Corporate Defaults: The Merton Model

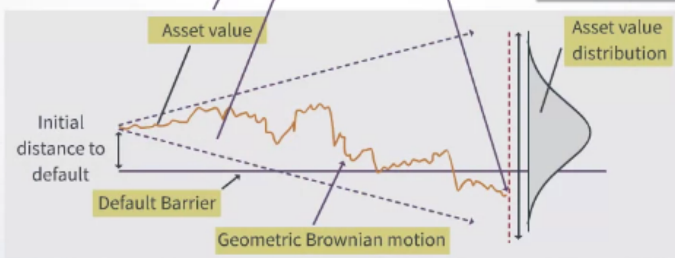
The **Merton model** of corporate defaults (1974-present) is the most popular modeling framework, used as a benchmark for many studies.

The firm is run by equity holders. At time T, they pay the face value of the debt D if the firm (asset) value is larger than D, and keep the remaining amount. If the firm value at time T is less than D, bond holders take over, and recover a "recovery" value V_T, while equity holders get nothing:

$$B_T = \min(V_T, D) = D - \max\,(D - V_T, 0)$$

$$S_T = \max\,(V_T - D,\ 0)$$

Limited liability of equity holders



Merton Model as a Structural Default Model

Default probability in the Merton model:

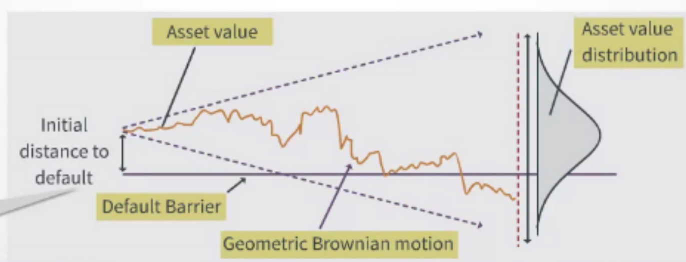
$$\Pr(default) = \mathbb{E}\left[\mathbb{I}_{V_T<D}\right] = \Pr(V_T < D) = N(-d_2)$$

Probabilistic structural model

$$d_2 = \frac{\log\dfrac{V_t}{D} + \left(r - \dfrac{\sigma_V^2}{2}\right)(T-t)}{\sigma_V\sqrt{(T-t)}}$$

Depends only on the Assets/Debt ratio and asset volatility

The barrier is fixed and static



Idea of BSM Model

- Way to determine how many stocks you should have in your replicating portfolio, in each scenario for the future,

  so that the Total Portfolio Value will ALWAYS be zero in the future, no matter what happens with the stock price.

- This is possible ONLY if your time steps are infinitesimal.

  For this very special setting, the Black-Scholes model finds a UNIQUE option price and UNIQUE number of shares you should have in your replicating portfolio NOW.

Binomial Model: Optimal Replication

Arbitrage pricing: price the stock by constructing a replicating portfolio of a stock and a bond.

Stock option

$$C_0 \begin{cases} \xrightarrow{\;p\;} C_1(uS) = \max(uS - K, 0) \\ \xrightarrow{\;1-p\;} C_1(dS) = \max(dS - K, 0) \end{cases}$$

Portfolio

$$\Pi_0 \begin{cases} \xrightarrow{\;p\;} \Pi_1(uS) \\ \xrightarrow{\;1-p\;} \Pi_1(dS) \end{cases}$$

$$C_0 = \Pi_0 = \Delta S + \theta B$$
$$C_1(uS) = \Pi_1(uS) = \Delta uS + \theta(1+r)B$$
$$C_1(dS) = \Pi_1(dS) = \Delta dS + \theta(1+r)B$$

$$\Delta = \frac{C_1(uS) - C_1(dS)}{uS - dS}$$
$$\theta B = \frac{uC_1(uS) - dC_1(dS)}{(1+r)(u-d)}$$

Choosing the **control** (Delta) this way completely **eliminates risk** of the option!
This happens **only** for the binomial model in discrete time, and for the BSM in continuous time!

Discrete-time BSM Model

We start with a discrete-time version of the BSM model. The problem of option hedging and pricing in this formulation amounts to a **sequential risk minimization**. To define risk in an option, we follow a local risk minimization approach (Föllmer and Schweizer (1994), Potters and Bouchaud (2001), Kapoor et. al. (2010), Grau (2007)).

We take the view of a seller of a European option (e.g. a put option) with maturity T and the terminal payoff of $H_T(S_T)$. To hedge the option, the seller use the proceeds of the sale to set up a replicating (hedge) portfolio Π_t made of the stock S_t and a risk-free bank deposit B_t. The value of hedge portfolio at any time $t \leq T$ is

$$\Pi_t = u_t S_t + B_t \qquad \text{Eq: 1}$$

where u_t is a stock position at time t, taken to hedge risk in the option.

Hedge Portfolio Evaluation

The replicating portfolio tries to exactly match the option price in all possible future states of the world. Start at $t = T$:

$$\Pi_T = H_T(S_T) \quad \text{Eq: 2}$$

This sets a terminal condition for Π_T at $t = T$.

To find B_t for previous times $t < T$, we impose the *self-financing constraint* which requires that all future changes in the hedge portfolio should be funded from an initially set bank account.

$$u_t S_{t+1} + e^{r\Delta t} B_t = u_{t+1} S_{t+1} + B_{t+1} \quad \text{Eq: 3}$$

This can be expressed as a recursive relation for B_t at any time $t < T$ using its value at the next time instance:

$$B_t = e^{-r\Delta t}\left[B_{t+1} + (u_{t+1} - u_t)\, S_{t+1}\right], \quad t = T-1, \ldots, 0 \quad \text{Eq: 4}$$

Hedge Portfolio Evaluation

Plugging this into Eq.(1) produces a recursive relation for Π_t in terms of its values at later times, which can therefore be solved backward in time, starting from $t = T$ with the terminal condition (2), and continued all the way to the current time $t = 0$:

$$\Pi_t = e^{-r\Delta t}\left[\Pi_{t+1} - u_t \Delta S_t\right], \quad \Delta S_t = S_{t+1} - e^{r\Delta t} S_t \quad \text{Eq: 5}$$

Eqs.(4) and (5) imply that both B_t and Π_t are not measurable at any $t < T$, as they depend on the future. Respectively, their values today B_0 and Π_0 will be random quantities with some distributions. We can compute them using Monte Carlo!

Hedge Portfolio Evaluation with Monte Carlo

For any given hedging strategy $\{u_t\}_{t=0}^{T}$, these distributions can be estimated using Monte Carlo simulation:

- **Forward pass :** Simulate N paths of the underlying $S_1 \to S_2 \to \ldots \to S_N$,
- **Backward pass :** Evaluate Π_t going backward on each path.

As the choice of a hedge strategy does not affect the evolution of the underlying, such simulation of forward paths should only be performed once. Alternatively, we could use real historical data for stock prices, together with a pre-determined hedging strategy $\{u_t\}_{t=0}^{T}$ and a terminal condition (2).

But first, we need a hedge strategy u_t to implement this Monte Carlo!

Optimal Hedging in the Discrete-Time BSM Model

Our information set $\mathcal{F}_t$ is a set of all simulated MC paths. The optimal hedge $u^{\star}(S_t)$ is obtained by minimization of variance of Π_t across all MC paths at time t:

$$\begin{aligned} u_t^{\star}(S_t) &= arg \min_u \mathrm{Var}\left[\Pi_t | \mathcal{F}_t\right] \\ &= arg \min_u \mathrm{Var}\left[\Pi_{t+1} - u_t \Delta S_t | \mathcal{F}_t\right] \end{aligned} \qquad \boxed{\text{Eq: 8}}$$

- The first form in (Eq: 8) implies that all uncertainty in Π_t is due to uncertainty of the bank cash amount B_t .
- An optimal hedge should minimize the cost of hedge capital at each time step t.

Optimal Hedging in the Discrete-Time BSM Model

The optimal hedge $u^{\star}(S_t)$:

$$\begin{aligned} u_t^{\star}(S_t) &= \arg\min_u \mathrm{Var}\left[\Pi_t | \mathcal{F}_t\right] \\ &= \arg\min_u \mathrm{Var}\left[\Pi_{t+1} - u_t \Delta S_t | \mathcal{F}_t\right] \end{aligned} \qquad \boxed{\text{Eq: 9}}$$

The optimal hedge can be found analytically by setting the derivative of (Eq: 9) to zero. This gives

$$u_t^{\star}(S_t) = \frac{\mathrm{Cov}\left(\Pi_{t+1}, \Delta S_t | \mathcal{F}_t\right)}{\mathrm{Var}\left(\Delta S_t | \mathcal{F}_t\right)}, \quad t = T-1, \ldots, 0 \qquad \boxed{\text{Eq: 10}}$$

How one computes one-step expectations depends on whether we deal with a continuous or a discrete state space.

Option Pricing in Discrete-Time BSM

Using the equation for Π_t and the tower law of conditional expectations, we obtain

$$\begin{aligned} \hat{C}_t &= \mathbb{E}_t\left[\Pi_t\right] \\ &= \mathbb{E}_t\left[e^{-r\Delta t}\Pi_{t+1}\right] - u_t(S_t)\,\mathbb{E}_t\left[\Delta S_t\right] \\ &= \mathbb{E}_t\left[e^{-r\Delta t}\mathbb{E}_{t+1}\left[\Pi_{t+1}\right]\right] - u_t(S_t)\,\mathbb{E}_t\left[\Delta S_t\right] \\ &= \mathbb{E}_t\left[e^{-r\Delta t}\hat{C}_{t+1}\right] - u_t(S_t)\,\mathbb{E}_t\left[\Delta S_t\right], \quad t = T-1, \ldots, 0 \end{aligned} \qquad \boxed{\text{Eq: 12}}$$

Fair Option Pricing in Discrete-Time BSM

The dealer needs to charge some risk premium on top of the mean price $\hat{C}_0$ when selling the option, as she has to compensate for risk that the actual amount B_0 for a specific future scenario will be larger than its present expected value $\mathbb{E}_0\left[B_0\right]$.

One possible specification of a risk premium is to add the cumulative expected discounted variance of the hedge portfolio along all time steps $t = 0, \ldots, N$, with a risk-aversion parameter λ:

$$C_0^{(ask)}(S, u) = \mathbb{E}_0\left[\Pi_0 + \lambda \sum_{t=0}^{T} e^{-rt}\mathrm{Var}_t\left[\Pi_t\right] \middle| S_0 = S, u_0 = u\right] \qquad \boxed{\text{Eq: 13}}$$

The option seller should minimize this option price to be competitive.

Maximization Problem for Option Pricing

The problem of minimization of a fair (to the dealer) option price (13) can be equivalently expressed as the problem of maximization of its negative $V_t = -C_t^{(ask)}$, where

$$V_t(S_t) = \mathbb{E}_t\left[-\Pi_t - \lambda \sum_{t'=t}^{T} e^{-r(t'-t)} \mathrm{Var}_{t'}\left[\Pi_{t'}\right]\right] \quad \boxed{\text{Eq: 14}}$$

Hedging and Pricing in the BS Limit

The framework presented above provides a smooth transition to the strict BS limit $\Delta t \to 0$. In this limit, the BSM model dynamics under the physical measure $\mathbb{P}$ is described by a continuous-time Geometric Brownian motion with a drift μ and volatility σ:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \quad \boxed{\text{Eq: 15}}$$

where W_t is a standard Brownian motion.

Hedging in the BS Limit

Consider first the optimal hedge strategy

$$u_t^{\star}(S_t) = \frac{\mathrm{Cov}\left(\Pi_{t+1}, \Delta S_t \middle| \mathcal{F}_t\right)}{\mathrm{Var}\left(\Delta S_t \middle| \mathcal{F}_t\right)} = \frac{\mathrm{Cov}\left(C_{t+1}, \Delta S_t \middle| \mathcal{F}_t\right)}{\mathrm{Var}\left(\Delta S_t \middle| \mathcal{F}_t\right)}$$

in the BS limit $\Delta t \to 0$

Using the first-order Taylor expansion

$$\hat{C}_{t+1} = C_t + \frac{\partial C_t}{\partial S_t} \Delta S_t + O\left(\Delta t\right) \quad \boxed{\text{Eq: 16}}$$

in (Eq: 10), we obtain

$$u_t^{BS}\left(S_t\right) = \lim_{\Delta t \to 0} u_t^{\star}(S_t) = \frac{\partial C_t}{\partial S_t} \quad \boxed{\text{Eq: 17}}$$

which is the correct optimal hedge in the continuous-time BSM model.

Pricing in the BS Limit

We had the recursive formula for the option price

$$\hat{C}_t = \mathbb{E}_t \left[e^{-r\Delta t} \hat{C}_{t+1} \middle| \mathcal{F}_t \right] - u_t(S_t)\, \mathbb{E}_t \left[\Delta S_t \middle| \mathcal{F}_t \right], \quad t = T-1, \ldots, 0$$

First compute the limit of the second term in the equation for $\hat{C}_t$:

$$\begin{aligned} \lim_{\Delta t \to 0} u_t(S_t)\, \mathbb{E}_t \left[\Delta S_t \right] &= \lim_{dt \to 0} u_t^{BS} S_t (\mu - r) dt \\ &= \lim_{dt \to 0} (\mu - r) S_t \frac{\partial C_t}{\partial S_t} dt \end{aligned} \qquad \boxed{\text{Eq: 19}}$$

To evaluate the first term in the equation for $\hat{C}_t$, we use the second-order Taylor expansion:

$$\begin{aligned} \hat{C}_{t+1} &= C_t + \frac{\partial C_t}{\partial t} dt + \frac{\partial C_t}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 C_t}{\partial S_t^2} \left(dS_t \right)^2 + \ldots \\ &= C_t + \frac{\partial C_t}{\partial t} dt + \frac{\partial C_t}{\partial S_t} S_t \left(\mu dt + \sigma dW_t \right) \\ &+ \frac{1}{2} \frac{\partial^2 C_t}{\partial S_t^2} S_t^2 \left(\sigma^2 dW_t^2 + 2 \mu \sigma dW_t dt \right) + O\left(dt^2 \right) \end{aligned} \qquad \boxed{\text{Eq: 20}}$$

Pricing in the BS Limit

We had the recursive formula for the option price

$$\hat{C}_t = \mathbb{E}_t \left[e^{-r\Delta t} \hat{C}_{t+1} \middle| \mathcal{F}_t \right] - u_t(S_t)\, \mathbb{E}_t \left[\Delta S_t \middle| \mathcal{F}_t \right], \quad t = T-1, \ldots, 0$$

Use (Eqs.19) and (Eqs.20) for two terms in this expression, use $\mathbb{E}\left[dW_t \right] = 0$ and $\mathbb{E}\left[dW_t^2 \right] = dt$, and simplify. We see that the stock drift μ drops out from the problem.

This becomes the celebrated **Black-Scholes equation** in the limit $dt \to 0$:

$$\frac{\partial C_t}{\partial t} + r S_t \frac{\partial C_t}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} - r C_t = 0 \qquad \boxed{\text{Eq: 21}}$$

Therefore, if the world is lognormal, both our hedging and pricing formulae coincide with the Black-Scholes model in the strict limit $\Delta t \to 0$.