodynamics (QED) coupled to two dimensional quantum gravity through the lens of dynamical triangulations. We first derive a novel duality between our proposed setup and an Ising model residing on a curved background, building on an earlier duality between *free* fermions and Ising spins on curved spacetimes. We then pursue a second, unrelated approach wherein we utilise the integrability of lower dimensional, flat-space QED to construct *solvable* flat space lattice analogs, which we aim to later extend to solvable *curved* space lattices, and ultimately, curved space QED in the continuum. Along the way, we also explore the proliferation of the famous chiral anomaly of QED to lattice discretisations, with an eye towards implications for curved space QED.

I. INTRODUCTION

The search for a quantum theory of gravity has always ranked amongst the most awe-inspiring quests ever undertaken by the human mind. With a selection of unresolved theoretical issues (such as the singularities pervading general relativity (GR)) and growing phenomenological tensions (for instance the problem of the cosmological constant), a clear understanding of quantum gravity is now steadily ceasing to be merely the speculative desires of a few, and is working our way through to more hands-on physics.

String theory [1, 2] is, at least by popular consensus, our best model for a 'theory of everything'. Even so, its status as a comprehensible and viable theory is far from established. The major technical stumbling blocks include an unclear non-perturbative framework [3] and a lack of clarity on how infrared (IR) physics actually *emerges* from a stringy base [4, 5]. However, even after setting aside these mathematical whinging (and the far more troubling lack of experimental evidence!), we see that string theory stills suffer from some philosophical shortcomings, primary of which is that we get a *lot* more than we ordered. Large extra dimensions, supersymmetry and a potpourri of undetected particles are all indispensable to string theories which could potentially describe our universe. While such themes align with the popular guiding notion of an elegant universe, they are well at odds with another long held maxim - that of Occam's razor, which argues for pushing forward theories on the basis of *simplicity*, rather than beauty.

Part of the reason for the shift towards strings lies in its (relatively) higher calculability, a fact exemplified by the emergence of the Einstein equations from string field theory [1, 2] and recent advances in black hole entropy [6–9]. Nevertheless, even string theory is nowhere near providing concrete predictions and it would be foolish to throw out other quantum gravity theories at this stage. Coupled with the disturbing (apparent) absence of supersymmetry at string-predicted energy scales of $E \sim \mathcal{O}(10)$TeV, such as those achieved at the LHC,[10–13] and the aforementioned simplicity arguments, alternative quantum theories of gravity are definitely an avenue to explore.

If we are to believe in the wisdom of Occam, we are inevitably led to the result that GR *must* be thought of as the infrared (IR) limit of a well-posed QFT. How must we reconcile this with perturbative non-renormalisability? Weinberg [14] proposed a possible answer which now goes by the name of *asymptotic safety*[15–17]. The crux is that while GR might not be *perturbatively* renormalisable, it might nevertheless make perfect sense as a *non-perturbative* quantum field theory. As an *extremely* down to earth analog, we may understand this in terms of (nearly) globally well defined functions which nevertheless do not possess Taylor series with an infinitely large convergence radius. The real function $1/(1-x)$ is one such example. This function allows a Taylor expansion around $x = 0$, with a radius of convergence of 1, with its formal Taylor series diverging for all values outside this range. Of course, this certainly does not imply that the function *itself* is nonsensical outside its domain of convergence, only that we must relax our perturbative treatments in order to get sensible answers. Similarly, while Feynman expansions about a given background might not furnish finite results, this failure *could* be interpreted as a flaw in our methodology rather than in the theory as such. This intuition can be made precise using the language of renormalisation group flow. Worded this way, asymptotic safety states that while perturbatively non-renormalisable, GR may nevertheless be the IR limit of a flow originating from an *ultraviolet (UV) fixed point*, so that any couplings remain finite even as we scale up the energy.

While asymptotic safety does provide a possible resolution to the quantum gravity puzzle, we are not out of the woods yet. Perturbation theory is by far the predominant method for concrete computations in QFT.

While analytic or partly analytic non-perturbative techniques such as the conformal bootstrap or holographic dualities are not without use, they are arguably still in their nascent stages and are limited to rather special (often supersymmetric) QFTs. Additionally, *precise* computations are usually not possible with these techniques even within their range of validity. How must we then even *attempt* to answer the question of whether GR is asymptotically safe? One possibility is, of course, to appeal to lattice regularisations, arguably the sole non-perturbative method with the power to at least *potentially* yield concrete results. On a more foundational level, lattice approaches could serve as starting points for rigorous formulations of (at least Euclidean) QFTs (i.e. by defining the QFT as a critical point of a renormalisation group flow initiated by varying the lattice spacing a). As far as string alternatives go, lattice formulations therefore possess both theoretical and engineering advantages, and are therefore worth at least some investigation.

Dynamical Triangulations, proposed in the 1990s by Ambjorn[18–23], are one such formulation of lattice quantum gravity utilising the path integral approach to field theory. Much as in usual lattice field theory, where the underlying flat spacetime is replaced by a hypercubic lattice with matter fields living on its vertices or links, dynamical triangulations replace Lorentzian manifolds with discrete analogs known as *triangulations* (for instance, a tetrahedron is a simple example of a triangulation of the two-sphere S^2). Again, matter fields reside on the vertices or links of triangulations and the path integral over Lorentzian manifolds is replaced by a sum over relevant triangulations, rendering the regulated path integral finite. As with usual lattice theory, continuum behaviour is then gauged by identifying combinations parameters that yield a sensible (scale invariant) continuum limit.

Abiding by our guiding theme of keeping the mathematical structure as terse and close to reality as possible, 3+1D is the most suitable arena for hunting for RG fixed points. However, such searches are prone to problems both at the level of the setup and in the actual numerics. While progress is nevertheless being made in these areas, it is advisable to bend our guiding principle a little bit and search for fixed points in 1+1D, a more tractable endeavour for several reasons. First, classical gravity is trivial in 1+1 dimensions. Second, at least a few of the numerous problems of setup, while daunting in 3+1D (especially with regards to the addition of fermionic fields), have been tackled already in 1+1D [18, 20]. Lastly, theories in two spacetime dimensions are far more amenable to analytical treatment via techniques from both formal field theory and condensed matter. Significant progress [24–29], both analytical and numerical, has been made in the case of pure 2D/1+1D gravity and 2D/1+1D gravity coupled to *either* scalars, spinors or vectors. The hope is that developments in lower dimensions could pave the way for a better understanding in the more physical realm of 3+1D.

Project Aims: The aim of this project, and possible continuations, is to attempt to add to the present body of results in two spacetime dimensions (Euclidean and causal). In particular, this project details the first steps towards analytically probing the critical limits of a system comprising Dirac fermions coupled to two dimensional gravity and a $U(1)$ gauge field. In addition to serving as the natural extension of *individually* placing fields of various spins atop a curved background, gauge theories are the backbone of standard model physics, so an understanding of how gauge fields impact criticality in two dimensions is, we believe, a necessary first step towards a deeper understanding of 'real world quantum gravity', at least within the framework of dynamical triangulations.

At the other end of the spectrum, Quantum Electrodynamics(QED) in 1+1D - better known as the Schwinger model [30, 31] - is a rare example of a *solvable* quantum field theory. Additionally, the Schwinger model boasts a variety of *non-perturbative* phenomena[32, 33] - confinement, a chiral anomaly and a Higgs mechanism, to name a few, and it would be interesting to delve into how the introduction of geometry affects these phenomenon. In this work, we focus specifically on the second of these - the chiral anomaly - and outline how chiral anomalies can be computed on a flat lattice in a manner which could be extended to dynamical triangulations. We aim to explore the effects of curvature on anomalies in the near future.

This work is organised as follows: In section II, we outline the formalism of dynamical triangulations, with emphasis on two dimensional systems. Following this, in section III, we elucidate major developments in the case of fermionic fields and vector fields living on two dimensional triangulations. The fermionic sector, which can be mapped to a system of Ising spins living on *dual* triangulations, will be of particular interest to us. Specifically, we develop an analogous mapping between the Schwinger model and spin systems in section IV. We also discuss physical insights that can be drawn from this correspondence. While beautiful in its own right, our mapping does not yield analytical results. Keeping the magic of the Schwinger model in mind, we therefore attack the problem from the field theoretic side. We begin by briefly reviewing the necessary *continuum* theory in section V. Naïve interpolations to the lattice are fraught with issues, a theme will clarify in section VI. Continuing and concluding, we also present a possible resolution of these issues in section for lattices with a *flat* geometry and outline brief discussion on qualitative aspects of the chiral anomaly on flat lattices.

II. DYNAMICAL TRIANGULATIONS - QUANTUM GRAVITY WITH NO STRINGS ATTACHED!

Although initially concocted in a purely *geometric* fashion, the desire to incorporate gravity into a quantum framework historically shifted towards a dual perspective, with relativists persisting with geometry, and field theorists viewing gravity as a usual spin-2 field, albeit one with highly intricate relations to spacetime coordinates. (It is interesting to note that this duality now has found far more literal interpretations with the advent of AdS/CFT and related correspondences!). The fundamental issue with discretising gravity as a *field theory* is that the traditional field viewpoint simply does not gel well with the usual lattice methodology. Most continuum theories can be translated to the lattice by simply demanding that we define the relevant quantum fields *only* at *discrete* locations in spacetime. Spacetime *itself* need not be discretised. However, in the case of GR, the intertwining between the gravitational field and the underlying spacetime coordinates makes such a separation undesirable. If we nevertheless wish to continue down this path, new difficulties arise. Discretisations usually require a manifest set of coordinates, so that discretising gravity in such a fashion would conflict with the sacrosanct theme of diffeomorphism invariance. Perhaps most worrisome of all though is the fact that one can show that, in such setups, lattice parameters need not *necessarily* correspond to ultraviolet (UV) cutoffs, so that the resulting theories may not even be divergence free!

Thankfully, we can bypass all these issues by simply reverting to the geometric foundations of GR. Viewing spacetime as a *generic* Lorentzian manifold with the gravitational field serving as the metric, we see that discretising gravity effectively reduces to understanding how to discretise *manifolds*. *This* is a problem that is well-known to geometers and topologists, and the most popular solution involves *triangulation* - replacing a given n-manifold in question to a collection of equilateral n-simplices (The n-simplex $[x_0, x_1, ..., x_n]$ is the smallest n dimensional convex set containing the real points $\{x_0, x_1, ..., x_n\}$), with their edges glued together. It can be shown that *topological* aspects of manifolds can be captured by the number of vertices, edges and higher dimensional extensions present in their triangulations. For instance, the genus (i.e. number of holes) of a 2-manifold equals $(2-V-E+F)/2$ [34] for *any* of its triangulations, with V, E and F respectively representing the number of vertices, edges and faces. On the other hand, it can also be shown that angles (or higher dimensional extensions) between constituent equilateral simplices can be used to model the *geometry* of associated manifolds.

To summarise, we can discretise generic n-manifolds by appropriate *triangulations*. One can in principle establish a one-to-one correspondence between n-manifolds with a given *geometry* and triangulations with a *fixed* side length (a) comprising a *fixed* number of n-simplices (Again, this is loosely because once the side lengths and number of simplices are specified, the angles between simplices encode the geometry completely and uniquely). Therefore a 'sum' over all possible manifold configurations (i.e. the measure $\int \mathcal{D}g_{\mu\nu}$ in the gravitational path integral) can be replaced by a sum over *triangulations* with a fixed number N of equilateral simplices of a given side length a (i.e. $\sum_{\{N,a\}_{\text{fixed}}}$). More specifically, the source free generating functional can be replaced by a discrete, *finite* sum over triangulations i.e.

$$\mathcal{Z} = \int \mathcal{D}g_{\mu\nu} e^{iS_{EH}[g]} \rightarrow \sum_{\{\tau\}} e^{iS_\tau} \tag{1}$$

where $S_{EH}[g]$ is the Einstein–Hilbert action, τ denotes triangulations with a fixed total number of simplices and side lengths and S_τ is an appropriate discretisation of S_{EH} for the triangulation τ [1]. In this manner the gravitational path integral can be rendered finite (with a serving as a regulator). Needless to say, matter fields can be placed atop the triangulations (at the centres or edges of the simplices) to obtain a lattice discretisation of a continuum system with both matter fields and gravity.

A final point concerning the metric signature is in order. Lattice field theories normally reside atop discrete *Euclidean* backgrounds, primarily because Wick rotations furnish a natural interpretation of the resulting *Euclidean* path integral as a genuine partition function (this is unlike the more physical causal path integral which is inherently complex, so that ties to probability distributions are more obscure). This reasoning is more of a convenience than a necessity however [37] and theoretical issues regarding Wick rotations in *generic* spacetimes have inspired triangulation formulations that are inherently causal. Appropriately called *causal* dynamical triangulations (CDTs), temporal structures in this paradigm are imposed by a combination of topological and geometric choices. Specifically, the topology is assumed toroidal, with the arrow of time directed along the 'longitudinal circle' describing the torus and each triangulation thereof is foliated into appropriately chosen (transverse) 'constant time slices'. Currently, CDTs and *Euclidean* dynamical triangulations (EDT) are equally in vogue, with no serious conflicts between results on either side. In this work, we will primarily work with EDTs as our base, but our

[1] Such constructions are the subject of *Regge Calculus*[35, 36]. For our purposes, since the two dimensional continuum action is trivial, S_τ can be simply referenced to zero so long as we do not alter our topology

results are (qualitatively) largely independent of the existence of a causal structure. We will nevertheless make clear any minor differences, as and when they arise.

III. LIFE ON THE LATTICE - TRIANGULATED FERMIONS AND GAUGE FIELDS

The upshot of using minimalistic formulations such as triangulations (and more generically, asymptotic safety) is that the questions we must pose are correspondingly diminished in number and reduced in complexity. In particular, establishing quantum gravity theories within the triangulations framework is essentially now just a yes or no question - i.e. *Do we, or do we not, have a critical scaling for our lattice theory?* While recent results hint at slightly more positive perspectives, the current consensus is that it is *unlikely* that asymptotically flat *pure* GR in 3+1D flows from a UV fixed point (although results in more cosmologically relevant spaces, such as de-Sitter background are more promising [38, 39]). Are we then at an impasse? Certainly not! While we *do* see that a *pure* Einstein–Hilbert action in four spacetime dimensions does not yield an asymptotically safe quantum field theory [2], but our Universe is certainly more than just gravity! Within the framework of dynamical triangulations, matter fields do much more than simply making our Universe a less boring place! Formally, the addition of matter fields enlarges the abstract 'theory space', but more importantly enhances its structure. Amongst other things, these enhancements could, in principle, involve the creation of *new fixed points* - fixed points which could source new classes of asymptotically safe theories. That is, while GR *itself* might not be asymptotically safe, appending matter fields could well provide the required fix. This idea gains traction when we note that the relative dearth of fixed points in theory space requires us to fine-tune of parameters to obtain sensible theories. This fine-tuning could potentially provide solutions to naturalness or hierarchy problems.

While nominally straightforward, appending matter fields to triangulations is a computationally expensive procedure, at least in 3+1D. The associated numerical complexities have thus far proved difficult to overcome. While steady progress is being made on this front [40, 41], concrete results are, for now, well out of reach. We are thus led into the abyss that befalls almost all ambitious theories - we must sacrifice reality for results and switch to working with simplified 'toy' models with a greater chance of solvability. In our case, this simplification amounts to a reduction in dimension - systems comprising matter fields atop two (Euclidean or causal) dimensional triangulations boast an enhanced solvability, stemming from the following reasons:

1. Classical GR in two space(time) dimensions is trivial - the Einstein–Hilbert action reduces to a mere surface term [42].
2. Field theories are more amenable to analytic techniques in two dimensions. [3]
3. The reduction in dimension significantly reduces numerical complexity.

As usual, the hope is that understanding the workings of these simpler systems will lend perspectives on systems more connected to reality.

Progress in two space(time) dimensions has been much more satisfactory on both analytic and numeric fronts. Criticality has been observed in systems comprising triangulations coupled to scalar fields (alternatively a cosmological constant) or gauge fields, both abelian and non-abelian [28, 29]. That said, arguably the most prized result is the *analytic* derivation of the critical coupling for a setup of Majorana fermions coupled to Euclidean triangulations (Extensions to Dirac fermions are also known [26]). The culmination of several seminal efforts, the key insight into this result was the discovery of a remarkable duality [43] between the triangulated fermions system and a solvable setup comprising Ising spins living on *dual* triangulations (a 'pseudo-lattice' obtained by replacing vertices of the triangulation by links and links of the triangulation by vertices).

While undoubtedly intriguing, these results nevertheless are only the first step in a thorough understanding of field theories atop two dimensional triangulations. In particular, each of the setups mentioned placed only a *single* matter field on the lattice. Ideally, we would next like to consider sets of *mutually interacting* matter fields atop dynamical triangulations. Additionally, since our universe is pervaded by Yang–Mills theories [44], dressing triangulations with the constituents of an interacting gauge theory is the natural follow up. The object of this work is to *analytically* address (at least in part) the simplest of these possibilities, that of quantum electrodynamics (QED) coupled to two dimensional gravity via triangulations.

QED in two *flat* spacetime dimensions (commonly referred to as the *Schwinger model*[30, 31]) is a field theorists dream! Simple enough to be exactly solvable, it is nevertheless complex enough to admit a wealth of non-perturbative phenomena, such as confinement[32], mass

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[2] Strictly speaking, this is one of two possible inferences, the other being that the dynamical triangulations paradigm is itself flawed, but we (naturally!) refrain from pursuing this alternative in this work.

[3] There is no *single* concrete reason for this. Rather, the reduced dimensionality induces simplifications on several of the standard mathematical structures one could place atop this base space (topological, differentiable, algebraic etc.). Different techniques then exploit the simplicity of one or more of these *specific* structures.

generation[45] and a chiral anomaly[33]. From the point of view of obtaining interesting physics, this means that we have a lot to look forward to. As far as *approaching* the problem goes, two (radically different) directions are visibly suggestive, and we shall explore them both in due course. First, it is worth asking if the aforementioned duality between triangulated fermions and Ising spins has an analogue even when $U(1)$ gauge fields are added to the mix. This is certainly not a trivial question, given that the original correspondence was itself quite delicate and simultaneously difficult to unearth. Nonetheless, even a partial correspondence would have huge ramifications.
Second, one might naturally hope that the integrability of the Schwinger in the *continuum* translates to solvable systems on the lattice. Assuming that this *does* pan out for flat lattices, we could attempt to construct solvable systems of *triangulated* Schwinger-like systems which presumably model, in the continuum limit, a viable theory of 'quantum geometro-electrodynamics' in two spacetime dimensions. This latter approach is (clearly) more involved, and, in this work, we shall confine ourselves to an analysis of the 'flat-space half' of the procedure. We hope to extend our results to triangulations in the near future.
In the rest of this section, we shall enumerate relevant features of *free* triangulated fermions/gauge fields alluded to earlier. As promised, we shall then use these details to construct a spin-system correspondence to triangulated Schwinger in the following section, where we shall also assess the implications of such a correspondence.

A. Triangulated Gauge Fields

For the pure gauge half of our proposed triangulated Schwinger action, we appeal to the work of Candido et al. [29]. This subsection does not contain any novel details, but merely aims to unwrap the expressions obtained in [29] while also serving as a gentle introduction to the basic lattice gauge constructs that will make frequent appearances in following sections.
The dynamics of a pure *continuum, Lie algebra valued* gauge field A_μ is governed by the usual free Yang–Mills action

$$S_{\text{YM}} \equiv -\frac{1}{4g^2}\int d\mathbf{x}\text{Tr}(F_{\mu\nu}F^{\mu\nu}), \tag{2}$$

where g is the Yang–Mills coupling, and the field tensor F is defined as

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \tag{3}$$

so that the action is manifestly invariant under the following local action of the gauge group $\mathcal{G}$ -

$$A_\mu \to \Omega A_\mu \Omega^{-1} + i\Omega\partial_\mu\Omega^{-1}, \quad \Omega \in \mathcal{G}. \tag{4}$$

While four-vector potentials serve as the best choice for generalised coordinates in the continuum, it turns out that local gauge invariance can be better enforced on the lattice via the so called *link variables.* These Lie *group* valued objects $U_\mu(n)$ are the lattice analogues of Wilson lines and live on the *links* of the lattice. With Latin indices representing spacetime coordinates and Greek indices denoting vector components, the link variables transform under local lattice gauge transforms as $U_\mu(n) \to \Omega_n U_\mu(n)\Omega^{-1}(n+\hat{\mu})$ (where $\Omega(n) \in \mathcal{G}\forall n$). It is therefore not hard to see that the product $\mathcal{W}_U$ of a sequence of M connected link variables (i.e the underlying sequence of links form a chain) $\{U_{\mu_1}(n1), U_{\mu_2}(n_1), ..., U_{\mu_M}(n_M)\}$ transforms as $\mathcal{W}_U \to \Omega(n_1)\mathcal{W}_U\Omega^{-1}(n_M)$. Making use of a routine trick in continuum gauge theory, we see that $\text{Tr}(\mathcal{W}_U)$ is a gauge invariant object - it is in fact simply the lattice analog of a Wilson *loop.* On a *standard* hypercubic lattice, it turns out that a lattice discretisation of Yang–Mills theory can be formulated entirely in terms of link variables, via the *Wilson* action [46]

$$S_W = -\frac{2}{g^2}\sum_{\pi\in\mathcal{P}}\text{Re}(\text{Tr}(\pi) - 1), \tag{5}$$

where the sum is over $\mathcal{P}$, the set of *plaquette variables* - the 'simplest' conceivable Wilson loops, in that their underlying (closed) link sequences do not enclose even a single vertex. Although superficially distant from its continuum partner, the Wilson action can indeed be shown, via the replacement $U_\mu(n) \to e^{iaA_\mu(n)}$, to reproduce the Yang–Mills action eq. (2) in the naïve $a \to 0$ limit. The ability to express the action in terms of 'physical' (gauge invariant) observables is an added bonus.
The action 5 remains, for the most part, unchanged even upon the addition of a curved background. Specifically, the Lagrangian remains *nominally* invariant (although the definition of the field tensor requires substituting partials with covariant derivatives, as per the minimal coupling prescription), and the *symbolic* alteration to the action comes from the inclusion of a non-trivial measure factor $\sqrt{-g}$ (where g is the metric determinant). It is evident that any respectable discretisation of gauge fields interacting with gravity must imbibe this 'volume factor'. Since the notion of an integral over spacetime translates to a sum over vertices in the lattice viewpoint, it would seem reasonable to incorporate local volume factors via *weighted* sums over vertices. Candido et al. posit that coordination numbers of vertices serve as valid lattice realisations of the measure factor $\sqrt{-g}$. This is loosely because the volume 'surrounding' a given vertex (which encodes, at the lattice level, the continuum measure factor) is proportional to the number of simplices sharing said vertex. But this is just the coordination number of the vertex in question. They also propose appending local factors of the square of the inverse of the coordination number to each element in the sum in the action eq. (5). The argument here is that a plaquette variable π_v centred at a vertex v with spacetime labels x_v and coordination n_v can (loosely) be tied to the continuum gauge field as $\pi_v \sim e^{in_v a A_\mu(x_v)}$ (i.e, the exponent of the

mean field times the loop length), so that transiting from the continuum to the lattice suggests the replacement $A_\mu(x_v) \to\sim (\pi_v - 1)/n_v$. Since the field tensor is (to leading order) linear in the gauge fields and the action is quadratic in the field tensor, we require *two* factors of n_v in the lattice action.
Making these replacements, Candido et al. proposed the following action for gauge fields atop a triangulation τ

$$S_\tau = -\frac{2}{g^2}\sum_v \frac{\mathrm{Re}(\mathrm{Tr}(\pi_v)) - 1}{n_v}, \tag{6}$$

with the sum ranging over vertices, with corresponding plaquette variables π_v and coordination numbers n_v. As stated earlier, the backreaction of the gauge fields on the underlying gravitational field is captured by a sum over triangulations with a given topology, augmented by a causal structure in the case of CDTs. Typically, the topology is taken to be spherical for EDTs and, as mentioned earlier, toroidal for CDTs.

B. Triangulated Fermion Fields

In general, fermions are the hardest type of particles to model using field theoretic approaches. This is primarily due to their rather esoteric transformation properties under Lorentz transformations. This is apparent, although not stressed enough, even at the level of flat spacetime where it is solely the magic of the gamma matrices that allows us to link spinorial representations to vectorial ones without much hassle. These links do not however survive the transition to curved space, wherein the local diffeomorphism group (i.e.$GL_n(\mathbb{R})$) does not admit spinorial representations. Even so, these *local* issues can be circumvented by appealing to tetrad based formalisms. In doing so however, we run into another pitfall, this time a *global* one - we must ensure that our base manifold admits a covering by a collection of local tetrad frames satisfying a compatibility condition. This compatibility condition is geared to ensure that the usual ambiguities associated with 2π rotations of the spin tetrads do not lead to inconsistencies. It can be shown that the existence of a suitable cover is tied to the topology of the base space. While necessary and sufficient conditions for the existence of such a cover have been established for generic four dimensional (pseudo) Riemannian base spaces, it is still a murky matter to construct lattice realisations of these conditions. Issues associated with such constructions have proved serious enough to (temporarily) halt progress in 3+1D systems of triangulated fermions.
Thankfully, lowering the dimensionality of spacetime is once again enough to render the problem tractable. In two dimensions, orientability suffices as an indicator of whether or not the manifold admits a spin structure. Lattice formulations imbibing orientation are not too tricky to devise, and dynamical triangulations in this vein were explored in great detail by Midgal [43], and subsequently by Burda and collaborators in a series of seminal works [24–29]. These studies all centred around a novel extension of the standard Dirac–Wilson action to randomly triangulated surfaces. Before addressing said extension, a few words on the usual ‘hypercubic’ regularisations of spinor fields are in order.
The Dirac action in Minkowski space reads

$$S_{\mathrm{D}} = \int d\mathbf{x} i\overline{\psi}(\gamma^\mu\partial_\mu - m)\psi \tag{7}$$

where ψ denotes the fermionic (Dirac) field, $\overline{\psi}$ the adjoint, and the gamma matrices γ_μ satisfy the usual anticommutation relations $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. Naïvely discretising this on a lattice (post a Wick rotation) with spacing a, we expect to obtain an action resembling

$$S_{\mathrm{N}} = \sum_{n,\mu} \frac{\overline{\psi}(n)\gamma^\mu\partial_\mu(\psi(n+\hat{\mu}) - \psi(n-\hat{\mu}))}{2a} \tag{8}$$

Irritatingly enough, this action does not yield the ‘correct’ particle spectrum at the lattice level. Explicitly, the particle content of the above lattice theory is not limited to the free fermions that we see in the $a \to 0$, but also contains a set of ‘doublers’ of mass $m \sim \mathcal{O}(1/a)$, which cease to be dynamical in the small a limit. While we do not expect lattice realisations of quantum fields to be picture perfect, a collection of fictitious particles is clearly too much! Accordingly, preferred discretisations of the Dirac action typically add in an extra few terms to the lattice action which effectively kill off the doublers while also maintaining a sensible continuum limit. Prime among these actions is the Dirac–Wilson scheme, which proposes the following ‘doubler free’ action

$$S_{\mathrm{DW}} = \sum_{N,\mu\in\{-4,4\}} \overline{\psi}(n)(m + \frac{4}{a})\psi(n) + \frac{\overline{\psi}(n)(1-\gamma_\mu)\psi(n+\hat{\mu})}{2a}. \tag{9}$$

where $\gamma_{-\mu} \equiv \gamma_\mu$. It is this action that was modified to suit triangulations by Burda et al. Appropriate modifications require A) shifting to a more tetrad-oriented, ‘coordinate-free’ basis, and B) making sense of the spin-structure alluded to earlier for generic topologies. In gist, the requisite action atop a triangulation τ is

$$S = \sum_{i,j} \overline{\psi}_i D_{ij}\psi_j, \tag{10}$$

where, in the spirit of enforcing coordinate invariance, we have indexed spinors with a ‘generic’ Latin index. The sum is over oriented links $< i, j >$ and

$$D_{ij} = \frac{1}{2} - KH_{ij}, \tag{11}$$

where K (for reasons that will later become evident) is the referred to as the hopping parameter and H, the hopping operator is defined as

$$H_{ij} \equiv \frac{(1 + n^i_{ij}.\gamma)}{2}\mathcal{U}_{ij}. \tag{12}$$

In the above expression, n^i_{ij} is the vector originating from the centre of the i$^{\text{th}}$ simplex and terminating at the centre of the j$^{\text{th}}$ and $\mathcal{U}_{ij}$ is the lattice complement of the continuum spin connection. U_{ij} is a local construct that nevertheless encodes the global geometry of the triangulation.

Having outlined the construction of the object underpinning fermionic dynamics atop random triangulations, it still remains to be seen whether these lattice dynamics have a sensible continuum limit. This is indeed true and the discovery of *analytic* expressions for the relevant critical parameters should rank as one of the most impressive achievements in the field. As mentioned earlier, this analyticity owes its existence to a duality between triangulated fermions and triangulated Ising spins.

In brief, this duality is established by means of the *hopping parameter expansion* (HPE), a formal perturbative series for the Euclidean partition function akin to the vacuum bubble expansion in the continuum. Intuitively, we may build such an expansion by morphing the continuum integral over vacuum bubbles to a sum over the set $\mathcal{L}$ of *unoriented loops* L over the lattice. The contribution of each loop to sum various geometrically with its length $S(L)$, which one naturally anticipates from the sum over paths imagery and the algebra of amplitudes. We thus have

$$\mathcal{Z} \propto \sum_{L \in \mathcal{L}} K^{S(L)}, \tag{13}$$

where K, the hopping parameter, represents the 'propagation amplitude' across an arbitrary link of the lattice. Massaging the HPE into a form more traditionally encountered in statistical physics, we obtain

$$\mathcal{Z} \propto \sum_{L \in \mathcal{L}} e^{S(L)\ln(K)}. \tag{14}$$

We can then discern the duality by recognising that, modulo a shift in the vacuum energy, this is *precisely* the partition function generated by a Hamiltonian spin system sitting on the *dual* triangulation. In this spin system, each pair of adjacent anti-parallel spins contribute an energy of $\ln(K)$ while parallelly aligned spins do not bear any energy costs whatsoever. Visually, the loops constituting the sum on the fermionic side map to *domain walls* - contours separating blocks of parallel spins - on the Ising side of the correspondence. As a second addition to the fermion-Ising dictionary, we see that the hopping parameter (or more precisely its logarithm) serves as the inverse temperature, β, for the Ising side of the duality.

As the quintessential statistical model, critical parameters for Ising systems are directly computable in a variety of settings. As it turns out, random triangulations constitute one such family of solvable settings, so that knowledge of the critical inverse temperature is all that we need to analytically evaluate the critical value of the hopping parameter K_{crit}.

Having reviewed some of the developments in the field, we will, in the following section, proceed with our own analysis, dressing triangulations with Schwinger fermions and studying in depth the properties of the resulting mix of fields.

IV. SCHWINGER ON THE LATTICE

Our Schwinger setup comprises Dirac fermions and a $U(1)$ gauge field over a collection of random Euclidean triangulations [4] of a *spherical* topology. Keeping the underlying topology simple allows us to concentrate our attentions on the effects of *geometry*, which is our primary motivation. Of course, since the two do go hand-in-hand, richer topologies certainly warrant separate studies. These will be the subject of future explorations, although we shall make some generic comments in passing.

Piecing together the two halves of the action reviewed earlier, we see that the EDT action for a given triangulation τ comprises a now *gauged* fermionic half S^τ_F and a free gauge half S^τ_G.

$$S^\tau_{\text{Schwinger}} = S^\tau_F + S^\tau_G. \tag{15}$$

The fermionic half of the action remains *nominally* equivalent to the free case eq. (10) discussed earlier. Similarly, the hopping expansion of the Dirac Wilson operator D also retains its original, ungauged form eq. (11). However, the hopping operator H eq. (12) is *now* modified by the link variables U_{ij}

$$\mathcal{H}^\tau_{ij} = \frac{(1 + n_{ij}.\gamma)\mathcal{U}^\tau_{ij}\Omega_{ij}}{2}. \tag{16}$$

For the *specific* case of $U(1)$, the exponential map, which links Lie groups to their respective Lie algebras, is rather trivial (it is just the usual complex exponential function), so that we may equally well express the link variables in terms of *algebra* valued link fields $\alpha_{ij} \in [0, 2\pi)$ via the group-algebra mapping $U_{ij} = e^{i\alpha_{ij}}$.

Next, for brevity, we define the variable $\gamma \equiv \frac{2}{g^2}$, in terms of which the Candido gauge action eq. (6) may be rewritten as

$$S_G = -\gamma \sum_v \frac{\text{Re}(\pi_v)}{n_v} + f_\tau, \tag{17}$$

where we have packaged all the 'plaquette-independent' term into a single object $f_\tau \equiv \gamma \Sigma_v \frac{1}{n_v}$. f_τ depends solely

[4] Again, this is merely for simplicity in computations, the qualitative features of our analysis are not affected by signature choices

on the geometry of the triangulation and factors out of the functional integral over $U(1)$ fields. For this reason, we can (and will) ignore factors of f_τ during intermediate computations and reintroduce them only towards the end. The full *partition function*, Z, is now given by [5]

$$Z = \sum_\tau \int (\prod_{i<j} d\alpha_{ij})(\prod_k d\overline{\psi}_k d\psi_k) e^{-(S_F^\tau + S_G^\tau)}. \tag{18}$$

We can integrate over the fermionic degrees of freedom to get a fermionic determinant which we can then expand in K, the hopping parameter. This expansion can be reinterpreted as a sum over loops on the dual lattice (indeed, this is how the HPE is formally derived even in the free case, we merely sketched out plausibility arguments earlier). Happily, it can be shown [26] that the addition of gauge fields merely adds a multiplicative factor to each term of the free HPE. Indeed, It is not difficult to show that the multiplicative factor for a given loop is obtained by multiplying out the cosines of the link variables residing on said loop (i.e., for example, the multiplicative factor for a loop traversing three links $\{l_1, l_2, l_3\}$ respectively hosting the algebra valued link variables $\{\alpha_1, \alpha_2, \alpha_3\}$ is $\cos(\alpha_1)\cos(\alpha_2)\cos(\alpha_3)$.). Therefore, we may express the partition function restricted to a single triangulation τ as

$$Z_\tau \propto \int (\prod_{i<j} d\alpha_{ij})(\sum_l C(l)) e^{\gamma \sum_v \frac{\mathrm{Re}(\pi_v)}{n_v}} \tag{19}$$

$$= \int (\prod_{i<j} d\alpha_{ij})(\sum_l C(l))(\prod_v e^{\gamma \frac{\mathrm{Re}(\pi_v)}{n_v}}) \tag{20}$$

where $C(l)$ represents the contribution of a loop l to the partition function (i.e. the fermionic contribution eq. (13) times the aforementioned multiplicative factor). The proportionality constant is inherited from the free fermionic portion of $C(l)$. Since this factor is independent of l itself, and since the partition function is really only defined upto an overall multiplicative factor, we loose no generality in replacing the proportionality with equalities.

[5] It may seem that we have been blasé with our functional measure, since functional integrals over gauge fields traditionally make use of the Haar measure. However, for the *specific* case of $U(1)$, the Haar measure coincides with the trivial $d\alpha$ measure, so there are no formal issues to bother with.

Expanding out the exponentials, this yields

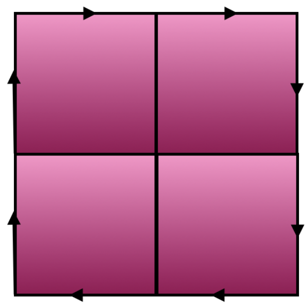
$$\begin{aligned} Z_\tau = \int (\prod_{i<j} d\alpha_{ij})(\sum_l C(l)) \\ \Big(\sum_{i_1, i_2 \ldots i_M} \left(\frac{\gamma}{2}\right)^I \big(\frac{(\pi_1 + \pi_1^*)^{i_1}}{n_1^{i_1} i_1!} \ldots \frac{(\pi_M + \pi_M^*)^{i_M}}{n_M^{i_M} i_M!}\Big) \\ = \int (\prod_{i<j} d\alpha_{ij})(\sum_l C(l)) \\ \Big(\sum_{\substack{i_1, i_2 \ldots i_M \\ k_1, k_2 \ldots k_M}} \left(\frac{\gamma}{2}\right)^I \Big(\binom{i_1}{k_1} \frac{\pi_1^{i_1 - 2k_1}}{n_1^{i_1} i_1!} \ldots \binom{i_M}{k_M} \frac{\pi_M^{i_M - 2k_M}}{n_M^{i_M} i_M!}\Big), \end{aligned} \tag{21}$$

where $I = \sum_{l=1}^M i_l$, M is the total number of vertices, and we have utilised the unitarity of the plaquette variables. Additionally, while the k indices should in principle run from 0 through the appropriate i indices, we will extend their range to $\mathbb{Z}$ with the understanding that the factorial of a negative integer is to be taken to be $\pm\infty$. Such extensions, while partly nominal, do not affect the sum and will not lead to any issues later on. On the contrary, they will simplify both notation and computation. From the form of the plaquette variables and the structure of the expansion, it is clear that the link variables α_{ij} only appear in the integrand as complex exponents (i.e., as $e^{in\alpha_{ij}}$) for integral n (the cosines in the loop factors can be expanded to yield exponents). The required integrals are therefore quite trivial and all the complexity now resides in the combinatorics of the expansion. In fact, the Fourier relation $\int_0^{2\pi} e^{in\theta} = 2\pi\delta_{n0}$ ensures that *most* of the terms of the above series simply yield zero on integrating over the link variables. However, as there are a good many links, keeping track of *individual* link variables can become quite cumbersome, so possible improvements could arise from translating the above relation into the language of *loops* instead. Since the partition function explicitly contains a sum over loops, stemming from the HPE, it is conceivable that utilising loops as our fundamental variables will greatly simplify our algebra.

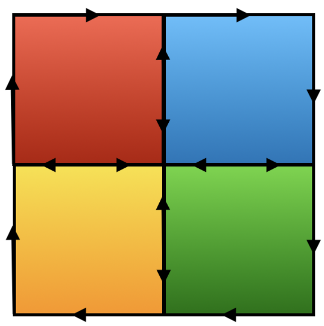
A little further inspection reveals that *plaquettes*, rather than *generic* loops,serve as the ideal summands for our expansions. Plaquettes are, after all, just loops (albeit 'special' ones), so their utilisation gels neatly with the HPE portion of eq. (21). Furthermore, plaquette variables *also* serve as 'generators' for the free gauge portion of the expansion eq. (21). Incorporating *generic* loop variables into the free gauge half of $\mathcal{Z}$ is, by contrast, not as simple. Plaquettes are therefore *the* way to go. Of course, since the HPE carries a sum over *generic* loops, so that our proposed switch necessitates a better understanding of how arbitrary loops can be represented in terms of the more 'fundamental' plaquettes.

To this effect, we introduce the notion of a 'plaquette expansion'. Pictorially, an oriented loop l can be 'expanded' as a 'sum' of k oriented plaquettes $\{P_1, P_2 \ldots P_k\}$

as $l = `\sum_{i=1}^{k}{}' P_i$ if the 'boundary' of the region spanned by the k plaquettes is simply the loop l with the plaquette links in the 'interior' of said region cancelling out one another due to opposing orientations. Figure 1b depicts this for the more symmetric case of a regular square lattice.



(a)



(b)

FIG. 1: Plaquette expansions: Observe that the 2×2 loop enclosing the similarly tiled squares in figure (a) is the 'sum' of the four 1×1 plaquettes, each enclosing a differently tiled square, in figure (b)

By construction, the *global* structure of the loops, which imbibes the topological aspects of our triangulations, must now be encoded in the algebra of plaquette expansions. Reversing this train of thought, we see that imposing topological constraints on our triangulations must in turn impact the algebra of plaquette expansions. Such influences are best traced out using the language of homology groups. Indeed, the loose 'sum' ('+') that we have utilised thus far can be formalised as the addition operation between elements of a chain group. In the interests of progress however, we will not stagnate with details and will instead adopt a blasé approach. Specifically, we will simply state the results that we will require, with the understand that our conjectures present straightforward proofs using standard algebraic topology. We therefore posit the following three (loosely worded) statements:

1. Assuming a *simply connected* topology, Each non self-intersecting loop on the dual lattice can be expressed as an (oriented) sum of plaquettes Indeed, this result is *not* true for triangulations of spaces with a non-zero genus, for instance, the two torus T^2.

2. Assuming a *connected* topology, every *contractible*, non self-intersecting loop in the HPE can be expressed as an (oriented) sum of plaquettes in *precisely* two ways in which each plaquette appears at most once. We will call such expansions 'fundamental' expansions.

3. Let P_v denote one of the two oriented plaquettes 'enclosing' the vertex v. In the sense of plaquette expansions, the second plaquette 'enclosing' v is thus naturally represented as $-P_v$. Assuming an *orientable* topology, we claim that it is possible to orient each plaquettes (i.e, to decide, for each vertex v, *which* of its surrounding plaquettes to designate P_v, and which to designate $-P_v$) so that $\sum_v P_v = 0$. In what follows, we shall implicitly assume such a family of orientations.

4. Assuming an *orientable,connected topology)* The only combinations of plaquettes that sum to zero are the trivial combination and those of the kind mentioned in point 3 above.

We are now in a position to restate the Fourier relations for the link variables in terms of loops. The portion of the loop factor stemming from the HPE is, for a specific loop l (modulo the 'free fermion factor') just the product of cosines of link variables lying on the loop. On substituting trigonometric factors with exponentials, we see that the key to switching from links to loops lies in understanding that this 'gauge factor' can *equally* well be obtained by multiplying out the plaquette variables corresponding to those plaquettes that appear in the plaquette expansion of l.[6] Intuitively, the link variables in the 'interior' of the region spanned by the relevant plaquette cancel out due to opposing orientations so that only the link variables residing on the 'boundary' of the region (i.e. l) remain. (Again, see Figure 1b for a graphical depiction). It is interesting to note that the abelian nature of the $U(1)$ gauge group is crucial for this remodelling of the 'gauge factor'. Therefore, the loop factor for a length n loop l, with a plaquette expansion $l = \sum_k c_k P_k, c_k \in \mathbb{Z}$ (we now drop the quotes over plaquette sums) takes the form

$$C(l) = K^n \pi_1^{c_1} \pi_2^{c_2} ... \pi_M^{c_M} . \tag{22}$$

From eq. (21), the contribution of a given loop to the full partition function is thus its HPE portion ($C(l)$,

[6] Strictly speaking, there is *some* work to be done - we in fact get *one* of 2^N exponential factors for an N link loop. Each cosine in the loop factor contributes *two* exponential factors, so that an N length loop yields the mentioned 2^N terms. However, elementary (non-topological) features of the plaquette expansion eventually kill off all but one of these factors - the one that emerges from multiplying out the plaquette variables atop the links of the loop.

which is the free fermion factor times the gauge factor) appended to a sum over 'plaquette factors' arising from the free gauge action. Again, most of the terms in this sum will not survive the integral over the link variables. Our task is simply to identify and collect the terms that *do* survive.

Now, in words, the Fourier relation simply states that a given term (i.e. the product of a specific loop factor and a specific plaquette factor) in the expansion will survive the integral over a specific link variable if the term does not depend on this link variable. Consequently, the only terms that remain after all the integrals have been performed are the terms that have *no* explicit dependence whatsoever on *any* of the link variables. From points 3 and 4 on our earlier list of conjectures, it is clear that this can happen if a term satisfies one of the two conditions below:

1. The plaquette factor of the term is the product of inverses of link variables appearing in one the fundamental expansions of the loop.

2. The plaquette factor of the term is the product of inverses of link variables appearing in one the fundamental expansions of the loop *and* an integer exponent of the product of *all* oriented link variables (i.e. $(\pi_1\pi_2...\pi_M)^y$).

Formally, the former point is just the latter with y set to zero, but it is useful to distinguish the two cases. Crudely put, a term contributes *only* if the plaquette factors cancel out the gauge factors in $C(l)$ *up to* a global exponent of the product of all the plaquette variables.

Having ascertained which terms in the series expansion yield non-zero contributions, it remains to calculate the coefficients of these terms. Generically, we we would like to find the coefficients of plaquette factors $\pi_1^{c_1}\pi_2^{c_2}...\pi_M^{c_M}$ in the plaquette expansion of the gauge action. Since the expansion eq. (21) factors into a product of series, one for each plaquette, the coefficients undergo a similar factorisation and take the form $f(c_1)f(c_2)...f(c_M)$ for some as yet undetermined function f. To find f, we may restrict ourselves to the series for any individual plaquette, say the first one. The series corresponding to the first plaquette is

$$\sum_{\substack{i_1\in\mathbb{Z}^+\cup\{0\}\\ k_1\in\mathbb{Z}}} (\frac{\gamma}{2n_1})^{i_1}\binom{i_1}{k_1}\frac{\pi_1^{i_1-2k_1}}{i_1!} = \sum_{j_1\in\mathbb{Z}} f(j_1)\pi_1^{j_1}. \quad (23)$$

Precise expressions for f can be obtained by comparing coefficients on either side. For this purpose, it serves to consider separate cases for even and odd j_1. For even $j_1 = 2m_1, m_1 \in \mathbb{Z}$, we find that only even i_1 contributions occur and we have

$$f(j_1) = f(2m_1) = \sum_{p_1\in\mathbb{Z}+\cup\{0\}} (\frac{\gamma}{2n_1})^{2p_1}\binom{2p_1}{p_1-m_1}\frac{1}{(2p_1)!}, \quad (24)$$

where we have replaced our summation variable i_1 with $p_1 \equiv \frac{i_1}{2}$ and eliminated k_1. This can be further simplified as follows

$$\begin{aligned} f(j_1) = f(2m_1) &= \sum_{p_1\in\mathbb{Z}+\cup\{0\}} (\frac{\gamma}{2n_1})^{2p_1}\binom{2p_1}{p_1-m_1}\frac{1}{(2p_1)!} \\ &= \sum_{p_1\in\mathbb{Z}+\cup\{0\}} \frac{(\frac{\gamma}{2n_1})^{2p_1}}{(p_1-m_1)!(p_1+m_1)!} \\ &= \sum_{q_1\in\mathbb{Z}+\cup\{0\}} \frac{(\frac{\gamma}{2n_1})^{2q_1+2m_1}}{(q_1)!(q_1+2m_1)!} \\ &= \sum_{q_1\in\mathbb{Z}+\cup\{0\}} \frac{(\frac{\gamma}{2n_1})^{2q_1+j_1}}{(q_1)!(q_1+j_1)!}, \end{aligned} \quad (25)$$

where $q_1 \equiv p_1 - m_1$ and we have been liberal with our interpretation of factorials of negative integers as $\pm\infty$. *The final expression is precisely the Frobenius series for the modified Bessel functions of the first kind* $I_{j_1}(\frac{\gamma}{n_1})$. Therefore, for even j_1, we have $f(j_1) = I_{j_1}(\frac{\gamma}{n_1})$. A similar computation follows through for the case when j_1 is odd. Not unsurprisingly, the result is unchanged, so that we have $f(j_1) = I_{j_1}(\frac{\gamma}{n_1})$ for all integers j_1.

Reverting to the generic case of M plaquettes and utilising our earlier results, we can *finally* obtain the contribution of a loop with a fundamental expansion comprising plaquettes $l_1, l_2...l_L$. Call this set of plaquettes $\mathcal{P}_L$ and its complement $\mathcal{Q}_L$. Furthermore, let $n(p)$ denote the coordination number of the plaquette p. Then, the loop l of length $L(l)$ contributes

$$\begin{aligned} X^{n(l_1)+n(l_2)...+n(l_L)} &\sum_{y\in\mathbb{Z}}\Big(\prod_{p\in\mathcal{P}_L} I_y(\frac{\gamma}{n(p)})\prod_{q\in\mathcal{Q}_L} I_{1+y}(\frac{\gamma}{n(q)})\Big) \\ &= X^{L(l)}\sum_y F_y(l), \end{aligned} \quad (26)$$

where $F_y(l) \equiv \prod_{p\in\mathcal{P}_L} I_y(\frac{\gamma}{n(p)})\prod_{q\in\mathcal{Q}_L} I_{1+y}(\frac{\gamma}{n(q)})$ encapsulates all the complexities introduced by the gauge field. The partition function restricted to a given triangulation τ is this

$$Z_\tau = \sum_l\sum_y X^{L(l)}F_y(l). \quad (27)$$

It is worth poring over the physical implications of this result. First, we note that unlike free fermions, but like the free $U(1)$ gauge field, the partition function shows an *explicit* dependence on the discretised curvature variables (the coordination numbers of the vertices). The gauge fields therefore 'bring out' the curvature of the underlying manifold. The effects of the *gauge* curvature on the other hand are (perturbatively) brought out by the HPE with the indices of the Bessel functions in the series serving (in a loose sense) as a measure of the winding numbers of the loop (or analogous well-defined variables on a compact manifold). It is important to note that while plaquette expansions played a pivotal role in *computing* the partition function, the final expressions for the loop factors do not make any references to the plaquettes. Rather, they depend on the positioning of the individual vertices on the 'inside' and 'outside' (or relevant compact analogs) of the loop in addition to the usual loop length. The $\mathcal{P}$ and $\mathcal{Q}$ regions, while introduced in the context of plaquette expansions *really* represent the 'interior/exterior' of the loop.

To map the triangulated Schwinger model to a spin system, we can play loose and swap the sum over triangulations with the sum over the Bessel index y. Doing this and performing some admittedly rather artificial manipulations, we obtain

$$\begin{aligned} Z_\tau &= \sum_l \sum_y X^{L(l)} F_y(l) \\ &= \sum_y \sum_l X^{L(l)} F_y(l) \\ &= \sum_y \sum_l e^{L(l)\log(X)+\log(F_y(l))}. \end{aligned} \tag{28}$$

Confining ourselves to a *single* y index, we see that the free fermion term can be interpreted as a contribution from a domain wall of a set of Ising spins living on the triangulation as usual. The gauge field are encoded in F_y. The crux is that because $F(y)$ factors into a product of terms, one for each vertex, the log of F_y splits into a *sum* of terms, one for each vertex. The contribution of each vertex depends on two features - first, the coordination number of the vertex and second, whether the vertex in question lies in the $\mathcal{P}$ zone or the $\mathcal{Q}$ zone (this is keeping in line with our interpretation of $\mathcal{P}/\mathcal{Q}$ as 'interior' and 'exterior' regions). With the loops representing domain walls, it is clear that these zones are simply the domains themselves. In particular, Ising spins 'pointing in one direction' contribute $\mathcal{P}$ factors while spins of the opposite orientation contribute $\mathcal{Q}$ factors. This is very reminiscent of interactions with an external magnetic field, with the correspondence becoming exact if we shift the energy by an overall constant factor. Additionally, we see that the 'magnetic field' seen by a vertex depends on the *coordination number* of this vertex.

In conclusion, for a *fixed* y and τ, the partition function is precisely that of an Ising model living on the triangulation in the presence of an external 'coordination dependent', y dependent magnetic field. The full partition function, in this viewpoint, takes the form of a *sum* over (infinitely many) Ising partition functions, each comprising the same sets of spins, but with different magnetic field strengths.

While visually pleasing, the presence of a countably infinite number of Ising systems on the spin side of the duality render it impossible to probe criticality for the Schwinger model. Each of the Ising systems described above reaches criticality at a *different* reduced temperature β. Consequently, it is impossible to predict the dynamics of a system whose partition function takes into account *each* of these systems in equal measure. The presence of a locally varying magnetic field for each individual Ising system is just the final nail in the coffin.

Nonetheless, a few points deserve mention. We begin with a trivial one - the transition to CDT. Topological issues aside (we will discuss these shortly), we see that the *algebra* underlying our computations is precisely the same, save that our initial action stored an extra factor of i. As a result, we may show that, barring a few external factors of i in out intermediate expressions, the key difference lies in the replacement of *modified* Bessel functions K_p with *regular* Bessel functions J_p (since we have $K_p(ix) \propto J_p(x)$). As such, the imposition of a causal structure, bar topology, only alters the *values* of the magnetic field strengths controlling our Ising systems, without affecting the qualitative structure of the expansion. This is a pleasing result, for this (again, *barring* topological considerations) validates our long held conjecture of Wick rotations not unduly affecting any underlying *physics*.

Less trivial are the effects of *topology*. While we cannot *precisely* survey the effects of the topology, there are nevertheless some hints that topological consequences can be quite influential! After all, while we did not stress on it earlier, our expansion of the partition function Z, post 'free gauge cancellations', retained *only* loop factors arising from *those loops which bore a plaquette expansion* in the first place. From conjecture 1 on our list, we see that the existence of an expansion pre-supposed the existence of a simply-connected topology. In fact, it is not too difficult to show that for *generic* topologies, the possibility of plaquette expansions are restrained to *contractible* loops. This is a highly intriguing result. Recall that the *free* fermion HPE was not choosy about the nature of its constituent loops. Indeed, *any* loop conceivable made the cut. The addition of gauge fields, by contrast, had induced an admission criterion of sorts. While quantitative adjustments to the HPE factors are only to be expected, the absence of entire *classes* of loops from the modified expansion is extremely curious. Indeed, topological effects feature prominently in gauge-theoretic studies, the most rudimentary example being the Aharanov-Bohm effect. While fascinating, we are still only at the stage of vague conjecture. It would be interesting to see if we can obtain anything more *concrete*, but this would constitute a completely different, highly involved line of thought well beyond our current scope. We therefore conclude at our present level.

Given that our rather ambitious aims translated to only modest chances of success, it should not be too surprising that our ideas did not *quantitatively* pan out. The *free*-fermion duality is evidently too delicate to survive the addition of a gauge field to the action, even though we could glean some qualitative insights from it. As a result, we must now change direction. As remarked earlier, the nature of our Schwinger system suggested *two* viable methodologies. Having explored the first of these in great detail, we now turn our attentions to the second one, that of exploiting the solvability of the Schwinger model in the flat space continuum. In the next section, we will briefly recap a section of well-established techniques for solving the Schwinger model. Following this, we will attempt a novel implementation of these techniques on the lattice in section VI.

V. THE FUJIKAWA METHOD

For historical reasons,the pedagogy of the Schwinger model is heavily biased towards operator based methods. Specifically, a large part of related literature utilises inherently Hamiltonian techniques [30, 31], wherein fermionic operators are mapped *non-locally* to bosonic counterparts, in terms of which the Hamiltonian manifestly decouples. Lagrangian perspectives were explored only much later, primarily by Fujikawa and collaborators [47–49]. Since dynamical triangulations are rooted in the path integral formulation, it is nevertheless the latter methodology which will prove fruitful for our purposes. We therefore give a brief account of Fujikawa's approach to solving the Schwinger model. In so doing, we will also highlight the salient features which we will extensively make use of for lattice analogues. As it turns out, working with a Lorentzian signature necessitates a careful micromanaging of negative and positive signs for numerous objects. Since this nonetheless has no effect on the underlying physics, we will henceforth always assume Euclidean backgrounds (and therefore, we will **not** distinguish between contravariant and covariant indices).
The Schwinger action is then given by

$$S_{\text{Schwinger}} = \int d\mathbf{x} i\overline{\psi}(\gamma^\mu(\partial_\mu - iA_\mu) - m)\psi. \tag{29}$$

From the form of the $U(1)$ gauge transformation $A_\mu \to A_\mu - \partial_\mu\Gamma$, it is clear that 'pure gauge configurations', with $A_\mu = \partial_\mu\chi$ for some scalar χ can easily be trivialised by an appropriate choice of Γ. Of course, not all gauge configurations are pure. Remarkably, it turns out that a *partial* trivialisation can nevertheless be obtained for *any* configuration of the gauge field. Specifically, any configuration A_μ in the Lorentz gauge $\partial_\mu A_\mu = 0$ satisfies

$$\gamma_\mu A_\mu = \gamma_\mu\gamma_5\partial_\mu\phi, \tag{30}$$

for some scalar ϕ. This partial trivialisation is a straightforward application of two key observations

1. For any 'nice enough' vector field A_μ, we have $A_\mu = \partial_\mu\chi + \epsilon_{\mu\nu}\partial_\nu\phi$ for some scalar fields ϕ and χ. This is simply an echo of the more familiar Helmholtz theorem in three space dimensions, which states that any vector field may be decomposed into a gradient and a curl. (Choosing the above mentioned Lorentz gauge is essentially tantamount to setting the gradient to zero).

2. The Dirac algebra in two dimensions satisfies

$$\gamma_5\gamma_\mu \propto \epsilon_{\mu\nu}\gamma_\nu. \tag{31}$$

It is then immediately clear that the interactions of any configuration A_μ with the spinor field ψ can be negated by first (partially) trivialising the configuration to a scalar ϕ, via eq. (30) and subsequently performing a local *chiral* transform

$$\psi \to e^{i\gamma_5\phi}\psi, \tag{32}$$

to completely 'gauge' away the ϕ term. In short, the transformations eq. (30) and eq. (32) transform the Schwinger action from a *coupled* vector-spinor system into a *decoupled* scalar-spinor system.
There is one catch however. The path integral formulation is bipartite - it includes not only the action, but also the functional measure. While normally sedate, the measure is in this case *not* invariant under the local chiral transform [47, 48]. The corrections induced by the measure add an extra scalar mass $m = \frac{e^2}{\pi}$ to the action, and it is this factor that sources the famous mass generation of the Schwinger model.
The anomalous mass generation aside, it is thus clear that the partial trivialisation mentioned earlier is the key theme underlying the integrability of the Schwinger model. It is therefore trivialisations of this kind that we must hunt for at the level of the lattice if we are to study triangulations of Schwinger systems. Operationally, the trivialisations themselves stem from the peculiarities of vector fields and the gamma matrix algebra in two dimensions. Any potential lattice solutions that we could envisage should hence make use of discretised analogues of these two themes. However, the lattice introduces artefacts of its own, and it turns out that interlacing these artefacts with our Schwinger themes is no straightforward matter, as we shall see in the next section.

VI. FUJIKAWA ON THE LATTICE

Before delving into the details, it is worth reiterating our methodology and pondering over a few lingering subtleties. Figure 2 schematically outlines our proposed procedure. Since *directly* tackling the problem of a gravitationally bound Schwinger system is out of question, we are forced to proceed in a roundabout fashion, from the flat space continuum, to hypercubic lattices, to triangulations, and ultimately, to curved continua. While circuitous, this route is certainly accessible via numerics. *Our* goal however, is to gain *analytic* insights into the workings of the systems in consideration. Indeed, simplicity reasons aside, this is the prime reason behind our choice of Schwinger.

There are in fact *two* analytic footholds that we wish to utilise. First, there is the solvability intrinsic to Schwinger which we would ideally like to use to decouple our previously constructed *lattice* Schwinger model, first with flat lattices, then with triangulations. *If* we are indeed able to accomplish this, we may then probe criticality by simply appealing to previous results on pure gauge/spinor fields atop triangulations, and it is here

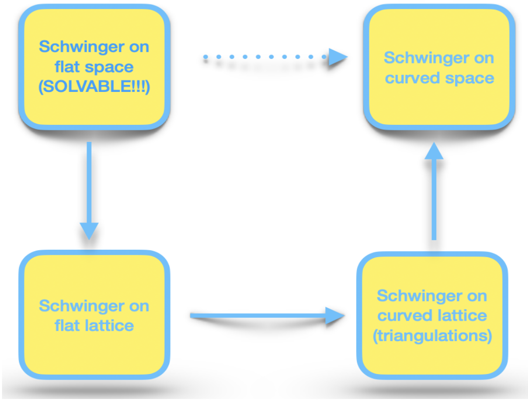


FIG. 2: Methodology

that the analytics of Burda et al. will play an indispensable role. Along the way, we can also use these two sets of analytics (those of Schwinger and Burda et al.) to probe the effects of curvature and gravitational back reactions on non-perturbative phenomena such as confinement and anomalies.

Unhappily, there is a possible issue of mismatching setups. More precisely, the work of Burda et al. is deeply ingrained onto the framework of Dirac–Wilson type discretisations. If we shift to a different lattice setup, say with staggered fermions, there is no guarantee that we can preserve the duality with Ising spins and, by extension, the analytic results on criticality that we so badly desire. On the other hand, as per our above procedure, we also require a lattice discretisation of the Schwinger model that preserves the separability of its continuum limit. Even if we *do* assume that such a family of integrable lattice models exists, this family may not bear any resemblance whatsoever to the Dirac–Wilson type discretisations that are essential for analytic criticality. A graphical description of this potential mismatch is shown in Figure 3. If such a mismatch *does* indeed exist, then

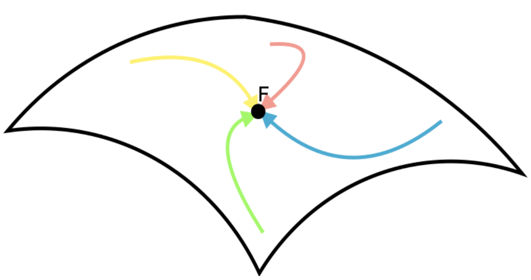


FIG. 3: An RG flow description of our methodology:Numerous curves flow towards the fixed point that is continuum Schwinger, including the red *integrability preserving* curve and the blue curve that is the Wilson discretisation of the Schwinger action. Our hope is that both curves *coincide* although there is a priori no *great* reason to suppose so.

short of completely reformulating Burda's results in more suitable discretisation schemes, we have no clear way forward. That said, the results that we *could* glean are reason enough to forge ahead, assuming that all is well and that there is *no* mismatch, at least for a while. Indeed, we will adopt this outlook going forward for the entirety of this work.

Qualitative contemplation aside, let us now dive into some concrete calculations. Our aims are two-fold. First, as we have reiterated to the point of weariness, we wish to demonstrate that the lattice Schwinger model we have constructed is separable into a decoupled pair of fields. Second (and less repetitive), we wish to study, at least qualitatively, the emergence of chiral anomalies. While triangulations are our primary setting in the long term, the complexities involved render a step wise course of action more feasible. Specifically, in this work, we will confine our attentions to the simpler problem of decoupling the (Euclidean) Schwinger model on a regular square lattice. Additionally, to merely *illustrate* the novel difficulties posed by lattice regularisations, we will regress to using the *naïve* gauged form of the action eq. (8), rather than the standard Dirac-Wilson action. This naïve action reads

$$S = \sum_{n,\mu} \frac{\overline{\psi(n)}(\gamma_\mu(\Omega_\mu(n)\psi(n+\hat{\mu}) - \Omega_{-\mu}(n)\psi(n-\hat{\mu}))}{2a}, \tag{33}$$

where Ωs are link variables as usual. Near the continuum limit, the link variables furnish expansions in terms of the gauge field A_μ, with $\Omega\mu(n) \sim 1 + iaA_\mu(n)$. Making this replacement and ignoring factors of a, we obtain

$$S \sim \sum_{n,\mu} \frac{\overline{\psi(n)}(\gamma_\mu(\psi(n+\hat{\mu}) - \psi(n-\hat{\mu}))}{2a} + \frac{\overline{\psi(n)}(\gamma_\mu(A_\mu(n)\psi(n+\hat{\mu}) + A_\mu(n-\hat{\mu})\psi(n-\hat{\mu}))}{2a}. \tag{34}$$

which seems more in tune with the continuum action. The kinetic terms are trivial, and the two gauge terms are nearly identically, bar a simple $\mu \to -\mu$ replacement. Without loss of generality, let us therefore confine our attentions to the first of these two gauge terms -$\Delta S_G \equiv \overline{\psi(n)}\gamma_\mu A_\mu(n)\psi(n+\hat{\mu})$ (where we again ignore factors of a). Recall that our strategy in the continuum involved trivialising gauge configurations via a 'Helmholtz-like' decomposition of arbitrary vector fields. We can mirror this decomposition on the lattice, by conjecturing a lattice decomposition of the form $A_\mu(n) = \chi(n+\hat{\mu}) - \chi(n) + \epsilon_{\mu\nu}(\phi(n+\hat{\nu}) - \phi(n))$ for some scalar fields χ and ϕ. The former can then be removed via a lattice gauge transformation, so that we are left with what is essentially a lattice analogue of a two dimensional 'curl'. Thus, we have

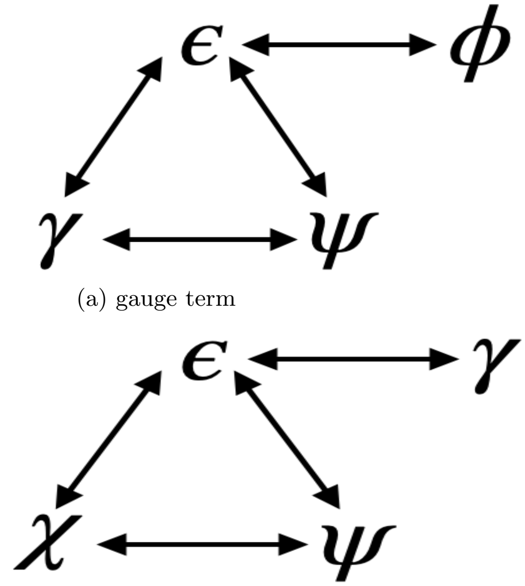
$$\Delta S_G = \epsilon_{\mu\nu}\overline{\psi(n)}\gamma_\mu(\phi(n+\hat{\nu}) - \phi(n))\psi(n+\hat{\mu}). \tag{35}$$

Mimicking Fujikawa once more, we then proceed with an infinitesimal chiral transformation of the form $\psi(n) \to$

$(1+i\gamma_5\beta(n)\psi(n)$. Employing the Dirac identity eq. (31), we see that the kinetic term in eq. (33) yields a *chiral addition* term δS_C of the form (trivial factors aside)

$$\Delta S_C = \epsilon_{\mu\nu}\overline{\psi(n)}\gamma_\nu\Big(\beta(n+\hat{\mu})-\beta(n)\Big)\psi(n+\hat{\mu}), \quad (36)$$

and it is *here* that we encounter the first of numerous lattice complications. While we have followed Fujikawa's methods almost to the letter, the results are *not* what we hoped for. Indeed, the chiral addition ΔS_C resembles the gauge term ΔS_G, with β replacing ϕ, but a couple of misplaced indices unfortunately render the two terms *slightly* different. As a result, the naïve lattice analogue of the Fujikawa approach does *not* decouple the lattice Schwinger action.
This curious failure of the Fujikawa method at the lattice level calls for some inspection. To better understand the cause of this issue, we employ a pictorial approach. The 'index diagrams' in Figures 4a and 4b organise the various objects appearing in the action in terms of shared indices. Specifically, two objects are linked by an arrow iff. they share a common index. The issue is now immediately apparent. Each of the two index diagrams contains an 'index triangle' formed by three of the four participating objects. From an algebraic standpoint, this means that the three objects in question share a common index! This seems to violate the standard rule of having an index appearing at most twice in any tensorial expression. However, this rule originates from the requirement of continuum Lorentz invariance, a requirement that is in fact broken on the lattice! The appearance of a threefold repeated index is a consequence of explicit Lorentz symmetry breaking at the lattice level. Indeed, we see that the spinor field typically does not host an index of its own in the continuum, and erasing the spinor from the two diagrams renders them indistinguishable. On the lattice however, derivatives are replaced by discrete differences and the spinor fields must take on indices to encode directionality of partial derivatives. This is the root of our problem.



(a) gauge term

(b) chiral addition term

FIG. 4: Index diagrams

There seems to be no simple fix available. For instance, an obvious first shots would involve modifying the chiral transformation to something more generic. However, since the Clifford algebra and the associated bilinears form a basis for the space of $2x2$ matrices, any *local* transformation of the fields may be incorporated into the generic transform $\psi \to (1+i\alpha_\mu\gamma_\mu + i\chi\gamma_5)\psi$. However, a couple of lines of algebra is enough to show that even an extended parameterisation of this form is not sufficient to solve the problem.
It is actually not surprising that first attempts of this form fail. The heart of the problem is, after all, the mismatch of indices. Correcting this requires a transformation that correlates *different* spatial/vectorial indices so that the transforms that we would require must possess at least a hint of *non-locality*, either in the spatial or the directional indices. For this reason, *smearing* transformations of the form .

$$\psi(n) \to \sum_{\{n_i\}} \alpha_i\psi(n_i), \quad (37)$$

are the next logical step. Here, $\{n_i\}$ collectively denote both the index n and its nearest neighbours and the α_is sum to 1. We may also include non-local extensions of chiral transforms in similarly smeared manners.
Sadly, transformations of this kind can be shown, after some cumbersome computer algebra, to be of no help. The issue is that while these transforms can be tuned to demolish the troublesome terms at the nearest neighbour level, they introduce *new* corrections of their own at the *next* nearest neighbour level for instance. We could therefore attempt to include such terms in our smearing process as well, and possibly more if need be. The requirement of all additional corrections nullifying one another then serves as a bootstrap condition. However, the space of allowed smearing transformations is simply *too* large to allow us to solve the resulting bootstrap equations. Even worse, it is not even clear whether the bootstrap equations *possess* solutions to begin with. In the even that they *do* exist, uniqueness issues could prove troublesome. It seems that our present approach is simply not tenable.
While we have not solved the problem yet, our failed attempts have given us some direction. Indeed, it is now clear that non-locality, at either the vertex or the link level, is a crucial ingredient for any potential solution. Additionally, we encountered a vast space of possibilities at the linear level, all of which were on the exact same footing. It is conceivable that the correct fix can only be determined by reverting to the non-linear level, where higher order corrections could considerably shrink

the space of viable possibilities. For this reason, we now abandon our top-down approach, wherein we aimed to mimic Fujikawa at the linear level and then bootstrap higher order corrections, and switch to a bottom-up approach, wherein we directly aim to tackle the non-linear problem in all its glory.

Of course, the core of this second attempt is not too different from our initial approach - we must *somehow* attempt to trivialise the electromagnetic interaction. The difference is that this interaction is now embodied by *group-valued* link variables rather than *algebraic* vector fields, which makes such trivialisations trickier. We proceed in an operational manner - assuming we *can* somehow trivialise the link variables, we would ideally *like* to remove the said trivialisations by a chiral transformation. But since chiral transformations necessarily take the form $\psi(n) \to e^{i\gamma_5\chi(n)}\psi(n)$, they can *only* negate terms involving the exponential of γ_5. Hence, our trivialisations *must* take this form! This straightforward argument therefore allows us to paraphrase our attempted trivialisation in a concrete fashion: *Given a vector field $A_\mu(n)$, does there exist a vector field $B_\mu(n)$ such that $e^{iA_\mu(n)} = e^{i\gamma_5 B_\mu(n)}$?*

To answer this, it is convenient to employ trigonometric functions instead of exponentials. Using the identity $\gamma_5^2 = 1$, we see that $e^{i\gamma_5 B_\mu(n)} = \cos(B_\mu(n)) + i\gamma_5\sin(B_\mu(n))$. Making use of this expansion and our now familiar Dirac identity eq. (31), we eventually obtain

$$A_\mu(n) = \frac{1}{i}\ln(\cos(B_\mu(n)) + \epsilon_{\mu\nu}\sin(B_\nu(n))). \tag{38}$$

This equation is rather striking! For one, the implicit summation over the ν indices demonstrates that, when viewed as *link* variables, the mapping from B to A is non-local. This is interesting, but not unexpected, we had already predicted this from our observations at the linear level. Nevertheless, eq. (38) provides a pleasing confirmation of our earlier insight.

However, there is still one attribute highlighted by eq. (38) that we could not easily (if at all!) have guessed at the linear level. Clearly, B *must* be inherently *complex*! This is quite puzzling, since chiral transforms in the continuum obviously involve only real fields. Clearly, *something* must happen in the $a \to 0$ limit to ensure a smooth transition to the continuum. We will not press this point for now, but we would like to stress the peculiarity of this result.

The natural continuation is to invert this expression to explicitly obtain B in terms of A. Before doing so, it is worth pointing out that we have thus far not made use of the gauge freedom available to us. However, since there is no *clear* choice that could simplify the above expressions immediately, we are better off first inverting them and *then* worrying about gauge transforms. The reader should remember that we are no longer trying to match Fujikawa's method precisely, so there is no cause for worry in not imposing the (discrete) Lorentz gauge immediately.

To invert the above equations, it is helpful, albeit ugly, to rewrite the equations with explicit indices. Doing so (and suppressing spatial indices for clarity), we have

$$\begin{aligned} e^{iA_1} &= \cos(B_1) + \sin(B_2), \\ e^{iA_2} &= \cos(B_2) - \sin(B_1). \end{aligned} \tag{39}$$

These equations can be *formally* inverted using standard trigonometric identities. In particular, using an intermediate substitution $B_3 \equiv B_1 - \frac{\pi}{2}$ we can switch between sin's and cos's to obtain

$$B_1 + B_2 = \frac{\pi}{2} + 2\tan^{-1}\left(\frac{e^{i(A_1 - A_2)}}{2}\right). \tag{40}$$

Substituting this back into either equation, it is possible, although cumbersome, to obtain (none too pretty!) explicit solutions for B_1 and B_2. It must be emphasised that our solutions are merely formal, and some extra fine-tuning is needed to ensure that our complex arguments do not pose issues.

In principle, our next step would be to verify that our B function solution may be expressed as the 'lattice gradient of a lattice scalar field', i.e. that $B_\mu(n) = C(n+\hat{\mu}) - C(n)$ for some C. Unfortunately, given the complexity of the B solutions, establishing the veracity of this claim was not possible in the alloted time frame. Nevertheless, the very existence of the Fujikawa method in the continuum is strong evidence in favour of our conjecture, which we aim to justify a short while after the submission of this work. Should our claim hold true, a simple chiral transform will be enough to destroy our B field, so that we will *finally* have achieved a decoupling of Fujikawa on a flat lattice. We would like to emphasise the novelty of this result, especially in view of the fact that no such decoupling has (to our knowledge) been achieved despite the prominence of the continuum method, which emerged a fairly long time ago (the original articles date from the 1980s).

As mentioned, our next step would be to extend our results to the continuum. Here, we expect topological artefacts to intervene (for instance, the existence and uniqueness of our 'Helmholtz-like' decomposition is quite dependent on the topology of the space considered), thereby modelling the effects of geometry (and/or topology) on Schwinger, as we had initially desired. Even if it is ultimately not possible to extend our results to the continuum analytically, we may still assume that numerical computations will be able to shed some light on the matter.

We conclude with a short qualitative mention of the chiral anomaly. The continuum chiral anomaly emerges from the invariance of the functional measure $\mathcal{D}\psi$ under local chiral transforms. However, at a lattice level, chiral transforms, even local ones, involve the exponential of the γ_5 matrix, a *manifestly* traceless matrix. Since exponentials map traces to determinants, and since the latter serve as appropriate Jacobians for the functional (lattice) measure, it is not difficult to see that *lattice* chiral transformations do *not* stem from the measure.

At a lattice level, this discrepancy arises from the fact that even the *massless* Dirac-Wilson action is *not* chirally invariant (this is because of the doubler-destroying terms, which break the invariance of the naı"ve action). It is this lack of invariance which sources he chiral anomaly. As it turns out that the Dirac–Wilson action is not well suited (analytically or numerically) for dealing with 'chirally oriented' lattice studies of gauge theories. Typically, lattice theorists use modified, annoyingly complex actions embodying a *modified* version of chiral symmetry [50] (it is not possible to construct a doubler free lattice action with manifest chiral symmetry, thanks to the famous no-go theorem of Nielson and Ninomiya [51]). However, we believe that our novel lattice Fujikawa technology might be enough to build Dirac–Wilson based understandings of the chiral anomaly for the *specific* case of Schwinger. This line of thought constitutes a second, related direction for our work in the near future.

VII. CONCLUSIONS

The *broad* aim of this project was to *analytically* probe the dynamics of coupled gauge fields and fermions atop an interacting gravitational background using the methodology of dynamical triangulations. Our system - the triangulated Schwinger model - was chosen so as to maximise our chances of obtaining exact results, and our methods centred around adapting what we may call the 'solvable' aspects of precursor models to our own needs. Specifically, our proposed methodology followed two unrelated routes:

1. We attempted to extend an exact duality between *free* triangulated *free* fermions and a triangulated spin model to incorporate $U(1)$ *gauged* fermions.

2. We tried to trace back the integrability of the *continuum* Schwinger model and establish establish *exactly* solvable hypercubic lattice discretisations. Proceeding down this path, we hoped to conform these models to *triangulated* lattices and investigate the continuum limit, thereby gaining insights into a *true* 'quantum geometro-electrodynamics' system.

As regards the first route, we were indeed able to establish a relation between the Schwinger partition function and that of a (collection of) modified triangulated Ising model. While fascinating in its own right, this correspondence was rather cosmetic, offering little analytic benefits. That said, this mapping certainly has its uses. For instance, we were able to gain some interesting insights into the the effects of the underlying topology. Additionally, while we did not venture into such depths, asymptotic expansions utilising our gauged HPE may shed some lights on the effects of gauge fields and geometry on fermionic dynamics. Additionally, it is possible that the perturbative expansions we have developed could be of numerical interest.

We believe that the complexity of our 'duality' is a reflection of both the speciality of free triangulated the underlying complexity of the Schwinger system itself. The triangulated fermionic system is in a sort of 'Goldilocks zone' wherein the stars align just well enough to allow a perfect duality to manifest. Even minor shifts away from this system, as we have demonstrated, spoil the resulting dualities to the extent that all the analytic niceties vanish. Consequently, spin-system correspondences are *probably* not the best way to go about.

On the other hand, our second route seems quite promising. This direction is admittedly far more circuitous, with multiple dead-ends, several of which we ran into head on! Specifically, we began with a 'top-down' *flat* lattice approach when constructing analogues of continuum Schwinger, wherein we started at the 'linear level' in gauge link variables. We tried to fix a 'correct' lattice regularisation at this linear level by mimicking the continuum Fujikawa approach as closely as possible. Had this panned out, our next step would have been to peruse full non-linear extensions using some sort of bootstrap methodology. As we say however, lattice regularisations came with their own family of demons which we attempted to refute using novel non-local extensions of standard lattice transformations and a variety of hitherto unseen smearing techniques. Ultimately however, it became clear that the flaw was in our flow - the information available to us at linear order was simply *not* enough to pinpoint a unique lattice discretisation that solved all our pressing issues.

We then switched to a bottom-up approach, aiming to *first* construct solvable models and *then* link them to the continuum and have already seen evidence for good prospects. In particular, we were able to partially unearth what seems to be a novel plausible replacement for the Fujikawa method on flat lattices. Along the way, we also outlined some unique perspectives on the chiral anomaly in our lattice approach. We outlined a qualitative conjecture on the emergence of the chiral anomaly for both the naïve and doubler free lattice versions of Dirac theory from a subtle counting issue and a chirally varying Lagrangian respectively. This is potentially quite intriguing since prior studies involving the chiral anomaly have centred around transformation properties of the functional measure under chiral transforms.

Unfortunately, with the project being far more more nuanced that initially imagined and the time spent working our way out of the labyrinth referenced above greatly shortening an already constrained time frame, we were not able to piece together our partial fixes into a full-fledged solution. That said, we *now* believe that the work carried out as part of this project serves as the 'developmental phase' of a much broader endeavour, in which we have evaded numerous pitfalls to lay out what seems to be the most promising approach to a theory of continuum 'quantum-geometroelectrodynamics', at least from the viewpoint of triangulations.

As already mentioned, future work along these lines will involve ironing out the issues currently present in our proposed flat-lattice decoupled Schwinger system. We will then look to broaden this system to accommodate randomly triangulated lattices, which we hope will allow us to establish the existence of critical points in parameter space. Alongside this, we will strive to improve on our intriguing, yet incomplete study of the lattice chiral anomaly. If all goes well, we will be capable of working out the effects of geometry on the anomaly which will serve as a fantastic probe of the *physics* of interacting quantum gravity systems.

Of course, the above list of goals is itself forms only a small part of an even greater venture, this time involving the entire dynamical triangulations community - that of better understanding the physics of two dimensional triangulated quantum systems, a venture which will no doubt be an important stepping stone to our quest to better understand the quantum theory that underpins our Universe.

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