Geometry

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Contents

Chapter 1

Topology

1.1 Qutoient Map

Definition 1.1.1. Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is said to be a quotient map provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

Definition 1.1.2. If X is a space and A is a set and if $p : X \to A$ is a surjective map, then there exists exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map; it is called the quotient topology induced by p.

The topology $\mathcal{T}$ is of course defined by letting it consist of those subsets U of A such that $p^{-1}(U)$ is open in X. It is easy to check that $\mathcal{T}$ is a topology. The sets $\varnothing$ and A are open because $p^{-1}(\varnothing) = \varnothing$ and $p^{-1}(A) = X$. The other two conditions follow from the equations

$$p^{-1}\left(\bigcup_{\alpha\in J} U_\alpha\right) = \bigcup_{\alpha\in J} p^{-1}\left(U_\alpha\right)$$
$$p^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \bigcap_{i=1}^{n} p^{-1}\left(U_i\right).$$

Proposition 1.1.3. We say that a subset C of X is saturated with respect to the surjective map $p : X \to Y$ if C contains every set $p^{-1}(\{y\})$ that it intersects. Thus C is saturated if it equals the complete inverse image of a subset of Y. Then surjective map $p : X \to Y$ is quotient map if and only if it is continuous and maps saturated open(or closed) sets of X to open(closed) sets of Y.

Proposition 1.1.4. $p : X \to Y$ is a surjective continuous map that is either open or closed, then p is a quotient map.

Theorem 1.1.5 (universal proptery of quotient map)**.** Let $p : X \to Y$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$, for $y \in Y$. Then g induces a unique map $f : Y \to Z$ such that $f \circ p = g$. The induced map f is continuous if and only if g is continuous; f is a quotient map if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, dashed, "f"] & Z \end{tikzcd}$$

Definition 1.1.6. Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a quotient space of X.

Theorem 1.1.7. Let $g : X \to Z$ be a surjective continuous map. Let X^* be the following collection of subsets of X :

$$X^* = \left\{g^{-1}(\{z\}) \mid z \in Z\right\}.$$

Give X^* the quotient topology.

(1) The map g induces a bijective continuous map $f : X^* \to Z$, which is a homeomorphism if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"] \arrow[dr, "g"] & \\ X^* \arrow[r, "f"] & Z \end{tikzcd}$$

(2) If Z is Hausdorff, so is X^*.

Proposition 1.1.8. $f : X \to Y$ is a quotient map, Z is a locally compact Hausdorff space, then $f \times \mathrm{id} : X \times Z \to Y \times Z$ is a quotient map.

1.2 Fundamental Group

Definition 1.2.1. If f and f' are continuous maps of the space X into the space Y, we say that f is homotopic to f' if there is a continuous map $F : X \times I \to Y$ such that

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = f'(x)$$

for each x. (Here $I = [0, 1]$.) The map F is called a homotopy between f and f'. If f is homotopic to f', we write $f \simeq f'$. If $f \simeq f'$ and f' is a constant map, we say that f is nulhomotopic.

Now we consider the special case in which f is a path in X. Recall that if $f : [0, 1] \to X$ is a continuous map such that $f(0) = x_0$ and $f(1) = x_1$, we say that f is a path in X from x_0 to x_1. We also say that x_0 is the initial point, and x_1 the final point, of the path f. In this chapter, we shall for convenience use the interval $I = [0, 1]$ as the domain for all paths.

If f and f' are two paths in X, there is a stronger relation between them than mere homotopy. It is defined as follows:

Two paths f and f', mapping the interval $I = [0, 1]$ into X, are said to be path homotopic if they have the same initial point x_0 and the same final point x_1, and if there is a continuous map $F : I \times I \to X$ such that

$$\begin{aligned} F(s, 0) &= f(s) \quad \text{and} \quad F(s, 1) = f'(s), \\ F(0, t) &= x_0 \quad\ \ \text{and} \quad F(1, t) = x_1, \end{aligned}$$

for each $s \in I$ and each $t \in I$. We call F a path homotopy between f and f'.

Proposition 1.2.2. The relations $\simeq$ and $\simeq p$ are equivalence relations.

Proof: Let us verify the properties of an equivalence relation. Given f, it is trivial that $f \simeq f$; the map $F(x, t) = f(x)$ is the required homotopy. If f is a path, F is a path homotopy.

Given $f \simeq f'$, we show that $f' \simeq f$. Let F be a homotopy between f and f'. Then $G(x, t) = F(x, 1 - t)$ is a homotopy between f' and f. If F is a path homotopy, so is G.

Suppose that $f \simeq f'$ and $f' \simeq f''$. We show that $f \simeq f''$. Let F be a homotopy between f and f', and let F' be a homotopy between f' and f''. Define $G : X \times I \to Y$ by the equation

$$G(x, t) = \begin{cases} F(x, 2t) & \text{for } t \in \left[0, \frac{1}{2}\right], \\ F'(x, 2t - 1) & \text{for } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

The map G is well defined, since if $t = \frac{1}{2}$, we have $F(x, 2t) = f'(x) = F'(x, 2t - 1)$. Because G is continuous on the two closed subsets $X \times \left[0, \frac{1}{2}\right]$ and $X \times \left[\frac{1}{2}, 1\right]$ of $X \times I$, it is continuous on all of $X \times I$, by the pasting lemma. Thus G is the required homotopy between f and f''.

Definition 1.2.3. If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, we define the product $f * g$ of f and g to be the path h given by the equations

$$h(s) = \begin{cases} f(2s) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ g(2s - 1) & \text{for } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

The function h is well-defined and continuous. It is a path in X from x_0 to x_2. We think of h as the path whose first half is the path f and whose second half is the path g.

The product operation on paths induces a well-defined operation on path-homotopy classes, defined by the equation

$$[f] * [g] = [f * g].$$

To verify this fact, let F be a path homotopy between f and f' and let G be a path homotopy between g and g'. Define

$$H(s,t) = \begin{cases} F(2s,t) & \text{for } s \in \left[0, \frac{1}{2}\right], \\ G(2s-1,t) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Because $F(1,t) = x_1 = G(0,t)$ for all t, the map H is well-defined. You can check that H is the required path homotopy between $f * g$ and $f' * g'$.

Example 1.2.4. let A be any convex subspace of $\mathbb{R}^n$, Let f and g be any two maps of a space X into A. It is easy to see that f and g are homotopic; the map

$$F(x,t) = (1-t)f(x) + tg(x)$$

is a homotopy between them. It is called a straight-line homotopy because it moves the point $f(x)$ to the point $g(x)$ along the straight-line segment joining them.

If f and g are paths from x_0 to x_1, then F will be a path homotopy.

Proposition 1.2.5. If $k : X \to Y$ is a continuous map, and if F is a path homotopy in X between the paths f and f', then $k \circ F$ is a path homotopy in Y between the paths $k \circ f$ and $k \circ f'$.

Proposition 1.2.6. The operation $*$ has the following properties:

(1) (Associativity) If $[f] * ([g] * [h])$ is defined, so is $([f] * [g]) * [h]$, and they are equal.

(2) (Right and left identities) Given $x \in X$, let e_x denote the constant path $e_x : I \to X$ carrying all of I to the point x. If f is a path in X from x_0 to x_1, then

$$[f] * [e_{x_1}] = [f] \quad \text{and} \quad [e_{x_0}] * [f] = [f].$$

(3) (Inverse) Given the path f in X from x_0 to x_1, let $\bar{f}$ be the path defined by $\bar{f}(s) = f(1-s)$. It is called the reverse of f. Then

$$[f] * [\bar{f}] = [e_{x_0}] \quad \text{and} \quad [\bar{f}] * [f] = [e_{x_1}].$$

Proof: (1), (2) and (3) follow from the fact that

Notice that $I = [0,1]$ is convex, we can construct path-homotopy between different paths in $I = [0,1]$ to prove (1), (2) and (3) respectively.

Definition 1.2.7. Let X be a space; let x_0 be a point of X. A path in X that begins and ends at x_0 is called a loop based at x_0. The set of path homotopy classes of loops based at x_0, with the operation $*$, is called the fundamental group of X relative to the base point x_0. It is denoted by $\pi_1(X, x_0)$.

Proposition 1.2.8. Let α be a path in X from x_0 to x_1. We define a map

$$\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

by the equation

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha].$$

The map $\hat{\alpha}$ is well-defined and a group isomorphism.

Theorem 1.2.9. A space X is said to be simply connected if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial (one-element) group for some $x_0 \in X$, and hence for every $x_0 \in X$. We often express the fact that $\pi_1(X, x_0)$ is the trivial group by writing $\pi_1(X, x_0) = 0$.

In a simply connected space X, any two paths having the same initial and final points are path homotopic.

Proof: Let α and β be two paths from x_0 to x_1. Then $\alpha * \bar{\beta}$ is defined and is a loop on X based at x_0. Since X is simply connected, this loop is path homotopic to the constant loop at x_0. Then

$$[\alpha * \bar{\beta}] * [\beta] = [e_{x_0}] * [\beta],$$

from which it follows that $[\alpha] = [\beta]$.

Definition 1.2.10. Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map such that $h(x_0) = y_0$. Define

$$h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

by the equation

$$h_*([f]) = [h \circ f].$$

The map h_* is well-defined by Proposition 1.2.5, and this map h_* is called the homomorphism induced by h, relative to the base point x_0.

Proposition 1.2.11 (Functorial property)**.** If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.

If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X with Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.

Example 1.2.12. A subset A of $\mathbb{R}^n$ is said to be star convex if for some point a_0 of A, all the line segments joining a_0 to other points of A lie in A. Show that if A is star convex, A is simply connected.

Proposition 1.2.13. $f : X \to Y$ is a continous map, α is a path from x_0 to x_1. And assume $y_0 = f(x_0), y_1 = f(x_1)$, we have the following commute diagram

$$\begin{tikzcd}
\pi(X, x_0) \arrow[r, "f_{*,x_0}"'] \arrow[d, "\hat{\alpha}"] & \pi(Y, y_0) \arrow[d, "\hat{\beta}"'] \\
\pi(X, x_1) \arrow[r, "f_{*,x_1}"] & \pi(Y, y_1)
\end{tikzcd}$$

where β is the path $f \circ \alpha$.

1.3 Covering space

Definition 1.3.1. Let $p : E \to B$ be a continuous surjective map. The open set U of B is said to be evenly covered by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into slices.

Let $p : E \to B$ be continuous and surjective. If every point b of B has a neighborhood U that is evenly covered by p, then p is called a covering map, and E is said to be a covering space of B.

Proposition 1.3.2. If B is connected, $p : E \to B$ is a covering map, then $p^{-1}(b), b \in B$ have the same If B is connected, $p : E \to B$ is a covering map, then $p^{-1}(b), b \in B$ have the same caridinity. We call it the number of sheets of p.

Proposition 1.3.3. Covering map is open map, hence a quotient map.

Proof:

Definition 1.3.4. A map between topological spaces is called proper if the preimage of every compact set is compact.

Lemma 1.3.5. Let Y be locally compact Hausdorff spaces. A proper continuous map $p : X \to Y$ is closed.

Proof: Let $C \subset X$ be closed. Let $y \in Y - f(C)$. Since Y is locally compact, y has a neighborhood V with compact closure. Since f is proper, $f^{-1}(\bar{V})$ is compact in X. Let $E = C \cap f^{-1}(\bar{V})$, E is compact; thus, $f(E)$ is compact. Since Y is Hausdorff, $f(E)$ is closed. Let $\tilde{V} = V - f(E)$. $\tilde{V}$ is a neighborhood of y disjoint from $f(C)$ as desired.

Proposition 1.3.6 (Sufficient Conditions for Properness)**.** Suppose X and Y are topological spaces, and $F : X \to Y$ is a continuous map.

(1) If X is compact and Y is Hausdorff, then F is proper.

(2) If F is a closed map with compact fibers, then F is proper.

(3) If F is a topological embedding with closed image, then F is proper.

(4) If Y is Hausdorff and F has a continuous left inverse (i.e., a continuous map $G : Y \to X$ such that $G \circ F = \mathrm{Id}_X$), then F is proper.

(5) If F is proper and $A \subseteq X$ is a subset that is saturated with respect to F, then $F|_A : A \to F(A)$ is proper.

Theorem 1.3.7. It $p : E \to B$ and $p' : E' \to B'$ are covering maps, then

$$p \times p' : E \times E' \to B \times B'$$

is a covering map.

Definition 1.3.8 (lifting)**.** Let $p : E \to B$ be a map. If f is a continuous mapping of some space X into B, a lifting of f is a continuous map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

Lemma 1.3.9 (lifting of path)**.** Let $p : E \to B$ be a covering map, let $p(e_0) = b_0$. Any path $f : [0,1] \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f}$ in E beginning at e_0.

Proof: Cover B by open sets $(U_i)_{i \in I}$ each of which is evenly covered by p. Find a subdivision of $[0,1]$, say $s_0, \ldots, s_n$, such that for each i the set $f([s_i, s_{i+1}])$ lies in some open set U_i. (Here we use the Lebesgue number lemma.) We define the lifting $\tilde{f}$ step by step.

First, define $\tilde{f}(0) = e_0$. Then, supposing $\tilde{f}(s)$ is defined for $0 \leq s \leq s_i$, we define $\tilde{f}$ on $(s_i, s_{i+1}]$ as follows: The set $f([s_i, s_{i+1}])$ lies in some open set U_i that is evenly covered by p. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices; each set V_α is mapped homeomorphically onto U by p. Now $\tilde{f}(s_i)$ lies in only one of these sets, say in V_0. Define $\tilde{f}(s)$ for $s \in (s_i, s_{i+1}]$ by the equation

$$\tilde{f}(s) = (p \mid V_0)^{-1}(f(s)).$$

Uniquness: trivial.

Lemma 1.3.10 (lifting of path homotopy)**.** Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let the map $F : I \times I \to B$ be continuous, with $F(0,0) = b_0$. There is a unique lifting of F to a continuous map

$$\tilde{F} : I \times I \to E$$

such that $\tilde{F}(0,0) = e_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.

Proof: The proof of existence and uniquness is similiar to the existence and uniquness of lift of path.

Now suppose that F is a path homotopy. We wish to show that $\tilde{F}$ is a path homotopy. The map F carries the entire left edge $0 \times I$ of I^2 into a single point b_0 of B. Because $\tilde{F}$ is a lifting of F, it carries this edge into the set $p^{-1}(b_0)$. But this set has the discrete topology as a subspace of E. Since $0 \times I$ is connected and $\tilde{F}$ is continuous, $\tilde{F}(0 \times I)$ is connected and thus must equal a one-point set. Similarly, $\tilde{F}(1 \times I)$ must be a one-point set. Thus $\tilde{F}$ is a path homotopy.

Theorem 1.3.11. Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let f and g be two paths in B from b_0 to b_1; let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths in E beginning at e_0. If f and g are path homotopic, then $\tilde{f}$ and $\tilde{g}$ end at the same point of E and are path homotopic.

Proof: By Lemma 1.3.9 and Lemma 1.3.10.

Theorem 1.3.12. Let $p : E \to B$ be a covering map; let $b_0 \in B$. Choose e_0 so that $p(e_0) = b_0$. Given an element $[f]$ of $\pi_1(B, b_0)$, let $\tilde{f}$ be the lifting of f to a path in E that begins at e_0. Let $\phi([f])$ denote the end point $\tilde{f}(1)$ of $\tilde{f}$. Then ϕ is a well-defined set map

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0).$$

We call ϕ the lifting correspondence derived from the covering map p. It depends of course on the choice of the point e_0.

Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. If E is path connected, then the lifting correspondence

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

is surjective. If E is simply connected, it is bijective.

Proof: If E is path connected, then, given $e_1 \in p^{-1}(b_0)$, there is a path $\tilde{f}$ in E from e_0 to e_1. Then $f = p \circ \tilde{f}$ is a loop in B at b_0, and $\phi([f]) = e_1$ by definition.

Suppose E is simply connected. Let $[f]$ and $[g]$ be two elements of $\pi_1(B, b_0)$ such that $\phi([f]) = \phi([g])$. Let $\tilde{f}$ and $\tilde{g}$ be the liftings of f and g, respectively, to paths in E that begin at e_0; then $\tilde{f}(1) = \tilde{g}(1)$. Since E is simply connected, there is a path homotopy $\tilde{F}$ in E between $\tilde{f}$ and $\tilde{g}$. Then $p \circ \tilde{F}$ is a path homotopy in B between f and g.

Example 1.3.13. Fundamental group of $\mathbb{S}^1 \simeq \mathbb{Z}$

Proof: Let $p : \mathbb{R} \to S^1 : x \mapsto e^{2\pi i x}$, let $e_0 = 0$, and let $b_0 = p(e_0) = 1$. Then $p^{-1}(b_0)$ is the set $\mathbb{Z}$ of integers. Since $\mathbb{R}$ is simply connected, the lifting correspondence

$$\phi : \pi_1\left(S^1, b_0\right) \to \mathbb{Z}$$

is bijective.

Given $[f]$ and $[g]$ in $\pi_1(B, b_0)$, let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths on $\mathbb{R}$ beginning at 0 . Let $n = \tilde{f}(1)$ and $m = \tilde{g}(1)$; then $\phi([f]) = n$ and $\phi([g]) = m$, by definition. Let $\tilde{\tilde{g}}$ be the path

$$\tilde{\tilde{g}}(s) = n + \tilde{g}(s)$$

on $\mathbb{R}$. Because $p(n+x) = p(x)$ for all $x \in \mathbb{R}$, the path $\tilde{\tilde{g}}$ is a lifting of g; it begins at n. Then the product $\tilde{f} * \tilde{\tilde{g}}$ is defined, and it is the lifting of $f * g$ that begins at 0 , as you can check. The end point of this path is $\tilde{\tilde{g}}(1) = n + m$. Then by definition,

$$\phi([f] * [g]) = n + m = \phi([f]) + \phi([g]).$$

Definition 1.3.14 (equivalence of covering map)**.** Let $p : E \to B$ and $p' : E' \to B$ be covering maps. They are said to be equivalent if there exists a homeomorphism $h : E \to E'$ such that $p = p' \circ h$. The homeomorphism h is called an equivalence of covering maps or an equivalence of covering spaces.

Proposition 1.3.15. Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$.

(1) The homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism.

(2) Let $H = p_*(\pi_1(E, e_0))$. The lifting correspondence ϕ induces an injective map

$$\Phi : \pi_1(B, b_0) / H \to p^{-1}(b_0)$$

of the collection of right cosets of H into $p^{-1}(b_0)$, which is bijective if E is path connected.

(3) If f is a loop in B based at b_0, then $[f] \in H$ if and only if f lifts to a loop in E based at e_0.

Proof: (1): Suppose $\tilde{h}$ is a loop in E at e_0, and $p_*([\tilde{h}])$ is the identity element. Let F be a path homotopy between $p \circ \tilde{h}$ and the constant loop. If $\tilde{F}$ is the lifting of F to E such that $\tilde{F}(0,0) = e_0$, then by Lemma 1.3.10, $\tilde{F}$ is a path homotopy between $\tilde{h}$ and the constant loop at e_0.

(2): Let $h \in \pi_1(B, b_0)$ and $\tilde{h}$ be the lift of h and f be an element in $\pi_1(E, e_0)$ then f is a lift of $p \circ f$. Notice that $p \circ (f * \tilde{h}) = (p \circ f) * (p \circ \tilde{h}) = (p \circ f) * h$, then $f * \tilde{h}$ is a lift of $(p \circ f) * h$. Hence Φ is well-defined. If E is path connected, then Φ is surjective by Theorem 1.3.12.

Injectivity of Φ means that $\phi([f]) = \phi([g])$ if and only if $[f] \in H * [g]$ which follows from the definition of H.

(3) Trivial.

Remark 1.3.16. In the following theorems, the statement that $p : E \to B$ is a covering map will include the assumption that E and B are locally path connected and path connected.

Theorem 1.3.17. Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let $f : Y \to B$ be a continuous map, with $f(y_0) = b_0$. Suppose Y is path connected and locally path connected. The map f can be lifted to a map $\tilde{f} : Y \to E$ such that $\tilde{f}(y_0) = e_0$ if and only if

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0)).$$

Furthermore, if such a lifting exists, it is unique.

Proof:

Theorem 1.3.18. Let $p : E \to B$ and $p' : E' \to B$ be covering maps; let $p(e_0) = p'(e_0') = b_0$. There is an equivalence $h : E \to E'$ such that $h(e_0) = e_0'$ if and only if the groups

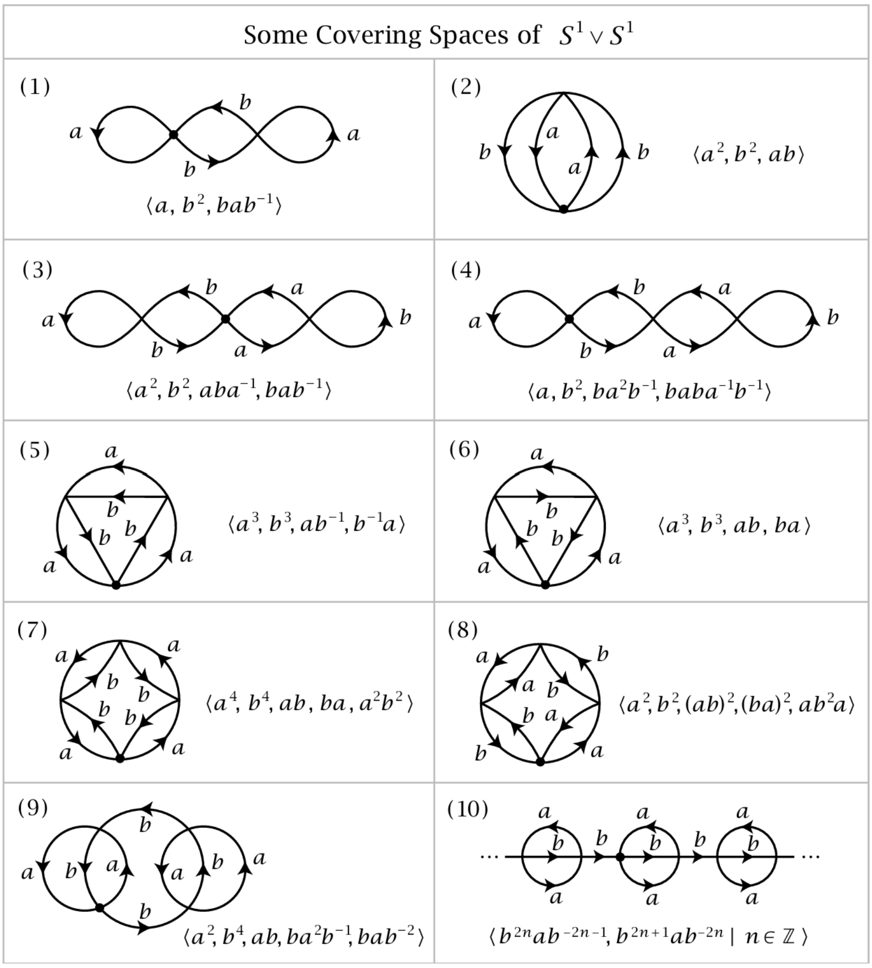
$$H_0 = p_*(\pi_1(E, e_0)) \quad \text{and} \quad H_0' = p_*'(\pi_1(E', e_0'))$$

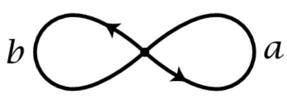
are equal. If h exists, it is unique.

Theorem 1.3.19. Let $p : E \to B$ and $p' : E' \to B$ be covering maps; let $p(e_0) = p'(e_0') = b_0$. The covering maps p and p' are equivalent if and only if the subgroups

$$H_0 = p_*(\pi_1(E, e_0)) \quad \text{and} \quad H_0' = p_*'(\pi_1(E', e_0'))$$

of $\pi_1(B, b_0)$ are conjugate.





Example 1.3.20 (covering spaces of $S^1 \bigvee S^1$).

Corollary 1.3.21. Free group of free 2 has free subgroup of rank 3, 4, 5 and countable caridinity.

Definition 1.3.22 (universal covering space)**.** Suppose $p : E \to B$ is a covering map, with $p(e_0) = b_0$. If E is simply connected, then E is called a universal covering space of B.

Definition 1.3.23 (semilocally simply connected)**.** A space B is said to be semilocally simply connected if for each $b \in B$, there is a neighborhood U of b such that the homomorphism

$$i_* : \pi_1(U, b) \to \pi_1(B, b)$$

induced by inclusion is trivial.

Theorem 1.3.24 (existence of covering space)**.** Let B be path connected, locally path connected, and semilocally simply connected. Let $b_0 \in B$. Given a subgroup H of $\pi_1(B, b_0)$, there exists a covering map $p : E \to B$ and a point $e_0 \in p^{-1}(b_0)$ such that

$$p_*(\pi_1(E, e_0)) = H$$

Theorem 1.3.25. Given $p : E \to B$ with $p(e_0) = b_0$, let F be the set $F = p^{-1}(e_0)$. Let

$$\Phi : \pi_1(B, b_0) / H_0 \to F$$

be the lifting correspondence; it is a bijection. Define also a correspondence

$$\Psi : \mathcal{C}(E, p, B) \to F$$

by setting $\Psi(h) = h(e_0)$ for each covering transformation $h : E \to E$. Since h is uniquely determined once its value at e_0 is known, the correspondence Ψ is injective.

The image of the map Ψ equals the image under Φ of the subgroup $N(H_0)/H_0$ of $\pi_1(B, b_0)/H_0$ and the bijection

$$\Phi^{-1} \circ \Psi : \mathcal{C}(E, p, B) \to N(H_0)/H_0$$

is an isomorphism of groups.

1.4 Seifert-van Kampen

Theorem 1.4.1 (Seifert-van Kampen, version 1)**.** Suppose $X = U \cup V$, where U and V are open sets of X. Suppose that $U \cap V$ is path connected, and that $x_0 \in U \cap V$. Let i and j be the inclusion mappings of U and V, respectively, into X. Then the images of the induced homomorphisms

$$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0) \quad \text{and} \quad j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$$

generate $\pi_1(X, x_0)$.

Theorem 1.4.2 (Seifert-van Kampen, version 2)**.** Let $X = U \cup V$, where U and V are open in X; assume U, V, and $U \cap V$ are path connected; let $x_0 \in U \cap V$. Let H be a group, and let

$$\phi_1 : \pi_1(U, x_0) \longrightarrow H \quad \text{and} \quad \phi_2 : \pi_1(V, x_0) \longrightarrow H$$

be homomorphisms. Let i_1, i_2, j_1, j_2 be the homomorphisms indicated in the following diagram, each induced by inclusion.

$$\begin{tikzcd}
& \pi_1(U, x_0) \arrow[d, "j_1"] \arrow[dr, bend left, "\phi_1"] & \\
\pi_1(U\cap V, x_0) \arrow[ur, "i_1"] \arrow[r] \arrow[dr, "i_2"'] & \pi_1(X, x_0) \arrow[r, dashed, "\Phi"] & H \\
& \pi_1(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

If $\phi_1 \circ i_1 = \phi_2 \circ i_2$, then there is a unique homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that $\Phi \circ j_1 = \phi_1$ and $\Phi \circ j_2 = \phi_2$.

Theorem 1.4.3 (Seifert-van Kampen, version 3)**.** Assume the hypotheses of the preceding theorem. Let

$$j : \pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

be the homomorphism of the free product that extends the homomorphisms j_1 and j_2 induced by inclusion. Then j is surjective, and its kernel is the least normal subgroup N of the free product that contains all elements represented by words of the form

$$\left(i_1(g)^{-1}, i_2(g)\right),$$

for $g \in \pi_1(U \cap V, x_0)$.

Proof: Step 1: j is surjective. By Theorem 1.4.1.

Step 2: $N \subset \ker j$.

Step 3: $N \supset \ker j$ which follows from the following diagram:

$$
\begin{tikzcd}
 & \pi_1(U,x_0) \arrow[d, "j_1"] \arrow[dr, "\Phi_1"] & \\
\pi_1(U\cap V,x_0) \arrow[ur, bend left, "i_1"] \arrow[dr, bend right, "i_2"'] & \pi_1(X,x_0) \arrow[r, shift left, "\Phi"] & \pi_1(U,x_0) * \pi_1(V,x_0)/N \arrow[l, shift left, "\tilde{j}"] \\
 & \pi_1(V,x_0) \arrow[u, "j_2"'] \arrow[ur, "\Phi_2"'] &
\end{tikzcd}
$$

N generated by $(i_1(g)^{-1}, i_2(g))$, $g \in \pi_1(U\cap V)$.

$\tilde{j}$ induced by $j: \pi_1(U,x_0) * \pi_1(V,x_0) \longrightarrow \pi_1(X,x_0)$

j induced by j_1 and j_2

1.5 Homotopy equivalence

Definition 1.5.1 (Homotopy equivalence)**.** Let $f : X \to Y$ and $g : Y \to X$ be continuous maps. Suppose that the map $g \circ f : X \to X$ is homotopic to the identity map of X, and the map $f \circ g : Y \to Y$ is homotopic to the identity map of Y. Then the maps f and g are called homotopy equivalences, and each is said to be a homotopy inverse of the other.

Lemma 1.5.2 (Homotopy invariance of fundamental group homomorphism)**.** Let $h, k : (X, x_0) \to (Y, y_0)$ be continuous maps. If h and k are homotopic, and if the image of the base point x_0 of X remains fixed at y_0 during the homotopy, then the homomorphisms h_* and k_* are equal.

Proof: The proof is immediate. By assumption, there is a homotopy $H : X \times I \to Y$ between h and k such that $H(x_0, t) = y_0$ for all t. It follows that if f is a loop in X based at x_0, then the composite

$$I \times I \xrightarrow{f \times \mathrm{id}} X \times I \xrightarrow{H} Y$$

is a homotopy between $h \circ f$ and $k \circ f$; it is a path homotopy because f is a loop at x_0 and H maps $x_0 \times I$ to y_0.

Proposition 1.5.3 (correspondence between path-way connected components)**.** There's a one-to-one correspondence between path-way connected components of X and path-way connected components of Y.

Definition 1.5.4 (retraction)**.** If $A \subset X$, a retraction of X onto A is a continuous map $r : X \to A$ such that $r|_A$ is the identity map of A. If such a map r exists, we say that A is a retract of X.

Proposition 1.5.5. If A is a retract of X, then the homomorphism of fundamental groups induced by inclusion $j : A \to X$ is injective.

Definition 1.5.6 (deformation retract)**.** Let A be a subspace of X. We say that A is a deformation retract of X if the identity map of X is homotopic to a map that carries all of X into A, such that each point of A remains fixed during the homotopy.

Proof: A is a deformation retract of X if and only if there's $H : X \times I \to X$ such that $H(x, 0) = x$ for all $x \in X$, $H(a, 1) = a$ for all $a \in A$ and $H(x, 1) \in A$ for all $x \in X$.

Proposition 1.5.7. $r : X \to A$ gives a deformation retract from X to A if and only if there's homotopy $H(x, t)$ between id_X and $i \circ r$. We call such $H(x, t)$ also a deformation retract.

Definition 1.5.8 (contractible space)**.** We call a topological space contractible if it is homotopy equivalent to a one-point space.

Proposition 1.5.9. X is contractible if and only if $\mathrm{id} : X \to X$ is nulhomotopic.

Example 1.5.10. X is a topological space, cone of X is the space $X \times [0, 1]/ \sim$, where $\sim$ means the points in $X \times \{1\}$ are all equivalent. In general, we denote it by CX. Also, it's easy to say that $X \times \{1\}$ with subspace topology in CX is isomorphic to X.

Lemma 1.5.11. $f : X \to Y$ is a quotient map, $A \subset X$. $r : X \to A$ is a deformation retract, and we denote the homotopy between $i \circ r : X \to X$, and $\mathrm{id} : X \to X$ by $H(x,t)$. If $f(x) = f(y)$ implies $f(H(x,t)) = f(H(y,t))$, $f(A)$ is a deformation retract of Y.

Proof:

$$\begin{tikzcd} X \times I \arrow[r, "H"] \arrow[d, "f\times\mathrm{id}"'] & X \arrow[d, "f"] \\ Y \times I \arrow[r, dashed, "G"'] & Y \end{tikzcd}$$

By Proposition 1.1.8, $f \times \mathrm{id}$ is a quotient map, then by Proposition 1.1.5, there's a uniquness continous map G makes the diagram commute. Then it's easy to check $y \mapsto G(y,1)$ is a deformation retract of Y onto $f(B)$.

Proposition 1.5.12. CX is contractible.

Proof: Consider

$$H : (X \times I) \times I \to X \times I$$
$$(x,t,s) \mapsto (x, (1-s)t + s \cdot 1)$$

and use Lemma 1.5.11.

Proposition 1.5.13. Show that a map $f : X \to Y$ is nullhomotopic if and only if it extends to a map $F : CX \to Y$.

Proof: "only if": Consider $H : X \times I \to Y$ is a homotopy between f and a constant map, then by Proposition 1.1.5, H induce a continous map F between CX and Y which makes the following diagram commute

$$\begin{tikzcd} X \times I \arrow[d, "\pi"] \arrow[dr, "H"] & \\ X \times I/\sim \arrow[r, dashed, "F"] & Y \end{tikzcd}$$

"if": Consider the composition $H : X \times I \xrightarrow{\pi} CX \xrightarrow{F} Y$

Proposition 1.5.14. Let A be a deformation retract of X; let $x_0 \in A$. Then the inclusion

$$j : (A, x_0) \to (X, x_0)$$

induces an isomorphism of fundamental groups.

Proof: By Example 1.3.13.

Theorem 1.5.15. Let $h : S^1 \to X$ be a continuous map. Then the following conditions are equivalent:

(1) h is nulhomotopic.

(2) h extends to a continuous map $k : B^2 \to X$.

(3) $(h_{x_0})_*$ is the trivial homomorphism of fundamental groups for some $x_0 \in \mathbb{S}^1$

Proof: (1) $\Leftrightarrow$ (2): By Proposition 1.5.13.

(1) $\Rightarrow$ (3): trivial.

(3) $\Rightarrow$ (1): By proposition 1.2.13, we may assume $x_0 = 1$. Since h_* is trivial, there's $H : I \times I \to X$ such that $H(x,0) = h(1), H(x,1) = h(\mathrm{e}^{2\pi \mathrm{i} x})$. Let $p_0 : I \to S^1 \quad x \mapsto \mathrm{e}^{2\pi \mathrm{i} x}$, then $p_0 \times \mathrm{id} : I \times I \to S^1 \times I$ is a quotient map, hence by Proposition 1.1.5, there's $\tilde{H}$ making the following diagram commute

$$\begin{tikzcd} I \times I \arrow[d, "p_0 \times \mathrm{id}"] \arrow[dr, "H"] & \\ S^1 \times I \arrow[r, dashed, "\tilde{H}"] & X \end{tikzcd}$$

and $\tilde{H}$ is what we need.

Corollary 1.5.16. Identity map $\mathbb{S}^1 \to \mathbb{S}^1$ are not nulhomotopic.

Theorem 1.5.17. Given a continuous map $v : \mathbb{S}^1 \to \mathbb{R}^2 - \{0\}$, there exists a point of S^1 where the vector field points directly inward and a point of S^1 where it points directly outward.

Theorem 1.5.18 (Brouwer fixed-point theorem for the disc)**.** If $f : B^2 \to B^2$ is continuous, then there exists a point $x \in B^2$ such that $f(x) = x$.

Example 1.5.19 (Fundamental theorem of Algebra)**.** A polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

of degree $n > 0$ with complex coefficients has at least one complex root.

Proof: Step 1 : Let $f : \mathbb{S}^1 \to \mathbb{S}^1 : z \mapsto z^n$. Then by Theorem 1.3.12, the induced group homomorphism $f_* : \pi(\mathbb{S}^1, 1) \to \pi(\mathbb{S}^1, 1)$ is injective.

Step 2 : We show that if $g : \mathbb{S}^1 \to \mathbb{R}^2 - 0$ is the map $g(z) = z^n$, then g is not nulhomotopic.

Let $j : \mathbb{S}^1 \to \mathbb{R}^2 - \nvdash$ be inclusion. Notice that $j \circ f = g$, then by Theorem 1.5.5, g_* is injective. Hence g is not nulhomotopic.

Step 3 : Now we prove a stronger case of the theorem. Given a polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

we assume that

$$|a_{n-1}| + \cdots + |a_1| + |a_0| < 1$$

and show that the equation has a root lying in the unit ball B^2. Notice that if we replace x by cx for a sufficiently large $c > 0$, we can obtain the original Fundemental Theorem of Algebra. Assume it has no such root. Then we can define a map $k : B^2 \to \mathbb{R}^2 - \nvdash$ by the equation

$$k(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

Let h be the restriction of k to S^1. Because h extends to a map of the unit ball into $\mathbb{R}^2 - \nvdash$, the map h is nulhomotopic.

On the other hand, we shall define a homotopy F between h and the map g of Step 2; since g is not nulhomotopic, we have a contradiction. We define $F : S^1 \times I \to \mathbb{R}^2 - \nvdash$ by the equation

$$F(z,t) = z^n + t\left(a_{n-1}z^{n-1} + \cdots + a_0\right).$$

$F(z,t)$ never equals $\nvdash$ because

$$\begin{aligned} |F(z,t)| &\geq |z^n| - \left|t\left(a_{n-1}z^{n-1} + \cdots + a_0\right)\right| \\ &\geq 1 - t\left(\left|a_{n-1}z^{n-1}\right| + \cdots + |a_0|\right) \\ &= 1 - t\left(|a_{n-1}| + \cdots + |a_0|\right) > 0. \end{aligned}$$

Example 1.5.20.

$$\begin{aligned} r : \mathbb{R}^n - \{0\} &\to \mathbb{S}^{n-1} \\ x &\mapsto \frac{x}{||x||} \end{aligned}$$

gives us a deformation retract of $\mathbb{R}^n - \{0\}$ onto $\mathbb{S}^{n-1}$.

Proposition 1.5.21. *$f, g \in C(X, S^n)$, if f and g satisfies $f(x) \neq -g(x), \forall x \in X$, then $f \simeq g$.*

Proof: Consider

$$H(x,t) = \frac{(1-t)f(x) + tg(x)}{||(1-t)f(x) + tg(x)||}$$

Proposition 1.5.22. *$f, g \in C(X, E)$, E is a convex subgroup of $\mathbb{R}^n$, then $f \simeq g$. by the homotopic map $H(x,t) = (1-t)f(x) + tg(x)$.*

Theorem 1.5.23. *Let $h, k : X \to Y$ be continuous maps; let $h(x_0) = y_0$ and $k(x_0) = y_1$. If h and k are homotopic, there is a path α in Y from y_0 to y_1 such that $k_* = \hat{\alpha} \circ h_*$. Indeed, if $H : X \times I \to Y$ is the homotopy between h and k, then α is the path $\alpha(t) = H(x_0, t)$.*

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "h_*"] \arrow[dr, "k_*"'] & \pi_1(Y, y_0) \arrow[d, "\hat{\alpha}"] \\ & \pi_1(Y, y_1) \end{tikzcd}$$

Proof: Let $f : I \to X$ be a loop in X based at x_0. We must show that

$$k_*([f]) = \hat{\alpha}\left(h_*([f]\right).$$

This equation states that $[k \circ f] = [\bar{\alpha}] * [h \circ f] * [\alpha]$, or equivalently, that

$$[\alpha] * [k \circ f] = [h \circ f] * [\alpha].$$

This is the equation we shall verify. To begin, consider the loops f_0 and f_1 in the space $X \times I$ given by the equations

$$f_0(s) = (f(s), 0) \quad \text{and} \quad f_1(s) = (f(s), 1).$$

Consider also the path c in $X \times I$ given by the equation

$$c(t) = (x_0, t).$$

Then $H \circ f_0 = h \circ f$ and $H \circ f_1 = k \circ f$, while $H \circ c$ equals the path α. Let $F : I \times I \to X \times I$ be the map $F(s,t) = (f(s), t)$. Consider the following paths in $I \times I$, which run along the four edges of $I \times I$:

$$\beta_0(s) = (s, 0) \quad \text{and} \quad \beta_1(s) = (s, 1),$$
$$\gamma_0(t) = (0, t) \quad \text{and} \quad \gamma_1(t) = (1, t).$$

Then $F \circ \beta_0 = f_0$ and $F \circ \beta_1 = f_1$, while $F \circ \gamma_0 = F \circ \gamma_1 = c$. The broken-line paths $\beta_0 * \gamma_1$ and $\gamma_0 * \beta_1$ are paths in $I \times I$ from $(0,0)$ to $(1,1)$; since $I \times I$ is convex, there is a path homotopy G between them. Then $F \circ G$ is a path homotopy in $X \times I$ between $f_0 * c$ and $c * f_1$. And $H \circ (F \circ G)$ is a path homotopy in Y between

$$(H \circ f_0) * (H \circ c) = (h \circ f) * \alpha \quad \text{and}$$
$$(H \circ c) * (H \circ f_1) = \alpha * (k \circ f),$$

Corollary 1.5.24. Let $h, k : X \to Y$ be homotopic continuous maps; let $h(x_0) = y_0$ and $k(x_0) = y_1$. If h_* is injective, or surjective, or trivial, so is k_*.

Corollary 1.5.25. Let $h : X \to Y$. If h is nulhomotopic, then h_* is the trivial homomorphism.

Corollary 1.5.26. Let $f : X \to Y$ be continuous; let $f(x_0) = y_0$. If f is a homotopy equivalence, then

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

is an isomorphism.

Proof: Let $g : Y \to X$ be a homotopy inverse for f. Consider the maps

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1),$$

where $x_1 = g(y_0)$ and $y_1 = f(x_1)$. We have the corresponding induced homomorphisms:

$$\pi_1(X, x_0) \xrightarrow{(f_{x_0})_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(X, x_1) \xrightarrow{(f_{x_1})_*} \pi_1(Y, y_1)$$

Now

$$g \circ f : (X, x_0) \longrightarrow (X, x_1)$$

is by hypothesis homotopic to the identity map, so there is a path α in X such that

$$(g \circ f)_* = \hat{\alpha} \circ (i_X)_* = \hat{\alpha}.$$

It follows that $(g \circ f)_* = g_* \circ (f_{x_0})_*$ is an isomorphism. Hence g_* is surjective. Similarly, because $f \circ g$ is homotopic to the identity map i_Y, the homomorphism $(f \circ g)_* = (f_{x_1})_* \circ g_* = \hat{\beta}$ for some path β in Y. Hence g_* is injective.

Corollary 1.5.27. For $n \geq 2$, $\mathbb{S}^n$ is simply connected.

Proof: Let $p = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ and $q = (0, \ldots, 0, -1)$ be the "north pole" and the "south pole" of S^n, respectively.

Step 1. We show that if $n \geq 1$, the punctured sphere $S^n - p$ is homeomorphic to $\mathbb{R}^n$. Define $f : (S^n - p) \to \mathbb{R}^n$ by the equation

$$f(x) = f(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}} (x_1, \ldots, x_n).$$

One checks that f is a homeomorphism by showing that the map $g : \mathbb{R}^n \to (S^n - p)$ given by

$$g(y) = g(y_1, \ldots, y_n) = (t(y) \cdot y_1, \ldots, t(y) \cdot y_n, 1 - t(y)),$$

where $t(y) = 2/\left(1 + \|y\|^2\right)$, is a right and left inverse for f. Note that the reflection map $(x_1, \ldots, x_{n+1}) \to (x_1, \ldots, x_n, -x_{n+1})$ defines a homeomorphism of $S^n - p$ with $S^n - q$, so the latter is also homeomorphic to $\mathbb{R}^n$.

Step 2. We prove the theorem. Let U and V be the open sets $U = S^n - p$ and $V = S^n - q$ of S^n.

Note first that for $n \geq 1$, the sphere S^n is path connected. This follows from the fact that U and V are path connected (being homeomorphic to $\mathbb{R}^n$) and have the point $(1, 0, \ldots, 0)$ of S^n in common.

Now we show that for $n \geq 2$, the sphere S^n is simply connected. The spaces U and V are simply connected, being homeomorphic to $\mathbb{R}^n$. Their intersection equals $S^n - p - q$, which is homeomorphic under stereographic projection to $\mathbb{R}^n - \nvdash$. The latter space is path connected, for every point of $\mathbb{R}^n - \nvdash$ can be joined to a point of S^{n-1} by a straight-line path, and S^{n-1} is path connected if $n \geq 2$. Then by Theorem 1.4.1, we finish the proof.

Proposition 1.5.28. $\pi_1(X \times Y, x_0 \times y_0)$ is isomorphic with $\pi_1(X, x_0) \times \pi_1(Y, y_0)$. In particular, let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projection mappings. If we use the base points indicated in the statement of the theorem, we have induced homomorphisms

$$\begin{aligned} p_* &: \pi_1(X \times Y, x_0 \times y_0) \longrightarrow \pi_1(X, x_0), \\ q_* &: \pi_1(X \times Y, x_0 \times y_0) \longrightarrow \pi_1(Y, y_0). \end{aligned}$$

We define a homomorphism

$$\Phi : \pi_1(X \times Y, x_0 \times y_0) \longrightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

by the equation

$$\Phi([f]) = p_*([f]) \times q_*([f]) = [p \circ f] \times [q \circ f].$$

then Φ is an isomorphism.

Proposition 1.5.29. $\mathbb{RP}^n \simeq \mathbb{S}^n / \{x \sim -x\} \simeq \mathbb{D}^n / \mathbb{S}^{n-1}$ where the second isomorphism is given by

$$(z_1, \ldots, z_n) \in \mathbb{D}^n / \mathbb{S}^{n-1} \mapsto [z_1, \ldots, z_n, \sqrt{1 - z_1^2 - \cdots - z_n^2} \quad] \in \mathbb{RP}^n$$

Example 1.5.30. $\pi(\mathbb{RP}^n, y) \simeq \mathbb{Z}/2\mathbb{Z}$

Proof: Step 1: $p : \mathbb{S}^n \to \mathbb{RP}^n \quad (x_0, \ldots, x_n) \mapsto [x_0, \ldots, x_n]$ is a surjective continuous open map.

Step 2: p is a covering map.

It follows from the fact the $\mathbb{RP}^n \simeq \mathbb{S}^n/\sim$, where $\simeq$ is the equivalence relation defined by $x \sim y$ iff $x = \pm y$.

Step 3: $\pi(\mathbb{RP}^n, y) \simeq \mathbb{Z}/2\mathbb{Z}$

By Corollary 1.5.27 and Theorem 1.3.12.

1.6 Classifiction of Compact Surface

Definition 1.6.1 (polygonal representation of topological space)**.**

Lemma 1.6.2. Let X be a Hausdorff space; let A be a closed path-connected subspace of X. Suppose that there is a continuous map $h : B^2 \to X$ that maps $B^2 - \mathbb{S}^1$ bijectively onto $X - A$ and maps $\mathbb{S}^1$ into A. Let $p \in S^1$ and let $a = h(p)$; let $k : (S^1, p) \to (A, a)$ be the map obtained by restricting h. Then the homomorphism

$$i_* : \pi_1(A, a) \longrightarrow \pi_1(X, a)$$

induced by inclusion is surjective, and its kernel is the least normal subgroup of $\pi_1(A, a)$ containing the image of $k_* : \pi_1(S^1, p) \to \pi_1(A, a)$.

Proposition 1.6.3. Let X be the wedge of the circles $S_1, \ldots, S_n$; let p be the common point of these circles. Then $\pi_1(X, p)$ is a free group. If f_i is a loop in S_i that represents a generator of $\pi_1(S_i, p)$, then the loops $f_1, \ldots, f_n$ represent a system of free generators for $\pi_1(X, p)$.

Proof: By Theorem 1.4.3

Corollary 1.6.4. Let P be a polygonal region; let

$$w = (a_{i_1})^{\epsilon_1} \cdots (a_{i_n})^{\epsilon_n}$$

be a labelling scheme for the edges of P. Let X be the resulting quotient space; let $\pi : P \to X$ be the quotient map. If π maps all the vertices of P to a single point x_0 of X, and if $a_1, \ldots, a_k$ are the distinct labels that appear in the labelling scheme, then $\pi_1(X, x_0)$ is isomorphic to the quotient of the free group on k generators $\alpha_1, \ldots, \alpha_k$ by the least normal subgroup containing the element

$$(\alpha_{i_1})^{\epsilon_1} \cdots (\alpha_{i_n})^{\epsilon_n}.$$

Definition 1.6.5. Let X be a Hausdorff space that is the union of the subspaces $S_1, \ldots, S_n$, each of which is homeomorphic to the unit circle S^1. Assume that there is a point p of X such that $S_i \cap S_j = \{p\}$ whenever $i \neq j$. Then X is called the wedge of the circles $S_1, \ldots, S_n$.

Definition 1.6.6 (dunce cap)**.** Let n be a positive integer with $n > 1$. Let $r : S^1 \to S^1$ be rotation through the angle $2\pi/n$, mapping the point $(\cos\theta, \sin\theta)$ to the point $(\cos(\theta + 2\pi/n), \sin(\theta + 2\pi/n)\ )$. Form a quotient space X from the unit ball B^2 by identifying each point x of S^1 with the points $r(x), r^2(x), \ldots, r^{n-1}(x)$. We call it the n-fold dunce cap.

Proof: By Theorem 1.6.2.

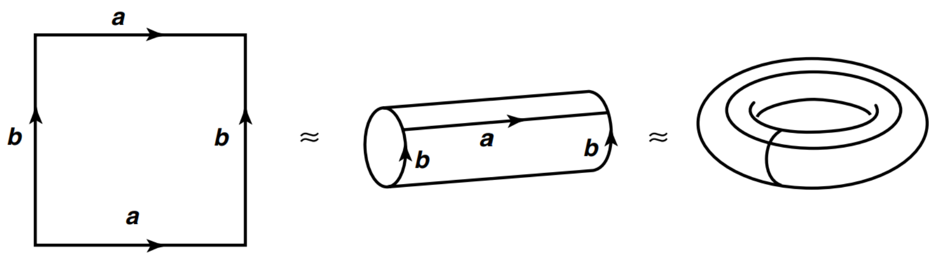


Figure 1.1: torus

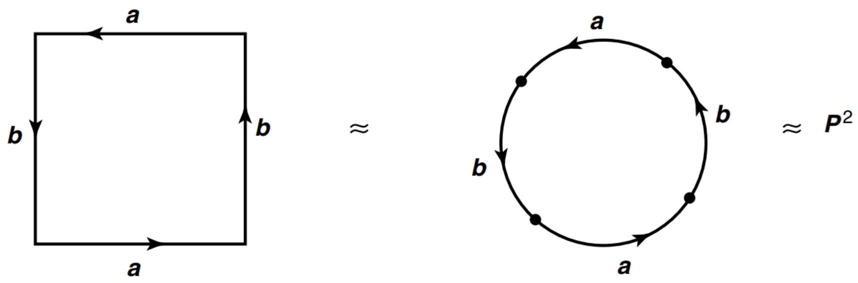


Figure 1.2: projective space

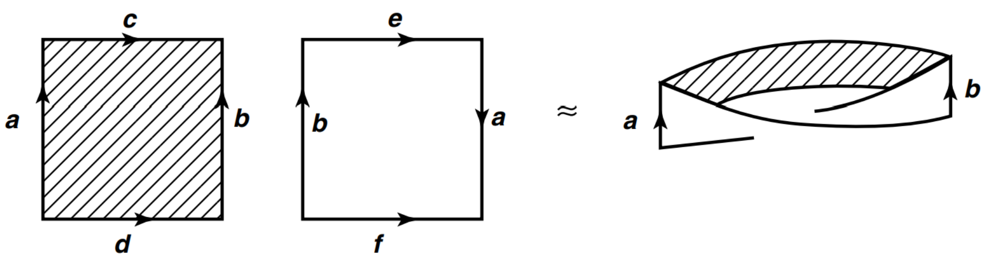


Figure 1.3: Mobious band

Proposition 1.6.7. Let X be the space obtained from a finite collection of polygonal regions by pasting edges together according to some labelling scheme. Then X is a compact Hausdorff space.

Theorem 1.6.8. Consider the space obtained from a $4n$-sided polygonal region P by means of the labelling scheme

$$\left(a_1b_1a_1^{-1}b_1^{-1}\right)\left(a_2b_2a_2^{-1}b_2^{-1}\right)\cdots\left(a_nb_na_n^{-1}b_n^{-1}\right).$$

This space is called the n-fold connected sum of tori, or simply the n-fold torus, and denoted $T\#\ldots\#T$.

Theorem 1.6.9. Let X denote the n-fold torus. Then $\pi_1(X, x_0)$ is isomorphic to the quotient of the free group on the $2n$ generators $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$ by the least normal subgroup containing the element

$$[\alpha_1, \beta_1][\alpha_2, \beta_2] \cdots [\alpha_n, \beta_n]$$

where $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$, as usual.

Theorem 1.6.10. Let $m > 1$. Consider the space obtained from a $2m$-sided polygonal region P in the plane by means of the labelling scheme

$$(a_1 a_1)(a_2 a_2) \cdots (a_m a_m)$$

This space is called the m-fold connected sum of projective planes, or simply the $\boldsymbol{m}$-fold projective plane, and denoted $P^2 \# \ldots \# P^2$.

Theorem 1.6.11. Let X denote the m-fold projective plane. Then $\pi_1(X, x_0)$ is isomorphic to the quotient of the free group on m generators $\alpha_1, \ldots, \alpha_m$ by the least normal subgroup containing the element

$$(\alpha_1)^2 (\alpha_2)^2 \cdots (\alpha_m)^2 .$$

Definition 1.6.12 (elementary operations on schemes)**.** We now list a number of elementary operations that can be performed on a labelling scheme $w_1, \ldots, w_m$ without affecting the resulting quotient space X.

(1) Cut. One can replace the scheme $w_1 = y_0 y_1$ by the scheme $y_0 c^{-1}$ and $c y_1$, provided c does not appear elsewhere in the total scheme and y_0 and y_1 have length at least two.

(2) Paste. One can replace the scheme $y_0 c^{-1}$ and $c y_1$ by the scheme $y_0 y_1$, provided c does not appear elsewhere in the total scheme.

(3) Relabel. One can replace all occurrences of any given label by some other label that does not appear elsewhere in the scheme. Similarly, one can change the sign of the exponent of all occurrences of a given label a; this amounts to reversing the orientations of all the edges labelled " a ". Neither of these alterations affects the pasting map.

(4) Permute. One can replace any one of the schemes w_i by a cyclic permutation of w_i. Specifically, if $w_i = y_0 y_1$, we can replace w_i by $y_1 y_0$. This amount to renumbering the vertices of the polygonal region P_i so as to begin with a different vertex; it does not affect the resulting quotient space.

(5) Flip. One can replace the scheme

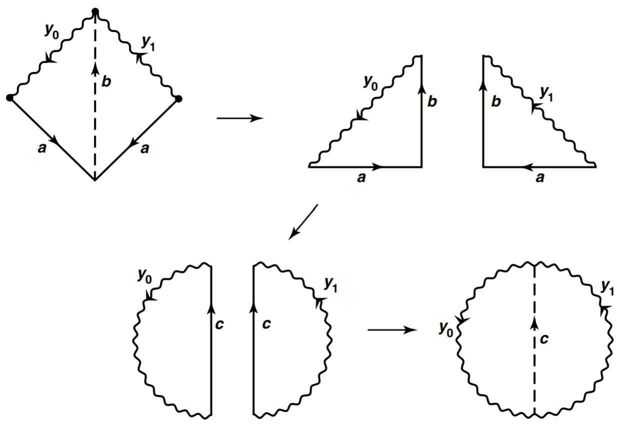
$$w_i = (a_{i_1})^{\epsilon_1} \cdots (a_{i_n})^{\epsilon_n}$$

by its formal inverse

$$w_i^{-1} = (a_{i_n})^{-\epsilon_n} \cdots (a_{i_1})^{-\epsilon_1} .$$

This amounts simply to "flipping the polygonal region P_i over.". The order of the vertices is reversed, and so is the orientation of each edge. The quotient space X is not affected.

(6) Cancel. One can replace the scheme $w_i = y_0aa^{-1}y_1$ by the scheme y_0y_1, provided a does not appear elsewhere in the total scheme and both y_0 and y_1 have length at least two.



Example 1.6.13. The Klein bottle K is the space obtained from the labelling scheme $aba^{-1}b$ is isomorphic to the 2-fold projective plane $P^2\#P^2(aabb)$ through the following elementary operations:

$$\begin{aligned} aba^{-1}b &\longrightarrow abc^{-1} \text{ and } ca^{-1}b \quad \text{by cutting} \\ &\longrightarrow c^{-1}ab \text{ and } b^{-1}ac^{-1} \quad \text{by permuting the first and flipping the second} \\ &\longrightarrow c^{-1}aac^{-1} \quad \text{by pasting} \\ &\longrightarrow aacc \end{aligned}$$

Theorem 1.6.14. Suppose $w_1, \ldots, w_k$ is a labelling scheme for the polygonal regions $P_1, \ldots, P_k$. If each label appears exactly twice in this scheme, we call it a proper labelling scheme. Note the important fact: If one applies any elementary operation to a proper scheme, one obtains another proper scheme.

Theorem 1.6.15 (classification)**.** Let X be the quotient space obtained from a polygonal region in the plane by pasting its edges together in pairs. Then X is homeomorphic either to S^2, to the n-fold torus T_n, or to the m-fold projective plane P_m.

Proof: Let w be the labelling scheme by which one forms the space X from the polygonal region P. Then w is a proper scheme of length least 4 . It suffice to show that w is equivalent(through elementary operations) to one of the following schemes: $aa^{-1}bb^{-1}$, $abab$, $(a_1a_1)(a_2a_2)\cdots(a_ma_m)$ with $m \geq 2$, $\left(a_1b_1a_1^{-1}b_1^{-1}\right)\left(a_2b_2a_2^{-1}b_2^{-1}\right)\cdots\left(a_nb_na_n^{-1}b_n^{-1}\right)$ with $n \geq 1$.

Theorem 1.6.16 (classification)**.** If X is a compact connected surface, then X is homeomorphic to a space obtained from a polygonal region in the plane by pasting the edges together in pairs.

1.7 Homology Group

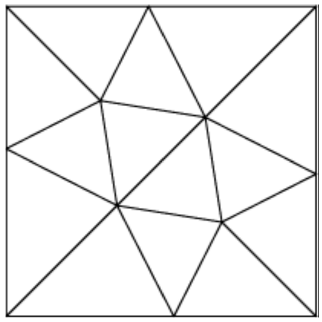


Figure 1.4: triangulation of torus

Chapter 2

Differential Manifold and Riemannian Geometry

2.1 Foundation

Definition 2.1.1. An n-dimensional topological manifold with boundary is a second-countable Hausdorff space M in which every point has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to a (relatively) open subset of $\mathbb{H}^n$. An open subset $U \subseteq M$ together with a map $\varphi : U \to \mathbb{R}^n$ that is a homeomorphism onto an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ will be called a chart for M, just as in the case of manifolds. When it is necessary to make the distinction, we will call (U, φ) an interior chart if $\varphi(U)$ is an open subset of $\mathbb{R}^n$ (which includes the case of an open subset of $\mathbb{H}^n$ that does not intersect $\partial\mathbb{H}^n$), and a boundary chart if $\varphi(U)$ is an open subset of $\mathbb{H}^n$ such that $\varphi(U) \cap \partial\mathbb{H}^n \neq \varnothing$. A boundary chart whose image is a set of the form $B_r(x) \cap \mathbb{H}^n$ for some $x \in \partial\mathbb{H}^n$ and $r > 0$ is called a coordinate half-ball.

A point $p \in M$ is called an interior point of M if it is in the domain of some interior chart. It is a boundary point of M if it is in the domain of a boundary chart that sends p to $\partial\mathbb{H}^n$. The boundary of M (the set of all its boundary points) is denoted by ∂M; similarly, its interior, the set of all its interior points, is denoted by Int M.

Proposition 2.1.2 (Topological Invariance of the Boundary)**.** If M is a topological manifold with boundary, then each point of M is either a boundary point or an interior point, but not both. Thus ∂M and $\operatorname{Int} M$ are disjoint sets whose union is M.

Proposition 2.1.3. Let M be a topological n-manifold with boundary. Int M is an open subset of M and a topological n-manifold without boundary and ∂M is a closed subset of M and a topological $n-1$ manifold without boundary.

Definition 2.1.4. To see how to define a smooth structure on a manifold with boundary, recall that a map from an arbitrary subset $A \subseteq \mathbb{R}^n$ to $\mathbb{R}^k$ is said to be smooth if in a neighborhood of each point of A it admits an extension to a smooth map defined on an open subset of $\mathbb{R}^n$. Thus, if U is an open subset of $\mathbb{H}^n$, a map $F : U \to \mathbb{R}^k$ is smooth if for each $x \in U$, there exists an open subset $\widetilde{U} \subseteq \mathbb{R}^n$ containing x and a smooth map $\widetilde{F} : \widetilde{U} \to \mathbb{R}^k$ that agrees with F on

$\widetilde{U} \cap \mathbb{H}^n$. If F is such a map, the restriction of F to $U \cap \operatorname{Int} \mathbb{H}^n$ is smooth in the usual sense. By continuity, all partial derivatives of F at points of $U \cap \partial\mathbb{H}^n$ are determined by their values in Int $\mathbb{H}^n$, and therefore in particular are independent of the choice of extension.

Proposition 2.1.5. U is a open subset of $\mathbb{H}^n$. $F : U \to \mathbb{R}^k$ is smooth if and only if F is continuous, $F|_{U \cap \operatorname{Int} \mathbb{H}^n}$ is smooth, and the partial derivatives of $F|_{U \cap \operatorname{Int} \mathbb{H}^n}$ of all orders have continuous extensions to all of U.

Definition 2.1.6. Let M, N be smooth manifolds, and let $F : M \to N$ be any map. We say that F is a smooth map if for every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$. If M and N are smooth manifolds with boundary, smoothness of F is defined in exactly the same way, with the usual understanding that a map whose domain is a subset of $\mathbb{H}^n$ is smooth if it admits an extension to a smooth map in a neighborhood of each point, and a map whose codomain is a subset of $\mathbb{H}^n$ is smooth if it is smooth as a map into $\mathbb{R}^n$.

Definition 2.1.7. Suppose M is a topological space, and let $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ be an arbitrary open cover of M, indexed by a set A. A partition of unity subordinate to $\mathcal{X}$ is an indexed family $(\psi_\alpha)_{\alpha \in A}$ of continuous functions $\psi_\alpha : M \to \mathbb{R}$ with the following properties:

(1) $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in A$ and all $x \in M$.

(2) $\operatorname{supp} \psi_\alpha \subseteq X_\alpha$ for each $\alpha \in A$.

(3) The family of supports $(\operatorname{supp} \psi_\alpha)_{\alpha \in A}$ is locally finite, meaning that every point has a neighborhood that intersects supp ψ_α for only finitely many values of α.

(4) $\sum_{\alpha \in A} \psi_\alpha(x) = 1$ for all $x \in M$.

Theorem 2.1.8 (Existence of Partitions of Unity)**.** Suppose M is a smooth manifold with or without boundary, and $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ is any indexed open cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{X}$.

Definition 2.1.9 (Extension Lemma for Smooth Functions)**.** Suppose M is a smooth manifold with or without boundary, $A \subseteq M$ is a closed subset, and $f : A \to \mathbb{R}^k$ is a smooth function. For any open subset U containing A, there exists a smooth function $\widetilde{f} : M \to \mathbb{R}^k$ such that $\left.\widetilde{f}\right|_A = f$ and $\operatorname{supp} \tilde{f} \subseteq U$.

Proposition 2.1.10 (Smooth Manifold Chart Lemma)**.** Let M be a set, and suppose we are given a collection $\{U_\alpha\}$ of subsets of M together with maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, such that the following properties are satisfied:

(1) For each α, φ_α is a bijection between U_α and an open subset $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$.

(2) For each α and β, the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$.

(3) Whenever $U_\alpha \cap U_\beta \neq \varnothing$, the map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth.

(4) Countably many of the sets U_α cover M.

(5) Whenever p, q are distinct points in M, either there exists some U_α containing both p and q or there exist disjoint sets U_α, U_β with $p \in U_\alpha$ and $q \in U_\beta$.

Then M has a unique smooth manifold structure such that each $(U_\alpha, \varphi_\alpha)$ is a smooth chart.

Lemma 2.1.11 (existence of bump functions)**.** If M is a topological space, $A \subseteq M$ is a closed subset, and $U \subseteq M$ is an open subset containing A, a continuous function $\psi : M \to \mathbb{R}$ is called a bump function for A supported in U if $0 \leq \psi \leq 1$ on M, $\psi \equiv 1$ on A, and $\operatorname{supp} \psi \subseteq U$.

Let M be a smooth manifold with or without boundary. For any closed subset $A \subseteq M$ and any open subset U containing A, there exists a smooth bump function for A supported in U.

Example 2.1.12 (smooth structure on vector space)**.** Let V be a finite-dimensional real vector space. Any norm on V determines a topology, which is independent of the choice of norm. With this topology, V is a topological n manifold, and has a natural smooth structure defined as follows. Each (ordered) basis $(E_1, \ldots, E_n)$ for V defines a basis isomorphism $E : \mathbb{R}^n \to V$ by

$$E(x) = \sum_{i=1}^{n} x^i E_i.$$

This map is a homeomorphism, so (V, E^{-1}) is a chart. If $\left(\widetilde{E}_1, \ldots, \widetilde{E}_n\right)$ is any other basis and $\widetilde{E}(x) = \sum_j x^j \widetilde{E}_j$ is the corresponding isomorphism, then there is some invertible matrix $\left(A_i^j\right)$ such that $E_i = \sum_j A_i^j \widetilde{E}_j$ for each i. The transition map between the two charts is then given by $\widetilde{E}^{-1} \circ E(x) = \tilde{x}$, where $\tilde{x} = (\widetilde{x}^1, \ldots, \tilde{x}^n)$ is determined by

$$\sum_{j=1}^{n} \widetilde{x}^j \widetilde{E}_j = \sum_{i=1}^{n} x^i E_i = \sum_{i,j=1}^{n} x^i A_i^j \widetilde{E}_j.$$

It follows that $\tilde{x}^j = \sum_i A_i^j x^i$. Thus, the map sending x to $\tilde{x}$ is an invertible linear map and hence a diffeomorphism, so any two such charts are smoothly compatible. The collection of all such charts thus defines a smooth structure, called the standard smooth structure on V.

Definition 2.1.13. Let M be a smooth n-manifold with or without boundary, and let $p \in M$. Then T_pM is an n-dimensional vector space, and for any smooth chart $(U, (x^i))$ containing p, the coordinate vectors $\partial / \partial x^1|_p, \ldots, \partial / \partial x^n|_p$ form a basis for T_pM.

2.2 Immersions and Submersions

Proposition 2.2.1. Suppose $F : M \to N$ is a smooth map and $p \in M$. If dF_p is surjective, then p has a neighborhood U such that $F|_U$ is a submersion. If dF_p is injective, then p has a neighborhood U such that $F|_U$ is an immersion.

Proof: If we choose any smooth coordinates for M near p and for N near $F(p)$, either hypothesis means that Jacobian matrix of F in coordinates has full rank at p. Since the set of $m \times n$ matrices of full rank is an open subset of $\mathrm{M}(m \times n, \mathbb{R})$(where $m = \dim M$ and $n = \dim N$), so by continuity, the Jacobian of F has full rank in some neighborhood of p.

Theorem 2.2.2 (Inverse Function Theorem for Manifolds)**.** Suppose M and N are smooth manifolds, and $F: M \to N$ is a smooth map. If $p \in M$ is a point such that dF_p is invertible, then there are connected neighborhoods U_0 of p and V_0 of $F(p)$ such that $\left.F\right|_{U_0}: U_0 \to V_0$ is a diffeomorphism.

Corollary 2.2.3. $F: M \to N$ is a local diffeomorphism if and only if it is both a smothh immersion and smooth submersion.

Corollary 2.2.4. $F: M \to N$ smooth map, $\dim M = \dim N$. F is diffeomorphism if and only if F is bijective and both smooth immersion and submersion.

Theorem 2.2.5 (Rank Theorem)**.** Suppose M and N are smooth manifolds of dimensions m and n, respectively, and $F: M \to N$ is a smooth map with constant rank r. For each $p \in M$ there exist smooth charts (U, φ) for M centered at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subseteq V$, in which F has a coordinate representation of the form

$$\hat{F}\left(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m\right) = \left(x^1, \ldots, x^r, 0, \ldots, 0\right).$$

In particular, if F is a smooth submersion, this becomes

$$\widehat{F}\left(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m\right) = \left(x^1, \ldots, x^n\right),$$

and if F is a smooth immersion, it is

$$\hat{F}\left(x^1, \ldots, x^m\right) = \left(x^1, \ldots, x^m, 0, \ldots, 0\right).$$

Theorem 2.2.6 (Global Rank Theorem)**.** Let M and N be smooth manifolds, and suppose $F: M \to N$ is a smooth map of constant rank.

(1) If F is surjective, then it is a smooth submersion.

(2) If F is injective, then it is a smooth immersion.

(3) If F is bijective, then it is a diffeomorphism.

Definition 2.2.7 (embedded submanifold)**.** Suppose M is a smooth manifold with or without boundary. An embedded submanifold of M is a subset $S \subseteq M$ that is a manifold (without boundary) in the subspace topology, endowed with a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is a smooth embedding.

If S is an embedded submanifold of M, the difference $\dim M - \dim S$ is called the codimension of S in M, and the containing manifold M is called the ambient manifold for S. An embedded hypersurface is an embedded submanifold of codimension 1 . The empty set is an embedded submanifold of any dimension.

Theorem 2.2.8 (smooth structure of boundary)**.** Theorem 5.11. If M is a smooth n-manifold with boundary, then with the subspace topology, ∂M is a topological $(n-1)$-dimensional manifold (without boundary), and has a unique smooth structure such that it is a properly embedded submanifold of M.

Proposition 2.2.9. Suppose M is a smooth manifold with or without boundary and $S \subseteq M$ is an embedded submanifold. Then S is properly embedded if and only if it is a closed subset of M.

Definition 2.2.10. If M is a smooth manifold with boundary and $p \in \partial M$, it is intuitively evident that the vectors in T_pM can be separated into three classes: those tangent to the boundary, those pointing inward, and those pointing outward. Formally, we make the following definition. If $p \in \partial M$, a vector $v \in T_pM - T_p\partial M$ is said to be inwardpointing if for some $\varepsilon > 0$ there exists a smooth curve $\gamma : [0, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$, and it is outward-pointing if there exists such a curve whose domain is $(-\varepsilon, 0]$. The following proposition gives another characterization of inward-pointing and outward-pointing vectors, which is usually much easier to check.

For the following results, manifolds mentioned are all without boundary.

Definition 2.2.11 (regular submanifold/local slice criterion)**.** A subset S of a manifold(or satisfying local slice criterion) N of dimension n is a regular submanifold of dimension k if for every $p \in S$ there is a coordinate neighborhood $(U, \phi) = (U, x^1, \ldots, x^n)$ of p in the maximal atlas of N such that $U \cap S$ is defined by the vanishing of $n - k$ of the coordinate functions. By renumbering the coordinates, we may assume that these $n - k$ coordinate functions are $x^{k+1}, \ldots, x^n$. We call such a chart (U, ϕ) in N an adapted chart relative to S. On $U \cap S, \phi = \left(x^1, \ldots, x^k, 0, \ldots, 0\right)$. Let

$$\phi_S : U \cap S \to \mathbb{R}^k$$

be the restriction of the first k components of ϕ to $U \cap S$, that is, $\phi_S = \left(x^1, \ldots, x^k\right)$.

Proposition 2.2.12. Let S be a regular submanifold of N of dimension k, then S with subspace topology is a topological manifold of dimension k. And there's a unique smooth structure on S such that S is a k-dimensional embedded submanifold of M.

Proof: Clearly, S is a k-dimensional topological manifold. It's easy to check that for the collection of all adapted charts $\{U, \phi\}$ relative to S , $\{U \cap S, \phi_S\}$ form an atlas of S.

Conversely, if $F : S \to M$ is a smooth embedding, there's (U, ϕ) and $(V, \psi) = (V, y^1, \ldots, y^m)$ such that $\psi \circ F \circ \phi^{-1}$ is given by

$$(x^1, \ldots, x^k) \mapsto (x^1, \ldots, x^k, 0, \ldots, 0) \in \mathbb{R}^m$$

Let $V_1 = \psi^{-1}(\psi(V) \cap \phi(U) \times \mathbb{R}^{n-k})$ and $U = S \cap W$, W open in M. We have $U = S \cap W = S \cap (V_1 \cap W) = \left\{v \in V_1 \cap W : y^{k+1}(v) = \cdots = y^m(v) = 0\right\}$. Hence local chart of S is given by adapted chart, which implies the smooth structure on S is unique.

Theorem 2.2.13 (Constant-rank level set theorem)**.** Let M and N be smooth manifolds, and let $\Phi : M \to N$ be a smooth map with constant rank r. Each level set of Φ is a properly embedded submanifold of codimension r in M.

Proof: Notice that $\Phi^{-1}(c)$ is regular submanifold and closed submanifold, hence it can be given a unique smooth structure such that $\Phi^{-1}(c)$ is a properly embedded submanifold of codimension r in M.

Proposition 2.2.14. Let M and N be smooth manifolds, and let $\Phi : M \to N$ be a smooth map with constant rank r. Let S be a level set of Φ at $c \in N$, then by Theorem 2.2.13, S be a properly embedded submanifold of M. We have an exact sequence

$$0 \to T_pS \to T_pM \xrightarrow{(\mathrm{d}\Phi)_p} T_cN$$

for all p in S.

2.3 Vector Field

Proposition 2.3.1. Let M be a smooth manifold with or without boundary, and let $X : M \to TM$ be a rough vector field. The following are equivalent:

(1) X is smooth.

(2) For every $f \in C^\infty(M)$, the function Xf is smooth on M.

Lemma 2.3.2 (Extension of vector field)**.** Let M be a smooth manifold with or without boundary, and let $A \subseteq M$ be a closed subset. Suppose X is a smooth vector field along A. Given any open subset U containing A, there exists a smooth global vector field $\tilde{X}$ on M such that $\tilde{X}\Big|_A = X$ and $\operatorname{supp}\tilde{X} \subseteq U$.

Definition 2.3.3. We can also consider mixed (k, m)-tensors on V, that is, multilinear functions defined on the product $V \times \cdots \times V \times V^* \times \cdots \times V^*$ of k copies of V and m copies of V^*. A (k, m)-tensor is then k times covariant and m times contravariant on V. The space of all (k, m)-tensors on V is denoted by $\mathcal{T}^{k,m}(V^*, V)$.

Definition 2.3.4. A (k, m)-tensor field is a map that to each point $p \in M$ assigns a tensor $T \in \mathcal{T}^{k,m}\left(T_p^*M, T_pM\right)$.

Definition 2.3.5. The space of (k, m)-tensor fields is clearly a vector space, since linear combinations of (k, m)-tensors are still (k, m)-tensors. If W is a coordinate neighborhood of M, we know that $\left\{(dx^i)_p\right\}$ is a basis for T_p^*M and that $\left\{\left(\frac{\partial}{\partial x^i}\right)_p\right\}$ is a basis for T_pM. Hence, the value of a (k, m)-tensor field T at a point $p \in W$ can be written as the tensor

$$T_p = \sum a_{i_1\cdots i_k}^{j_1\cdots j_m}(p)\left(dx^{i_1}\right)_p \otimes \cdots \otimes \left(dx^{i_k}\right)_p \otimes \left(\frac{\partial}{\partial x^{j_1}}\right)_p \otimes \cdots \otimes \left(\frac{\partial}{\partial x^{j_m}}\right)_p$$

where the $a_{i_1\cdots i_k}^{j_1\cdots j_m} : W \to \mathbb{R}$ are functions which at each $p \in W$ give us the components of T_p relative to these bases of T_p^*M and T_pM. Just as we did with vector fields, we say that a tensor field is smooth if all these functions are smooth for all coordinate systems of the maximal atlas. Again, we only need to consider the coordinate systems of an atlas, since all overlap maps are differentiable.

Proposition 2.3.6. M is a smooth manifold with or without boundary, there's a one-to-one bijection between vector field $\mathfrak{X}(M)$ on M and linear operaters $C^\infty(M) \to C^\infty(M)$ satisfying Leibniz rule.

Proposition 2.3.7. Given two differentiable vector fields $X, Y \in \mathfrak{X}(M)$ on a smooth manifold M(with or without boundary), there exists a unique differentiable vector field $Z \in \mathfrak{X}(M)$ such that

$$Z \cdot f = (X \circ Y - Y \circ X) \cdot f$$

for every differentiable function $f \in C^\infty(M)$. And we denote Z by $[X, Y]$.

Proof: It suffice to check $[X, Y]$ satisfies Leibniz rule and use Proposition 2.3.6

Theorem 2.3.8 (Coordinate Formula for the Lie Bracket)**.** Let X, Y be smooth vector fields on a smooth manifold M with or without boundary, and let

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$$

and

$$Y = \sum_{i=1}^{n} Y^i \frac{\partial}{\partial x^i}$$

be the coordinate expressions for X and Y in terms of some smooth local coordinates $(U, x^1, \ldots, x^n)$ for M. Then $[X, Y]$ has the following coordinate expression:

$$[X, Y] = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$

Proposition 2.3.9. Given $X, Y, Z \in \mathfrak{X}(M)$, we have:

(1) Bilinearity: for any $\alpha, \beta \in \mathbb{R}$,

$$[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$$
$$[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z]$$

(2) Antisymmetry:

$$[X, Y] = -[Y, X]$$

(3) Jacobi identity:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0;$$

Definition 2.3.10. A vector space V equipped with an antisymmetric bilinear map $[\cdot, \cdot]$: $V \times V \to V$ (called a Lie bracket) satisfying the Jacobi identity is called a Lie algebra. A linear map $F : V \to W$ between Lie algebras is called a Lie algebra homomorphism if $F([v_1, v_2]) = [F(v_1), F(v_2)]$ for all $v_1, v_2 \in V$. If F is bijective then it is called a Lie algebra isomorphism.

Lemma 2.3.11 (Extension Lemma for Vector Fields on Submanifolds)**.** M is a smooth manifold with or without boundary and $S \subseteq M$ is an embedding submanifold. Given $X \in \mathfrak{X}(S)$, then there is a smooth vector field Y on a neighborhood of S in M such that $X = Y|_S$.

2.4 Differential form

Definition 2.4.1. Let $(U, x^1, \ldots, x^n)$ be a chart on a manifold M. A-form $\omega = \sum a_I dx^I$ on U is smooth(continous) if and only if the coefficient functions a_I are all smooth(continous) on U.

Definition 2.4.2 (Characterization of a smooth k-form)**.** Let ω be a k-form on a manifold M. The following are equivalent:

(1) The k-form ω is C^∞ on M.

(2) The manifold M has an atlas such that on every chart $(U, \phi) = (U, x^1, \ldots, x^n)$ in the atlas, the coefficients a_I of $\omega = \sum a_I dx^I$ relative to the coordinate frame $\left\{dx^I\right\}_{I \in \mathcal{J}_{k,n}}$ are all C^∞.

(3) On every chart $(U, \phi) = (U, x^1, \ldots, x^n)$ on M, the coefficients a_I of $\omega = \sum a_I dx^I$ relative to the coordinate frame $\left\{dx^I\right\}_{I \in \mathcal{J}_{k,n}}$ are all C^∞.

(4) For any k smooth vector fields $X_1, \ldots, X_k$ on M, the function $\omega\left(X_1, \ldots, X_k\right)$ is C^∞ on M.

Remark 2.4.3. (1), (2) and (3) in above conditions are also equivalent if we replace "smooth" by "continuous".

Definition 2.4.4 (pullback of differential form)**.** Now suppose $F: N \to M$ is a C^∞ map of manifolds. At each point $p \in N$, the differential

$$F_{*,p} : T_p N \to T_{F(p)} M$$

is a linear map of tangent spaces, and so there is a pullback map

$$\left(F_{*,p}\right)^* : A_k\left(T_{F(p)} M\right) \to A_k\left(T_p N\right).$$

If ω is a k-form on M, then its pullback $F^*\omega$ is the k-form on N defined pointwise by $\left(F^*\omega\right)_p = \left(F_{*,p}\right)^*\left(\omega_{F(p)}\right)$ for all $p \in N$.

2.5 Orientation

Definition 2.5.1. Let V be a finite dimensional vector space and consider two ordered bases $\beta = \{b_1, \ldots, b_n\}$ and $\beta' = \{b_1', \ldots, b_n'\}$. There is a unique linear transformation $S: V \to V$ such that $b_i' = S b_i$ for every $i = 1, \ldots, n$. We say that the two bases are equivalent if $\det S > 0$. This defines an equivalence relation that divides the set of all ordered bases of V into two equivalence classes. An orientation for V is an assignment of a positive sign to the elements of one equivalence class and a negative sign to the elements of the other. The sign assigned to a basis is called its orientation and the basis is said to be positively oriented or negatively oriented according to its sign. It is clear that there are exactly two possible orientations for V.

We always take $[e_1, \ldots, e_n]$ as an representative to the positive orientation of $\mathbb{R}^n$.

Definition 2.5.2. An isomorphism $A : V \to W$ between two oriented vector spaces carries equivalent ordered bases of V to equivalent ordered bases of W. Under this case, isomorphism A is said to be orientation-preserving.

An orientation of a smooth manifold consists of a choice of orientations for all tangent spaces T_pM. If $\dim M = n \geq 1$, these orientations have to fit together smoothly, meaning that for each point $p \in M$ there exists a local chart $(U, \varphi) = (U, x^1, \ldots, x^n)$ around p such that

$$T_pM \to \mathbb{R}^n, \quad \sum a_i \frac{\partial}{\partial x^i} \mapsto (a_1, \ldots, a_n)$$

is order-preserving.

Proposition 2.5.3. If a smooth manifold M is connected and orientable then it admits precisely two orientations.

Proof:

Proposition 2.5.4. A smooth manifold M is orientable if and only if there exists an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ for which all the overlap maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ has positive Jacobi determinant for all $p \in \varphi_\alpha(U_\alpha \cap U_\beta)$.

Proof: For every $p \in M$, take $(U_\alpha, \varphi_\alpha) \in \mathcal{A}$ such that $p \in U_\alpha$. Fix an orientation on T_pM to be the equivalent class of the basis

$$(\left.\frac{\partial}{\partial x_1}\right|_p, \left.\frac{\partial}{\partial x_2}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p)$$

, then the map

$$p \mapsto [(\left.\frac{\partial}{\partial x_1}\right|_p, \left.\frac{\partial}{\partial x_2}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p)]$$

is well-defined and gives an orientation on M.

Proposition 2.5.5. *M is a connected manifold, M is orientable if and only if M has a nowhere-vanishing n-form.*

Proof: Suppose $(X_1, \ldots, X_n)$ is an orientable on M, for all $p \in M$, there exists a parameterization $(U, \varphi) = (U, x^1, \ldots, x^n)$ around p such that

$$T_pM \to \mathbb{R}^n, \quad \sum a_i \frac{\partial}{\partial x^i} \mapsto (a_1, \ldots, a_n)$$

is order-preserving. Hence $\mathrm{d}x^1 \wedge \ldots \mathrm{d}x^n(X_1, \ldots, X_n) > 0$ for all $p \in U$. For all p, consider such local chart, we get an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$. Consider $\{\rho_\alpha\}_{\alpha \in A}$ be partition of union.

Let

$$\omega_p = \sum_{\rho_\alpha(p) \neq 0} \rho_\alpha(p) \mathrm{d}x_\alpha^1 \wedge \ldots \mathrm{d}x_\alpha^n$$

Then $\omega(X_1, \ldots, X_n) \neq 0$, which implies ω is a nowhere-vanishing n-form.

Conversely, consider ω be a nowhere-vanishing n-form, take a connected local chart $(U, x^1, \ldots, x^n)$ and assume that $\omega(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}) > 0$, otherwise we can replace x^1 by $-x^1$.

Definition 2.5.6. Two oriented atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ and $\{V_\beta, \psi_\beta\}_\beta$ on a manifold M are said to be equivalent if the transition functions

$$\phi_\alpha \circ \psi_\beta^{-1} : \psi_\beta^{-1}(U_\alpha \cap V_\beta) \to \phi_\beta^{-1}(U_\alpha \cap V_\beta)$$

have positive Jacobian determinant for all α, β.

Proposition 2.5.7. M is an oriented connected manifold, then there's a ono-to-one correspondence between equivanlece classes of oriented atlases on M.

Proof: Two oriented atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ and $\{V_\beta, \psi_\beta\}_\beta$ on a manifold M are equivalent, they give the same orientations on M.

Two oriented atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ and $\{V_\beta, \psi_\beta\}_\beta$ on a manifold M are inequivalent, they give different orientations on M.

And also notice that if $\{(U_\alpha, x^1, \ldots, x^n)\}_{\alpha\in A}$ is an oriented atalas, then $\{(U_\alpha, -x^1, \ldots, x^n)\}_{\alpha\in A}$ is also an oriented atalas which is not equivalent to previous one.

Theorem 2.5.8. If ω is a nowhere-vanishing n-form(not need to be smooth) on a connected oriented manifold M, $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ is an oriented atlas. For every $p \in M$ and every coordinate neighborhood $(U, \phi) = (u, x^1, \ldots, x^n)$ of p, ω can be written as $w(q) = f(q)dx^1 \wedge \ldots dx^n, q \in U$. Define

$$g(p) = \begin{cases} 1 & \text{if} \quad f(p) > 0 \\ -1 & \text{if} \quad f(p) < 0 \end{cases} \tag{2.1}$$

g is well-defined since the atlas is oriented. We have g is constant function on M.

2.6 Integration on manifold

Let M be a oriented manifold M with an oriented atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$, ω be a compactly support continuous n-form on M. Suppose $\operatorname{supp}\omega \subset U_\alpha$ and $(U_\alpha, \phi_\alpha) = (u, x^1, \ldots, x^n)$.Then ω has a local expression on U_α written as $w(q) = f(q)dx^1 \wedge \ldots dx^n, q \in U_\alpha$, define

$$\int_M \omega = \int_{\phi_\alpha(U_\alpha)} f \circ \phi_\alpha^{-1} \tag{2.2}$$

Notice that this definition is independent of the choice of the chart in oriented atlas since if $(U_\beta, \phi_\beta) = (U_\alpha, y^1, \ldots, y^n)$ is another chart in oriented atlas such that $\operatorname{supp}\omega \subset U_\beta$

$$\begin{aligned}
\int_M \omega &= \int_{\phi_\alpha(U_\alpha)} f \circ \phi_\alpha^{-1} \\
&= \int_{\phi_\alpha(U_\alpha \cap U_\beta)} f \circ \phi_\alpha^{-1} \\
&= \int_{\phi_\beta(U_\alpha \cap U_\beta)} f \circ \phi_\beta^{-1} |\det D_x(\phi_\alpha \circ \phi_\beta^{-1})| && (D_x \text{ is Jacobian matrix}) \\
&= \int_{\phi_\beta(U_\alpha \cap U_\beta)} f \circ \phi_\beta^{-1} \det D_x(\phi_\alpha \circ \phi_\beta^{-1}) && (\text{proposition of oriented atlas})
\end{aligned}$$

Notice that the last identity agrees with the integration defined by $(U_\beta, \phi_\beta) = (U_\alpha, y^1, \ldots, y^n)$, hence the integration is well-defined.

In general, if ω is arbitrary compactly supported n-form on M, we need a lemma.

Lemma 2.6.1. Let $\{\rho_\alpha\}_{\alpha\in A}$ be a collection of functions on M and ω a continous k-form on M with compact support. If the collection $\{\rho_\alpha : M \to \mathbb{R}\}_{\alpha\in A}$ of supports is locally finite, then $\rho_\alpha\omega \equiv 0$ for all but finte α.

Proof: Let $p \in \operatorname{supp}\omega$. Since $\operatorname{supp}\rho_\alpha$ is locally fnite, there is a neighborhood W_p of p in M htat intersects only finitely many of the sets $\operatorname{supp}\rho_\alpha$. The collection $\{W_p | p \in M\}$ is an open covering of $\operatorname{supp}\omega$. Take a finite subcovering $W_1, \ldots, W_p$, then we have only finte many $\operatorname{supp}\rho_\alpha$ intersects $\bigcup_{i=1}^n W_i$. In partical, $\rho_\alpha\omega \equiv 0$ for all but finte α.

Let $\{\rho_\alpha\}_{\alpha\in A}$ to be the smooth partition of unity dominated to oriented atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$. Let U_i, $i = 1, \ldots, n$ be all the charts such that $\rho_\alpha\omega$ are nontrival. Define

$$\int_M \omega = \sum_{i=1}^n \int_M \rho_i w$$

where the integration of $\rho_i w$ on M is belong to the case we have already defined.

To check the above identity is well defined, let $\{V_\beta, \psi_\beta\}_\beta$ be another oriented atlas equivalent to $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$. $\{\chi_\beta\}_\beta$ be the smooth partition of unity of M dominated by $\{V_\beta, \psi_\beta\}_\beta$ and let V_i, $i = 1, \ldots m$ be all the charts such that $\chi_\alpha\omega$ are nontrival. Notice that

$$\sum_{i=1}^n \int_M \rho_i w = \sum_{i=1}^n \int_M \rho_i \sum_{j=1}^m \chi_j w = \sum_{i=1}^n \sum_{j=1}^m \int_M \rho_i \chi_j w$$

Since $\{V_\beta, \psi_\beta\}_\beta$ is equivalent to $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$, the integration of $\rho_i\chi_j\omega$ defined by U_i is identical to the integretion defined by V_j. Hence $\int_M \omega$ is well-defined.

Now we assume M is connected and oriented.

Since there's a smooth nowhere-vanishing n-form ω such that the coefficient of the local expression under the given oriented atlas is always positive. Take a function f in the space of compactly supported function $C_c(M)$, then $f\omega$ is a compactly supported continous n-form. Hence the linear functional

$$I : f \to \int_M f\omega$$

is a positive linear functional on $C_c(M)$. By Riesz Representation Theroem, there's a Radon measure μ on M such that

$$I(f) = \int f \mathrm{d}\mu$$

for all $f \in C_c(M)$.

2.7 Affine Connection

Definition 2.7.1. A tensor $g \in \mathcal{T}^2\left(T_p^* M\right)$ is said to be

(1) symmetric if $g(v, w) = g(w, v)$ for all $v, w \in T_pM$;

(2) nondegenerate if $g(v,w)=0$ for all $w\in T_pM$ implies $v=0$;

(3) positive definite if $g(v,v)>0$ for all $v\in T_pM\backslash\{0\}$.

A covariant 2-tensor field g is said to be symmetric, nondegenerate or positive definite if g_p is symmetric, nondegenerate or positive definite for all $p\in M$. If $x:V\to\mathbb{R}^n$ is a local chart, we have

$$g=\sum_{i=1}^{n}\sum_{j=1}^{n}g_{ij}dx^i\otimes dx^j$$

in V, where

$$g_{ij}=g\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right).$$

It is easy to see that g is symmetric, nondegenerate or positive definite if and only if the matrix (g_{ij}) has these properties.

Definition 2.7.2. A Riemannian metric on a smooth manifold M is a symmetric positive definite smooth covariant 2-tensor field g. A smooth manifold M equipped with a Riemannian metric g is called a Riemannian manifold, and is denoted by (M,g).

Definition 2.7.3. An oriented manifold is an orientable manifold together with a choice of an orientation. A map $F:M\to N$ between two oriented manifolds with the same dimension is said to be orientation-preserving if $F_{*,p}$ is orientation-preserving at all points $p\in M$.

The Riemannian metric is also denoted by form of inner product $p\mapsto\langle\,,\,\rangle_p$.

Proposition 2.7.4. Let (N,g) be a Riemannian manifold and $F:M\to N$ an immersion($F_{*,p}$ is injective for all $p\in M$). Then for $u,v\in T_pM$, define $(u,v)_p=(F_{*,p}(u),F_{*,p}(v))$ is a Riemannian metric in M (called the induced metric).

Example 2.7.5. The standard metric on

$$S^n=\left\{x\in\mathbb{R}^{n+1}\mid\|x\|=1\right\}$$

is the metric induced on S^n by the Euclidean metric on $\mathbb{R}^{n+1}$. A chart $(U,\varphi)=(U,x^1,\ldots,x^n)$

$$U=\left\{x\in S^n\mid x^{n+1}>0\right\}$$

and

$$\varphi^{-1}\left(x_1,\ldots,x_n\right)=\left(x_1,\ldots,x_n,\sqrt{1-\left(x_1\right)^2-\cdots-\left(x_n\right)^2}\right),\left(x_1,\ldots,x_n\right)\in B(0,1)\subset\mathbb{R}^n$$

and the corresponding coefficients of the metric tensor are

$$g_{ij}=\left\langle\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right\rangle=\delta_{ij}+\frac{x^ix^j}{1-\left(x^1\right)^2-\cdots-\left(x^n\right)^2}.$$

Definition 2.7.6. Let M be a differentiable manifold. An affine connection on M is a map $\nabla:\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ such that

(1) $\nabla_{fX+gY}Z=f\nabla_XZ+g\nabla_YZ$;

(2) $\nabla_X(Y+Z)=\nabla_XY+\nabla_XZ$

(3) $\nabla_X(fY)=(X\cdot f)Y+f\nabla_XY$ for all $X,Y,Z\in\mathfrak{X}(M)$ and $f,g\in C^\infty(M,\mathbb{R})$ (we write $\nabla_XY:=\nabla(X,Y)$).

The vector field ∇_XY is sometimes known as the covariant derivative of Y along X.

Proposition 2.7.7. Let ∇ be an affine connection on M, let $X, Y \in \mathfrak{X}(M)$ and $p \in M$. Then $(\nabla_X Y)_p \in T_pM$ depends only on X_p and values of Y on a neighborhood of p.

Proof: Use the existence of bump function on arbitrary manifold.

Definition 2.7.8. $(W, \varphi) = (W, x^1, \ldots, x^n)$ is a local chart on some open set $W \subset M$ and

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}, \quad Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial x^i}$$

on this set, we have

$$\nabla_X Y = \sum_{i=1}^n \left(X \cdot Y^i + \sum_{k=1}^n \sum_{j=1}^n \Gamma^i_{jk} X^j Y^k \right) \frac{\partial}{\partial x^i}$$

where the n^3 differentiable functions $\Gamma^i_{jk} : W \to \mathbb{R}$, called the Christoffel symbols, are defined by

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \sum_{i=1}^n \Gamma^i_{jk} \frac{\partial}{\partial x^i}.$$

Remark 2.7.9. Above concept is well-defined due to Lemma 2.1.11 and Lemma 2.3.2.

Lemma 2.7.10. Show that $(\nabla_X Y)_p$ actually depends only on X_p and the values of Y along any curve tangent to X_p. More precisely, suppose that $\gamma : (-\varepsilon, \varepsilon) \to M$ is a curve with $\gamma(0) = p$ and $\dot\gamma(0) = X_p$, and suppose Y and $\tilde{Y}$ are vector fields that agree along γ. Show that $(\nabla_{X_p} Y)_p = (\nabla_{X_p} \tilde{Y})_p$.

Proof: Notice that

$$X(Y^i)(p) = \sum_{j=1}^n X^j(p) \frac{\partial Y^i}{\partial x^j}\bigg|_p = (Y^i \circ \gamma)'(0)$$

Hence value of $\nabla_{X_p} Y$ at p depends only on a curve γ such that $\gamma(0) = p, \gamma'(0) = X_p$

Definition 2.7.11. A curve γ is a smooth function $c : I \to M$, an vector field along γ is smooth map $X : I \to TM$ such that $X(t) \in T_{c(t)}M$, $c'(t)$ is defined to be $c_*(\frac{\mathrm{d}}{\mathrm{d}t})$.

Definition 2.7.12 (covariant derivative)**.** Let ∇ be an affine connection on M. $C^\infty_\gamma(M)$ is all the vector field along γ. For each curve $\gamma : I \to M, \nabla$ determines a unique operator

$$\frac{D}{\mathrm{d}t} : C^\infty_\gamma(M) \to C^\infty_\gamma(M)$$

satisfying the following properties:

(1) Linearity over $\mathbb{R}$:

$$\frac{D}{\mathrm{d}t}(aV + bW) = a\frac{D}{\mathrm{d}t}V + b\frac{D}{\mathrm{d}t}W \quad \text{for } a, b \in \mathbb{R}$$

(2) Product rule:

$$\frac{D}{\mathrm{d}t}(fV) = \dot{f}V + f\frac{D}{\mathrm{d}t}V \quad \text{for } f \in C^\infty(I)$$

(3) For any $V \in C_\gamma^\infty(M)$, $t_0 \in I$, if there's a interval $t_0 \in (t_0-\varepsilon, t_0+\varepsilon)$, such that for all vector field X satisfying $X(\gamma(t)) = V(t)$ for all $t \in (t_0-\varepsilon, t_0+\varepsilon)$, we have

$$\frac{DV}{\mathrm{d}t}(t_0) = (\nabla_{\dot\gamma(t_0)}X)_{\gamma(t_0)}.$$

$\frac{DV}{\mathrm{d}t}$ is called the covariant derivative of V along γ.

Proof: Uniqueness: By using the trick of bump function, we can get that $\frac{DV}{\mathrm{d}t}(t_0)$ depends only on the value of V on a small neighborhood of t_0.

Take a local chart $(U, x^1, \ldots, x^n)$, consider the local expression of V on a neighborhood of t_0

$$V(t) = \sum_{k=1}^n \alpha_k(t) \frac{\partial}{\partial x^k}\bigg|_{\gamma(t)}, t \in (t_0-\epsilon, t_0+\epsilon)$$

By existence of bump function, we may assume $\alpha_k(t)$ can be extended to I, and $\frac{\partial}{\partial x^k}$ can be extended to a vector field X_k on M. Then $X_k(\gamma(t))$ be a vector field along γ which agrees with $\frac{\partial}{\partial x^k}|_{\gamma(t)}$ in $(t_0-\epsilon, t_0+\epsilon)$. Then,

$$\frac{DV}{\mathrm{d}t}(t_0) = \sum_{k=1}^m \alpha_k'(t_0) \frac{\partial}{\partial x^k}\bigg|_{\gamma(t_0)} + \alpha_k(t_0)\frac{D}{\mathrm{d}t}(X_k(\gamma(t)))(t_0)$$

Since $X_k(\gamma(t))$ satisfies condition (3), we have

$$\frac{DV}{\mathrm{d}t}(t_0) = \sum_{k=1}^m \alpha_k'(t_0) \frac{\partial}{\partial x^k}\bigg|_{\gamma(t_0)} + \alpha_k(t_0)(\nabla_{\dot\gamma(t_0)}X_k)_{\gamma(t_0)}.$$

Hence, $\frac{DV}{\mathrm{d}t}(t_0)$ only denpends on V, ∇ and γ.

Existence: From the proof of uniqueness, we can get there's a neighborhood $t_0 \in J = (t_0-\varepsilon, t_0+\varepsilon)$ of t_0 and a local chart $(U, x^1, \ldots, x^n)$ such that $\gamma(J) \subset U$ and for $k = 1, \ldots, m$, $\alpha_k(t)$ can be extended to smooth function on I and $\frac{\partial}{\partial x^k}$ can be extended to global vector field. If

$$\gamma'(t) = \sum_{k=1}^n \beta_k(t) \frac{\partial}{\partial x^k}\bigg|_{\gamma(t)}$$

We may assume $\beta_k(t)$ can also be extended to $C^\infty(I)$. Then we define

$$\begin{aligned}\frac{DV}{\mathrm{d}t}(t) &= \sum_{k=1}^m \alpha_k'(t) \frac{\partial}{\partial x^k}\bigg|_{\gamma(t)} + \alpha_k(t)(\nabla_{\dot\gamma(t)}\frac{\partial}{\partial x^k})_{\gamma(t)}, \quad t \in (t_0-\varepsilon, t_0+\varepsilon)\\ &= \sum_{k=1}^m \alpha_k'(t) \frac{\partial}{\partial x^k}\bigg|_{\gamma(t)} + \sum_{k=1}^m \alpha_k(t) \sum_{i=1}^m\sum_{j=1}^m \beta_i(t)\Gamma_{ik}^j(\gamma(t))\frac{\partial}{\partial x^j}\bigg|_{\gamma(t)}\end{aligned}$$

Then this is well-defined follows from the uniqueness. And it's esay to check $\frac{DV}{\mathrm{d}t}(t)$ satisfies (1),(2),(3).

Lemma 2.7.13 (Existence and Uniqueness for Linear ODEs). Let $(a, b) = I \subset \mathbb{R}$ be an interval, and for $1 \le i, j \le n$ let $a_{ij} : I \to \mathbb{R}$ be arbitrary smooth(or C^1) functions. And we denote $A(t) = (a_{ij}(t))$, let $x(t) = (x_1(t), \ldots, x_n(t))^T$then linear initial-value problem

$$\begin{aligned} x'(t) &= = A(t)x(t),\\ x(t_0) &= x_0 \in \mathbb{R}^n\end{aligned}$$

has a unique smooth(or C^1) solution on all of I for any $t_0 \in I$.

Proposition 2.7.14. Let M be a differentiable manifold with an affine connection ∇. A vector field V along a curve $c : I \to M$ is called parallel when $\frac{DV}{dt} = 0$, for all $t \in I$.

Let M be a differentiable manifold with an affine connection ∇. Let $c : I \to M$ be a differentiable curve in M and $V_{t_0} \in T_{c(t_0)}M$ Then there exists a unique parallel vector field V along c, such that $V(t_0) = V_{t_0}$, $V(t)$ is called the parallel transport of $V(t_o)$ along c.

Proof: We only need to prove the case when $c(I) \subset U$ where $(U, x^1, \dots, x^n)$ be a local chart. Assume

$$c'(t) = \sum_{k=1}^{n} b_k(t) \frac{\partial}{\partial x^k}\bigg|_{c(t)}$$

$$V(t) = \sum_{k=1}^{n} a_k(t) \frac{\partial}{\partial x^k}\bigg|_{c(t)}$$

Then $\mathrm{d}V/\mathrm{d}t = 0$ is equivalent to

$$0 = \sum_{i=1}^{n} \left(a_k'(t) + \sum_{j=1}^{n} \sum_{k=1}^{n} \Gamma_{jk}^{i}(c(t)) b_j(t) a_k(t) \right) \left(\frac{\partial}{\partial x^i} \right)_{c(t)}$$

By Lemma 2.7.13, we complete the proof.

Definition 2.7.15. A connection ∇ in a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ is said to be compatible with the metric if

$$X \cdot \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Proposition 2.7.16. Let M be a Riemannian manifold. A connection ∇ on M is compatible with a metric if and only if for any vector fields V and W along the differentiable curve $c : I \to M$ we have

$$\frac{d}{dt}\langle V, W\rangle = \left\langle \frac{DV}{dt}, W \right\rangle + \left\langle V, \frac{DW}{dt} \right\rangle, \quad t \in I.$$

Definition 2.7.17. The torsion operator of a connection ∇ on M is the operator $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

for all $X, Y \in \mathfrak{X}(M)$. The connection is said to be symmetric if $T(X, Y) = 0$ for all X, Y.

Obviouly, the term $[X, Y]$ is an correction term to make $T(X, Y)$ to be $C^\infty(M)$-bilinear, so that $T(X, Y)_p$ only depends on X_p, Y_p.

Proposition 2.7.18. $T(X, Y)$ depends only on X_p and Y_p by trick of bump function. Hence, T is the $(2,1)$-tensor field on M given in local coordinates by

$$T = \sum_{i,j,k=1}^{n} \left(\Gamma_{ij}^k - \Gamma_{ji}^k\right) dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k}$$

Proof: Considering a coordinate chart $x : W \subset M \to \mathbb{R}^n$, we have

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \sum_{k=1}^{n} \Gamma_{ji}^k \frac{\partial}{\partial x^k}.$$

Hence

$$T(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x^k} - \sum_{k=1}^{n} \Gamma_{ji}^k \frac{\partial}{\partial x^k}.$$

Corollary 2.7.19. A connection ∇ is symmetric if and only if for any local chart $(U, x^1, \dots, x^n)$, $\Gamma_{ij}^k(p) = \Gamma_{ji}^k(p)$ for all $p \in U$.

Definition 2.7.20 (Levi-Civita)**.** If $(M, \langle \cdot, \cdot \rangle)$ is a Riemannian manifold then there exists a unique connection ∇ on M which is symmetric and compatible with $\langle \cdot, \cdot \rangle$. In local coordinates $(x^1, \dots, x^n)$, the Christoffel symbols for this connection are

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

where $(g^{ij}) = (g_{ij})^{-1}$.

Proof: Let $X, Y, Z \in \mathfrak{X}(M)$. If the Levi-Civita connection exists then we must have

$$\begin{aligned} X \cdot \langle Y, Z \rangle &= \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \\ Y \cdot \langle X, Z \rangle &= \langle \nabla_Y X, Z \rangle + \langle X, \nabla_Y Z \rangle \\ -Z \cdot \langle X, Y \rangle &= - \langle \nabla_Z X, Y \rangle - \langle X, \nabla_Z Y \rangle, \end{aligned}$$

as ∇ is compatible with the metric. Moreover, since ∇ is symmetric, we must also have

$$\begin{aligned} -\langle [X, Z], Y \rangle &= - \langle \nabla_X Z, Y \rangle + \langle \nabla_Z X, Y \rangle; \\ -\langle [Y, Z], X \rangle &= - \langle \nabla_Y Z, X \rangle + \langle \nabla_Z Y, X \rangle; \\ \langle [X, Y], Z \rangle &= \langle \nabla_X Y, Z \rangle - \langle \nabla_Y X, Z \rangle. \end{aligned}$$

Adding these six equalities, we obtain the Koszul formula

$$\begin{aligned} 2 \langle \nabla_X Y, Z \rangle =& X \cdot \langle Y, Z \rangle + Y \cdot \langle X, Z \rangle - Z \cdot \langle X, Y \rangle \\ & - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle + \langle [X, Y], Z \rangle. \end{aligned}$$

Existence: define $\nabla_X Y$ locally by

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

Definition 2.7.21 (geodesic)**.** The curve c is called a geodesic of the connection ∇ if $\dot{c}$ is parallel along c, i.e. if

$$\frac{D\dot{c}}{dt}(t) = 0$$

for all $t \in I$.

Proposition 2.7.22. Now we are going to determine the local equations satisfied by a geodesic γ in a system of coordinates $(U, \varphi) = (U, x^1, \ldots, x^n)$ about $\gamma(t_0)$. In U, assume

$$\varphi(\gamma(t)) = (x_1(t), \ldots, x_n(t)) .$$

Then

$$\frac{d\gamma}{dt} = \sum_{i=1}^{n} x_i'(t) \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)}$$

Hence, γ will be a geodesic if and only if

$$0 = \frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = \sum_k \left(\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx_i}{dt}\frac{dx_j}{dt}\right)\frac{\partial}{\partial x^k}.$$

Hence the second order system

$$\frac{d^2 x_k}{dt^2} + \sum_{i=1}^{n}\sum_{j=1}^{n} \Gamma_{ij}^k(\gamma(t))\frac{dx_i}{dt}\frac{dx_j}{dt} = 0, \quad k = 1, \ldots, n$$

yields the desired equation.

Any differentiable curve $t \to \gamma(t)$ in M determines a curve $t \to \left(\gamma(t), \frac{d\gamma}{dt}(t)\right)$ in TM. If γ is a geodesic then, on TU, the curve

$$t \to \left(x_1(t), \ldots, x_n(t), \frac{dx_1(t)}{dt}, \ldots, \frac{dx_n(t)}{dt}\right)$$

satisfies the system

$$\frac{dx_k}{dt} = y_k, k = 1, \ldots, n$$

$$\frac{dy_k}{dt} = -\sum_{i=1}^{n}\sum_{j=1}^{n} \Gamma_{ij}^k(\gamma(t)) y_i y_j \quad k = 1, \ldots, n$$

Lemma 2.7.23. If $c(t)$ is a curve such that $c'(t_0) \neq 0$, then by Constant Rank Theorem, we can always find a small interval J contain t_0 and a chart (U, φ) contain $c(t_0)$ s.t the curve $c : I \to M$ has the following representation

$$\varphi(c(t)) = (t, 0, \ldots, 0) \quad \forall t \in J$$

Hence the vector field along $c' : I \to TM$ has representation on $J \subset I$

$$c'(t) = \frac{\partial}{\partial x^1}\bigg|_{c(t)}$$

where $c(J) \subset U$, and therefore induced by a global vector field on M on some small neighborhood of t_0.

Lemma 2.7.24. Let V be an open subset of $\mathbb{R}^n, p_0$ a point in V, and $f : V \to \mathbb{R}^n$ a C^∞ function. There's an interval (a, b) such that the differential equation

$$dy/dt = f(y), \quad y(0) = p_0,$$

has a unique C^∞ solution $y : (a, b) \to V$.

Proposition 2.7.25. For any two geodesics, $\gamma_1 : I_1 \to M$ and $\gamma_2 : I_2 \to M$, if $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1'(a) = \gamma_2'(a)$, for some $a \in I_1 \cap I_2$, then $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$.

Proof: Suppose $\gamma, \sigma : I \to M$ are geodesics defined on an open interval with $\gamma(t_0) = \sigma(t_0)$ and $\dot{\gamma}(t_0) = \dot{\sigma}(t_0)$. By the uniqueness part of the ODE theorem(Lemma 2.7.24), they agree on some neighborhood of t_0. Let β be the supremum of numbers b such that they agree on $[t_0, b]$. If $\beta \in I$, then by continuity $\gamma(\beta) = \sigma(\beta)$ and $\dot{\gamma}(\beta) = \dot{\sigma}(\beta)$, and applying local uniqueness in a neighborhood of β, we conclude that they agree on a slightly larger interval, which is a contradiction. Arguing similarly to the left of t_0, we conclude that they agree on all of I.

Definition 2.7.26. For every $p \in M$ and every $v \in T_pM$, consider all the geodesic $\gamma_i : I_i \to M, i \in W$ such that $\gamma(0) = p, \gamma'(0) = v$. Denote the union of all these interval to be I. Then $\gamma : I \to M(\ \gamma(t_0) = \gamma_i(t_0)$ if $t_0 \in I_i)$ is still a geodesic such that $\gamma(0) = p, \gamma'(0) = v$. We call it maximal geodesic.

Proposition 2.7.27 (homogeneity of geodesics)**.** For any $v \in T_pM$ and $c, t \in \mathbb{R}$,

$$\gamma_{cv}(t) = \gamma_v(ct),$$

whenever either side is defined.

2.8 Exponential Map

Definition 2.8.1. Let (M, g) be a Riemannian manifold. For every $p \in M$, let $\mathcal{D}(p)$ (or simply, $\mathcal{D}$) be the subset of T_pM given by

$$\mathcal{D}(p) = \{v \in T_pM \mid \gamma_v(1) \quad \text{is defined }\}$$

where γ_v is the unique maximal geodesic with initial conditions $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. The exponential map is the map, $\exp_p : \mathcal{D}(p) \to M$, given by

$$\exp_p(v) = \gamma_v(1)$$

Proposition 2.8.2 (Properties of the Exponential Map)**.** Let (M, g) be a Riemannian manifold

(1) $D(p)$ is open and star-shape with respect to 0, and

$$\bigcup_{p \in M} D(p)$$

open in TM.

(2) the curve

$$t \mapsto \exp_p(tv), \quad tv \in \mathcal{D}(p)$$

gives the maximal geodesic γ_v through p such that $\gamma_v'(0) = v$.

(3) The exponential map

$$(p, v) \in \bigcup_{p \in M} D(p) \subset TM \mapsto \exp_p(v)$$

is smooth.

Proof: (2): Notice that $tv \in D(p)$ iff $\gamma_v(t)$ is defined. Hence $\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$

Proposition 2.8.3. If M is a compact Riemannian manifold, then for each $p \in M$ and $v \in T_pM$, there is a geodesic $c : \mathbb{R} \to M$ with $c(0) = p, \dot{c}(0) = v$. In other words, compact Riemannian manifold is geodesically complete.

Definition 2.8.4. A Riemannian manifold, (M, g), is geodesically complete iff $\mathcal{D}(p) = T_pM$, for all $p \in M$, that is, iff the exponential, $\exp_p(v)$, is defined for all $p \in M$ and for all $v \in T_pM$.

Proposition 2.8.5 (normal neighborhood)**.** (M, g) is a Riemannian manifold, consider T_pM as a normed vector space. Give T_pM a differential structrue as follow: for any local chart $(U, x^1, \ldots, x^n)$ around p , the map

$$\varphi_p : T_pM \to \mathbb{R}^n \quad \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i} \mapsto (a^1, \ldots, a^n)$$

is an homomorphism, and it's easy to check (T_pM, φ_p) are compatible with each other.

(1) If $p \in M$, then

$$(d \exp_p)_0 : T_0 (T_pM) \to T_pM$$

is nonsingular at the origin of T_pM. Consequently, $\exp_p$ is a local diffeomorphism.

(2) Any open neighborhood $\mathcal{U}$ of $p \in M$ that is the diffeomorphic image under $\exp_p$ of a star-shaped open neighborhood of $0 \in T_pM$ as in the preceding lemma is called a normal neighborhood of p. If $\varepsilon > 0$ is such that $\exp_p$ is a diffeomorphism on the ball $B_\varepsilon(0) \subset T_pM$ (where the radius of the ball is measured with respect to the norm defined by g), then the image set $\exp_p(B_\varepsilon(0))$ is called a geodesic ball in M. Also, if the closed ball $\bar{B}_\varepsilon(0)$ is contained in an open set $\mathcal{V} \subset T_pM$ on which $\exp_p$ is a diffeomorphism, then $\exp_p\left(\bar{B}_\varepsilon(0)\right)$ is called a closed geodesic ball, and $\exp_p\left(\partial \bar{B}_\varepsilon(0)\right)$ is called a geodesic sphere.

Proof: (1): The differentials are nonsingular follows from the homogeneity property of geodesics given an important identification of tangent spaces. Let $I_0 : T_pM \to T_0T_pM$ be the canonical isomorphism, i.e., $I_0(v) = \frac{d}{dt}(tv)\big|_{t=0}$. Recall that if $v \in D(p)$, then $c_v(t) = c_{tv}(1)$ for all $t \in [0, 1]$. Thus,

$$\begin{aligned}
(d \exp_p)_0 (I_0(v)) &= \frac{d}{dt} \exp_p(tv)\bigg|_{t=0} \\
&= \frac{d}{dt} c_{tv}(1)\bigg|_{t=0} \\
&= \frac{d}{dt} c_v(t)\bigg|_{t=0} \\
&= \dot{c}_v(0) \\
&= v
\end{aligned}$$

Definition 2.8.6 (normal coordinates)**.** An orthonormal basis $\{E_i\}$ for T_pM gives an isomorphism $E : \mathbb{R}^n \to T_pM$ by $E(x^1, \ldots, x^n) = x^i E_i$. If $\mathcal{U}$ is a normal neighborhood of p, we can combine this isomorphism with the exponential map to get a coordinate chart

$$\varphi := E^{-1} \circ \exp_p^{-1} : \mathcal{U} \to \mathbb{R}^n.$$

Any such coordinates are called (Riemannian) normal coordinates centered at p. Given $p \in M$ and a normal neighborhood $\mathcal{U}$ of p, there is a one-to-one correspondence between normal coordinate charts and orthonormal bases at p.

In any normal coordinate chart centered at p, define the radial distance function r by

$$r(x) := \left(\sum_i \left(x^i\right)^2 \right)^{1/2},$$

and the unit radial vector field $\partial/\partial r$ by

$$\frac{\partial}{\partial r} := \sum_{i=1}^{n} \frac{x^i}{r} \frac{\partial}{\partial x^i}\bigg|_q , q \neq p$$

Proposition 2.8.7 (Properties of Normal Coordinates)**.** Let M be a Riemannian manifold and ∇ the Levi–Civita connection on M. Let $(\mathcal{U}, \varphi) = (\mathcal{U}, x^1, \ldots, x^n)$ be any normal coordinate chart centered at p.

(1) For any $V = V^i \frac{\partial}{\partial x^i} \in T_pM$, the geodesic γ_V starting at p with initial velocity vector V is represented in normal coordinates by the radial line segment

$$\varphi(\gamma_V(t)) = \left(tV^1, \ldots, tV^n\right)$$

as long as γ_V stays within $\mathcal{U}$.

(2) The coordinates of p are $(0, \ldots, 0)$.

(3) The components of the metric at p are $g_{ij} = \delta_{ij}$.

(4) Any Euclidean ball $\{x : r(x) < \varepsilon\}$ contained in $\mathcal{U}$ is a geodesic ball in M.

(5) At any point $q \in \mathcal{U} - p, \partial/\partial r$ is the velocity vector of the unit speed geodesic from p to q, and therefore has unit length with respect to g.

(6) The first partial derivatives of g_{ij} and the Christoffel symbols vanish at p.

Proof: (3) Notice that

$$\frac{\partial f}{\partial x^i} = \frac{\partial (f \circ \exp_p)(x_i E_i)}{\partial x_i} = \frac{\partial (f \circ \gamma_{E_i}(x_i))}{\partial x_i} = E_i$$

(6) Notice that

$$\varphi(\gamma_V(t)) = \left(tV^1, \ldots, tV^n\right)$$

Consider the geodesic equation, we have for $k = 1, \ldots, n$

$$0 = \frac{d^2 x_k}{dt^2} + \sum_{i=1}^{n} \sum_{j=1}^{n} \Gamma_{ij}^{k}(p) \frac{dx_i}{dt} \frac{dx_j}{dt} = \sum_{i=1}^{n} \sum_{j=1}^{n} \Gamma_{ij}^{k}(p) V_i V_j$$

Since $(\exp_p)^{-1}(U)$ is open and star-shaped at 0, it's easy to see that $\Gamma_{ij}^{k}(p) = 0$.

Since ∇ is compatible with Riemannian metric g, $\frac{\partial g_{ij}}{\partial x_i}(p) = 0$

Theorem 2.8.8 (Gauss's lemma)**.** Let $p \in M$ and $v \in D(p) \subset T_pM$, and consider the canonical isomorphism:

$$T_pM \to T_v(T_pM)$$
$$\tau \mapsto I'_\tau(0)$$

where $I(t) = v + \tau t, t \in (-\varepsilon, \varepsilon)$.

Then for $w \in T_pM$

$$\langle (d\exp_p)_v(v), (d\exp_p)_v(w) \rangle_{\exp_p(v)} = \langle v, w \rangle_p$$

2.9 Volume Form

2.10 Curvature

Definition 2.10.1. Curvature R of a connection ∇ is a correspondence that to each pair of vector fields $X, Y \in \mathfrak{X}(M)$ associates the map $R(X, Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Proposition 2.10.2. (1) $R(fX_1 + gX_2, Y) Z = fR(X_1, Y) Z + gR(X_2, Y) Z$,

(2) $R(X, fY_1 + gY_2) Z = fR(X, Y_1) Z + gR(X, Y_2) Z$,

(3) $R(X, Y)(fZ_1 + gZ_2) = fR(X, Y)Z_1 + gR(X, Y)Z_2$, for all vector fields $X, X_1, X_2, Y, Y_1, Y_2, Z, Z_1, Z_2, \in \mathfrak{X}(M)$ and all smooth functions $f, g \in C^\infty(M)$

Corollary 2.10.3. By using bump function, it can be shown that $R(X, Y)(Z)$ only depends on X_p, Y_p, Z_p.

Corollary 2.10.4. $R(X, Y)$ defines a $(3, 1)$-tensor(called the Riemann tensor), and its local expression on $(U, x^1, \ldots, x^n)$ is

$$R = \sum_{i,j,k,l=1}^{n} R_{ijk}^{l} dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l}$$

where

$$R_{ijk}^{l} = \frac{\partial \Gamma_{jk}^{l}}{\partial x^i} - \frac{\partial \Gamma_{ik}^{l}}{\partial x^j} + \sum_{m=1}^{n} \Gamma_{jk}^{m} \Gamma_{im}^{l} - \sum_{m=1}^{n} \Gamma_{ik}^{m} \Gamma_{jm}^{l}.$$

Proof: Since

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = \sum_{l=1}^{n} R^l_{ijk}\frac{\partial}{\partial x^l}.$$

Notice that $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$, we have

$$\begin{aligned}
R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} &= \nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k} \\
&= \nabla_{\frac{\partial}{\partial x^i}}\left(\sum_{m=1}^{n}\Gamma^m_{jk}\frac{\partial}{\partial x^m}\right) - \nabla_{\frac{\partial}{\partial x^j}}\left(\sum_{m=1}^{n}\Gamma^m_{ik}\frac{\partial}{\partial x^m}\right) \\
&= \sum_{m=1}^{n}\left(\frac{\partial}{\partial x^i}\cdot\Gamma^m_{jk} - \frac{\partial}{\partial x^j}\cdot\Gamma^m_{ik}\right)\frac{\partial}{\partial x^m} + \sum_{l,m=1}^{n}\left(\Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}\right)\frac{\partial}{\partial x^l}
\end{aligned}$$

$$= \sum_{l=1}^{n}\left(\frac{\partial\Gamma^l_{jk}}{\partial x^i} - \frac{\partial\Gamma^l_{ik}}{\partial x^j} + \sum_{m=1}^{n}\Gamma^m_{jk}\Gamma^l_{im} - \sum_{m=1}^{n}\Gamma^m_{ik}\Gamma^l_{jm}\right)\frac{\partial}{\partial x^l},$$

and so

$$R^l_{ijk} = \frac{\partial\Gamma^l_{jk}}{\partial x^i} - \frac{\partial\Gamma^l_{ik}}{\partial x^j} + \sum_{m=1}^{n}\Gamma^m_{jk}\Gamma^l_{im} - \sum_{m=1}^{n}\Gamma^m_{ik}\Gamma^l_{jm}.$$

Theorem 2.10.5 (Bianchi identity)**.** If M is a manifold with a symmetric connection then the associated curvature satisfies

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

Proof: This property is a direct consequence of the Jacobi identity of vector fields. Indeed,

$$\begin{aligned}
&R(X,Y)Z + R(Y,Z)X + R(Z,X)Y \\
&= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X \\
&\quad + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y \\
&= \nabla_X\left(\nabla_Y Z - \nabla_Z Y\right) + \nabla_Y\left(\nabla_Z X - \nabla_X Z\right) + \nabla_Z\left(\nabla_X Y - \nabla_Y X\right) \\
&\quad - \nabla_{[X,Y]}Z - \nabla_{[Y,Z]}X - \nabla_{[Z,X]}Y
\end{aligned}$$

and so, since the connection is symmetric, we have

$$\begin{aligned}
&R(X,Y)Z + R(Y,Z)X + R(Z,X)Y \\
&\quad = \nabla_X[Y,Z] + \nabla_Y[Z,X] + \nabla_Z[X,Y] - \nabla_{[Y,Z]}X - \nabla_{[Z,X]}Y - \nabla_{[X,Y]}Z \\
&\quad = [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.
\end{aligned}$$

Definition 2.10.6. We will assume from this point on that (M,g) is a Riemannian manifold and ∇ its Levi-Civita connection.

$$R(X,Y,Z,W) := g(R(X,Y)Z,W).$$

Again, we can show that $R(X,Y,Z,W)$ only depends on X_p, Y_p, Z_p, W_p. We can define a new covariant 4-tensor, it's local expression on $(U, x^1, \ldots, x^n)$ can be written as

$$R = \sum_{i,j,k,l=1}^{n} R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$

where

$$R_{ijkl} = g\left(R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) = g\left(\sum_{m=1}^{n} R_{ijk}{}^{m} \frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^l}\right) = \sum_{m=1}^{n} R_{ijk}{}^{m} g_{ml}.$$

Proposition 2.10.7. If X, Y, Z, W are vector fields in M and ∇ is the Levi-Civita connection, then

(1) $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$;

(2) $R(X, Y, Z, W) = -R(Y, X, Z, W)$;

(3) $R(X, Y, Z, W) = -R(X, Y, W, Z)$;

(4) $R(X, Y, Z, W) = R(Z, W, X, Y)$.

Proof: (3): Now, using the fact that the Levi-Civita connection is compatible with the metric, we have

$$X \cdot \langle \nabla_Y Z, Z \rangle = \langle \nabla_X \nabla_Y Z, Z \rangle + \langle \nabla_Y Z, \nabla_X Z \rangle$$

and

$$[X, Y] \cdot \langle Z, Z \rangle = 2 \left\langle \nabla_{[X,Y]} Z, Z \right\rangle .$$

Hence,

$$\begin{aligned} R(X, Y, Z, Z) =& \langle \nabla_X \nabla_Y Z, Z \rangle - \langle \nabla_Y \nabla_X Z, Z \rangle - \left\langle \nabla_{[X,Y]} Z, Z \right\rangle \\ =& X \cdot \langle \nabla_Y Z, Z \rangle - \langle \nabla_Y Z, \nabla_X Z \rangle - Y \cdot \langle \nabla_X Z, Z \rangle \\ &+ \langle \nabla_X Z, \nabla_Y Z \rangle - \frac{1}{2}[X, Y] \cdot \langle Z, Z \rangle \\ =& \frac{1}{2} X \cdot (Y \cdot \langle Z, Z \rangle) - \frac{1}{2} Y \cdot (X \cdot \langle Z, Z \rangle) - \frac{1}{2}[X, Y] \cdot \langle Z, Z \rangle \\ =& \frac{1}{2}[X, Y] \cdot \langle Z, Z \rangle - \frac{1}{2}[X, Y] \cdot \langle Z, Z \rangle = 0. \end{aligned}$$

Definition 2.10.8. Let Π be a 2-dimensional subspace of T_pM and let X_p, Y_p be two linearly independent elements of Π. Then, the sectional curvature of Π is defined as

$$K(\Pi) := -\frac{R\left(X_p, Y_p, X_p, Y_p\right)}{\|X_p\|^2 \|Y_p\|^2 - \langle X_p, Y_p \rangle^2}$$

Definition 2.10.9. A Riemannian manifold is called isotropic at a point $p \in M$ if its sectional curvature is a constant K_p for every section $\Pi \subset T_pM$. Moreover, it is called isotropic if it is isotropic at all points. Note that every 2-dimensional manifold is trivially isotropic. Its sectional curvature $K(p) := K_p$ is called the Gauss curvature.

Proposition 2.10.10. The Riemannian curvature tensor at p is uniquely determined by the values of the sectional curvatures of sections (that is, 2-dimensional subspaces) of T_pM.

Proof: Let us consider two covariant 4-tensors R_1, R_2 on T_pM satisfying the symmetry properties, then the tensor $T := R_1 - R_2$ also satisfies these symmetry properties. We will see that, if the values $R_1(X_p, Y_p, X_p, Y_p)$ and $R_2(X_p, Y_p, X_p, Y_p)$ agree for every $X_p, Y_p \in T_pM$ (that is, if $T(X_p, Y_p, X_p, Y_p) = 0$ for every $X_p, Y_p \in T_pM$), then $R_1 = R_2$ (that is, $T \equiv 0$). Indeed, for all vectors $X_p, Y_p, Z_p \in T_pM$, we have

$$\begin{aligned} 0 &= T(X_p + Z_p, Y_p, X_p + Z_p, Y_p) = T(X_p, Y_p, Z_p, Y_p) + T(Z_p, Y_p, X_p, Y_p) \\ &= 2T(X_p, Y_p, Z_p, Y_p) \end{aligned}$$

and so

$$\begin{aligned} 0 &= T(X_p, Y_p + W_p, Z_p, Y_p + W_p) = T(X_p, Y_p, Z_p, W_p) + T(X_p, W_p, Z_p, Y_p) \\ &= T(Z_p, W_p, X_p, Y_p) - T(W_p, X_p, Z_p, Y_p) \end{aligned}$$

that is, $T(Z_p, W_p, X_p, Y_p) = T(W_p, X_p, Z_p, Y_p)$. Hence T is invariant by cyclic permutations of the first three elements and so, by the Bianchi identity, we have $3T(X_p, Y_p, Z_p, W_p) = 0$

Proposition 2.10.11. M is isotropic at p with sectional curvature K_p if and only if there's $(U, x^1, \ldots, x^n)$ a coordinate system around p, such that the coefficients of the Riemannian curvature tensor at p are given by

$$R_{ijkl}(p) = -K_p(g_{ik}g_{jl} - g_{il}g_{jk})$$

Proof: We first define a covariant 4-tensor A on T_pM as

$$A := \sum_{i,j,k,l=1}^{n} -K_p(g_{ik}g_{jl} - g_{il}g_{jk})\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$

It's easy to check that A satisfies the symmetry properties of Proposition 2.10.7. Moreover,

$$\begin{aligned} A(X_p, Y_p, X_p, Y_p) &= \sum_{i,j,k,l=1}^{n} -K_p(g_{ik}g_{jl} - g_{il}g_{jk})\, X_p^i Y_p^j X_p^k Y_p^l \\ &= -K_p\left(\langle X_p, X_p\rangle \langle Y_p, Y_p\rangle - \langle X_p, Y_p\rangle^2\right) \\ &= R(X_p, Y_p, X_p, Y_p) \end{aligned}$$

Proposition 2.10.12. M is isotropic at p with sectional curvature K_p if there's an orthonormal basis(not necessarily standard) $\{X_1, \ldots, X_n\}$ of T_pM such that

$$R(X_i, Y_j, X_k, Y_l) = \begin{cases} -K_p \|X_i\|^2 \|X_j\|^2 & i = k, j = l \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.10.13. A Riemannian manifold is called a manifold of constant curvature if it is isotropic and K_p is the same at all points of M.

Definition 2.10.14. $(M,\langle\cdot,\cdot\rangle)$ be a Riemannian manifold, N is a regular submanifold of M, then $f:N\to M$ be a immersion which induces a Riemannian metric on N. Hence we denote N with its Riemannian metric by $(N,\langle\langle\cdot,\cdot\rangle\rangle)$. Consider T_pN as a subvector space of T_pM

$$T_pM = T_pN \oplus (T_pN)^{\perp}.$$

Therefore, every element v of T_pM can be written uniquely as $v=v^{\top}+v^{\perp}$, where $v^{\top}\in T_pN$ is the tangential part of v and $v^{\perp}\in (T_pN)^{\perp}$ is the normal part of v. Let $\widetilde{\nabla}$ and ∇ be the Levi-Civita connections of $(M,\langle\cdot,\cdot\rangle)$ and $(N,\langle\langle\cdot,\cdot\rangle\rangle)$, respectively.

Let X,Y be two vector fields on N and take $\tilde{X},\tilde{Y}$ to be the vector field on a neighborhood of N such that $\tilde{X}|_N = X, \tilde{Y}|_N = Y$ by Lemma 2.3.11. By Lemma 2.7.10, it's easy to show that $\left(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}\right)_p$ depends only on X_p and Y for all $p\in N$. Hence $\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}$ is a well-defined vector field on N, which is independent of the choice of extension. We define the second fundamental form of N as

$$B(X,Y) := \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \nabla_X Y.$$

Proposition 2.10.15. Let $(M,\langle\cdot,\cdot\rangle)$ be a Riemannian manifold with Levi-Civita connection $\widetilde{\nabla}$, and let $(N,\langle\langle\cdot,\cdot\rangle\rangle)$ be a submanifold with the induced metric and Levi-Civita connection ∇. Show that

$$\nabla_X Y = \left(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}\right)^{\top}$$

for all $X,Y\in\mathfrak{X}(N)$, where $\tilde{X},\tilde{Y}$ are any extensions of X,Y to $\mathfrak{X}(M)$ by Lemma 2.3.11. Hence $B(X,Y)_p = (\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y})_p^{\perp}\in (T_pN)^{\perp}$.

Proposition 2.10.16. $B(X,Y)$ is symmetric and bilinear and $B(X,Y)_p$ only depends on values of X_p and Y_p.

Definition 2.10.17. Using the second fundamental form, we can define, for each vector $n_p\in (T_pN)^{\perp}$, a symmetric bilinear map $H_{n_p}:T_pN\times T_pN\to\mathbb{R}$ through

$$H_{n_p}(X_p,Y_p) = \langle B(X_p,Y_p), n_p\rangle.$$

The corresponding quadratic form is often called the second fundamental form of f at p along the vector n_p.

Definition 2.10.18 (The Weingarten Equation)**.** Since H_{n_p} is bilinear, there exists a unique linear map $S_{n_p}:T_pN\to T_pN$ satisfying

$$\left\langle\left\langle S_{n_p}(X_p), Y_p\right\rangle\right\rangle = H_{n_p}(X_p,Y_p) = \langle B(X_p,Y_p), n_p\rangle$$

for all $X_p,Y_p\in T_pN$.

If n is a vector field on a neighborhood of N such that $n_p\in (T_pN)^{\perp}$ for all $p\in N$, then we have

$$\begin{aligned}\langle\langle S_n(X),Y\rangle\rangle = \langle B(X,Y),n\rangle &= \left\langle \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \nabla_X Y, n\right\rangle\\ &= \left\langle \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, n\right\rangle = \widetilde{X}\cdot\langle\widetilde{Y},n\rangle - \left\langle \widetilde{Y}, \widetilde{\nabla}_{\widetilde{X}}n\right\rangle\\ &= \left\langle -\widetilde{\nabla}_{\widetilde{X}}n, \widetilde{Y}\right\rangle = \left\langle\left\langle -\left(\widetilde{\nabla}_{\widetilde{X}}n\right)^{\top}, Y\right\rangle\right\rangle.\end{aligned}$$

Hence

$$\langle\langle S_{n_p}(X_p), Y_p\rangle\rangle = \left\langle\left\langle -\left(\widetilde{\nabla}_{\widetilde{X}} n\right)_p^{\top}, Y_p\right\rangle\right\rangle$$

for all $Y_p \in T_pN$, therefore

$$S_{n_p}(X_p) = -\left(\widetilde{\nabla}_{\widetilde{X}} n\right)_p^{\top}.$$

Theorem 2.10.19 (Remarkable Theorem)**.** Let p be a point in N, let X_p and Y_p be two linearly independent vectors in $T_pN \subset T_pM$ and let $\Pi \subset T_pN \subset T_pM$ be the 2-dimensional subspace generated by these vectors. Let $K^N(\Pi)$ and $K^M(\Pi)$ denote the corresponding sectional curvatures in N and M, respectively. Then

$$K^N(\Pi) - K^M(\Pi) = \frac{\langle B(X_p, X_p), B(Y_p, Y_p)\rangle - \|B(X_p, Y_p)\|^2}{\|X_p\|^2\|Y_p\|^2 - \langle X_p, Y_p\rangle^2}$$

Definition 2.10.20 (Poincare ball model)**.** $\mathbb{B}^n_R$ is the ball of radius R in $\mathbb{R}^n$, with the metric given in coordinates $(u^1, \ldots, u^n)$ by

$$4R^4\frac{(du^1)^2 + \cdots + (du^n)^2}{(R^2 - |u|^2)^2}.$$

Definition 2.10.21 (Poincare half-space model)**.** $\mathbb{H}^n_R$ is the upper half-space in $\mathbb{R}^n$ defined in coordinates $(x^1, \ldots, x^{n-1}, y)$ by $\{y > 0\}$, with the metric

$$R^2\frac{(dx^1)^2 + \cdots + (dx^{n-1})^2 + dy^2}{y^2}.$$

Definition 2.10.22 (hyperboloid model)**.** $\mathbb{H}^n_R$ is the "upper sheet" $\{\tau > 0\}$ of the twosheeted hyperboloid in $\mathbb{R}^{n+1}$ defined in coordinates $(\xi^1, \ldots, \xi^n, \tau)$ by the equation $\tau^2 - |\xi|^2 = R^2$, with the metric induced by the Minkowski metric $m = (d\xi^1)^2 + \cdots + (d\xi^n)^2 - (d\tau)^2$ on $\mathbb{R}^{n+1}$.

Proposition 2.10.23. For any fixed $R > 0$, the three models defined above are all mutually isometric.

Proposition 2.10.24 (curvature of poincare half-plane)**.** Consider on $\mathbb{H}^n$ the metric

$$g_{ij} = \frac{\delta_{ij}}{F^2}$$

where F is a positive differentiable function on $\mathbb{H}^n$. Write $g^{ij} = F^2\delta_{ij}$ to denote the inverse matrix of g_{ij}, and put $\log F = f$. Under these conditions, denoting $\frac{\partial f}{\partial x_j} = f_j$, we have

$$\frac{\partial g_{ik}}{\partial x_j} = -\delta_{ik}\frac{2}{F^3}F_j = -2\frac{\delta_{ik}}{F^2}f_j.$$

To calculate the Christoffel symbols, observe that

$$\begin{aligned}
\Gamma^k_{ij} &= \frac{1}{2}\sum_m \left\{\frac{\partial}{\partial x_i}g_{jm} + \frac{\partial}{\partial x_j}g_{mi} - \frac{\partial}{\partial x_m}g_{ij}\right\} g^{mk} \\
&= \frac{1}{2}\left\{\frac{\partial}{\partial x_i}g_{jk} + \frac{\partial}{\partial x_j}g_{ki} - \frac{\partial}{\partial x_k}g_{ij}\right\} F^2 \\
&= -\delta_{jk}f_i - \delta_{ki}f_j + \delta_{ij}f_k;
\end{aligned}$$

therefore we can conclude that if all three indices are distinct, $\Gamma_{ij}^k = 0$, while if two indices are equal, we have

$$\Gamma_{ij}^i = -f_j, \quad \Gamma_{ii}^j = f_j, \quad \Gamma_{ij}^j = -f_i, \quad \Gamma_{ii}^i = -f_i.$$

To calculate the coefficients of the curvature, observe that

$$\begin{aligned} -R_{ijij} &= -\sum_\ell R_{iji}^\ell g_{\ell j} = -R_{iji}^j g_{jj} = -R_{iji}^j \frac{1}{F^2} \\ &= \frac{1}{F^2}\left\{\sum_\ell \Gamma_{ii}^\ell \Gamma_{j\ell}^j - \sum_\ell \Gamma_{ji}^\ell \Gamma_{i\ell}^j + \frac{\partial}{\partial x_j}\Gamma_{ii}^j - \frac{\partial}{\partial x_i}\Gamma_{ji}^j\right\}. \end{aligned}$$

Since $\frac{\partial}{\partial x_j}\Gamma_{ii}^j = f_{jj}$ and $\frac{\partial}{\partial x_i}\Gamma_{ji}^j = -f_{ii}$, we obtain

$$\begin{aligned} -F^2 R_{ijij} &= -\sum_{l, l\neq i,j} f_\ell f_\ell + f_i^2 - f_j^2 - f_i^2 + f_j^2 + f_{jj} + f_{ii} \\ &= -\sum_\ell f_\ell^2 + f_i^2 + f_j^2 + f_{ii} + f_{jj}. \end{aligned}$$

In addition, $R_{ijk\ell} = 0$ if all four indices are distinct, and if any three indices are distinct, we have: $R_{ijk}^i = -f_k f_j - f_{kj}, \quad R_{ijk}^j = f_i f_k + f_{ki}, \quad R_{ijk}^k = 0.$

Let $F = x_n^2$, we have $R_{ijkl} = 0$ if $(i,j,k,l) \neq= (i,j,i,j)$ and the sectional curvature with respect to the plane generated by $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}$ is

$$\begin{aligned} K_p &= -\frac{R_{ijij}}{g_{ii}g_{jj}} = -R_{ijij}F^4 \\ &= \left(-\sum_l f_l^2 + f_i^2 + f_j^2 + f_{ii} + f_{jj}\right) F^2 = -1 \end{aligned}$$

Chapter 3

Riemann Surface

3.1 Foundation

Let X be a connected, Hausdorff topological space, which is locally homeomorphism to a open subset of $\mathbb{C}$.

Definition 3.1.1. A complex chart on X is a homeomorphism $\varphi : U \to V$ of an open subset $U \subseteq X$ onto an open subset $V \subseteq \mathbb{C}$. A complex atlas on X is an open cover $\mathfrak{A} = \{(U_i, \phi_i)\}_{i \in I}$ of X by complex charts such that the transition maps

$$\varphi_i \circ \varphi_j^{-1}\Big|_{\varphi_j(U_i \cap U_j)} : \varphi_j\left(U_i \cap U_j\right) \to \varphi_i\left(U_i \cap U_j\right)$$

Proposition 3.1.2. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas. If two charts (V, ψ) and (W, σ) are both compatible with the atlas $\{(U_\alpha, \phi_\alpha)\}$, then they are compatible with each other.

Proof: Let $p \in V \cap W$. We need to show that $\sigma \circ \psi^{-1}$ is holomorphic at $\psi(p)$. Since $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for $M, p \in U_\alpha$ for some α. Then p is in the triple intersection $V \cap W \cap U_\alpha$.

Hence, $\sigma \circ \psi^{-1} = (\sigma \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \psi^{-1})$ is holomorphic on $\psi\left(V \cap W \cap U_\alpha\right)$, hence at $\psi(p)$. Since p was an arbitrary point of $V \cap W$, this proves that $\sigma \circ \psi^{-1}$ is holomorphic on $\psi(V \cap W)$.

Corollary 3.1.3. For every atlas on X, there's a unique maximal atlas containing it.

Definition 3.1.4 (Riemann Surface)**.** A complex structrue on X is a maximal atlas on X. We call X with a complex structrue on X a Riemann Surface.

Example 3.1.5 (complex plane)**.** The complex structure is defined by the atlas $\{\mathrm{id} : \mathbb{C} \to \mathbb{C}\}$.

Example 3.1.6. Let $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and we introduce the following topology. A subset of $\widehat{\mathbb{C}}$ is open if it is either an open subset of $\mathbb{C}$ or it is of the form $U \cup \{\infty\}$, where $U \subseteq \mathbb{C}$ is the complement of a compact subset of $\mathbb{C}$. With this topology $\widehat{\mathbb{C}}$ is a compact Hausdorff topological space, homeomorphic to the 2-sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ via the stereographic projection. Let $U_1 := \mathbb{C}$ and $U_2 := \mathbb{C}^* \cup \{\infty\}$. Let $\varphi_1 := \mathrm{id} : U_1 \to \mathbb{C}$ and let $\varphi_2 : U_2 \to \mathbb{C}$ be defined by $\varphi_2(z) = 1/z$ if $z \in \mathbb{C}^*$ and $\varphi_2(\infty) = 0$. Then φ_1, φ_2 are homeomorphisms.

Example 3.1.7. Let X be a Riemann surface. Let $Y \subseteq X$ be an open connected subset. Then Y is a Riemann surface in a natural way. An atlas is formed by all complex charts $\varphi : U \to V$ on X with $U \subseteq Y$.

Definition 3.1.8. Let X, Y be Riemann surfaces. A continuous map $f : X \to Y$ is called holomorphic if for every pair of charts $\varphi_1 : U_1 \to V_1$ on X and $\varphi_2 : U_2 \to V_2$ on Y with $f(U_1) \subseteq U_2$,

$$\varphi_2 \circ f \circ \varphi_1^{-1} : V_1 \to V_2$$

is holomorphic. A map $f : X \to Y$ is a biholomorphism if there is a holomorphic map $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Two Riemann surfaces are called isomorphic if there is a biholomorphism between them.

Proposition 3.1.9. Let N and M be Riemann Surfaces, and $F : N \to M$ a continuous map. The following are equivalent:

(1) The map $F : N \to M$ is holomorphic.

(2) There are atlases $\mathfrak{U}$ for N and $\mathfrak{V}$ for M such that for every chart (U, ϕ) in $\mathfrak{U}$ and (V, ψ) in $\mathfrak{V}$, with $\phi(U) \subset V$, the map

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$$

is holomorphic.

Definition 3.1.10. Let X be a Riemann surface, and let $Y \subseteq X$ be an open subset. A function $f : Y \to \mathbb{C}$ is holomorphic if for every chart $\varphi : U \to V$ on X the function $f \circ \varphi^{-1} : \varphi(U \cap Y) \to \mathbb{C}$ is holomorphic. We denote by $\mathcal{O}(Y)$ the set of all holomorphic functions on Y. Clearly, $\mathcal{O}(Y)$ is a $\mathbb{C}$-algebra.

Theorem 3.1.11. Let U be an open subset of a Riemann surface. Let $a \in U$. If $f \in \mathcal{O}(U - \{a\})$ is bounded in a neighborhood of a, then there is $F \in \mathcal{O}(U)$ such that $F|_{U-\{a\}} = f$.

Theorem 3.1.12. Let X, Y be Riemann surfaces. Let $f_1, f_2 : X \to Y$ be holomorphic maps which coincide on a set $A \subseteq X$ with a limit point in X. Then $f_1 = f_2$.

Theorem 3.1.13. Let X, Y be Riemann surfaces and let $f : X \to Y$ be a non-constant holomorphic map. Let $a \in X$ and $b = f(a)$. Then there is an integer $k \geq 1$ and charts $\varphi : U \to V$ on X and $\psi : U' \to V'$ on Y such that $a \in U, \varphi(a) = 0, b \in U', \psi(b) = 0, f(U) \subseteq U'$ and $\psi \circ f \circ \varphi^{-1} : V \to V' : z \mapsto z^k$.

The number k is called the multiplicity of f at a and denoted by $m_a(f)$. The multiplicity is independent of the choice of the charts.

Theorem 3.1.14. Let $f : X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Then f is open.

Theorem 3.1.15. Let $f : X \to Y$ be an injective holomorphic map between Riemann surfaces. Then f is a biholomorphism from X to $f(X)$.

Lemma 3.1.16. X is a topological space, E is a subset of X. Then E has no limit point in X if and only if E is closed and every points in E is isolated.

Definition 3.1.17. Let X be a Riemann surface and let Y be an open subset of X. A meromorphic function on Y is a holomorphic function $f : Y' \to \mathbb{C}$, where Y' is an open subset of Y such that $Y \backslash Y'$ contains only isolated points and

$$\lim_{x \to a} |f(x)| = \infty \qquad \text{for all } a \in Y \backslash Y'.$$

The points of $Y \backslash Y'$ are called the poles of f. The set of all meromorphic functions on Y is denoted by $\mathscr{M}(Y)$. It is easy to see that $\mathscr{M}(Y)$ is a $\mathbb{C}$-algebra.

Theorem 3.1.18. Let X be a Riemann surface and let $f \in \mathscr{M}(X)$. For each pole a of f define $f(a) := \infty$. The resulting map $f : X \to \widehat{\mathbb{C}}$ is holomorphic. Conversely, let $f : X \to \widehat{\mathbb{C}}$ be holomorphic. Then f is either identically equal to ∞ or $f^{-1}(\infty)$ consists of isolated points and $f : X \backslash f^{-1}(\infty) \to \mathbb{C}$ is meromorphic on X.

Proof: By Lemma 3.1.16

Definition 3.1.19. Let $p : Y \to X$ be a non-constant holomorphic map between Riemann surfaces. A point $y \in Y$ is called a branch point of p if there is no neighborhood of y on which p is injective, or equivalently, if $m_y(p) \geq 2$. We say that p is unbranched if it has no branch points.

Definition 3.1.20. Let X, Y, Z be topological spaces and let $p : Y \to X$ and $f : Z \to X$ be continuous maps. A lifting of f over p is a continuous map $g : Z \to Y$ such that $f = p \circ g$.

Theorem 3.1.21. Let X, Y, Z be Riemann surfaces. Let $p : Y \to X$ be an unbranched holomorphic map and let $f : Z \to X$ be holomorphic. Then every lifting $g : Z \to Y$ of f is holomorphic.

Proof: Notice that p is locally a biholomorphic map, so we have every lift g is holomorphic.

Theorem 3.1.22. Let X, Y be compact Riemann surfaces, and let $f : X \to Y$ be a non-constant holomorphic map. Then f is surjective, and for all $y \in Y$,

$$g : y \mapsto \sum_{x \in f^{-1}(y)} m_x(y)$$

is a constant map. We call the value of g the degree of f.

Chapter 4

Complex Geometry

Chapter 5

Lie Group

5.1 Foundation