Natural Sciences Tripos

Part III Astrophysics

Cosmological Perturbations in a Novel Theory of Gravity

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Abstract

Despite being one of the most successful physical theories of the twentieth century, Einstein's General Relativity (GR) is well accepted as being incomplete. GR fails to integrate with Quantum Mechanics at high energies and provides no physical motivation for a dark energy constant, Λ, which is widely understood to be the cause of our Universe's accelerated expansion. This has generated strong interest in alternatives to GR, one such theory being Poincaré Gauge Theory (PGT), which has the potential to address both aforementioned issues. The interest of this report is the cosmological perturbation theory of the Metrical Analogue (MA); a projection of PGT onto a bi-scalar tensor theory. We show that a non-canonical square-root term in the MA Lagrangian, $\sqrt{|J_\mu J^\mu|}$, possesses subtle perturbative behaviour that disables the spatial perturbations of one of the scalar fields. In turn we show that this provides a stability condition for the torsion field that physically motivates dark energy in the theory. Finally, we show that this stability condition gives rise to Einstein-like perturbed field equations, with stable modifications to the Bardeen potentials.

1 Introduction

The most widely accepted cosmological model of the Universe is the ΛCDM model. It is the simplest theory to describe the following observed properties of the universe: the presence and structure of the Cosmic Microwave Background (CMB), Large Scale Structure (LSS) of galaxy distribution, light element abundancies and the late time accelerated expansion of the universe Condon and Matthews [2018]. ΛCDM in conjunction with a model of cosmic inflation in the very early universe serves as the bedrock of our understanding of modern-day cosmology Pajer [2023]. One of the key principles of the ΛCDM model is that Einstein's theory of General Relativity (GR) is the correct theory of gravity on cosmological scales and that behaviour not explained by GR is caused by other physical phenomena. For example, galaxy formation and behaviour are dependent on dark matter and late time acceleration is caused by a cosmological constant (Λ) or dark energy Wechsler and Tinker [2018], Peebles and Ratra [2003]. There are many theories that focus on the origin of these phenomena from other branches of physics, for example: acceleration caused by new types of bosons from string theory and dark matter being composed of new particles such as axions from particle physics Dutta and Maharana [2021], Duffy and Van Bibber [2009]. However, potential discrepancies between the ΛCDM model and observations Verde et al. [2019], Handley [2021], Di Valentino et al. [2020], Bullock and Boylan-Kolchin [2017], Riess [2020], Riess et al. [2019], Aghanim et al. [2020] have generated interest in extended theories of gravity which contain additional parameters beyond GR to explain these phenomena.

1.1 Horndeski Theory

Lovelock's Theorem restricts the available choices of extended gravity Clifton et al. [2012], Capozziello and De Laurentis [2011]; it states that the only obtainable second order Euler-Lagrange expression from a scalar action of the type $\mathcal{L} = \mathcal{L}(g_{\mu\nu})$ is

$$E^{\mu\nu} = \alpha\sqrt{-g}\left[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right] + \lambda\sqrt{-g}g^{\mu\nu}. \tag{1}$$

Note this is the same as Einstein's original field equations with constant of proportionality α and a cosmological constant λ. A common approach to circumvent this theorem is to add scalar fields to the action, coupling them to the metric $g_{\mu\nu}$. The most generalised form of this approach is Horndeski Theory Kobayashi et al. [2011], Langlois and Noui [2016]. The general theory of interest in this report is similar to Horndeski theory, however instead of one additional scalar field there are two, making it a bi-scalar tensor theory. Additionally further non-canonical terms in the Lagrangian distinguish it from Horndeski Theory. We will motivate the theory in section 1.2, but for the moment we simply introduce it. The theory is summarised in the form of a Lagrangian density termed the Metrical Analogue (MA) Barker et al. [2020a]

$$\begin{aligned}\mathcal{L}_{\text{MA}} = &\left[\tfrac{1}{2}{M_{\text{Pl}}}^2\upsilon_2 + \sigma_3\phi^2 + \tfrac{1}{2}(\sigma_3 - \sigma_2)\psi^2\right]R \\ &+ 12\left[\sigma_3 X^{\phi\phi} + \tfrac{1}{2}(\sigma_3 - \sigma_2)X^{\psi\psi}\right] + \sqrt{|J_\mu J^\mu|} \\ &+ \tfrac{3}{4}{M_{\text{Pl}}}^2\left[(\alpha_0 + \upsilon_2)\phi^2 - (\alpha_0 - 4\upsilon_1)\psi^2\right] \\ &+ \tfrac{3}{2}\left(\sigma_3\phi^4 - 2\sigma_2\phi^2\psi^2 + \sigma_3\psi^4\right) + L_{\text{m}}(\Phi; g),\end{aligned} \tag{2a}$$

$$J_\mu \equiv 4\sigma_1\psi^3\partial_\mu(\phi/\psi) - {M_{\text{Pl}}}^2(\alpha_0 + \upsilon_2)\partial_\mu\phi, \quad X^{\phi\phi} \equiv -g^{\mu\nu}\frac{1}{2}\partial_\mu\phi\partial_\nu\phi, \tag{2b}$$

where ϕ and ψ are scalar fields, R is the Ricci scalar, M_{Pl} is the Planck mass and $L_{\text{m}}(\Phi; g)$ is the matter contribution to the Lagrangian. Φ represents tensorial matter fields such as bosons. $\alpha_0, \sigma_1, \sigma_2, \sigma_3, \upsilon_1$ and υ_2 are all dimensionless coupling constants which can further restrict the theory to a specific case. The metric signature is $(+, -, -, -)$ and the Riemann tensor is defined as

$$R^\sigma{}_{\rho\mu\nu} = 2\partial_{[\mu}\Gamma^\sigma_{\nu]\rho} + 2\Gamma^\sigma_{[\mu|\lambda|}\Gamma^\lambda_{\nu]\rho} \tag{3a}$$

with

$$\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^\nu} + \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right), \tag{3b}$$

where $g_{\mu\nu}$ is the metric tensor.

Of particular note here is the square-root term $\sqrt{|J_\mu J^\mu|}$, a non-canonical term not typically found in Lagrangia. It possesses peculiar 'parasitic' behaviour and is called the Cuscuton Afshordi et al. [2007b,a], Boruah et al. [2017], Mansoori and Molaee [2023], Bhattacharyya et al. [2018]. As is seen later on it introduces unusual effects.

1.2 Origins in Poincaré Gauge Theory

As promised, we now turn to understanding the motivation for 2a, for which it is necessary to understand its origin in Poincaré Gauge Theory (PGT). PGT is a ten-parameter theory of gravity based upon the Poincaré symmetry group $G = T_4 \rtimes \text{SO}^+(1,3)$. It is generated by extending the Yang–Mills approach of internal symmetries to spacetime symmetries. This naturally leads to introducing Riemann–Cartan geometry on the spacetime manifold Kibble [1961], Utiyama [1956], Sciama [1964]. For a review of PGT see Obukhov [2018]. GR is non-perturbatively renormalisable, which means it is not currently possible to successfully integrate GR with Quantum Field Theory (QFT) at high energies Pajer [2023], Shomer [2007]. A key motivation for considering PGT is that for certain parameter choices in the Lagrangian shown below, the theory has the potential to be renormalisable. There are currently 58 known cases which are power-counting renormalisable (PCR) when linearised near Minkowski spacetime. One of the cases corresponds to the theory discussed in this report Lin et al. [2020]:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{PGT}} = & -\lambda\mathcal{R} + (r_4 + r_5)\,\mathcal{R}^{AB}\mathcal{R}_{AB} \\
& + (r_4 - r_5)\,\mathcal{R}^{AB}\mathcal{R}_{BA} + \left(\frac{r_1}{3} + \frac{r_2}{6}\right)\mathcal{R}^{ABCD}\mathcal{R}_{ABCD} \\
& + \left(\frac{2r_1}{3} - \frac{2r_2}{3}\right)\mathcal{R}^{ABCD}\mathcal{R}_{ACBD} \\
& + \left(\frac{r_1}{3} + \frac{r_2}{6} - r_3\right)\mathcal{R}^{ABCD}\mathcal{R}_{CDAB} \\
& + \left(\frac{\lambda}{4} + \frac{t_1}{3} + \frac{t_2}{12}\right)\mathcal{T}^{ABC}\mathcal{T}_{ABC} \\
& + \left(-\frac{\lambda}{2} - \frac{t_1}{3} + \frac{t_2}{6}\right)\mathcal{T}^{ABC}\mathcal{T}_{BCA} \\
& + \left(-\lambda - \frac{t_1}{3} + \frac{2t_3}{3}\right)\mathcal{T}_{B}{}^{AB}\mathcal{T}_{CA}{}^{C},
\end{aligned}
\tag{4}
$$

where

$$
\mathcal{R}^{AB}{}_{\mu\nu} = 2(\partial_{[\mu}A^{AB}{}_{\nu]} + A^{A}{}_{E[\mu}A^{EB}{}_{\nu]}), \tag{5a}
$$

$$
\mathcal{T}^{A}{}_{\mu\nu} = 2(\partial_{[\mu}b^{A}{}_{\nu]} + A^{A}{}_{E[\mu}b^{E}{}_{\nu]}) \tag{5b}
$$

are the translational and rotational field strengths respectively. Compare 5a with 3a, notice that when torsion as defined in 5b goes to zero, 5a and 3a are describing the same Riemannian curvature. The translational gauge field is $h_A{}^{\mu}$, its inverse is $b^A{}_{\mu}$ such that $b^A{}_{\mu}h_B{}^{\mu} = \delta^A_B$ and $b^A{}_{\mu}h_A{}^{\nu} = \delta^{\nu}_{\mu}$. $A^{AB}{}_{\mu} = -A^{BA}{}_{\mu}$ is the gauge field corresponding to Lorentz transformations. The capital Latin indices are for the local Lorentz frame and the Greek indices correspond to the coordinate frame. The dimensionful λ and t and the dimensionless r parameters restrict the theory analogous to the dimensionless parameters in 2a.

The bi-scalar tensor Lagrangian 2a is created by mapping the PGT Lagrangian 4 to a cosmological background. This means converting the field strengths of Riemann–Cartan geometry, $\mathcal{T}^{A}{}_{\mu\nu}, \mathcal{R}^{AB}{}_{\mu\nu}$, to the analogous term in purely Riemannian geometry, $R^{\sigma}{}_{\rho\mu\nu}$. PGT contains a total of 40 Degrees of Freedom (DoF), 16 tetrad DoF and 24 spin connection DoF. In addition to the 10 DoF of the metric in GR, the isotropy of cosmology restricts PGT (specifically the torsion tensor $\mathcal{T}^{A}{}_{\mu\nu}$) to just 2 DoF; a 0^+ and 0^- mode respectively referred to as ϕ and ψ Tsamparlis [1979]. These are the scalar fields in 2a. Evidently information present in the PGT theory is lost when translated to the bi-scalar tensor theory, additionally only the PGT theory has the potential to be non-perturbatively renormalisable, the MA does not posses this property. However the MA does retain key phenomenological behaviour from PGT: an emergent dark energy component and a dark radiation component that could alleviate possible Hubble tension Barker et al. [2020a] Barker et al. [2020b].

1.3 Emergent Dark Energy

Of the 58 PCR cases, the theory considered in this report is the one specified by the following parameter choices in the 2a

$$\alpha_0 = 0, \quad \sigma_3 = 0, \quad \sigma_2 = \sigma_1, \quad \upsilon_2 = -\frac{4}{3}. \tag{6}$$

Going forward all references to the MA mean this specific theory. Taking variational derivates of 2a with respect to ϕ and ψ yields their equations of motion and taking it with respect to $g_{\mu\nu}$ gives field equations analogous to the Einstein Field Equations (EFE). Assuming flat FLRW spacetime and a matter dominated Universe the solutions to ϕ and ψ equations yield the following values

$$\phi = (1 - \sqrt{3})H, \tag{7}$$

$$\psi = \frac{M_{\mathrm{Pl}}}{\sqrt{3}\sqrt{-\sigma_1}}. \tag{8}$$

When the ϕ and ψ solutions are substituted into the field equations (under the same assumptions) the EFE with a dark energy component are recovered

$$G_{\mu\nu} + g_{\mu\nu} {M_{\mathrm{Pl}}}^2 \upsilon_1/\sigma_1 = 0. \tag{9}$$

It is important to note here that the constant ψ has taken on the interpretation of a torsional condensate proportional to an emergent dark energy constant, ${M_{\mathrm{Pl}}}^2\upsilon_1/\sigma_1$ Barker et al. [2020a]. This constant is equivalent to λ in 1. So to summarise: PGT, motivated by its healthier interaction with QFT compared with GR, when mapped to a bi-scalar tensor theory leads to background field equations similar to GR with a physically motivated emergent dark energy constant.

At this point we take the opportunity to define the relevant constants. The Planck mass defined as $M_{\mathrm{Pl}} = \sqrt{\hbar c/G}$, with a value of $2.176\,434 \times 10^{-8}$kg, where $c = 2.997\,924\,58 \times 10^{8}$ ms^{-1} is the speed of light, $\hbar = 1.054\,571\,68 \times 10^{-34}$Js is the reduced Planck constant and $G = 6.674\,30 \times 10^{-11}\mathrm{Nm}^2\mathrm{kg}^{-2}$ is Newton's gravitational constant Newell et al. [2019].

1.4 Aims

Observed temperature anisotropies in the CMB and the nature of LSS are very well described by the cosmological perturbation theory of GR Condon and Matthews [2018], Peter [2013], Fasiello et al. [2022]. Consequently, any theory of extended gravity should match GR in its ability to explain CMB and LSS. Therefore this report has the following aims:

- Assess the perturbative stability of the MA around its background solutions. This is crucial to determine the physical viability of the emergent dark energy component.
- Obtain the perturbed field equations for the MA and assess their stability.

- Make comparisons between the perturbed field equations of the MA and GR and characterise any differences.

The structure of the report is as follows: in Section 2 the details of the approach are discussed, in Section 3 the form of the covariant equations are considered. Sections 4 and 5 are concerned with the square root Cuscuton term $\sqrt{|J_\mu J^\mu|}$ and its consequences. Finally Section 6 looks at the recovery of Einstein-like field equations.

2 Approach

2.1 xAct, xPert and xPand

The equations of interest in this project can be large and unwieldy, so attempting to carry out calculations by hand would be prohibitively difficult. Consequently computer algebra was employed to obtain the perturbation equations. All calculations were carried out in Mathematica using a selection of tools from the xAct suite of packages. Namely *xTensor* for defining and setting up tensor algebra, *xPert* for general spacetime perturbations and *xPand* for cosmological perturbations Brizuela et al. [2009], Pitrou et al. [2013], Martin-Garcia [2004].

Before examining how the perturbation equations are obtained, let us first consider some of the perturbation theory underpinning the *xPert* and *xPand* packages. This will aid in understanding the output from the code in these packages and the mathematical framework in which they operate.

Consider two manifolds $\overline{\mathcal{M}}$, the background manifold and $\mathcal{M}$, the perturbed manifold. The two manifolds are related by a diffeomorphism $\phi\colon\ \overline{\mathcal{M}} \to \mathcal{M}$. One can translate tensor objects from one manifold to the other using the pull-back function $\phi^\star$, or push-forward function $\phi_\star$ and their respective inverses. This then allows for the description of the metric on the perturbed manifold, $\boldsymbol{g}$, in terms of the metric on the background manifold $\bar{\boldsymbol{g}}$ Pitrou et al. [2013], Bloomfield [2013]

$$\phi^\star(\boldsymbol{g}) = \bar{\boldsymbol{g}} + \Delta[\bar{\boldsymbol{g}}] = \bar{\boldsymbol{g}} + \sum_{n=1}^{\infty} \frac{\Delta^n[\bar{\boldsymbol{g}}]}{n!} \,. \tag{10}$$

From this expression of the metric on the perturbed manifold the perturbed connection components, Riemann tensor and other curvature tensors can all be obtained. See Brizuela et al. [2009], Pitrou et al. [2013] for more details about this process.

Implementation of perturbation theory in a general manifold is handled through use of the *xPert* package. Here is an illustrative example of how one would obtain the perturbed Ricci Scalar:

- First using *xTensor* we define the necessary geometric objects:

In[1] := `DefManifold[ M, 4, {`α, β, μ, ν, λ, σ`} ];`

In[2] := `DefMetric[-1, g[-`α`,-`β`], CD, {";","`$\bar{\nabla}$`"}, PrintAs->"`$\bar{\mathrm{g}}$`"];`

where `M` represents the 4-dimensional background Manifold, `CD` defines the covariant derivative and $\bar{\mathrm{g}}$ is the background metric.

- Then using *xPert* we can define the metric perturbations, `dg`, to the background metric $\bar{\mathrm{g}}$:

  In[3] := `DefMetricPerturbation[ g, dg, `ε` ];`

- Now we can obtain the perturbed Ricci scalar to first order in terms of quantities defined on the background manifold:

  In[4] := `ExpandPerturbation@Perturbed[ RicciScalarCD[], 1 ]`

  $$Out[4] := \; R[\bar{\nabla}] - \varepsilon\, dg^{1\alpha\beta} R[\bar{\nabla}]_{\alpha\beta} + \varepsilon\, \bar{\nabla}_\beta \bar{\nabla}_\alpha dg^{1\alpha\beta} - \varepsilon\, \bar{\nabla}_\beta \bar{\nabla}^\beta dg^{1\alpha}{}_\alpha$$

  Here ε is the perturbative expansion parameter and the 1 superscript denotes the order of the metric perturbation, e.g. dg^1 is the first order perturbation.

xPand builds upon the general perturbative approach developed in *xPert* by specialising perturbations to a range of homogenous cosmological space-time backgrounds Pitrou et al. [2013]. It does so through the 3+1 formalism where the background manifold is foliated (sliced) into 3-dimensional spatial hypersurfaces Gourgoulhon [2007], Malik and Wands [2009] with an orthogonal timelike vector $\bar{\boldsymbol{n}}$ and induced metric $\bar{\boldsymbol{h}}$ of the spatial hypersurface which obey the following relations

$$\bar{g}_{\mu\nu} = \bar{h}_{\mu\nu} - \bar{n}_\mu \bar{n}_\nu\,, \qquad \bar{h}_{\mu\nu}\bar{n}^\mu = 0, \qquad \bar{h}^\mu{}_\rho \bar{h}^\rho{}_\nu = \bar{h}^\mu{}_\nu\,.$$

There are a range of background cosmologies available to choose from in *xPand*, however for the purposes of this report flat FLRW was chosen. This is necessary as the MA has only been developed for a flat FLRW universe; for a curved universe it is probable that the MA would have to be modified to agree with PGT. It is also worth noting that *xPand* works in the conformal frame and that time derivatives refer to conformal time. Now let us look at a straightforward example of how *xPand* can be used to obtain cosmological perturbations:

- As before we employ *xTensor* for the geometric preliminaries:

```
DefManifold[ M, 4, {α, β, μ, ν} ];
DefMetric[ -1, g[-α,-β], CD, {";", "∇̄"} ];
```

- Now we use the `SetSlicing` function to assign the cosmology we wish to work in:

```
SetSlicing[ g, n, h, cd, {"|", "D"}, "FLFlat" ];
```

- We define our metric perturbations with the following commands:

```
DefMetricFields[ g, dg, h ]
DefMatterFields[ u, dg, h ]
```

- *xPand* also allows us to define additional quantities in the 3+1 formalism using the `DefProjectedTensor` function. Finally we define a function `MyToxPand` from `ToxPand` which condenses several functions needed for perturbations into one easy to use function:

```
MyToxPand[expr, gauge, order ] = ToxPand[expr, dg, u, du, h, gauge, order]
```

`expr` and `order` naturally denote the expression you wish to perturb and the perturbation order you want. The `gauge` input allows the user to specify a gauge if desired, no gauge can be chosen by inputing "`AnyGauge`". We are now ready to do cosmological perturbations.

- Here is the perturbed Ricci scalar in flat FLRW spacetime:

In[5] := `MyToxPand[RicciScalar[], "AnyGauge", 1]`

$$
\begin{aligned}
Out[5] := 6\,\mathcal{H}^2 + 6\,\mathcal{H}' + \varepsilon\big(&-12\,\mathcal{H}^{2\;(1)}\phi - 12\,\mathcal{H}'^{\;(1)}\phi - 6\,\mathcal{H}^{\;(1)}\phi' - 18\,\mathcal{H}^{\;(1)}\psi' - 6\;^{(1)}\psi'' \\
&+ 6\,\mathcal{H}\,D_\alpha D^{\alpha\;(1)}B - 2\,D_\alpha D^{\alpha\;(1)}B' + 6\,\mathcal{H}\,D_\alpha D^{\alpha\;(1)}E' + 2\,D_\alpha D^{\alpha\;(1)}E'' \\
&- 2\,D_\alpha D^{\alpha\;(1)}\phi + 4\,D_\alpha D^{\alpha\;(1)}\psi\big)
\end{aligned}
$$

where D_α represents the spatial derivative on the hypersurface. E and B are scalar perturbations of the metric as defined in the 3+1 formalism and ϕ and ψ are the Bardeen potentials Uggla and Wainwright [2011]. Note going forward the Bardeen potentials will be relabelled so as not to be confused with the scalar fields in the MA

$$\phi \to X, \quad \psi \to Y. \tag{11}$$

2.2 Obtaining desired equations

Using the variational derivative tool of *xPert*, `VarD`, we can obtain the covariant field equations and the ϕ and ψ equations of motion directly from the MA. See Appendix for full equations.

The size of these equations demonstrates the power and necessity of computer algebra, obtaining them by hand let alone perturbing would be a daunting task. While the detail of these equations may seem impenetrable at the moment, they can be understood at a broad strokes level in far simpler forms through considerations of the original Lagrangian.

Due to the linear nature of the variational derivative method we are free to split the Lagrangian into additive sections, vary them separately and recombine the outputs to obtain the same equations as if we had varied the total Lagrangian. If we split the Cuscuton term, $\sqrt{|J_\mu J^\mu|}$ from the MA and then vary the remainder, the equations become far simpler:

- Contribution to the ϕ-equation from $\mathcal{L}_{\mathrm{MA}} - \sqrt{|J_\mu J^\mu|}$

$$L_\phi = -2\phi\big({M_{\mathrm{Pl}}}^2 + 3\sigma_1 {\psi}^2\big). \tag{12}$$

- Contribution to the ψ-equation from $\mathcal{L}_{\mathrm{MA}} - \sqrt{|J_\mu J^\mu|}$

$$L_\psi = \psi\big(6{M_{\mathrm{Pl}}}^2 \upsilon_1 - \sigma_1\big(6{\phi}^2 + R\,[\bar{\nabla}]\,\big)\big) + 6\sigma_1\big(\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}^{\mathrm{a1}}\psi\big). \tag{13}$$

- Contribution to the field equations from $\mathcal{L}_{\mathrm{MA}} - \sqrt{|J_\mu J^\mu|}$

$$\begin{aligned} L_{bc} =& \frac{1}{12}\big(-2\big(4{M_{\mathrm{Pl}}}^2 + 3\sigma_1{\psi}^2\big)R\,[\bar{\nabla}]_{bc} + \bar{g}_{bc}\big(6{\phi}^2\big({M_{\mathrm{Pl}}}^2 + 3\sigma_1{\psi}^2\big) + 4{M_{\mathrm{Pl}}}^2 R\,[\bar{\nabla}] + \\ & {\psi}^2\big(-18{M_{\mathrm{Pl}}}^2\upsilon_1 + 3\sigma_1 R\,[\bar{\nabla}]\,\big) - 12\sigma_1\psi\big(\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}^{\mathrm{a1}}\psi\big) + 6\sigma_1\big(\bar{\nabla}_{\mathrm{a1}}\psi\big)\big(\bar{\nabla}^{\mathrm{a1}}\psi\big)\big) + \\ & 6\sigma_1\big(-4\big(\bar{\nabla}_b\psi\big)\big(\bar{\nabla}_c\psi\big) + \psi\big(\bar{\nabla}_b\bar{\nabla}_c\psi + \bar{\nabla}_c\bar{\nabla}_b\psi\big)\big)\big). \end{aligned} \tag{14}$$

Evidently the vast majority of the complexity of the full covariant equations comes from the Cuscuton. When varied by itself the Cuscuton returns terms of the form C/g^3 for the ϕ and ψ equations and $\boldsymbol{C}/g$ for the field equations. The Cuscuton terms can then be added to 12, 13 and 14 to create the full equations.

The field equations are:

$$\boldsymbol{L} + \boldsymbol{C}/g = 0 \tag{15}$$

where the ***bold*** denotes the fact this is a tensor equation,

the ϕ equation is

$$L_\phi + C_\phi/g^3 = 0, \tag{16}$$

and the ψ equation is

$$L_\psi + C_\psi/g^3 = 0. \tag{17}$$

Here C and g refer to the contributions from the Cusctuon and the various L refer to the contributions form the remainder of the MA. It is worth noting that the g in all three expressions is the same and is

$$g \equiv \big[\big({M_{\rm Pl}}^2+3\sigma_1{\psi}^2\big)^2\big(\big(\bar{\nabla}_{\rm a1}\phi\big)\big(\bar{\nabla}^{\rm a1}\phi\big)\big)+ \\ 3\sigma_1\phi\psi\big(-2\big({M_{\rm Pl}}^2+3\sigma_1{\psi}^2\big)\big(\big(\bar{\nabla}_{\rm a1}\psi\big)\big(\bar{\nabla}^{\rm a1}\phi\big)\big)+3\sigma_1\phi\psi\big(\big(\bar{\nabla}_{\rm a1}\psi\big)\big(\bar{\nabla}^{\rm a1}\psi\big)\big)\big)\big]^{1/2}. \tag{18}$$

This understanding of the form of the covariant equations will aid in perturbing them, which is the concern of Sections 3 and 4.

3 Why g can't be in the denominator

Separating the Cuscuton from the rest of the Lagrangian then varying the terms separately leads to covariant equations 15, 16 and 17. The background solutions, $\psi = constant$ and $\phi \propto H$, are valid solutions to the following equations for $n \geq 0$:

$$g^n\boldsymbol{L}+\boldsymbol{C}g^{n-1}=0, \tag{19a}$$
$$g^nL_\phi+C_\phi g^{n-3}=0, \tag{19b}$$
$$g^{n+1}L_\psi+C_\psi g^{n-2}=0. \tag{19c}$$

Consequently, we are free to perturb any of these equations around the background values.

Let us consider the $n = 0$ case for the field equations. On the background $\boldsymbol{C} \to g^2$ and so the field equations become $\boldsymbol{L} = 0$. This is exactly equation 9 which gives the Friedmann equations with dark energy.

Now let us consider the first order perturbation for n=0

$$\delta\boldsymbol{L}+\frac{1}{\bar{g}^2}(-\delta g\bar{\boldsymbol{C}}+\bar{g}\delta\boldsymbol{C})=0, \tag{20}$$

where the over-bar denotes a quantity on the background. As $\delta\boldsymbol{C} \propto \bar{g}(\delta g + A\nabla\delta\psi)$ the factors of $\bar{g}$ divide out and the equation becomes

$$\delta\boldsymbol{L} \propto \delta g + B\nabla\delta\psi. \tag{21}$$

This perturbed equation, like its 0th order counterpart, is completely free of any singularities, and returns Einstein-like perturbations sourced by scalar field terms. We can arrive at this conclusion without making any assumptions about the behaviour of $\delta\psi$ or $\delta\phi$.

However at second order we encounter a problem, the second order field equations are

$$^{(2)}\boldsymbol{L}+\frac{1}{g^3}(2\delta g^2\bar{\boldsymbol{C}}-(\delta^{(2)}g)\bar{\boldsymbol{C}}\bar{g}-2\delta\boldsymbol{\delta C}\bar{g}+\boldsymbol{\delta}^{(2)}\boldsymbol{C}\bar{g}^2)=0. \tag{22}$$

The g terms do not all cancel out in the fraction i.e. you would have terms like $2\delta g^2/g$ to contend with. Which, without specifying certain behaviour of δg, would be a singular term on the background. Even if ϕ and ψ were only near their background values this perturbed scalar term would dominate over the tensor perturbations from $\boldsymbol{L}$, disabling the Einstein-like character of the perturbations. We will continue to encounter this problem at all higher orders. This problem would naturally be present in the $n = 0$ ϕ- and ψ-equations as well. So, without a careful prescription for all perturbative orders of the equation we would encounter singularities.

To avoid this problem all we should have to do is choose n for our starting equations such that g is in the numerator only. However carefully considering the behaviour of g as it is perturbed reveals another problem, discussed in Section 4.

4 The $g = \sqrt{J_a J^a}$ term introduces potential singularities

The problem of entering powers of the background value of g, $\bar{g}$, into the denominators of terms occurs even when we start with g in the numerator. The reader is invited to refer to 18 which is really $\sqrt{J_a J^a}$ where

$$J_a = \left({M_{\mathrm{Pl}}}^2 + 3\sigma_1 \psi^2\right)\bar{\nabla}_{\mathrm{a}}\phi - 3\sigma_1 \phi\psi\bar{\nabla}_{\mathrm{a}}\psi. \tag{23}$$

Before we dive into the direct perturbation of g, let us understand why it contains singular terms when perturbed. Consider the following perturbed expressions $\sqrt{(\tilde{\phi})^2}$, $\sqrt{\tilde{\phi}}$:

$$\sqrt{(\tilde{\phi})^2} \approx \sqrt{(\phi)^2} + \frac{\epsilon\left({}^{(1)}\phi\right)\sqrt{(\phi)^2}}{(\phi)} + \frac{\epsilon^2\left({}^{(2)}\phi\right)\sqrt{(\phi)^2}}{2(\phi)} + \frac{\epsilon^3\left({}^{(3)}\phi\right)\sqrt{(\phi)^2}}{6(\phi)} \tag{24a}$$

$$\sqrt{\tilde{\phi}} \approx \epsilon^2\Big(- \frac{\left({}^{(1)}\phi\right)^2}{8(\phi)^{3/2}} + \frac{{}^{(2)}\phi}{4\sqrt{(\phi)}}\Big) + \epsilon^3\Big(\frac{\left({}^{(1)}\phi\right)^3}{16(\phi)^{5/2}} - \frac{\left({}^{(1)}\phi\right)\left({}^{(2)}\phi\right)}{8(\phi)^{3/2}} + \frac{{}^{(3)}\phi}{12\sqrt{(\phi)}}\Big) + \frac{\epsilon\left({}^{(1)}\phi\right)}{2\sqrt{(\phi)}} + \sqrt{(\phi)}. \tag{24b}$$

Clearly the perturbation of $\sqrt{\phi^2}$ is just the perturbation of ϕ and is stable as $\phi \to 0$, whereas the perturbation of $\sqrt{\phi}$ is clearly singular as $\phi \to 0$. The behaviour of $g = \sqrt{J_a J^a}$ sits in between $24a$ and $24b$.

To explain what is meant by this consider the way computer algebra naively attempts to perturb $\sqrt{x^2}$. At second order the peturbative algorithm would generate the following terms

$$\sqrt{(\tilde{x})^2}_2 = \frac{(\delta^{(1)}x)^2}{\sqrt{x^2}} + \frac{x\delta^{(2)}x}{2\sqrt{x^2}} - \frac{(\delta^{(1)}x)^2}{\sqrt{x^2}}. \tag{25}$$

Clearly in this example the $(\delta^{(1)}x)^2$ terms cancel to reproduce the result for x perturbed to second order. However $\sqrt{J_a J^a}$ only permits time-dependent perturbations when it is sent to a specific spacetime, not spatial or metric perturbations. This means that only the time-dependent perturbations are cancelled in the way $(\delta^{(1)}x)^2$ terms are. The metric and spatial perturbations will enter into the system and carry extra terms in their denominators which cause them to be singular/dominant.

In order to understand the nature of the dominant terms introduced by g let us consider a simpler but illustrative and similar example $\sqrt{\nabla_a\psi\nabla^a\psi}$. The second order perturbation of $\sqrt{\nabla_a\psi\nabla^a\psi}$ in flat FRW spacetime is:

$$\begin{aligned}
&\delta(\sqrt{\nabla_a\psi\nabla^a\psi}) \approx \qquad (26)\\
&-\frac{1}{2(\acute{\psi})^2}\sqrt{-\frac{(\acute{\psi})^2}{(a)^2}}\big(-\big({}^{(2)}\acute{\psi}\big)(\acute{\psi})+\big({}^{(1)}B_a\big)\big({}^{(1)}B^a\big)(\acute{\psi})^2+2\big({}^{(1)}\acute{\psi}\big)(\acute{\psi})\big({}^{(1)}X\big)-\\
&3(\acute{\psi})^2\big({}^{(1)}X\big)^2+(\acute{\psi})^2\big({}^{(2)}X\big)+2\big({}^{(1)}B^a\big)(\acute{\psi})^2\big(\bar{D}_a{}^{(1)}B\big)+2\big({}^{(1)}B^a\big)(\acute{\psi})\big(\bar{D}_a{}^{(1)}\psi\big)+\\
&(\acute{\psi})^2\big(\bar{D}_a{}^{(1)}B\big)\big(\bar{D}^{a(1)}B\big)+2(\acute{\psi})\big(\bar{D}_a{}^{(1)}\psi\big)\big(\bar{D}^{a(1)}B\big)+\big(\bar{D}_a{}^{(1)}\psi\big)\big(\bar{D}^{a(1)}\psi\big)\big),
\end{aligned}$$

where $\acute{\psi}$ refers to the derivative of ψ with respect to conformal time. Of particular concern is the last term $\big(\bar{D}_a{}^{(1)}\psi\big)\big(\bar{D}^{a(1)}\psi\big)\frac{1}{2a(\acute{\psi})}$. The $\acute{\psi}$ is analogous to the FRW value of g in our theory $\bar{g} = \big(\acute{\phi}\big({M_{\rm Pl}}^2 + 3\sigma_1\psi^2\big) - 3\sigma_1\phi\psi\acute{\psi}\big)$ which is 0 on the background, this leads to $\big(\bar{D}_a{}^{(1)}\psi\big)\big(\bar{D}^{a(1)}\psi\big)$ becoming the dominant term. This is the exact same term that becomes dominant when we perturb $\sqrt{J_a J^a}$ and leads to the conclusion that the perturbed ψ field cannot vary spatially.

Note that odd powers of g greater than one also have this problem; consider g^3 which is actually $\sqrt{J_a J^a}(J_a J^a)$. While perturbations of the $J_a J^a$ component will never introduce terms into the denominator, the $\sqrt{J_a J^a}$ component will. We would only be able to avoid this behaviour if every g term was to an even power.

This is never the case. All the equations, no matter the value of n, contain odd powers of g. They will all contain a square root term and eventually lead to a singularity/dominating term at a given perturbative level.

To see why the dominant term from g is $\big(\bar{D}_a{}^{(1)}\psi\big)\big(\bar{D}^{a(1)}\psi\big)$ let us consider the form of g and cosmological perturbations in more detail.

We might expect that alongside $\big(\bar{D}_a{}^{(1)}\psi\big)\big(\bar{D}^{a(1)}\psi\big)$ we would have a $\big(\bar{D}_a{}^{(1)}\phi\big)\big(\bar{D}^{a(1)}\phi\big)$ term. However the nature of g leads to this not being the case.

Recall $J_a = \big({M_{\rm Pl}}^2+3\sigma_1\psi^2\big)\bar{\nabla}_{\rm a}\phi-3\sigma_1\phi\psi\bar{\nabla}_{\rm a}\psi$. The $\bar{\nabla}_{\rm a}\phi$ term is coupled to $\big({M_{\rm Pl}}^2+3\sigma_1\psi^2\big)$ which goes to zero on the background. This is why the first pair of perturbed spatial derivatives we see are for ψ, the spatial ϕ terms are suppressed by the factor of $\big({M_{\rm Pl}}^2+3\sigma_1\psi^2\big)$

until you go to higher order.

Another consideration is perturbed metric terms. Like the perturbed spatial derivatives they will not be produced by background expressions and therefore we expect them to be dominant too. However at lower orders they are suppressed; perturbed metric terms will be coupled to the background value of g meaning they will not be the leading dominant term. As an example let us look at a metric perturbation from our earlier illustrative model: $\left({}^{(1)}B_a\right)\left({}^{(1)}B^a\right)(\acute{\psi})^2$. We can clearly see this term is coupled to $(\acute{\psi})$ which in this toy model is being set to zero on the background analogous to $\bar{g}$.

At higher orders we would expect both spatial ϕ perturbation pairs, $\left(\bar{D}_a{}^{(1)}\phi\right)\left(\bar{D}^{a(1)}\phi\right)$, and metric perturbations like $\left({}^{(1)}B_a\right)\left({}^{(1)}B^a\right)$ to appear as dominant terms. Given that the spatial ψ perturbations would already have been set to zero at lower orders we might expect ϕ spatial perturbations and metric perturbations not to interact with the spatial ψ perturbations. However direct verification of this particular result and an assessment of the exact character of the spatial ϕ and metric perturbations is not currently feasible due to the computational limits of *xPand* when trying to obtain higher order perturbed terms.

Evidently, more investigation into higher order perturbations is needed to isolate the exact behaviour of the dynamical perturbed terms, especially for ϕ, and any potential interactions they might have with scalar perturbed metric terms. Additionally it is entirely possible that when ϕ and ψ are not near their background values, the full numerical simulations of the perturbed cosmological equations do not evolve ϕ and ψ towards their background values. However understanding these phenomena is beyond the current scope of this report.

Instead let us turn our attention to reconsidering the equations of motion and field equations where we suppress the spatial components of ψ, motivated by the results of the lower order perturbed equations, and naively consider this a sufficient stability condition. This is the focus of the next section.

5 Perturbations with suppressed spatial components

Starting with this set of covariant equations let us consider their perturbation with the spatial component of ψ suppressed

$$g\boldsymbol{L} + \boldsymbol{C} = 0, \tag{27a}$$
$$g^3 L_\phi + C_\phi = 0, \tag{27b}$$
$$g^3 L_\psi + C_\psi = 0. \tag{27c}$$

Note that we still have to motivate the suppression of the spatial component from these equations, and as it is impossible to try and obtain two competing solutions from one equation simultaneously we use 27*b* to switch off the spatial ψ perturbations and consider equations 27*c* and 27*a* under this condition. Let us consider 27*c* equation at first order

$$3(\delta g)g^2 L_\psi + g^3 \delta L_\psi + \delta C_\psi = 0. \tag{28}$$

On the background g and δC_ψ both become zero, killing the whole first order equation. However $\delta C_\psi / g^2$ is convergent and non zero at the background level, hence if we divide through by g^2 and then impose background conditions the equation becomes

$$3(\delta g)L_\psi + \delta C_\psi / g^2 = 0. \tag{29}$$

Let us consider the $\delta C_\psi / g^2$ and δg terms which are of the form

$$\delta C_\psi / g^2 \propto \acute{\mathcal{H}}\left(2\mathcal{H}^2\left({}^{(1)}\psi\right) - 2\acute{\mathcal{H}}\left({}^{(1)}\psi\right) + \mathcal{H}\left({}^{(1)}\acute{\psi}\right)\right) - \frac{1}{9}\mathcal{H}^2 \bar{D}_a \bar{D}^{a(1)}\psi, \tag{30a}$$

$$\delta g \propto 2\mathcal{H}^2\left({}^{(1)}\psi\right) - 2\acute{\mathcal{H}}\left({}^{(1)}\psi\right) + \mathcal{H}\left({}^{(1)}\acute{\psi}\right). \tag{30b}$$

Remember we are suppressing the ψ spatial perturbations and so the $\bar{D}_a \bar{D}^{a(1)}\psi$ term in $\delta C_\psi / g^2$ is zero. This leads to $\delta C_\psi / g^2 \propto \delta g$ and an ordinary differential equation (ODE) in conformal time $\delta g = 0$ satisfying the first order perturbed ψ equation. Let us examine this ODE.

Relabelling ${}^{(1)}\psi \to y$, the ODE becomes

$$2(\mathcal{H}^2 - \mathcal{H}')y + \mathcal{H}\delta y' = 0. \tag{31}$$

The original un-perturbed equations of motion were for matter-only and dark-energy-only epochs, for both of which the $\delta g = 0$ equation is the same, so now let us consider 31 in both those epochs.

In the matter dominated epoch the scale factor becomes Weinberg [2008]

$$a \propto t^{2/3}. \tag{32}$$

The definition of conformal time $a d\tau = dt$, gives

$$a \propto \tau^2 \tag{33}$$

$$\mathcal{H} = \frac{2}{\tau} \tag{34}$$

and 31 becomes

$$y' = -\frac{6}{\tau}y. \tag{35}$$

This yields $\delta\psi \propto \tau^{-6}$ meaning perturbations in the ψ field decay away in the matter dominated epoch.

Now doing the same for the dark energy dominated era, the scale factor is Weinberg [2008]

$$a \propto e^{H_0 t}. \tag{36}$$

In conformal time this gives

$$a \propto \frac{1}{\tau} \tag{37}$$
$$\mathcal{H} = -\frac{1}{\tau}. \tag{38}$$

However this renders $\mathcal{H}^2 - \mathcal{H}' = 0$ leaving us with $y' = 0$ and ${}^{(1)}\psi =$ constant. So in the matter dominated era perturbations to the background torsion field decay away. Then as the Universe becomes more dominated by the torsion field itself, what is left of these perturbations freezes and becomes constant, however in both regimes the system is stable.

Now we turn our attention to the field equations. At the first perturbed order we have

$$\delta g \boldsymbol{L} + g \boldsymbol{\delta L} + \boldsymbol{\delta C} = 0. \tag{39}$$

Imposing the background values of ϕ and ψ would completely kill 39, i.e. set all terms to zero. So let us now consider the second order equation

$$(\delta^{(2)} g)\boldsymbol{L} + \delta g \boldsymbol{\delta L} + g \boldsymbol{\delta}^{(2)}\boldsymbol{L} + \boldsymbol{\delta}^{(2)}\boldsymbol{C} = 0. \tag{40}$$

As we have suppressed the perturbed spatial derivatives of ψ, $\delta^{(2)} g$ no longer contains dominating terms and so we are free to impose background conditions giving us

$$\boldsymbol{\delta L} = \boldsymbol{\delta}^{(2)}\boldsymbol{C}/(\delta g). \tag{41}$$

We might be worried that $\boldsymbol{\delta}^{(2)}\boldsymbol{C}/(\delta g)$ would be divergent if we were to impose the ODE condition from the first order perturbed ψ equation. However let us examine $\boldsymbol{\delta}^{(2)}\boldsymbol{C}$

$$\begin{split}\boldsymbol{\delta}^{(2)}\boldsymbol{C} \propto \bar{h}_{bc}\big(4\mathcal{H}^4\big({}^{(1)}\psi\big)^2 + 4\acute{\mathcal{H}}^2\big({}^{(1)}\psi\big)^2 + 4\mathcal{H}^3\big({}^{(1)}\psi\big)\big({}^{(1)}\acute{\psi}\big) - 4\mathcal{H}\acute{\mathcal{H}}\big({}^{(1)}\psi\big)\big({}^{(1)}\acute{\psi}\big) + \\ \mathcal{H}^2\big(-8\acute{\mathcal{H}}\big({}^{(1)}\psi\big)^2 + \big({}^{(1)}\acute{\psi}\big)^2 - \big(\bar{D}_a{}^{(1)}\psi\big)\big(\bar{D}^{a(1)}\psi\big)\big)\big) + \mathcal{H}\big(-2\mathcal{H}^2\big({}^{(1)}\psi\big)\big(\bar{n}_c\big(\bar{D}_b{}^{(1)}\psi\big) + \\ \bar{n}_b\big(\bar{D}_c{}^{(1)}\psi\big)\big) + 2\acute{\mathcal{H}}\big({}^{(1)}\psi\big)\big(\bar{n}_c\big(\bar{D}_b{}^{(1)}\psi\big) + \bar{n}_b\big(\bar{D}_c{}^{(1)}\psi\big)\big) + \mathcal{H}\big(\big(\bar{D}_b{}^{(1)}\psi\big)\big(-\bar{n}_c\big({}^{(1)}\acute{\psi}\big) + \\ \bar{D}_c{}^{(1)}\psi\big) + \bar{n}_b\big(\bar{n}_c\big(\bar{D}_a{}^{(1)}\psi\big)\big(\bar{D}^{a(1)}\psi\big) - \big({}^{(1)}\acute{\psi}\big)\big(\bar{D}_c{}^{(1)}\psi\big)\big)\big)\big)\big).\end{split} \tag{42}$$

As before we suppress the perturbed spatial variation of ψ and we find 42 becomes

$$\begin{split}\boldsymbol{\delta}^{(2)}\boldsymbol{C} \propto \bar{h}_{bc}\big(4\mathcal{H}^4\big({}^{(1)}\psi\big)^2 + 4\acute{\mathcal{H}}^2\big({}^{(1)}\psi\big)^2 + 4\mathcal{H}^3\big({}^{(1)}\psi\big)\big({}^{(1)}\acute{\psi}\big) - 4\mathcal{H}\acute{\mathcal{H}}\big({}^{(1)}\psi\big)\big({}^{(1)}\acute{\psi}\big) + \\ \mathcal{H}^2\big(-8\acute{\mathcal{H}}\big({}^{(1)}\psi\big)^2 + {}^{(1)}\acute{\psi}\big)^2\big),\end{split} \tag{43}$$

which is simply $\boldsymbol{\delta}^{(2)}\boldsymbol{C} \propto \bar{h}_{bc}(\delta g)^2$. This means we are free to impose the ODE condition from the ψ equation and arrive at

$$\boldsymbol{\delta L} = 0. \tag{44}$$

Remember that the $\boldsymbol{L}$ contains the Einstein tensor and will source similar perturbations to those from classical GR. We now examine these perturbations in detail in the next section.

Before moving on we should note the conspicuous absence of any perturbed ϕ terms in the perturbed ψ equation. They are also not present in the field equations in the next section. The reason they are not present is the same reason that $\left(\bar{D}_a{}^{(1)}\psi\right)\left(\bar{D}^{a(1)}\psi\right)$ is the first dominant term: the ϕ perturbations are coupled to terms that go to zero on the background. Consequently they will only enter into the perturbation equations at higher orders and as before these equations are not obtainable currently due to the computational limits of *xPand*.

6 Analysing the perturbed field equations

Written in full $\boldsymbol{\delta L} = 0$ is

$$\begin{aligned}
&{M_{\mathrm{Pl}}}^3 \upsilon_1 a^2 \Big(2M_{\mathrm{Pl}}\left({}^{(1)}E_{bc}\right) - M_{\mathrm{Pl}}\left({}^{(1)}B_c\right)\bar{n}_b - M_{\mathrm{Pl}}\left({}^{(1)}B_b\right)\bar{n}_c + 2\sqrt{3}\sqrt{-\sigma_1}\,\bar{h}_{bc}\left({}^{(1)}\psi\right) - \\
&2\sqrt{3}\sqrt{-\sigma_1}\,\bar{n}_b\bar{n}_c\left({}^{(1)}\psi\right) - 2M_{\mathrm{Pl}}\bar{n}_b\bar{n}_c\left({}^{(1)}X\right) - 2M_{\mathrm{Pl}}\bar{h}_{bc}\left({}^{(1)}Y\right)\Big)\tfrac{1}{2\sigma_1} - \tfrac{1}{12}M_{\mathrm{Pl}}\Big(6M_{\mathrm{Pl}}\left({}^{(1)}E''_{bc}\right) + \\
&12M_{\mathrm{Pl}}\left({}^{(1)}E'_{bc}\right)\mathcal{H} - 12M_{\mathrm{Pl}}\left({}^{(1)}E_{bc}\right)\mathcal{H}^2 - 24M_{\mathrm{Pl}}\left({}^{(1)}E_{bc}\right)\mathcal{H}' + 6M_{\mathrm{Pl}}\left({}^{(1)}B_c\right)\mathcal{H}^2\bar{n}_b + 12M_{\mathrm{Pl}}\left({}^{(1)}B_c\right)\mathcal{H}'\bar{n}_b + \\
&6M_{\mathrm{Pl}}\left({}^{(1)}B_b\right)\mathcal{H}^2\bar{n}_c + 12M_{\mathrm{Pl}}\left({}^{(1)}B_b\right)\mathcal{H}'\bar{n}_c - 72\sqrt{-\sigma_1}\,\bar{h}_{bc}\mathcal{H}^2\left({}^{(1)}\psi\right) + 52\sqrt{3}\sqrt{-\sigma_1}\,\bar{h}_{bc}\mathcal{H}^2\left({}^{(1)}\psi\right) + \\
&8\sqrt{3}\sqrt{-\sigma_1}\,\bar{h}_{bc}\mathcal{H}'\left({}^{(1)}\psi\right) + 72\sqrt{-\sigma_1}\,\mathcal{H}^2\bar{n}_b\bar{n}_c\left({}^{(1)}\psi\right) - 60\sqrt{3}\sqrt{-\sigma_1}\,\mathcal{H}^2\bar{n}_b\bar{n}_c\left({}^{(1)}\psi\right) + \\
&4\sqrt{3}\sqrt{-\sigma_1}\,\bar{h}_{bc}\mathcal{H}\left({}^{(1)}\psi'\right) - 12\sqrt{3}\sqrt{-\sigma_1}\,\mathcal{H}\bar{n}_b\bar{n}_c\left({}^{(1)}\psi'\right) + 4\sqrt{3}\sqrt{-\sigma_1}\,\bar{h}_{bc}\left({}^{(1)}\psi''\right) + 12M_{\mathrm{Pl}}\bar{h}_{bc}\mathcal{H}^2\left({}^{(1)}X\right) + \\
&24M_{\mathrm{Pl}}\bar{h}_{bc}\mathcal{H}'\left({}^{(1)}X\right) + 12M_{\mathrm{Pl}}\bar{h}_{bc}\mathcal{H}\left({}^{(1)}X'\right) + 12M_{\mathrm{Pl}}\bar{h}_{bc}\mathcal{H}^2\left({}^{(1)}Y\right) + 24M_{\mathrm{Pl}}\bar{h}_{bc}\mathcal{H}'\left({}^{(1)}Y\right) + 24M_{\mathrm{Pl}}\bar{h}_{bc}\mathcal{H}\left({}^{(1)}Y'\right) - \\
&36M_{\mathrm{Pl}}\mathcal{H}\bar{n}_b\bar{n}_c\left({}^{(1)}Y'\right) + 12M_{\mathrm{Pl}}\bar{h}_{bc}\left({}^{(1)}Y''\right) + 3M_{\mathrm{Pl}}\bar{n}_c\left(\bar{D}_a\bar{D}^{a(1)}B_b\right) + 3M_{\mathrm{Pl}}\bar{n}_b\left(\bar{D}_a\bar{D}^{a(1)}B_c\right) - \\
&6M_{\mathrm{Pl}}\left(\bar{D}_a\bar{D}^{a(1)}E_{bc}\right) - 4\sqrt{3}\sqrt{-\sigma_1}\,\bar{h}_{bc}\left(\bar{D}_a\bar{D}^{a(1)}\psi\right) + 4\sqrt{3}\sqrt{-\sigma_1}\,\bar{n}_b\bar{n}_c\left(\bar{D}_a\bar{D}^{a(1)}\psi\right) + 6M_{\mathrm{Pl}}\bar{h}_{bc}\left(\bar{D}_a\bar{D}^{a(1)}X\right) - \\
&6M_{\mathrm{Pl}}\bar{h}_{bc}\left(\bar{D}_a\bar{D}^{a(1)}Y\right) + 12M_{\mathrm{Pl}}\bar{n}_b\bar{n}_c\left(\bar{D}_a\bar{D}^{a(1)}Y\right) - 6M_{\mathrm{Pl}}\mathcal{H}\left(\bar{D}_b{}^{(1)}B_c\right) - 3M_{\mathrm{Pl}}\left(\bar{D}_b{}^{(1)}B'_c\right) + \\
&4\sqrt{3}\sqrt{-\sigma_1}\,\mathcal{H}\bar{n}_c\left(\bar{D}_b{}^{(1)}\psi\right) - 4\sqrt{3}\sqrt{-\sigma_1}\,\bar{n}_c\left(\bar{D}_b{}^{(1)}\psi'\right) - 12M_{\mathrm{Pl}}\mathcal{H}\bar{n}_c\left(\bar{D}_b{}^{(1)}X\right) - \\
&12M_{\mathrm{Pl}}\bar{n}_c\left(\bar{D}_b{}^{(1)}Y'\right) - 6M_{\mathrm{Pl}}\mathcal{H}\left(\bar{D}_c{}^{(1)}B_b\right) - 3M_{\mathrm{Pl}}\left(\bar{D}_c{}^{(1)}B'_b\right) + 4\sqrt{3}\sqrt{-\sigma_1}\,\mathcal{H}\bar{n}_b\left(\bar{D}_c{}^{(1)}\psi\right) - \\
&4\sqrt{3}\sqrt{-\sigma_1}\,\bar{n}_b\left(\bar{D}_c{}^{(1)}\psi'\right) - 12M_{\mathrm{Pl}}\mathcal{H}\bar{n}_b\left(\bar{D}_c{}^{(1)}X\right) - 12M_{\mathrm{Pl}}\bar{n}_b\left(\bar{D}_c{}^{(1)}Y'\right) + 4\sqrt{3}\sqrt{-\sigma_1}\left(\bar{D}_c\bar{D}_b{}^{(1)}\psi\right) - \\
&6M_{\mathrm{Pl}}\left(\bar{D}_c\bar{D}_b{}^{(1)}X\right) + 6M_{\mathrm{Pl}}\left(\bar{D}_c\bar{D}_b{}^{(1)}Y\right)\Big) = 0.
\end{aligned}$$

Please refer to the end of section 2.1 and 11 for details of notation in the perturbed field equations.

We have imposed the Newtonian gauge which reduces all scalar perturbations to the Bardeen potentials. This helps simplify the equations but in this form they are still intimidating. Fortunately the tools of *xPand* allow us to separate this equation into its [$\texttt{Time}, \texttt{Time}$], [$\texttt{Time}, \texttt{Space}$], [$\texttt{Space}, \texttt{Time}$] and [$\texttt{Space}, \texttt{Space}$] components along the lines of the 3+1 formalism. This will allow for easier interpretation and more direct comparison with the analogous equations from the Einstein Tensor:

- Here is the [$\texttt{Time}, \texttt{Time}$] component from the MA field equations with the perturbed spatial component of ψ suppressed

$$3M_{\text{Pl}}\big(-\sigma_1\big)^{3/2}\big(3\mathcal{H}\big({}^{(1)}\acute{Y}\big) - \bar{D}_a\bar{D}^{a(1)}Y\big) = 3{M_{\text{Pl}}}^2\upsilon_1 a^2\big(-\sqrt{3}\sigma_1\big({}^{(1)}\psi\big)+ M_{\text{Pl}}\sqrt{-\sigma_1}\big({}^{(1)}X\big)\big) + {\sigma_1}^2\big(3\big(-6+5\sqrt{3}\big)\mathcal{H}^2\big({}^{(1)}\psi\big) + 3\sqrt{3}\mathcal{H}\big({}^{(1)}\acute{\psi}\big)\big). \tag{45}$$

- Here is the GR equivalent

$$3\mathcal{H}\big({}^{(1)}\acute{Y}\big) - \big(\bar{D}_a\bar{D}^{a(1)}Y\big) = 0. \tag{46}$$

The [$\texttt{Time}, \texttt{Space}$] and [$\texttt{Space}, \texttt{Time}$] components of the field equations are identical and consequently only one need be analysed.

- Here is the [$\texttt{Time}, \texttt{Space}$] component from the MA field equations with the perturbed spatial component of ψ suppressed

$$3M_{\text{Pl}}\sigma_1\big(2\big({}^{(1)}B_c\big)\big(\mathcal{H}^2 + 2\acute{\mathcal{H}}\big) + \bar{D}_a\bar{D}^{a(1)}B_c - 4\big(\mathcal{H}\big(\bar{D}_c{}^{(1)}X\big) + \bar{D}_c{}^{(1)}\acute{Y}\big)\big) = 6{M_{\text{Pl}}}^3\upsilon_1 a^2\big({}^{(1)}B_c\big). \tag{47}$$

- Here is the GR equivalent

$$-\big({}^{(1)}B_c\big)\big(\mathcal{H}^2 + 2\acute{\mathcal{H}}\big) - \frac{1}{2}\big(\bar{D}_a\bar{D}^{a(1)}B_c\big) + 2\big(\mathcal{H}\big(\bar{D}_c{}^{(1)}X\big) + \bar{D}_c{}^{(1)}\acute{Y}\big) = 0. \tag{48}$$

- Here is the [$\texttt{Space}, \texttt{Space}$] component from the MA field equations with the perturbed spatial component of ψ suppressed

$$
\begin{aligned}
&-\frac{1}{2}M_{\mathrm{Pl}}\big({}^{(1)}E''_{bc}+2\big({}^{(1)}E'_{bc}\big)\mathcal{H}-2\big({}^{(1)}E_{bc}\big)\mathcal{H}^2-4\big({}^{(1)}E_{bc}\big)\mathcal{H}' \\
&+2\bar{h}_{bc}\mathcal{H}^2\big({}^{(1)}X\big)+4\bar{h}_{bc}\mathcal{H}'\big({}^{(1)}X\big)+2\bar{h}_{bc}\mathcal{H}\big({}^{(1)}X'\big)+2\bar{h}_{bc}\mathcal{H}^2\big({}^{(1)}Y\big) \\
&+4\bar{h}_{bc}\mathcal{H}'\big({}^{(1)}Y\big)+4\bar{h}_{bc}\mathcal{H}\big({}^{(1)}Y'\big)+2\bar{h}_{bc}\big({}^{(1)}Y''\big)-\bar{D}_a\bar{D}^{a(1)}E_{bc}+\bar{h}_{bc}\big(\bar{D}_a\bar{D}^{a(1)}X\big)- \\
&\bar{h}_{bc}\big(\bar{D}_a\bar{D}^{a(1)}Y\big)-\mathcal{H}\big(\bar{D}_b{}^{(1)}B_c\big)-\frac{1}{2}\big(\bar{D}_b{}^{(1)}B'_c\big)-\mathcal{H}\big(\bar{D}_c{}^{(1)}B_b\big)- \\
&\frac{1}{2}\big(\bar{D}_c{}^{(1)}B'_b\big)-\bar{D}_c\bar{D}_b{}^{(1)}X+\bar{D}_c\bar{D}_b{}^{(1)}Y\big) \\
&= \\
&\frac{1}{3\sigma_1}\big(3M_{\mathrm{Pl}}{}^2\upsilon_1 a^2\big(M_{\mathrm{Pl}}\big({}^{(1)}E_{bc}\big)+\bar{h}_{bc}\big(\sqrt{3}\sqrt{-\sigma_1}\big({}^{(1)}\psi\big)-M_{\mathrm{Pl}}\big({}^{(1)}Y\big)\big)\big)+ \\
&\big(-\sigma_1\big)^{3/2}\big(\bar{h}_{bc}\big(\big(-18+13\sqrt{3}\big)\mathcal{H}^2\big({}^{(1)}\psi\big)+\sqrt{3}\mathcal{H}\big({}^{(1)}\psi'\big)+\sqrt{3}\big(2\mathcal{H}'\big({}^{(1)}\psi\big)+{}^{(1)}\psi''\big)\big)\big).
\end{aligned}
\tag{49}
$$

- Here is the GR equivalent

$$
\begin{aligned}
&{}^{(1)}E''_{bc}+2\big({}^{(1)}E'_{bc}\big)\mathcal{H}-2\big({}^{(1)}E_{bc}\big)\mathcal{H}^2-4\big({}^{(1)}E_{bc}\big)\mathcal{H}'+2\bar{h}_{bc}\mathcal{H}^2\big({}^{(1)}X\big)+ \\
&4\bar{h}_{bc}\mathcal{H}'\big({}^{(1)}X\big)+2\bar{h}_{bc}\mathcal{H}\big({}^{(1)}X'\big)+2\bar{h}_{bc}\mathcal{H}^2\big({}^{(1)}Y\big)+4\bar{h}_{bc}\mathcal{H}'\big({}^{(1)}Y\big)+ \\
&4\bar{h}_{bc}\mathcal{H}\big({}^{(1)}Y'\big)+2\bar{h}_{bc}\big({}^{(1)}Y''\big)-\bar{D}_a\bar{D}^{a(1)}E_{bc}+\bar{h}_{bc}\big(\bar{D}_a\bar{D}^{a(1)}X\big)-\bar{h}_{bc}\big(\bar{D}_a\bar{D}^{a(1)}Y\big)- \\
&\mathcal{H}\big(\bar{D}_b{}^{(1)}B_c\big)-\frac{1}{2}\big(\bar{D}_b{}^{(1)}B'_c\big)-\mathcal{H}\big(\bar{D}_c{}^{(1)}B_b\big)-\frac{1}{2}\big(\bar{D}_c{}^{(1)}B'_b\big)-\bar{D}_c\bar{D}_b{}^{(1)}X+\bar{D}_c\bar{D}_b{}^{(1)}Y=0.
\end{aligned}
\tag{50}
$$

The first thing to observe in equations 45, 47 and 49 is that the MA reproduces all of the same terms as GR. The extra terms the MA generates are of two types: terms with a pre-factor of $M_{\mathrm{Pl}}{}^2\upsilon_1/\sigma_1$ and terms with a pre-factor of $\sqrt{-\sigma_1}/M_{\mathrm{Pl}}$. If we return to the the un-perturbed Field equations on the background

$$G_{\mu\nu}+g_{\mu\nu}M_{\mathrm{Pl}}{}^2\upsilon_1/\sigma_1=0 \tag{51}$$

where here g is the metric, we notice that the constant $M_{\mathrm{Pl}}{}^2\upsilon_1/\sigma_1$ is acting as a dark energy component Λ, as discussed in section 1.3. The value of Λ is determined to be very small Copeland et al. [2006], Riess et al. [2004], so terms coupled to it in the perturbed field equations will be dominated by the rest of the equation and can be safely ignored.

We may now turn our attention to the second type of term, those coupled to $\sqrt{-\sigma_1}/M_{\mathrm{Pl}}$. Immediately we can see that in the [`Time`, `Space`] and [`Space`, `Time`] equations there are no terms of the second type and they are completely identical to the GR equivalent aside from a scalar perturbation coupled to dark energy.

The [`Time`, `Time`] and [`Space`, `Space`] equations do contain terms of the second type which depend only on the ψ field and are as follows:

- scalar field component of [`Time`, `Time`]

$$3\big(-6+5\sqrt{3}\big)\mathcal{H}^2\big({}^{(1)}\psi\big)+3\sqrt{3}\mathcal{H}\big({}^{(1)}\acute{\psi}\big), \tag{52}$$

- scalar field component of [`Space`, `Space`]

$$\bar{h}_{bc}\Big(\big(-18+13\sqrt{3}\big)\mathcal{H}^2\big({}^{(1)}\psi\big)+\sqrt{3}\mathcal{H}\big({}^{(1)}\acute{\psi}\big)+\sqrt{3}\big(2\acute{\mathcal{H}}\big({}^{(1)}\psi\big)+{}^{(1)}\psi''\big)\Big). \tag{53}$$

To consider how these terms might affect the Bardeen potentials let us work in a simplified regime were we suppress the spatial variations of the Bardeen potentials and then allow 52 and 53 to source the potentials.

Working in the matter dominated epoch the 52 and 53 both simply become proportional to τ^{-8}. Concerning ourselves solely with variations in conformal time the [`Time`, `Time`] equation reduces to

$$\mathcal{H}\big({}^{(1)}\acute{Y}\big)\propto\tau^{-8}, \tag{54}$$

giving

$${}^{(1)}Y=b_1\tau^{-6}+b_2 \tag{55}$$

where b_1, b_2 are constants of integration. The [`Space`, `Space`] equation allows us to determine the other Bardeen potential ${}^{(1)}X$ which takes a similar form

$${}^{(1)}X=c_1\tau^{-6}+c_2. \tag{56}$$

Again c_1, c_2 are constants of integration. If we apply this same prescription to the GR field equations we would obtain the following result

$${}^{(1)}Y=d_1, \tag{57}$$

$${}^{(1)}X=d_2. \tag{58}$$

While it is important to remember that these solutions do not describe the full behaviour of the Bardeen potentials as we have suppressed their spatial components, it appears that the ψ perturbations are sourcing an additional decaying part to the potential solutions. Naturally full numerical solutions will be required to assess the full character of the equations and quantify any deviation from the ΛCDM model; this would be done with cosmological Boltzmann codes such as *CLASS* Blas et al. [2011] or *CAMB* Lewis and Bridle [2002]. However, the relative simplicity of this model compared with other scalar-tensor theories, due to the suppression of the spatial dynamical behaviour of the ψ field, may give this model an advantage. Especially if high precision measurements of the CMB and LSS give tight constraints upon deviation from ΛCDM predictions.

In this section the tools of *xPand* have allowed us to simplify and separate the field equations for analysis. In a regime where the spatial components of the scalar fields are suppressed Einstein-like field equations are recovered. Solving in a conformal-time only system leads to solutions similar to GR with an additional decaying component to the Bardeen potentials in the Newtonian gauge.

7 Concluding remarks

In this report computer algebra has been employed extensively to obtain covariant equations of motion from an extended gravity theory and examine their cosmological perturbations. This theory is particularly interesting because of the inclusion of the non-canonical Cuscuton, which is known to exhibit unusual behaviour Afshordi et al. [2007b,a], Boruah et al. [2017], Mansoori and Molaee [2023], Bhattacharyya et al. [2018]. It has been demonstrated that when a term deriving from this Cuscuton in the Lagrangian, g as defined in 18, sits in the denominator of these equations, character reminiscent of GR is suppressed. Further it has been shown that even when in the numerator, g possesses a unique perturbative character that, in conjunction with its coupled structure, suppresses the spatial component of the perturbed torsion field as shown in 26. This is crucial for validating the theory as the torsion condensate field ψ only acts like dark energy when it remains on or near its constant value. The resulting time-only perturbations have been shown to be stable around their background value, as described by 31. When proposing models such as $2a$ we have extended Einstein's theory of gravity with extra degrees of freedom as permitted by Lovelock's theorem. Typically, we would expect these extra degrees of freedom to interfere with the cosmological perturbation theory. Surprisingly this is not what we find: the non-canonical structure of the MA leads to the recovery of Einstein-like perturbed field equations, 45 47 49. These field equations have a simpler extension than traditional scalar tensor theories due to the suppression of the spatial components of the scalar field, giving relations 55 and 56 for the Bardeen potentials.

It is important to remember the full context of this theory. The metrical analogue was obtained through a mapping process of the full PGT theory to a Horndeski-like theory. Consequently, the complete perturbative picture of the theory will only be fully assessed when realised in the Riemann-Cartan geometry where the PGT Lagrangian resides. The complex and lengthy nature of the equations involved, even at the level of the bi-scalar tensor theory, has run up against the computational limits of the *xPand* tool. This invites further work to obtain the perturbation equations at higher levels, with an aim to identify the effect of the Cuscuton on the character of ϕ field and metric perturbations. Full numerical simulations of both Cuscuton perturbations away from the background and the field equations with full character are natural next steps to further determine the stability of the theory.

However, with these caveats in mind, it is worth noting that the Metrical Analogue 2a merits investigating even without considerations of PGT. The MA's ability to address dark energy and Hubble tension are strong enough phenomenological motivations alone to make the theory worthy of interest. Even if one were to argue that the MA seems contrived, many theories of extended gravity, such as Relativistic Modified Newtonian Dynamics (see equation 5 in Skordis and Złośnik [2021]) which is far more Frankensteinian than the MA, are given serious attention despite the fact their only justification is phenomenological. The fact the MA has its origins in PGT serves only to further justify the value in studying it.

This report has demonstrated the power of computer algebra to explore and analyse intensely complex equations, that would otherwise be too difficult for a human to assess. With the tools of computer algebra the following results have been obtained:

- We have demonstrated that the non-canonical Cuscuton has the atypical effect of suppressing the spatial perturbations of the torsion field.
- This leads to a conformal time dependent only ODE giving rise to stable perturbations around the torsion field.
- These two results combined lead to the remarkable recovery of Einstein-like perturbed field equations, with only minor modifications to the Bardeen potentials.

8 Acknowledgements

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Appendix

The covariant equation of motion for the ϕ field

$$\tfrac{64}{27}\big(\big({M_{\mathrm{Pl}}}^2+3\sigma_1{\psi}^2\big)^2\big(\big(\bar{\nabla}_{\mathrm{a1}}\phi\big)\big(\bar{\nabla}^{\mathrm{a1}}\phi\big)\big)+3\sigma_1\phi\psi\big(-2\big({M_{\mathrm{Pl}}}^2+3\sigma_1{\psi}^2\big)\big(\big(\bar{\nabla}_{\mathrm{a1}}\psi\big)\big(\bar{\nabla}^{\mathrm{a1}}\phi\big)\big)$$
$$+3\sigma_1\phi\psi\big(\big(\bar{\nabla}_{\mathrm{a1}}\psi\big)\big(\bar{\nabla}^{\mathrm{a1}}\psi\big)\big)\big)\big)^{3/2}\big(-2\phi\big({M_{\mathrm{Pl}}}^2+3\sigma_1{\psi}^2\big)\big)$$
$$-\tfrac{256}{81}\big(\big({M_{\mathrm{Pl}}}^2+3\sigma_1{\psi}^2\big)^2\big(\big(\bar{\nabla}_{\mathrm{a1}}\phi\big)\big(\bar{\nabla}^{\mathrm{a1}}\phi\big)\big)\big(\big({M_{\mathrm{Pl}}}^2+$$
$$3\sigma_1{\psi}^2\big)^2\big(\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}^{\mathrm{a1}}\phi\big)+12\sigma_1\psi\big({M_{\mathrm{Pl}}}^2+3\sigma_1{\psi}^2\big)\big(\bar{\nabla}_{\mathrm{a1}}\psi\big)\big(\bar{\nabla}^{\mathrm{a1}}\phi\big)-3\sigma_1\phi\big({M_{\mathrm{Pl}}}^2\psi\big(\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}^{\mathrm{a1}}\psi\big)+$$
$$3\sigma_1{\psi}^3\big(\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}^{\mathrm{a1}}\psi\big)+{M_{\mathrm{Pl}}}^2\big(\bar{\nabla}_{\mathrm{a1}}\psi\big)\big(\bar{\nabla}^{\mathrm{a1}}\psi\big)+12\sigma_1{\psi}^2\big(\bar{\nabla}_{\mathrm{a1}}\psi\big)\big(\bar{\nabla}^{\mathrm{a1}}\psi\big)\big)\big)-$$
$$\big({M_{\mathrm{Pl}}}^2+3\sigma_1{\psi}^2\big)^3\big(\bar{\nabla}^{\mathrm{a1}}\phi\big)\big(3\sigma_1\psi\big(\bar{\nabla}_{\mathrm{a1}}\phi\big)\big(\bar{\nabla}_b\psi\big)+$$
$${M_{\mathrm{Pl}}}^2\big(\bar{\nabla}_b\bar{\nabla}_{\mathrm{a1}}\phi\big)+3\sigma_1{\psi}^2\big(\bar{\nabla}_b\bar{\nabla}_{\mathrm{a1}}\phi\big)\big)\big(\bar{\nabla}^b\phi\big)-27{\sigma_1}^3{\phi}^3{\psi}^2\big({M_{\mathrm{Pl}}}^2\big(\bar{\nabla}_{\mathrm{a1}}\psi\big)\big(\bar{\nabla}^{\mathrm{a1}}\psi\big)\big(\big(\big(\bar{\nabla}_{\mathrm{a1}}\psi\big)\big(\bar{\nabla}^{\mathrm{a1}}\psi\big)\big)-$$
$$\big(\bar{\nabla}_b\psi\big)\big(\bar{\nabla}^b\psi\big)\big)+3\sigma_1{\psi}^2\big(\bar{\nabla}_{\mathrm{a1}}\psi\big)\big(\bar{\nabla}^{\mathrm{a1}}\psi\big)\big(4\big(\big(\bar{\nabla}_{\mathrm{a1}}\psi\big)\big(\bar{\nabla}^{\mathrm{a1}}\psi\big)\big)-\big(\bar{\nabla}_b\psi\big)\big(\bar{\nabla}^b\psi\big)\big)$$

$$+ M_{\rm Pl}{}^2\psi\big(\big(\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\psi\big)\big)\big(\bar\nabla_{\rm a1}\bar\nabla^{\rm a1}\psi\big) - \big(\bar\nabla^{\rm a1}\psi\big)\big(\bar\nabla_b\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^b\psi\big)\big) +$$
$$3\sigma_1\psi^3\big(\big(\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\psi\big)\big)\big(\bar\nabla_{\rm a1}\bar\nabla^{\rm a1}\psi\big) - \big(\bar\nabla^{\rm a1}\psi\big)\big(\bar\nabla_b\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^b\psi\big)\big)\big) +$$
$$9\sigma_1{}^2\phi^2\psi\big(M_{\rm Pl}{}^2+3\sigma_1\psi^2\big)\big(3\sigma_1\psi^2\big(\bar\nabla_{\rm a1}\psi\big)\big(4\big(\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\psi\big)\big)\big(\bar\nabla^{\rm a1}\phi\big)+8\big(\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\phi\big)\big)\big(\bar\nabla^{\rm a1}\psi\big)-$$
$$3\big(\bar\nabla^{\rm a1}\phi\big)\big(\bar\nabla_b\psi\big)\big(\bar\nabla^b\psi\big)\big) + 2M_{\rm Pl}{}^2\big(\bar\nabla_{\rm a1}\psi\big)\big(\big(\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\phi\big)\big)\big(\bar\nabla^{\rm a1}\psi\big) - \big(\bar\nabla^{\rm a1}\phi\big)\big(\bar\nabla_b\psi\big)\big(\bar\nabla^b\psi\big)\big) +$$
$$M_{\rm Pl}{}^2\psi\big(\big(\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\psi\big)\big)\big(\bar\nabla_{\rm a1}\bar\nabla^{\rm a1}\phi\big) + 2\big(\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\phi\big)\big)\big(\bar\nabla_{\rm a1}\bar\nabla^{\rm a1}\psi\big) -$$
$$\big(\big(\bar\nabla_{\rm a1}\bar\nabla_b\psi\big)\big(\bar\nabla^{\rm a1}\phi\big) + \big(\bar\nabla^{\rm a1}\psi\big)\big(\bar\nabla_b\bar\nabla_{\rm a1}\phi\big) + \big(\bar\nabla^{\rm a1}\phi\big)\big(\bar\nabla_b\bar\nabla_{\rm a1}\psi\big)\big)\big(\bar\nabla^b\psi\big)\big) +$$
$$3\sigma_1\psi^3\big(\big(\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\psi\big)\big)\big(\bar\nabla_{\rm a1}\bar\nabla^{\rm a1}\phi\big) + 2\big(\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\phi\big)\big)\big(\bar\nabla_{\rm a1}\bar\nabla^{\rm a1}\psi\big) - \big(\big(\bar\nabla_{\rm a1}\bar\nabla_b\psi\big)\big(\bar\nabla^{\rm a1}\phi\big) +$$
$$\big(\bar\nabla^{\rm a1}\psi\big)\big(\bar\nabla_b\bar\nabla_{\rm a1}\phi\big) + \big(\bar\nabla^{\rm a1}\phi\big)\big(\bar\nabla_b\bar\nabla_{\rm a1}\psi\big)\big)\big(\bar\nabla^b\psi\big)\big)\big)$$
$$- 3\sigma_1\phi\big(M_{\rm Pl}{}^2 + 3\sigma_1\psi^2\big)^2\big(- M_{\rm Pl}{}^2\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\phi\big)\big(\bar\nabla_b\psi\big)\big(\bar\nabla^b\phi\big) +$$
$$3\sigma_1\psi^2\big(\bar\nabla^{\rm a1}\phi\big)\big(8\big(\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\phi\big)\big)\big(\bar\nabla_{\rm a1}\psi\big) - \big(\bar\nabla_b\psi\big)\big(2\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^b\phi\big) + \big(\bar\nabla_{\rm a1}\phi\big)\big(\bar\nabla^b\psi\big)\big)\big) +$$
$$M_{\rm Pl}{}^2\psi\big(2\big(\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\phi\big)\big)\big(\bar\nabla_{\rm a1}\bar\nabla^{\rm a1}\phi\big) - \big(\bar\nabla^{\rm a1}\phi\big)\big(\big(\bar\nabla_b\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^b\phi\big) + \big(\bar\nabla_{\rm a1}\bar\nabla_b\phi + \bar\nabla_b\bar\nabla_{\rm a1}\phi\big)\big(\bar\nabla^b\psi\big)\big)\big) +$$
$$\psi^3\big(6\sigma_1\big(\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\phi\big)\big)\big(\bar\nabla_{\rm a1}\bar\nabla^{\rm a1}\phi\big) - 3\sigma_1\big(\bar\nabla^{\rm a1}\phi\big)\big(\big(\bar\nabla_b\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^b\phi\big) + \big(\bar\nabla_{\rm a1}\bar\nabla_b\phi +$$
$$\bar\nabla_b\bar\nabla_{\rm a1}\phi\big)\big(\bar\nabla^b\psi\big)\big)\big)\big)\big) = 0$$

The covariant field equations

$$\tfrac{8}{3}\big[\big(M_{\rm Pl}{}^2 + 3\sigma_1\psi^2\big)^2\big(\big(\bar\nabla_{\rm a1}\phi\big)\big(\bar\nabla^{\rm a1}\phi\big)\big) + 3\sigma_1\phi\psi\big(- 2\big(M_{\rm Pl}{}^2 + 3\sigma_1\psi^2\big)\big(\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\phi\big)\big) +$$
$$3\sigma_1\phi\psi\big(\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\psi\big)\big)\big)\big]^{1/2} \times \tfrac{1}{12}\big(- 2\big(4M_{\rm Pl}{}^2 + 3\sigma_1\psi^2\big)R\left[\bar\nabla\right]_{bc} + \bar g_{bc}\big(6\phi^2\big(M_{\rm Pl}{}^2 + 3\sigma_1\psi^2\big) +$$
$$4M_{\rm Pl}{}^2R\left[\bar\nabla\right] + \psi^2\big(- 18M_{\rm Pl}{}^2\upsilon_1 + 3\sigma_1R\left[\bar\nabla\right]\big) - 12\sigma_1\psi\big(\bar\nabla_{\rm a1}\bar\nabla^{\rm a1}\psi\big) + 6\sigma_1\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\psi\big)\big) +$$
$$6\sigma_1\big(- 4\big(\bar\nabla_b\psi\big)\big(\bar\nabla_c\psi\big) + \psi\big(\bar\nabla_b\bar\nabla_c\psi + \bar\nabla_c\bar\nabla_b\psi\big)\big)\big)$$
$$+\tfrac{16}{9}\big(-\bar g_{bc}\big(\big(M_{\rm Pl}{}^2+3\sigma_1\psi^2\big)^2\big(\big(\bar\nabla_{\rm a1}\phi\big)\big(\bar\nabla^{\rm a1}\phi\big)\big)+3\sigma_1\phi\psi\big(-2\big(M_{\rm Pl}{}^2+3\sigma_1\psi^2\big)\big(\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\phi\big)\big)+$$
$$3\sigma_1\phi\psi\big(\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\psi\big)\big)\big)\big)+\big(\big(M_{\rm Pl}{}^2+3\sigma_1\psi^2\big)\big(\bar\nabla_b\phi\big)-3\sigma_1\phi\psi\big(\bar\nabla_b\psi\big)\big)\big(\big(M_{\rm Pl}{}^2+3\sigma_1\psi^2\big)\big(\bar\nabla_c\phi\big)-$$
$$3\sigma_1\phi\psi\big(\bar\nabla_c\psi\big)\big)\big) = 0$$

The covariant equation of motion for the ψ field

$$\big(\big(M_{\rm Pl}{}^2 + 3\sigma_1\psi^2\big)^2\big(\big(\bar\nabla_{\rm a1}\phi\big)\big(\bar\nabla^{\rm a1}\phi\big)\big) +$$
$$3\sigma_1\phi\psi\big(- 2\big(M_{\rm Pl}{}^2 + 3\sigma_1\psi^2\big)\big(\big(\bar\nabla_{\rm a1}\psi\big)\big(\bar\nabla^{\rm a1}\phi\big)\big) +$$

$$3\sigma_1\phi\psi((\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\psi))))^{3/2}\psi(6M_{\mathrm{Pl}}{}^2v_1 - \sigma_1(6\phi^2 + R[\bar{\nabla}])) + 6\sigma_1(\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}^{\mathrm{a1}}\psi)$$
$$\psi(3(M_{\mathrm{Pl}}{}^2+3\sigma_1\psi^2)^3((\bar{\nabla}_{\mathrm{a1}}\phi)(\bar{\nabla}^{\mathrm{a1}}\phi))(\bar{\nabla}_{\mathrm{a1}}\phi)(\bar{\nabla}^{\mathrm{a1}}\phi)+\phi(M_{\mathrm{Pl}}{}^2+3\sigma_1\psi^2)^2(((\bar{\nabla}_{\mathrm{a1}}\phi)(\bar{\nabla}^{\mathrm{a1}}\phi))((M_{\mathrm{Pl}}{}^2+$$
$$3\sigma_1\psi^2)(\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}^{\mathrm{a1}}\phi)-6\sigma_1\psi(\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\phi))-(\bar{\nabla}^{\mathrm{a1}}\phi)(M_{\mathrm{Pl}}{}^2(\bar{\nabla}_b\bar{\nabla}_{\mathrm{a1}}\phi)(\bar{\nabla}^b\phi)+3\sigma_1\psi^2(\bar{\nabla}_b\bar{\nabla}_{\mathrm{a1}}\phi)(\bar{\nabla}^b\phi)+$$
$$3\sigma_1\psi(\bar{\nabla}_{\mathrm{a1}}\phi)(6((\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\phi))+(\bar{\nabla}_b\psi)(\bar{\nabla}^b\phi))))-27\sigma_1{}^3\phi^4\psi^2((\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\psi)(((\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\psi))-$$
$$(\bar{\nabla}_b\psi)(\bar{\nabla}^b\psi)) + \psi(((\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\psi))(\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}^{\mathrm{a1}}\psi) - (\bar{\nabla}^{\mathrm{a1}}\psi)(\bar{\nabla}_b\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^b\psi))) +$$
$$9\sigma_1{}^2\phi^3\psi(2M_{\mathrm{Pl}}{}^2(\bar{\nabla}_{\mathrm{a1}}\psi)(((\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\phi))(\bar{\nabla}^{\mathrm{a1}}\psi) - (\bar{\nabla}^{\mathrm{a1}}\phi)(\bar{\nabla}_b\psi)(\bar{\nabla}^b\psi)) -$$
$$3\sigma_1\psi^2(\bar{\nabla}_{\mathrm{a1}}\psi)(2((\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\psi))(\bar{\nabla}^{\mathrm{a1}}\phi)-2((\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\phi))(\bar{\nabla}^{\mathrm{a1}}\psi)+3(\bar{\nabla}^{\mathrm{a1}}\phi)(\bar{\nabla}_b\psi)(\bar{\nabla}^b\psi))+$$
$$M_{\mathrm{Pl}}{}^2\psi(((\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\psi))(\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}^{\mathrm{a1}}\phi)+2((\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\phi))(\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}^{\mathrm{a1}}\psi) - ((\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}_b\psi)(\bar{\nabla}^{\mathrm{a1}}\phi)+$$
$$(\bar{\nabla}^{\mathrm{a1}}\psi)(\bar{\nabla}_b\bar{\nabla}_{\mathrm{a1}}\phi) + (\bar{\nabla}^{\mathrm{a1}}\phi)(\bar{\nabla}_b\bar{\nabla}_{\mathrm{a1}}\psi))(\bar{\nabla}^b\psi)) +$$
$$3\sigma_1\psi^3(((\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\psi))(\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}^{\mathrm{a1}}\phi) + 2((\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\phi))(\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}^{\mathrm{a1}}\psi) -$$
$$((\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}_b\psi)(\bar{\nabla}^{\mathrm{a1}}\phi) + (\bar{\nabla}^{\mathrm{a1}}\psi)(\bar{\nabla}_b\bar{\nabla}_{\mathrm{a1}}\phi) + (\bar{\nabla}^{\mathrm{a1}}\phi)(\bar{\nabla}_b\bar{\nabla}_{\mathrm{a1}}\psi))(\bar{\nabla}^b\psi))) -$$
$$3\sigma_1\phi^2(M_{\mathrm{Pl}}{}^2 + 3\sigma_1\psi^2)(M_{\mathrm{Pl}}{}^2(\bar{\nabla}_{\mathrm{a1}}\psi)(((\bar{\nabla}_{\mathrm{a1}}\phi)(\bar{\nabla}^{\mathrm{a1}}\phi))(\bar{\nabla}^{\mathrm{a1}}\psi) - (\bar{\nabla}^{\mathrm{a1}}\phi)(\bar{\nabla}_b\psi)(\bar{\nabla}^b\phi)) -$$
$$3\sigma_1\psi^2(3((\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\psi))(\bar{\nabla}_{\mathrm{a1}}\phi)(\bar{\nabla}^{\mathrm{a1}}\phi)+4((\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\phi))(\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\phi)-((\bar{\nabla}_{\mathrm{a1}}\phi)(\bar{\nabla}^{\mathrm{a1}}\phi))$$
$$(\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\psi) + 2(\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\phi)(\bar{\nabla}_b\psi)(\bar{\nabla}^b\phi) + (\bar{\nabla}_{\mathrm{a1}}\phi)(\bar{\nabla}^{\mathrm{a1}}\phi)(\bar{\nabla}_b\psi)(\bar{\nabla}^b\psi)) +$$
$$M_{\mathrm{Pl}}{}^2\psi(2((\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\phi))(\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}^{\mathrm{a1}}\phi)+((\bar{\nabla}_{\mathrm{a1}}\phi)(\bar{\nabla}^{\mathrm{a1}}\phi))(\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}^{\mathrm{a1}}\psi)-(\bar{\nabla}^{\mathrm{a1}}\phi)((\bar{\nabla}_b\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^b\phi)+$$
$$(\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}_b\phi+\bar{\nabla}_b\bar{\nabla}_{\mathrm{a1}}\phi)(\bar{\nabla}^b\psi)))+3\sigma_1\psi^3(2((\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^{\mathrm{a1}}\phi))(\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}^{\mathrm{a1}}\phi)+((\bar{\nabla}_{\mathrm{a1}}\phi)(\bar{\nabla}^{\mathrm{a1}}\phi))(\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}^{\mathrm{a1}}\psi)-$$
$$(\bar{\nabla}^{\mathrm{a1}}\phi)((\bar{\nabla}_b\bar{\nabla}_{\mathrm{a1}}\psi)(\bar{\nabla}^b\phi) + (\bar{\nabla}_{\mathrm{a1}}\bar{\nabla}_b\phi + \bar{\nabla}_b\bar{\nabla}_{\mathrm{a1}}\phi)(\bar{\nabla}^b\psi))))) = 0$$

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