Overview of Advanced RL in Finance:

Week 3: Beyond RL

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Brooklyn NY 2018

In these notes:

- **RL-inspired model for market dynamics**
- **Plan:**
  - RL-based market model
  - Role of frictions
  - Corporate defaults

Market dynamics and IRL

"Quantum Equilibrium-Disequilibrium" (QED) model

Discrete-time dynamics:

$$X_{t+\Delta t} = (1 + r_t \Delta t)(X_t - cX_t \Delta t + u_t \Delta t)$$
$$r_t = r_f + \mathbf{w}\mathbf{z}_t - \mu u_t + \frac{\sigma}{\sqrt{\Delta t}} \varepsilon_t$$
$$u_t = \phi X_t + \lambda X_t^2$$

Here c is a dividend rate.

"Quantum Equilibrium-Disequilibrium" (QED) model

Discrete-time dynamics:

$$\begin{aligned} X_{t+\Delta t} &= (1 + r_t \Delta t)(X_t - c X_t \Delta t + u_t \Delta t) \\ r_t &= r_f + \mathbf{w} \mathbf{z}_t - \mu u_t + \frac{\sigma}{\sqrt{\Delta t}} \varepsilon_t \\ u_t &= \phi X_t + \lambda X_t^2 \end{aligned} \tag{1}$$

In the limit $\Delta t \to dt$, this produces the "Quantum Equilibrium-Disequilibrium" (QED) model:

$$dX_t = \kappa X_t \left(\frac{\theta}{\kappa} - X_t - \frac{g}{\kappa} X_t^2 \right) dt + \sigma X_t \left(dW_t + \mathbf{w} \mathbf{z}_t \right) \tag{2}$$

where

$$g = \mu \lambda, \quad \kappa = \mu \phi - \lambda, \quad \theta = r_f - c + \phi \tag{3}$$

The GBM limit in the QED model

If there is no money exchange between the market and the outside, i.e. $\phi = \lambda = 0$ and hence $g = 0$ and $\kappa = 0$, we formally recover the GBM model:

$$dX_t = \left(r_f + \mathbf{w}\mathbf{z}_t\right) X_t dt + \sigma X_t dW_t \tag{4}$$

The same GBM dynamics are obtained if $\phi \neq 0$, but instead we take a limit of zero friction $\mu = 0, \lambda = 0$.

The GMR limit in the QED model

If $\mu > 0$ and $\phi \neq 0$ but $\lambda = 0$ (and hence $g = 0$), the QED model reduces to the Geometric Mean Reversion (GMR) model (with signals):

$$dX_t = \kappa X_t \left(\frac{\theta}{\kappa} - X_t \right) dt + X_t \left(\sigma dW_t + \mathbf{w} \mathbf{z}_t \right) \tag{5}$$

The GMR model without signals $\mathbf{z}_t$ was studied by Dixit and Pindyck, and Ewald and Yang.
If we also take the noiseless limit $\sigma = 0$ and $\mathbf{z}_t = 0$, we obtain the "Verhulst limit".

Control question

Select all correct answers:

1. The GBM model is recovered from the QED model in the limit $g = 0$.
2. The GBM model is recovered from the QED model in the limit $\kappa = 0, g = 0$.
3. The GMR model is recovered from the GBM model in the limit $g = 0$.
4. The GMR model is recovered from the QED model in the limit $g = 0$.

Correct answers: 2, 4.

Diffusion in a potential: the Langevin equation

Langevin equaiton

- Bacheliier (1900): a model of Brownian motion (free diffusion), applied to the stock market modeling (ABM model)
- Louis Bachelier $\rightarrow$ Andrey Kolmogorov $\rightarrow$ Paul Levy $\rightarrow$ Leonard Savage $\rightarrow$ Paul Samuelson (GBM, 1965)
- Einstein (1905) - diffusion for Brownian particles.
- Paul Langevin (1908) - simplified approach to diffusion in a force potential (e.g. inter-molecular forces)
- Fokker and Planck (1920s) - Brownian motion in a force potential

Langevin Equation

Langevin equation

$$\ddot{x} + \gamma \dot{x} + U'(x) = \sqrt{\frac{2\gamma k T}{M}} \dot{W} \tag{6}$$

Here x is a particle position, M is its mass, γ is a dissipation constant, $U(x)$ is a (generally non-linear) potential force, and $\dot{W}$ is a Gaussian white noise

Langevin Equation

Langevin equation

$$\ddot{x} + \gamma \dot{x} + U'(x) = \sqrt{\frac{2\gamma k T}{M}} \dot{W} \tag{7}$$

Here x is a particle position, M is its mass, γ is a dissipation constant, $U(x)$ is a (generally non-linear) potential force, and $\dot{W}$ is a Gaussian white noise

Example: a potential U created by light particles of mass m at thermal equilibrium at temperature T with a Maxwell distribution of velocities

$$f(v) = \sqrt{\frac{m}{2\pi k T}} \exp\left\{ -\frac{m v^2}{2 k T} \right\} \tag{8}$$

(See e.g. Schuss, "Theory and applications of stochastic processes")

Langevin diffusion in a potential: example

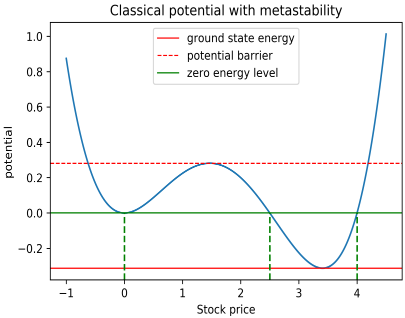


Figure: A classical potential with two metastable states.

Langevin Equation: the phase space representation

Langevin equation

$$\ddot{X} + \gamma \dot{X} + U'(X) = \sqrt{2\varepsilon\gamma}\,\dot{W} \tag{9}$$

can also be written in a phase space representation

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\gamma v - U'(x) + \sqrt{\frac{2\gamma kT}{M}}\,\dot{W} \end{aligned} \tag{10}$$

A solution of the LE is a two-dimensional process for the pair $(x(t), \dot{x}(t))$. (See Schuss)

The Overdamped Langevin Equation

The overdamped limit $\gamma \to \infty$ (the Smoluchowski limit) of the Langevin equation:

$$\gamma \dot{X} + U'(X) = \sqrt{\frac{2\gamma k T}{M}} \dot{W} \qquad (11)$$

How to obtain: use the phase space representation and and scale time $t = \gamma s$ (see Schuss for details)

Example of overdamped Langevin dynamics in ML: Stochastic Gradient Descent!

Example: Smoluchowski limit of a free Brownian particle

A free Brownian particle corresponds to a motion without a potential, i.e. $U = 0$. The Smoluchowski limit of the Langevin equation produces

$$\gamma \dot{X} = \sqrt{\frac{2\gamma k T}{M}} \dot{W} \tag{12}$$

or

$$dx = \sqrt{\frac{2\gamma k T}{M}} dW_t \equiv \sqrt{2D} dW_t \tag{13}$$

where D is Einstein's diffusion coefficient

Langevin dynamic vs Ito diffusion

Langevin dynamics

$$dX_t = -\frac{\partial U(X_t)}{\partial X_t} dt + \sigma(X_t) d\xi_t \tag{14}$$

where ξ_t is a noise term (a Gaussian noise $\xi_t = W_t$ or 'colored' noise).

Ito diffusion for the GBM model

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{15}$$

The classical potential for the GBM model is

$$U_{GBM}(X) = -\frac{\mu}{2} X^2 \tag{16}$$

This is a potential of an *inverted* oscillator.

The classical potential for the GBM model

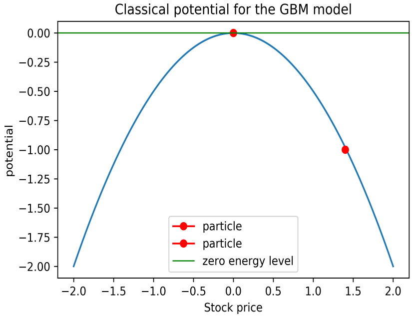


Figure: A classical potential for the GBM model.

Control question

Select all correct answers:

1. Stochastic Gradient Descent (SGD) is an example of free Ito diffusion without a potential.
2. The SGD is better described by a jump-diffusion process where jumps happens on outliers in the data.
3. The parameter evolution in the SGD is described by the Langevin equation where the potential is given by the loss function.
4. The overdamped Langevin equation $\gamma \dot{X} + U'(X) = \sigma \dot{W}_t$ is obtained in the large friction limit $\gamma \to \infty$ of the Brownian motion.

Correct answers: 3,4.

Classical dynamics

Classical potential for the QED model

We had the SDE for the QED model:

$$dX_t = \kappa X_t \left(\frac{\theta}{\kappa} - X_t - \frac{g}{\kappa} X_t^2 \right) dt + \sigma X_t \left(dW_t + \mathbf{wz}_t \right) \tag{17}$$

The classical potential $U(x)$ for the QED model is therefore

$$U(x) = -\frac{1}{2}\theta x^2 + \frac{1}{3}\kappa x^3 + \frac{1}{4} g x^4 \tag{18}$$

This is a potential of a quartic oscillator.

Parametrization of a quartic potential

Another parametrization in terms of parameters a, b defining zeros of the potential:

$$U(x) = -\frac{1}{2}\theta x^2 \left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{b}\right) \tag{19}$$

The relation between two sets of parameters:

$$\frac{\kappa}{\theta} = \frac{3}{2}\frac{a+b}{ab}, \quad \frac{g}{\theta} = -\frac{2}{ab} \tag{20}$$

Quartic potential

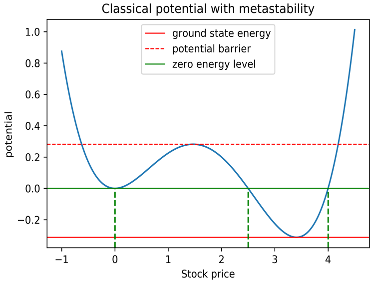


Figure: A classical quartic potential with two metastable states with $\theta < 0$ and $a, b > 0$

Another parametrization:

$$U(x) = -\frac{1}{2}\theta x^2 \left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{b}\right) \qquad (21)$$

Quartic potential in log-space

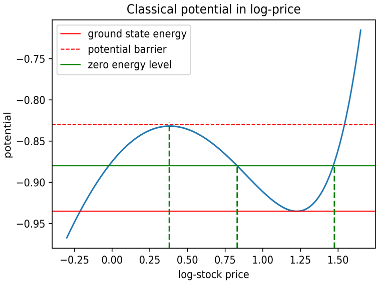


Figure: A classical quartic potential with a metastable states with $\theta < 0$ in the log-space $y = log(x)$

Another parametrization:

$$U(y) = -\theta y + \kappa e^{y} + \frac{1}{2} g e^{2y} \tag{22}$$

Quartic potential: metastability at zero

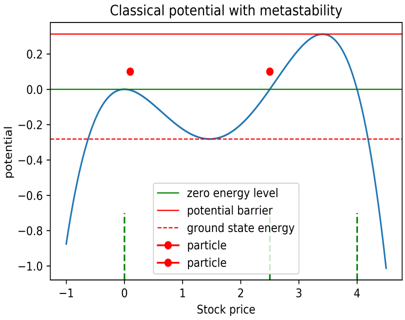


Figure: A classical quartic potential with $\theta > 0$ and $a, b > 0$ with a metastable state at $x = 0$

Another parametrization in terms of zero location points a and b:

$$U(x) = -\frac{1}{2}\theta x^2 \left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{b}\right) \qquad (23)$$

Quartic potential: instability at zero

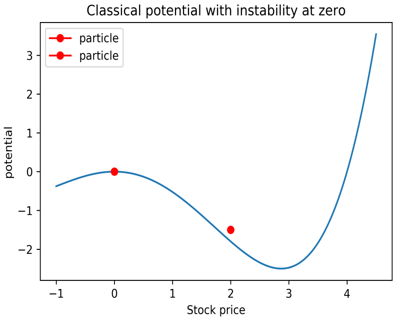


Figure: A classical quartic potential with $\theta > 0$ and $a < 0, b > 0$ with a unstable state at $x = 0$

Another parametrization in terms of zero location points a and b:

$$U(x) = -\frac{1}{2}\theta x^2 \left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{b}\right) \qquad (24)$$

Control question

Select all correct answers:

1. Quartic potential is non-singular (i.e. it is finite for any finite real- or complex-valued argument).
2. Quartic potential in the log-space is given by a fourth degree polynomial in $y = \log x$.
3. If we set $a = b$ in the classical potential $U(x)$, the resulting potential will only have two extrema, instead of three.
4. The only singularity of the quartic potential in the log-space $y = \log x$ is at a negative infinity.

Correct answers: 1, 4.

Potential minima and Newton's law

Classical minima

The classic potential (18) has three extrema at

$$\bar{x}_0 = 0, \quad \bar{x}_{1,2} = \frac{-\kappa \pm \sqrt{\kappa^2 + 4g\theta}}{2g} \qquad (25)$$

where $\bar{x}_1$ and $\bar{x}_2$ correspond to the plus and minus signs, respectively. The first extremum $\bar{x}_0 = 0$ a degenerate solution: not only $U'(\bar{x}_0)' = 0$, but also $U''(\bar{x}_0) = 0$. It is called a natural boundary. Once the price touches the zero level $x = 0$, the system will stay in this state forever.

Classical minima: expansion for small g

For small values $g \to 0$, we obtain the following expressions for the extrema $\bar{x}_{1,2}$:

$$\begin{aligned}
\bar{x}_1 &= \frac{\theta}{\kappa}\left(1 - \frac{g\theta}{\kappa^2}\right) + O(g^2) \\
\bar{x}_2 &= -\frac{\kappa}{g} - \frac{\theta}{\kappa}\left(1 - \frac{g\theta}{\kappa^2}\right) + O(g^2)
\end{aligned} \tag{26}$$

Note that the first root $\bar{x}_1$ is non-perturbative in κ and perturbative in g, while the second root in non-perturbative in both κ and g.

The classical potential for the QED model

A particle with energy E can move in a classically allowed region where the sum of kinetic and potential energy equals E:

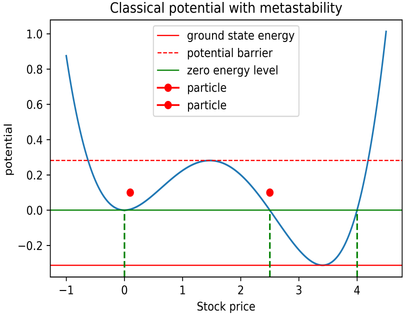


Figure: Classical motion in the QED potential.

Classical Newtonian mechanics

The Newton second law (mass $m = 1$ times the acceleration $a \equiv \ddot{x}$ equals force $F(x) = -U'(x)$

$$\ddot{x} = -U'(x) = \theta x - \kappa x^2 - g x^3 \qquad (27)$$

The $\mathcal{CPT}$ symmetry of Newtonian mechanics

The Newton second law

$$\ddot{x} = -U'(x) = \theta x - \kappa x^2 - g x^3 \tag{28}$$

The $\mathcal{CPT}$ symmetry of the Newtonian mechanics:

$$\begin{aligned} &\mathcal{C}\text{-parity:} \quad \kappa \to -\kappa \\ &\mathcal{P}\text{-parity:} \quad x \to -x \\ &\mathcal{T}\text{-parity (Time reversal):} \quad t \to -t \end{aligned} \tag{29}$$

Eq. (28) is separately symmetric with respect to the time reversal $\mathcal{T}$ and the joint $\mathcal{CP}$-inversion. As a consequence, it is also invariant with respect to a simultaneous $\mathcal{CPT}$ transformation.

Control question

Select all correct answers:

1. The word 'non-perturbative' means that a corresponding parameter is fixed and not subject to changes.
2. A model is non-perturbative in a parameter θ if dependence of observables on θ cannot be obtained as a result of a regular perturbation theory in small values of θ.
3. The Newtonian mechanics is invariant under reflection of time because the Lagrangian does not explicitly depend on time.
4. The Newtonian mechanics is invariant under reflection of time because it contains the second derivative with respect to time. Under the time reversal, it stays the same.

Correct answers: 2, 4.

Classical dynamics: the Lagrangian and the Hamiltonian

Energy conservation

The total energy E that is equal to the sum of the kinetic energy $K = \frac{m\dot{x}^2}{2}$ and the potential energy $U(x)$ is a constant in time:

$$E \equiv \frac{m\dot{x}^2}{2} + U(x) \tag{30}$$

Can write it as follows:

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}\left[E - U(x)\right]} \tag{31}$$

This is is a differential equation that we can integrate:

$$t = t(x) = \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx}{\sqrt{E - U(x)}} + \text{const} \tag{32}$$

The classical motion is only allowed in a region wher $U(x) < E$.

Bounded and unbounded classical motion

As $K = \frac{m\dot{x}^2}{2} \geq 0$, *turning points* of a potential $U(x)$ are those points where $K = 0$ and hence

$$\text{Turning points: } U(x) = E \tag{33}$$

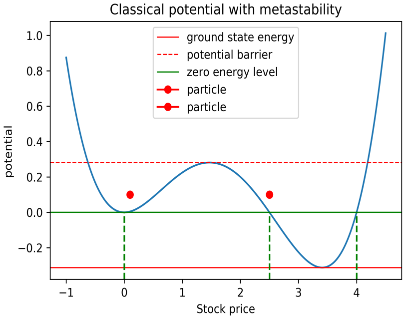


Figure: Classical motion in the QED potential.

The classical potential for the QED model

If for a given E we have two turning points $x_1(E)$ and $x_2(E)$, then the period of classical oscillation in a potential well is

$$T(E) = \sqrt{2m} \int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{E - U(x)}} \tag{34}$$

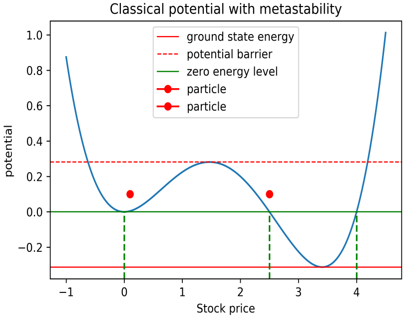


Figure: Classical motion in the QED potential.

The Hamiltonian principle of the least action

The action S and the Lagrangian $\mathcal{L}$:

$$S = \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t) dt = \int_{t_1}^{t_2} \left[\frac{m\dot{x}^2}{2} - U(x) \right] dt \tag{35}$$

The Hamiltonian principle:

$$\begin{aligned} \delta S &= \delta \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t) dt \\ &= \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial x} \delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}} \delta \dot{x} \right] dt = 0 \end{aligned} \tag{36}$$

This produces the Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0 \tag{37}$$

The action for a fixed energy

Using conservation of energy E, we can express the momentum $p = m\dot{x}$ in terms of energy and the potential energy

$$H = \frac{1}{2}p^2 + U(y) = E \quad \Leftrightarrow \quad p = \sqrt{2\left[E - U(y)\right]} \tag{38}$$

Substituting this into (35), we obtain

$$S = \int_{y_0}^{y_f} \sqrt{2\left[E - U(y)\right]} dy \tag{39}$$

From classical mechanics to quantum mechanics

The role of action:

$$S(E) = \int_{y_0}^{y_f} \sqrt{2\left[E - U(y)\right]}dy \qquad (40)$$

- In *classical* mechanics: **one** path from y_0 to y_f determined by the Lagrange equation, the action along the path is (40)
- In *quantum* mechanics: **infinite** number of paths y_0 to y_f, each path has the probability (weight)

$$p(E, path) \sim \exp\left(\frac{i}{\hbar} S(E, path)\right) \qquad (41)$$

Control question

Select all correct answers:

1. In classical mechanics, a particle moves from an initial to final point along a single trajectory.
2. In quantum mechanics, a particle in a sense moves from one point to another along an infinite number of paths all at once.
3. The Hamilton principle of the least action produces the Lagrange equation of motion that defines a trajectory of a classical particle.
4. In quantum mechanics, a trajectory of a particle is determined by the quantum Lagrange equation.

Correct answers: 1, 2, 3.

Langevin equation and Fokker-Planck equations

Langevin dynamic and the Fokker-Planck equation

Langevin equation is a path-wise SDE. If we want to study statistical properties of the stochastic systems, we can instead use equations for a probability distribution of the system. This produces the Fokker-Planck equation (the forward Kolmogorov equation):

$$\begin{aligned} \dot{p}(x, t|x_0) &= \frac{\partial}{\partial x}\left[U'(x)p(x, t|x_0)\right] \\ &+ \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x)p(x, t|x_0)\right] \end{aligned} \qquad (42)$$

Initial conditions:

$$\lim_{t \to t_0} p(x, t|x_0) = \delta(x - x_0)$$

The Fokker-Planck equation from the Langevin equation

Calculate the time derivative of the mean value of some functional $f(X)$:

$$\begin{aligned}\frac{d}{dt}\langle f(X)\rangle &= \langle \frac{d}{dt} f(X)\rangle = \langle -U'(X_t) f_x + \frac{1}{2}\sigma^2(X_t) f_{xx}\rangle \\ &= \int \left(-U'(X_t) f_x + \frac{1}{2}\sigma^2(x) f_{xx}\right) p(x|x_0) dx\end{aligned}$$

On the other hand, we can compute it differently:

$$\frac{d}{dt}\langle f(X)\rangle = \int f(x) \frac{\partial p(x|x_0)}{\partial t} dx \qquad (43)$$

Integrating by parts in the first relation, we have

$$\frac{d}{dt}\langle f(X)\rangle = \int f(x) \left(U'(x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\sigma^2(x)\right) p(x|x_0) dx \qquad (44)$$

The Fokker-Planck equation from the Langevin equation

Therefore, the two expressions should be the same:

$$\int f(x) \frac{\partial p(x|x_0)}{\partial t} dx = \int f(x) \left(U'(x) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \sigma^2(x) \right) p(x|x_0) dx \tag{45}$$

Because $f(x)$ is arbitrary, we obtain the FPE:

$$\begin{aligned} \frac{\partial p(x,t|x_0)}{\partial t} &= \frac{\partial}{\partial x} \left[U'(x) p(x,t|x_0) \right] \\ &+ \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[\sigma^2(x) p(x,t|x_0) \right] \end{aligned}$$

Natural boundary at $x = 0$ in the FPE

An absorbing boundary condition at $x = 0$: a collapse of the process:

$$\lim_{x \to 0} p(x, t | x_0) = 0 \tag{46}$$

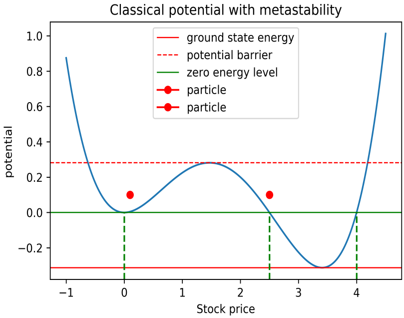


Figure: For a non-negative process with $x \geq 0$, a particle that touches $x = 0$ is 'absorbed': the system collapses.

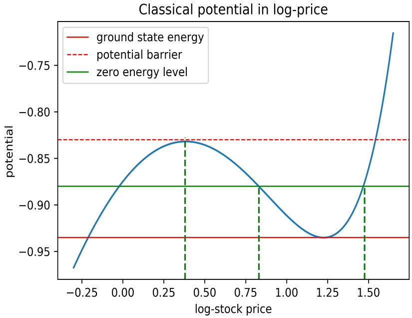
The FPE equation in log-space

The FPE in the log-price space $y = \log x$:

$$\frac{\partial p(y,t|y_0)}{\partial t} = \frac{\partial}{\partial y}\left[U'(y)p(y,t|y_0)\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}\left[p(y,t|y_0)\right] \quad (47)$$

Here the potential $U(y)$ is

$$U(y) = -\theta y + \kappa e^y + \frac{1}{2} g e^{2y}$$



The FPE equation in log-space: the noisless limit

When $\sigma = 0$, the FPE in the log-space produces

$$\dot{y} = -U'(y) \tag{48}$$

This produces

$$\frac{dU(y)}{dt} = U'(y)\dot{y} = -\left[U'(y)\right]^2 \leq 0 \tag{49}$$

The particle $y(t)$ always moves to minimize $U(y)$, only stops when $U'(y) = 0$.

Control question

Select all correct answers:

1. The Fokker-Planck equation (FPE) is a
Stochastic Differential Equation (SDE).
2. The Fokker-Planck equation (FPE) is a
Partial Differential Equation (PDE).
3. The FPE equation is a first-order equation,
therefore it requires one boundary condition.
4. The FPE equation is a second-order equation,
therefore it requires two boundary conditions.

Correct answers: 2, 4.

Tbe Fokker-Planck equation and quantum mechanics

The FPE equation in log-space

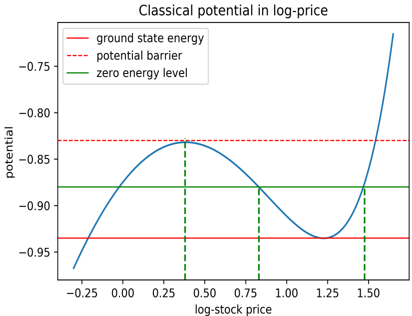
The FPE in the log-price space $y = \log x$:

$$\frac{\partial p(y,t|y_0)}{\partial t} = \frac{\partial}{\partial y}\left[U'(y)p(y,t|y_0)\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}\left[p(y,t|y_0)\right] \tag{50}$$

Here the potential $U(y)$ is

$$U(y) = -\theta y + \kappa e^y + \frac{1}{2} g e^{2y}$$

It can lead to the Kramer escape:

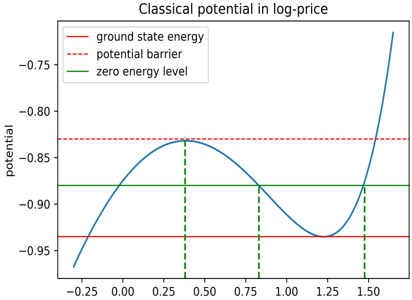


The FPE equation: stationary and quasi-stationary distributions

Stationary solution of the FPE equaiton

$$p(y, t|y_0) = \frac{1}{Z} \exp\left(-\frac{2U(y)}{\sigma^2}\right) \tag{51}$$

where Z is a normalization constant. When there is is metastability, it shows as divergence of the normalization constant Z. A metastable state can decay through thermal fluctuations.



The Fokker-Planck equation and the Schrödinger equation

Assume that volatility is constant, $\sigma(x) = \sigma$.
Make the following ansatz for the FPE:

$$\tilde{p}(x, t|x_0) = e^{-\frac{1}{\sigma^2} U(x)} K(y, t|y_0) \qquad (52)$$

Using this in Eq.(46), we obtain an *imaginary* time Schrödinger equation for $K(y, t|y_0)$:

$$-\sigma^2 \frac{\partial K(y, t|y_0)}{\partial t} = HK(y, t|y_0) \qquad (53)$$

where H is the Hamiltonian

$$\begin{aligned} H &= -\frac{\sigma^4}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left(U'(x)\right)^2 - \frac{1}{2}\sigma^2 U''(x) \\ &\equiv -\frac{\sigma^4}{2}\frac{\partial^2}{\partial x^2} + V(x) \end{aligned} \qquad (54)$$

where $V(x)$ is an equivalent quantum-mechanical potential.

The Schrödinger equation and Supersymmetry

Supersymmetry (SUSY) of the Schrödinger equation (53)

$$H = A^{+} A \tag{55}$$

where

$$A = \frac{1}{\sqrt{2}} \left[\sigma^2 \frac{\partial}{\partial y} + U' \right], \quad A^{+} = \frac{1}{\sqrt{2}} \left[-\sigma^2 \frac{\partial}{\partial y} + U' \right] \tag{56}$$

Operators A, A^{+} are sometimes called supercharge generators, and the function U_y is called the superpotential. The supersymmetric partner Hamiltonian H_{+} is obtained by swapping their order:

$$H_{+} = AA^{+} = -\frac{\sigma^4}{2} \frac{\partial^2}{\partial y^2} + \frac{1}{2} \left(U' \right)^2 + \frac{1}{2} \sigma^2 U'' \tag{57}$$

The unbroken and broken SUSY

Due to supersymmetry, if Ψ_n is an eigenvector of H with an eigenvalue E_n, than the state $A\Psi_n$ will be an eigenstate of H_+ with the same eigenvalue E_n:

$$H_+ A\Psi_n = AA^+ A\Psi_n = AH\Psi_n = AE_n\Psi_n = E_n A\Psi_n \tag{58}$$

Meaning: that all eigenstates except a 'vacuum' state with energy $E_0 = 0$ (if it exists - see below) should be degenerate in energy with eigenstates of the SUSY partner Hamiltonian H_+.

SUSY can be unbroken or spontaneously broken. If the energy of the ground state is larger than zero, than SUSY is spontaneously broken:

$$\begin{aligned} \text{Unbroken SUSY:} \quad & A\Psi_0 = 0 \cdot \Psi_0 = 0 \;\; (E_0 = 0) \\ \text{Broken SUSY:} \quad & A\Psi_0 = E_0\Psi_0, \;\; E_0 > 0 \end{aligned} \tag{59}$$

Escape from a metastable state

Large stock drops and defaults can be thought as tunneling through a potential barrier. The classical transition state theory gives the probability of a particle jumping over a barrier as a product of two factors: the Arrhenius factor B and a pre-factor A. The Arrhenius factor is

$$B = \exp\left(-E_b/kT\right) \tag{60}$$

where E_b is the barrier height, and T is the temperature. The pre-factor A for a 1D well is given by the frequency ω_0 of oscillations at the bottom of the well:

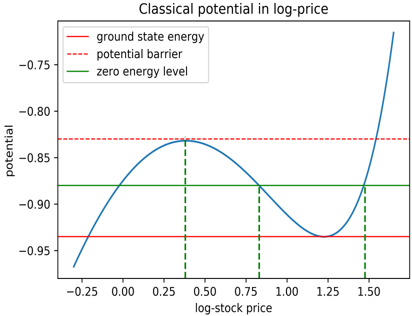
$$A = \frac{\omega_0}{2\pi} \tag{61}$$

Tunneling in QM: imaginary time and imaginary action:

$$S(E) = \int_{y_0}^{y_f} \sqrt{2\left[E - U(y)\right]}dy \tag{62}$$

Escape by tunneling and divergence of perturbation theory

- Tunneling is a *non-perturbative* effect: it can't be obtained as an expansion in small values of κ and g around a model with a 'trivial vacuum' $\bar{x} = 0$.
- *Divergence of perturbative series* and tunneling have the same origin.
- This is similar to Dyson's divergence of Quantum Electro-Dynamics.



Summary

- RL/IRL can be used not only to compute specific quantity, but also to build models themselves.
- The model we presented in the previous course can be both re-derived and improved using methods from physics
- Analysis of different symmetries of the problem play a key role
- Symmetries determine the nature of phase transitions
- For you course project: re-estimate the QED model with non-zero g.

Control question

Select all correct answers:

1. The FPE can be transformed to a Schrödinger equation by a substitution $P(y,t) = P_0(y)K(y,y)$ where P_0 is the stationary distribution.
2. The FPE can be transformed to a Schrödinger equation by a substitution $P(y,t) = \sqrt{P_0(y)}K(y,y)$ where P_0 is the stationary distribution.
3. Tunneling is a process of random transformation of profits into losses.
4. Tunneling is a process of passage through a potential barrier that is activated by noise.

Correct answers: 2, 4.