Number Theory 2: Arithmetic function

Erzhuo Wang

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Youth STEM Academy

Introduction

In the section, we will study four kinds of arithmetic functions(算术函数/数论函数). They are

- $\tau(n)$ = the number of positive divisors of n.(除数函数)
- $\sigma(n)$ = the sum of the positive divisors of n.(除数和函数)
- $\phi(n)$ = the number of positive integers less than n that are relatively prime to n.(欧拉函数)
- $\nu_p(n)$ = the exponent of the largest power of p.(n 唯一因子分解中 p 的幂次)

$\sigma(n)$

If the positive integer n has the prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, then the sum of the positive divisors of n is given by

$$\sigma(n) = \frac{p_1^{k_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{k_2+1} - 1}{p_2 - 1} \cdot \frac{p_3^{k_3+1} - 1}{p_3 - 1} \cdots \frac{p_m^{k_m+1} - 1}{p_m - 1}$$

Theorem

If the positive integer n has the prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, then the number of positive divisors of n is given by

$$\tau(n) = (k_1 + 1)(k_2 + 1) \cdots (k_m + 1).$$

Exercise

How many ordered positive integer pairs (a, b) satisfying $ab = 100$?
有多少个正整数 (a, b) 对满足 $ab = 100$.

Odd $\tau(n)$

Proposition

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Proof: A natural number n has an odd number of divisors exactly when n is a perfect square. Indeed, if n has prime factorization $p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, then $\tau(n) = (k_1 + 1)(k_2 + 1) \cdots (k_m + 1)$ is odd only if each of its factors is odd, and this happens exactly when each of the exponents $k_1, k_2, \ldots k_m$ are even. That means that n is a perfect square.

Theorem

If the positive integer n has prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, then the number of positive integers less than n that are relatively prime to n is given by

$$\begin{aligned}\phi(n) &= n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_m}\right)\\ &= \left(p_1^{k_1}-p_1^{k_1-1}\right)\left(p_2^{k_2}-p_2^{k_2-1}\right)\cdots\left(p_m^{k_m}-p_m^{k_m-1}\right)\end{aligned}$$

Example

For example, $\phi(7) = 6, \phi(10) = 4$, and $\phi(p) = p - 1$ if p is prime.

floor function(高斯函数)

Definition

$\lfloor x \rfloor =$ 不超过 x 的最大整数.

Proposition

m 是一个正整数, $1, 2, 3 \ldots, n$ 中恰有 $\left\lfloor \frac{n}{m} \right\rfloor$ 个数被 m 整除.

Example

$1, 2, 3 \ldots, 100$ 中恰有 $\left\lfloor \frac{100}{3} \right\rfloor = 33$ 个数被 3 整除.

Legendre's formula(勒让德公式)

Theorem

For any prime number p and any positive integer n, let $\nu_p(n)$ be the exponent of the largest power of p that divides n (that is, the p-adic valuation of n). Then

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$$

where $\lfloor x \rfloor$ is the floor function.

Exercise

Question (AMC 8, 2017-19)

For any positive integer M, the notation $M!$ denotes the product of the integers 1 through M . What is the largest integer n for which 5^n is a factor of the sum $98! + 99! + 100!$?
计算 $\nu_5(98! + 99! + 100!)$.

Exercise

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Proof: Factoring out $98! + 99! + 100!$, we have $98!(1 + 99 + 99 \times 100)$, which is $98!(10000)$. And $98!$ has $\left\lfloor \frac{98}{5} \right\rfloor + \left\lfloor \frac{98}{25} \right\rfloor = 19 + 3 = 22$ factors of 5.

Homework

Question (AMC 8, 2018-18)

How many positive factors does 23,232 have?

23232 有多少个正因子?

Question

How many positive cubes divide 3!5!7!?

有多少个完全立方数 (形如 $n = k^3, k \geq 1$) 能整除 3!5!7!.

Question

What's is the largest power of 2 that divides the number $K = 75! - 71!$.

计算 $\nu_2(75! - 71!)$.