Cosmological Perturbation Theory of Scalar Modes in Poincaré Gauge Theory

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Cosmological perturbation theory of scalar modes is carried out in the context of Poincaré gauge theory. This analysis extends upon previous work through its application to general Riemann–Cartan spacetime. Decomposition under the restriction of purely perturbative torsion, is shown to give rise to the Bardeen potentials along with novel gauge invariant quantities associated with torsion. When applied to field equations, the results of general relativity are reproduced within Einstein–Cartan theory before showing that linear torsion does not propagate in a strongly-coupled $\mathcal{R} + \mathcal{R}^2$ modified gravity theory (extending the notion of strong-coupling to FLRW backgrounds).

Word Count: 4997 excluding the abstract, appendices, captions and references
Statement: Except where specific reference is made to the work of others, this work is original and has not been already submitted either wholly or in part to satisfy any degree requirement at this or any other university

I. INTRODUCTION

After its original formulation by Einstein, General Relativity (GR) was found to be invariant under local transformations within the Poincaré group [1]. This group consists of translational and rotational Lorentz transformations and from Noether's theorem, symmetries within this group lead to the covariantly conserved quantity of rotational field strength (referred to as the Riemann tensor within GR) [2]. When extended to theories containing torsion, notably Einstein–Cartan (EC) theory, this invariance was shown to be maintained [3]. In addition, torsion itself (equivalently known as the translational field strength) is shown to be a covariantly conserved quantity [2].

This gauge invariance was determined in retrospect, with GR itself being established from the use of the (strong) equivalence principle much earlier [4]. Following the success of gauge theoretic descriptions of fundamental interactions throughout the twentieth century [5], Poincaré gauge theories (PGT) were developed with invariance under the associated gauge transformations at their heart (rather than as a result of the equivalence principle). In enforcing Lagrangian invariance under local transformations, the U(1) group was shown to produce a description of electromagnetism before this result was generalised to the SU(2) group by Yang and Mills [6]. Enforcing the same condition upon the Poincaré group leads to the emergence of gravity. Of particular interest in PGT's development, were ways in which to develop a theory of gravity that, unlike GR, would: be amenable to with quantization [7–9]; not include classical singularities [10–12]; and be able to be unified with other fundamental interactions [2].

Recent PGT formulations of gravity have been shown to address the issue known as the Hubble *tension* [13–15], and so are currently of particular interest [16–18]. This cosmological problem concerns the evolution of the homogeneous, isotropic Universe — first mathematically described by Friedman in 1922 (using Einstein's equations of GR) [19]. Later observational evidence gave rise to Hubble's law, in which the Hubble parameter (encoded as the Hubble *constant* H_0 when evaluated today) dictates the rate of the Universe's expansion [20]. The Hubble *tension* concerns the discrepancies in the value of H_0 obtained from attempted measurements, with this mismatch claimed to reach the level of $4 - 5\sigma$ [21, 22]. Low value inferences of this quantity are the result of measurements from early-Universe sources, such as those found from the Cosmic Microwave Background (CMB) using WMAP [23] and Planck [24] satellites. Late-Universe sources, however, give rise to high measurements of this same parameter — examples of which include: Cepheid-calibrated supernovae [25], strong gravitational lensing [22], neutron star mergers [26] and tip of red giant branch stars [27, 28].

The increase in recession velocity over time suggested by these measurements, indicates that the Universe's expansion is accelerating. Although a fascinating conclusion, fundamental mechanisms for this are not described by the standard model of ΛCDM cosmology (referencing the cosmological constant Λ and Cold Dark Matter), whose basis lies in GR [24, 29, 30].

These contemporary PGT approaches promisingly describe the progression of the background but have not yet been shown to fully account for the behaviour and evolution of perturbations to this system, ultimately leading to the formation of large scale structure. Analysis of these variations (upon the Friedmann–Lemaître–Robertson–Walker (FLRW) background) is well described by cosmological perturbation theory, originally posed by Lifshitz in the context of GR [31] before becoming the basis of extensive further study [32, 33]. In particular, this has proven to be very successful in accurately predicting the evolution of temperature anisotropies in the CMB [34, 35]. Useful pedagogical and synoptical descriptions of cosmological perturbation theory have been outlined in [36, 37].

Thus far, cosmological perturbation theory has undergone very limited investigation in the context of PGT. This investigation has been restricted to highly simplified (impoverished geometry) versions of PGT, with particular focus on teleparallel theories (in Weitzenböck spacetime T_4, following convention used in fig. 1) [38–41]. These are a subset of gravity theories in which curvature vanishes while torsion remains. Although useful in these restricted contexts, perturbations to PGT in U_4 have not yet been investigated and are the subject of our following analysis.

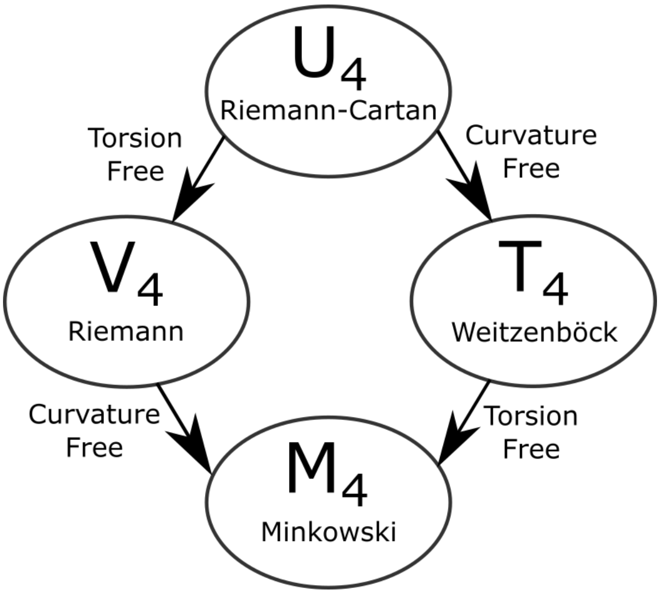

Figure 1. Classification of various spacetimes: Riemann–Cartan (U_4), (pseudo-)Riemann (V_4), (teleparallel) Weitzenböck (T_4) and (*flat*) Minkowski (M_4). The relationship between these four spacetimes are also shown through taking curvature and torsion to zero. Notation taken from Blagojević [2].

Also of interest in such PGTs, is the emergence of the strong-coupling problem [42–49]. This issue regards the absence of linearised modes in the field equations of strongly-coupled theories [50, 51]. In particular, Lagrangians of the form

$$\mathcal{L} = -\tfrac{1}{2}m^2B^2 + B^2(\mathcal{D}B)^2 + \mathcal{O}(B^6), \tag{1}$$

succumb to this problem — upon linearisation of the general field B (with indices omitted), there exist no quadratic order kinetic terms (indicated by the derivative $\mathcal{D}$). The resulting field equations cannot contain propagating linear modes, despite the Lagrangian allowing for a propagating field at higher orders. For such theories, the process of linearisation is itself seen to be invalid [52].

The subsequent report outlines the PGT development of scalar perturbation theory, novelly within Riemann–Cartan spacetime. A mathematical basis to PGT is first given in section II before perturbations to the relevant gauge fields are carried out in section III. Application of these decompositions in the construction of field equations is explored in section IV, preceding the summary of work given in section V. Conventions, further information and details of symbolic computation are contained within appendices A to C.

II. PGT FRAMEWORK

PGT is intimately related to transformations between global coordinate frames and local (*flat*) Lorentz frames. The mathematical formulation of this requires gauge fields to transform between these frames, where indices are related to frames as outlined in appendix A. This is accomplished by the introduction of the translational gauge field $b^i{}_\mu$ and its associated inverse translational gauge field $h_i{}^\mu$. These operate by

$$\begin{aligned} A_{\dots\mu\dots} &\equiv b^i{}_\mu A_{\dots i\dots}, & A^{\dots\nu\dots} &\equiv h_j{}^\nu A^{\dots j\dots}, \\ A_{\dots i\dots} &\equiv h_i{}^\mu A_{\dots\mu\dots}, & A^{\dots j\dots} &\equiv b^j{}_\nu A^{\dots\nu\dots}, \end{aligned} \tag{2}$$

where A is a general tensor. These two fields are related by the identities

$$b^i{}_\mu h_i{}^\nu \equiv \delta^\nu_\mu, \tag{3a}$$
$$b^i{}_\mu h_j{}^\mu \equiv \delta^i_j. \tag{3b}$$

A important consequence of these relations is in the formation of the global (curvature containing) metric, and its inverse

$$g_{\mu\nu} \equiv b^i{}_\mu b^j{}_\nu \eta_{ij} \ , \quad g^{\mu\nu} \equiv h_i{}^\mu b_j{}^\nu \eta^{ij}. \tag{4}$$

These fields contain no built-in symmetries and so can be represented by general 4×4 matrices. This shows the translational gauge fields to contain 16 Degrees of Freedom (DoF). Due to their inverse relationship, these 16 DoF are shared between both $b^i{}_\mu$ and $h_i{}^\mu$.

Full description of the system requires the formulation of a covariantly transforming derivative. In GR, this operation is carried out with the use of the metric connection $\Gamma^\mu{}_{\nu\rho}$ as follows

$$\nabla_\mu \equiv \partial_\mu + \Gamma^\nu{}_{\mu\rho} f^\rho{}_\nu, \tag{5}$$

where f is the representation of the generators associated with the subject of the covariant derivative. The analogue of this within PGT requires the use of the rotational gauge field $\mathcal{A}^{ij}{}_{\mu}$ and the representation of the subject's Lorentz spin generators Σ_{ij}, given by

$$\mathcal{D}_{\mu} \equiv \partial_{\mu} + \tfrac{1}{2}\mathcal{A}^{ij}{}_{\mu}\Sigma_{ij}. \tag{6}$$

This gauge field is antisymmetric in its first two indices and so can be seen to contain 24 DoF — six associated with the first two indices, for each of the four possible values taken by the third index. These two gauge fields together fully describe PGT in U_4, shown to be a system of 40 DoF.

Another integral part of this description is its link to geometric structure. Firstly, the translational gauge field can be identified with the geometric object known as the tetrad. This defines the transformation rules for four orthogonal basis 4-vector fields $\boldsymbol{e}$, applicable to general geometries,

$$\boldsymbol{e}_{\mu} \equiv b^{i}{}_{\mu}\hat{\boldsymbol{e}}_{i}\ , \quad \hat{\boldsymbol{e}}_{i} \equiv h_{i}{}^{\mu}\boldsymbol{e}_{\mu}. \tag{7}$$

The rotational gauge field can, using the previously referenced analogy to the metric connection of GR, be identified with the geometric object known as the spin connection. Due to this direct correspondence, the gauge fields are, from this point, known synonymously with their geometric counterparts. In particular, this geometric interpretation allows general PGT to be identified with Riemann–Cartan spacetime U_4 [2]. Within this spacetime, torsion emerges and can be associated with the of the metric connection

$$\mathcal{T}^{\mu}{}_{\nu\rho} \equiv \Gamma^{\mu}{}_{\nu\rho} - \Gamma^{\mu}{}_{\rho\nu}. \tag{8}$$

As touched upon in section I, the covariantly conserved quantities of PGT can be used to construct a Lagrangian and so are an integral part of the formulation of any PGT. The first of these is the Riemann–Cartan tensor $\mathcal{R}^{ij}{}_{\mu\nu}$, expanded in terms of the tetrad and spin connection as

$$\mathcal{R}^{ij}{}_{\mu\nu} \equiv \partial_{\mu}\mathcal{A}^{ij}{}_{\nu} - \partial_{\nu}\mathcal{A}^{ij}{}_{\mu} + \mathcal{A}^{i}{}_{k\mu}\mathcal{A}^{kj}{}_{\nu} - \mathcal{A}^{i}{}_{k\nu}\mathcal{A}^{kj}{}_{\mu}. \tag{9}$$

The torsion tensor defined in eq. (8) is the second covariantly conserved quantity and can be similarly expressed in terms of the gauge fields as

$$\mathcal{T}^{\mu}{}_{\nu\rho} \equiv h_{i}{}^{\mu}(\mathcal{D}_{\nu}b^{i}{}_{\rho} - \mathcal{D}_{\rho}b^{i}{}_{\nu}). \tag{10}$$

With this mathematical outline established, the tools for further study in the following sections have been developed.

III. DECOMPOSITION AND TRANSFORMATION OF THE GAUGE FIELDS

Although cosmological perturbation theory has been studied from the mid-twentieth century [31], its connection with gauge transformations and gauge invariant quantities was only identified decades after its onset, notably by Bardeen [33]. In this procedure, the decomposed fields of choice are subject to gauge transformations to identify how each part of the decomposition transforms. From these transformation laws, gauge invariant combinations can be formed which (when chosen appropriately) correspond to observable quantities.

This work has been extensively carried out in the context of GR. In this case, the metric is the field of choice and its perturbed form upon a FLRW background is well-known. The subsequent gauge transformations give rise to the emergence of the Bardeen potentials Φ and Ψ; these are outlined in more depth in section III B.

This decomposition can be done by breaking down the chosen field into scalar, vector and tensor components. However, the focus of this report is the linearised, scalar part of this decomposition as the vector and tensor modes are significantly more difficult to study — moreover, in GR they can be appropriately neglected all together [53]. This analysis is yet to be studied in the case of PGT in U_4.

A. Decomposition of Poincaré Gauge Transformations

The first step in this procedure is the decomposition of the gauge transformations themselves. As the Poincaré group concerns both translations and Lorentz rotations, the operator forms of both of these actions must be broken down.

Translations in time and space are carried out by the Jacobian operator, represented using partial derivatives (shown explicitly in eq. (11)). This is given by the identity transformation plus the infinitesimal gauge perturbation ξ^{μ} as

$$\frac{\partial x^{\nu}}{\partial x'^{\mu}} = \delta^{\nu}_{\mu} + \partial_{\mu}\xi^{\nu}. \tag{11}$$

This is a common treatment of Jacobian transformations [36] but the Lorentz rotations require an unstudied decomposition. These are represented by the two index object $\Lambda^{i}{}_{j}$, which can be expressed as the identity plus the infinitesimal (antisymmetric) perturbative 4-rotation $R^{i}{}_{j}$,

$$\Lambda^{i}{}_{j} = \delta^{i}_{j} + R^{i}{}_{j}. \tag{12}$$

Considering only the scalar decompositions of these gauge parameters allows for eqs. (11) and (12) to be expanded, in terms of the gauge angles $\{T, L, P, Q\}$, in the following arrangement

$$\xi^{\mu} = n^{\mu}T + \partial^{\bar{\mu}}L, \tag{13a}$$

$$R_{ij} = n_i\partial_{\bar{j}}P - n_j\partial_{\bar{i}}P + \varepsilon_{\bar{i}\bar{j}}{}^{\bar{k}}\partial_{\bar{k}}Q. \tag{13b}$$

It is important to consider the physical meanings of these gauge angles. T and L can be simply interpreted as shifts in time and space respectively, seen by the time directionality of the first term in eq. (13a) (along n^{μ}) and the second term's spatial gradient. On the other hand, P and Q require more thought to describe. The purely spatial parts of R_{ij} can be represented by an antisymmetric 3×3 matrix associated with spatial rotations, with Q describing the degree of this transformation. The mixed time-space components of eq. (13b) can similarly be associated with the remaining part of the Lorentz rotation, namely the Lorentz boosts. This shows P to be the measure of infinitesimal Lorentz boosts. For further clarity, the geometric interpretation of these gauge angles is shown in fig. 2.

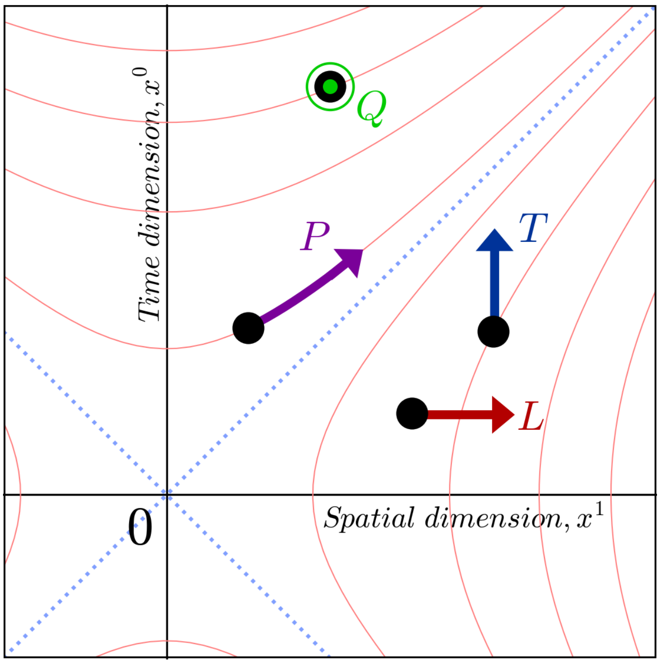

Figure 2. Graphical representation of the action of the gauge angles $\{T, L, P, Q\}$ on a Minkowski diagram. The dotted blue lines indicate the border of the light (null) cone and the narrow red lines are hyperbolae corresponding to surfaces of constant spacetime interval. The transformations associated with each gauge angle are shown by their associated direction of action, using arrows from arbitrarily placed black markers. Jacobian translations in time and space are shown in blue and red respectively, while Lorentz-boosts are indicated by purple. Green represents the spatial rotations, shown with the use of an out-of-page symbol to indicate rotation about the time-axis.

B. Treatment of the Tetrad

With the gauge transformations decomposed in terms of scalar gauge angles, the break down of the gauge fields themselves can be tackled. This treatment is first applied to the tetrad.

1. Decomposition

The approach taken in this report is to linearly decompose the tetrad into background b_0 and perturbed δb components. Representing these on an FLRW background then gives the $b = b_0 + \delta b$ form of the scalar decomposition, where

$$(b_0)^i{}_{\mu} = a(t)\delta^i_{\mu}\ , \tag{14a}$$

$$(\delta b)^i{}_{\mu} = a(t)[n^i n_{\mu}A + n^i\partial_{\bar{\mu}}B + n_{\mu}\partial^{\bar{i}}C + \delta^{\bar{i}}_{\bar{\mu}}D + (\partial^{\bar{i}}\partial_{\bar{\mu}} + \frac{1}{3}\delta^{\bar{i}}_{\bar{\mu}}\nabla^2)E + \varepsilon^{\bar{i}}{}_{\bar{\mu}\bar{k}}\partial^{\bar{k}}F]. \tag{14b}$$

Equation (14b) shows that the tetrad perturbations can be split into the following parts: purely temporal (A), mixed temporal and spatial (B and C), spatial trace (D), traceless symmetric (E) and antisymmetric (F). These six scalar perturbations are accompanied by four divergence-free vectors and a traceless, transverse tensor. Together, these fully describe the tetrad, with the scalars, vectors and tensor containing one, two and two DoF each — this accounts for the total $6(1)+4(2)+1(2) = 16$ DoF within the tetrad (as required in section II). The vector and tensor perturbations, however, are not considered as part of this report. The above form of the tetrad can also be expressed as eq. (15).

J^P	Tensor form	Scalar expansion	
0^+	$b^{\perp}{}_{\perp}$	$a(1+A)$	(15a)
0^+	$b^{\bar{i}}{}_{\bar{i}}$	$3a(1+D)$	(15b)
1^-	$b^{\bar{i}}{}_{\perp}$	$a\partial^{\bar{i}}C$	(15c)
1^-	$b^{\perp}{}_{\bar{i}}$	$a\partial_{\bar{i}}B$	(15d)
1^+	$b_{[\bar{i}\bar{j}]}$	$-a\varepsilon_{\bar{i}\bar{j}\bar{k}}\partial^{\bar{k}}F$	(15e)
2^+	$b_{(\bar{i}\bar{j})} - \frac{1}{3}\eta_{\bar{i}\bar{j}}b^{\bar{k}}{}_{\bar{k}}$	$a\left[\partial_{\bar{j}}\partial_{\bar{i}} + \frac{1}{3}\eta_{\bar{i}\bar{j}}\nabla^2\right]E$	(15f)

Here J^P are the spin-parity modes, chosen to more easily identify how each scalar varies under gauge transformations.

2. Behaviour Under Gauge Transformations

With the decomposition of the tetrad outlined above, its behaviour under gauge transformations can now be studied. This requires the tetrad transformation law

$$(b')^{i}{}_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \Lambda^{i}{}_{k} b^{k}{}_{\nu}, \tag{16}$$

where $'$ denotes the transformed field. This induced change is derived using the form of the Lie derivative of the field itself [54], taken to linear order. Using the linearisations outlined in section III A, eq. (16) can be reexpressed to linear order as

$$b' = b_0 + \delta b + b_0 \partial \xi + b_0 R, \tag{17}$$

where indices are omitted for simplicity. The gauge transformation of the tetrad can be carried out using the decomposition outlined in section III A. Once transformed, the spin-parity modes of the tetrad are re-identified in eq. (18).

J^P	Tensor form	Scalar expansion	
0^+	$(b')^{\perp}{}_{\perp}$	$a(1 + A + \dot{T} + \mathcal{H}T)$	(18a)
0^+	$(b')^{\bar{i}}{}_{\bar{i}}$	$3a(1 + D - \frac{1}{3}\nabla^2 L + \mathcal{H}T)$	(18b)
1^-	$(b')^{\bar{i}}{}_{\perp}$	$a\partial^{\bar{i}}(C - P + \dot{L})$	(18c)
1^-	$(b')^{\perp}{}_{\bar{i}}$	$a\partial_{\bar{i}}(B + P + T)$	(18d)
1^+	$(b')_{[\bar{i}\bar{j}]}$	$-a\varepsilon_{\bar{i}\bar{j}\bar{k}}\partial^{\bar{k}}(F + Q)$	(18e)
2^+	$(b')_{(\bar{i}\bar{j})} - \frac{1}{3}\eta_{\bar{i}\bar{j}}(b')^{\bar{k}}{}_{\bar{k}}$	$a\left[\partial_{\bar{j}}\partial_{\bar{i}} + \frac{1}{3}\eta_{\bar{i}\bar{j}}\nabla^2\right](E + L)$	(18f)

Direct comparison between eqs. (15) and (18) shows how each perturbative scalar transforms. These are specified by

$$\begin{aligned} &A \to A + \dot{T} + \mathcal{H}T, && B \to B + P + T, \\ &C \to C - P + \dot{L}, && D \to D - \tfrac{1}{3}\nabla^2 L + \mathcal{H}T, \\ &E \to E + L, && F \to F + Q. \end{aligned} \tag{19}$$

With this result, gauge invariant combinations of these scalars can be identified (by eliminating the gauge angles). Crucially, the Bardeen potentials Ψ and Φ are seen to be invariant and given by

$$\Psi = A - \mathcal{H}(B + C - \dot{E}) - (\dot{B} + \dot{C} - \ddot{E}), \tag{20a}$$

$$\Phi = -D + \mathcal{H}(B + C - \dot{E}) - \tfrac{1}{3}\nabla^2 E, \tag{20b}$$

where the conventions used in eq. (14) lead to differences in sign when compared with pedagogy. The invariance of the Bardeen potentials is an encouraging reproduction of this analysis, as well as supporting the form of the tetrad decomposition used.

3. *The Inverse Tetrad and Metric*

Following this result, other important fields related to the tetrad can be contructed. The first of these is the inverse tetrad, whose form can be determined from eq. (3). Substitution with the known form of the tetrad into these equations gives the $h = h_0 + \delta h$ form of the inverse tetrad, where

$$(h_0)_i{}^{\mu} = \tfrac{1}{a}\delta_i^{\mu}\,, \tag{21a}$$

$$\begin{aligned}(\delta h)_i{}^{\mu} = &-\tfrac{1}{a}[n^i n_\mu A + n^i \partial_{\bar{\mu}} C + n_\mu \partial^{\bar{i}} B + \delta^{\bar{i}}_{\bar{\mu}} D, \\ &+ (\partial^{\bar{i}}\partial_{\bar{\mu}} + \tfrac{1}{3}\delta^{\bar{i}}_{\bar{\mu}}\nabla^2)E - \varepsilon^{\bar{i}}{}_{\bar{\mu}\bar{k}}\partial^{\bar{k}} F].\end{aligned} \tag{21b}$$

Together, the tetrad and inverse tetrad are used to construct the general perturbed metric. As mentioned in appendix A, this is a vital ingredient in the construction of this theory, allowing for contraction of indices in general coordinate frames. This field is assembled using eq. (4) and yields eq. (22).

J^P	Tensor form	Scalar expansion	
0^+	$g_{\perp\perp}$	$a^2(1 + 2A)$	(22a)
0^+	$g^{\bar{\mu}}{}_{\bar{\mu}}$	$3a^2(1 + 2D)$	(22b)
1^-	$g_{\bar{\mu}\perp}$	$a^2\partial_{\bar{\mu}}(B + C)$	(22c)
1^-	$g_{\perp\bar{\nu}}$	$a^2\partial_{\bar{\nu}}(B + C)$	(22d)
2^+	$g_{(\bar{\mu}\bar{\nu})} - \frac{1}{3}\eta_{\bar{\mu}\bar{\nu}}g^{\bar{\rho}}{}_{\bar{\rho}}$	$2a^2(\partial_{\bar{\nu}}\partial_{\bar{\mu}} + \frac{1}{3}\eta_{\bar{\mu}\bar{\nu}}\nabla^2)E$	(22e)

This agrees with the form of the linearised metric upon a FLRW background (up to parameter labelling) found in the literature [36]. The agreement of this field with its known form is a vital reproduction of this formulation and is used extensively in the following work.

C. Treatment of the Spin Connection

With the successful reproduction of the Bardeen potentials and general metric, this same procedure can be applied to the remaining gauge field in PGT, the spin connection. An important part of this procedure lies in the separation of the spin connection into its curvature and torsion associated portions. This is done using the Ricci rotation coefficients (RRC) and contortion, denoted by Δ and K respectively, with the simple relationship [2]

$$\mathcal{A}_{ij\mu} = \Delta_{ij\mu} + K_{ij\mu}. \tag{23}$$

Treatment of these two objects is separately considered prior to their behaviour under gauge transformations.

1. *Ricci Rotation Coefficients*

The RRC represent the purely curvature-associated components of the spin connection. With the use of the previously decomposed tetrad and metric, the RRC can be expressed as

$$\Delta_{ij\mu} = \frac{1}{2}(c_{ijm} - c_{mij} + c_{jmi})b^m{}_\mu, \tag{24a}$$

$$c_{ijm} = \eta_{il}c^l{}_{\nu\sigma}h_j{}^\nu h_m{}^\sigma, \tag{24b}$$

$$c^i{}_{\mu\nu} = \partial_\mu b^i{}_\nu - \partial_\nu b^i{}_\mu, \tag{24c}$$

as taken from Blagojević [2]. The identities in eq. (3) then allow for this to be more succinctly written as

$$\Delta_{ij\mu} = 2h_{[i|}{}^\nu \eta_{|j]l}\partial_{[\mu|}b^l{}_{|\nu]} - \eta_{ml}b^m{}_\mu h_i{}^\nu h_j{}^\sigma \partial_{[\nu|}b^l{}_{|\sigma]}. \tag{25}$$

Along with the decompositions outlined in section III, this allows the RRC to be represented in terms of the scalars $\{A, ..., F\}$. Due to its lengthy nature, this is not shown in full here, but its contribution to the spin connection is given in eq. (32). An interesting result of this expansion is the non-zero value of the background RRC

$$(\Delta_0)^{[\perp\bar{k}]}{}_{\bar{k}} = 3\mathcal{H}, \tag{26a}$$

$$(\Delta_0)^{ij}{}_\mu = n^{[i}\delta^{\bar{j}]}_{\bar{k}} b^{\bar{k}}{}_{\bar{\mu}}\ \mathcal{H}. \tag{26b}$$

As a check of physical compatibility, the dimensions of this quantity are seen to be that of energy. This is expected from the form of the spin connection as defined in eq. (6), in which the derivative take the dimensions of inverse length (equivalent to energy). This is an encouraging check that the procedure produces physically suitable results.

2. Contortion

The contortion represents the torsion-containing elements of the spin connection and can be given in terms of the torsion $\mathcal{T}$, defined in eq. (8), as

$$K^\mu{}_{\nu\sigma} = \frac{1}{2}(\mathcal{T}^\mu{}_{\nu\sigma} - \mathcal{T}_\sigma{}^\mu{}_\nu + \mathcal{T}_{\nu\sigma}{}^\mu). \tag{27}$$

Further expansion of the torsion requires its separation into irreducible components (irreps) of the SO(+1, 3) group; these are given by tensor, vector and pseudo-vector parts, respectively ${}^{(1)}\mathcal{T}^{\mu[\nu\sigma]}$, ${}^{(2)}\mathcal{T}^\mu$ and ${}^{(3)}\mathcal{T}^\mu$ (irreps). The specific arrangement of irreps chosen here is

$$\mathcal{T}^\mu{}_{\nu\sigma} = {}^{(1)}\mathcal{T}^\mu{}_{\nu\sigma} + 2{}^{(2)}\mathcal{T}_{[\nu|}\delta^\mu_{|\sigma]} + \varepsilon^{\rho\mu}{}_{\nu\sigma}{}^{(3)}\mathcal{T}_\rho. \tag{28}$$

From this stage, scalar decomposition of the contortion can be performed. For the following analysis, the torsion is taken to be first order perturbative (vanishing at background order). The case involving a torsionful background (torsion condensate) is touched upon in appendix B.

Due to its symmetries (antisymmetric in its last two indices), the contortion is seen to contain 24 DoF, similarly determined as with the spin connection in section II. These are accounted for through eight scalars, six divergence free vectors and two traceless, transverse tensors; 8(1) + 6(2) + 2(2) = 24 DoF. For the theory considered in this report, the vector and tensor components are taken to be vanishing while the eight scalars $\{\alpha, \beta, W, \mathcal{Z}, U, V, X, Y\}$ are related to the previously presented irreps through

$$\mathcal{T}^\mu{}_{\nu\sigma} = {}^{(1)}\mathcal{T}^\mu{}_{\nu\sigma} + 2{}^{(2)}\mathcal{T}_{[\nu|}\delta^\mu_{|\sigma]} + \varepsilon^{\rho\mu}{}_{\nu\sigma}{}^{(3)}\mathcal{T}_\rho, \tag{29a}$$

$$\begin{aligned}{}^{(1)}\mathcal{T}^\mu{}_{\nu\sigma} = {} & n^\mu n_\nu \partial_{\bar{\sigma}}\alpha + n^\mu \varepsilon^{\bar{\lambda}}{}_{\bar{\nu}\bar{\sigma}}\partial_{\bar{\lambda}}\beta \\ & + n_\nu(\partial^{\bar{\mu}}\partial_{\bar{\sigma}} + \tfrac{1}{3}\delta^{\bar{\mu}}_{\bar{\sigma}}\nabla^2)W \\ & + (\varepsilon^{\bar{\mu}}{}_{\bar{\sigma}\bar{\lambda}}\partial_{\bar{\nu}} + \varepsilon_{\bar{\nu}\bar{\sigma}\bar{\lambda}}\partial^{\bar{\mu}})\partial^{\bar{\lambda}}\mathcal{Z},\end{aligned} \tag{29b}$$

$${}^{(2)}\mathcal{T}_\mu = n_\mu \dot{U} + \partial_{\bar{\mu}}V, \tag{29c}$$

$${}^{(3)}\mathcal{T}_\mu = n_\mu \dot{X} + \partial_{\bar{\mu}}Y. \tag{29d}$$

The decomposition of the $0^\pm$, $1^\pm$ and 2^+ modes are well studied in the context of cosmological perturbation theory [55]. However, the form of the scalar representation of the 2^- mode is not often considered in the literature. Its structure, therefore, requires deeper explanation.

This can be approached by looking at the degrees of freedom contained within the 2^- mode, beginning with three spatial indices contributing 27 DoF. From this point, the required antisymmetry of the final two indices eliminates 9 DoF. As well as this, enforcing a traceless condition in the first two indices (and by connection between the first and last indices) eliminates a further three DoF. Finally, the cyclic condition of the torsion, given by

$$\mathcal{T}_{\mu\nu\sigma} + \mathcal{T}_{\sigma\mu\nu} + \mathcal{T}_{\nu\sigma\mu} = 0, \tag{30}$$

removes another ten DoF — three where all indices are equal, six as a result of two equal indices and the remaining one from all unequal indices. Together, these conditions restrict the 2^- mode to the expected five DoF.

Constructing a scalar representation of such an object is accomplished by first forming the general scalar representation of a three index, spatial tensor (composed of only derivatives and alternating tensors). This is given by

$$\begin{aligned} & c_1\delta_{\bar{\mu}\bar{\nu}}\partial_{\bar{\sigma}}\mathcal{Z} + c_2\delta_{\bar{\sigma}(\bar{\mu}}\partial_{\bar{\nu})}\mathcal{Z} + c_3\partial_{\bar{\sigma}}\partial_{\bar{\nu}}\partial_{\bar{\mu}}\mathcal{Z} \\ & \quad + c_4\varepsilon_{\bar{\nu}\bar{\sigma}\bar{\rho}}\partial^{\bar{\rho}}\partial_{\bar{\mu}}\mathcal{Z},\end{aligned} \tag{31}$$

where c_n are constants. Imposition of the above symmetry conditions on eq. (31) requires $c_3 = 0$, whilst c_1 and c_2 form a set of singular coupled equations. The only prefactor not forced to vanish is c_4, giving the scalar form of the 2^- mode used in eq. (29b).

As with the tetrad in section III B 2, the spin connection can be decomposed into its spin-parity modes prior to gauge transformation, in eq. (32).

J^P	Tensor form	Scalar expansion	
0^+	$\mathcal{A}^{\bar{i}\perp}{}_{\bar{i}}$	$3[\mathcal{H}(1-A+D)+\dot{D}+\frac{1}{3}\nabla^2 C+\dot{U}]$	(32a)
0^-	$\mathcal{A}^{\bar{i}\bar{j}}{}_{\bar{k}}\varepsilon^{\bar{k}}{}_{\bar{i}\bar{j}}$	$2\nabla^2 F-3\dot{X}$	(32b)
1^+	$\mathcal{A}^{\bar{i}\bar{j}}{}_{\perp}$	$\frac{1}{2}[\varepsilon^{\bar{i}\bar{j}}{}_{\bar{k}}\partial^{\bar{k}}(2\dot{F}++\beta-Y)+(\partial^{\bar{j}}\partial^{\bar{i}}+\frac{1}{3}\eta^{\bar{i}\bar{j}}\nabla^2)W]$	(32c)
1^-	$\mathcal{A}^{\bar{k}\bar{i}}{}_{\bar{k}}$	$2\partial^{\bar{i}}[-\mathcal{H}B+D+\frac{1}{3}\nabla^2 E+V]$	(32d)
1^+	$\mathcal{A}_{\perp[\bar{i}\bar{j}]}$	$\frac{1}{2}\varepsilon_{\bar{i}\bar{j}\bar{k}}\partial^{\bar{k}}[2\mathcal{H}F-\beta-Y]$	(32e)
1^-	$\mathcal{A}^{\bar{i}\perp}{}_{\perp}$	$\partial^{\bar{i}}[-A+\dot{B}+\mathcal{H}(B+C)+\frac{1}{2}\alpha-V]$	(32f)
2^+	$\mathcal{A}_{\perp(\bar{i}\bar{j})}-\frac{1}{3}\eta_{\bar{i}\bar{j}}\mathcal{A}^{\bar{k}\perp}{}_{\bar{k}}$	$[\partial_{\bar{j}}\partial_{\bar{i}}+\frac{1}{3}\eta_{\bar{i}\bar{j}}\nabla^2](C-\mathcal{H}E-\dot{E}-W)$	(32g)
2^-	$\mathcal{A}_{\bar{i}(\bar{j}\bar{k})}-\frac{1}{3}\eta_{\bar{i}\bar{k}}\mathcal{A}_{\bar{l}\bar{j}}{}^{\bar{l}}-\frac{1}{6}\varepsilon_{\bar{i}\bar{j}\bar{k}}\varepsilon_{\bar{l}\bar{a}}{}^{\bar{b}}\mathcal{A}^{\bar{l}\bar{a}}{}_{\bar{b}}$	$[\eta_{\bar{j}\bar{k}}\partial_{\bar{i}}+\frac{1}{6}\eta_{\bar{i}\bar{k}}\partial_{\bar{j}}-\frac{1}{2}\eta_{\bar{i}\bar{j}}\partial_{\bar{k}}](\mathcal{H}B-D-\frac{1}{3}\nabla^2 E-V)$ $+\frac{1}{6}\varepsilon_{\bar{i}\bar{j}\bar{k}}(3\dot{X}-2\nabla^2 F)+\varepsilon_{\bar{i}(\bar{j}\vert\bar{l}}\partial^{\bar{l}}\partial_{\vert\bar{k})}(F+Z)$	(32h)

From the background outlined in these subsections, the transformation of the spin connection can now be undertaken.

3. Behaviour Under Gauge Transformations

Similarly to the tetrad in section III B 2, the induced change in the spin connection is found by demanding that the covariant derivative (as defined in eq. (6)) transforms appropriately. Following this treatment gives the transformation law [54]

$$(\mathcal{A}')^{ij}{}_{\mu}=\frac{\partial x^{\nu}}{\partial x'^{\mu}}(\Lambda^{i}{}_{k}\Lambda^{j}{}_{l}\mathcal{A}^{kl}{}_{\nu}-\Lambda^{jk}\partial_{\nu}\Lambda^{i}{}_{k}). \tag{33}$$

Linearising eq. (33), using the gauge decompositions outlined in section III A, gives the expanded form of the transformed spin connection. As with the tetrad treatment (eq. (17)), this expansion can be expressed, with its indices omitted, as

$$\mathcal{A}'=\mathcal{A}_0+\delta\mathcal{A}+\mathcal{A}_0\partial\xi+\mathcal{A}_0 R+\partial R, \tag{34}$$

where $\mathcal{A}_0$ and $\delta\mathcal{A}$ represent the background and first order perturbative parts of the spin connection respectively. Gauge transformations to the spin connection can then be carried out, and once again represented in terms of its mode decomposition below.

J^P	Tensor form	Scalar expansion	
0^+	$(\mathcal{A}')^{\bar{i}\perp}{}_{\bar{i}}$	$3[\mathcal{H}(1-A+D)+\dot{D}+\frac{1}{3}\nabla^2 C+\dot{U}-\frac{1}{3}\mathcal{H}\nabla^2 L-\frac{1}{3}\nabla^2 P]$	(35a)
0^-	$(\mathcal{A}')^{\bar{i}\bar{j}}{}_{\bar{k}}\varepsilon^{\bar{k}}{}_{\bar{i}\bar{j}}$	$2\nabla^2(F+Q)-3\dot{X}$	(35b)
1^+	$(\mathcal{A}')^{\bar{i}\bar{j}}{}_{\perp}$	$\frac{1}{2}[\varepsilon^{\bar{i}\bar{j}}{}_{\bar{k}}\partial^{\bar{k}}(2\dot{F}+\beta-Y+2\dot{Q})+(\partial^{\bar{j}}\partial^{\bar{i}}+\frac{1}{3}\eta^{\bar{i}\bar{j}}\nabla^2)(W+P)]$	(35c)
1^-	$(\mathcal{A}')^{\bar{k}\bar{i}}{}_{\bar{k}}$	$2\partial^{\bar{i}}(-\mathcal{H}B+D+\frac{1}{3}\nabla^2 E+V-\mathcal{H}P)$	(35d)
1^+	$(\mathcal{A}')_{\perp[\bar{i}\bar{j}]}$	$\frac{1}{2}\varepsilon_{\bar{i}\bar{j}\bar{k}}\partial^{\bar{k}}(2\mathcal{H}F-\beta-Y+2\mathcal{H}Q)$	(35e)
1^-	$(\mathcal{A}')^{\bar{i}\perp}{}_{\perp}$	$\partial^{\bar{i}}[-A+\dot{B}+\mathcal{H}(B+C)+\frac{1}{2}\alpha-V+\dot{P}+\mathcal{H}\dot{L}]$	(35f)
2^+	$(\mathcal{A}')_{\perp(\bar{i}\bar{j})}-\frac{1}{3}\eta_{\bar{i}\bar{j}}(\mathcal{A}')^{\bar{k}\perp}{}_{\bar{k}}$	$[\partial_{\bar{j}}\partial_{\bar{i}}+\frac{1}{3}\eta_{\bar{i}\bar{j}}\nabla^2](C-\mathcal{H}E-\dot{E}-W-P-\mathcal{H}L)$	(35g)
2^-	$(\mathcal{A}')_{\bar{i}(\bar{j}\bar{k})}-\frac{1}{3}\eta_{\bar{i}\bar{k}}(\mathcal{A}')_{\bar{l}\bar{j}}{}^{\bar{l}}-\frac{1}{6}\varepsilon_{\bar{i}\bar{j}\bar{k}}\varepsilon_{\bar{l}\bar{a}}{}^{\bar{b}}(\mathcal{A}')^{\bar{l}\bar{a}}{}_{\bar{b}}$	$[\eta_{\bar{j}\bar{k}}\partial_{\bar{i}}+\frac{1}{6}\eta_{\bar{i}\bar{k}}\partial_{\bar{j}}-\frac{1}{2}\eta_{\bar{i}\bar{j}}\partial_{\bar{k}}](\mathcal{H}B-D-\frac{1}{3}\nabla^2 E-V+\mathcal{H}P)$ $+\frac{1}{6}\varepsilon_{\bar{i}\bar{j}\bar{k}}[3\dot{X}-2\nabla^2(F+Q)]+\varepsilon_{\bar{i}(\bar{j}\vert\bar{l}}\partial^{\bar{l}}\partial_{\vert\bar{k})}(F+\mathcal{Z}+Q)$	(35h)

Comparison between eqs. (32) and (35) requires more thought than in the case of the tetrad in section III B. This is firstly done with the curvature-associated perturbative scalars to give the following relations

$$\begin{aligned} &A\to A, && B\to B+P,\\ &C\to C-P+\dot{L}, && D\to D-\tfrac{1}{3}\nabla^2 L,\\ &E\to E+L, && F\to F+Q. \end{aligned} \tag{36}$$

An important result of this is the continued gauge invariance of the Bardeen potentials Ψ and Φ (as defined in eq. (20)). As well as this, the omission of the gauge angle T suggests that the spin connection is entirely invariant under infinitesimal temporal translations.

The torsion-associated perturbative scalars similarly undergo gauge transformations of the form

$$\begin{aligned} &\alpha\to\alpha+4\dot{P}, && \beta\to\beta+\dot{Q},\\ &W\to W+P, && \mathcal{Z}\to\mathcal{Z}+Q,\\ &\dot{U}\to\dot{U}-\tfrac{1}{3}\nabla^2 P, && V\to V+\dot{P},\\ &X\to X, && Y\to Y-\dot{Q}-2\mathcal{H}Q. \end{aligned} \tag{37}$$

These previously unstudied transformation relations allow for the construction of a variety of possible gauge invariant quantities. Of particular interest for the coming analysis are the two novel combinations of torsion-associated scalars

$$\Omega=\dot{U}-\dot{V}+\tfrac{1}{4}\dot{\alpha}+\tfrac{1}{3}\nabla^2 W, \tag{38a}$$

$$\Theta=2\mathcal{H}\beta+\dot{\beta}-\dot{X}+\dot{Y}. \tag{38b}$$

An interesting remark at this point is related to the unresolved scalars F and $\mathcal{Z}$. As seen in section IV, these scalars do not present themselves as physical observables in the field equations, and so these unresolved gauge transformations do not result in any inconsistencies in the theory. In addition, the newly constructed scalar representation of the 2^- mode is similarly shown to be excluded from the subsequent work.

Although not directly appearing in the later discussions of modified gravity theories (section IV), there exist further gauge invariant quantities that can be identified from these scalar transformations, given by

$$\begin{aligned} \Upsilon&=C-\dot{E}+W,\\ \Pi&=V-\dot{W},\\ \Xi&=\tfrac{1}{4}\nabla^2\alpha+3\ddot{U}. \end{aligned} \tag{39}$$

Notably, these can be formed using both curvature- and torsion-associated scalars.

IV. FIELD EQUATIONS

With the decomposition and transformation of the gauge fields now carried out, their applications in the construction of field equations can be investigated. A standard method of building field equations is using the Lagrangian formulation. This requires the construction of the Lagrangian (density) $\mathcal{L}$, which itself is made up of contractions of covariantly conserved fields. This ensures that $\mathcal{L}$, and in turn its associated action, possess the intended symmetries of the system. In the context of gravity theories, this approach has been taken since the emergence of GR, notably by Hilbert [56]. A useful attribute of this approach is the ease of modifying theories by adding and varying terms in the Lagrangian.

Upon construction of a suitable Lagrangian, the field equations are ascertained using variational principles, such that it follows Hamilton's principle of stationary action. The study of these approaches is extensive, with helpful work concerning so-called $f(\mathcal{R})$ gravity theories compiled by Quiros [57].

Within PGT, there exist two gauge fields (the tetrad and spin connection) with which to vary a Lagrangian. Subsequently, this gives rise to two distinct sets of field equations, referred to here as the spin-torsion and stress-energy equations.

In the presence of matter, the Lagrangian gains the matter-associated term $\mathcal{L}_{\mathrm{M}}$ which gives rise to further structure within the field equations. When looking at the effects this has on the field equations, the additional terms are given by

$$\sigma^{\mu}{}_{ij} \equiv -\frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta \mathcal{A}^{ij}{}_{\mu}}, \tag{40a}$$

$$\mathsf{T}^{\mu}{}_{\nu} \equiv -\frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta b^{k}{}_{\mu}} b^{k}{}_{\nu} \equiv h_{k}{}^{\mu} \frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta h_{k}{}^{\nu}}, \tag{40b}$$

where $\sigma^{\mu}{}_{ij}$ and $\mathsf{T}^{\mu\nu}$ represent the matter-spin and stress-energy tensors respectively. Contributions from the spin of matter are far less significant than those associated with stress-energy effects for the large-scale system studied here [58, 59]. These results come from the study of Weyssenhoff fluids, in which torsion is taken to be an emergent property of the spin of matter itself, in turn giving a physical source of torsion as allowed for in U_4 systems [58, 60]. Therefore, the matter-spin tensor is taken to vanish in the following analysis.

Although not further detailed within this report, the formulation of PGT field equations is captured in more depth by Blagojević [2].

A. Reproduction of General Relativity

Before tackling modified theories of gravity, the work conducted up to this point is examined in the case of known linearised cosmological field equations. The obvious choice for this is that of GR, which has been studied repeatedly, and throughout this research useful pedagogical sources for comparison have included lecture notes by Fergusson [36] and Challinor [61]. These field equations are generally constructed via variation, with respect to the metric, of the Einstein–Hilbert action

$$S_{\mathrm{EC}} \equiv \int \mathcal{L}_{\mathrm{EC}}\, \mathrm{d}^4 x, \quad \mathcal{L}_{\mathrm{EC}} \equiv \frac{1}{2\kappa}\sqrt{|g|}\,\mathcal{R}, \tag{41}$$

where $\mathcal{R}$ is the Ricci–Cartan scalar. However, GR can also be reproduced by variation of the same action with respect to the gauge fields of PGT, using the definition of the Riemann–Cartan field strength in eq. (9) (EC theory).

1. *Spin-Torsion Equations*

The major difference between GR and general PGT is that the geometric concept of torsion vanishes within GR, moving from U_4 to V_4 spacetimes. Another non-geometric interpretation of this can be seen by considering the spin-torsion equations of EC. In order to do this, the form of the torsion tensor as defined in eq. (10) is used. Helpful in the simplification of this expression is the closed form of the of the covariant derivative of the tetrad, given by

$$\mathcal{D}_{\mu} b^{i}{}_{\nu} = \partial_{\mu} b^{i}{}_{\nu} + \mathcal{A}^{i}{}_{j\mu} b^{j}{}_{\nu}. \tag{42}$$

With this background and the vanishing spin-matter tensor, the spin-torsion equations simply require that the contortion itself must also vanish — thus corresponding to the previously described geometric interpretation. The independence of the contortion's spin-parity modes enforces each mode to, in turn, equal zero. By considering the form of the scalar decomposition of the torsion (outlined in section III C 2), this requirement reduces simply to

$$\alpha = \beta = W = U = V = X = Y = \mathcal{Z} = 0. \tag{43}$$

With this, the spin connection takes a significantly simpler form whilst the tetrad is unaltered. The subsequent stress-energy equations similarly take a simpler form.

2. *Stress-Energy Equations*

In order to form the stress-energy equations of EC, the relationships between the Riemann–Cartan tensor, Ricci–Cartan tensor and Ricci–Cartan scalar must be explicitly given ($\mathcal{R}^{ij\mu\nu}$, $\mathcal{R}^{\mu\nu}$ and $\mathcal{R}$ respectively). This requires careful application of the index transformation laws detailed in section II to give the relations

$$\mathcal{R}_{\mu\nu} \equiv g_{\mu\sigma} h_{i}{}^{\rho} h_{j}{}^{\sigma} \mathcal{R}^{ij}{}_{\rho\nu}, \tag{44a}$$

$$\mathcal{R} \equiv h_{i}{}^{\mu} h_{j}{}^{\nu} \mathcal{R}^{ij}{}_{\mu\nu}. \tag{44b}$$

With these objects defined, the stress-energy field equations can be constructed. Their form is very well-known, and can be written using the Einstein tensor $G_{\mu\nu}$ and the previously defined stress-energy tensor $\mathsf{T}^{\mu\nu}$ as

$$G_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \mathcal{R} = \kappa \mathsf{T}_{\mu\nu}. \tag{45}$$

Together, this information yields the linearised field equations exclusively in terms of the scalars $\{A, ..., F\}$. Once these are constructed, the identification of the gauge invariant Bardeen potentials simplifies their appearance further. To display this result, eq. (45) is separated into its spin-parity modes as well as zeroth and first order parts, below in eq. (46)

Zeroth order

J^P	Tensor form	Scalar expansion	
0^+	$G_{\perp\perp}$	$3\mathcal{H}^2$	(46a)
0^+	$G^{\bar{\mu}}{}_{\bar{\mu}}$	$3(\mathcal{H}^2+2\dot{\mathcal{H}})$	(46b)
1^-	$G_{\perp\bar{\mu}}$	0	(46c)
2^+	$G_{\bar{\mu}\bar{\nu}}-\frac{1}{3}\eta_{\bar{\mu}\bar{\nu}}G^{\bar{\sigma}}{}_{\bar{\sigma}}$	0	(46d)

First order

J^P	Tensor form	Scalar expansion	
0^+	$G_{\perp\perp}$	$2\nabla^2\Phi-6\mathcal{H}\dot{\Phi}$	(46e)
0^+	$G^{\bar{\mu}}{}_{\bar{\mu}}$	$6\ddot{\Phi}+6(\mathcal{H}^2+2\dot{\mathcal{H}})(\Phi+\Psi)$	(46f)
1^-	$G_{\perp\bar{\mu}}$	$2\partial_{\bar{\mu}}(\dot{\Phi}+\mathcal{H}\Psi)+12\mathcal{H}\dot{\Phi}+6\mathcal{H}\dot{\Psi}$ $+\,2\nabla^2(\Psi-\Phi)$	(46g)
2^+	$G_{\bar{\mu}\bar{\nu}}-\frac{1}{3}\eta_{\bar{\mu}\bar{\nu}}G^{\bar{\sigma}}{}_{\bar{\sigma}}$	$[\partial_{\bar{\nu}}\partial_{\bar{\mu}}+\frac{1}{3}\eta_{\bar{\mu}\bar{\nu}}\nabla^2](\Phi-\Psi)$	(46h)

An important conclusion to draw from these results is their agreement with the well-known linearised GR cosmological field equations. This also shows the decomposition and formation of the Bardeen potentials outlined in section III B leads to the expected results when applied to EC.

B. $\mathcal{R}^2$ Augmentation to EC

After accurately reproducing GR, the same procedure can be applied to novel gravity theories. The theory of interest here is that of an $\mathcal{R}^2$ augmentation to EC (unstudied in the context of cosmological perturbation theory). This theory is named such due to the addition of the final term in the Lagrangian

$$\mathcal{L}_{\rm DT}=\frac{\alpha_0}{2\kappa}\sqrt{|g|}\mathcal{R}+\alpha_5\mathcal{R}_{[\mu\nu]}\mathcal{R}^{[\mu\nu]}, \tag{47}$$

where the subscript refers to the present dynamical torsion [62]. Here, α_0 and α_5 are dimensionless scaling parameters such that GR is retrieved when α_0 and α_5 are set to unity and zero respectively. Although not immediately obvious, the second term in eq. (47) is purely a torsionful addition to the Lagrangian. This can be seen by considering the torsion-free case in which the Ricci–Cartan tensor is symmetric. Therefore, the non-vanishing contribution to the antisymmetric part of the Ricci–Cartan tensor must be purely torsionful. From this information it can be seen that when α_0 is set to zero, the Lagrangian produces an exclusively torsion-based theory of gravity. As previously mentioned, these such systems have been investigated using PGT in the form of teleparallel theories (in T_4 spacetime), in some cases leading to a Teleparallel Equivalent of General Relativity (TEGR) [38, 41]. The full form of the Lagrangian in eq. (47) is, therefore, seen to be a theory between that of EC and TEGR.

Following the background disucssion of this theory, its linearised field equations can be constructed.

1. Spin-Torsion Equations

The spin-torsion equations are given in their simplified form in eq. (48). This contains the contortion as the α_0 contribution (GR), along with contributions from the α_5 modified gravity terms — these include mixed torsion and Riemann–Cartan terms, as well as covariant derivatives of the Riemann–Cartan tensor. All the symbols used reference tensors whose meanings have been previously defined, to give

$$\begin{aligned}\frac{\alpha_0}{\kappa}K^{\mu}{}_{ij}+4\alpha_5\Big(&\mathcal{T}^{\mu}{}_{[i|l}\mathcal{R}^{lm}{}_{|j]m}-\mathcal{T}^{\mu}{}_{[i|l}\mathcal{R}_{|j]m}{}^{lm}+\mathcal{T}^{\nu}{}_{\nu l}h_{[i|}{}^{\mu}\mathcal{R}_{|j]m}{}^{lm}\\&-\mathcal{T}^{\nu}{}_{\nu l}h_{[i|}{}^{\mu}\mathcal{R}^{lm}{}_{|j]m}+\mathcal{T}^{\nu}{}_{[i|\nu}\mathcal{R}_{|j]m}{}^{\mu m}-\mathcal{T}^{\nu}{}_{[i|\nu}\mathcal{R}^{\mu m}{}_{|j]m}\\&+\mathcal{D}_{[i|}\mathcal{R}_{|j]m}{}^{\mu m}-\mathcal{D}_{[i|}\mathcal{R}^{\mu m}{}_{|j]m}+h_{[i|}{}^{\mu}\mathcal{D}_k\mathcal{R}^{km}{}_{|j]m}-h_{[i|}{}^{\mu}\mathcal{D}_k\mathcal{R}_{|j]m}{}^{km}\Big)=\sigma^{\mu}{}_{ij}.\end{aligned} \tag{48}$$

After manipulation, eq. (48) can be rewritten in terms of the field $B_{\mu\nu}$, defined as [63]

$$\begin{aligned}\tfrac{1}{\sqrt{\kappa}}B_{\mu\nu}\equiv\ &2\mathcal{D}_{\sigma}{}^{(1)}\mathcal{T}^{\sigma}{}_{[\mu\nu]}-2\mathcal{D}_{[\mu}{}^{(2)}\mathcal{T}_{\nu]}\\&+\tfrac{3}{2}\varepsilon^{\sigma\lambda}{}_{\mu\nu}\mathcal{D}_{[\sigma}{}^{(3)}\mathcal{T}_{\lambda]}+2^{(2)}\mathcal{T}_{\sigma}{}^{(1)}\mathcal{T}^{\sigma}{}_{[\mu\nu]}\\&+3\varepsilon^{\sigma\lambda\rho}{}_{[\mu|}{}^{(3)}\mathcal{T}_{\sigma}{}^{(1)}\mathcal{T}_{|\nu]\lambda\rho}.\end{aligned} \tag{49}$$

Although not immediately obvious, the resulting equations take the form of those associated with eq. (1). This shows the system to be strongly-coupled and in turn any linearisation should suffer from the strong-coupling problem outlined in section I.

To expand eq. (48), a closed form of the Riemann–Cartan tensor's covariant derivative is required (as with the tetrad in eq. (42)). Not found in the literature, this was expanded in the case of the four-Roman index Riemann–Cartan tensor as

$$\begin{aligned}\mathcal{D}_k\mathcal{R}_i{}^l{}_{jl} = \partial_k\mathcal{R}_i{}^l{}_{jl} - \mathcal{A}^s{}_{ik}\mathcal{R}_s{}^l{}_{jl} + \mathcal{A}^l{}_{sk}\mathcal{R}_i{}^s{}_{jl} \\ - \mathcal{A}^s{}_{jk}\mathcal{R}_i{}^l{}_{sl} - \mathcal{A}^s{}_{lk}\mathcal{R}_i{}^l{}_{js}.\end{aligned} \tag{50}$$

As with the treatment of the spin-torsion equations in the previous section IV A 1, the matter-spin tensor $\sigma^{\mu ij}$ is taken to be negligible. This gives a set of conditions imposed on the torsion-associated scalars (and gauge invariant quantities of section III), more simply expressed by decomposing the tensor into its constituent independent modes (each of which must independently vanish) in eq. (51).

J^P	Tensor form	Scalar expansion	
0^+	$\sigma^{\bar{\mu}}{}_{\perp\bar{\mu}}$	$-\frac{6\alpha_0}{\kappa}\Omega + \frac{8\alpha_5}{a^2}\nabla^2\Omega$	(51a)
0^-	$\varepsilon_{\bar{\lambda}}{}^{\bar{\mu}\bar{\nu}}\sigma^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}}$	$\frac{6\alpha_0}{\kappa}\Theta - \frac{8\alpha_5}{a^2}\nabla^2\Theta$	(51b)
1^+	$\sigma^{\perp}{}_{\bar{\mu}\bar{\nu}}$	$-\frac{4\alpha_5\mathcal{H}}{a^2}\varepsilon_{\bar{\mu}\bar{\nu}\bar{\sigma}}\partial^{\bar{\sigma}}\Theta$	(51c)
1^-	$\sigma^{\bar{\nu}}{}_{\bar{\mu}\bar{\nu}}$	$\frac{8\alpha_5}{a^2}\partial_{\bar{\mu}}(\dot{\Omega} - \mathcal{H}\Omega)$	(51d)
1^+	$\sigma_{[\bar{\mu}\|\perp\|\bar{\nu}]}$	$\frac{2\alpha_5\mathcal{H}}{a^2}\varepsilon_{\bar{\mu}\bar{\nu}\bar{\sigma}}\partial^{\bar{\sigma}}\Theta$	(51e)
1^-	$\sigma^{\perp}{}_{\perp\bar{\mu}}$	$\frac{8\alpha_5\mathcal{H}}{a^2}\partial_{\bar{\mu}}\Omega$	(51f)
2^+	$\sigma_{(\bar{\mu}\|\perp\|\bar{\nu})} - \frac{1}{3}\eta_{\bar{\mu}\bar{\nu}}\sigma^{\bar{\lambda}}{}_{\perp\bar{\lambda}}$	$\frac{4\alpha_5}{a^2}(\partial_{\bar{\nu}}\partial_{\bar{\mu}} + \frac{1}{3}\eta_{\bar{\mu}\bar{\nu}}\nabla^2)\Omega$	(51g)
2^-	$\sigma_{(\bar{\mu}\|\bar{\nu}\|\bar{\sigma})} - \frac{1}{3}\eta_{\bar{\mu}\bar{\sigma}}\sigma^{\bar{\lambda}}{}_{\bar{\nu}\bar{\lambda}} - \frac{1}{6}\varepsilon_{\bar{\mu}\bar{\nu}\bar{\sigma}}\varepsilon_{\bar{\lambda}}{}^{\bar{\rho}\bar{\tau}}\sigma^{\bar{\lambda}}{}_{\bar{\rho}\bar{\tau}}$	$\varepsilon_{\bar{\mu}\bar{\nu}\bar{\sigma}}(\frac{\alpha_0}{\kappa}\Theta + \frac{4\alpha_5}{3a^2}\nabla^2\Theta)$	(51h)
(*All modes vanish,* $\sigma_{\mu\nu\sigma} = 0$)		$+\frac{2\alpha_5}{a^2}(\eta_{\bar{\nu}\bar{\sigma}}\partial_{\bar{\mu}} - \frac{1}{3}\eta_{\bar{\mu}\bar{\sigma}}\partial_{\bar{\nu}} - 2\eta_{\bar{\mu}\bar{\nu}}\partial_{\bar{\sigma}})(\mathcal{H}\Omega - \dot{\Omega})$	

Although discussions of these conditions could be given here, they are more relevant once the form of the stress-energy equations is also known. Therefore, the following treatment lies in determining these stress-energy equations.

2. *Stress-Energy Equations*

The stress-energy equations associated with this $\mathcal{R}+\mathcal{R}^2$ theory are given here, in their simplified form, as

$$\begin{aligned}\frac{\alpha_0}{\kappa}G_{\mu\nu} + 2\alpha_5\Big(2\mathcal{R}_{\mu\sigma\nu\rho}\mathcal{R}^{[\rho|\tau|\sigma]}{}_{\tau} \\ + 2g_{\mu\theta}\mathcal{R}_{\sigma\rho\nu}{}^{\sigma}\mathcal{R}^{[\rho|\tau|\theta]}{}_{\tau} \\ + g_{\mu\nu}\mathcal{R}_{\sigma}{}^{\rho\sigma\tau}\mathcal{R}_{[\rho|\lambda|\tau]}{}^{\lambda}\Big) = \mathsf{T}_{\mu\nu},\end{aligned} \tag{52}$$

where G and T are the Einstein and stress-energy tensors as previously defined. The $\mathcal{R}^2$ augmentation appears as the collection of terms within parentheses in eq. (52). Further expansion gives the expected GR result (multiplied by a scale factor of α_0) with an additional contribution from the α_5 terms. Deviations from the GR eq. (46) are given in the form of a mode decomposition of the stress-energy tensor in eq. (53).

J^P	Tensor form	Scalar expansion	
0^+	$\delta\mathsf{T}_{\perp\perp}$	$\frac{6\alpha_0\mathcal{H}}{\kappa}\Omega$	(53a)
0^+	$\delta\mathsf{T}^{\bar{\mu}}{}_{\bar{\mu}}$	$\frac{6\alpha_0}{\kappa}(\mathcal{H}\Omega + \dot{\Omega})$	(53b)
1^-	$\delta\mathsf{T}_{\bar{\mu}\perp}$	$-\left(\frac{2\alpha_0}{\kappa} + \frac{8\alpha_5\mathcal{H}^2}{a^2}\right)\partial_{\bar{\mu}}\Omega$	(53c)
1^-	$\delta\mathsf{T}_{\perp\bar{\nu}}$	$\left(\frac{2\alpha_0}{\kappa} + \frac{8\alpha_5\dot{\mathcal{H}}}{a^2}\right)\partial_{\bar{\nu}}\Omega$	(53d)
1^+	$\delta\mathsf{T}_{[\bar{\mu}\bar{\nu}]}$	$\left(\frac{\alpha_0}{2\kappa} + \frac{2\alpha_5(\mathcal{H}^2+\dot{\mathcal{H}})}{a^2}\right)\varepsilon_{\bar{\mu}\bar{\nu}\bar{\sigma}}\partial^{\bar{\sigma}}\Theta$	(53e)
2^+	$\delta[\mathsf{T}_{(\bar{\mu}\bar{\nu})} - \frac{1}{3}\eta_{\bar{\mu}\bar{\nu}}\mathsf{T}^{\bar{\sigma}}{}_{\bar{\sigma}}]$	0	(53f)

When α_5 is set to zero, eqs. (51a) and (51b) simply show $\Omega = \Theta = 0$. Consequently, all variations to the stress-energy tensor, in eq. (53), vanish to exactly reproduce GR. This simple check of consistency allows the following unstudied part of the analysis to be undertaken.

Another critical element of this discussion is the notion of propagating torsion — firstly considered in the case of *flat* Minkowski spacetime where $\mathcal{H} = 0$. Under this restriction, the divergence of eq. (51d) shows $\nabla^2\dot{\Omega} = 0$. Substitution of this result into the time derivative of eq. (51a) then shows the perturbative quantity Ω is constant in time ($\dot{\Omega} = 0$). Using this result, eq. (51h) reduces to

$$\frac{\alpha_0}{\kappa}\Theta + \frac{4\alpha_5}{3a^2}\nabla^2\Theta = 0. \tag{54}$$

When considered in conjunction with eq. (51b), this shows that the perturbative quantity $\Theta = -\Theta$ and so, in turn, vanishes entirely. Together, these results show that torsion perturbations upon a Minkowski background do not propagate. This supports quantum field theoretic (QFT) analysis in which linearised torsion is known to be non-propagating upon a Minkowski background [64].

In the case of the general, expanding Universe, the same analysis can be applied to eq. (51). The exact procedure of relating eqs. (51b), (51d) and (51h) once again shows $\Theta = 0$. In addition, eq. (51)a and eq. (51)f show that the remaining quantity Ω must also vanish entirely. Applying this to the stress-energy eq. (53), again, shows all deviations from GR vanish. These interpretations suggest that linear perturbative torsion is non-propagating within such an $\mathcal{R}^2$ modified theory, in the cases of both the *flat* Minkowski background and the expanding Universe.

These effects are the consequence of the strong-coupling problem as referenced in section I. For the system studied here, this arises due to the coupling with torsion-associated degrees of freedom [65, 66], found in eq. (29). As expected from previous work, the above analysis shows torsion to be non-propagating upon the Minkowski background. However, this result has been arrived at in a novel way, using a (simpler) cosmological perturbative approach as opposed to the existing QFT procedure [64]. This cosmological perturbative method is, therefore, seen to be superior to the existing means of probing the strong-coupling problem (notably for systems upon non-Minkowski backgrounds). In addition, this conclusion has newly shown that the strong-coupling problem extends to the case of the expanding (FLRW) Universe — by offering another avenue through with to investigate the strong-coupling problem, the importance of the above analysis is seen.

V. CONCLUDING REMARKS

Cosmological perturbation theory is yet to be attempted using PGT upon general Riemann–Cartan spacetime, and so is the focus of this work. This is first tackled by carrying out a scalar decomposition of the two gauge fields contained within PGT (the tetrad and spin connection) to linear order. In particular, only the case of perturbative (first order) torsion is considered. Gauge transformations of these fields result in the formation of invariant combinations of these perturbative scalars. The curvature-associated perturbations lead to the reproduction of the Bardeen potentials, while the torsion-associated perturbations motivate the formation of novel gauge invariant quantities.

The form of these scalar decompositions is validated through the derivation of the EC linearised, cosmological field equations. The chosen decompositions of the tetrad and spin connection are shown to reproduce GR, after which they are applied to an $\mathcal{R}^2$ modified gravity theory containing torsion. For this latter system, the torsion-associated modes are shown to be non-propagating. In particular, the strong-coupling problem is seen to arise in both the cases of Minkowski and (novelly) expanding FLRW backgrounds.

Opportunities for developments to this theory lie in its formulation upon a torsion condensate. Aside from this, applications to further modified gravity theories (such as those not suffering from the strong-coupling problem) and extensions to vector and tensor decomposition of the gauge fields are also of interest.

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Appendix A: Conventions Used

The conventions used throughout the above analysis are briefly outlined here.

In the formulation of a local theory of the Poincaré group, there requires different treatments of the global frame and local Lorentz frames. These are distinguished by the use of Greek $\{\mu, \nu, \sigma, ...\}$ and Roman $\{i, j, k, ...\}$ spacetime indices that can take on values in the range $[0, 3]$ — the former representing the global coordinate frame, while local, *flat* Lorentz frames are indicated by the latter. Symmetric and antisymmetric combinations of indices are denoted using () and [] respectively. An important consequence of this representation is the method used in lowering and raising indices. For example, this is carried out using the general (curvature containing) metric $g_{\mu\nu}$ in the global frame, whilst the locally flat frame uses the Minkowski metric η_{ij}. In relation to this, the 'West Coast' signature of the metric is used here such that the time and space components are positive and negative respectively — these are summarised by $(+, -, -, -)$ in the $1 + 3$ dimensions setup considered here.

In the selection of temporal and spatial components of different tensors, the definition of a time unit vector is required. The role of the is taken by $[n_\mu] = (1, 0, 0, 0)$ throughout the above analysis. It also allows for the definitions

$$A_{\ldots\perp\ldots} := A_{\ldots\mu\ldots} n^\mu \tag{A1a}$$

$$A_{\ldots\bar{\mu}\ldots} := A_{\ldots\nu\ldots}(\delta^\nu_\mu - n^\nu n_\mu) \tag{A1b}$$

where a general tensor $A_{\ldots\mu\ldots}$ has had its temporal and spatial components selected, and referred to using the symbols $\perp$ and $^{-}$ respectively. Also of use to note, is the relationship between the four dimensional alternating tensor and the three dimensional (spatial) Levi–Civita symbol given by

$$\varepsilon_{\bar{\mu}\bar{\nu}\bar{\sigma}} = \varepsilon_{\mu\nu\sigma}{}^{\rho}\, n_\rho. \tag{A2}$$

Calligraphic script is used when referring to fields such as the Riemman–Cartan tensor, torsion tensor and spin connection ($\mathcal{R}$, $\mathcal{T}$ and $\mathcal{A}$ respectively). Whilst scalars involved in the decomposition, discussed in section III, are expressed using the standard script $\{A, B, ...\}$.

When referring to the time coordinate t, the conformal time is used — related to the cosmic time τ by $d\tau = a(t)dt$ where a is the scale factor of the Universe. Derivatives with respect to this conformal time are denoted by a dot $\dot{}$, and in turn give the conformal Hubble number as $\mathcal{H} \equiv \frac{\dot{a}}{a}$.

In addition, natural units are used in which $c = \hbar = 1$ and κ denotes the Einstein gravitational constant, equal to $8\pi G$. Consequently, the dimensions of use are length and energy (equivalent to inverse length) — for example, time and distance have dimensions of length while derivatives have dimensions of energy.

Appendix B: Torsion Condensates

As briefly mentioned in the main body of the report (section I), PGT formulations of gravity upon a torsion condensate have been a compelling area of recent study. In particular, such theories have been shown to motivate the appearance of a late-Universe cosmological constant and in turn address the Hubble *tension* [16, 17]. This is done through the consideration of *dark radiation*, the mediating interaction between dark matter modes (similar to its baryonic counterpart of electromagnetism). The inclusion of *dark radiation* is shown to be able to account for the mismatch in inferred measurements of the Hubble *constant*, and in turn describe the Universe's (observed) expansion [67–69].

The term *torsion condensate* refers to a system in which there exists a non-zero contribution from the torsion at the background (zeroth order) level. This addition complicates the approach taken when concerning the gauge transformations of section III. Therefore, below is given a brief account of how exactly such a procedure would be tackled in the case of a torsion condensate.

As torsion is only present in the formulation of the rotational gauge field, this change is relevant to its behaviour under gauge transformations as given by eq. (34), reprinted here as

$$\mathcal{A}' = \mathcal{A}_0 + \delta\mathcal{A} + \mathcal{A}_0\partial\xi + \mathcal{A}_0 R + \partial R. \tag{B1}$$

Further expansion of the spin connection into the RRC and contortion can then be carried out to give

$$\mathcal{A}_0 = \Delta_0 + K_0 \tag{B2a}$$
$$\delta\mathcal{A} = \delta\Delta + \delta K \tag{B2b}$$

where subscript 0 and prefix δ represent the zeroth and first order contributions respectively. The (previously not considered) presence of K_0 leads to the extra complexity in the study of the condensate.

The exact form of the background contribution of the torsion is related to the vector and pseudo-vector components of the torsion only (referred to as ${}^{(2)}T^{\mu}$ and ${}^{(3)}T^{\mu}$ respectively, as in section III C 2). Although the values of these vary depending upon the theory being investigated, the most relevant to the above condensate theory are those of Constant Torsion Emergent Gravity (CTEG) [18, 70, 71].

Within CTEG, the value of the pseudo-vector torsion is taken to be constant. In particular, only the time component of this 4-vector is non-zero, referred to as the pseudo-scalar torsion ${}^{(3)}T^{\perp}$. The non-zero contribution to the vector torsion is, similarly, in its time component only. After computation, this value has been shown to evolve with the expansion of the Universe according to ${}^{(2)}T^{\perp} \propto \mathcal{H}$ [16]. The proportionality constant in this expression is dependent upon the dominant matter-fluid of the Universe [70].

Gauge transformation analysis of this modified system would include an additional two scalars (with known values). The inclusion of these, would give rise to new gauge invariant quantities and in turn, significantly vary the field equations of any gravity theories containing torsion (altering sections III and IV of the above report). However, the tools outlined in this report make up the necessary background to tackle such CTEG theories.

Appendix C: Symbolic Computation

Throughout this work, there was required significant tensorial manipulation which would be very laborious if carried out by hand. A vital part of this analysis was then carried out with the help of the symbolic computation software `Mathematica` [72], along with the tensor manipulation package `xAct: Efficient tensor computer algebra for the Wolfram Language` [73–86]. This is a commonly used tool in research, having been applied to varying fields of study, including: perturbation theory [87], modified gravity theories [88] and stellar collapse [89].

In order to avoid lengthy component algebra, eq. (A1) are integral in this setup and used extensively in these computational scripts. These allow for spatial and temporal components of objects to be identified in a more general sense (with respect to a general time four-vector n^μ) when compared with explicit component manipulation methods, such as those making use of the `xCoba` packages.

Examplar scripts used in formulation of the above theory are then detailed in the subsections below. Two-column formatting is used here as a space-saving measure.

1. Tetrad Decomposition and Gauging

The script below is associated with calculations contained within section III B.

```
(*Loading the xAct packages required for tensor
    manipulation*)
<< xAct`xTras`
(*Loading the TexAct package that allows for
    outputs to be given directly in LaTeX format*)
<< xAct`TexAct`
$PrePrint = ScreenDollarIndices;
(*Deinition of base manifold and metric -- taken to
     be flat as curvature and torsion are being
    manually added*)
DefManifold[M, 4, {i, j, k, l, m, p, q, r, s, t}];
DefMetric[-1, g[-i, -j], CD, {",", "\[PartialD]"},
  PrintAs -> "\[Eta]", FlatMetric -> True];
(*Canonically ordering CDs*)
SortCovDsStart[CD];

(*Definition of the time vector*)
DefTensor[n[i], M];
(*Setting n[i] to be normal*)
AutomaticRules[n, MakeRule[{n[i] n[-i], 1}]];
(*Setting the derivative of n[i] to zero*)
AutomaticRules[n, MakeRule[{CD[i][n[j]], 0}]];

(*Defining the tetrad*)
DefTensor[B[i, j], M];
(*Definition of a perturbation parameter*)
DefTensor[\[Epsilon][], M];
(*Splitting up the tetrad into background and
    perturbed components*)
DefTensor[B0[i, j], M];
DefTensor[\[Delta]B[i, j], M];
(*Can now give the expression for the tetrad
    decomposed into these components*)
AutomaticRules[B,
  MakeRule[{B[i, j], B0[i, j] + \[Epsilon][] \[
      Delta]B[i, j]}]];
(*Outputting the untransformed tetrad expanded thus
     far*)
B[i, -j]

(*Defining physical properties of the system: scale
     factor and small tetrad perturbations*)
DefTensor[a[], M];
DefTensor[As[], M, PrintAs -> "A"];
DefTensor[Bs[], M, PrintAs -> "B"];
DefTensor[Cs[], M, PrintAs -> "C"];
DefTensor[Ds[], M, PrintAs -> "D"];
DefTensor[Es[], M, PrintAs -> "E"];
DefTensor[Fs[], M, PrintAs -> "F"];
(*Setting the time derivative of the scale factor
    to the Hubble number H, while setting its
    spatial derivatives to vanishing*)
DefTensor[H[], M];
AutomaticRules[a, MakeRule[{CD[i][a[]], n[i] H
    []}]];
(*Expressing the background and perturbed tetrad in
     terms of these scalars only*)
AutomaticRules[B0, MakeRule[{B0[i, -j], a[] delta[i
    , -j]}]];
AutomaticRules[\[Delta]B, MakeRule[{\[Delta]B[i, -j
    ], a[] ( n[i] n[-j] As[]
      + n[i] CD[-k][Bs[]] (delta[k, -j] - n[k] n[-
          j])
      + n[-j] CD[k][Cs[]] (delta[-k, i] - n[-k] n[
          i])
      + Ds[] delta[k, -l] (delta[-k, i] - n[-k] n[
          i]) (delta[l, -j] - n[l] n[-j])
      + CD[k][CD[-l][Es[]]] (delta[-k, i] - n[-k]
          n[i]) (delta[l, -j] - n[l] n[-j])
      - 1/3 delta[k, -l] (delta[-k, i] - n[-k] n[i
          ]) (delta[l, -j] - n[l] n[-j]) CD[m][CD
          [-p][Es[]]] (delta[p, -m] - n[p] n[-m])
      + epsilong[l, -m, p, r] n[-r] (delta[-l, i]
          - n[-l] n[i]) (delta[m, -j] - n[m] n[-j
          ]) CD[-q][Fs[]] (delta[q, -p] - n[q] n[-
          p]))}]];
(*Outputting the untransformed tetrad expanded thus
     far*)
CollectTensors[B[i, -j]]


(*Can now decompose this into the appropriate modes
    *)

        (*0+ spin-parity mode #1*)
DefTensor[plus0n1[], M];
AutomaticRules[plus0n1, MakeRule[{plus0n1[], n[-i]
    n[j] B[i, -j]}]];
CollectTensors[plus0n1[]]

        (*0+ spin-parity mode #2*)
DefTensor[plus0n2[], M];
AutomaticRules[plus0n2,
  MakeRule[{plus0n2[],
    B[j, -k] (delta[k, -i] - n[k] n[-i]) (delta[i,
        -j] - n[i] n[-j])}]];
CollectTensors[plus0n2[]]

        (*1- spin-parity mode #1*)
DefTensor[minus1n1[i], M];
AutomaticRules[minus1n1,
  MakeRule[{minus1n1[i],
    n[j] B[k, -j] (delta[-k, i] - n[-k] n[i])}]];
CollectTensors[minus1n1[i]]

        (*1- spin-parity mode #2*)
```

```
DefTensor[minus1n2[i], M];
AutomaticRules[minus1n2,
  MakeRule[{minus1n2[-i],
    n[-j] B[j, -k] (delta[k, -i] - n[k] n[-i])}]];
CollectTensors[minus1n2[-i]]

        (*1+ spin-parity mode*)
DefTensor[plus1[i, j], M];
AutomaticRules[plus1,
  MakeRule[{plus1[-i, -j],
    1/2 (B[-k, -l] - B[-l, -k]) (delta[k, -i] - n[k
        ] n[-i]) (delta[l, -j] - n[l] n[-j])}]];
CollectTensors[plus1[-i, -j]]

        (*2+ spin-parity mode*)
DefTensor[plus2[i, j], M];
AutomaticRules[plus2,
  MakeRule[{plus2[-i, -j],
    1/2 (B[-k, -l] + B[-l, -k]) (delta[k, -i] - n[k
        ] n[-i]) (delta[l, -j] - n[l] n[-j])
    - 1/3 delta[-k, -l] (delta[k, -i] - n[k] n[-i])
          (delta[l, -j] - n[l] n[-j]) plus0n2[]}]];
CollectTensors[plus2[-i, -j]]


(*Defining the gauged tetrad*)
DefTensor[gaugeB[i, j], M];
(*Defining guage variables*)
DefTensor[\[Xi][i], M];
DefTensor[R[i, j], M];
(*Expression for how the tetrad varies under gauge
    transformations*)
AutomaticRules[gaugeB,
  MakeRule[{gaugeB[i, -j],
    B0[i, -j] + \[Epsilon][] (\[Delta]B[i, -j] + B0
        [i, -k] CD[-j][\[Xi][k]] + B0[k, -j] R[i, -
        k])}]];

(*Decomposition of Jacobian part into scalars*)
DefTensor[T[], M];
DefTensor[L[], M];
AutomaticRules[\[Xi],
  MakeRule[{\[Xi][i],
    n[i] T[] + (delta[i, -j] - n[i] n[-j]) CD[j][L
        []]}]];
(*Decomposition of Lorentz part into scalars*)
DefTensor[P[], M];
DefTensor[Q[], M];
AutomaticRules[R,
  MakeRule[{R[-i, -j], (n[-i] CD[-k][P[]] (delta[k,
       -j] - n[k] n[-j]) -
       n[-j] CD[-k][P[]] (delta[k, -i] - n[k] n[-i
          ]))
    + (delta[m, -i] - n[m] n[-i])
       (delta[p, -j] - n[p] n[-j])
       (delta[k, -q] - n[k] n[-q])
       (delta[s, -k] - n[s] n[-k])
       epsilong[-m, -p, q, r] n[-r] CD[-s][Q[]]}]];

(*Determining how the scale factor transforms under
     gauge transformations*)
AtoH = MakeRule[{a[], (1 + \[Epsilon][] H[] T[]) a
    []}];
(*Rule to reduce expressions to linear order*)
Linearise1 = MakeRule[{\[Epsilon][]^2, 0}];
Linearise2 = MakeRule[{CD[i][\[Epsilon][]], 0}];

(*Output gauge transformed tetrad only in terms of
    scalars*)
CollectTensors[gaugeB[i, -j] /. AtoH] /. Linearise1
     /. Linearise2

(*Redefining the gauge transformed tetrad as the
    expression above [to save on computing time]*)
DefTensor[ExpGaugeB[i, j], M];
AutomaticRules[ExpGaugeB,
  MakeRule[{ExpGaugeB[i, -j],
    a[] xAct`xTensor`delta[i, -j] +
     a[] xAct`xTensor`delta[i, -j] Ds[] \[Epsilon
        ][] +
     a[] As[] n[i] n[-j] \[Epsilon][] -
     a[] Ds[] n[i] n[-j] \[Epsilon][] +
     a[] xAct`xTensor`delta[i, -j] H[] T[] \[
        Epsilon][] +
     a[] n[-j] \[Epsilon][] CD[i][Cs[]] -
     a[] n[-j] \[Epsilon][] CD[i][P[]] +
     a[] n[i] \[Epsilon][] CD[-j][Bs[]] +
     a[] n[i] \[Epsilon][] CD[-j][P[]] +
     a[] n[i] \[Epsilon][] CD[-j][T[]] +
     a[] \[Epsilon][] CD[-j][CD[i][Es[]]] +
     a[] \[Epsilon][] CD[-j][CD[i][L[]]] -
     a[] n[i] n[-j] n[k] \[Epsilon][] CD[-k][Bs[]]
        -
     a[] n[i] n[-j] n[k] \[Epsilon][] CD[-k][Cs[]]
        -
     a[] n[-j] n[k] \[Epsilon][] CD[-k][CD[i][Es
        []]] -
     a[] n[i] n[k] \[Epsilon][] CD[-k][CD[-j][Es
        []]] -
     a[] n[i] n[k] \[Epsilon][] CD[-k][CD[-j][L[]]]
        -
     1/3 a[] xAct`xTensor`delta[i, -j] \[Epsilon][]
         CD[-k][
       CD[k][Es[]]] +
     1/3 a[] n[i] n[-j] \[Epsilon][] CD[-k][CD[k][
        Es[]]] +
     1/3 a[] xAct`xTensor`delta[i, -j] n[k] n[l] \[
        Epsilon][] CD[-l][
       CD[-k][Es[]]] +
     2/3 a[] n[i] n[-j] n[k] n[l] \[Epsilon][] CD[-
        l][CD[-k][Es[]]] -
     a[] epsilong[i, -j, -k, -l] n[k] \[Epsilon][]
        CD[l][Fs[]] -
     a[] epsilong[i, -j, -k, -l] n[k] \[Epsilon][]
        CD[l][Q[]]}]];


(*Can now decompose this into the appropriate modes
     [transformed version of those before]*)

        (*0+ spin-parity mode #1*)
DefTensor[Gplus0n1[], M];
AutomaticRules[Gplus0n1,
  MakeRule[{Gplus0n1[], n[-i] n[j] ExpGaugeB[i, -j
      ]}]];
CollectTensors[Gplus0n1[]]

        (*0+ spin-parity mode #2*)
DefTensor[Gplus0n2[], M];
AutomaticRules[Gplus0n2,
  MakeRule[{Gplus0n2[],
    ExpGaugeB[
      j, -k] (delta[k, -i] - n[k] n[-i]) (delta[i,
         -j] - n[i] n[-j])}]];
CollectTensors[Gplus0n2[]]

        (*1- spin-parity mode #1*)
DefTensor[Gminus1n1[i], M];
AutomaticRules[Gminus1n1,
  MakeRule[{Gminus1n1[i],
    n[j] ExpGaugeB[k, -j] (delta[-k, i] - n[-k] n[i
       ])}]];
CollectTensors[Gminus1n1[i]]

        (*1- spin-parity mode #2*)
DefTensor[Gminus1n2[i], M];
AutomaticRules[Gminus1n2,
  MakeRule[{Gminus1n2[-i],
    n[-j] ExpGaugeB[j, -k] (delta[k, -i] - n[k] n[-
       i])}]];
CollectTensors[Gminus1n2[-i]]

        (*1+ spin-parity mode*)
DefTensor[Gplus1[i, j], M];
AutomaticRules[Gplus1,
  MakeRule[{Gplus1[-i, -j],
    1/2 (ExpGaugeB[-k, -l] - ExpGaugeB[-l, -k]) (
        delta[k, -i] - n[k] n[-i]) (delta[l, -j] -
        n[l] n[-j])}]];
CollectTensors[Gplus1[-i, -j]]

        (*2+ spin-parity mode*)
DefTensor[Gplus2[i, j], M];
AutomaticRules[Gplus2,
  MakeRule[{Gplus2[-i, -j],
```

```
1/2 (ExpGaugeB[-k, -l] + ExpGaugeB[-l, -k]) (
     delta[k, -i] -
     n[k] n[-i]) (delta[l, -j] - n[l] n[-j]) -
  1/3 delta[-k, -l] (delta[k, -i] - n[k] n[-i])
     (delta[l, -j] -
        n[l] n[-j]) Gplus0n2[]}]];
CollectTensors[Gplus2[-i, -j]]
```

2. Spin Connection Decomposition and Gauging

The script below is associated with calculations contained within section III C.

```
(*Loading the xAct packages required for tensor
    manipulation*)
<< xAct`xTras`
(*Loading the TexAct package that allows for
    outputs to be given directly in LaTeX format*)
<< xAct`TexAct`
$PrePrint = ScreenDollarIndices;
(*Deinition of base manifold and metric -- taken to
     be flat as curvature and torsion are being
    manually added*)
DefManifold[M, 4, {i, j, k, l, m, p, q, r, s, t}];
DefMetric[-1, g[-i, -j], CD, {",", "\[PartialD]"},
  PrintAs -> "\[Eta]", FlatMetric -> True];
(*Canonically ordering CDs*)
SortCovDsStart[CD];

(*Definition of the time vector*)
DefTensor[n[i], M];
(*Setting n[i] to be normal*)
AutomaticRules[n, MakeRule[{n[i] n[-i], 1}]];
(*Setting the derivative of n[i] to zero*)
AutomaticRules[n, MakeRule[{CD[i][n[j]], 0}]];


(*Creation of the spin connection*)

(*Defining general spin connection*)
DefTensor[\[CapitalDelta][i, j, k], M];
DefTensor[K[i, j, k], M];
DefTensor[A[i, j, k], M];
AutomaticRules[A,
  MakeRule[{A[i, j, k], \[CapitalDelta][i, j, k] +
      K[i, j, k]}]];
A[-i, -j, -k]

(*Defining the tetrad and its inverse*)
DefTensor[b[i, j], M];
DefTensor[h[i, j], M];
(*Outlining the identity relationships of b and h*)
AutomaticRules[b, MakeRule[{b[i, -j] h[-i, k],
    delta[-j, k]}]];
AutomaticRules[b, MakeRule[{b[i, -j] h[-k, j],
    delta[i, -k]}]];
DefTensor[c[i, j, k], M];
AutomaticRules[\[CapitalDelta],
  MakeRule[{\[CapitalDelta][-i, -j, -k],
    1/2 (delta[-i, -l] c[l, -m, -p] h[-j, m] h[-q,
        p] -
       delta[-q, -l] c[l, -m, -p] h[-i, m] h[-j, p]
          +
       delta[-j, -l] c[l, -m, -p] h[-q, m] h[-i, p
          ]) b[q, -k]}]];
(*Expanding the RRC in terms of the tetrad, its
    inverse and the c[i,j,k] terms*)
A[-i, -j, -k]

(*Further expanding the c terms using the tetrad*)
AutomaticRules[c,
  MakeRule[{c[i, -j, -k], CD[-j][b[i, -k]] - CD[-k
      ][b[i, -j]]}]];
(*Again expanding the RRC in terms of only the
    tetrad and its inverse*)
A[-i, -j, -k]

(*Seperation of tetrad into background and
    decomposed parts*)
DefTensor[b0[i, j], M];
DefTensor[\[Delta]b[i, j], M];
(*Seperation of inverse tetrad into background and
    decomposed parts*)
DefTensor[h0[i, j], M];
DefTensor[\[Delta]h[i, j], M];
(*Including perturbation parameter epsilon*)
DefTensor[\[Epsilon][], M];
(*Setting the background to zeroth order and
    perturbations to first order*)
AutomaticRules[b,
  MakeRule[{b[i, j], b0[i, j] + \[Epsilon][] \[
       Delta]b[i, j]}]];
AutomaticRules[h,
  MakeRule[{h[i, j], h0[i, j] + \[Epsilon][] \[
       Delta]h[i, j]}]];

(*Expansion of the contorsion tensor in terms of
    the torsion tensor [taken to be first order in
    this model]*)
DefTensor[Tor[i, j, k], M];
AutomaticRules[K,
  MakeRule[{K[-i, -j, -k],
    1/2 (Tor[-i, -j, -k] + Tor[-j, -k, -i] -
        Tor[-k, -i, -j]) \[Epsilon][]}]];
(*Decomposing the torsion into tensor [1], vector
    [2] and axial-vector [3] components*)
DefTensor[T1[i, j, k], M];
DefTensor[T2[i], M];
DefTensor[T3[i], M];
AutomaticRules[Tor,
  MakeRule[{Tor[i, -j, -k],
    T1[i, -j, -k] + (T2[-j] delta[i, -k] - T2[-k]
         delta[i, -j]) +
     T3[-l] epsilong[l, i, -j, -k]}]];
(*Further decomposing these modes in terms of only
    scalars -- time and space derivatives of
    scalars*)
DefTensor[\[Alpha][], M];
DefTensor[\[Beta][], M];
DefTensor[W[], M];
DefTensor[Z[], M];
AutomaticRules[T1, MakeRule[{T1[i, -j, -k],
    n[i] n[-j] (delta[l, -k] - n[l] n[-k]) CD[-l
         ][\[Alpha][]]
     + n[i] n[q] epsilong[l, -j, -k, -q] CD[-l][\[
         Beta][]]
     + n[-j] (CD[l][CD[-m][W[]]] -
        1/3 delta[l, -m] CD[p][
          CD[-q][W[]]] (delta[q, -p] - n[q] n[-p]))
            (delta[-l, i] -
        n[-l] n[i]) (delta[m, -k] - n[m] n[-k])
     + epsilong[i, -k, -l, -m] n[m] CD[l][
        CD[-p][Z[]]] (delta[p, -j] - n[p] n[-j]) +
      epsilong[-j, -k, -l, -m] n[m] CD[l][
        CD[p][Z[]]] (delta[-p, i] - n[-p] n[i])}]];
DefTensor[U[], M];
DefTensor[V[], M];
AutomaticRules[T2,
  MakeRule[{T2[i],
    n[i] n[-j] CD[j][U[]] + (delta[i, -j] - n[i] n
        [-j]) CD[j][
       V[]]}]];
DefTensor[X[], M];
DefTensor[Y[], M];
AutomaticRules[T3,
  MakeRule[{T3[i],
    n[i] n[-j] CD[j][X[]] + (delta[i, -j] - n[i] n
        [-j]) CD[j][Y[]]}]];
```

```
(*Reexpanding spin connection in terms of the
    tetrad, inverse tetrad and the torsion-
    associated scalars up to linear order*)
Linearise1 = MakeRule[{\[Epsilon][]^2, 0}];
Linearise2 = MakeRule[{\[Epsilon][]^3, 0}];
Linearise3 = MakeRule[{\[Epsilon][]^4, 0}];
LineariseDiv = MakeRule[{CD[i][\[Epsilon][]], 0}];
CollectTensors[A[-i, -j, -k]] /. Linearise1 /.
    Linearise2 /.  Linearise3 /. LineariseDiv

(*Defining scalar parameters used in the expansion
    of the tetrad and its inverse: scale factor and
     scalar perturbations*)
DefTensor[a[], M];
DefTensor[As[], M, PrintAs -> "A"];
DefTensor[Bs[], M, PrintAs -> "B"];
DefTensor[Cs[], M, PrintAs -> "C"];
DefTensor[Ds[], M, PrintAs -> "D"];
DefTensor[Es[], M, PrintAs -> "E"];
DefTensor[Fs[], M, PrintAs -> "F"];
(*Setting the time derivative of the scale factor
    to the Hubble number H, while setting its
    spatial derivatives to vanishing*)
DefTensor[H[], M];
AutomaticRules[a, MakeRule[{CD[i][a[]], a[] n[i] H
    []}]];

(*Expanding the tetrad in terms of scalars*)
AutomaticRules[b0, MakeRule[{b0[i, j], a[] delta[i,
     j]}]];
AutomaticRules[\[Delta]b, MakeRule[{\[Delta]b[i, j
    ],
    a[] (n[i] n[j] As[]
       + n[i] CD[k][Bs[]] (delta[-k, j] - n[-k] n[j
           ])
       + n[j] CD[k][Cs[]] (delta[-k, i] - n[-k] n[i
           ])
       +
       delta[k,
         l] (delta[i, -k] - n[i] n[-k]) (delta[j, -
            l] -
          n[j] n[-l]) Ds[]
       + (delta[i, -k] - n[i] n[-k]) (delta[j, -l]
           -
          n[j] n[-l]) (CD[k][CD[l][Es[]]] -
          1/3 delta[k, l] CD[p][
            CD[q][Es[]]] (delta[-p, -q] - n[-p] n[-
                q]))
       + (delta[i, -m] - n[i] n[-m]) (delta[j, -p]
           -
          n[j] n[-p]) (delta[s, -q] - n[s] n[-q])
               epsilong[m, p,
         q, -r] n[r] CD[-s][Fs[]]) }]];

(*Expanding the inverse tetrad in terms of scalars
    *)
AutomaticRules[h0, MakeRule[{h0[i, j], 1/a[] delta[
    i, j]}]];
AutomaticRules[\[Delta]h, MakeRule[{\[Delta]h[i, j
    ],
    1/a[] (-n[i] n[j] As[]
       - n[i] CD[k][Cs[]] (delta[-k, j] - n[-k] n[j
           ])
       - n[j] CD[k][Bs[]] (delta[-k, i] - n[-k] n[i
           ])
       -
       delta[k,
         l] (delta[i, -k] - n[i] n[-k]) (delta[j, -
            l] -
          n[j] n[-l]) Ds[]
       - (delta[i, -k] - n[i] n[-k]) (delta[j, -l]
           -
          n[j] n[-l]) (CD[k][CD[l][Es[]]] -
          1/3 delta[k, l] CD[p][
            CD[q][Es[]]] (delta[-p, -q] - n[-p] n[-
                q]))
       + (delta[i, -m] - n[i] n[-m]) (delta[j, -p]
           -
          n[j] n[-p]) (delta[s, -q] - n[s] n[-q])
               epsilong[m, p,
         q, -r] n[r] CD[-s][Fs[]]) }]];
```

```
(*Output full first order, un-gauged spin
    connection*)
CollectTensors[Out[52]]

(*Redefining the (expanded) spin connection as the
    above expression \
[to save computing time]*)
DefTensor[expA[i, j, k], M];
AutomaticRules[expA,
  MakeRule[{expA[-i, -j, -k], -g[-j, -k] H[] n[-i]
      +
     g[-i, -k] H[] n[-j] + As[] g[-j, -k] H[] n[-i]
          \[Epsilon][] -
     Ds[] g[-j, -k] H[] n[-i] \[Epsilon][] -
     As[] g[-i, -k] H[] n[-j] \[Epsilon][] +
     Ds[] g[-i, -k] H[] n[-j] \[Epsilon][] -
     n[-j] n[-k] \[Epsilon][] CD[-i][As[]] +
     g[-j, -k] H[] \[Epsilon][] CD[-i][Bs[]] +
     H[] n[-j] n[-k] \[Epsilon][] CD[-i][Cs[]] -
     g[-j, -k] \[Epsilon][] CD[-i][Ds[]] +
     n[-j] n[-k] \[Epsilon][] CD[-i][Ds[]] -
     g[-j, -k] \[Epsilon][] CD[-i][V[]] +
     1/2 n[-j] n[-k] \[Epsilon][] CD[-i][\[Alpha
         ][]] +
     n[-i] n[-k] \[Epsilon][] CD[-j][As[]] -
     g[-i, -k] H[] \[Epsilon][] CD[-j][Bs[]] -
     H[] n[-i] n[-k] \[Epsilon][] CD[-j][Cs[]] +
     g[-i, -k] \[Epsilon][] CD[-j][Ds[]] -
     n[-i] n[-k] \[Epsilon][] CD[-j][Ds[]] +
     g[-i, -k] \[Epsilon][] CD[-j][V[]] -
     1/2 n[-i] n[-k] \[Epsilon][] CD[-j][\[Alpha
         ][]] +
     1/2 n[-k] \[Epsilon][] CD[-j][CD[-i][W[]]] +
     1/2 n[-i] n[-j] \[Epsilon][] CD[-k][\[Alpha
         ][]] -
     n[-j] \[Epsilon][] CD[-k][CD[-i][Cs[]]] +
     H[] n[-j] \[Epsilon][] CD[-k][CD[-i][Es[]]] +
     1/2 n[-j] \[Epsilon][] CD[-k][CD[-i][W[]]] +
     n[-i] \[Epsilon][] CD[-k][CD[-j][Cs[]]] -
     H[] n[-i] \[Epsilon][] CD[-k][CD[-j][Es[]]] -
     1/2 n[-i] \[Epsilon][] CD[-k][CD[-j][W[]]] -
     g[-j, -k] H[] n[-i] n[l] \[Epsilon][] CD[-l][
         Bs[]] +
     g[-i, -k] H[] n[-j] n[l] \[Epsilon][] CD[-l][
         Bs[]] -
     g[-j, -k] n[-i] n[l] \[Epsilon][] CD[-l][U[]]
         +
     g[-i, -k] n[-j] n[l] \[Epsilon][] CD[-l][U[]]
         +
     g[-j, -k] n[-i] n[l] \[Epsilon][] CD[-l][V[]]
         -
     g[-i, -k] n[-j] n[l] \[Epsilon][] CD[-l][V[]]
         -
     1/2 epsilong[-i, -j, -k, -m] n[l] n[m] \[
         Epsilon][]
      CD[-l][X[]] + 1/2 epsilong[-i, -j, -k, -m] n[
          l] n[m] \[Epsilon][]
      CD[-l][Y[]] -
     1/2 n[-i] n[-j] n[-k] n[l] \[Epsilon][] CD[-l
         ][\[Alpha][]] +
     n[-j] n[-k] n[l] \[Epsilon][] CD[-l][CD[-i][Bs
         []]] +
     n[-j] n[-k] n[l] \[Epsilon][] CD[-l][CD[-i][Cs
         []]] -
     H[] n[-j] n[-k] n[l] \[Epsilon][] CD[-l][CD[-i
         ][Es[]]] -
     n[-j] n[-k] n[l] \[Epsilon][] CD[-l][CD[-i][W
         []]] -
     n[-i] n[-k] n[l] \[Epsilon][] CD[-l][CD[-j][Bs
         []]] -
     n[-i] n[-k] n[l] \[Epsilon][] CD[-l][CD[-j][Cs
         []]] +
     H[] n[-i] n[-k] n[l] \[Epsilon][] CD[-l][CD[-j
         ][Es[]]] +
     n[-j] n[l] \[Epsilon][] CD[-l][CD[-k][CD[-i][
         Es[]]]] -
     n[-i] n[l] \[Epsilon][] CD[-l][CD[-k][CD[-j][
         Es[]]]] +
     1/3 g[-j, -k] H[] n[-i] \[Epsilon][] CD[-l][CD
         [l][Es[]]] -
     1/3 g[-i, -k] H[] n[-j] \[Epsilon][] CD[-l][CD
         [l][Es[]]] +
```

```
      1/6 g[-j, -k] n[-i] \[Epsilon][] CD[-l][CD[l][
          W[]]] -
      1/6 g[-i, -k] n[-j] \[Epsilon][] CD[-l][CD[l][
          W[]]] -
      1/6 g[-i, -j] n[-k] \[Epsilon][] CD[-l][CD[l][
          W[]]] +
      1/6 n[-i] n[-j] n[-k] \[Epsilon][] CD[-l][CD[l
          ][W[]]] +
      1/3 g[-j, -k] \[Epsilon][] CD[-l][CD[l][CD[-i
          ][Es[]]]] -
      1/3 n[-j] n[-k] \[Epsilon][] CD[-l][CD[l][CD[-
          i][Es[]]]] -
      1/3 g[-i, -k] \[Epsilon][] CD[-l][CD[l][CD[-j
          ][Es[]]]] +
      1/3 n[-i] n[-k] \[Epsilon][] CD[-l][CD[l][CD[-
          j][Es[]]]] -
      1/2 epsilong[-i, -j, -k, -l] \[Epsilon][] CD[l
          ][Y[]] -
      1/3 g[-j, -k] H[] n[-i] n[l] n[m] \[Epsilon][]
           CD[-m][
        CD[-l][Es[]]] +
      1/3 g[-i, -k] H[] n[-j] n[l] n[m] \[Epsilon][]
           CD[-m][
        CD[-l][Es[]]] -
      1/6 g[-j, -k] n[-i] n[l] n[m] \[Epsilon][] CD
          [-m][CD[-l][W[]]] +
      1/6 g[-i, -k] n[-j] n[l] n[m] \[Epsilon][] CD
          [-m][CD[-l][W[]]] +
      1/6 g[-i, -j] n[-k] n[l] n[m] \[Epsilon][] CD
          [-m][CD[-l][W[]]] +
      1/3 n[-i] n[-j] n[-k] n[l] n[m] \[Epsilon][]
          CD[-m][
        CD[-l][W[]]] -
      1/3 g[-j, -k] n[l] n[m] \[Epsilon][] CD[-m][
        CD[-l][CD[-i][Es[]]]] -
      2/3 n[-j] n[-k] n[l] n[m] \[Epsilon][] CD[-m][
        CD[-l][CD[-i][Es[]]]] +
      1/3 g[-i, -k] n[l] n[m] \[Epsilon][] CD[-m][
        CD[-l][CD[-j][Es[]]]] +
      2/3 n[-i] n[-k] n[l] n[m] \[Epsilon][] CD[-m][
        CD[-l][CD[-j][Es[]]]] +
      epsilong[-j, -k, -l, -m] H[] n[-i] n[l] \[
          Epsilon][] CD[m][
        Fs[]] - epsilong[-i, -k, -l, -m] H[] n[-j] n
             [
        l] \[Epsilon][] CD[m][Fs[]] -
      1/2 epsilong[-j, -k, -l, -m] n[-i] n[
        l] \[Epsilon][] CD[m][\[Beta][]] +
      1/2 epsilong[-i, -k, -l, -m] n[-j] n[
        l] \[Epsilon][] CD[m][\[Beta][]] +
      1/2 epsilong[-i, -j, -l, -m] n[-k] n[
        l] \[Epsilon][] CD[m][\[Beta][]] -
      epsilong[-j, -k, -l, -m] n[l] \[Epsilon][] CD[
          m][CD[-i][Z[]]] +
      epsilong[-i, -j, -l, -m] n[l] \[Epsilon][] CD[
          m][CD[-k][Fs[]]] +
      epsilong[-i, -j, -l, -m] n[l] \[Epsilon][] CD[
          m][CD[-k][Z[]]] +
      epsilong[-j, -k, -m, -p] n[-i] n[l] n[m] \[
          Epsilon][]
       CD[p][CD[-l][Z[]]] -
      epsilong[-i, -j, -m, -p] n[-k] n[l] n[m] \[
          Epsilon][]
       CD[p][CD[-l][Z[]]]}]];


(*Decomposition of ungauged A into spin - parity
    modes*)
(*Can now decompose this into the appropriate modes
    *)

        (*0+ spin-parity mode*)
DefTensor[plus0[], M];
AutomaticRules[plus0,
  MakeRule[{plus0[],
    expA[i, j, -k] n[-j] (delta[l, -i] - n[l] n[-i
        ]) (delta[k, -l] -
       n[k] n[-l])}]];
CollectTensors[plus0[]]

        (*0- spin-parity mode*)
DefTensor[minus0[], M];
AutomaticRules[minus0,
  MakeRule[{minus0[], expA[i, j, -k] epsilong[l, k,
       -i, -j] n[-l]}]];
CollectTensors[minus0[]]

        (*1+ spin-parity mode #1*)
DefTensor[plus1n1[i, j], M];
AutomaticRules[plus1n1,
  MakeRule[{plus1n1[i, j],
    expA[k, l, -m] n[
      m] (delta[i, -k] - n[i] n[-k]) (delta[-l, j]
          - n[-l] n[j])}]];
CollectTensors[plus1n1[i, j]]

        (*1- spin-parity mode #1*)
DefTensor[minus1n1[i], M];
AutomaticRules[minus1n1, MakeRule[{minus1n1[i],
    expA[j, k, -l]
     (delta[i, -k] - n[i] n[-k])
     (delta[l, -j] - n[l] n[-j])}]];
CollectTensors[minus1n1[i]]

        (*1+ spin-parity mode #2*)
DefTensor[plus1n2[i, j], M];
AutomaticRules[plus1n2,
  MakeRule[{plus1n2[-j, -k],
    1/2 n[i] (expA[-i, -l, -m] - expA[-i, -m, -l])
        (delta[-j, l] -
       n[-j] n[l]) (delta[-k, m] - n[-k] n[m])}]];
CollectTensors[plus1n2[-i, -j]]

        (*1- spin-parity mode #2*)
DefTensor[minus1n2[i], M];
AutomaticRules[minus1n2,
  MakeRule[{minus1n2[i],
    expA[l, j, -k] n[-j] n[k] (delta[-l, i] - n[-l]
         n[i])}]];
CollectTensors[minus1n2[i]]

        (*2+ spin-parity mode*)
DefTensor[plus2[i, j], M];
AutomaticRules[plus2,
  MakeRule[{plus2[-i, -j],
    1/2 (expA[-k, -l, -m] + expA[-k, -m, -l]) n[
       k] (delta[-i, l] - n[-i] n[l]) (delta[-j, m]
           - n[-j] n[m])
     + 1/
      3 delta[-k, -l] (delta[-i, k] - n[-i] n[k]) (
          delta[-j, l] -
       n[-j] n[l]) plus0[]}]];
CollectTensors[plus2[-i, -j]]

        (*2- spin-parity mode [Not needed in
            determining gauge invariant quantites
            ]*)
DefTensor[minus2[i, j, k], M];
AutomaticRules[minus2,
  MakeRule[{minus2[-i, -j, -k],
    1/2 (expA[-l, -m, -p] + expA[-l, -p, -m]) (
        delta[l, -i] -
       n[l] n[-i]) (delta[m, -j] - n[m] n[-j]) (
            delta[p, -k] -
       n[p] n[-k])
     - 1/
      3 delta[-m, -p] (delta[m, -i] - n[m] n[-i]) (
          delta[p, -k] -
       n[p] n[-k]) minus1n1[-j]
     - 1/6 epsilong[-i, -j, -k, l] n[-l] minus0
         []}]];
CollectTensors[minus2[-i, -j, -k]]



(*Gauging the spin connection*)

(*Rules to select zeroth and first order parts of
    the spin connection*)
Select0Order = MakeRule[{\[Epsilon][], 0}];
Select1Order = MakeRule[{\[Epsilon][], 1}];

(*0th order part of spin connection*)
expA[-i, -j, -k] /. Select0Order

(*First order part of spin connection*)
```

```
expA[-i, -j, -k] - % /. Select1Order

(*Splitting of the spin connection into background
    and perturbed \
parts for use in the gauge transformation*)
DefTensor[A0[i, j, k], M];
DefTensor[\[Delta]A[i, j, k], M];
DefTensor[gaugeA[i, j, k], M];

(*Defining guage transformation*)
DefTensor[\[Xi][i], M];
DefTensor[R[i, j], M];
AutomaticRules[gaugeA,
  MakeRule[{gaugeA[i, j, -k],
    A0[i, j, -k] + \[Epsilon][] (\[Delta]A[i, j, -k
        ] +
        A0[i, j, -l] CD[-k][\[Xi][l]] + R[j, -l] A0
            [i, l, -k] +
        R[i, -l] A0[l, j, -k] - CD[-k][R[i, j]])
            }]];

(*Decomposition of the gauge transformation into
    associated gauge \
angles*)
        (*Decomposition of Jacobian-associated part
            into scalars*)
DefTensor[T[], M];
DefTensor[L[], M];
AutomaticRules[\[Xi],
  MakeRule[{\[Xi][i],
    n[i] T[] + (delta[i, -j] - n[i] n[-j]) CD[j][L
        []]}]];
        (*Decomposition of Lorentz rotation-
            associated part into scalars*)
DefTensor[P[], M];
DefTensor[Q[], M];
AutomaticRules[R,
  MakeRule[{R[-i, -j], (n[-i] CD[-k][
        P[]] (delta[k, -j] - n[k] n[-j]) -
      n[-j] CD[-k][P[]] (delta[k, -i] - n[k] n[-i
          ])) + (delta[
        m, -i] - n[m] n[-i])
      (delta[p, -j] - n[p] n[-j])
      (delta[k, -q] - n[k] n[-q])
      (delta[s, -k] - n[s] n[-k])
      epsilong[-m, -p, q, r] n[-r] CD[-s][Q[]]}]];

(*Expanding the guage transformation in terms of
    the background spin connection, the
    perturbation to the spin connection and the
    gauge angles*)
CollectTensors[gaugeA[i, j, -k]] /. Linearise1 /.
    Linearise2 /.
  Linearise3 /. LineariseDiv

(*Expansion of the background and perturbed spin
    connection in terms of the tetrad and torsion
    scalars [A,...,F and \[Tau],..,\[Lambda]] --
    using the previously found expressions for A0
    and \[Delta]A*)
AutomaticRules[A0,
  MakeRule[{A0[-i, -j, -k], -g[-k, -j] H[] n[-i] +
      g[-k, -i] H[] n[-j]}]];
AutomaticRules[\[Delta]A,
  MakeRule[{\[Delta]A[-i, -j, -k],
    As[] g[-k, -j] H[] n[-i] - Ds[] g[-k, -j] H[] n
        [-i] -
     As[] g[-k, -i] H[] n[-j] + Ds[] g[-k, -i] H[]
         n[-j] -
     n[-j] n[-k] CD[-i][As[]] + g[-k, -j] H[] CD[-i
         ][Bs[]] +
     H[] n[-j] n[-k] CD[-i][Cs[]] - g[-k, -j] CD[-i
         ][Ds[]] +
     n[-j] n[-k] CD[-i][Ds[]] - g[-k, -j] CD[-i][V
         []] +
     1/2 n[-j] n[-k] CD[-i][\[Alpha][]] + n[-i] n[-
         k] CD[-j][As[]] -
     g[-k, -i] H[] CD[-j][Bs[]] - H[] n[-i] n[-k]
         CD[-j][Cs[]] +
     g[-k, -i] CD[-j][Ds[]] - n[-i] n[-k] CD[-j][Ds
         []] +
     g[-k, -i] CD[-j][V[]] - 1/2 n[-i] n[-k] CD[-j
         ][\[Alpha][]] +
     1/2 n[-k] CD[-j][CD[-i][W[]]] +
     1/2 n[-i] n[-j] CD[-k][\[Alpha][]] - n[-j] CD
         [-k][CD[-i][Cs[]]] +
      H[] n[-j] CD[-k][CD[-i][Es[]]] + 1/2 n[-j] CD
         [-k][CD[-i][W[]]] +
      n[-i] CD[-k][CD[-j][Cs[]]] - H[] n[-i] CD[-k
         ][CD[-j][Es[]]] -
     1/2 n[-i] CD[-k][CD[-j][W[]]] -
     g[-k, -j] H[] n[-i] n[l] CD[-l][Bs[]] +
     g[-k, -i] H[] n[-j] n[l] CD[-l][Bs[]] -
     g[-k, -j] n[-i] n[l] CD[-l][U[]] +
     g[-k, -i] n[-j] n[l] CD[-l][U[]] +
     g[-k, -j] n[-i] n[l] CD[-l][V[]] -
     g[-k, -i] n[-j] n[l] CD[-l][V[]] -
     1/2 epsilong[-i, -j, -k, -m] n[l] n[m] CD[-l][
         X[]] +
     1/2 epsilong[-i, -j, -k, -m] n[l] n[m] CD[-l][
         Y[]] -
     1/2 n[-i] n[-j] n[-k] n[l] CD[-l][\[Alpha][]]
         +
     n[-j] n[-k] n[l] CD[-l][CD[-i][Bs[]]] +
     n[-j] n[-k] n[l] CD[-l][CD[-i][Cs[]]] -
     H[] n[-j] n[-k] n[l] CD[-l][CD[-i][Es[]]] -
     n[-j] n[-k] n[l] CD[-l][CD[-i][W[]]] -
     n[-i] n[-k] n[l] CD[-l][CD[-j][Bs[]]] -
     n[-i] n[-k] n[l] CD[-l][CD[-j][Cs[]]] +
     H[] n[-i] n[-k] n[l] CD[-l][CD[-j][Es[]]] +
     n[-j] n[l] CD[-l][CD[-k][CD[-i][Es[]]]] -
     n[-i] n[l] CD[-l][CD[-k][CD[-j][Es[]]]] +
     1/3 g[-k, -j] H[] n[-i] CD[-l][CD[l][Es[]]] -
     1/3 g[-k, -i] H[] n[-j] CD[-l][CD[l][Es[]]] +
     1/6 g[-k, -j] n[-i] CD[-l][CD[l][W[]]] -
     1/6 g[-k, -i] n[-j] CD[-l][CD[l][W[]]] -
     1/6 g[-j, -i] n[-k] CD[-l][CD[l][W[]]] +
     1/6 n[-i] n[-j] n[-k] CD[-l][CD[l][W[]]] +
     1/3 g[-k, -j] CD[-l][CD[l][CD[-i][Es[]]]] -
     1/3 n[-j] n[-k] CD[-l][CD[l][CD[-i][Es[]]]] -
     1/3 g[-k, -i] CD[-l][CD[l][CD[-j][Es[]]]] +
     1/3 n[-i] n[-k] CD[-l][CD[l][CD[-j][Es[]]]] -
     1/2 epsilong[-i, -j, -k, -l] CD[l][Y[]] -
     1/3 g[-k, -j] H[] n[-i] n[l] n[m] CD[-m][CD[-l
         ][Es[]]] +
     1/3 g[-k, -i] H[] n[-j] n[l] n[m] CD[-m][CD[-l
         ][Es[]]] -
     1/6 g[-k, -j] n[-i] n[l] n[m] CD[-m][CD[-l][W
         []]] +
     1/6 g[-k, -i] n[-j] n[l] n[m] CD[-m][CD[-l][W
         []]] +
     1/6 g[-j, -i] n[-k] n[l] n[m] CD[-m][CD[-l][W
         []]] +
     1/3 n[-i] n[-j] n[-k] n[l] n[m] CD[-m][CD[-l][
        W[]]] -
     1/3 g[-k, -j] n[l] n[m] CD[-m][CD[-l][CD[-i][
         Es[]]]] -
     2/3 n[-j] n[-k] n[l] n[m] CD[-m][CD[-l][CD[-i
         ][Es[]]]] +
     1/3 g[-k, -i] n[l] n[m] CD[-m][CD[-l][CD[-j][
         Es[]]]] +
     2/3 n[-i] n[-k] n[l] n[m] CD[-m][CD[-l][CD[-j
         ][Es[]]]] +
     epsilong[-j, -k, -l, -m] H[] n[-i] n[l] CD[m][
         Fs[]] -
     epsilong[-i, -k, -l, -m] H[] n[-j] n[l] CD[m][
         Fs[]] -
     1/2 epsilong[-j, -k, -l, -m] n[-i] n[l] CD[m
         ][\[Beta][]] +
     1/2 epsilong[-i, -k, -l, -m] n[-j] n[l] CD[m
         ][\[Beta][]] +
     1/2 epsilong[-i, -j, -l, -m] n[-k] n[l] CD[m
         ][\[Beta][]] -
     epsilong[-j, -k, -l, -m] n[l] CD[m][CD[-i][Z
         []]] +
     epsilong[-i, -j, -l, -m] n[l] CD[m][CD[-k][Fs
         []]] +
     epsilong[-i, -j, -l, -m] n[l] CD[m][CD[-k][Z
         []]] +
     epsilong[-j, -k, -m, -p] n[-i] n[l] n[m] CD[p
         ][CD[-l][Z[]]] -
     epsilong[-i, -j, -m, -p] n[-k] n[l] n[m] CD[p
         ][CD[-l][Z[]]]}]];

(*Guage-transformed spin connection fully expanded
    in terms of scalars*)
```

```
CollectTensors[Out[109]]

(*Once again, redefining the transformed spin
    connection as the above expression [to save
    computing time]*)
DefTensor[ExpGaugeA[i, j, k], M];
AutomaticRules[ExpGaugeA,
  MakeRule[{ExpGaugeA[i, j, -k], -xAct`xTensor`
      delta[j, -k] H[] n[i] +
      xAct`xTensor`delta[i, -k] H[] n[j] +
     As[] xAct`xTensor`delta[j, -k] H[] n[i] \[
         Epsilon][] -
     xAct`xTensor`delta[j, -k] Ds[] H[] n[i] \[
         Epsilon][] -
     As[] xAct`xTensor`delta[i, -k] H[] n[j] \[
         Epsilon][] +
     xAct`xTensor`delta[i, -k] Ds[] H[] n[j] \[
         Epsilon][] -
     n[j] n[-k] \[Epsilon][] CD[i][As[]] +
     xAct`xTensor`delta[j, -k] H[] \[Epsilon][] CD[
         i][Bs[]] +
     H[] n[j] n[-k] \[Epsilon][] CD[i][Cs[]] -
     xAct`xTensor`delta[j, -k] \[Epsilon][] CD[i][
         Ds[]] +
     n[j] n[-k] \[Epsilon][] CD[i][Ds[]] +
     xAct`xTensor`delta[j, -k] H[] \[Epsilon][] CD[
         i][P[]] -
     H[] n[j] n[-k] \[Epsilon][] CD[i][P[]] -
     xAct`xTensor`delta[j, -k] \[Epsilon][] CD[i][V
         []] +
     1/2 n[j] n[-k] \[Epsilon][] CD[i][\[Alpha][]]
         +
     n[i] n[-k] \[Epsilon][] CD[j][As[]] -
     xAct`xTensor`delta[i, -k] H[] \[Epsilon][] CD[
         j][Bs[]] -
     H[] n[i] n[-k] \[Epsilon][] CD[j][Cs[]] +
     xAct`xTensor`delta[i, -k] \[Epsilon][] CD[j][
         Ds[]] -
     n[i] n[-k] \[Epsilon][] CD[j][Ds[]] -
     xAct`xTensor`delta[i, -k] H[] \[Epsilon][] CD[
         j][P[]] +
     H[] n[i] n[-k] \[Epsilon][] CD[j][P[]] +
     xAct`xTensor`delta[i, -k] \[Epsilon][] CD[j][V
         []] -
     1/2 n[i] n[-k] \[Epsilon][] CD[j][\[Alpha][]]
         +
     1/2 n[-k] \[Epsilon][] CD[j][CD[i][W[]]] +
     1/2 n[i] n[j] \[Epsilon][] CD[-k][\[Alpha][]]
         -
     n[j] \[Epsilon][] CD[-k][CD[i][Cs[]]] +
     H[] n[j] \[Epsilon][] CD[-k][CD[i][Es[]]] +
     H[] n[j] \[Epsilon][] CD[-k][CD[i][L[]]] +
     n[j] \[Epsilon][] CD[-k][CD[i][P[]]] +
     1/2 n[j] \[Epsilon][] CD[-k][CD[i][W[]]] +
     n[i] \[Epsilon][] CD[-k][CD[j][Cs[]]] -
     H[] n[i] \[Epsilon][] CD[-k][CD[j][Es[]]] -
     H[] n[i] \[Epsilon][] CD[-k][CD[j][L[]]] -
     n[i] \[Epsilon][] CD[-k][CD[j][P[]]] -
     1/2 n[i] \[Epsilon][] CD[-k][CD[j][W[]]] -
     xAct`xTensor`delta[j, -k] H[] n[i] n[l] \[
         Epsilon][]
      CD[-l][Bs[]] +
     xAct`xTensor`delta[i, -k] H[] n[j] n[l] \[
         Epsilon][]
      CD[-l][Bs[]] -
     xAct`xTensor`delta[j, -k] H[] n[i] n[l] \[
         Epsilon][]
      CD[-l][P[]] +
     xAct`xTensor`delta[i, -k] H[] n[j] n[l] \[
         Epsilon][]
      CD[-l][P[]] -
     xAct`xTensor`delta[j, -k] n[i] n[l] \[Epsilon
         ][] CD[-l][U[]] +
     xAct`xTensor`delta[i, -k] n[j] n[l] \[Epsilon
         ][] CD[-l][U[]] +
     xAct`xTensor`delta[j, -k] n[i] n[l] \[Epsilon
         ][] CD[-l][V[]] -
     xAct`xTensor`delta[i, -k] n[j] n[l] \[Epsilon
         ][] CD[-l][V[]] -
     1/2 epsilong[i, j, -k, -m] n[l] n[m] \[Epsilon
         ][] CD[-l][X[]] +
     1/2 epsilong[i, j, -k, -m] n[l] n[m] \[Epsilon
         ][] CD[-l][Y[]] -
     1/2 n[i] n[j] n[-k] n[l] \[Epsilon][] CD[-l
         ][\[Alpha][]] +
     n[j] n[-k] n[l] \[Epsilon][] CD[-l][CD[i][Bs
         []]] +
     n[j] n[-k] n[l] \[Epsilon][] CD[-l][CD[i][Cs
         []]] -
     H[] n[j] n[-k] n[l] \[Epsilon][] CD[-l][CD[i][
         Es[]]] -
     n[j] n[-k] n[l] \[Epsilon][] CD[-l][CD[i][W
         []]] -
     n[i] n[-k] n[l] \[Epsilon][] CD[-l][CD[j][Bs
         []]] -
     n[i] n[-k] n[l] \[Epsilon][] CD[-l][CD[j][Cs
         []]] +
     H[] n[i] n[-k] n[l] \[Epsilon][] CD[-l][CD[j][
         Es[]]] +
     n[j] n[l] \[Epsilon][] CD[-l][CD[-k][CD[i][Es
         []]]] -
     n[i] n[l] \[Epsilon][] CD[-l][CD[-k][CD[j][Es
         []]]] +
     1/3 xAct`xTensor`delta[j, -k] H[] n[i] \[
         Epsilon][] CD[-l][
       CD[l][Es[]]] -
     1/3 xAct`xTensor`delta[i, -k] H[] n[j] \[
         Epsilon][] CD[-l][
       CD[l][Es[]]] +
     1/6 xAct`xTensor`delta[j, -k] n[i] \[Epsilon
         ][] CD[-l][
       CD[l][W[]]] -
     1/6 xAct`xTensor`delta[i, -k] n[j] \[Epsilon
         ][] CD[-l][
       CD[l][W[]]] -
     1/6 g[i, j] n[-k] \[Epsilon][] CD[-l][CD[l][W
         []]] +
     1/6 n[i] n[j] n[-k] \[Epsilon][] CD[-l][CD[l][
        W[]]] +
     1/3 xAct`xTensor`delta[j, -k] \[Epsilon][] CD
         [-l][
       CD[l][CD[i][Es[]]]] -
     1/3 n[j] n[-k] \[Epsilon][] CD[-l][CD[l][CD[i
         ][Es[]]]] -
     1/3 xAct`xTensor`delta[i, -k] \[Epsilon][] CD
         [-l][
       CD[l][CD[j][Es[]]]] +
     1/3 n[i] n[-k] \[Epsilon][] CD[-l][CD[l][CD[j
         ][Es[]]]] -
     1/2 epsilong[i, j, -k, -l] \[Epsilon][] CD[l][
         Y[]] -
     1/3 xAct`xTensor`delta[j, -k] H[] n[i] n[l] n[
       m] \[Epsilon][] CD[-m][CD[-l][Es[]]] +
     1/3 xAct`xTensor`delta[i, -k] H[] n[j] n[l] n[
         m] \[Epsilon][]
      CD[-m][CD[-l][Es[]]] -
     1/6 xAct`xTensor`delta[j, -k] n[i] n[l] n[m]
         \[Epsilon][]
      CD[-m][CD[-l][W[]]] +
     1/6 xAct`xTensor`delta[i, -k] n[j] n[l] n[m]
         \[Epsilon][] CD[-m][
       CD[-l][W[]]] +
     1/6 g[i, j] n[-k] n[l] n[m] \[Epsilon][] CD[-m
         ][CD[-l][W[]]] +
     1/3 n[i] n[j] n[-k] n[l] n[m] \[Epsilon][] CD
         [-m][CD[-l][W[]]] -
     1/3 xAct`xTensor`delta[j, -k] n[l] n[m] \[
         Epsilon][]
      CD[-m][CD[-l][CD[i][Es[]]]] -
     2/3 n[j] n[-k] n[l] n[m] \[Epsilon][] CD[-m][
       CD[-l][CD[i][Es[]]]] +
     1/3 xAct`xTensor`delta[i, -k] n[l] n[m] \[
         Epsilon][] CD[-m][
       CD[-l][CD[j][Es[]]]] +
     2/3 n[i] n[-k] n[l] n[m] \[Epsilon][] CD[-m][
       CD[-l][CD[j][Es[]]]] +
     epsilong[j, -k, -l, -m] H[] n[i] n[l] \[
         Epsilon][] CD[m][Fs[]] -
     epsilong[i, -k, -l, -m] H[] n[j] n[l] \[
         Epsilon][] CD[m][Fs[]] +
     epsilong[j, -k, -l, -m] H[] n[i] n[l] \[
         Epsilon][] CD[m][Q[]] -
     epsilong[i, -k, -l, -m] H[] n[j] n[l] \[
         Epsilon][] CD[m][Q[]] -
     1/2 epsilong[j, -k, -l, -m] n[i] n[
       l] \[Epsilon][] CD[m][\[Beta][]] +
```

```
        1/2 epsilong[i, -k, -l, -m] n[j] n[
          l] \[Epsilon][] CD[m][\[Beta][]] +
        1/2 epsilong[i, j, -l, -m] n[-k] n[
          l] \[Epsilon][] CD[m][\[Beta][]] -
        epsilong[j, -k, -l, -m] n[l] \[Epsilon][] CD[m
             ][CD[i][Z[]]] +
        epsilong[i, j, -l, -m] n[l] \[Epsilon][] CD[m
             ][CD[-k][Fs[]]] +
        epsilong[i, j, -l, -m] n[l] \[Epsilon][] CD[m
             ][CD[-k][Q[]]] +
        epsilong[i, j, -l, -m] n[l] \[Epsilon][] CD[m
             ][CD[-k][Z[]]] +
        epsilong[j, -k, -m, -p] n[i] n[l] n[m] \[
             Epsilon][] CD[p][
           CD[-l][Z[]]] -
        epsilong[i, j, -m, -p] n[-k] n[l] n[m] \[
             Epsilon][] CD[p][
           CD[-l][Z[]]]}]];


(*Decomposition of gauged A into spin - parity
     modes*)
          (*0+ spin-parity mode*)
DefTensor[Gplus0[], M];
AutomaticRules[Gplus0,
  MakeRule[{Gplus0[],
    ExpGaugeA[i,
      j, -k] n[-j] (delta[l, -i] - n[l] n[-i]) (
          delta[k, -l] -
       n[k] n[-l])}]];
CollectTensors[Gplus0[]]

          (*0- spin-parity mode*)
DefTensor[Gminus0[], M];
AutomaticRules[Gminus0,
  MakeRule[{Gminus0[],
    ExpGaugeA[i, j, -k] epsilong[l, k, -i, -j] n[-l
         ]}]];
CollectTensors[Gminus0[]]

          (*1+ spin-parity mode #1*)
DefTensor[Gplus1n1[i, j], M];
AutomaticRules[Gplus1n1,
  MakeRule[{Gplus1n1[i, j],
    ExpGaugeA[k, l, -m] n[
      m] (delta[i, -k] - n[i] n[-k]) (delta[-l, j]
          - n[-l] n[j])}]];
CollectTensors[Gplus1n1[i, j]]

          (*1- spin-parity mode #1*)
DefTensor[Gminus1n1[i], M];
AutomaticRules[Gminus1n1, MakeRule[{Gminus1n1[i],
     ExpGaugeA[j, k, -l]
       (delta[i, -k] - n[i] n[-k])
       (delta[l, -j] - n[l] n[-j])}]];
CollectTensors[Gminus1n1[i]]

          (*1+ spin-parity mode #2*)
DefTensor[Gplus1n2[i, j], M];
AutomaticRules[Gplus1n2,
  MakeRule[{Gplus1n2[-j, -k],
    1/2 n[i] (ExpGaugeA[-i, -l, -m] -
         ExpGaugeA[-i, -m, -l]) (delta[-j, l] -
         n[-j] n[l]) (delta[-k, m] - n[-k] n[m])}]];
CollectTensors[Gplus1n2[-i, -j]]

          (*1- spin-parity mode #2*)
DefTensor[Gminus1n2[i], M];
AutomaticRules[Gminus1n2,
  MakeRule[{Gminus1n2[i], ExpGaugeA[i, j, -k] n[-j]
         n[k]},
    MetricOn -> All, ContractMetrics -> True]];
CollectTensors[Gminus1n2[i]]

          (*2+ spin-parity mode*)
DefTensor[Gplus2[i, j], M];
AutomaticRules[Gplus2,
  MakeRule[{Gplus2[-i, -j],
    1/2 (ExpGaugeA[-k, -l, -m] + ExpGaugeA[-k, -m,
          -l]) n[
         k] (delta[-i, l] - n[-i] n[l]) (delta[-j, m]
          - n[-j] n[m]) +
      1/3 delta[-k, -l] (delta[-i, k] - n[-i] n[k])
           (delta[-j, l] -
          n[-j] n[l]) Gplus0[]}]];
CollectTensors[Gplus2[-i, -j]]

(*2- spin-parity mode*)
DefTensor[Gminus2[i, j, k], M];
AutomaticRules[Gminus2,
  MakeRule[{Gminus2[-i, -j, -k],
    1/2 (ExpGaugeA[-l, -m, -p] +
          ExpGaugeA[-l, -p, -m]) (delta[l, -i] -
          n[l] n[-i]) (delta[m, -j] - n[m] n[-j]) (
              delta[p, -k] -
          n[p] n[-k])
      - 1/
        3 delta[-m, -p] (delta[m, -i] - n[m] n[-i]) (
            delta[p, -k] -
          n[p] n[-k]) Gminus1n1[-j]
      - 1/6 epsilong[-i, -j, -k, l] n[-l] Gminus0
            []}]];
CollectTensors[Gminus2[-i, -j, -k]]
```

3. Formulation of the Field Equations

The script below is associated with calculations contained within section IV A. The calculations contained within section IV B are carried out using a similar script to this. However, their lengthy nature is not included here due to their similarity with the script below.

```
(*Loading the xAct packages required for tensor
     manipulation*)
<< xAct`xTras`
(*Loading the TexAct package that allows for
     outputs to be given directly in LaTeX format*)
<< xAct`TexAct`
$PrePrint = ScreenDollarIndices;
(*Deinition of base manifold and metric -- taken to
       be flat as curvature and torsion are being
     manually added*)
DefManifold[M, 4, {i, j, k, l, m, p, q, r, s, t}];
DefMetric[-1, g[-i, -j], CD, {",", "\[PartialD]"},
  PrintAs -> "\[Eta]", FlatMetric -> True];
(*Canonically ordering CDs*)
SortCovDsStart[CD];

(*Definition of the time vector*)
DefTensor[n[i], M];
(*Setting n[i] to be normal*)
AutomaticRules[n, MakeRule[{n[i] n[-i], 1}]];
(*Setting the derivative of n[i] to zero*)
AutomaticRules[n, MakeRule[{CD[i][n[j]], 0}]];


(*Creation of the spin connection*)

(*Defining general spin connection*)
DefTensor[\[CapitalDelta][i, j, k], M];
DefTensor[K[i, j, k], M];
DefTensor[A[i, j, k], M];
AutomaticRules[A,
  MakeRule[{A[i, j, k], \[CapitalDelta][i, j, k] +
       K[i, j, k]}]];
A[-i, -j, -k]

(*Defining the tetrad and its inverse*)
```

```
DefTensor[b[i, j], M];
DefTensor[h[i, j], M];
(*Outlining the identity relationships of b and h*)
AutomaticRules[b, MakeRule[{b[i, -j] h[-i, k],
    delta[-j, k]}]];
AutomaticRules[b, MakeRule[{b[i, -j] h[-k, j],
    delta[i, -k]}]];
DefTensor[c[i, j, k], M];
AutomaticRules[\[CapitalDelta],
  MakeRule[{\[CapitalDelta][-i, -j, -k],
    1/2 (delta[-i, -l] c[l, -m, -p] h[-j, m] h[-q,
        p] -
       delta[-q, -l] c[l, -m, -p] h[-i, m] h[-j, p]
          +
       delta[-j, -l] c[l, -m, -p] h[-q, m] h[-i, p
          ]) b[q, -k]}]];
(*Expanding the RRC in terms of the tetrad, its
    inverse and the c[i,j,k] terms*)
A[-i, -j, -k]

(*Further expanding the c terms using the tetrad*)
AutomaticRules[c,
  MakeRule[{c[i, -j, -k], CD[-j][b[i, -k]] - CD[-k
      ][b[i, -j]]}]];
(*Again expanding the RRC in terms of only the
    tetrad and its inverse*)
A[-i, -j, -k]

(*Seperation of tetrad into background and
    decomposed parts*)
DefTensor[b0[i, j], M];
DefTensor[\[Delta]b[i, j], M];
(*Seperation of inverse tetrad into background and
    decomposed parts*)
DefTensor[h0[i, j], M];
DefTensor[\[Delta]h[i, j], M];
(*Including perturbation parameter epsilon*)
DefTensor[\[Epsilon][], M];
(*Setting the background to zeroth order and
    perturbations to first order*)
AutomaticRules[b,
  MakeRule[{b[i, j], b0[i, j] + \[Epsilon][] \[
      Delta]b[i, j]}]];
AutomaticRules[h,
  MakeRule[{h[i, j], h0[i, j] + \[Epsilon][] \[
      Delta]h[i, j]}]];

(*Expansion of the contorsion tensor in terms of
    the torsion tensor [taken to be first order in
    this model]*)
DefTensor[Tor[i, j, k], M];
AutomaticRules[K,
  MakeRule[{K[-i, -j, -k],
    1/2 (Tor[-i, -j, -k] + Tor[-j, -k, -i] -
       Tor[-k, -i, -j]) \[Epsilon][]}]];
(*Decomposing the torsion into tensor [1], vector
    [2] and axial-vector [3] components*)
DefTensor[T1[i, j, k], M];
DefTensor[T2[i], M];
DefTensor[T3[i], M];
AutomaticRules[Tor,
  MakeRule[{Tor[i, -j, -k],
    T1[i, -j, -k] + (T2[-j] delta[i, -k] - T2[-k]
        delta[i, -j]) +
     T3[-l] epsilong[l, i, -j, -k]}]];
(*Further decomposing these modes in terms of only
    scalars -- time and space derivatives of
    scalars*)
DefTensor[\[Alpha][], M];
DefTensor[\[Beta][], M];
DefTensor[W[], M];
DefTensor[Z[], M];
AutomaticRules[T1, MakeRule[{T1[i, -j, -k],
    n[i] n[-j] (delta[l, -k] - n[l] n[-k]) CD[-l
        ][\[Alpha][]]
     + n[i] n[q] epsilong[l, -j, -k, -q] CD[-l][\[
         Beta][]]
     + n[-j] (CD[l][CD[-m][W[]]] -
        1/3 delta[l, -m] CD[p][
          CD[-q][W[]]] (delta[q, -p] - n[q] n[-p]))
              (delta[-l, i] -
        n[-l] n[i]) (delta[m, -k] - n[m] n[-k])
     + epsilong[i, -k, -l, -m] n[m] CD[l][
          CD[-p][Z[]]] (delta[p, -j] - n[p] n[-j]) +
        epsilong[-j, -k, -l, -m] n[m] CD[l][
          CD[p][Z[]]] (delta[-p, i] - n[-p] n[i])}]];
DefTensor[U[], M];
DefTensor[V[], M];
AutomaticRules[T2,
  MakeRule[{T2[i],
    n[i] n[-j] CD[j][U[]] + (delta[i, -j] - n[i] n
        [-j]) CD[j][
       V[]]}]];
DefTensor[X[], M];
DefTensor[Y[], M];
AutomaticRules[T3,
  MakeRule[{T3[i],
    n[i] n[-j] CD[j][X[]] + (delta[i, -j] - n[i] n
        [-j]) CD[j][Y[]]}]];

(*Reexpanding spin connection in terms of the
    tetrad, inverse tetrad and the torsion-
    associated scalars up to linear order*)
Linearise1 = MakeRule[{\[Epsilon][]^2, 0}];
Linearise2 = MakeRule[{\[Epsilon][]^3, 0}];
Linearise3 = MakeRule[{\[Epsilon][]^4, 0}];
LineariseDiv = MakeRule[{CD[i][\[Epsilon][]], 0}];
CollectTensors[A[-i, -j, -k]] /. Linearise1 /.
    Linearise2 /.   Linearise3 /. LineariseDiv

(*Defining scalar parameters used in the expansion
    of the tetrad and its inverse: scale factor and
     scalar perturbations*)
DefTensor[a[], M];
DefTensor[As[], M, PrintAs -> "A"];
DefTensor[Bs[], M, PrintAs -> "B"];
DefTensor[Cs[], M, PrintAs -> "C"];
DefTensor[Ds[], M, PrintAs -> "D"];
DefTensor[Es[], M, PrintAs -> "E"];
DefTensor[Fs[], M, PrintAs -> "F"];
(*Setting the time derivative of the scale factor
    to the Hubble number H, while setting its
    spatial derivatives to vanishing*)
DefTensor[H[], M];
AutomaticRules[a, MakeRule[{CD[i][a[]], a[] n[i] H
    []}]];

(*Expanding the tetrad in terms of scalars*)
AutomaticRules[b0, MakeRule[{b0[i, j], a[] delta[i,
     j]}]];
AutomaticRules[\[Delta]b, MakeRule[{\[Delta]b[i, j
    ],
    a[] (n[i] n[j] As[]
       + n[i] CD[k][Bs[]] (delta[-k, j] - n[-k] n[j
           ])
       + n[j] CD[k][Cs[]] (delta[-k, i] - n[-k] n[i
           ])
       +
       delta[k,
         l] (delta[i, -k] - n[i] n[-k]) (delta[j, -
            l] -
         n[j] n[-l]) Ds[]
       + (delta[i, -k] - n[i] n[-k]) (delta[j, -l]
          -
         n[j] n[-l]) (CD[k][CD[l][Es[]]] -
         1/3 delta[k, l] CD[p][
           CD[q][Es[]]] (delta[-p, -q] - n[-p] n[-
              q]))
       + (delta[i, -m] - n[i] n[-m]) (delta[j, -p]
          -
         n[j] n[-p]) (delta[s, -q] - n[s] n[-q])
              epsilong[m, p,
        q, -r] n[r] CD[-s][Fs[]])}]];

(*Expanding the inverse tetrad in terms of scalars
    *)
AutomaticRules[h0, MakeRule[{h0[i, j], 1/a[] delta[
    i, j]}]];
AutomaticRules[\[Delta]h, MakeRule[{\[Delta]h[i, j
    ],
    1/a[] (-n[i] n[j] As[]
      - n[i] CD[k][Cs[]] (delta[-k, j] - n[-k] n[j
          ])
      - n[j] CD[k][Bs[]] (delta[-k, i] - n[-k] n[i
          ])
      -
```

```
    delta[k,
      l] (delta[i, -k] - n[i] n[-k]) (delta[j, -
         l] -
       n[j] n[-l]) Ds[]
   - (delta[i, -k] - n[i] n[-k]) (delta[j, -l]
       -
       n[j] n[-l]) (CD[k][CD[l][Es[]]] -
       1/3 delta[k, l] CD[p][
         CD[q][Es[]]] (delta[-p, -q] - n[-p] n[-
             q]))
   + (delta[i, -m] - n[i] n[-m]) (delta[j, -p]
       -
       n[j] n[-p]) (delta[s, -q] - n[s] n[-q])
           epsilong[m, p,
      q, -r] n[r] CD[-s][Fs[]])}]];

(*Output full first order, un-gauged spin
    connection*)
CollectTensors[Out[52]]

(*Redefining the (expanded) spin connection as the
    above expression \
[to save computing time]*)
DefTensor[expA[i, j, k], M];
AutomaticRules[expA,
  MakeRule[{expA[-i, -j, -k], -g[-j, -k] H[] n[-i]
      +
     g[-i, -k] H[] n[-j] + As[] g[-j, -k] H[] n[-i]
         \[Epsilon][] -
     Ds[] g[-j, -k] H[] n[-i] \[Epsilon][] -
     As[] g[-i, -k] H[] n[-j] \[Epsilon][] +
     Ds[] g[-i, -k] H[] n[-j] \[Epsilon][] -
     n[-j] n[-k] \[Epsilon][] CD[-i][As[]] +
     g[-j, -k] H[] \[Epsilon][] CD[-i][Bs[]] +
     H[] n[-j] n[-k] \[Epsilon][] CD[-i][Cs[]] -
     g[-j, -k] \[Epsilon][] CD[-i][Ds[]] +
     n[-j] n[-k] \[Epsilon][] CD[-i][Ds[]] -
     g[-j, -k] \[Epsilon][] CD[-i][V[]] +
     1/2 n[-j] n[-k] \[Epsilon][] CD[-i][\[Alpha
         ][]] +
     n[-i] n[-k] \[Epsilon][] CD[-j][As[]] -
     g[-i, -k] H[] \[Epsilon][] CD[-j][Bs[]] -
     H[] n[-i] n[-k] \[Epsilon][] CD[-j][Cs[]] +
     g[-i, -k] \[Epsilon][] CD[-j][Ds[]] -
     n[-i] n[-k] \[Epsilon][] CD[-j][Ds[]] +
     g[-i, -k] \[Epsilon][] CD[-j][V[]] -
     1/2 n[-i] n[-k] \[Epsilon][] CD[-j][\[Alpha
         ][]] +
     1/2 n[-k] \[Epsilon][] CD[-j][CD[-i][W[]]] +
     1/2 n[-i] n[-j] \[Epsilon][] CD[-k][\[Alpha
         ][]] -
     n[-j] \[Epsilon][] CD[-k][CD[-i][Cs[]]] +
     H[] n[-j] \[Epsilon][] CD[-k][CD[-i][Es[]]] +
     1/2 n[-j] \[Epsilon][] CD[-k][CD[-i][W[]]] +
     n[-i] \[Epsilon][] CD[-k][CD[-j][Cs[]]] -
     H[] n[-i] \[Epsilon][] CD[-k][CD[-j][Es[]]] -
     1/2 n[-i] \[Epsilon][] CD[-k][CD[-j][W[]]] -
     g[-j, -k] H[] n[-i] n[l] \[Epsilon][] CD[-l][
         Bs[]] +
     g[-i, -k] H[] n[-j] n[l] \[Epsilon][] CD[-l][
         Bs[]] -
     g[-j, -k] n[-i] n[l] \[Epsilon][] CD[-l][U[]]
         +
     g[-i, -k] n[-j] n[l] \[Epsilon][] CD[-l][U[]]
         +
     g[-j, -k] n[-i] n[l] \[Epsilon][] CD[-l][V[]]
         -
     g[-i, -k] n[-j] n[l] \[Epsilon][] CD[-l][V[]]
         -
     1/2 epsilong[-i, -j, -k, -m] n[l] n[m] \[
         Epsilon][]
      CD[-l][X[]] + 1/2 epsilong[-i, -j, -k, -m] n[
          l] n[m] \[Epsilon][]
      CD[-l][Y[]] -
     1/2 n[-i] n[-j] n[-k] n[l] \[Epsilon][] CD[-l
         ][\[Alpha][]] +
     n[-j] n[-k] n[l] \[Epsilon][] CD[-l][CD[-i][Bs
         []]] +
     n[-j] n[-k] n[l] \[Epsilon][] CD[-l][CD[-i][Cs
         []]] -
     H[] n[-j] n[-k] n[l] \[Epsilon][] CD[-l][CD[-i
         ][Es[]]] -
     n[-j] n[-k] n[l] \[Epsilon][] CD[-l][CD[-i][W
         []]] -
     n[-i] n[-k] n[l] \[Epsilon][] CD[-l][CD[-j][Bs
         []]] -
     n[-i] n[-k] n[l] \[Epsilon][] CD[-l][CD[-j][Cs
         []]] +
     H[] n[-i] n[-k] n[l] \[Epsilon][] CD[-l][CD[-j
         ][Es[]]] +
     n[-j] n[l] \[Epsilon][] CD[-l][CD[-k][CD[-i][
         Es[]]]] -
     n[-i] n[l] \[Epsilon][] CD[-l][CD[-k][CD[-j][
         Es[]]]] +
     1/3 g[-j, -k] H[] n[-i] \[Epsilon][] CD[-l][CD
         [l][Es[]]] -
     1/3 g[-i, -k] H[] n[-j] \[Epsilon][] CD[-l][CD
         [l][Es[]]] +
     1/6 g[-j, -k] n[-i] \[Epsilon][] CD[-l][CD[l][
         W[]]] -
     1/6 g[-i, -k] n[-j] \[Epsilon][] CD[-l][CD[l][
         W[]]] -
     1/6 g[-i, -j] n[-k] \[Epsilon][] CD[-l][CD[l][
         W[]]] +
     1/6 n[-i] n[-j] n[-k] \[Epsilon][] CD[-l][CD[l
         ][W[]]] +
     1/3 g[-j, -k] \[Epsilon][] CD[-l][CD[l][CD[-i
         ][Es[]]]] -
     1/3 n[-j] n[-k] \[Epsilon][] CD[-l][CD[l][CD[-
         i][Es[]]]] -
     1/3 g[-i, -k] \[Epsilon][] CD[-l][CD[l][CD[-j
         ][Es[]]]] +
     1/3 n[-i] n[-k] \[Epsilon][] CD[-l][CD[l][CD[-
         j][Es[]]]] -
     1/2 epsilong[-i, -j, -k, -l] \[Epsilon][] CD[l
         ][Y[]] -
     1/3 g[-j, -k] H[] n[-i] n[l] n[m] \[Epsilon][]
          CD[-m][
       CD[-l][Es[]]] +
     1/3 g[-i, -k] H[] n[-j] n[l] n[m] \[Epsilon][]
          CD[-m][
       CD[-l][Es[]]] -
     1/6 g[-j, -k] n[-i] n[l] n[m] \[Epsilon][] CD
         [-m][CD[-l][W[]]] +
     1/6 g[-i, -k] n[-j] n[l] n[m] \[Epsilon][] CD
         [-m][CD[-l][W[]]] +
     1/6 g[-i, -j] n[-k] n[l] n[m] \[Epsilon][] CD
         [-m][CD[-l][W[]]] +
     1/3 n[-i] n[-j] n[-k] n[l] n[m] \[Epsilon][]
         CD[-m][
       CD[-l][W[]]] -
     1/3 g[-j, -k] n[l] n[m] \[Epsilon][] CD[-m][
       CD[-l][CD[-i][Es[]]]] -
     2/3 n[-j] n[-k] n[l] n[m] \[Epsilon][] CD[-m][
       CD[-l][CD[-i][Es[]]]] +
     1/3 g[-i, -k] n[l] n[m] \[Epsilon][] CD[-m][
       CD[-l][CD[-j][Es[]]]] +
     2/3 n[-i] n[-k] n[l] n[m] \[Epsilon][] CD[-m][
       CD[-l][CD[-j][Es[]]]] +
     epsilong[-j, -k, -l, -m] H[] n[-i] n[l] \[
         Epsilon][] CD[m][
       Fs[]] - epsilong[-i, -k, -l, -m] H[] n[-j] n
           [
       l] \[Epsilon][] CD[m][Fs[]] -
     1/2 epsilong[-j, -k, -l, -m] n[-i] n[
       l] \[Epsilon][] CD[m][\[Beta][]] +
     1/2 epsilong[-i, -k, -l, -m] n[-j] n[
       l] \[Epsilon][] CD[m][\[Beta][]] +
     1/2 epsilong[-i, -j, -l, -m] n[-k] n[
       l] \[Epsilon][] CD[m][\[Beta][]] -
     epsilong[-j, -k, -l, -m] n[l] \[Epsilon][] CD[
         m][CD[-i][Z[]]] +
     epsilong[-i, -j, -l, -m] n[l] \[Epsilon][] CD[
         m][CD[-k][Fs[]]] +
     epsilong[-i, -j, -l, -m] n[l] \[Epsilon][] CD[
         m][CD[-k][Z[]]] +
     epsilong[-j, -k, -m, -p] n[-i] n[l] n[m] \[
         Epsilon][]
      CD[p][CD[-l][Z[]]] -
     epsilong[-i, -j, -m, -p] n[-k] n[l] n[m] \[
         Epsilon][]
      CD[p][CD[-l][Z[]]]}]];
```

```
(*Decomposition of ungauged A into spin - parity
    modes*)
(*Can now decompose this into the appropriate modes
    *)

        (*0+ spin-parity mode*)
DefTensor[plus0[], M];
AutomaticRules[plus0,
  MakeRule[{plus0[],
    expA[i, j, -k] n[-j] (delta[l, -i] - n[l] n[-i
        ]) (delta[k, -l] -
      n[k] n[-l])}]];
CollectTensors[plus0[]]

        (*0- spin-parity mode*)
DefTensor[minus0[], M];
AutomaticRules[minus0,
  MakeRule[{minus0[], expA[i, j, -k] epsilong[l, k,
      -i, -j] n[-l]}]];
CollectTensors[minus0[]]

        (*1+ spin-parity mode #1*)
DefTensor[plus1n1[i, j], M];
AutomaticRules[plus1n1,
  MakeRule[{plus1n1[i, j],
    expA[k, l, -m] n[
      m] (delta[i, -k] - n[i] n[-k]) (delta[-l, j]
          - n[-l] n[j])}]];
CollectTensors[plus1n1[i, j]]

        (*1- spin-parity mode #1*)
DefTensor[minus1n1[i], M];
AutomaticRules[minus1n1, MakeRule[{minus1n1[i],
    expA[j, k, -l]
     (delta[i, -k] - n[i] n[-k])
     (delta[l, -j] - n[l] n[-j])}]];
CollectTensors[minus1n1[i]]

        (*1+ spin-parity mode #2*)
DefTensor[plus1n2[i, j], M];
AutomaticRules[plus1n2,
  MakeRule[{plus1n2[-j, -k],
    1/2 n[i] (expA[-i, -l, -m] - expA[-i, -m, -l])
        (delta[-j, l] -
      n[-j] n[l]) (delta[-k, m] - n[-k] n[m])}]];
CollectTensors[plus1n2[-i, -j]]

        (*1- spin-parity mode #2*)
DefTensor[minus1n2[i], M];
AutomaticRules[minus1n2,
  MakeRule[{minus1n2[i],
    expA[l, j, -k] n[-j] n[k] (delta[-l, i] - n[-l]
         n[i])}]];
CollectTensors[minus1n2[i]]

        (*2+ spin-parity mode*)
DefTensor[plus2[i, j], M];
AutomaticRules[plus2,
  MakeRule[{plus2[-i, -j],
    1/2 (expA[-k, -l, -m] + expA[-k, -m, -l]) n[
       k] (delta[-i, l] - n[-i] n[l]) (delta[-j, m]
           - n[-j] n[m])
      + 1/
       3 delta[-k, -l] (delta[-i, k] - n[-i] n[k]) (
          delta[-j, l] -
        n[-j] n[l]) plus0[]}]];
CollectTensors[plus2[-i, -j]]

        (*2- spin-parity mode [Not needed in
             determining gauge invariant quantites
             ]*)
DefTensor[minus2[i, j, k], M];
AutomaticRules[minus2,
  MakeRule[{minus2[-i, -j, -k],
    1/2 (expA[-l, -m, -p] + expA[-l, -p, -m]) (
        delta[l, -i] -
        n[l] n[-i]) (delta[m, -j] - n[m] n[-j]) (
          delta[p, -k] -
        n[p] n[-k])
      - 1/
       3 delta[-m, -p] (delta[m, -i] - n[m] n[-i]) (
          delta[p, -k] -
        n[p] n[-k]) minus1n1[-j]
      - 1/6 epsilong[-i, -j, -k, l] n[-l] minus0
          []}]];
CollectTensors[minus2[-i, -j, -k]]



(*Gauging the spin connection*)

(*Rules to select zeroth and first order parts of
    the spin connection*)
Select0Order = MakeRule[{\[Epsilon][], 0}];
Select1Order = MakeRule[{\[Epsilon][], 1}];

(*0th order part of spin connection*)
expA[-i, -j, -k] /. Select0Order

(*First order part of spin connection*)
expA[-i, -j, -k] - % /. Select1Order

(*Splitting of the spin connection into background
    and perturbed \
parts for use in the gauge transformation*)
DefTensor[A0[i, j, k], M];
DefTensor[\[Delta]A[i, j, k], M];
DefTensor[gaugeA[i, j, k], M];

(*Defining guage transformation*)
DefTensor[\[Xi][i], M];
DefTensor[R[i, j], M];
AutomaticRules[gaugeA,
  MakeRule[{gaugeA[i, j, -k],
    A0[i, j, -k] + \[Epsilon][] (\[Delta]A[i, j, -k
        ] +
        A0[i, j, -l] CD[-k][\[Xi][l]] + R[j, -l] A0
            [i, l, -k] +
        R[i, -l] A0[l, j, -k] - CD[-k][R[i, j]])
            }]];

(*Decomposition of the gauge transformation into
    associated gauge \
angles*)
        (*Decomposition of Jacobian-associated part
            into scalars*)
DefTensor[T[], M];
DefTensor[L[], M];
AutomaticRules[\[Xi],
  MakeRule[{\[Xi][i],
    n[i] T[] + (delta[i, -j] - n[i] n[-j]) CD[j][L
        []]}]];
        (*Decomposition of Lorentz rotation-
           associated part into scalars*)
DefTensor[P[], M];
DefTensor[Q[], M];
AutomaticRules[R,
  MakeRule[{R[-i, -j], (n[-i] CD[-k][
          P[]] (delta[k, -j] - n[k] n[-j]) -
        n[-j] CD[-k][P[]] (delta[k, -i] - n[k] n[-i
            ])) + (delta[
          m, -i] - n[m] n[-i])
       (delta[p, -j] - n[p] n[-j])
       (delta[k, -q] - n[k] n[-q])
       (delta[s, -k] - n[s] n[-k])
       epsilong[-m, -p, q, r] n[-r] CD[-s][Q[]]}]];

(*Expanding the guage transformation in terms of
    the background spin connection, the
    perturbation to the spin connection and the
    gauge angles*)
CollectTensors[gaugeA[i, j, -k]] /. Linearise1 /.
    Linearise2 /.
  Linearise3 /. LineariseDiv

(*Expansion of the background and perturbed spin
    connection in terms of the tetrad and torsion
    scalars [A,...,F and \[Tau],..,\[Lambda]] --
    using the previously found expressions for A0
    and \[Delta]A*)
AutomaticRules[A0,
  MakeRule[{A0[-i, -j, -k], -g[-k, -j] H[] n[-i] +
      g[-k, -i] H[] n[-j]}]];
AutomaticRules[\[Delta]A,
  MakeRule[{\[Delta]A[-i, -j, -k],
```

```
As[] g[-k, -j] H[] n[-i] - Ds[] g[-k, -j] H[] n
    [-i] -
 As[] g[-k, -i] H[] n[-j] + Ds[] g[-k, -i] H[]
    n[-j] -
 n[-j] n[-k] CD[-i][As[]] + g[-k, -j] H[] CD[-i
    ][Bs[]] +
 H[] n[-j] n[-k] CD[-i][Cs[]] - g[-k, -j] CD[-i
    ][Ds[]] +
 n[-j] n[-k] CD[-i][Ds[]] - g[-k, -j] CD[-i][V
    []] +
 1/2 n[-j] n[-k] CD[-i][\[Alpha][]] + n[-i] n[-
    k] CD[-j][As[]] -
 g[-k, -i] H[] CD[-j][Bs[]] - H[] n[-i] n[-k]
    CD[-j][Cs[]] +
 g[-k, -i] CD[-j][Ds[]] - n[-i] n[-k] CD[-j][Ds
    []] +
 g[-k, -i] CD[-j][V[]] - 1/2 n[-i] n[-k] CD[-j
    ][\[Alpha][]] +
 1/2 n[-k] CD[-j][CD[-i][W[]]] +
 1/2 n[-i] n[-j] CD[-k][\[Alpha][]] - n[-j] CD
    [-k][CD[-i][Cs[]]] +
 H[] n[-j] CD[-k][CD[-i][Es[]]] + 1/2 n[-j] CD
    [-k][CD[-i][W[]]] +
 n[-i] CD[-k][CD[-j][Cs[]]] - H[] n[-i] CD[-k
    ][CD[-j][Es[]]] -
 1/2 n[-i] CD[-k][CD[-j][W[]]] -
 g[-k, -j] H[] n[-i] n[l] CD[-l][Bs[]] +
 g[-k, -i] H[] n[-j] n[l] CD[-l][Bs[]] -
 g[-k, -j] n[-i] n[l] CD[-l][U[]] +
 g[-k, -i] n[-j] n[l] CD[-l][U[]] +
 g[-k, -j] n[-i] n[l] CD[-l][V[]] -
 g[-k, -i] n[-j] n[l] CD[-l][V[]] -
 1/2 epsilong[-i, -j, -k, -m] n[l] n[m] CD[-l][
    X[]] +
 1/2 epsilong[-i, -j, -k, -m] n[l] n[m] CD[-l][
    Y[]] -
 1/2 n[-i] n[-j] n[-k] n[l] CD[-l][\[Alpha][]]
    +
 n[-j] n[-k] n[l] CD[-l][CD[-i][Bs[]]] +
 n[-j] n[-k] n[l] CD[-l][CD[-i][Cs[]]] -
 H[] n[-j] n[-k] n[l] CD[-l][CD[-i][Es[]]] -
 n[-j] n[-k] n[l] CD[-l][CD[-i][W[]]] -
 n[-i] n[-k] n[l] CD[-l][CD[-j][Bs[]]] -
 n[-i] n[-k] n[l] CD[-l][CD[-j][Cs[]]] +
 H[] n[-i] n[-k] n[l] CD[-l][CD[-j][Es[]]] +
 n[-j] n[l] CD[-l][CD[-k][CD[-i][Es[]]]] -
 n[-i] n[l] CD[-l][CD[-k][CD[-j][Es[]]]] +
 1/3 g[-k, -j] H[] n[-i] CD[-l][CD[l][Es[]]] -
 1/3 g[-k, -i] H[] n[-j] CD[-l][CD[l][Es[]]] +
 1/6 g[-k, -j] n[-i] CD[-l][CD[l][W[]]] -
 1/6 g[-k, -i] n[-j] CD[-l][CD[l][W[]]] -
 1/6 g[-j, -i] n[-k] CD[-l][CD[l][W[]]] +
 1/6 n[-i] n[-j] n[-k] CD[-l][CD[l][W[]]] +
 1/3 g[-k, -j] CD[-l][CD[l][CD[-i][Es[]]]] -
 1/3 n[-j] n[-k] CD[-l][CD[l][CD[-i][Es[]]]] -
 1/3 g[-k, -i] CD[-l][CD[l][CD[-j][Es[]]]] +
 1/3 n[-i] n[-k] CD[-l][CD[l][CD[-j][Es[]]]] -
 1/2 epsilong[-i, -j, -k, -l] CD[l][Y[]] -
 1/3 g[-k, -j] H[] n[-i] n[l] n[m] CD[-m][CD[-l
    ][Es[]]] +
 1/3 g[-k, -i] H[] n[-j] n[l] n[m] CD[-m][CD[-l
    ][Es[]]] -
 1/6 g[-k, -j] n[-i] n[l] n[m] CD[-m][CD[-l][W
    []]] +
 1/6 g[-k, -i] n[-j] n[l] n[m] CD[-m][CD[-l][W
    []]] +
 1/6 g[-j, -i] n[-k] n[l] n[m] CD[-m][CD[-l][W
    []]] +
 1/3 n[-i] n[-j] n[-k] n[l] n[m] CD[-m][CD[-l][
    W[]]] -
 1/3 g[-k, -j] n[l] n[m] CD[-m][CD[-l][CD[-i][
    Es[]]]] -
 2/3 n[-j] n[-k] n[l] n[m] CD[-m][CD[-l][CD[-i
    ][Es[]]]] +
 1/3 g[-k, -i] n[l] n[m] CD[-m][CD[-l][CD[-j][
    Es[]]]] +
 2/3 n[-i] n[-k] n[l] n[m] CD[-m][CD[-l][CD[-j
    ][Es[]]]] +
 epsilong[-j, -k, -l, -m] H[] n[-i] n[l] CD[m][
    Fs[]] -
 epsilong[-i, -k, -l, -m] H[] n[-j] n[l] CD[m][
    Fs[]] -
 1/2 epsilong[-j, -k, -l, -m] n[-i] n[l] CD[m
    ][\[Beta][]] +
 1/2 epsilong[-i, -k, -l, -m] n[-j] n[l] CD[m
    ][\[Beta][]] +
 1/2 epsilong[-i, -j, -l, -m] n[-k] n[l] CD[m
    ][\[Beta][]] -
 epsilong[-j, -k, -l, -m] n[l] CD[m][CD[-i][Z
    []]] +
 epsilong[-i, -j, -l, -m] n[l] CD[m][CD[-k][Fs
    []]] +
 epsilong[-i, -j, -l, -m] n[l] CD[m][CD[-k][Z
    []]] +
 epsilong[-j, -k, -m, -p] n[-i] n[l] n[m] CD[p
    ][CD[-l][Z[]]] -
 epsilong[-i, -j, -m, -p] n[-k] n[l] n[m] CD[p
    ][CD[-l][Z[]]]}]];

(*Guage-transformed spin connection fully expanded
   in terms of scalars*)
CollectTensors[Out[109]]

(*Once again, redefining the transformed spin
   connection as the above expression [to save
   computing time]*)
DefTensor[ExpGaugeA[i, j, k], M];
AutomaticRules[ExpGaugeA,
  MakeRule[{ExpGaugeA[i, j, -k], -xAct`xTensor`
      delta[j, -k] H[] n[i] +
     xAct`xTensor`delta[i, -k] H[] n[j] +
    As[] xAct`xTensor`delta[j, -k] H[] n[i] \[
      Epsilon][] -
    xAct`xTensor`delta[j, -k] Ds[] H[] n[i] \[
      Epsilon][] -
    As[] xAct`xTensor`delta[i, -k] H[] n[j] \[
      Epsilon][] +
    xAct`xTensor`delta[i, -k] Ds[] H[] n[j] \[
      Epsilon][] -
    n[j] n[-k] \[Epsilon][] CD[i][As[]] +
    xAct`xTensor`delta[j, -k] H[] \[Epsilon][] CD[
      i][Bs[]] +
    H[] n[j] n[-k] \[Epsilon][] CD[i][Cs[]] -
    xAct`xTensor`delta[j, -k] \[Epsilon][] CD[i][
      Ds[]] +
    n[j] n[-k] \[Epsilon][] CD[i][Ds[]] +
    xAct`xTensor`delta[j, -k] H[] \[Epsilon][] CD[
      i][P[]] -
    H[] n[j] n[-k] \[Epsilon][] CD[i][P[]] -
    xAct`xTensor`delta[j, -k] \[Epsilon][] CD[i][V
      []] +
    1/2 n[j] n[-k] \[Epsilon][] CD[i][\[Alpha][]]
      +
    n[i] n[-k] \[Epsilon][] CD[j][As[]] -
    xAct`xTensor`delta[i, -k] H[] \[Epsilon][] CD[
      j][Bs[]] -
    H[] n[i] n[-k] \[Epsilon][] CD[j][Cs[]] +
    xAct`xTensor`delta[i, -k] \[Epsilon][] CD[j][
      Ds[]] -
    n[i] n[-k] \[Epsilon][] CD[j][Ds[]] -
    xAct`xTensor`delta[i, -k] H[] \[Epsilon][] CD[
      j][P[]] +
    H[] n[i] n[-k] \[Epsilon][] CD[j][P[]] +
    xAct`xTensor`delta[i, -k] \[Epsilon][] CD[j][V
      []] -
    1/2 n[i] n[-k] \[Epsilon][] CD[j][\[Alpha][]]
      +
    1/2 n[-k] \[Epsilon][] CD[j][CD[i][W[]]] +
    1/2 n[i] n[j] \[Epsilon][] CD[-k][\[Alpha][]]
      -
    n[j] \[Epsilon][] CD[-k][CD[i][Cs[]]] +
    H[] n[j] \[Epsilon][] CD[-k][CD[i][Es[]]] +
    H[] n[j] \[Epsilon][] CD[-k][CD[i][L[]]] +
    n[j] \[Epsilon][] CD[-k][CD[i][P[]]] +
    1/2 n[j] \[Epsilon][] CD[-k][CD[i][W[]]] +
    n[i] \[Epsilon][] CD[-k][CD[j][Cs[]]] -
    H[] n[i] \[Epsilon][] CD[-k][CD[j][Es[]]] -
    H[] n[i] \[Epsilon][] CD[-k][CD[j][L[]]] -
    n[i] \[Epsilon][] CD[-k][CD[j][P[]]] -
    1/2 n[i] \[Epsilon][] CD[-k][CD[j][W[]]] -
    xAct`xTensor`delta[j, -k] H[] n[i] n[l] \[
      Epsilon][]
     CD[-l][Bs[]] +
    xAct`xTensor`delta[i, -k] H[] n[j] n[l] \[
      Epsilon][]
```

```
 CD[-l][Bs[]] -
xAct`xTensor`delta[j, -k] H[] n[i] n[l] \[
      Epsilon][]
 CD[-l][P[]] +
xAct`xTensor`delta[i, -k] H[] n[j] n[l] \[
      Epsilon][]
 CD[-l][P[]] -
xAct`xTensor`delta[j, -k] n[i] n[l] \[Epsilon
      ][] CD[-l][U[]] +
xAct`xTensor`delta[i, -k] n[j] n[l] \[Epsilon
      ][] CD[-l][U[]] +
xAct`xTensor`delta[j, -k] n[i] n[l] \[Epsilon
      ][] CD[-l][V[]] -
xAct`xTensor`delta[i, -k] n[j] n[l] \[Epsilon
      ][] CD[-l][V[]] -
1/2 epsilong[i, j, -k, -m] n[l] n[m] \[Epsilon
      ][] CD[-l][X[]] +
1/2 epsilong[i, j, -k, -m] n[l] n[m] \[Epsilon
      ][] CD[-l][Y[]] -
1/2 n[i] n[j] n[-k] n[l] \[Epsilon][] CD[-l
      ][\[Alpha][]] +
n[j] n[-k] n[l] \[Epsilon][] CD[-l][CD[i][Bs
      []]] +
n[j] n[-k] n[l] \[Epsilon][] CD[-l][CD[i][Cs
      []]] -
H[] n[j] n[-k] n[l] \[Epsilon][] CD[-l][CD[i][
      Es[]]] -
n[j] n[-k] n[l] \[Epsilon][] CD[-l][CD[i][W
      []]] -
n[i] n[-k] n[l] \[Epsilon][] CD[-l][CD[j][Bs
      []]] -
n[i] n[-k] n[l] \[Epsilon][] CD[-l][CD[j][Cs
      []]] +
H[] n[i] n[-k] n[l] \[Epsilon][] CD[-l][CD[j][
      Es[]]] +
n[j] n[l] \[Epsilon][] CD[-l][CD[-k][CD[i][Es
      []]]] -
n[i] n[l] \[Epsilon][] CD[-l][CD[-k][CD[j][Es
      []]]] +
1/3 xAct`xTensor`delta[j, -k] H[] n[i] \[
      Epsilon][] CD[-l][
  CD[l][Es[]]] -
1/3 xAct`xTensor`delta[i, -k] H[] n[j] \[
      Epsilon][] CD[-l][
  CD[l][Es[]]] +
1/6 xAct`xTensor`delta[j, -k] n[i] \[Epsilon
      ][] CD[-l][
  CD[l][W[]]] -
1/6 xAct`xTensor`delta[i, -k] n[j] \[Epsilon
      ][] CD[-l][
  CD[l][W[]]] -
1/6 g[i, j] n[-k] \[Epsilon][] CD[-l][CD[l][W
      []]] +
1/6 n[i] n[j] n[-k] \[Epsilon][] CD[-l][CD[l][
     W[]]] +
1/3 xAct`xTensor`delta[j, -k] \[Epsilon][] CD
      [-l][
  CD[l][CD[i][Es[]]]] -
1/3 n[j] n[-k] \[Epsilon][] CD[-l][CD[l][CD[i
      ][Es[]]]] -
1/3 xAct`xTensor`delta[i, -k] \[Epsilon][] CD
      [-l][
  CD[l][CD[j][Es[]]]] +
1/3 n[i] n[-k] \[Epsilon][] CD[-l][CD[l][CD[j
      ][Es[]]]] -
1/2 epsilong[i, j, -k, -l] \[Epsilon][] CD[l][
      Y[]] -
1/3 xAct`xTensor`delta[j, -k] H[] n[i] n[l] n[
  m] \[Epsilon][] CD[-m][CD[-l][Es[]]] +
1/3 xAct`xTensor`delta[i, -k] H[] n[j] n[l] n[
     m] \[Epsilon][]
 CD[-m][CD[-l][Es[]]] -
1/6 xAct`xTensor`delta[j, -k] n[i] n[l] n[m]
      \[Epsilon][]
 CD[-m][CD[-l][W[]]] +
1/6 xAct`xTensor`delta[i, -k] n[j] n[l] n[m]
      \[Epsilon][] CD[-m][
  CD[-l][W[]]] +
1/6 g[i, j] n[-k] n[l] n[m] \[Epsilon][] CD[-m
      ][CD[-l][W[]]] +
1/3 n[i] n[j] n[-k] n[l] n[m] \[Epsilon][] CD
      [-m][CD[-l][W[]]] -
1/3 xAct`xTensor`delta[j, -k] n[l] n[m] \[
      Epsilon][]
 CD[-m][CD[-l][CD[i][Es[]]]] -
2/3 n[j] n[-k] n[l] n[m] \[Epsilon][] CD[-m][
  CD[-l][CD[i][Es[]]]] +
1/3 xAct`xTensor`delta[i, -k] n[l] n[m] \[
      Epsilon][] CD[-m][
  CD[-l][CD[j][Es[]]]] +
2/3 n[i] n[-k] n[l] n[m] \[Epsilon][] CD[-m][
  CD[-l][CD[j][Es[]]]] +
epsilong[j, -k, -l, -m] H[] n[i] n[l] \[
      Epsilon][] CD[m][Fs[]] -
epsilong[i, -k, -l, -m] H[] n[j] n[l] \[
      Epsilon][] CD[m][Fs[]] +
epsilong[j, -k, -l, -m] H[] n[i] n[l] \[
      Epsilon][] CD[m][Q[]] -
epsilong[i, -k, -l, -m] H[] n[j] n[l] \[
      Epsilon][] CD[m][Q[]] -
1/2 epsilong[j, -k, -l, -m] n[i] n[
  l] \[Epsilon][] CD[m][\[Beta][]] +
1/2 epsilong[i, -k, -l, -m] n[j] n[
  l] \[Epsilon][] CD[m][\[Beta][]] +
1/2 epsilong[i, j, -l, -m] n[-k] n[
  l] \[Epsilon][] CD[m][\[Beta][]] -
epsilong[j, -k, -l, -m] n[l] \[Epsilon][] CD[m
      ][CD[i][Z[]]] +
epsilong[i, j, -l, -m] n[l] \[Epsilon][] CD[m
      ][CD[-k][Fs[]]] +
epsilong[i, j, -l, -m] n[l] \[Epsilon][] CD[m
      ][CD[-k][Q[]]] +
epsilong[i, j, -l, -m] n[l] \[Epsilon][] CD[m
      ][CD[-k][Z[]]] +
epsilong[j, -k, -m, -p] n[i] n[l] n[m] \[
      Epsilon][] CD[p][
 CD[-l][Z[]]] -
epsilong[i, j, -m, -p] n[-k] n[l] n[m] \[
      Epsilon][] CD[p][
  CD[-l][Z[]]]}]];


(*Decomposition of gauged A into spin - parity
     modes*)
        (*0+ spin-parity mode*)
DefTensor[Gplus0[], M];
AutomaticRules[Gplus0,
  MakeRule[{Gplus0[],
    ExpGaugeA[i,
      j, -k] n[-j] (delta[l, -i] - n[l] n[-i]) (
          delta[k, -l] -
       n[k] n[-l])}]];
CollectTensors[Gplus0[]]

        (*0- spin-parity mode*)
DefTensor[Gminus0[], M];
AutomaticRules[Gminus0,
  MakeRule[{Gminus0[],
    ExpGaugeA[i, j, -k] epsilong[l, k, -i, -j] n[-l
         ]}]];
CollectTensors[Gminus0[]]

        (*1+ spin-parity mode #1*)
DefTensor[Gplus1n1[i, j], M];
AutomaticRules[Gplus1n1,
  MakeRule[{Gplus1n1[i, j],
    ExpGaugeA[k, l, -m] n[
      m] (delta[i, -k] - n[i] n[-k]) (delta[-l, j]
          - n[-l] n[j])}]];
CollectTensors[Gplus1n1[i, j]]

        (*1- spin-parity mode #1*)
DefTensor[Gminus1n1[i], M];
AutomaticRules[Gminus1n1, MakeRule[{Gminus1n1[i],
     ExpGaugeA[j, k, -l]
      (delta[i, -k] - n[i] n[-k])
      (delta[l, -j] - n[l] n[-j])}]];
CollectTensors[Gminus1n1[i]]

        (*1+ spin-parity mode #2*)
DefTensor[Gplus1n2[i, j], M];
AutomaticRules[Gplus1n2,
  MakeRule[{Gplus1n2[-j, -k],
    1/2 n[i] (ExpGaugeA[-i, -l, -m] -
```

```
        ExpGaugeA[-i, -m, -l]) (delta[-j, l] -
        n[-j] n[l]) (delta[-k, m] - n[-k] n[m])}]];
CollectTensors[Gplus1n2[-i, -j]]

        (*1- spin-parity mode #2*)
DefTensor[Gminus1n2[i], M];
AutomaticRules[Gminus1n2,
  MakeRule[{Gminus1n2[i], ExpGaugeA[i, j, -k] n[-j]
        n[k]},
    MetricOn -> All, ContractMetrics -> True]];
CollectTensors[Gminus1n2[i]]

        (*2+ spin-parity mode*)
DefTensor[Gplus2[i, j], M];
AutomaticRules[Gplus2,
  MakeRule[{Gplus2[-i, -j],
    1/2 (ExpGaugeA[-k, -l, -m] + ExpGaugeA[-k, -m,
          -l]) n[
        k] (delta[-i, l] - n[-i] n[l]) (delta[-j, m]
              - n[-j] n[m]) +
      1/3 delta[-k, -l] (delta[-i, k] - n[-i] n[k])
          (delta[-j, l] -
        n[-j] n[l]) Gplus0[]}]];
CollectTensors[Gplus2[-i, -j]]

(*2- spin-parity mode*)
DefTensor[Gminus2[i, j, k], M];
AutomaticRules[Gminus2,
  MakeRule[{Gminus2[-i, -j, -k],
    1/2 (ExpGaugeA[-l, -m, -p] +
        ExpGaugeA[-l, -p, -m]) (delta[l, -i] -
        n[l] n[-i]) (delta[m, -j] - n[m] n[-j]) (
            delta[p, -k] -
        n[p] n[-k])
      - 1/
        3 delta[-m, -p] (delta[m, -i] - n[m] n[-i]) (
            delta[p, -k] -
        n[p] n[-k]) Gminus1n1[-j]
      - 1/6 epsilong[-i, -j, -k, l] n[-l] Gminus0
            []}]];
CollectTensors[Gminus2[-i, -j, -k]]
```