Algebra

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Contents

Chapter 1

Abstract Algebra

1.1 Linear Algebra

Theorem 1.1.1 (Newton's Formulas)**.** Let $f(x)$ be a monic polynomial of degree n with roots $\alpha_1, \ldots, \alpha_n$. Let s_i be the elementary symmetric function of degree i in the roots and define $s_i = 0$ for $i > n$. Let $p_i = \alpha_1^i + \cdots + \alpha_n^i, i \geq 0$, be the sum of the i^{th} powers of the roots of $f(x)$. Prove Newton's Formulas:

$$
\begin{aligned}
&p_1 - s_1 = 0 \\
&p_2 - s_1 p_1 + 2s_2 = 0 \\
&p_3 - s_1 p_2 + s_2 p_1 - 3s_3 = 0 \\
&\quad \vdots \\
&p_i - s_1 p_{i-1} + s_2 p_{i-2} - \cdots + (-1)^{i-1} s_{i-1} p_1 + (-1)^i i s_i = 0
\end{aligned}
$$

Definition 1.1.2 (resultant)**.** Let F be a field and let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ and $g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$ be two polynomials in $F[x]$. It's easy to say that a necessary and sufficient condition for $f(x)$ and $g(x)$ to have a common root (or, equivalently, a common divisor in $F[x]$) is the existence of a polynomial $a(x) \in F[x]$ of degree at most $m-1$ and a polynomial $b(x) \in F[x]$ of degree at most $n-1$ with $a(x)f(x) = b(x)g(x)$.

Write $a(x)$ and $b(x)$ explicitly as polynomials, then quating coefficients in the equation $a(x)f(x) = b(x)g(x)$ gives a system of $n+m$ linear equations for the coefficients of $a(x)$ and $b(x)$. Then this system has a nontrivial solution (hence $f(x)$ and $g(x)$ have a common zero) if

and only if the determinant

$$R(f,g) = \begin{vmatrix} a_n & a_{n-1} & \dots & a_0 & & & & \\ & a_n & a_{n-1} & \dots & a_0 & & & \\ & & a_n & a_{n-1} & \dots & a_0 & & \\ & & & \ddots & & & & \\ & & & & a_n & a_{n-1} & \dots & a_0 \\ b_m & b_{m-1} & \dots & b_0 & & & & \\ & b_m & b_{m-1} & \dots & b_0 & & & \\ & & b_m & b_{m-1} & \dots & b_0 & & \\ & & & \ddots & & & & \\ & & & & b_m & b_{m-1} & \dots & b_0 \end{vmatrix}$$

is zero. Here $R(f,g)$, called the resultant of the two polynomials, is the determinant of an $(n+m)\times(n+m)$ matrix R with m rows involving the coefficients of $f(x)$ and n rows involving the coefficients of $g(x)$.

Definition 1.1.3 (discriminant of polynomial)**.** $f(x) \in F[x]$ be a monic polynomial, $\alpha_1, \dots, \alpha_n \in \bar{F}$ are roots of $f(x)$. Then

$$D(f) = \prod_{i<j} (\alpha_i - \alpha_j)^2$$

Proposition 1.1.4.

$$D = (-1)^{n(n-1)/2} R\left(f, f'\right)$$

where D is the discriminant of $f(x)$.

Theorem 1.1.5. Let R be a Principal Ideal Domain, let M be a free R-module of finite rank n and let N be a submodule of M. Then

(1) N is free of rank $m, m \leq n$ and

(2) there exists a basis $y_1, y_2, \dots, y_n$ of M so that $a_1 y_1, a_2 y_2, \dots, a_m y_m$ is a basis of N where $a_1, a_2, \dots, a_m$ are nonzero elements of R with the divisibility relations

$$a_1 \left| a_2 \right| \cdots \mid a_m$$

Theorem 1.1.6 (Fundamental Theorem, Existence: Invariant Factor Form)**.** Let R be a P.I.D. and let M be a finitely generated R-module.

(1) Then M is isomorphic to the direct sum of finitely many cyclic modules. More precisely,

$$M \cong R^r \oplus R/\left(a_1\right) \oplus R/\left(a_2\right) \oplus \cdots \oplus R/\left(a_m\right)$$

for some integer $r \geq 0$ and nonzero elements $a_1, a_2, \dots, a_m$ of R which are not units in R and which satisfy the divisibility relations

$$a_1 \left| a_2 \right| \cdots \mid a_m$$

(2) M is torsion free if and only if M is free.

(3) In the decomposition in (1),

$$\mathrm{Tor}(M) \cong R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_m).$$

Theorem 1.1.7 (Fundamental Theorem, Existence: Elementary Divisor Form)**.** Let R be a P.I.D. and let M be a finitely generated R-module. Then M is the direct sum of a finite number of cyclic modules whose annihilators are either (0) or generated by powers of primes in R, i.e.,

$$M \cong R^r \oplus R/(p_1^{\alpha_1}) \oplus R/(p_2^{\alpha_2}) \oplus \cdots \oplus R/(p_t^{\alpha_t})$$

where $r \geq 0$ is an integer and $p_1^{\alpha_1}, \ldots, p_t^{\alpha_t}$ are positive powers of (not necessarily distinct) primes in R.

Theorem 1.1.8. Let R be a P.I.D.

(1) Two finitely generated R-modules M_1 and M_2 are isomorphic if and only if they have the same free rank and the same list of invariant factors.

(2) Two finitely generated R-modules M_1 and M_2 are isomorphic if and only if they have the same free rank and the same list of elementary divisors.

Corollary 1.1.9. Finite subgroup of a multiplication group of a field is cyclic.

1.2 Group Theory

1.2.1 Subgroup

Proposition 1.2.1. G be a group, H and K be subgroup of G, then

$$|HK| = \frac{|H||K|}{|H \cap K|}$$

1.2.2 Free Group

Definition 1.2.2. Let G be a group. If $\{G_\alpha\}_{\alpha \in J}$ is a family of subgroups of G, we say (as before) that these groups generate G if every element x of G can be written as a finite product of elements of the groups G_α. This means that there is a finite sequence $(x_1, \ldots, x_n)$ of elements of the groups G_α such that $x = x_1 \cdots x_n$. Such a sequence is called a word (of length n) in the groups G_α; it is said to represent the element x of G.

Definition 1.2.3. A word representing x of the form $(y_1, \ldots, y_m)$, where no group G_α contains both y_i and y_{i+1}, and where $y_i \neq 1$ for all i is called a reduced word.

Definition 1.2.4 (free product)**.** Let G be a group, let $\{G_\alpha\}_{\alpha\in J}$ be a family of subgroups of G that generates G. Suppose that $G_\alpha \cap G_\beta$ consists of the identity element alone whenever $\alpha \neq \beta$. We say that G is the free product of the groups G_α if for each $x \in G$, there is only one reduced word in the groups G_α that represents x. In this case, we write

$$G = \prod_{\alpha\in J}^{*} G_\alpha,$$

or in the finite case, $G = G_1 * \cdots * G_n$.

Proposition 1.2.5. Suppose the groups G_α generate G, where $G_\alpha \cap G_\beta = \{1\}$ for $\alpha \neq \beta$. In order for G to be the free product of these groups, it suffices to know that the representation of 1 by the empty word is unique.

Proposition 1.2.6. Let G be a group; let $\{G_\alpha\}$ be a family of subgroups of G. If G is the free product of the groups G_α, then G satisfies the following condition: Given any group H and any family of homomorphisms $h_\alpha : G_\alpha \to H$, there exists a homomorphism $h : G \to H$ whose restriction to G_α equals h_α, for each α. Furthermore, h is unique.

Definition 1.2.7 (external free product)**.** Let $\{G_\alpha\}_{\alpha\in J}$ be an indexed family of groups. Suppose that G is a group, and that $i_\alpha : G_\alpha \to G$ is a family of monomorphisms, such that G is the free product of the groups $i_\alpha(G_\alpha)$. Then we say that G is the external free product of the groups G_α, relative to the monomorphisms i_α.

Proposition 1.2.8 (existence of free product/coproduct in category of Group)**.** Given a family $\{G_\alpha\}_{\alpha\in J}$ of groups, there exists a group G and a family of monomorphisms $i_\alpha : G_\alpha \to G$ such that G is the free product of the groups $i_\alpha(G_\alpha)$.

Proof: We define a word (of length n) in the elements of the groups G_α to be an n-tuple $w = (x_1, \ldots, x_n)$ of elements of $\bigcup G_\alpha$. It is called a reduced word if $\alpha_i \neq \alpha_{i+1}$ for all i, where α_i is the index such that $x_i \in G_{\alpha_i}$, and if for each i, x_i is not the identity element of G_{α_i}. We define the empty set to be the unique reduced word of length zero.

Let W denote the set of all reduced words in the elements of the groups G_α. Let $P(W)$ denote the set of all bijective functions $\pi : W \to W$. Then $P(W)$ is itself a group, with composition of functions as the group operation. We shall obtain our desired group G as a subgroup of $P(W)$.

Step 1: For each index α and each $x \in G_\alpha$, we define a set map $\pi_x : W \to W$. It will satisfy the following conditions:

(1) If $x = 1_\alpha$, the identity element of G_α, then π_x is the identity map of W.

(2) If $x, y \in G_\alpha$ and $z = xy$, then $\pi_z = \pi_x \circ \pi_y$.

We proceed as follows: Let $x \in G_\alpha$. For notational purposes, let $w = (x_1, \ldots, x_n)$ denote the general nonempty element of W, and let α_1 denote the index such that $x_1 \in G_{\alpha_1}$. If $x \neq 1_\alpha$, define π_x as follows:

$$\pi_x(\varnothing) = (x),$$

$$\pi_x(w) = (x, x_1, \ldots, x_n), \text{if } \alpha_1 \neq \alpha$$
$$\pi_x(w) = (xx_1, \ldots, x_n), \text{if } \alpha_1 = \alpha, x_1 \neq x^{-1},$$
$$\pi_x(w) = (x_2, \ldots, x_n), \text{ if } \alpha_1 = \alpha \text{ and } x_1 = x^{-1}$$

If $x = 1_\alpha$, define π_x to be the identity map of W.

Step 2: We show that if $x, y \in G_\alpha$ and $z = xy$, then $\pi_z = \pi_x \circ \pi_y$ and $x \mapsto \pi_x$ is injective.

Step 3: Let G be the subgroup of $P(W)$ generated by the groups $G'_\alpha = i_\alpha(G_\alpha)$. We show that G is the free product of the groups G'_α.

First, we show that $G'_\alpha \cap G'_\beta$ consists of the identity alone if $\alpha \neq \beta$. Let $x \in G_\alpha$ and $y \in G_\beta$; we suppose that neither π_x nor π_y is the identity map of W and show that $\pi_x \neq \pi_y$. But this is easy, for $\pi_x(\varnothing) = (x)$ and $\pi_y(\varnothing) = (y)$, and these are different words. Second, we show that no nonempty reduced word

$$w' = (\pi_{x_1}, \ldots, \pi_{x_n})$$

in the groups G'_α represents the identity element of G. Let α_i be the index such that $x_i \in G_{\alpha_i}$; then $\alpha_i \neq \alpha_{i+1}$ and $x_i \neq 1_{\alpha_i}$ for each i. We compute

$$\pi_{x_1}(\pi_{x_2}(\cdots(\pi_{x_n}(\varnothing)))) = (x_1, \ldots, x_n),$$

so the element of G represented by w' is not the identity element of $P(W)$.

Proposition 1.2.9. Let $G = G_1 * G_2$, where G_1 is the free product of the subgroups $\{H_\alpha\}_{\alpha \in J}$ and G_2 is the free product of the subgroups $\{H_\beta\}_{\beta \in K}$. If the index sets J and K are disjoint, then G is the free product of the subgroups $\{H_\gamma\}_{\gamma \in J \cup K}$.

Proposition 1.2.10 (extension property)**.** Let $\{G_\alpha\}$ be a family of groups; let G be a group, let $i_\alpha : G_\alpha \to G$ be a family of homomorphisms. The following statement are equivalent:

(1) If each i_α is a monomorphism and G is the free product of the groups $i_\alpha(G_\alpha)$

(2) (coproduct)Given a group H and a family of homomorphisms $h_\alpha : G_\alpha \to H$, there exists a homomorphism $h : G \to H$ such that $h \circ i_\alpha = h_\alpha$ for each α.

Furthermore, h is unique if one of above statements holds.

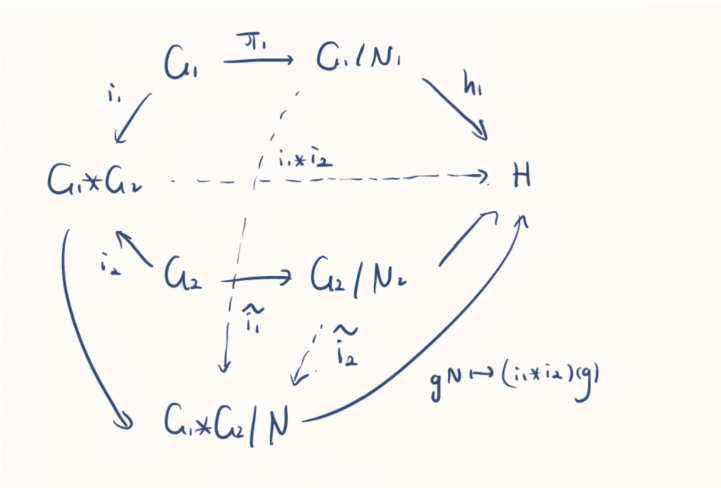
Proposition 1.2.11 (Uniqueness of free products)**.** Let $\{G_\alpha\}_{\alpha \in J}$ be a family of groups. Suppose G and G' are groups and $i_\alpha : G_\alpha \to G$ and $i'_\alpha : G_\alpha \to G'$ are families of monomorphisms, such that the families $\{i_\alpha(G_\alpha)\}$ and $\{i'_\alpha(G_\alpha)\}$ generate G and G', respectively. If both G and G' have the extension property, then there is a unique isomorphism $\phi : G \to G'$ such that $\phi \circ i_\alpha = i'_\alpha$ for all α.

Proposition 1.2.12. If S is a subset of G, one can consider the intersection N of all normal subgroups of G that contain S. It is easy to see that N is itself a normal subgroup of G; it is called the least normal subgroup of G that contains S. It can also be shown that the subgroup generated by $\cup_{g \in G} g^{-1} S g$ is the least normal subgroup of G that contains S.

Let $G = G_1 * G_2$. Let N_i be a normal subgroup of G_i, for $i = 1, 2$. If N is the least normal subgroup of G that contains N_1 and N_2, then

$$G/N \cong (G_1/N_1) * (G_2/N_2).$$

Proof: By Proposition 1.2.11 and Propostion 1.2.10, it suffice to show $G_1/N_1 \to G/N, G_2/N_2 \to G/N$ satisfies extension property.



Definition 1.2.13 (free group)**.** Let $\{a_\alpha\}$ be a family of elements of a group G. Suppose each a_α generates an infinite cyclic subgroup G_α of G. If G is the free product of the groups $\{G_\alpha\}$, then G is said to be a free group, and the family $\{a_\alpha\}$ is called a system of free generators for G.

In this case, for each element x of G, there is a unique reduced word in the elements of the groups G_α that represents x. This says that if $x \neq 1$, then x can be written uniquely in the form

$$x = (a_{\alpha_1})^{n_1} \cdots (a_{\alpha_k})^{n_k},$$

Definition 1.2.14. Let $\{a_\alpha\}_{\alpha \in J}$ be an arbitrary indexed family. Let G_α denote the set of all symbols of the form a_α^n for $n \in \mathbb{Z}$. We make G_α into a group by defining

$$a_\alpha^n \cdot a_\alpha^m = a_\alpha^{n+m}.$$

Then a_α^0 is the identity element of G_α, and a_α^{-n} is the inverse of a_α^n. We denote a_α^1 simply by a_α. The external free product of the groups $\{G_\alpha\}$ is called the free group on the elements a_α.

Theorem 1.2.15. Given G, suppose we are given a family $\{a_\alpha\}_{\alpha \in J}$ of generators for G. Let F be the free group on the elements $\{a_\alpha\}$. Then the obvious map $h(a_\alpha) = a_\alpha$ of these elements into G extends to a homomorphism $h : F \to G$ that is surjective. If N equals the kernel of h, then $F/N \cong G$. Each element of N is called a relation on F, and N is called the relations subgroup. We can specify N by giving a set of generators for N. But since N is normal in F, we can also specify N by a smaller set. Specifically, we can specify N by giving a family $\{r_\beta\}$ of elements of F such that these elements and their conjugates generate N, that is, such that N is the least normal subgroup of F that contains the elements r_β. In this case, we call the family $\{r_\beta\}$ a complete set of relations for G.

Definition 1.2.16. The abelianization G_{ab} of G is the group defined by

$$G_{ab} = G/[G, G],$$

where $[G, G]$ is the (normal) subgroup generated by commutators.

Proposition 1.2.17. $f : G \to H$ is a surjective group homomorphism, $\tilde{f} : G_{ab} \to H_{ab}$ is the natural homomorphism induced by f. Let $i : G \to G_{ab}$ be the natrual projection. Then $\mathrm{Ker}\tilde{f} = i(\mathrm{Ker}f)$

Example 1.2.18. Consider

(1) $G = \langle a_1, a_2, \ldots, a_n | a_1^2 \ldots a_n^2 = e \rangle$

(2) $H = \langle a_1, b_1, a_2, b_2 \ldots, a_n, b_n | [a_1, b_1] \ldots [a_n, b_n] = e \rangle$

Show that $G_{ab} \simeq \mathbb{Z}^{n-1} \times \mathbb{Z}/2\mathbb{Z}$, and $H_{ab} \simeq \mathbb{Z}^{2n}$

Proof: For (1), consider the surjective homomorphism $F = F(a_1, \ldots, a_n) \to G$, which induces a surjective homomorphism $\tilde{f} : F_{ab} \to G_{ab}$. Then by Proposition 1.2.17, G_{ab} is isomorphic to $\mathbb{Z}a_1 \oplus \ldots \mathbb{Z}a_n/(2(a_1 + \cdots + a_n))$.

1.3 Ring Theory

1.3.1 General Concepts

Definition 1.3.1 (division ring)**.**

1.3.2 Commutative Ring

Definition 1.3.2. A zero-divisor in a ring A is an element x for which there exists $y \neq 0$ in A such that $xy = 0$.

Definition 1.3.3. An ideal which is maximal among all proper ideals is called a maximal ideal; an ideal m of A is maximal if and only if A/m is a field.

Theorem 1.3.4. If I is a proper ideal then there exists at least one maximal ideal containing I.

Definition 1.3.5. A ring A is an integral domain (or simply a domain) if $A \neq 0$, and A has no zero-divisors other than 0.

Definition 1.3.6. A field F is an integral doamin such that every non-zero element in F is invertible.

Definition 1.3.7. A proper ideal($\neq A$) P of A for which A/P is an integral domain is called a prime ideal. In other words, P is prime if it satisfies:

(1) $P \neq A$.

(2) $x, y \in \Rightarrow xy \in P$ for $x, y \in A$.

A field is an integral domain, so that a maximal ideal is prime.

Proposition 1.3.8. There is a one-to-one order-preserving correspondence between the ideals J of A which contain I, and the ideals A/I. More precisely,we can say there are two bijection

$$\{\text{ideals of } A \text{ that contain } I\} \longleftrightarrow \{\text{ideals of } A/I\}$$

$$\{\text{prime ideals of } A \text{ that contain } I\} \longleftrightarrow \{\text{prime ideals of } A/I\}$$

given by the correspondences

$$J \longrightarrow J/I = \bar{J}$$

$$\pi^{-1}(\bar{J}) \longleftarrow \bar{J}$$

where π be the natural homomorphism from A to A/I.

Definition 1.3.9. A subset S of A is multiplicative if it satisfies:

(1) $x, y \in S \Rightarrow xy \in S$.

(2) $1 \in S$.

Definition 1.3.10. If I is an ideal of A then the set of elements of A, some power of which belongs to I, is an ideal of A.This set is called the radical of I, and is sometimes written $\sqrt{I}$.

Theorem 1.3.11. the radical $\sqrt{I}$ of I is the intersection of all prime ideals containing I.

Proof:

Lemma 1.3.12. Let S be a multiplicative set and I an ideal disjoint from S, then there exists a prime ideal containing I and disjoint from S.

Proof: If I is an ideal disjoint from S,then the set of ideals containing I and disjoint from S has a maximal element. If P is an ideal which is maximal among ideals disjoint from S then P is prime. For if $x, y \notin P, xy \in P$, then since $P + xA$ and $P + yA$ both meet S, the product $(P + xA)(P + yA)$ also meets S. However, $(P + xA)(P + yA) \subset P + xyA$,a contradiction!

If $x \notin \sqrt{I}$, $S_x = x^n : n \geq 0$ be a multiplicative subset. By lemma 1.3.12, we can find a prime ideal which contains I disjoint from S_x.

Definition 1.3.13. In particular if we take $I = (0)$ then $\sqrt{(0)}$ is the set of all nilpotent elements of A, and is called the nilradical of A; we will write $\text{nil}(A)$ for this. When $\text{nil}(A) = 0$ we say that A is reduced, For any ring A we write A_{red} for $A/\text{nil}(A)$ is of course reduced.

Definition 1.3.14. The intersection of all maximal ideals of a ring $A \neq 0$ is called the Jacobson radical, or simply the radical of A and written $\text{rad}(A)$.

Proposition 1.3.15. $x \in \text{rad}(A)$ if and only if $1 + xy$ is a unit in A for all $y \in A$.

Definition 1.3.16. A ring having just one maximal ideal is called a local ring, and a (non-zero) ring having only finitely many maximal ideals a semilocal ring. We often express the fact that A is a local ring with maximal ideal m by saying that (A, m) is a local ring; if this happens then the field $k = A/m$ is called the residue field of A. We will say that (A, m, k) is a local ring to mean that A is a local ring, $m = \text{rad}(A)$ and $k = A/m$.

Theorem 1.3.17. If $I_1, I_2, ..., I_n$ are ideals which are coprime(i.e. $I_i + I_j = A$ for all $i \neq j$) in pairs then $I_1 I_2 \dots I_n = I_1 \cap I_2 \cdots \cap I_n$

Theorem 1.3.18 (Chinese Reminder Theorem)**.** If $I_1, \dots, I_n$ are ideals which are coprime in pairs then

$$A/I_1 \times \cdots \times A/I_n \simeq A/(I_1 \dots I_n)$$

and the isomorphism map is given by

$$a + I_1 \dots I_n \to (a + I_1, \dots, a + I_n)$$

Theorem 1.3.19 (Prime Avoidance)**.** (1) Let $P_1, \dots P_n$ be prime ideals and let I be an ideal contained in $\bigcup_{i=1}^{n} P_i$. Then $I \subset P_i$ for some $1 \leq i \leq n$.

(2) Let P be a prime ideal. $P \supset I_1 \dots I_n$, then $P \supset I_i$ for some $1 \leq i \leq n$.

Proof: (2):If $P \supset IJ$ and $P \not\supseteq I$, there's $a \in I$ such that $a \notin P$. Since $P \supset IJ$, for all $b \in J$, $ab \in P$, then $b \in P$. Hence we have $P \supset J$.

Definition 1.3.20. Let R be an integral domain. Suppose $r \in R$ is nonzero and is not a unit. Then r is called irreducible in R if whenever $r = ab$ with $a, b \in R$, at least one of a or b must be a unit in R. Otherwise r is said to be reducible.The nonzero element $p \in R$ is called prime in R if the ideal (p) generated by p is a prime ideal.Two elements a and b of R differing by a unit are said to be associate in R.

Proposition 1.3.21. In an integral domain, a prime element is always irreducible.

Definition 1.3.22 (U.F.D)**.** A Unique Factorization Domain is an integral domain R in which every nonzero element $r \in R$ which is not a unit has the following two properties:

(1) r can be written as a finite product of irreducibles p of R: $r = p_1 \dots p_n$

(2) the decomposition in (1) is unique up to associates.

Proposition 1.3.23. A intergral domain R is U.F.D if and only if every irreducible element is prime and there's no infinite sequence (a_n) in R satisfying: $a_i | a_{i+1}$, a_i and a_j are not associate.

Definition 1.3.24 (P.I.D)**.** A Principal Ideal Domain is an integral domain in which every ideal is principal.

Proposition 1.3.25. Every Principal Ideal Domain is a Unique Factorization Domain.

Proposition 1.3.26. If F is a field, then $F[x]$ is a Principal Ideal Domain.

Lemma 1.3.27 (Gauss' Lemma)**.** Let R be a Unique Factorization Domain with field of fractions F and let $p(x) \in R[x]$. If $p(x)$ is reducible in $F[x]$ then $p(x)$ is reducible in $R[x]$. More precisely, if $p(x) = A(x)B(x)$ for some nonconstant polynomials $A(x), B(x) \in F[x]$, then there are nonzero elements $r, s \in F$ such that $rA(x) = a(x)$ and $sB(x) = b(x)$ both lie in $R[x]$ and $p(x) = a(x)b(x)$ is a factorization in $R[x]$.

Corollary 1.3.28. Let R be a Unique Factorization Domain, let F be its field of fractions and let $p(x) \in R[x]$. Suppose the greatest common divisor of the coefficients of $p(x)$ is 1. Then $p(x)$ is irreducible in $R[x]$ if and only if it is irreducible in $F[x]$. In particular, if $p(x)$ is a monic polynomial that is irreducible in $R[x]$, then $p(x)$ is irreducible in $F[x]$.

Proposition 1.3.29. If R is a U.F.D, then $R[x]$ is a U.F.D.

Proof: By Proposition 1.3.26, Lemma 1.3.27 and Corollary 1.3.28.

Corollary 1.3.30 (Eisenstein's criterion)**.** R is a U.F.D., $p(x) = a_n x^n + \cdots + a_1 x^1 + a_0 \in R[x]$ is a irreducible polynomial if there's some $p \in R$ such that $p \nmid a_n, p \mid a_{n-1} \ldots, a_0, p^2 \nmid a_0$

Proposition 1.3.31. Let A be a ring and let $A[x]$ be the ring of polynomials with coefficients in A. $A[[x]]$ be the ring of formal power series. Let $f = a_0 + a_1 x + \cdots + a_n x^n \in A[x], g = \sum_{n=0}^{\infty} b_n x^n$. Prove that

(1) f is a unit in $A[x] \Leftrightarrow a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent.

(2) f is nilpotent $\Leftrightarrow a_0, a_1, \ldots, a_n$ are nilpotent.

(3) f is a zero-divisor $\Leftrightarrow$ there exists $a \neq 0$ in A such that $af = 0$.(which implies if A is a domain, $A[x]$ is a domain).

(4) g is a unit in $A[[x]] \Leftrightarrow b_0$ is a unit in A.

(5) g is nilpotent, then b_n are all nilpotent.

Definition 1.3.32 (The Ring of Formal Laurent Series)**.** The ring of formal Laurent series in x with coefficients in R is denoted by $R((x))$, and is defined as follows. The elements of $R((x))$ are infinite expressions of the form

$$f(x) = a_r x^r + a_{r+1} x^{r+1} + a_{r+2} x^{r+2} + \cdots$$

in which $r \in \mathbb{Z}$ and $a_n \in R$ for all $n \geq r$.

Proposition 1.3.33. If R is a field then $R((x))$ is a field.

Proof: Consider a nonzero $f(x) = \sum_{n=I(f)}^{\infty} a_n x^n$ in $R((x))$. Then $a_{I(f)} \neq 0$ so that it is invertible in R, since R is a field. We may write $f(x) = x^{I(f)} g(x)$ with $g(x) = \sum_{n=0}^{\infty} a_{n+I(f)} x^n$, so that $g(x)$ is a formal power series in $R[[x]]$. The coefficient of x^0 in $g(x)$ is $a_{I(f)}$ and, it follows that $g(x)$ is invertible in $R[[x]]$, and hence in $R((x))$. Let $h(x) := x^{-I(f)} g^{-1}(x)$. Then

$$f(x)h(x) = x^{I(f)} g(x) x^{-I(f)} g^{-1}(x) = 1$$

so that $h(x) = f^{-1}(x)$ and $f(x)$ is invertible in $R((x))$. Therefore, $R((x))$ is a field.

1.4 Field Theory

1.4.1 Basic Concepts

Theorem 1.4.1. Let $p(x) \in F[x]$ be an irreducible polynomial of degree n over the field F and let K be the field $F[x]/(p(x))$. Let $\theta = x \bmod (p(x)) \in K$. Then the elements

$$1, \theta, \theta^2, \ldots, \theta^{n-1}$$

are a basis for K as a vector space over F, so the degree of the extension is n, i.e., $[K : F] = n$. Hence

$$K = \left\{a_0 + a_1\theta + a_2\theta^2 + \cdots + a_{n-1}\theta^{n-1} \mid a_0, a_1, \ldots, a_{n-1} \in F\right\}$$

consists of all polynomials of degree $< n$ in θ.

Definition 1.4.2. Let K be an extension of the field F and let S be a subset of K. Then the smallest subfield of K containing both F and the elements $s \in S$, denoted $F(S)$ is called the field generated by S over F. If the field K is generated by a single element α over $F, K = F(\alpha)$, then K is said to be a simple extension of F and the element α is called a primitive element for the extension.

Theorem 1.4.3. Let F be a field and let $p(x) \in F[x]$ be an irreducible polynomial. Suppose K is an extension field of F containing a root α of $p(x) : p(\alpha) = 0$. Let $F(\alpha)$ denote the subfield of K generated over F by α. Then

$$F(\alpha) \cong F[x]/(p(x))$$

Suppose that $p(x)$ is of degree n. Then

$$F(\alpha) = \left\{a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{n-1}\alpha^{n-1} \mid a_0, a_1, \ldots, a_{n-1} \in F\right\} \subseteq K$$

Theorem 1.4.4. Let $\varphi : F \xrightarrow{\sim} F'$ be an isomorphism of fields. Let $p(x) \in F[x]$ be an irreducible polynomial and let $p'(x) \in F'[x]$ be the irreducible polynomial obtained by applying the map φ to the coefficients of $p(x)$. Let α be a root of $p(x)$ (in some extension of F) and let β be a root of $p'(x)$ (in some extension of F'). Then there is an isomorphism

$$\sigma : F(\alpha) \xrightarrow{\sim} F'(\beta)$$
$$\alpha \longmapsto \beta$$

mapping α to β and extending φ, i.e., such that σ restricted to F is the isomorphism φ.

In the following statements, we always assume F be a field and let K be an extension of F, $\alpha, \beta \in K$ be an element.

Definition 1.4.5. The element $\alpha \in K$ is said to be algebraic over F if α is a root of some nonzero polynomial $f(x) \in F[x]$. If α is not algebraic over F, then α is said to be transcendental over F. The extension K/F is said to be algebraic if every element of K is algebraic over F.

Let α be algebraic over F. Then there is a unique monic irreducible polynomial $m_{\alpha,F}(x) \in F[x]$ which has α as a root. A polynomial $f(x) \in F[x]$ has α as a root if and only if $m_{\alpha,F}(x)$ divides $f(x)$ in $F[x]$.

Theorem 1.4.6. Let α be algebraic over the field F and let $F(\alpha)$ be the field generated by α over F. Then

$$F(\alpha) \cong F[x]/\left(m_\alpha(x)\right)$$

so that in particular

$$[F(\alpha) : F] = \deg m_\alpha(x) = \deg \alpha,$$

i.e., the degree of α over F is the degree of the extension it generates over F.

Proposition 1.4.7. The element $\alpha \in K$ is algebraic over F if and only if the simple extension $F(\alpha)/F$ is finite. More precisely, if α is an element of an extension of degree n over F then α satisfies a polynomial of degree at most n over F and if α satisfies a polynomial of degree n over F then the degree of $F(\alpha)$ over F is at most n.

Definition 1.4.8. Let K_1 and K_2 be two subfields of a field K. Then the composite field of K_1 and K_2, denoted K_1K_2, is the smallest subfield of K containing both K_1 and K_2. Similarly, the composite of any collection of subfields of K is the smallest subfield containing all the subfields.

Proposition 1.4.9. $F(\alpha, \beta) = (F(\alpha))(\beta)$, i.e., the field generated over F by α and β is the field generated by β over the field $F(\alpha)$ generated by α. In general, if $a_1, \ldots, a_n$ be elements of K, then $F(a_1, \ldots, a_n) = ((F(a_1)(a_2)) \ldots)(a_n)$

Corollary 1.4.10. If $K \subset L \subset M$ are field extensions, $L/K, M/L$ are algebraic extensions, then M/K is algerbaic.

Definition 1.4.11 (spilting field)**.** The extension field K of F is called a splitting field for the polynomial $f(x) \in F[x]$ if $f(x)$ factors completely into linear factors (or splits completely) in $K[x]$ and $f(x)$ does not factor completely into linear factors over any proper subfield of K containing F.

Theorem 1.4.12. For any field F, if $f(x) \in F[x]$ then there exists an extension K of F which is a splitting field for $f(x)$.

Proof: We first show that there is an extension E of F over which $f(x)$ splits completely into linear factors by induction on the degree n of $f(x)$. If $n = 1$, then take $E = F$. Suppose now that $n > 1$. If the irreducible factors of $f(x)$ over F are all of degree 1 , then F is the splitting field for $f(x)$ and we may take $E = F$. Otherwise, at least one of the irreducible factors, say $p(x)$ of $f(x)$ in $F[x]$ is of degree at least 2 . Hence, there is an extension E_1 of F containing a root α of $p(x)$. Over E_1 the polynomial $f(x)$ has the linear factor $x - \alpha$. The degree of the

remaining factor $f_1(x)$ of $f(x)$ is $n-1$, so by induction there is an extension E of E_1 containing all the roots of $f_1(x)$. Since $\alpha \in E$, E is an extension of F containing all the roots of $f(x)$. Now let K be the intersection of all the subfields of E containing F which also contain all the roots of $f(x)$. Then K is a field which is a splitting field for $f(x)$.

Definition 1.4.13. The field $\bar{F}$ is called an algebraic closure of F if $\bar{F}$ is algebraic over F and if every polynomial $f(x) \in F[x]$ splits completely over $\bar{F}$ (so that $\bar{F}$ can be said to contain all the elements algebraic over F).

A field K is said to be algebraieally closed if every polynomial with coefficients in K has a root in K.

Theorem 1.4.14. Let $\bar{F}$ be an algebraic closure of F. Then F is algebraically closed.

Proof: By Corollary 1.4.10.

Lemma 1.4.15. Let $\varphi : F \xrightarrow{\sim} F'$ be an isomorphism of fields, $\bar{F}'$ be the algebraic closure of F', E/F is a algebraic extension, then there's $\sigma : E \to \bar{F}'$ ring homomorphism satisfying $\sigma|_F = \varphi$.

Proof: By Zorn's Lemma and Theorem 1.4.4.

Theorem 1.4.16. For any field F, algebraic closure of F exists and is unique up to isomorphism.

Proof: Existence: For each polynomial $f \in F[X]$, choose a splitting field E_f, and let

$$\Omega = \left(\bigotimes_{f \in F[x]} E_f \right) / M$$

where M is a maximal ideal. It is clear that Ω is a F-algebra and E_f can be embedded into Ω. Since f splits in E_f, it must also split in the larger field Ω. Then all the algebraic elements in Ω is therefore an algebraic closure of F.

Uniqueness: By Lemma 1.4.15.

1.4.2 Separable

In the following statements, F is a field, and we fix an algebraic closure of F and denote it by $\bar{F}$.

Definition 1.4.17 (separable)**.** A polynomial $f(x) \in F[x]$ is separable if $f(x)$ has no multiple root in $\bar{F}$.

Proposition 1.4.18. A polynomial $f(x)$ has a multiple root $\alpha \in \bar{F}$ if and only if α is also a root of $f'(x)$. In particular, $f(x)$ is separable if and only if it is relatively prime to its derivative: $(f(x), D_x f(x)) = 1$.

Remark 1.4.19. For any two polynomials $f(x), g(x) \in F[x]$, they have the same g.c.d in $F[x]$ and $\bar{F}[x]$ since Euclidean division doesn't change if we replace F by any extension field of F.

Definition 1.4.20. $\alpha \in \bar{F}$ is separable if $m_\alpha(x) \in F[x]$ is separable polynomial.

$F \subset E \subset \bar{F}$ are field extensions, E/F is a separable extension if for all $\alpha \in E$, α is separable.

Definition 1.4.21 (perfect field)**.** A field $F \subset \bar{F}$ is perfect if and only if every finite extension of F is separable.

Lemma 1.4.22. Let $p(x)$ be an irreducible polynomial over a field F of characteristic p. Then there is a unique integer $k \geq 0$ and a unique irreducible separable polynomial $p_{\text{sep}}(x) \in F[x]$ such that

$$p(x) = p_{sep}\left(x^{p^k}\right)$$

Hence $p(x) = \deg p_{sep} p^k$ and $\deg p_{sep}$ is called the separable degree, p^k is called the inseparable degree of P.

Proposition 1.4.23. A field F is perfect if and only if it is a field of characteristic 0 or a field of characteristic $p > 0$ such that every element has a p-th root.

Proof: '$\Longleftarrow$': case 1: If $\operatorname{chap} F = 0$, then by Proposition 1.4.18, F is perfect.

case 2: If $\operatorname{chap} F = p$, $\alpha \in \bar{F}$, and $p(x) = m_\alpha(x) \in F[x]$ is inseparable, by Lemma 1.4.22, there's irreducible polynomial $q(x)$ such that $p(x) = q(x^p)$. Hence

$$p(x) = a_m x^{pm} + \cdots + a_1 x^p + a_0 = b_m^p x^{pm} + \ldots b_1^p x^p + b_0^p = (b_m x^m + \ldots b_0)^p$$

where $b_i^p = a_i$ for $i = 0, \ldots m$. A contradiction!

'$\Longrightarrow$': if $\operatorname{chap} F = p$ and $\alpha \in \bar{F}$ is not a p-th root, consider $p(x) = x^p - \alpha$. Notice that $(p(x), p'(x)) = p(x)$, then $p(x)$ is inseparable. However, if $\beta \in \bar{F}$ is a root of $p(x)$, then $p(x) = x^p - \alpha = x^p - \beta^p = (x - \beta)^p$. If $p(x)$ is reducible in $F[x]$, $p(x) = a(x)b(x)$ where $\deg a(x), \deg b(x) < p$.

Notice that $a(x) = (x - \beta)^s, b(x) = (x - \beta)^t \in F[x]$ with $s + t = p$, then $\beta^s \in F, \beta^t \in F$. Hence by Bezout Theorem, we have $\beta^{(s,t)} = \beta \in F$ which contradict to the fact that α is not a p-th root. Hence $p(x)$ is irreducible inseparable polynomial, and contradict to the fact F is perfect!

Corollary 1.4.24. In the proof of above Proposition, we can obtain if $\operatorname{chap} F \neq 0$ and $p(x) = x^p - \alpha \in F[x]$, either $p(x)$ is irreducible or $p(x) = (x - \beta)^p$ for some $\beta \in F$.

Example 1.4.25. $\mathbb{Q}, \mathbb{F}_q$ are perfect fields and $\mathbb{F}_p(t)$ is not perfect field.

Definition 1.4.26. Given field extensions $F \subset E \subset \bar{F}$, E is called purely inseparable if for each $\alpha \in E$ the minimal polynomial of α over F has only one distinct root. It is easy to see that the following are equivalent:

(1) E/F is purely inseparable

(2) if $\alpha \in E$ is separable over F, then $\alpha \in F$

(3) if $\alpha \in E$, then $\alpha^{p^n} \in F$ for some n (depending on α), and $m_{\alpha,F}(x) = x^{p^n} - \alpha^{p^n}$.

Proposition 1.4.27. Given a field extension $K/F, u \in K$ is separable over F iff $F(u) = F(u^p)$.

Proof: We assume u is separable first. Then $F_1 = F(u^p) \subseteq F(u)$. Consider the polynomial $X^p - u^p \in F_1[X]$ and u is a root of it. Let P be the minimal polynomial of u over $F_1 \Rightarrow P \mid X^p - u^p$. But $X^p - u^p = (X-u)^p$. Thus $P = (X-u)^k$ for some integer k. Since P is separable and all roots of it are different, $P = X - u$. Hence $u \in F_1$. Then $F(u) = F(u^p)$.

On the other hand, we assume $F(u) = F(u^p)$. Let P be the minimal polynomial of u over F. If P is not separable, then $P(X) = P_1(X^p)$. Since P_1 is irreducible and $P(u^p) = 0, P_1$ is the minimal polynomial of u^p. Then $[F(u) : F] = [F(u^p) : F] = \deg P = \deg P_1 = p \cdot \deg P_1$. A contradiction! Hence P is separable and u is separable.

Proposition 1.4.28. Assume $[K : F] = d < \infty$. The following statements are equivalent.

(1) $F \subseteq K$ is separable.

(2) $K = F \cdot K^p$, where $K^p = \{k^p \mid k \in K\}$ a subfield of K since $\mathrm{char}(F) = \mathrm{char}(K) = p > 0$;

(3) There is a basis $\{e_1, \ldots, e_d\}$ of K over F such that $\{e_1^p, \ldots, e_d^p\}$ is still a basis.

Proposition 1.4.29. A simple algebraic extension $F(u)/F$ is separable iff u is separable over F.

Proof: If $P(X) \in F[X]$ is the minimal polynomial of u over $F, P(X) = \sum_k a_k X^k$ with $\deg(P) = n$, then $\{1, u, \ldots, u^{n-1}\}$ form a basis of $F(u)$ over F. We prove $\left\{1, u^p, \ldots, u^{p(n-1)}\right\}$ is a basis as well.

If this is true, from the Lemma 1.4.28, $F(u)/F$ is separable.

If this is not true, there will exist $\{b_k\}$ which are not all zero such that $\sum_k b_k u^{kp} = 0$. Let $P_1(X) = \sum_k b_k X^k$, with $\deg(P_1) \leq n-1$. $P_1(u^p) = 0$. Then $[F(u^p) : F] \leq \deg(P_1) \leq n-1$. But since u is separable, $F(u) = F(u^p), [F(u) : F] = [F(u^p) : F] = n$, a contradiction!

Proposition 1.4.30. $F \subseteq E \subseteq K$ are field extensions. K/F is separable iff E/F and K/E are separable.

Proof: We only prove the part of $\Leftarrow$. If $[K : F] < \infty$, $K = E \cdot K^p = (F \cdot E^p) \cdot K^p = F \cdot (E^p \cdot K^p) = F \cdot K^p$. Hence K/F is separable.

If $[K : F] = \infty, u \in K$ and $P_u \in E[X]$ is the minimal polynomial of u over E. $P_u(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_0$. Consider $F \subseteq F(a_0, \ldots, a_{n_1}) \subseteq E \subseteq E(u) \subseteq K$. Since E/F is separable, according to the part of $\Rightarrow$ of 1 ., we know $F(a_0, \ldots, a_{n-1})/F$ is separable. And since the minimal polynomial of u over $F(a_0, \ldots, a_{n_1})$ is just P_u, which is separable. Then $F(a_0, \ldots, a_{n_1}, u)/F(a_0, \ldots, a_{n_1})$ is separable. Since $[F(a_0, \ldots, a_{n_1}, u) : F] < \infty, F(a_0, \ldots, a_{n_1}, u)/F$ is separable and especially u is separable.

Definition 1.4.31. Given an algebraic extension K/F, all separable elements in K form a subfield containing F, which is denoted by K_s. Especially if $K = \bar{F}, \bar{F}_s$ is denoted by F_{sep} and called the separable closure. This motivates us to study K_s/F and K/K_s respectively, which is the task in the next subsection.

Proposition 1.4.32. If E/F and E'/F are separable, then $E \cdot E'/F$ is separable.

Proposition 1.4.33. Given a finite algebraic extension K/F, we have the following equation

$$[K : F]_s = \left|\mathrm{Hom}_F(K, \bar{F})\right|$$

More generally, we have

Corollary 1.4.34. Let $\varphi : F \xrightarrow{\sim} F'$ be an isomorphism of fields, $\bar{F}'$ be the algebraic closure of F', E/F is a finite separable extension, then there are exactly $[E : F]$ ways to give an extension $\sigma : E \to \bar{F}'$ ring homomorphism satisfying $\sigma|_F = \varphi$.

1.4.3 Normal

Definition 1.4.35. Let $F \subset E \subset \bar{F}$ be field extensions, we call E/F normal if for all $\alpha \in E$, all the roots of $m_\alpha(x)$ lie in E.

Definition 1.4.36. Let $F \subset E \subset \bar{F}$ be field extensions. Let $\mathrm{Aut}(E/F)$ be the collection of automorphisms of K which fix F.

Theorem 1.4.37. Let $F \subset E \subset \bar{F}$ be field extensions, the following statements are equivalent:

(1) E/F is normal.

(2) every F-algebra homomorphism from E to $\bar{F}$ is a F-algebra homomorphism from E to E.

Moreover, if $[E : F] < \infty$, then the above statements are equivalent to that E is a splitting field of some $p(x) \in F[x]$.

Proof: (1)$\Longrightarrow$(2) is clear.

(2)$\Longrightarrow$(1): By Lemma 1.4.15

Now suppose $[E : F] < \infty$. First we assume $F \subseteq E$ is normal and choose $u_1 \in E - F$. Then its minimal polynomial is P_{u_1} and $[E : F(u_1)] < [E : F]$. Next we choose $u_2 \in E - F(u_1)$. Continuing this process, we conclude $E = F(u_1, \ldots, u_n)$. Let $P = \prod_{i=1}^n P_{u_i}$, and then E is the splitting field of P.

On the other hand, if E is the splitting field of $P \in F[X]$ whose roots in $\bar{F}$ are $\{u_1, \ldots, u_n\}$. Then $E = F(u_1, \ldots, u_n)$. Consider an F-algebra homomorphism $\iota : F(u_1, \ldots, u_n) \to \bar{F}$, since $\iota(u_i)$ is a root of P as well, $\iota(u_i) \in E$. Hence $\iota(E) \subseteq E$.

Proposition 1.4.38. Given field extensions $F \subset E \subset \bar{F}$, then all F-algerba homomorphisms from E to E are in $\mathrm{Aut}(E/F)$ i.e. $\mathrm{Aut}(E/F) = \{F\text{-algebra homomorphism between } E \text{ and } E\}$

Proof: Given any F-algebra homorphism $\tau : E \to E$, we know it's injective and it' enough to prove it's surjective. We assume $u \in E$ and $P \in F[x]$ is its minimal polynomial over F. If $u_1, \ldots, u_n$ are its different roots in $\bar{F}$, we assume only $u_1, \ldots, u_r$ are in E. Then $u \in \{u_1, \ldots, u_r\}$. Since τ fixes $F, \tau(u_i)$ is also a root of P in E where $1 \leq i \leq r$. Then $\tau : \{u_1, \ldots, u_r\} \to \{u_1, .., u_r\}$. That τ is injective implies it's surjective on this subset as well, which means $\exists u_i, \tau(u_i) = u$.

Proposition 1.4.39. For field extensions $F \subseteq E \subseteq K \subseteq \bar{F}$, if K/F is normal then K/E is normal.

Proposition 1.4.40. If E/F and E'/F are normal, then $E \cdot E'/F$ is normal.

1.4.4 Galois

Definition 1.4.41. Let E/F be a finite extension. Then E is said to be Galois over F and E/F is a Galois extension if it is separable and normal.

Theorem 1.4.42 (Fundamental Theorem of Galois Theory)**.** $F \subset K \subset \bar{F}$ be field extensions. K/F be a Galois extension and set $G = \mathrm{Gal}(K/F)$. Then there is a bijection:

$$\{\text{subfield of } K \text{ containing } F\} \longleftrightarrow \{\text{subgroup of } G\}$$

given by the correspondences

$$E \longrightarrow \{\text{elements of } G \text{ fixing } E\}$$

$$\text{fix field of } H \longleftarrow H$$

which are inverse to each other. Under this correspondence,

(1) K/E is always Galois.

(2) there's a one-to-one correspondence:

$$\begin{array}{ccc} \{F\text{-algebra homomorphism between } E \text{ and } \bar{F}\} & & \\ \uparrow{\scriptstyle \sigma H \mapsto \sigma|_E} & \searrow^{\text{Extended by 1.4.15 and 1.4.38}} & \\ \{\text{left cosets of } H \text{ in } G\} & \xrightarrow{\sigma H \mapsto \sigma|_E} & \{\sigma|_E : \sigma \in G\} \end{array}$$

(3) (inclusion reversing) If E_1, E_2 correspond to H_1, H_2, respectively, then $E_1 \subseteq E_2$ if and only if $H_2 \leq H_1$

(4) $[K : E] = |H|$ and $[E : F] = [G : H]$

(5) For all $\sigma \in G$,

$$\sigma(E) \longleftrightarrow \sigma H \sigma^{-1}$$

In particular, by Theorem 1.4.37, E is normal(hence Galois) over F if and only if H is a normal subgroup in G. If this is the case, then the Galois group is isomorphic to the quotient group

$$\mathrm{Gal}(E/F) \cong G/H$$

(6) If E_1, E_2 correspond to H_1, H_2, respectively, then the intersection $E_1 \cap E_2$ corresponds to the group $(H_1, H_2\ )$ generated by H_1 and H_2 and the composite field E_1E_2 corresponds to the intersection $H_1 \cap H_2$.

In the following statements, we fix a algebraic closure of F, and K, F', K_1, K_2 containing F are subfield of $\bar{F}$.

Theorem 1.4.43. Suppose K/F is a Galois extension and F'/F is any extension. Then KF'/F' is a Galois extension, with Galois group

$$\mathrm{Gal}\,(KF'/F') \cong \mathrm{Gal}\,(K/K \cap F')$$

isomorphic to a subgroup of $\mathrm{Gal}(K/F)$.

Corollary 1.4.44. Suppose K/F is a Galois extension and F'/F is any finite extension. Then

$$[KF' : F] = \frac{[K : F]\,[F' : F]}{[K \cap F' : F]}$$

Theorem 1.4.45. Let K_1 and K_2 be Galois extensions of a field F. Then

(1) The intersection $K_1 \cap K_2$ is Galois over F.

(2) The composite K_1K_2 is Galois over F. The Galois group is isomorphic to the subgroup

$$H = \left\{(\sigma, \tau) | \sigma|_{K_1 \cap K_2} = \tau|_{K_1 \cap K_2}\right\}$$

of the direct product $\mathrm{Gal}\,(K_1/F) \times \mathrm{Gal}\,(K_2/F)$ consisting of elements whose restrictions to the intersection $K_1 \cap K_2$ are equal.

Corollary 1.4.46. $F \subseteq E \subseteq F_{sep} \subseteq \bar{F}$ be finite separable extension, there's Galois extension K_1 contains E(for example, the composite of the splitting fields of the minimal polynomials for a basis for E over F). Take S be the set of all the Galois extenison of F which contains E, then

$$\bar{E} = \bigcap_{K \in S} K = \bigcap_{K \in S} (K \cap K_1)$$

is acturally finite many intersection of Galois extenison of F which contains E by Fundamental Theorem of Galois Theory.

Hence, there's minimal Galois extension of F that contains E.

Corollary 1.4.47. If K/F is finite and separable, then K/F is simple. In particular, any finite extension of fields of characteristic 0 is simple.

Proposition 1.4.48. $\bar{F}/F_{sep}$ is pruely inseparable extension and F_{sep} is separable and normal extension.

Proof: By characterizations of purely inseparable extension and definition of normal extension.

Definition 1.4.49.

A subset $\{a_1, a_2, \ldots, a_n\}$ of E is called algebraically independent over F if there is no nonzero polynomial

$$f(x_1, x_2, \ldots, x_n) \in F[x_1, x_2, \ldots, x_n]$$

such that $f(a_1, a_2, \ldots, a_n) = 0$. An arbitrary subset S of E is called algebraically independent over F if every finite subset of S is algebraically independent. The elements of S are called independent transcendentals over F.

A transcendence base for E/F is a maximal subset (with respect to inclusion) of E which is algebraically independent over F.

The extension E/F has a transcendence base and any two transcendence bases of E/F have the same cardinality.

Definition 1.4.50. The cardinality of a transcendence base for E/F is called the transcendence degree of E/F.

Proposition 1.4.51. E/F be a field extension, $\alpha_1, \ldots, \alpha_n \in E$, $F_i = F(a_1, \ldots, a_i)$, then $\{\alpha_1, \ldots, \alpha_n\}$ is algebracally independent over F, if and only if α_i is transcendental over F_{i-1} for all $i = 1, \ldots, n$.

Chapter 2

Commutative Algebra

2.1 Module

Proposition 2.1.1. A R-module M can be view as a ring homomorphism from R to endmorphism ring of M(as an abelian group) which is in general not necessarily commutative:

$$R \to \mathrm{End}(M)$$
$$r \to (x \to rx)$$

Conversely, if M is an abelian group, Given a ring homomorphism $f : R \to End(M)$, we have

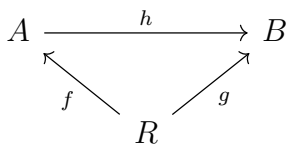
$$R \times M \to M$$
$$(r, m) \to f(r)m$$

is a R-module structure.

Remark 2.1.2. By Proposition 2.1.1, if we have a B-mdule M and a ring homomorphism $f : A \to B$, M has naturally a A-module structure.

Definition 2.1.3. $f : R \to B$ is a ring homomorphism, then B naturally has a R-module structure, we call B(with both a ring structure and A-module sturcte) a R-algebra.

And the morphism in R-algerba category between object $(A, f : R \to A)$ and $(B, g : R \to B)$, is the ring homomorphism $h : A \to B$ making the following diagram commute:



Definition 2.1.4 (annihilator)**.** Let A be a ring and M an A-module. Given submodules N, N' of M, the set $\{a \in A : aN' \subset N\}$ is an ideal of A, which we write $(N : N')_A$ Similarly, if I is an ideal then $\{x \in M : Ix \subset N\}$ is a submodule of M, which we write $(N : I)_M$.

For $a \in A$ we define $(N : a)_M$ to be $(N : (a))_M$.The ideal $(0 : M)_A$ is called the Annihilator of M, and written Ann(M). We can consider M as a module over $A/\mathrm{Ann}(M)$. If $\mathrm{Ann}(M) = 0$, we say that M is a faithful A-module. For $x \in M$, we write $\mathrm{Ann}(x) = \{a \in A : ax = 0\}$.

Definition 2.1.5 (finite module)**.** If M is finitely generated as an A-module, we say simply that M is a finite A-module, or is finite over A.

Definition 2.1.6 (finite ring homomorphism)**.** A ring map $R \to A$ is said to be finite if A is finite as an R-module.

Theorem 2.1.7 (Nakayama's lemma)**.** Let M be a finite A-module and I an ideal of A. If $M = IM$ then there exists $a \in A$ such that $aM = 0$ and $a \equiv 1 (\mathrm{mod}\, I)$. If in addition $I \subset rad(A)$, then $M = 0$.

Corollary 2.1.8. (A, m) be a Notherian local ring. If $A = mA$, then $A = 0$.

Corollary 2.1.9. Let A be a ring and I an ideal contained in $rad(A)$. Suppose that M is an A-module and $N \subset M$ a submodule such that M/N is finite over A. Then $M = N + IM$ implies $M = N$.

Proof: Consider the identity $M/N = I(M/N)$, then use Theorem 2.1.7.

In the following theorems, R is not necessarily be commutative, but we always assume R has an identity.

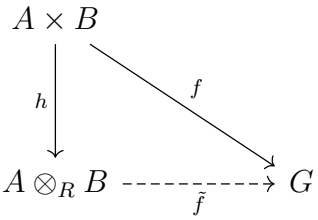
Definition 2.1.10. Let R be a ring, let A_R be a right R-module, let ${}_RB$ be a left R module, and let G be an (additive) abelian group. A function $f : A \times B \to G$ is called R-biadditive if, for all $a, a' \in A, b, b' \in B$, and $r \in R$, we have

$$\begin{aligned} f(a + a', b) &= f(a, b) + f(a', b), \\ f(a, b + b') &= f(a, b) + f(a, b'), \\ f(ar, b) &= f(a, rb). \end{aligned}$$

If R is commutative and A, B, and M are R-modules, then a function $f : A \times B \to M$ is called R-bilinear if f is R-biadditive and also

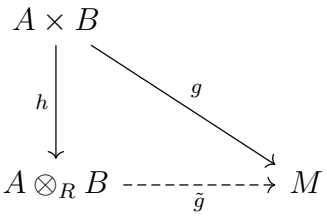
$$f(ar, b) = f(a, rb) = rf(a, b)$$

Definition 2.1.11 (Tensor product)**.** Given a ring R and modules A_R and ${}_RB$, then their tensor product is an abelian group $A \otimes_R B$ and an R-biadditive function $h : A \times B \to A \otimes_R B$



such that, for every abelian group G and every R-biadditive $f : A \times B \to G$, there exists a unique $\mathbb{Z}$-homomorphism $\tilde{f} : A \otimes_R B \to G$ making the following diagram commute.

Proposition 2.1.12. If R is a commutative ring and A, B are R-modules, then $A \otimes_R B$ is an R-module($r(a \otimes b) = (ra \otimes b)$), the function $h : A \times B \to A \otimes_R B$ is R-bilinear, and, for every R-module M and every R-bilinear function $g : A \times B \to M$, there exists a unique R-homomorphism $\tilde{g} : A \otimes_R B \to M$ making the following diagram commute.



Proposition 2.1.13. A is a ring, I is an ideal of A, M is a A-module, then $M \otimes_A (A/I) \simeq M/IM$ as A/I-module.

Proposition 2.1.14. If R is a ring, and ${}_{A,R}B$ are R-modules, then there are R-module isomorphisms:

$$A \otimes_R R \simeq A, \quad R \otimes_R B \simeq B$$

Theorem 2.1.15. If R and S are rings and A_R, ${}_RB_S$, S_C are (bi)modules, then there is an isomorphism:

$$(A \otimes_R B) \otimes_S C \simeq A \otimes_R (B \otimes_S C).$$

Theorem 2.1.16 (Commutativity)**.** If R is a commutative ring and M_R, ${}_RN$ are modules, then there is a R-isomorphism

$$\tau : M \otimes_R N \to N \otimes_R M$$

with $\tau : m \otimes n \mapsto n \otimes m$. The map τ is natural in the sense that the following diagram commutes:

$$\begin{array}{ccc} M \otimes_R N & \xrightarrow{\quad\tau\quad} & N \otimes_R M \\ \Big\downarrow{\scriptstyle f\otimes g} & & \Big\downarrow{\scriptstyle g\otimes f} \\ M' \otimes_R N' & \xrightarrow[\tau']{} & N' \otimes_R M' \end{array}$$

Theorem 2.1.17. Let R be a ring, A,$\{A_i\}_{i\in I}$ are right R-modules, B and $\{B_j\}_{j\in J}$ left R-modules. Then there are group isomorphisms:

$$\left(\bigoplus_{i\in I} A_i\right) \otimes_R B \simeq \bigoplus_{i\in I} (A_i \otimes_R B)$$
$$A \otimes_R \left(\bigoplus_{j\in J} B_j\right) \simeq \bigoplus_{j\in J} (A \otimes_R B_j)$$

Theorem 2.1.18 (Adjoint Associativity)**.** Let R and S be rings, let A be a right R-module, let B be an (R, S)-bimodule and let C be a right S-module. Then there is an natural bijection(acturally a isomorphism of abelian groups):

$$\mathrm{Hom}_S (A \otimes_R B, C) \cong \mathrm{Hom}_R (A, \mathrm{Hom}_S(B, C))$$

given by

$$\alpha : f \in \mathrm{Hom}_S (A \otimes_R B, C) \mapsto (a \mapsto (\Phi : b \mapsto f(a \otimes b)))$$

and

$$\beta : g \in \mathrm{Hom}_R (A, \mathrm{Hom}_S(B, C)) \mapsto (a \otimes b \mapsto g(a)(b))$$

Remark 2.1.19. 'natrual' in above theorem means: ${}_RB_S$ is a bi-module, then $(\_\_ \otimes_R B, \mathrm{Hom}_S(B, \_\_))$ is a adjoint pair between right R-module category and right S-module category.

Remark 2.1.20. (1) If ${}_RB_S$ is a bi-module, C is a right R-module, $\mathrm{Hom}_S(B, C)$ has a natrual right R-module sturct. Notice that we can define $fr(b) = f(rb)$, then $fr(bs) = f(r(bs)) = f((rb)s) = f(rb)s = (fr(b))s$,$f(r_1r_2)(b) = (fr_1)r_2(b)$. It makes $\mathrm{Hom}_S(B, C)$ to be a right R-module.

(2) If ${}_SB_R$ is a bi-module, C is a left S-module, then $\mathrm{Hom}_S(B, C)$ has a natrual left R-module sturct.

(3) If ${}_SB_R$ is a bi-module, C is a left S-module, then $B \otimes_R A$ has a natrual left S-module structure.

Proposition 2.1.21. If M is a left R-module, then there's left R-module isomorphism

$$\mathrm{Hom}_R(R, M) \simeq M$$

Theorem 2.1.22. Let R be a ring with. If A is a right R-module and F is a free left R-module with basis Y, thell every element u of $A \otimes_R F$ may be written uniquely in the form $u = \sum_{i=1}^{n} a_i \otimes y_i$, where $a_i \in A$ and the y_i are distinct elements of Y.

Theorem 2.1.23. If R is a ring with identity and A_R and ${}_RB$ are free R-modules with bases X and Y respectively, then $A \otimes_R B$ is a free (right) R-module$((a \otimes b)r = ar \otimes b)$ with basis $W = \{x \otimes y : x \in X, y \in Y\}$.

Proposition 2.1.24. If k is a commutative ring and A and B are k-algebras, then the tensor product $A \otimes_k B$ is a k-algebra if we define

$$(a \otimes b)\,(a' \otimes b') = aa' \otimes bb'.$$

Lemma 2.1.25 (The Short Five Lemma)**.** Let R be a ring and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & N' & \xrightarrow{u'} & N & \xrightarrow{v'} & N'' & \longrightarrow & 0 \end{array}$$

a commutative diagram of R-modules and R-module homomorphisms such that each row is a short exact sequence. Then

(1) α, γ monomorphisms $\Rightarrow \beta$ is a monomorphism(injective);

(2) α, γ epimorphisms $\Rightarrow \beta$ is an epimorphism(surjective);

(3) α, γ isomorphisms $\Rightarrow \beta$ is an isomorphism.

Definition 2.1.26 (Spilt exact sequence)**.** Let R be a ring and $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ a short exact sequence of R-module homomorphisms. Then the following conditions are equivalent:

(1) There is an R-module homomorphism $h : A_2 \to B$ with $gh = 1_{A_2}$;

(2) There is an R-module homomorphism $k : B \to A_1$ with $kf = 1_{A_1}$;

(3) the given sequence is isomorphic (with identity maps on A_1 and A_2) to the direct sum short exact sequence $0 \to A_1 \xrightarrow{l_1} A_1 \oplus A_2 \xrightarrow{\pi_2} A_2 \to 0$; in particular $B \simeq A_1 \oplus A_2$.

(4)

$$0 \to \mathrm{Hom}_R(D, A) \xrightarrow{\bar{f}} \mathrm{Hom}_R(D, B) \xrightarrow{\bar{g}} \mathrm{Hom}_R(D, C) \to 0$$

is a spilt exact sequence of abelian groups for all R-module D.

(5)

$$0 \leftarrow \mathrm{Hom}_R(A, J) \xleftarrow{\bar{f}} \mathrm{Hom}_R(B, J) \xleftarrow{\bar{g}} \mathrm{Hom}_R(C, J) \to 0$$

is a spilt exact sequence of abelian groups for all R-module D.

A short exact sequence that satisfies the equivalent conditions is said to be split or a split exact sequence.

Lemma 2.1.27 (Snake lemma)**.** Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & N' & \xrightarrow{u'} & N & \xrightarrow{v'} & N'' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of A-modules and homomorphisms, with the rows exact. Then there exists an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(f') & \xrightarrow{\bar{u}} & \mathrm{Ker}(f) & \xrightarrow{\bar{v}} & \mathrm{Ker}(f'') & & \\ & & & & & \searrow^{d} & & & \\ & & \mathrm{Coker}(f') & \xrightarrow[\bar{u}']{} & \mathrm{Coker}(f) & \xrightarrow[\bar{v}']{} & \mathrm{Coker}(f'') & \longrightarrow & 0 \end{array}$$

in which $\bar{u}, \bar{v}$ are restrictions of u, v, and $\bar{u}', \bar{v}'$ are induced by u', v'. The boundary homomorphism d is defined as follows: if $x'' \in \mathrm{Ker}\,(f'')$, we have $x'' = v(x)$ for some $x \in M$, and $v'(f(x)) = f''(v(x)) = 0$, hence $f(x) \in \mathrm{Ker}\,(v') = \mathrm{Im}\,(u')$, so that $f(x) = u'\,(y')$ for some $y' \in N'$. Then $d\,(x'')$ is defined to be the image of y' in Coker (f').

Proposition 2.1.28.

(1)

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C$$

is any short exact sequence of R-modules, if and only if for all R-module D

$$0 \to \mathrm{Hom}_R(D, A) \xrightarrow{\bar{f}} \mathrm{Hom}_R(D, B) \xrightarrow{\bar{g}} \mathrm{Hom}_R(D, C)$$

is an exact sequence of abelian groups.($\mathrm{Hom}(D, \square)$ is left exact in R-module category).

(2)

$$0 \leftarrow C \xleftarrow{g} B \xleftarrow{f} A$$

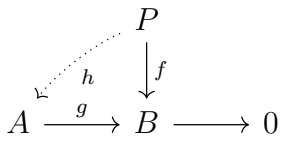
is any short exact sequence of R-modules, is any short exact sequence of R-modules, if and only if for all R-module D

$$0 \to \mathrm{Hom}_R(C, D) \xrightarrow{\bar{g}} \mathrm{Hom}_R(B, D) \xrightarrow{\bar{f}} \mathrm{Hom}_R(A, D)$$

is an exact sequence of abelian groups.($\mathrm{Hom}(\square, D)$ is left exact in the opposite category of R-mdoule category).

Definition 2.1.29 (Projective module)**.** Let R be a ring. The following conditions on an R-module P are equivalent.

(1) given a diagram as follow with row exact, there's h making the diagram commute.



(2) every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ is split exact.

(3) there is a free module F and an R-module K such that $F \cong K \oplus P$.(summand of free module)

(4) if $f : B \to C$ is any R-module epimorphism then $\bar{f} : \mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C)$ is an epimorphism of abelian groups;

(5) if

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is any short exact sequence of R-modules, then

$$0 \to \mathrm{Hom}_R(P, A) \xrightarrow{\bar{f}} \mathrm{Hom}_R(P, B) \xrightarrow{\bar{g}} \mathrm{Hom}_R(P, C) \to 0$$

is an exact sequence of abelian groups.($\mathrm{Hom}(P, \square)$ is exact in Category of R-mdoule)

Proposition 2.1.30. Every free module F over a ring R is projective.

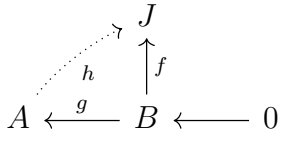
Proposition 2.1.31. Let R be a ring. A direct sum of R-modules $\sum_i P_i$ is projective if and only if each P_i is projective.

Proposition 2.1.32. If R is commutative then the tensor product of two projective R-modules (with a natural R-module structure) is projective.

Proof: By Adjoint Associativity.

Definition 2.1.33 (Injective module)**.** Let R be a ring, the following conditions on a R-module J are equivalent:

(1) given a diagram as follow with row exact, there's h making the diagram commute.



(2) every short exact sequence $0 \to J \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split exact.

(3) If J is a submodule of B, then there's submodule K such that $B = J \oplus K$.

(4) if $f : B \to C$ is any R-module monomorphism then $\bar{f} : \mathrm{Hom}_R(A, J) \leftarrow \mathrm{Hom}_R(B, J)$ is an epimorphism of abelian groups;

(5) if

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is any short exact sequence of R-modules, then

$$0 \leftarrow \mathrm{Hom}_R(A, J) \xleftarrow{\bar{f}} \mathrm{Hom}_R(B, J) \xleftarrow{\bar{g}} \mathrm{Hom}_R(C, J) \to 0$$

is an exact sequence of abelian groups.

(6) for every left ideal L of R, any R-module homomorphism $L \to J$ can be extended to $R \to J$(Baer's Criterion)

Proposition 2.1.34. A direct product of R-modules $\prod_{i\in I} J_i$ is injective ifand only if J_i is injective for every $J_i, i \in I$.

Proposition 2.1.35. If R is a P.I.D., then Q is injective if and only if $rQ = Q$ for every nonzero $r \in R$.

Proof: By Baer's Criterion.

Proposition 2.1.36. Suppose that D is a right R-module and that L, M and N are left R-modules. If

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0 \text{ is exact,}$$

then the associated sequence of abelian groups

$$D \otimes_R L \xrightarrow{1\otimes\psi} D \otimes_R M \xrightarrow{1\otimes\varphi} D \otimes_R N \longrightarrow 0 \quad \text{ is exact.}$$

Proposition 2.1.37. Let R be a ring and let M be an R-module. Then M is contained in an injective R-module.

Proposition 2.1.38. Any modules over a PID, it is a projective module if and only if it is a free module.

Definition 2.1.39 (Flat module)**.** Let A be a right R-module. Then the following are equivalent:

(1) For any left R-modules L, M, and N, if

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is a short exact sequence, then

$$0 \longrightarrow A \otimes_R L \xrightarrow{1\otimes\psi} A \otimes_R M \xrightarrow{1\otimes\varphi} A \otimes_R N \longrightarrow 0$$

is also a short exact sequence.

(2) For any left R-modules L and M, if $0 \to L \xrightarrow{\psi} M$ is an exact sequence of left R-modules (i.e., $\psi : L \to M$ is injective) then $0 \to A \otimes_R L \xrightarrow{1\otimes\psi} A \otimes_R M$ is an exact sequence of abelian groups (i.e., $1 \otimes \psi : A \otimes_R L \to A \otimes_R M$ is injective).

We say A is flat right R-module if A satisfies one of above conditions. Similarly, we can define left flat R-module.

Proposition 2.1.40. Projective modules over a commutative ring R are flat.

Example 2.1.41. $\mathbb{Q}/\mathbb{Z}$ is not flat.

Proof: Since $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z} \simeq \mathbb{Q}/\mathbb{Z}$, we have $\frac{1}{2} + \mathbb{Z} \otimes 1$ is non-zero. Consider a exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$$

, tensor the exact sequence with $\mathbb{Q}/\mathbb{Z}$. Notice that $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{1\otimes(\times 2)} \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$ is not injective since $\frac{1}{2} + \mathbb{Z} \otimes 1$ in its kernel. Hence $\mathbb{Q}/\mathbb{Z}$ is not flat.

Proposition 2.1.42. $\sum_{i\in I} A_i$ flat if and only if each $A_i, i \in I$ flat.

Proof: Since tensor product commute with direct sum.

Example 2.1.43.

	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Q}/\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Q}$
flat	✓	✓(By 2.7.8)	×(2.1.41)	✓(2.1.42)
projective	✓	×(By 2.1.38)	×	×(2.1.31)
injective	×(By 2.1.35)	✓(By 2.1.35)	✓(By 2.1.35)	×(2.1.34)

2.2 Specturm

Proposition 2.2.1. Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E.

(1) if a is the ideal generated by E, then $V(E) = V(a) = V(\sqrt{a})$.

(2) $V(\varnothing) = X$,$V((1)) = \varnothing$

(3) if $(E_i)_{i\in I}$ is any family of subsets of A, then

$$V(E_i)_{i\in I} = \bigcap_{i\in I} V(E_i)$$

(4) $V(I \cap J) = V(IJ) = V(I) \cup V(J)$ for any ideals I,J of A.

These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the Zariski topology. The topological space X is called the prime spectrum of A, and is written Spec(A).

Proof: By Theorem 1.3.19

Proposition 2.2.2. $X = \text{Spec}(A)$, $D(f) = X - V(f)$.

(1) $D(f)$ form a basis of X.

(2) $D(fg) = D(f) \cap D(g)$.

(3) $D(f)$ is quasi-compact.

(4) $D(f) = \varnothing \Leftrightarrow f$ is a unit.

(5) $D(f) = X \Leftrightarrow f$ is nilpotent.

(6) Open subset of X is quasi-compact iff it is a finite union of $D(f)$.

The sets X_f are called basic open sets of X=Spec$A$

Proposition 2.2.3. It is sometimes convenient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of X =Spec$A$. When thinking of x as a prime ideal of A, we denote it by P_x. Show that:

(1) the set $\{x\}$ is closed in Spec(A) if and only if P_x is maximal.

(2) $\overline{\{x\}} = V(P_x)$

(3) $\overline{\{x\}}$ dense in X if and only if P_x equals to all the intersection of prime ideals of A.

Definition 2.2.4. A topological space X is said to be irreducible if $X \neq \varnothing$ and satisfies the following three equivalent conditions:

(1) every pair of non-empty open sets intersects.

(2) every non-empty open set is dense in X.

(3) X is not a union of two closed, proper, non-empty sets.

Proposition 2.2.5. Let X be a topological space.

(1) If Y is an irreducible subspace of X, then the closure Y of Y in X is irreducible.

(2) Every irreducible subspace of X is contained in a maximal irreducible subspace.

(3) The maximal irreducible subspaces of X are closed and cover X. They are called the irreducible components of X.

Proposition 2.2.6. A is a ring, Spec$A$ is the specturm of A.

There is a one-to-one order-reversing correspondence between the radical ideals($\sqrt{I} = I$) and the closed subsets of Spec$A$. More precisely,we can say there are three bijections

$$\{\text{radical ideals of } A\} \longleftrightarrow \{\text{closed subset of Spec}A\}$$

$$\{\text{prime ideals }\} \longleftrightarrow \{\text{irreducible closed subset}\}$$

$$\{\text{minimal ideals }\} \longleftrightarrow \{\text{irreducible components}\}$$

given by the correspondences

$$I \longrightarrow V(I)$$

$$\bigcap_{P \in E} P \longleftarrow V(E)$$

Corollary 2.2.7. $X = \text{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal.

Proposition 2.2.8. Let $\varphi : A \to B$ be a ring homomorphism. Let X =Spec$A$ and Y =Spec$B$. Let ϕ to be the map:

$$\phi : \text{Spec}B \to \text{Spec}A$$
$$P \mapsto \varphi^{-1}(P)$$

(1) If $f \in A$, then $\phi^{-1}(D(f)) = D(\varphi(f))$,and hence ϕ is continuous.

(2) I is an ideal of A, $\phi^{-1}(V(I)) = V(\varphi(I))$.

(3) J is an ideal of B, $\overline{\phi(V(J))} = V(\phi(J))$

(4) If φ is surjective, then ϕ is a homeomorphism of Y onto the closed subset $V(\text{Ker}(\phi))$ of X.

Definition 2.2.9. Let X be an arbitrary topological space.

(1) A point $x \in X$ is called closed if the set $\{x\}$ is closed,

(2) We say that a point $\eta \in X$ is a generic point if $\overline{\{\eta\}} = X$.

(3) Let x and x' be two points of X. We say that x is a generization of x' or that x' is a specialization of x if $x' \in \overline{\{x\}}$.

(4) A point $x \in X$ is called a maximal point if its closure $\overline{\{x\}}$ is an irreducible component of X.

(5) Thus a point $\eta \in X$ is generic if and only if it is a generization of every point of X. As the closure of an irreducible set is again irreducible, the existence of a generic point implies that X is irreducible.

Proposition 2.2.10. If $X = \operatorname{Spec} A$ is the spectrum of a ring, then

(1) A point $x \in X$ is closed if and only if $\mathfrak{p}_x$ is a maximal ideal.

(2) A point x is a generization of a point x' (in other words, x' is a specialization of x) if and only if $\mathfrak{p}_x \subseteq \mathfrak{p}_{x'}$.

(3) A point $x \in X$ is a maximal point if and only if $\mathfrak{p}_x$ is a minimal prime ideal.

(4) A point $\eta \in X$ is a generic point of X if and only if $\mathfrak{p}_\eta$ is the unique minimal prime ideal. This exists if and only if the nilradical of A is a prime ideal.

Definition 2.2.11. A topological space is called Noetherian if it satisfies one of the following equivalent conditions

(1) descending chain of closed subsets becomes stationary.

(2) ascending chain of open subsets becomes stationary.

(3) every non-empty set of open subsets of X has a maximal element.

(4) every non-empty set of closed subsets of X has a minmal element.

Example 2.2.12. R is a Noetherian ring, then $X = \operatorname{Spec}(R)$ is a Notherian space.

Proof: By Theorem 2.2.6

Theorem 2.2.13 (Decomposition into irreducibles)**.** Let X be a Noetherian topological space.

(1) There exist a nonnegative integer n and closed, irreducible subsets $Z_1, ..., Z_n \subset X$ such that $X = Z_1 \cup \cdots \cup Z_n$ and $Z_i \nsubseteq Z_j$ for $i \neq j$.

(2) If $Z_1, ..., Z_n$ are closed, irreducible subsets satisfying (1), then every irreducible subset $Z \subset X$ is contained in some Z_i.

(3) If $Z_1, ..., Z_n \subset X$ are closed, irreducible subsets satisfying (1), then they are precisely the irreducible components of X. In particular, the Z_i are uniquely determined up to order.

Proposition 2.2.14. Let X be a Notherian topological space.

(1) Every subspace of X is noetherian.

(2) Every subset of X is quasi-compact (in particular, X is quasi-compact).

Proof: (1): Let $(Z_i)_i$ be a descending chain of closed subsets of a subspace Y. Then the closures $\bar{Z}_i$ of Z_i in X form a descending chain of closed subsets of X which becomes stationary by hypothesis. As we have $Z_i = Y \cap \bar{Z}_i$, this shows that the chain $(Z_i)_i$ becomes stationary as well. This proves (1).

(2): By (1) it suffices to show that X is quasi-compact. Let $(U_i)_i$ be an open covering of X and let $\mathcal{U}$ be the set of those open subsets of X that are finite unions of the subsets U_i. As X is noetherian, $\mathcal{U}$ has a maximal element V. Clearly $V = X$, otherwise there would exist an U_i such that $V \subsetneq V \cup U_i \in \mathcal{U}$. This shows that $(U_i)_i$ has a finite subcovering.

Corollary 2.2.15. A Notherian ring has only finite many minimal prime ideals.

Proof: By Example 2.2.15 and Theorem 2.2.13.

Proposition 2.2.16. Let X be a topological space that has a finite covering $X = \bigcup_{i=1}^{r} X_i$ by noetherian subspaces. Then X itself is noetherian.

Proposition 2.2.17. Let X be a topological space and let $X = \bigcup_{i \in I} U_i$ be an open covering of X by connected open subsets U_i.

(1) If X is not connected, then there exists a subset $\emptyset \neq J \subsetneq I$ such that for all $j \in J$, $i \in I \backslash J, U_j \cap U_i = \emptyset$.

(2) If X is connected, I is finite, and all the U_i are irreducible, then X is irreducible.

Example 2.2.18. Let $A = R[T]$, where R is a principal ideal domain. We assume that R is not a field.

Let $X = \operatorname{Spec} R[T]$. By Gauss's Lemma, the prime elements of $R[T]$ are either of the form p, where p is a prime element of R, or of the form f, where $f \in R[T]$ is a primitive polynomial which is irreducible in $\operatorname{Frac}(R)[T]$.

If $p \in R$ is a prime element, R/pR is a field. The closure $V(pR[T])$ of $\{pR[T]\}$ is homeomorphic to $\operatorname{Spec}(R/pR)[T]$, and $(R/pR)[T]$ is a principal ideal domain with infinitely many nonequivalent prime elements. We see that $(pR[T])$ is not a maximal ideal, and the prime ideals in $V(pR[T])$ different from $(pR[T])$ are the maximal ideals generated by p and f where $f \in R[T]$ is a polynomial such that its image in $(R/pR)[T]$ is irreducible.

As for the prime ideal genereted by f, if the leading coefficient of f is a unit in R, $R[T]/fR[T] \simeq R^{\deg f}$ by division with remainders in $\operatorname{Frac}(R)[T]$. Hence (f) is not a maximal ideal.

If P is a non-zero prime ideal with $P \cap R = 0$, then take $f \in P$ be a primitive irrducible polynomial. If $(f) \neq P$, take primitive irreducible polynomial $g \in P$ such that $f \nmid g$. By Bezout Theorem, there's non-zero element a in R such that $a \in R$, a contradiction!

Example 2.2.19. $\mathrm{Spec}(A \times B) = \mathrm{Spec}(A) \sqcup \mathrm{Spec}(B)$.

Proof: Take an element $\mathfrak{p}$ of $\mathrm{Spec}(A \times B)$ and define

$$\mathfrak{p}_A = \pi_A(\mathfrak{p}) \quad \text{and} \quad \mathfrak{p}_B = \pi_B(\mathfrak{p}),$$

where π_A and π_B are the projections. Note that $\mathfrak{p}_A$ is an ideal that satisfies $aa' \in \mathfrak{p}_A \Longrightarrow a \in \mathfrak{p}_A$ or $a' \in \mathfrak{p}_A$. Thus, if $\mathfrak{p}_A$ is proper, it is prime in A. Similarly for $\mathfrak{p}_B$. It follows that $\mathfrak{p}_A \times B$ (resp. $A \times \mathfrak{p}_B$) is prime when $\mathfrak{p}_A$ (resp. $\mathfrak{p}_B$) is proper.

Claim: One, and only one, of $\mathfrak{p}_A$ and $\mathfrak{p}_B$ is proper. Indeed. Suppose that $1 \notin \mathfrak{p}_A$. Pick $b \in \mathfrak{p}_B$. There exists $a \in A$ such that $(a, b) \in \mathfrak{p}$. Therefore, $(a, 1)(1, b) = (a, b) \in \mathfrak{p}$. But $(1, b) \notin \mathfrak{p}$, or we would have $1 \in \mathfrak{p}_A$, which is not. Hence, $(a, 1) \in \mathfrak{p}$, implying that $1 \in \mathfrak{p}_B$. In other words, not both of them are proper. Moreover, in case $\mathfrak{p}_A$ is proper, we also have $\mathfrak{p}_A \times B \subseteq \mathfrak{p}$. Conversely, if $(a, b) \in \mathfrak{p}$, then $a \in \mathfrak{p}_A$ and so $(a, b) \in \mathfrak{p}_A \times B$, i.e., equality is attained. Finally, suppose that $\mathfrak{p}_A = A$ and $\mathfrak{p}_B = B$. Then $(1, b) \in \mathfrak{p}$ for some $b \in B$ and $(a, 1) \in \mathfrak{p}$ for some $a \in A$. It follows that $(1, 1) = (1, 0)(1, b) + (0, 1)(a, 1) \in \mathfrak{p}$, which is impossible. The claim is now clear.

2.3 Chain conditions

Definition 2.3.1 (Notherian)**.** Ring(R-module) A is said to be Noetherian if it satisfies the following three equivalent conditions:

(1) Every non-empty set of ideals(submodules) in A has a maximal element.

(2) Every ascending chain of ideals(submodules) in A is stationary.

(3) Every ideal(submodule) is finitely generated.

Definition 2.3.2 (Artinian)**.** Ring(R-module) A is said to be Artinian if it satisfies the following three equivalent conditions:

(1) Every non-empty set of ideals(submodules) in A has a minimal element.

(2) Every decending chain of ideals(submodules) in A is stationary.

Theorem 2.3.3. Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be an exact sequence of A-modules. Then

1. M is Noetherian $\Leftrightarrow$ M' and M'' are Noetherian;
2. M is Artinian $\Leftrightarrow$ M' and M'' are Artinian.

Corollary 2.3.4. If $M_i (1 \leqslant i \leqslant n)$ are Noetherian (resp. Artinian) A-modules, so is

$$\bigoplus_{i=1}^{n} M_i$$

.

Proof: Apply Theorem 2.3.3 to the exact sequence

$$0 \to M_n \to \bigoplus_{i=1}^{n} M_i \to \bigoplus_{i=1}^{n-1} M_i \to 0$$

Corollary 2.3.5. Let A be a Noetherian (resp. Artinian) ring, M a finitely generated A-module. Then M is Noetherian (resp. Artinian).

Definition 2.3.6 (length of module)**.** Let R be a ring. For any R-module M we define the length of M over R by the formula

$$\lg_A(M) = \text{length}_R(M) = \sup \{n \mid \exists 0 = M_0 \subset M_1 \subset \ldots \subset M_n = M, M_i \neq M_{i+1}\}$$

In other words it is the supremum of the lengths of chains of submodules.

Proposition 2.3.7 (finite length)**.** If the length of M is finite, then every chain can be refined to a maximal chain and all the maximal chains have the same length.

Proposition 2.3.8 (Artinian Ring)**.** Let A be a ring. The following are equivalent:

(1) The ring A is Artinian, i. e. it satisfies the descending chain condition for ideals.

(2) $\lg_A(A) < \infty$.

(3) The ring A is noetherian, and every prime ideal is a minimal prime ideal.

(4) The ring A is noetherian, and the natural map

$$A \to \prod_{\mathfrak{p} \in \operatorname{Spec} A} A_{\mathfrak{p}}$$

is an isomorphism.

(5) The ring A is noetherian and for all prime ideals $\mathfrak{p}$ of A, the localization $A_{\mathfrak{p}}$ is Artinian.

If A is Artinian, A has only finitely many prime ideals.

Example 2.3.9. $A = \mathbb{C}[x, y]/(x^2, xy, y^2)$ is an Artinian ring with infinite many ideals.

Proof: If $\mathfrak{p}$ is a prime ideal contains $I = (x^2, xy, y^2)$, we have $\mathfrak{p} \supset (x, y)$. Since $\mathfrak{p} \neq \mathbb{C}[x, y]$, $\mathfrak{p} = (x, y)$. On the other hand, notice that $(x^2, xy, y^2, x + ay), a \in \mathbb{C}$ are distinct ideals cotains I.

Proposition 2.3.10. Let A be a ring and M an A-module. Then if M is a Noetherian module, $A/\mathrm{Ann}(M)$ is a Noetherian ring.

Proof: If we set $\bar{A} = A/\mathrm{Ann}(M)$ and view M as an $\bar{A}$-module, then the submodules of M as an A-module or $\bar{A}$-module coincide, so that M is also Noetherian as an $\bar{A}$-module. We can thus replace A by $\bar{A}$, and then $\mathrm{Ann}(M) = (0)$. Now letting $M = A\omega_1 + \cdots + A\omega_n$, we can embed A in M^n by means of the map $a \mapsto (a\omega_1, \ldots, a\omega_n)$. By Theorem 1, M^n is a Noetherian module, so that its submodule A is also Noetherian.

Theorem 2.3.11 (Hilbert basis theorem)**.** R is Notherian, then $R[x]$ and $R[[x]]$ are Notherian.

Corollary 2.3.12. Let B be a finitely-generated A-algebra. If A is Noetherian, then so is B.

Proof: By Hilbert basis theorem and Theorem 2.3.3.

2.4 Localization

Definition 2.4.1 (Localization of Ring)**.** Let R be a ring, and S a multiplicative subset. Define a relation on $R \times S$ by $(x, s) \sim (y, t)$ if there is $u \in S$ such that $xtu = ysu$.Denote by $S^{-1}R$ the set of equivalence classes, and by $x/$ the class of (x, s)

It is easy to check that $S^{-1}R$ is a ring, with $0/1$ for 0 and $1/1$ for 1. It is called the ring of fractions with respect to S or the localization at S.

Let $\varphi_S : R \to S^{-1}R$ be the map given by $\varphi_S(x) = x/1$. Then φ_S is a ring homomorphism between R and $S^{-1}R$

Example 2.4.2 (Localization at a prime ideal)**.** Let R be a ring, p be a prime ideal. Set $S_p := R - p$. We call the ring $S_p^{-1}R$ the localization of R at p, and set $R_p := S_p^{-1}R$, $\varphi_p = \varphi_{S_p}$.

Example 2.4.3 (Localization at a element)**.** Let R be a ring, $f \in R$. Set $S_f := \{f^n : n \geq 0\}$. We call the ring $S_f^{-1}R$ the localization of R at f, and set $R_f := S_f^{-1}R$ and $\varphi_f := \varphi_{S_f}$.

Example 2.4.4. Let $f : A \to B$ be a ring homomorphism, S be a multiplicative subset of A, then denote $f(S)$ is a multiplicative subset of B. Denote the localization at $f(S)$ by $S^{-1}B$. Respectively, if P is a prime ideal of A, denote the localization at $S = f(A - P)$ by B_P.

Proposition 2.4.5. Every ideal in $S^{-1}A$ of the form $S^{-1}I$.

Proof: Notice that if $\bar{I}$ is an ideal of $S^{-1}A$, then $S^{-1}\varphi_S^{-1}(\bar{I}) = \bar{I}$.

Proposition 2.4.6. A is Notherian, then $S^{-1}A$ is Notherian.

Proposition 2.4.7. Let R be a ring, S be a multiplicative subset of R,
$S^{-1}I = \{x/s : s \in I, s \in S\}$. Then $S^{-1}I$ is the ideal generated by $\varphi_S(I)$, and the following conditions are equivalent:

(1) $S^{-1}I = S^{-1}R$

(2) $I \cap S \neq \varnothing$

(3) $\varphi_S^{-1}(S^{-1}I) = R$

Proof: Obviously, $S^{-1}I$ is the ideal generated by $\varphi_S(I)$.
(1)⇒(2):Consider $1/1 \in S^{-1}I$.
(2)⇒(3):Take $a \in I \cap S$, notice that $a/a = 1/1$.
(3)⇒(1):Consider $1/1 \in S^{-1}I$.

Proposition 2.4.8. Let R be a ring, S be a multiplicative subset of R, there's a one-to-one order-preserving bijection:

$$\{P \in \mathrm{Spec}R : P \cap S = \varnothing\} \longleftrightarrow \mathrm{Spec}(S^{-1}R)$$

given by the following maps:

$$P \longrightarrow S^{-1}P$$

$$\varphi_S^{-1}(\bar{P}) \longrightarrow \overline{P} \in \mathrm{Spec}(S^{-1}R)$$

Proof: Step 1 (well-defined): If $P \in \mathrm{Spec}(R)$ and $P \cap S = \varnothing$, then $S^{-1}P$ is a prime of $S^{-1}R$.

Step 2 (injective): $\varphi_S^{-1}(S^{-1}P) = P$.

Step 3 (surjective): Let J be a prime ideal of $S^{-1}R$, then $P = \varphi_S^{-1}(J)$ is a prime ideal of R. We show that $S^{-1}P = J$. For all $x/s \in J$, since J is an ideal, $x/1 = x/s \times s/1 \in J$, hence $x \in P$ and $x/s \in S^{-1}P$. It is clear that $\varphi_S(\varphi_S^{-1}(J)) \subset J$. Hence, we have $J = S^{-1}P$.

Definition 2.4.9 (Localization of Module)**.** The construction of $S^{-1}A$ can be carried through with an A-module M in place of the ring A. Define a relation $=$ on $M \times S$ as follows: $(m, s) = (m', s')$ if and only if there's $t \in S$ such that $t(sm' - s'm) = 0$.

In particular, if P is a prime ideal of A, $S = A - P$, we call $M_P = S^{-1}M$ the localization at P.

Proposition 2.4.10. $S^{-1}M$ has both A-module structure and $S^{-1}A$-module structure by the natrual way:

$$S^{-1}A \times S^{-1}M \to S^{-1}M$$

$$(a/s, m/s_1) \to am/(ss_1)$$

$$A \times S^{-1}M \to S^{-1}M$$

$$(a, m/s_1) \to a/(ss_1)$$

Let $f : M \to N$ be an A-module homomorphism. Then it gives rise to an $S^{-1}A$-module and A-module homomorphism:

$$S^{-1}M \to S^{-1}N$$

$$m/s_1 \to f(m)/s$$

And, if $M \xrightarrow{f} N \xrightarrow{g} P$ is exact, then $S^{-1}M \xrightarrow{S^{-1}f} S^{-1}N \xrightarrow{S^{-1}g} S^{-1}P$ is exact.

Remark 2.4.11. It follows from Proposition 2.4.10 that if N is a submodule of M, the map $S^{-1}M \xrightarrow{S^{-1}f} S^{-1}M$ is injective, where $f : N \to M$ be the embeding. Therefore $S{-}1N$ can be regarded as a submodule of $S^{-1}M$.

Remark 2.4.12. If P is a prime ideal of A, $S = A - P$,$f : M \to N$ be a A-module homomorphism, we usually denote $S^{-1}f$ by f_P.

Proposition 2.4.13. If N, P are submodule of M,then

(1) $S^{-1}(N + P) = S^{-1}M + S^{-1}P$

(2) $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$

(3) the map $S^{-1}f : S^{-1}M \to S^{-1}(M/N)$ given by the natrual homomorphism $f : M \to M/N$ is an surjective.In particular,$S^{-1}M/S^{-1}N \simeq S^{-1}(M/N)$ as $S^{-1}A$-module and A-mdoule.

Theorem 2.4.14. Let M be an A-module. Then the $S^{-1}A$ modules $S^{-1}M$ and $S^{-1}A \otimes_A M$ are naturally isomorphic. The isomorphisc map is given by the bi-linear map:

$$S^{-1}A \times M \to S^{-1}M$$

$$\varphi : (a/s, m) \to am/s$$

and the universal property of tensor product.

Remark 2.4.15. 'natrually' in above theorem means: given two covariant functors:$S^{-1}A \otimes \_\_$ and $S^{-1}\_\_$, then the isomorphic map induced by φ induce a natrual transformation between these two functors.

Proposition 2.4.16 (localization commute with tensor product)**.** Let R be a ring, S a multiplicative subset, M, N modules. Show $S^{-1}(M \otimes_R N) \simeq S^{-1}M \otimes_{S^{-1}R} S^{-1}N$.

Proof:

$$S^{-1}(M \otimes_R N) \simeq S^{-1}R \otimes_R (M \otimes_R N) \simeq S^{-1}M \otimes_R N \simeq$$
$$(S^{-1}M \otimes_{S^{-1}A} S^{-1}A) \otimes_A N \simeq S^{-1}M \otimes_{S^{-1}R} S^{-1}N$$

Proposition 2.4.17 (M=0 is a local property)**.** Let M be an A-module. Then the following are equivalent:

(1) $M = 0$

(2) $M_P = 0$ for all prime ideals P.

(3) $M_m = 0$ for maximal ideals m.

Proposition 2.4.18 (injective homomorphism is a local property)**.** Let $f : M \to N$ be A-module homomorphism, $f_P : M_P \to N_P$ be homomorphism induced by prime ideal P. Then the following are equivalent:

(1) f is injective

(2) f_P is injective for all prime ideals P.

(3) f_m is injective for maximal ideals m.

Proposition 2.4.19 (flat is a local property)**.** Let $f : M \to N$ be A-module homomorphism, $f_P : M_P \to N_P$ be homomorphism induced by prime ideal P. Then the following are equivalent:

(1) f is flat A-module.

(2) f_P is flat A_P-module for all prime ideals P.

(3) f_m is flat A_m-module for all maximal ideals m.

Proposition 2.4.20. Let M be a finitely generated A-module, S a multiplicatively closed subset of A. Then $S^{-1}(\mathrm{Ann}(M)) = \mathrm{Ann}(S^{-1}M)$.

Definition 2.4.21 (support of a module)**.** Let A be a ring, M an A-module. The support of M is defined to be the set $\mathrm{Supp}(M) = \{P \in \mathrm{Spec}(A) : M_P \neq 0\}$.

Proposition 2.4.22. M is a R-module, I is an ideal of R.

(1) $M \neq 0 \Leftrightarrow \mathrm{Supp}(M) = \varnothing$

(2) $V(I) = \mathrm{Supp}(R/I)$

(3) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$.

(4) If M is finitely generated, then $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$

(5) If M, N are finitely generated, then $\mathrm{Supp}(M \otimes_A N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$.

(6) If $M = \sum_{i\in I} M_i$, then $\mathrm{Supp}(M) = \bigcap_{i\in I} \mathrm{Supp}(M_i)$

Proof:

(1):By Theorem 2.4.17

(2):By Proposition 2.4.13 and Proposition 2.4.7.

(3):By Theorem 2.4.10.

(4):Notice that $M_P \neq 0 \Leftrightarrow \mathrm{Ann}(M_P) \neq R_P$. Then it follows from Proposition 2.4.20 and Proposition 2.4.7.

(5):Since localization commute with tensor product, it suffice to show:

Lemma 2.4.23. M, N are finitely generated R-module, in which (R, m, k) be a local ring, $M \otimes_R N = 0$, then $M = 0$ or $N = 0$.

Proof of the lemma: Notice that $M \otimes_R R/m \simeq M/mM$. Hence, by Theorem 2.1.23, and Nakayama's lemma, define $M_k = M \otimes_R k$, it suffice to show $M_k \otimes_k N_k \simeq (M \otimes_R N)_k$ as k-vector space. By Theorem 2.1.22, we have an canonical isomorphism as vector space. □

(6):trivial.

Proposition 2.4.24 (universal property of localization)**.** Let $g : A \to B$ be a ring homomorphism such that $g(s)$ is a unit in B for all $s \in S$. Then there exists a unique ring homomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$.

Theorem 2.4.25. let A be a ring, $S \subset A$ a multiplicative set, I an ideal of A and $\bar{S}$ the image of S in A/I; then there's ring isomorphism

$$S^{-1}A/S^{-1}I \simeq \bar{S}^{-1}(A/I)$$

given by

$$a/s + S^{-1}I \mapsto a + I/(s + I)$$

In particular, if $\mathfrak{p}$ is a prime ideal of A then

$$A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \simeq (A/\mathfrak{p})_{\overline{A-\mathfrak{p}}}.$$

where $\mathfrak{p}A_{\mathfrak{p}}$ is the ideal generated by $\varphi_{\mathfrak{p}}(\mathfrak{p})$. The left-hand side is the residue field of the local ring $A_{\mathfrak{p}}$, whereas the right-hand side is the field of fractions of the integral domain $A/\mathfrak{p}$. This field is written $\kappa(\mathfrak{p})$ and called the residue field of $\mathfrak{p}$.

Proof: By theorem 2.4.13 and universal property of localization.

Theorem 2.4.26. Let A be a ring, $S \subset A$ a multiplicative set, and $f : A \longrightarrow S^{-1}A$ the canonical map. If B is a ring, with ring homomorphisms $g : A \longrightarrow B$ and $h : B \longrightarrow S^{-1}A$ satisfying

(1) $f = hg$

(2) for every $b \in B$ there exists $s \in S$ such that $g(s) \cdot b \in g(A)$

Then $S^{-1}A \simeq g(S)^{-1}B \simeq T^{-1}B$, where $T = \{t \in B \mid h(t) \text{ is a unit of } S^{-1}A\}$.

Proof: By universal property of localization and condition (1) and (2), there are ring homomorphisms:

$$\begin{aligned} S^{-1}A &\to g(S)^{-1}B \\ \varphi : a/s &\mapsto g(a)/g(s) \end{aligned}$$

$$\begin{aligned} g(S)^{-1}B &\to S^{-1}A \\ \psi : b/g(s) &\mapsto h(b) \cdot (1/s) \end{aligned}$$

such that $\varphi \circ \psi = \mathrm{id}, \psi \circ \varphi = \mathrm{id}$. Hence $S^{-1}A \simeq g(S)^{-1}B$.

Since $T \supset g(S)$, by universal property of localization, there are ring homomorphisms:

$$\begin{aligned} S^{-1}A &\to T^{-1}B \\ \varphi : a/s &\mapsto g(a)/g(s) \end{aligned}$$

$$\begin{aligned} T^{-1}B &\to S^{-1}A \\ \psi : b/t &\mapsto h(b)h(t)^{-1} \end{aligned}$$

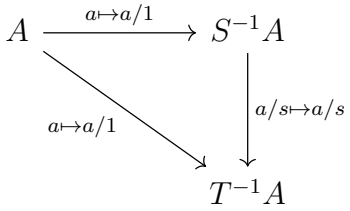
Notice that if $g(s_1)b = g(a_1), g(s_2) = tg(b_2)$, then $h(b)(s_1/1) = a_1/1, h(t)(s_2/1) = a_2/1$ and $\psi(b/t) = a_1/s_1 \cdot (a_2/s_2)^{-1}$. And it's easy to cheack that $\varphi(\psi(b/t)) = \varphi(a_1/s_1 \cdot (a_2/s_2)^{-1}) = g(a_1)/g(s_1) \cdot (g(a_2)/g(s_2))^{-1} = b/t$. Hence $S^{-1}A \simeq g(S)^{-1}B \simeq T^{-1}B$.

Corollary 2.4.27. If $\mathfrak{p}$ is a prime ideal of $A, S = A - \mathfrak{p}$ and B satisfies the conditions of the theorem, then setting $P = \mathfrak{p}A_{\mathfrak{p}} \cap B$ we have $A_{\mathfrak{p}} \simeq B_P$.

Proof: Under these circumstances the T in the theorem is exactly $B - P$ because $A_{\mathfrak{p}}$ is a local ring.

Corollary 2.4.28. If S and T are two multiplicative subsets of A with $S \subset T$, then writing T' for the image of T in $S^{-1}A$, we have $(T')^{-1}S^{-1}A \simeq T^{-1}A$.

Proof: Consider the following commutative diagram:



Proposition 2.4.29. Let A be a ring, and let $\operatorname{Spec} A = \bigcup_{i \in I} D(f_i)$ be a finite open covering by principal open subsets (i.e., the ideal generated by the f_i is all of A). Let M be an A-module. Then M is finitely generated if and only if for all i, the localization M_{f_i} is a finitely generated A_{f_i}-module.

Proof: Clearly, the condition is necessary. Now let M_{f_i} be finitely generated over A_{f_i}, for all i, say by elements $m_{ij}/(f_i)^{n_{ij}}, j = 1, \ldots, r_i, m_{ij} \in M$. Then the submodule $N \subseteq M$ generated by all the m_{ij} is a finitely generated A-module. For the A-module M/N, we therefore get that all localizations $(M/N)_{\mathfrak{p}} \cong M/N \otimes_A A_{\mathfrak{p}}$ at prime ideals $\mathfrak{p} \in \operatorname{Spec} A$ vanish. This is because, $\mathfrak{p} \in D(f_i)$, for all $m \in M$, $mf_i^s \in N$ for sufficiently large s, hence $m \otimes a/b = mf_i^s \otimes a/(bf_i^s) = 0$ for all $a/b \in A_{\mathfrak{p}}$. Then, by Theorem 2.4.17, $M/N = 0$.

Proposition 2.4.30. Let A be a ring and let B be an A-algebra. Let $f_1, \ldots, f_n \in B$ be elements generating the unit ideal (1), and such that for all i, the localization B_{f_i} is a finitely generated A-algebra. Then B is a finitely generated A-algebra.

Proof: By assumption, there exist $g_i \in B$ with $\sum_i g_i f_i = 1$. Furthermore, all the B_{f_i} are finitely generated, so we can find finitely many elements b_{ij} which generate B_{f_i} as an A-algebra. We write $b_{ij} = c_{ij}/f_i^m$ with $c_{ij} \in B$ and some $m \geq 0$, which we may assume to be independent of i, j.

Let C be the A-subalgebra of B generated by all elements g_i, f_i, c_{ij}. We will show that $C = B$, which of course implies that B is finitely generated. Let $b \in B$. For N sufficiently large, and all i, we have $f_i^N b \in C$. Because $\sum_i g_i f_i = 1$, we get that the f_i generate the unit ideal in C, so the same is true for $f_1^N, \ldots, f_n^N$, hence there exist $u_1, \ldots, u_n \in C$ such that $\sum_i u_i f_i^N = 1$. So we obtain $b = \left(\sum_i u_i f_i^N\right) b \in C$.

2.5 Intergral Extension

Definition 2.5.1. Let B be a ring, A a subring of B (so that $1 \in A$). An element x of B is said to be integral over A if x is a root of a monic polynomial with coefficients in A, that is if x satisfies an equation of the form

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

where the a_4 are elements of A.

Proposition 2.5.2. The following are equivalent:

(1) $x \in B$ is integral over A;

(2) $A[x]$ is a finitely generated A-module;

(3) $A[x]$ is contained in a subring C of B such that C is a finitely generated A-module;

(4) There exists a faithful $A[x]$-module M which is finitely generated as an A-module.

Proof: (4) implies (1): Assume $M = Am_1 + \cdots + Am_k$. Let $[xm_1, \ldots, xm_k] = [m_1, \ldots, m_k]R$. Then $(xI - R)m = 0$. Since $A[x]$-module is faithfal, then the statement follows from the following lemma

Lemma 2.5.3. Let $A = (a_{ij})$ be an $(r \times r)$ matrix with entries in an arbitrary ring, and let $A^* = \left(a_{ij}^*\right)$ be the adjoint matrix, i.e., $a_{ij}^* = (-1)^{i+j} \det(A_{ij})$, where the matrix A_{ij} is obtained from A by deleting the i-th column and the j-th row. Then one has

$$AA^* = A^*A = \det(A)E,$$

where E denotes the unit matrix of rank r. For any vector $x = (x_1, \ldots, x_r)$, this yields the implication

$$Ax = 0 \Longrightarrow (\det A)x = 0$$

Proposition 2.5.4. Let $x_i (1 \leqslant i \leqslant n)$ be elements of B, each integral over A. Then the ring $A[x_1, \ldots, x_n]$ is a finitely-generated A-module.

Corollary 2.5.5. The set C of elements of B which are integral over A is a subring of B containing A.

Corollary 2.5.6. Let $A \subseteq B \subseteq C$ be two ring extensions. If C is integral over B and B is integral over A, then C is integral over A.

Proof: For all $x \in C$, there's $f(y) = b_n y^n + \cdots + b_1 y_1 + b_0 \in B[y]$ such that x is a root of $f(y)$. Since x integral over B, x integral over $A[b_n, \ldots, b_0]$. By Proposition 2.5.2, $A[b_n, \ldots, b_0, x]$ is finitely generated $A[b_n, \ldots, b_0]$-module. By Proposition 2.5.4, $A[b_n, \ldots, b_0]$ is finite A-module, then $A[b_n, \ldots, b_0, x]$ is finite A-module. Hence by Proposition 2.5.2 again, x is integral over C.

Definition 2.5.7. The ring C containing all the integral elements in B is called the integral closure of A in B. If $C = A$, then A is said to be integrally closed in B. If $C = B$, the ring B is said to be integral over A.

Definition 2.5.8. If A is an integral domain with field of fractions K, then the integral closure $\bar{A}$ of A in K is called the normalization of A, and A is simply called integrally closed if $A = \bar{A}$.

Proposition 2.5.9. Let $A \subseteq B \subseteq C$ be rings. Suppose that A is Noetherian, that C is finitely generated as an A-algebra and that C is either finitely generated as a B-module or integral over B. Then B is finitely generated as an A-algebra.

Example 2.5.10. A U.F.D. is an integrally closed domaim.

Definition 2.5.11 (going-up and going down)**.** For a ring A and an A-algebra B, the following statement is called the going-up theorem: given two prime ideals $\mathfrak{p} \subset \mathfrak{p}'$ of A and a prime ideal P of B lying over $\mathfrak{p}$, there exists $P' \in \operatorname{Spec} B$ such that $P \subset P'$ and $P' \cap A = \mathfrak{p}'$.

Similarly, the going-down theorem is the following statement: given $\mathfrak{p} \subset \mathfrak{p}'$ and $P' \in \operatorname{Spec} B$ lying over $\mathfrak{p}'$, there exists $P \in \operatorname{Spec} B$ such that $P \subset P'$ and $P \cap A = \mathfrak{p}$.

Lemma 2.5.12. Suppose that $R \to S$ is an integral ring extension with $R \subset S$. Then $\varphi : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is surjective.

Proof: Let $\mathfrak{p} \subset R$ be a prime ideal. The localization $R_{\mathfrak{p}} \to S_{\mathfrak{p}}$ is injective (as localization is exact) and it's easy to check it is integral. Hence we may replace R, S by $R_{\mathfrak{p}}, S_{\mathfrak{p}}$ and we may assume R is local with maximal ideal $\mathfrak{m}$ and it suffices to show that $\mathfrak{m}S \neq S$ (By Algebraic Geometry 1.5.11). Let $R \subset S' \subset S$ where $S' = R[s_1, \ldots, x_n]$. Since $R \to S$ is integral, S' is finitely generated over R. Notice that $\mathfrak{m}S' = S'$, by Nakayama's Lemma, $S' = 0$, a contridiction!

Theorem 2.5.13. If $B \supset A$ is an extension ring which is integral over A then the going-up theorem holds.

Proof: We may replace R by $R/\mathfrak{p}$ and S by $S/\mathfrak{q}$. This reduces us to Lemma 2.5.12.

Proposition 2.5.14. A is a subring of B and B integral over A, then A is a field if and only if B is a field.

2.6 Dedekind Domain

Definition 2.6.1. A Noetherian, integrally closed integral domain in which every nonzero prime ideal is maximal($\dim = 1$) is called a Dedekind domain.

Proposition 2.6.2. Every non-zero ideal $\mathfrak{a}$ of a Dedekind domain $\mathcal{O}$ admits a factorization

$$\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$$

into nonzero prime ideals $\mathfrak{p}_i$ of $\mathcal{O}$ which is unique up to the order of the factors.

Moreover, $I_1 \mid I_2$ iff $I_1 \supset I_2$, $I_1 + I_2 = \gcd(I_1, I_2)$.

Proposition 2.6.3. The fractional ideals form an abelian group, the ideal group J_K of K. The identity element is $(1) = \mathcal{O}$, and the inverse of $\mathfrak{a}$ is

$$\mathfrak{a}^{-1} = \{x \in K \mid x\mathfrak{a} \subseteq \mathcal{O}\}$$

Definition 2.6.4 (fractional ideal)**.** Let A be an integral domain with field of fractions K. A fractional ideal I of A is an A-submodule I of K such that $I \neq 0$ and $\alpha I \subset A$ for some $0 \neq \alpha \in K$. The product of two fractional ideals is defined in the same way as the product of two ideals. If I is a fractional ideal of A we set $I^{-1} = \{\alpha \in K \mid \alpha I \subset A\}$; this is also a fractional ideal, and $II^{-1} \subset A$. In the particular case that $II^{-1} = A$ we say that I is invertible.

Proposition 2.6.5. An invertible fractional ideal of A is finitely generated as an A-module. Conversely, if A is Noetherian, every fractional ideal is finitely generated.

Proof: Let $1 = \sum a_i b_i$, where $a_i \in I, b_i \in I^{-1}$. Then $a_1, \ldots, a_n$ generate I.

Theorem 2.6.6. If A is a Dedekind domain, every fractional ideal in A is invertible.

Corollary 2.6.7. Every fractional ideal a admits a unique representation as a product

$$\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{\nu_p}$$

with $\nu_p \in \mathbb{Z}$ and $\nu_p = 0$ for almost all $\mathfrak{p}$. In other words, J_K is the free abelian group on the set of nonzero prime ideals $\mathfrak{p}$ of $\mathcal{O}$.

Definition 2.6.8 (ideal class group)**.** The fractional principal ideals $(a) = a\mathcal{O}, a \in K^*$, form a subgroup of the group of ideals J_K, which will be denoted P_K. The quotient group

$$Cl_K = J_K / P_K$$

is called the ideal class group, or class group for short, of K. Along with the group of units $\mathcal{O}^*$ of $\mathcal{O}$, it fits into the exact sequence

$$1 \longrightarrow \mathcal{O}^* \longrightarrow K^* \longrightarrow J_K \longrightarrow Cl_K \longrightarrow 1.$$

2.7 Flatness

Recall the definition of Falt modules.

Definition 2.7.1. Let R be a ring.

(1) An R-module M is called flat if whenever $N_1 \to N_2 \to N_3$ is an exact sequence of R-modules the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact as well.

(2) An R-module M is called faithfully flat if the complex of R-modules $N_1 \to N_2 \to N_3$ is exact if and only if the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact.

(3) A ring map $R \to S$ is called flat if S is flat as an R-module.

(4) A ring map $R \to S$ is called faithfully flat if S is faithfully flat as an R-module.

Proposition 2.7.2. An R-module M is faithfully flat if and only if

(1) M is flat

(2) for all complex $0 \to N_1 \to N_2 \to N_3 \to 0$, if the sequence $0 \to M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3 \to 0$ is exact, $0 \to N_1 \to N_2 \to N_3 \to 0$ is exact.

Proof: Notice that $N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3$ is exact if and only if $0 \to \mathrm{Im}f \to N_2 \to N_2/\ker g \to 0$ is exact. By definition of faithfully flat, it suffices to show that $0 \to M \otimes_R \mathrm{im}f \to M \otimes_R N_2 \to M \otimes_R N_2/\mathrm{Ker}g \to 0$ is exact. Use the fact M is flat to show this statement.

Proposition 2.7.3. Let R be a ring. Let $I, J \subset R$ be ideals. Let M be a flat R-module. Then $IM \cap JM = (I \cap J)M$.

Proof: Consider the exact sequence $0 \to I \cap J \to R \to R/I \oplus R/J$. Tensoring with the flat module M we obtain an exact sequence

$$0 \to (I \cap J) \otimes_R M \to M \to M/IM \oplus M/JM$$

Proposition 2.7.4 (direct limit preserves flat)**.** R be a ring and Let $\{M_i, \varphi_{ij} : M_i \to M_j\}, i \leq j$ be a direct system. If M_i is flat for all $i \in I$, then $\varinjlim M_i$ is flat.

Proof: Since tensor product is the left part of the adjoint pair (tensor, hom), it preserves direct limit. By Proposition 3.1.31, tensor product commutes with direct limit. Hence, $\varinjlim M_i$ is flact.

Theorem 2.7.5 (Transitivity)**.** $f : A \to B$ is a ring homomorphism, B is (faithfully flat) flat A-module, N is (faithfully flat) flat B-module, then N is (faithfully flat) flat over A.

Proof: By Theorem 2.1.14.

Proposition 2.7.6 (Base Change)**.** Suppose that M is (faithfully) flat over R and $R \to R'$ is a ring map. Then $M \otimes_R R'$ is (faithfully) flat over R'.

Proposition 2.7.7. Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-module, with C flat. Then A is flat if and only if B is flat.

Proof: By Long exact homology sequence, for any R-module D, there's exact sequence

$$0 \to \mathrm{Tor}_R^1(D, A) \to \mathrm{Tor}_R^1(D, B) \to 0$$

Theorem 2.7.8 (Localization)**.** $S^{-1}A$ is a flat A-module.

Proof: By Theorem 2.4.14.

Proposition 2.7.9. Let M be an R-module. The following are equivalent:

(1) M is flat over R.

(2) for every injection of R-modules $N \subset N'$ the map $N \otimes_R M \to N' \otimes_R M$ is injective.

(3) for every ideal $I \subset R$ the map $I \otimes_R M \to R \otimes_R M = M$ is injective.

(4) for every finitely generated ideal $I \subset R$ the map $I \otimes_R M \to R \otimes_R M = M$ is injective.

Proof: (4) implies (1): Firstly, every ideal of A is the direct limit of the finitely generated ideals contained in it, so that (4) implies (3).

Moreover, if N is an A-module and $N' \subset N$ a submodule, then since N is the direct limit of modules of the form $N' + F$, with F finitely generated, to prove that $N' \otimes M \longrightarrow N \otimes M$ is injective we can assume that $N = N' + A\omega_1 + \cdots + A\omega_n$. Then setting $N_i = N' + A\omega_1 + \cdots + A\omega_i$ (for $1 \leqslant i \leqslant n$), we need only show that each step in the chain

$$N' \otimes M \longrightarrow N_1 \otimes M \longrightarrow N_2 \otimes M \longrightarrow \cdots \longrightarrow N \otimes M$$

is injective, and finally that if $N = N' + A\omega$ then $N' \otimes M \longrightarrow N \otimes M$ is injective.

Now we set $I = \{a \in A \mid a\omega \in N'\}$. It's clear that I is the kernel of the morphism $A \to N/N', r \to r\omega + N'$, we have exact sequence

$$0 \to N' \longrightarrow N \longrightarrow A/I \to 0.$$

This induces a long exact sequence

$$\begin{array}{ccccc} 0 & \longleftarrow & & & \\ & & N' \otimes_A M & \longrightarrow & N \otimes_A M \longrightarrow A/I \otimes_A M \\ & & & & \longleftarrow \mathrm{Tor}_1^A(M, A/I) \end{array}$$

hence it is enough to prove that $\mathrm{Tor}_1^A(M, A/I) = 0$. For this consider the short exact sequence

$$0 \to I \longrightarrow A \longrightarrow A/I \to 0$$

and the induced long exact sequence

$$\begin{array}{ccccc} 0 & \longleftarrow & & & \\ & & I \otimes_A M & \longrightarrow & A \otimes_A M & \longrightarrow & A/I \otimes_A M \\ & & & & \operatorname{Tor}_1^A(M, A) = 0 & \longrightarrow & \operatorname{Tor}_1^A(M, A/I) \end{array}$$

since $I \otimes M \longrightarrow M$ is injective, $\operatorname{Tor}_1^A(M, A/I) = 0$.

Corollary 2.7.10. Let A be a ring and M an A-module. The following conditions are equivalent:

(1) M is flat;

(2) for every A-module N we have $\operatorname{Tor}_1^A(M, N) = 0$;

(3) $\operatorname{Tor}_1^A(M, A/I) = 0$ for every finitely generated ideal I.

Proof: For this consider the short exact sequence

$$0 \to I \longrightarrow A \longrightarrow A/I \to 0$$

and the induced long exact sequence

$$\begin{array}{ccccc} 0 & \longleftarrow & & & \\ & & I \otimes_A M & \longrightarrow & A \otimes_A M & \longrightarrow & A/I \otimes_A M \\ & & & & \operatorname{Tor}_1^A(M, A) = 0 & \longrightarrow & \operatorname{Tor}_1^A(M, A/I) \end{array}$$

Lemma 2.7.11. Let R be a ring. Let M be an R-module. The following are equivalent

(1) M is faithfully flat, and

(2) M is flat and for all R-module homomorphisms $\alpha : N \to N'$ we have $\alpha = 0$ if and only if $\alpha \otimes \mathrm{id}_M = 0$.

Proof: If $0 \to N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3 \to 0$ be a complex and the sequence $0 \to M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3 \to 0$ is exact. For all $x \in \operatorname{Ker} g$, consider $\alpha : R \to N_2/\operatorname{Im}(N_1), r \mapsto rx + \operatorname{Im}(N_1)$.

Proposition 2.7.12. Let M be a flat R-module. The following are equivalent:

(1) M is faithfully flat,

(2) for every nonzero R-module N, then tensor product $M \otimes_R N$ is nonzero,

(3) for all $\mathfrak{p} \in \mathrm{Spec}(R)$ the tensor product $M \otimes_R \kappa(\mathfrak{p})$ is nonzero, and

(4) for all maximal ideals $\mathfrak{m}$ of R the tensor product $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$ is nonzero.

Proof: (4) implies (1): If $0 \to N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3 \to 0$ be a complex and the sequence $0 \to M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3 \to 0$ is exact. Define $H = \mathrm{Ker}g/\mathrm{im}f$. Notice that $H \otimes_R M = 0$. If $H \neq 0$, take $0 \neq x \in H$, define $I = \{r \in R : rx = 0\}$. It's clear that $I \neq R$, so we can find a maximal ideal $\mathfrak{m}$ such that $I \subset \mathfrak{m}$. Consider the natural R-module homomorphism $R/I \to H, r \mapsto rx$. By flatness of M, $R/I \otimes_R M = M/IM = 0$. Hence, $M/\mathfrak{m}M = 0$, a contradiction!

Proposition 2.7.13. Let $R \to S$ be a flat ring map. The following are equivalent:

(1) $R \to S$ is faithfully flat,

(2) the induced map on Spec is surjective, and

(3) any closed point $x \in \mathrm{Spec}(R)$ is in the image of the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$.

Proof: By Algebraic Geometry Proposition 1.5.11, for prime ideal $\mathfrak{p}$ in S, the fiber of induced map in $\mathfrak{p}$ is non-empty if and only if $\mathrm{Spec}(S) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa(\mathfrak{p})) = \mathrm{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ is non-empty. Then this statement follows from Proposition 2.7.12.

Corollary 2.7.14. A flat local ring homomorphism of local rings is faithfully flat.

Corollary 2.7.15 (going-down theorem holds for flat morphism)**.** Let $\mathfrak{p} \subset \mathfrak{p}'$ be primes of R. Let $\mathfrak{q}' \subset S$ be a prime of S mapping to $\mathfrak{p}'$. Then there exists a prime $\mathfrak{q} \subset \mathfrak{q}'$ mapping to $\mathfrak{p}$.

Proof: Consider the flat local ring homomorphism $R_{\mathfrak{p}'} \to S_{\mathfrak{q}'}$.

2.8 Hilbert's Nullstellensatz

Theorem 2.8.1. Let k be a field, L an algebraic extension of k and $\alpha_1, \ldots, \alpha_n \in L$; then

(1) $k[\alpha_1, \ldots, \alpha_n] = k(\alpha_1, \ldots, \alpha_n)$.

(2) Write $\varphi : k[X_1, \ldots, X_n] \longrightarrow k(\alpha_1, \ldots, \alpha_n)$ for the homomorphism over k which maps X_i to α_i; then $\operatorname{Ker}\varphi$ is the maximal ideal generated by n elements of the form

$$f_1(X_1), f_2(X_1, X_2), \ldots, f_n(X_1, \ldots, X_n)$$

, where each f_i can be taken to be monic in X_i with coefficient in $k[X_1, \ldots, X_{i-1}]$

Proof: Let $g_i(X_i)$ be the monic minimal polynomial of α_i over $k(\alpha_1, \ldots, \alpha_{i-1})$, take a lift f_i of $g_i(X_i)$ in $k[X_1, \ldots, X_i]$ such that $\varphi(f_i) = g_i$. Then $\ker\varphi = (f_1, \ldots, f_n)$

Theorem 2.8.2. Let k be a field and domain A is an finitely generated k-algebra, if A is a field, then A is a finite extension of k.

Proof: Let $E = k[x_1, \ldots, x_n]$. If E is not algebraic over k, by Proposition 1.4.51, we can renumber the x_1 so that $x_1, \ldots, x_r$ are algebraically independent over k, where $r \geqslant 1$, and each of $x_{r+1}, \ldots, x_n$ is algebraic over the field $F = k(x_1, \ldots, x_r)$. Hence E is a finite algebraic extension of F and therefore finitely generated as an F-module. Applying Proposition 2.5.9 to $k \subseteq F \subseteq E$, it follows that F is a finitely generated k-algebra, say $F = k[y_1, \ldots, y_3]$. Each y_j is of the form f_j/g_j, where f_j and g_j are polynomials in $x_1, \ldots, x_r$. It contradicts to the fact that there are infinitely many irreducible polynomials in the ring $k[x_1, \ldots, x_r]$ (adapt Euclid's proof of the existence of infinitely many prime numbers).

Theorem 2.8.3. Let k be a field, and let m be any maximal ideal of the polynomial ring $k[X_1, \ldots, X_n]$; then the residue class field $k[X_1, \ldots, X_n]/m$ is algebraic over k. Hence m can be generated by n elements, and in particular if k is algebraically closed then m is of the form $m = (X_1 - \alpha_1, \ldots, X_n - \alpha_n)$ for $\alpha_i \in k$.

Proof: Set $k[X_1, \ldots, X_n]/m = K$, and write α_i for the image of X_i in K; then $K = k[\alpha_1, \ldots, \alpha_n]$. By the previous theorem, since K is a field it is algebraic over k, and then by Theorem 2.8.1, m is generated by n elements. If k is algebraically closed then $k = K$, so that each X_i is congruent modulo m to some $\alpha_i \in k$; then $(X_1 - \alpha_1, \ldots, X_n - \alpha_n) \subset m$. On the other hand $(X_1 - \alpha_1, \ldots, X_n - \alpha_n)$ is obviously a maximal ideal, so that equality must hold.

Theorem 2.8.4 (Hilbert's Nullstellensatz)**.** If k is algebracally closed, then

$$I(V(A)) = \sqrt{A}.$$

Proof: It is clear that $\sqrt{A} \subset I(V(A))$. The problem is to show the other inclusion. Put concretely this means the following: Let $A = (f_1, \ldots, f_m)$. If $g \in k[X_1, \ldots, X_n]$ satisfies:

$$\{f_1(a_1, \ldots, a_n) = \ldots = f_m(a_1, \ldots, a_n) = 0\} \Longrightarrow g(a_1, \ldots, a_n) = 0$$

then there is an integer ℓ and polynomials $h_1, \ldots, h_m$ such that

$$g^{\ell}(X) = \sum_{i=1}^{m} h_i(X) \cdot f_i(X).$$

To prove this, introduce the ideal

$$B = A \cdot k\left[X_1, \ldots, X_n, X_{n+1}\right] + \left(1 - g \cdot X_{n+1}\right)$$

in $k\left[X_1, \ldots, X_{n+1}\right]$ where $A \cdot k\left[X_1, \ldots, X_n, X_{n+1}\right]$ be the ideal generated by A. There are 2 possibilities: either B is a proper ideal, or $B = k\left[X_1, \ldots, X_{n+1}\right]$. In the first case, let M be a maximal ideal in $k\left[X_1, \ldots, X_{n+1}\right]$ containing B. By Theorem 2.8.3,

$$M = \left(X_1 - a_1, \ldots, X_n - a_n, X_{n+1} - a_{n+1}\right)$$

for some elements $a_i \in k$. Since M is the kernel of the homomorphism:

$$\begin{aligned} k\left[X_1, \ldots, X_n, X_{n+1}\right] &\longrightarrow k \\ f &\longmapsto f\left(a_1, \ldots, a_{n+1}\right), \end{aligned}$$

$B \subset M$ means that:

$$f_1\left(a_1, \ldots, a_n\right) = \ldots = f_m\left(a_1, \ldots, a_n\right) = 0 \tag{2.1}$$

and

$$1 = g\left(a_1, \ldots, a_n\right) \cdot a_{n+1}.$$

But by our assumptionon g, (2.1) implies that $g\left(a_1, \ldots, a_n\right) = 0$.A contradiction! Hence we can only conclude that the ideal B would not have been a proper ideal.

But then $1 \in B$. This means that there are polynomials $h_1, \ldots, h_m, h_{m+1} \in k\left[X_1, \ldots, X_{n+1}\right]$ such that:

$$\begin{aligned} 1 = &\sum_{i=1}^{m} h_i\left(X_1, \ldots, X_{m+1}\right) \cdot f_i\left(X_1, \ldots, X_n\right) \\ &+ \left(1 - g\left(X_1, \ldots, X_n\right) \cdot X_{n+1}\right) \cdot h_{m+1}\left(X_1, \ldots, X_{n+1}\right). \end{aligned}$$

Substituting g^{-1} for X_{n+1} in this formula, we get:

$$1 = \sum_{i=1}^{m} h_i\left(X_1, \ldots, X_n, 1/g\right) \cdot f_i\left(X_1, \ldots, X_n\right).$$

Clearing denominators, this gives:

$$g^{\ell}\left(X_1, \ldots, X_n\right) = \sum_{i=1}^{m} h_i^*\left(X_1, \ldots, X_n\right) \cdot f_i\left(X_1, \ldots, X_n\right)$$

for some new polynomials $h_i^* \in k\left[X_1, \ldots, X_n\right]$, i.e., $g \in \sqrt{A}$.

Corollary 2.8.5. If k is algebraically closed, then there is a one-to-one inclusion-reversing correspondence between algebraic sets(irreducible algebraic sets, points) in $\mathbb{A}^n$ and radical ideals(prime ideals, maximal ideals) in A, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$.

Proof: By Hilbert's Nullstellensatz.

2.9 Dimension Theory

Definition 2.9.1. Let X be a topological space; we consider strictly decreasing (or strictly increasing) chains $Z_0, Z_1, \ldots, Z_r$ of length r of irreducible closed subsets of X. The supremum of the lengths, taken over all such chains, is called the combinatorial dimension of X and denoted $\dim X$.

Remark 2.9.2. If X is a Noetherian space then there are no infinite strictly decreasing chains, but it can nevertheless happen that $\dim X = \infty$.

Proposition 2.9.3. Let Y be a subspace of X. If $S \subset Y$ is an irreducible closed subset of Y, then its closure in X is an irreducible closed subset $\bar{S} \subset X$ such that $\bar{S} \cap Y = S$. Hence, $\dim Y \leqslant \dim X$.

Definition 2.9.4. Let A be a ring. Define $\dim A = \dim \operatorname{Spec}(A)$.

Example 2.9.5. A is Artinian if and only if A is Notherian and $\dim A = 0$

Definition 2.9.6. If M is an A-module, we define the dimension of M by

$$\dim M = \dim(A/\operatorname{ann}(M)).$$

Proposition 2.9.7. If M is finitely generated then $\dim M$ is the combinatorial dimension of the closed subspace $\operatorname{Supp}(M) = V(\operatorname{ann}(M))$ of $\operatorname{Spec} A$.

Proof: By Proposition 2.2.8,

$$\dim M = \dim(A/\operatorname{ann}(M)) = \dim V(\operatorname{ann}(M))$$

Corollary 2.9.8. Let k be a field, then $\dim k[x_1, \ldots, x_n] = n$.

Proposition 2.9.9. A is an integral finitely generated k-algebra, then for $\mathfrak{p}$ a prime ideal in A,

$$\operatorname{height}\mathfrak{p} + \dim A/\mathfrak{p} = \dim A$$

Proposition 2.9.10. A be a ring, then $\dim A[T] \geq 1 + \dim A$. Moreover, if A is Noetherian, $\dim A[T] = 1 + \dim A$. In particular, if A is Noetherian, $\dim A[T_1, \ldots, T_n] = n + \dim A$.

Theorem 2.9.11 (Noether's Normalization Theorem)**.** Let k be a field. Let $S = k[x_1, \ldots, x_n]/I$ for some ideal I. If $I \neq (1)$, there exist $r \geq 0$, and $y_1, \ldots, y_r \in k[x_1, \ldots, x_n]$ such that

(1) the map $k[y_1, \ldots, y_r] \to S$ is injective, and

(2) the map $k[y_1, \ldots, y_r] \to S$ is finite. (That is, S is a finitely generated $k[y_1, \ldots, y_r]$-module)

In this case the integer r is the dimension of S. Moreover we may choose y_i to be in the $\mathbb{Z}$-subalgebra of $k[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$.

Corollary 2.9.12. Let k be a field and A an integral domain which is finitely generated over k. Define the transcendental degree of A to be transcendence degree of extension $\mathrm{Frac}(A)/k$.

$$\dim A = \mathrm{tr} \cdot \deg_k \mathrm{Frac} A$$

Proof: By Noether's Normalization Theorem, take $y_1, \ldots, y_r \in k[x_1, \ldots, x_n]$ such that the map $k[y_1, \ldots, y_r] \to S$ is injective and finite. By Proposition 2.5.2, A is integral over $k[y_1, \ldots, y_r]$. Hence, $\mathrm{Frac}(A)$ is algebraic over $k(y_1, \ldots, y_r)$. Hence, $r = \dim S = \deg_k \mathrm{Frac} A$.

Proposition 2.9.13. Let A be a finitely generated algebra over a field k and set $d = \dim A$. Assume that A is an integral domain. Let $0 \leq h(1) < h(2) < \cdots < h(r) \leq d$ be a sequence of integers. Let $\mathfrak{q}_{h(1)} \subsetneq \cdots \subsetneq \mathfrak{q}_{h(r)}$ be a chain of prime ideals of A such that $\dim V\left(\mathfrak{q}_{h(i)}\right) = d - h(i)$.

(1) There exists a finite injective k-algebra homomorphism $\varphi : k[T_1, \ldots, T_d] \to A$ such that $\varphi^{-1}\left(\mathfrak{q}_{h(i)}\right) = \left(T_1, \ldots, T_{h(i)}\right)$ for all $i = 1, \ldots, r$.

(2) For every homomorphism φ as in (1) the chain $\left(\mathfrak{q}_{h(i)}\right)_i$ can be completed to a chain of prime ideals $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_d$ of A such that $\varphi^{-1}(\mathfrak{q}_j) = (T_1, \ldots, T_j)$ for all $j = 1, \ldots, d$. In particular, any chain of prime ideals in A can be completed to a maximal chain of prime ideals and all maximal chains have the same length.

2.10 Graded Ring

Definition 2.10.1. A graded ring is a ring S, together with a decomposition $S = \bigoplus_{d\geqslant 0} S_d$ of S into a direct sum of abelian groups S_d, such that for any $d, e \geqslant 0$, $S_d \cdot S_e \subseteq S_{d+e}$. An element of S_d is called a homogeneous element of degree d. Thus any element of S can be written uniquely as a (finite) sum of homogeneous elements.

An ideal $\mathfrak{a} \subseteq S$ is a homogeneous ideal if $\mathfrak{a} = \bigoplus_{d\geq 0} (\mathfrak{a} \cap S_d)$.

Proposition 2.10.2. The sum, product, intersection, and radical of homogeneous ideals are homogeneous. To test whether a homogeneous ideal is prime, it is sufficient to show for any two homogeneous elements f, g, that $fg \in \mathfrak{a}$ implies $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$.

Proposition 2.10.3. Let S be a graded ring, let I be an homogeneous ideal in S and let $I_k = I \cap S_k$ for all $k \geq 0$. Then S/I is naturally a graded ring whose homogeneous component of degree k is isomorphic to S_k/I_k.

Proof: The map

$$S = \bigoplus_{k=0}^{\infty} S_k \longrightarrow \bigoplus_{k=0}^{\infty} (S_k/I_k)$$
$$(\ldots, s_k, \ldots) \longmapsto (\ldots, s_k \bmod I_k, \ldots)$$

is surjective with kernel $I = \bigoplus_{k=0}^{\infty} I_k$ and defines an isomorphism of graded rings.

Definition 2.10.4 (projective spectrum)**.** Let S be a graded ring. We denote by S_+ the ideal $\bigoplus_{d>0} S_d$. We define the set $\operatorname{Proj} S$ to be the set of all homogeneous prime ideals $\mathfrak{p}$, which do not contain all of S_+. If $\mathfrak{a}$ is a homogeneous ideal of S, we define the subset $V_+(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \supseteq \mathfrak{a}\}$. We have

(1) If $\mathfrak{a}$ and $\mathfrak{b}$ are homogeneous ideals in S, then $V(\mathfrak{ab}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

(2) If $\{\mathfrak{a}_i\}$ is any family of homogeneous ideals of S, then $V\left(\sum \mathfrak{a}_i\right) = \bigcap V\left(\mathfrak{a}_i\right)$.

Definition 2.10.5. A graded ring is a ring A endowed with a direct sum decomposition $A = \bigoplus_{d\geq 0} A_d$ of abelian groups such that $A_d A_e \subseteq A_{d+e}$ for all $d, e \geq 0$. Then A is an A_0-algebra. If R is any ring, a graded ring A is called graded R-algebra, if A_0 is an R-algebra (equivalently, A is an R-algebra and $RA_d = A_d$ for all d).

If A and B are graded R-algebras, a graded R-algebra homomorphism $\varphi : A \to B$ is an R-algebra homomorphism such that $\varphi\left(A_d\right) \subseteq B_d$ for all $d \geq 0$.

2.11 Completion

Let A be a ring and let $\mathfrak{a} \subset A$ be an ideal.

Definition 2.11.1. Let M be an A-module. The $\mathfrak{a}$-adic topology on M is the coarsest topology on M such that for all $m \in M$ the sets $m + \mathfrak{a}^n M$ for $n \geq 0$ form a neighborhood basis of m.

Proposition 2.11.2. The $\mathfrak{a}$-adic topology on M is Hausdorff if and only if

$$\bigcap_{n \geq 0} \mathfrak{a}^n M = 0$$

.

Proof: Trivial.

Proposition 2.11.3. Let $\varphi : A \to B$ be a ring homomorphism, let $\mathfrak{b} \subseteq B$ be an ideal and assume that there exists an integer $r \geq 1$ such that $\varphi(\mathfrak{a}^r) \subseteq \mathfrak{b}$. Then φ is continuous with respect to the $\mathfrak{a}$-adic topology on A and the $\mathfrak{b}$-adic topology on B.

Proof: Notice that if $\varphi(\mathfrak{a}^r) \subset \mathfrak{b}$, we have $\varphi(\mathfrak{a}^{rm}) \subset \mathfrak{b}^m$ for all $m \geq 1$.

Proposition 2.11.4. A is a ring and $\mathfrak{a}$ be an ideal of A. Every homomorphism of A-modules is continuous with respect to the $\mathfrak{a}$-adic topology.

Definition 2.11.5 (topological ring, topological module)**.**

Definition 2.11.6. For an A-module M we endow $M/\mathfrak{a}^n M$ with the discrete topology and call the topological A-module

$$\hat{M} = \varprojlim M/\mathfrak{a}^n M$$

the completion of M. For $M = A$ we obtain a topological ring $\hat{A}$, and $\hat{M}$ is a topological $\hat{A}$-module.

Definition 2.11.7. An A-module M is called $\mathfrak{a}$-adically complete if the canonical homomorphism $M \to \hat{M}$ is an isomorphism. In particular, A is said to be $\mathfrak{a}$-adically complete if the canonical homomorphism $A \to \varprojlim A/\mathfrak{a}^n A$ is an isomorphism.

If A is a local ring, we usually endow A (and every A-module M) with the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the maximal ideal of A. If A (or M) is $\mathfrak{m}$-adically complete, we simply say that A (or M) is complete.

Theorem 2.11.8 (Artin-Rees)**.** Let A be noetherian, $\mathfrak{a} \subseteq A$ an ideal, M a finitely generated A-module, and $N \subseteq M$ a submodule. Then there exists an integer $r > 0$ such that $\mathfrak{a}^n M \cap N = \mathfrak{a}^{n-r}(\mathfrak{a}^r M \cap N)$ for all $n > r$. In particular, the $\mathfrak{a}$-adic topology on M induces the $\mathfrak{a}$-adic topology on N.

Proof: Firstly, we show that if Artin-Rees Lemma holds, the $\mathfrak{a}$-adic topology on M induces the $\mathfrak{a}$-adic topology on N. Notice that $\mathfrak{a}^n N \subset \mathfrak{a}^n M \cap N$ and for all $m \geq 1$, take n such that $n - r = m$, we have $\mathfrak{a}^n M \cap N \subset \mathfrak{a}^{n-r}(\mathfrak{a}^r M \cap N) \subset \mathfrak{a}^m N$.

2.12 Local Ring

Proposition 2.12.1. If (A, m) is a local ring then the elements of A not contained in m are units; conversely a (non-zero) ring A whose non-units form an ideal m is a local ring with maximal ideal m.

Definition 2.12.2. Let $(R, m_R), (S, m_S)$ be two local rings. By definition, a local homomorphism of local rings is a ring homomorphism $f : R \to S$ such that $f(m_R) \subseteq m_S$.

Definition 2.12.3. If W is a set of generators of an A-module M which is minimal, in the sense that any proper subset of W does not generate M, then W is said to be a minimal basis of M.

Theorem 2.12.4. Let (A, m, k) be a local ring and M a finite A-module; set $\bar{M} = M/mM$. Now $\bar{M}$ is a finite-dimensional vector space over k, and we write n for its dimension. Then:

(1) If we take a basis $\{\bar{u}_1, \ldots, \bar{u}_n\}$ for $\bar{M}$ over k, and choose an inverse image $u_i \in M$ of each $\bar{u}_i$, then $\{u_1, \ldots, u_n\}$ is a minimal basis of M;

(2) conversely every minimal basis of M is obtained in this way, and so has n elements.

(3) If $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ are both minimal bases of M, and $v_i = \sum a_{ij} u_j$ with $a_{ij} \in A$ then $\det(a_{ij})$ is a unit of A, so that (a_{ij}) is an invertible matrix.

Proof:
(1) and (2): By Corollary 2.1.9
(3):By Proposition 2.12.1

Theorem 2.12.5 (Kaplansky)**.** Let (A, m) be a local ring; then a projective module M over A is free.

Proof: We only prove the case when M is finite. Choose a minimal basis $\omega_1, \ldots, \omega_n$ of M and define a surjective map $\varphi : F \longrightarrow M$ from the free module $F = Ae_1 \oplus \cdots \oplus Ae_n$ to M by $\varphi(\sum a_i e_i) = \sum a_i \omega_i$; if we set $K = \mathrm{Ker}(\varphi)$ then, from the minimal basis property(1),

$$\sum a_i \omega_i = 0 \Rightarrow a_i \in m \text{ for all } i.$$

Thus $K \subset \mathfrak{m}F$. Because M is projective, there exists $\psi : M \longrightarrow F$ such that $F = \psi(M) \oplus K$, and it follows that $K = mK$. On the other hand, K is a quotient of F, therefore finite over A, so that $K = 0$ by NAK and $F \simeq M$.

2.13 Associated Primes

Definition 2.13.1. Let A be a ring and let M be an A-module. A prime ideal $\mathfrak{p}$ of A is called an associated prime ideal of M if there exists an element $m \in M$ such that $\mathfrak{p} = \mathrm{Ann}(m) = \{a \in A : am = 0\}$. The set of associated prime ideals of M is denoted by $\mathrm{Ass}\ M$ or $\mathrm{Ass}_A M$.

Proposition 2.13.2. $\mathrm{Ass}\ M$ consists of those prime ideals $\mathfrak{p}$ such that M contains a submodule that is isomorphic to $A/\mathfrak{p}$.

Proof: If $\mathfrak{p} \in \mathrm{Ass}M$, for some $m \in M$, we have

$$A/\mathfrak{p} \simeq A/\{a \in A : am = 0\} \simeq (m).$$

If for some submodule N, $A/\mathfrak{p} \simeq N$ under A-modulo homomorphism φ, assume $\varphi(1) = n$, we have $\mathfrak{p} = \mathrm{Ann}(n)$.

Proposition 2.13.3. Let R be a ring. Let M be an R-module. Then $\mathrm{Ass}(M) \subset \mathrm{Supp}(M)$.

Proposition 2.13.4. For every multiplicative subset S of A, we have

$$\mathrm{Ass}\left(S^{-1}M\right) \supset \mathrm{Spec}\left(S^{-1}A\right) \cap \mathrm{Ass}(M)$$

where $\mathrm{Spec}\,(S^{-1}A) = \{\mathfrak{p} \subset A \text{ prime ideal} : \mathfrak{p} \cap S = \emptyset\}$.

Proposition 2.13.5. Let A be a noetherian ring and let M be an A-module.

(1) If $M \neq 0$, then $\mathrm{Ass}\ M \neq \emptyset$.

(2)
$$\bigcup_{\mathfrak{p} \in \mathrm{Ass}\ M} \mathfrak{p} = \{a \in A : \exists 0 \neq m \in M \text{ such that } am = 0\}$$

(3) For every multiplicative subset S of A, we have

$$\mathrm{Ass}\left(S^{-1}M\right) = \mathrm{Spec}\left(S^{-1}A\right) \cap \mathrm{Ass}(M)$$

where $\mathrm{Spec}\,(S^{-1}A) = \{\mathfrak{p} \subset A \text{ prime ideal} : \mathfrak{p} \cap S = \emptyset\}$.

(4) If M is finitely generated, then $\mathrm{Ass}\ M$ is finite,

$$\mathrm{Ass}\ M \subseteq \mathrm{Supp}\, M := \{\mathfrak{p} \subset \text{ A prime ideal } : M_{\mathfrak{p}} \neq 0\}$$

and $\mathrm{Ass}\ M$ and $\mathrm{Supp}\ M$ have the same minimal elements. In particular $\mathrm{Ass}\ A$ is finite, it contains the minimal prime ideals of A, and

$$\bigcup_{\mathfrak{p} \in \mathrm{Ass}\ A} \mathfrak{p}$$

is the set of zero divisors of A.

2.14 Finite and Finite Representation

Definition 2.14.1 (finite representation)**.** We say that an A-module M is of finite presentation if there exists an exact sequence of the form

$$A^p \longrightarrow A^q \longrightarrow M \to 0.$$

Lemma 2.14.2. Let $A \to B$ be a finite ring homomorphism. Then all fibers of the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ are finite .

Proof: Let $\mathfrak{p} \subset A$ be a prime ideal. We must show that the ring $B \otimes_A \kappa(\mathfrak{p})$ has only finitely many prime ideals. However, this ring is a finite-dimensional $\kappa(\mathfrak{p})$-vector space and thus it is Artinian (every ideal is also a $\kappa(\mathfrak{p})$-subvector space, and every descending chain of subvector spaces of a finite-dimensional vector space becomes stationary) and therefore its spectrum has only finitely many points by Algebra Proposition 2.3.8.

Proposition 2.14.3. Every A-module of finite presentation is of finite type. The converse holds if A is noetherian.

Proof: If M is a finite module over Noetherian ring A, we have a A-module surjective homomorphism $\varphi : A^n \to M$. Since A is Noetherian, A^n is also Noetherian. Hence, we have

$$M \simeq A^n / \mathrm{Ker}\varphi$$

where $\mathrm{Ker}\varphi$ is a finitely generted A-module. Assume $\mathrm{Ker}\varphi = A\alpha_1 + \ldots A\alpha_m$, define

$$\psi : A^m \to A^n, e_i \to \alpha_i$$

, we obtain an exact sequence $A^m \xrightarrow{\psi} A^n \xrightarrow{\varphi} M \to 0$.

Proposition 2.14.4. Let A be a ring, and suppose that M is an A-module of finite presentation. If

$$0 \to K \longrightarrow N \longrightarrow M \to 0$$

is an exact sequence and N is finitely generated then so is K.

Proof: By assumption there exists an exact sequence of the form $L_2 \xrightarrow{g} L_1 \xrightarrow{f} M \to 0$, where L_1 and L_2 are free modules of finite rank. From this we get the following commutative diagram

$$\begin{array}{ccccccccc} & & L_2 & \xrightarrow{f} & L_1 & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & K & \xrightarrow{\psi} & N & \xrightarrow{\varphi} & M & \longrightarrow & 0 \end{array}$$

If we write $N = A\xi_1 + \cdots + A\xi_n$, then there exist $v_i \in L_1$ such that $\varphi(\xi_i) = f(v_i)$. Set $\xi_i' = \xi_i - \alpha(v_i)$; then $\varphi(\xi_i') = 0$, so , that we can write $\xi_i' = \psi(\eta_i)$ with $\eta_i \in K$. Let us now prove that

$$K = \beta(L_2) + A\eta_1 + \cdots + A\eta_n.$$

For any $\eta \in K$, set $\psi(\eta) = \sum a_i \xi_i$, then

$$\psi\left(\eta - \sum a_i \eta_i\right) = \sum a_i \left(\xi_i - \xi_i'\right) = \alpha\left(\sum a_i v_i\right)$$

and since $0 = \varphi\alpha\left(\sum a_i v_i\right) = f\left(\sum a_i v_i\right)$, we can write $\sum a_i v_i = g(u)$ with $u \in L_2$. Now

$$\psi\beta(u) = \alpha g(u) = \alpha\left(\sum a_i v_i\right) = \psi\left(\eta - \sum a_i \eta_i\right)$$

so that $\eta = \beta(u) + \sum a_i \eta_i$, and this proves our assertion.

Chapter 3

Homological Algerba

3.1 Basic Definition in Category

Definition 3.1.1 (Category)**.** A category $\mathcal{C}$ consists of three ingredients: a class obj $(\mathcal{C})$ of objects, a set of morphisms $\mathrm{Hom}(A, B)$ for every ordered pair (A, B) of objects, and composition $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$, denoted by

$$(f, g) \mapsto gf$$

for every ordered triple A, B, C of objects. [We often write $f : A \to B$ or $A \xrightarrow{f} B$ instead of $f \in \mathrm{Hom}(A, B)$.] These ingredients are subject to the following axioms:

(1) the Hom sets are pairwise disjoint; that is, each $f \in \mathrm{Hom}(A, B)$ has a unique domain A and a unique target B;

(2) for each object A, there is an identity morphism $1_A \in \mathrm{Hom}(A, A)$ such that $f1_A = f$ and $1_B f = f$ for all $f : A \to B$;

(3) composition is associative: given morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, then

$$h(gf) = (hg)f$$

Definition 3.1.2 (Subcategory)**.** A category $\mathcal{S}$ is a subcategory of a category $\mathcal{C}$ if

(1) $\mathrm{obj}(\mathcal{S}) \subseteq \mathrm{obj}(\mathcal{C})$

(2) $\mathrm{Hom}_{\mathcal{S}}(A, B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A, B)$ for all $A, B \in \mathrm{obj}(\mathcal{S})$, where we denote Hom sets in $\mathcal{S}$ by $\mathrm{Hom}_{\mathcal{S}}(\square, \square)$,

(3) if $f \in \mathrm{Hom}_{\mathcal{S}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{S}}(B, C)$, then the composite $gf \in \mathrm{Hom}_{\mathcal{S}}(A, C)$ is equal to the composite $gf \in \mathrm{Hom}_{\mathcal{C}}(A, C)$,

(4) if $A \in \mathrm{obj}(\mathcal{S})$, then the identity $1_A \in \mathrm{Hom}_{\mathcal{S}}(A, A)$ is equal to the identity $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$. A subcategory $\mathcal{S}$ of $\mathcal{C}$ is a full subcategory if, for all $A, B \in \mathrm{obj}(\mathcal{S})$, we have $\mathrm{Hom}_{\mathcal{S}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$.

A subcategory $\mathcal{S}$ of $\mathcal{C}$ is a full subcategory if, for all $A, B \in \text{obj}(\mathcal{S})$, we have $\text{Hom}_{\mathcal{S}}(A, B) = \text{Hom}_{\mathcal{C}}(A, B)$.

Definition 3.1.3. For every category $\mathcal{C}$ the opposite category, denoted by $\mathcal{C}^{\text{opp}}$, is the category with the same objects as $\mathcal{C}$ and where for two objects X and Y of $\mathcal{C}^{\text{opp}}$ we set $\text{Hom}_{\mathcal{C}^{\text{opp}}}(X, Y) := \text{Hom}_{\mathcal{C}}(Y, X)$ with the obvious composition law.

Definition 3.1.4 (covariant functor)**.** If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a covariant functor $T : \mathcal{C} \to \mathcal{D}$ is a function such that

(1) if $A \in \text{obj}(\mathcal{C})$, then $T(A) \in \text{obj}(\mathcal{D})$,

(2) if $f : A \to A'$ in $\mathcal{C}$, then $T(f) : T(A) \to T(A')$ in $\mathcal{D}$,

(3) if $A \xrightarrow{f} A' \xrightarrow{g} A''$ in $\mathcal{C}$, then $T(A) \xrightarrow{T(f)} T(A') \xrightarrow{T(g)} T(A'')$ in $\mathcal{D}$ and

$$T(gf) = T(g)T(f),$$

(4) $T(1_A) = 1_{T(A)}$ for every $A \in \text{obj}(\mathcal{C})$.

Definition 3.1.5 (contravariant functor)**.** A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is by definition a covariant functor $F : \mathcal{C}^{\text{opp}} \to \mathcal{D}$, where $\mathcal{C}^{\text{opp}}$ is the opposite category of $\mathcal{C}$. Sometimes we use the notation $F : \mathcal{C} \to \mathcal{D}$ for a contravariant functor, in which case we explicitly state that F is contravariant.

Definition 3.1.6 (faithful functor)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is faithful (resp. full, resp. fully faithful) if the map $\text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{D}}(FX, FY\ )$ is an injection (resp. surjection, resp. bijection) for all $X, Y \in \text{Ob}(\mathcal{C})$.

Proposition 3.1.7. Let $F : \mathcal{C} \to \mathcal{D}$ is fully faithful functor.

(1) Let $f : X \to Y$ be a morphism of $\mathcal{C}$ such that Ff is an isomorphism. Then f is an isomorphism.

(2) Let X and Y be objects of $\mathcal{C}$ such that $FX \simeq FY$. Then $X \simeq Y$.

Definition 3.1.8 (isomorphism)**.** A morphism $f : A \to B$ in a category $\mathcal{C}$ is an isomorphism if there exists a morphism $g : B \to A$ in $\mathcal{C}$ with

$$gf = 1_A \quad \text{and} \quad fg = 1_B.$$

The morphism g is called the inverse of f.

Definition 3.1.9 (natural transformation)**.** Let $S, T : \mathcal{A} \to \mathcal{B}$ be covariant functors. A natural transformation $\tau : S \to T$ is a one-parameter family of morphisms in $\mathcal{B}$,

$$\tau = (\tau_A : SA \to TA)_{A \in \text{obj}(\mathcal{A})},$$

making the following diagram commute for all $f : A \to A'$ in $\mathcal{A}$:

$$\begin{array}{ccc} SA & \xrightarrow{\tau_A} & TA \\ {\scriptstyle Sf}\downarrow & & \downarrow{\scriptstyle Tf} \\ SA' & \xrightarrow[\tau_{A'}]{} & TA' \end{array}$$

A natural isomorphism is a natural transformation τ for which each τ_A is an isomorphism.

Proposition 3.1.10. Given functors $F, G, H : \mathcal{C} \to \mathcal{D}$ and natural transformations $\alpha : F \to G$ and $\beta : G \to H$, we have the (vertically) composite natural transformation $\beta\alpha : F \to H$. Functors $\mathcal{C} \to \mathcal{D}$ and natural transformations form a category Fun $(\mathcal{C}, \mathcal{D})$. Isomorphisms in this category are called natural isomorphisms. A natural transformation α is a natural isomorphism if and only if α_X is an isomorphism for every object X of $\mathcal{C}$.

Definition 3.1.11 (equivalence of categories)**.** An equivalence of categories is a functor $F : \mathcal{C} \to \mathcal{D}$ such that there exist a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\mathrm{id}_{\mathcal{C}} \simeq GF$ and $FG \simeq \mathrm{id}_{\mathcal{D}}$ The functors F and G are then called quasi-inverses of each other.

Definition 3.1.12 (essentially surjective)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is essentially surjective if for every object Y of $\mathcal{D}$, there exists an object X of $\mathcal{C}$ and an isomorphism $FX \simeq Y$.

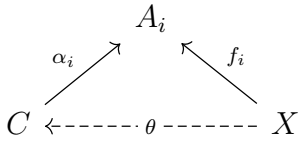
Proposition 3.1.13. A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.

Definition 3.1.14 (groupoid)**.** A category of which where every morphism is an isomorphism is called a groupoid.

Definition 3.1.15 (initial object)**.** An object A in a category $\mathcal{C}$ is called an initial object if, for every object X in $\mathcal{C}$, there exists a unique morphism $A \to X$. Any two initial objects in a category $\mathcal{C}$, should they exist, are isomorphic.

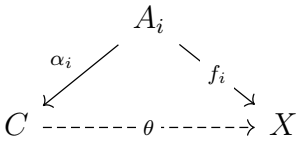
Definition 3.1.16 (terminal object)**.** An object Ω in a category $\mathcal{C}$ is called a terminal object if, for every object C in $\mathcal{C}$, there exists a unique morphism $X \to \Omega$.Any two terminal objects in a category $\mathcal{C}$, should they exist, are isomorphic.

Definition 3.1.17 (product,direct product in module)**.** Let $\mathcal{C}$ be a category, and let $(A_i)_{i\in I}$ be a family of objects in $\mathcal{C}$ indexed by a set I. A product is an ordered pair $\left(C, (p_i : C \to A_i)_{i\in I}\right)$, consisting of an object C and a family $(p_i : C \to A_i)_{i\in I}$ of projections, that is a solution to the following universal mapping problem: for every object X equipped with morphisms $f_i : X \to A_i$, there exists a unique morphism $\theta : X \to C$ making the diagram commute for each i.



Should it exist, a product is denoted by $\prod_{i\in I} A_i$, and it is unique to isomorphism, for it is a terminal object in a suitable category.

Definition 3.1.18 (coproduct,direct sum in module)**.** Let $\mathcal{C}$ be a category, and let $(A_i)_{i\in I}$ be a family of objects in $\mathcal{C}$ indexed by a set I. A coproduct is an ordered pair $\big(C, (\alpha_i : A_i \to C)_{i\in I}\big)$, consisting of an object C and a family $(\alpha_i : A_i \to C)_{i\in I}$ of morphisms, called injections, that is a solution to the following universal mapping problem: for every object X equipped with morphisms $(f_i : A_i \to X)_{i\in I}$, there exists a unique morphism $\theta : C \to X$ making the diagram commute for each i.



Should it exist, a coproduct is usually denoted by $\bigsqcup_{i\in I} A_i$ (the injections are not mentioned). A coproduct is unique to isomorphism, for it is an initial object in a suitable category.

Example 3.1.19 (coproduct in category of topological space)**.** $(X_i)_{i\in I}$ be a family of topological space, $f_i : X_i \to X$ be a family of continuous map. $\bigsqcup_{i\in I} A_i = \big\{(a_i, i) \in \big(\bigcup_{i\in I} A_i\big) \times I : a_i \in A_i\big\}$ be the disjoint union of $(X_i)_{\in I}$. Define U open in $\bigsqcup_{i\in I} A_i$ if and only if $f_i^{-1}(U)$ open in X_i for all $i \in I$. Then $\bigsqcup_{i\in I} A_i$ with continous maps $\alpha_i : a_i \mapsto (a_i, i)$ is the coproduct of a family of topological space.

Example 3.1.20 (coproduct in k-aglebra)**.** If F is a commutative ring and $(A_i)_{i\in I}$ is a family of F-algebra, we can define the tensor product of all these F-algebra

$$\bigotimes_{i\in I} A_i$$

to be the quotient of the F-vector space with basis $\prod_{i\in I} A_i$ by the subspace generated by elements of the form:

(1) $(x_i) + (y_i) - (z_i)$ with $x_j + y_j = z_j$ for one $j \in I$ and $x_i = y_i = z_i$ for all $i \neq j$

(2) $(x_j) - a\,(y_i)$ with $x_j = ay_j$ for one $j \in I$ and $x_i = y_i$ for all $i \neq j$

It can be made into a commutative F-algebra in an obvious fashion, and there are canonical homomorphisms

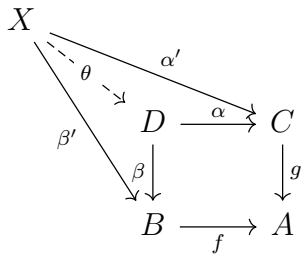
$$A_i \to \bigotimes_{i\in I} A_i$$

of F-algebras. Then by universal property of tensor product, the tensor product of all these F-algebra is the coproduct of A_i.

Example 3.1.21. Coproduct in the category of Group is the free product of groups.

Definition 3.1.22 (pushback/fibered product)**.** Given two morphisms $f : B \to A$ and $g : C \to A$ in a category $\mathcal{C}$, a pullback (or fibered product) is a triple (D, α, β) with $g\alpha = f\beta$ that is a solution to the universal mapping problem: for every (X, α', β') with $g\alpha' = f\beta'$, there exists a

unique morphism $\theta : X \to D$ making the diagram commute.

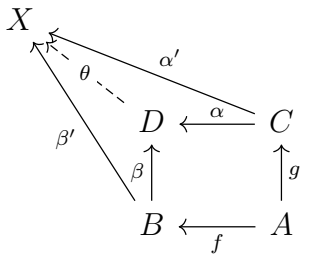


The pullback is often denoted by $B \times_A C$. Pullbacks, when they exist, are unique to isomorphism, for they are terminal objects in a suitable category.

Example 3.1.23 (fibered product in category of topological space)**.** A, B, C be topological spaces, $f : B \to A, g : C \to A$ be continuous maps, $D = \{(b, c) \in B \times C : f(b) = g(c)\}$ be the fibered product of $f : B \to A$ and $g : C \to A$.

Example 3.1.24 (fibered product in category of Set)**.** A, B, C be sets, $f : B \to A, g : C \to A$ be maps, $D = \{(b, c) \in B \times C : f(b) = g(c)\}$ be the fibered product of $f : B \to A$ and $g : C \to A$.

Definition 3.1.25 (pushout/fibered coproduct)**.** Given two morphisms $f : A \to B$ and $g : A \to C$ in a category $\mathcal{C}$, a pushout (or fibered sum) is a triple (D, α, β) with $\beta g = \alpha f$ that is a solution to the universal mapping problem: for every triple (Y, α', β') with $\beta' g = \alpha' g$, there exists a unique morphism $\theta : D \to Y$ making the diagram commute. The pushout is often denoted by $B \cup_A C$.



Pushouts are unique to isomorphism when they exist, for they are initial objects in a suitable category.

Example 3.1.26. In category of Commutative Rings, $f : A \to B, g : A \to B$ be ring homomorphism, then the pushout is given by tensor product of A-algebra B and A-algebra C and homorphism:

$$\begin{aligned} \beta : & B \to B \otimes_A C & \qquad \alpha : & C \to B \otimes_A C \\ & b \mapsto b \otimes 1 & & c \mapsto 1 \otimes c \end{aligned}$$

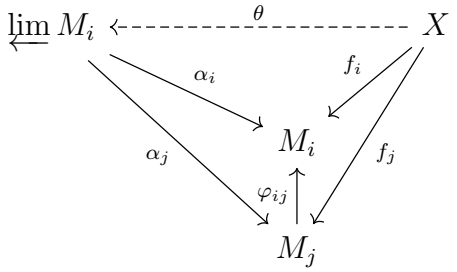
Definition 3.1.27. A inverse system is a functor from opposite category of the category induced by a given directed partially ordered set to a category $\mathcal{C}$. A morphism between inverse system is a natrual transformation between inverse system.

Conversely, a inverse system is a functor from the category induced by a given directed partially ordered set to a category $\mathcal{C}$. A morphism between inverse system is the natrual transformation between direct system.

Definition 3.1.28 (inverse limit)**.** Let $(M_i, \varphi_{ij} : M_i \leftarrow M_j), i \leq j$ be an inverse system, The inverse limit (also called projective limit or limit) is an object $\varprojlim M_i$ and a family of projections $\left(\alpha_i : \varprojlim M_i \to M_i\right)_{i\in I}$ such that:

(1) $\varphi_{ij}\alpha_j = \alpha_i$ whenever $i \preceq j$,

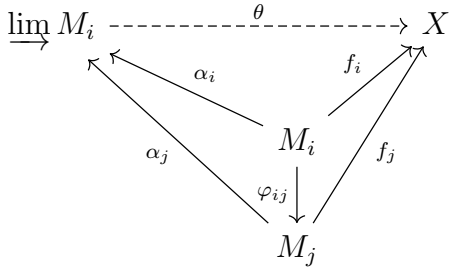
(2) for every $X \in \mathrm{obj}(\mathcal{C})$ and all morphisms $f_i : X \to M_i$ satisfying $\varphi_i^j f_j = f_i$ for all $i \preceq j$, there exists a unique morphism $\theta : X \to \varprojlim M_i$ making the diagram commute.



Definition 3.1.29 (direct limit)**.** Let $\{M_i, \varphi_{ij} : M_i \to M_j\}, i \leq j$ be a direct system. The direct limit (also called inductive limit or colimit) is an object $\varinjlim M_i$ and insertion morphisms $\left(\alpha_i : M_i \to \varinjlim M_i\right)_{i\in I}$.

(1) $\alpha_j\varphi_{ij} = \alpha_i$ whenever $i \preceq j$,

(2) Let $X \in \mathrm{obj}(\mathcal{C})$, and let there be given morphisms $f_i : M_i \to X$ satisfying $f_j\varphi_j^i = f_i$ for all $i \preceq j$. There exists a unique morphism $\theta : \varinjlim M_i \to X$ making the diagram commute.



Remark 3.1.30. If A, B are direct systems whose direct limit exist, and $F : A \to B$ is a natrual transformation. We have a natrual morphism between direct limit induced by F.

Proposition 3.1.31 (direct limit is exact functor)**.** Let $\left\{A_i, \alpha_j^i\right\}, \left\{B_i, \beta_j^i\right\}$, and $\left\{C_i, \gamma_j^i\right\}$ be direct systems of left R-modules over I. If $r : \left\{A_i, \alpha_j^i\right\} \to \left\{B_i, \beta_j^i\right\}$ and $s : \left\{B_i, \beta_j^i\right\} \to \left\{C_i, \gamma_j^i\right\}$ are morphisms of direct systems, and if

$$0 \to A_i \xrightarrow{r_i} B_i \xrightarrow{s_i} C_i \to 0$$

is exact for each $i \in I$, then there is an exact sequence

$$0 \to \varinjlim A_i \xrightarrow{r} \varinjlim B_i \xrightarrow{s} \varinjlim C_i \to 0$$

Proposition 3.1.32 (inverse limit is left exact functor)**.** In ${}_R\,\mathrm{Mod}$, let $r : \{A_i, \alpha_j^i\} \to \{B_i, \beta_j^i\}$ and $s : \{B_i, \beta_j^i\} \to \{C_i, \gamma_j^i\}$ be morphisms of inverse systems. If

$$0 \to A_i \xrightarrow{r_i} B_i \xrightarrow{s_i} C_i$$

is exact for each $i \in I$, prove that there are homomorphisms $\overleftarrow{r}, \overleftarrow{s}$ given by the universal property of inverse limits, and an exact sequence

$$0 \to \varprojlim A_i \xrightarrow{r} \varprojlim B_i \xrightarrow{r} \varprojlim C_i$$

Definition 3.1.33. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be covariant functors. The ordered pair (F, G) is an adjoint pair if, for each $C \in \mathrm{obj}(\mathcal{C})$ and $D \in \mathrm{obj}(\mathcal{D})$, there are bijections

$$\tau_{C,D} : \mathrm{Hom}_{\mathcal{D}}(FC, D) \to \mathrm{Hom}_{\mathcal{C}}(C, GD)$$

such that the following diagram commute:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(FC, D) & \xrightarrow{(Ff)^*} & \mathrm{Hom}_{\mathcal{D}}(FC', D) \\
\Big\downarrow{\scriptstyle \tau_{C,D}} & & \Big\downarrow{\scriptstyle \tau_{C',D}} \\
\mathrm{Hom}_{\mathcal{C}}(C, GD) & \xrightarrow[f^*]{} & \mathrm{Hom}_{\mathcal{C}}(C', GD)
\end{array}$$

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(FC, D) & \xrightarrow{(g)^*} & \mathrm{Hom}_{\mathcal{D}}(FC, D') \\
\Big\downarrow{\scriptstyle \tau_{C,D}} & & \Big\downarrow{\scriptstyle \tau_{C,D'}} \\
\mathrm{Hom}_{\mathcal{C}}(C, GD) & \xrightarrow[(Gg)_*]{} & \mathrm{Hom}_{\mathcal{C}}(C, GD')
\end{array}$$

Example 3.1.34 (Hom and Tensor)**.** If $B = {}_RB_S$ is a bimodule, $\square \otimes_R B : \mathrm{Mod}_R \to \mathrm{Mod}_S$ and $\mathrm{Hom}_S(B, \square) : \mathrm{Mod}_S \to \mathrm{Mod}_R$ be two functors. then $(\square \otimes_R B, \mathrm{Hom}_S(B, \square))$ is an adjoint pair. Similarly, if $B = {}_SB_R$ is a bimodule, $B \otimes_R \square :_R \mathrm{Mod} \to_S \mathrm{Mod}$ and $\mathrm{Hom}_S(B, \square) :_S \mathrm{Mod} \to_R \mathrm{Mod}$ be two functors. then $(B \otimes_R \square, \mathrm{Hom}_S(B, \square))$ is an adjoint pair.

Example 3.1.35 (Free and Forget)**.**

Example 3.1.36 (Induced Representation)**.** G is a finite group, H be a subgroup of G, then $\mathbb{C}[G]$ be a $(\mathbb{C}[G], \mathbb{C}[H])$ bi-module, functor $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} \square :_{\mathbb{C}[H]} \mathrm{Mod} \to_{\mathbb{C}[G]} \mathrm{Mod}$ and functor $\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], \square)$ be an adjoint pair, since $\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], \square) \simeq \mathrm{Res}^{\mathbb{C}[G]}_{\mathbb{C}[H]}$(Restriction from $\mathbb{C}[G]$-module to $\mathbb{C}[H]$-module), we have $(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} \square, \mathrm{Res}^{\mathbb{C}[G]}_{\mathbb{C}[H]})$ is an adjoint pair.

Proposition 3.1.37. Let (F, G) be an adjoint pair offunctors, where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. Then F preserves direct limits and G preserves inverse limits.

Proposition 3.1.38. If F preserve inverse limit, then F preserve kernel. If F preserve direct limit, then G preserve cokernel.

Proof: Trivial.

Definition 3.1.39. Now let $\mathcal{C}$ be an arbitrary categorye Define for every object X of $\mathcal{C}$ the functor

$$h_X : \mathcal{C}^{\text{opp}} \to (\text{ Sets }),$$
$$S \mapsto h_X(S) := \text{Hom}_{\mathcal{C}}(S, X),$$
$$(u : S' \to S) \mapsto (h_X(u) : h_X(S) \to h_X(S'), x \mapsto x \circ u).$$

If $f : X \to Y$ is a morphism in $\mathcal{C}$, for every object S the composition

$$h_f(S) : h_X(S) \to h_Y(S), \quad g \mapsto f \circ g$$

defines a morphism $h_f : h_X \to h_Y$ of functors. We obtain a (covariant) functor $X \mapsto h_X$ from $\mathcal{C}$ to the category $\widehat{\mathcal{C}}$ of functors $\mathcal{C}^{\text{opp}} \to$ (Sets).

Lemma 3.1.40 (Yoneda Lemma)**.** The functor $X \mapsto h_X$ induces a bijection

$$\text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\widehat{\mathcal{C}}}(h_X, h_Y).$$

In other words, $X \mapsto h_X$ is a fully faithful functor $\mathcal{C} \to \widehat{\mathcal{C}}$.

Proof: The inverse map is give by

$$F \in \text{Hom}_{\widehat{\mathcal{C}}}(h_X, h_Y) \to F(X) : \text{Hom}_{\mathcal{C}}(X, X) \to \text{Hom}_{\mathcal{C}}(X, Y) \to F(X)(\text{id}_X) \in \text{Hom}_{\mathcal{C}}(X, Y)$$

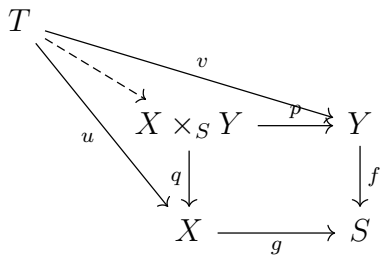
3.2 Fiber product

Definition 3.2.1. Assume $\mathcal{C}$ is a category. A morphism $h : T \to S$ in $\mathcal{C}$ is called an S-object. Sometimes we will simply write T instead of $h : T \to S$. The morphism h is called the structure morphism of T. Recall that for two S-objects $h : T \to S$ and $f : X \to S$ in $\mathcal{C}$ we denote by $\text{Hom}_S(T, X)$ the morphisms $w : T \to X$ such that $f \circ w = h$. These morphisms are called S-morphisms. In this way S-objects and S-morphisms form a category that we will denote by $\mathcal{C}/S$. Usually we write $X_S(T)$ instead of $\text{Hom}_S(T, X)$ and call $X_S(T)$ the set of T-valued points of X (over S). Note that the object id_S in $\mathcal{C}/S$ is a final object.

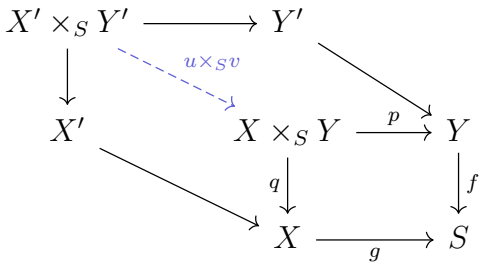
Proposition 3.2.2. Then $(X \times_S Y, p, q)$ is the unique (up to unique isomorphism) triple such that for all morphisms $h : T \to S$ the map

$$\text{Hom}_S(T, X \times_S Y) \to \text{Hom}_S(T, X) \times \text{Hom}_S(T, Y),$$
$$w \mapsto (p \circ w, q \circ w)$$

is a bijection. In other words, the fiber product of $f : X \to S$ and $g : Y \to S$ in $\mathcal{C}$ is the same as the product of the S-objects f and g in $\mathcal{C}/S$.



Proposition 3.2.3. The fiber product is functorial in the following sense: If X, Y, X', and Y' are S-objects and $u : X \to X', v : Y \to Y'$ are S-morphisms, then there exists a unique morphism, denoted $u \times_S v$, such that the following diagram is commutative



Definition 3.2.4 (cartesian)**.**

Yoneda Lemma has the following corollaries.

Proposition 3.2.5. Let S be an object in $\mathcal{C}$ and let X, Y, and Z be S-objects. There are isomorphisms (functorial in X, Y, and Z), called canonical,

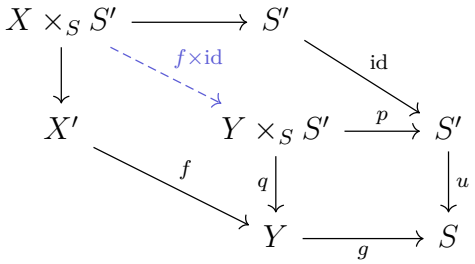
$$X \times_S S \xrightarrow{\sim} X,$$
$$X \times_S Y \xrightarrow{\sim} Y \times_S X,$$
$$(X \times_S Y) \times_S Z \xrightarrow{\sim} X \times_S (Y \times_S Z),$$

given on T-valued points, for $h : T \to S$ any S-object, by

$$X_S(T) \times S_S(T) \xrightarrow{\sim} X_S(T), \qquad (x, h) \mapsto x,$$
$$X_S(T) \times Y_S(T) \xrightarrow{\sim} Y_S(T) \times X_S(T), \qquad (x, y) \mapsto (y, x),$$
$$(X_S(T) \times Y_S(T)) \times Z_S(T) \xrightarrow{\sim} X_S(T) \times (Y_S(T) \times Z_S(T)), \quad ((x, y), z) \mapsto (x, (y, z)).$$

Definition 3.2.6 (base change)**.** Let $\mathcal{C}$ be a category in which arbitrary fiber products exist (e.g., the category of schemes), and let $u : S' \to S$ be a morphism in $\mathcal{C}$. If $X \to S$ is an S-object, $X \times_S S'$ is an S'-object via the second projectionby. It is called the inverse image or the base change of X by u.

If $Y \to S$ is a second S-object and $f : X \to Y$ an S-morphism, the morphism $f \times_S \mathrm{id} : X \times_S S' \to Y \times_S S'$ induced by the following diagram



is a morphism of S'-objects. Hence, we obtain a covariant functor

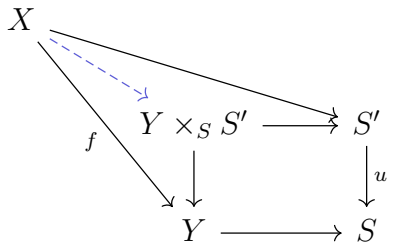
$$u^* : \mathcal{C}/S \to \mathcal{C}/S'$$

from the category of S-objects in $\mathcal{C}$ to the category of S'-objects. This functor is called base change by u.

Definition 3.2.7 (adjoint pair regarding the base change u)**.** $u : S' \to S$ be a morphism, $u^* : \mathcal{C}/S \to \mathcal{C}/S'$ be the base change functor. Now we define a functor $v^* : \mathcal{C}/S' \to \mathcal{C}/S$ by $(X \to S') \mapsto (X \to S' \to S)$. Then (v^*, u^*) is an adjoint pair. For X be a S'-object and Y be a S object. The isomorphism

$$\mathrm{Hom}_S(X, Y) \simeq \mathrm{Hom}_{S'}(X, Y \times_S S')$$

is given by mutually inverse maps



and natural projection $Y \times_S S' \to Y$.

Definition 3.2.8. Let $u : X \to S$ be an S-object. The morphism

$$\Delta_{X/S} := \Delta_u := (\mathrm{id}_X, \mathrm{id}_X)_S : X \to X \times_S X$$

is called the diagonal (morphism) of X over S.

Definition 3.2.9. Let $f : X \to Y$ be a morphism of S-objects. The morphism

$$\Gamma_f := (\mathrm{id}_X, f)_S : X \to X \times_S Y$$

is called the graph (morphism) of f.

Definition 3.2.10. Let $f, g : X \to Y$ be two S-morphisms. An S-object K together with an S-morphism $i : K \to X$ is called equalizer (or difference kernel) of f and g if for all S-objects T the map $i(T)$ yields a bijection

$$K_S(T) \xrightarrow{\sim} \{x \in X_S(T); f(T)(x) = g(T)(x)\}$$

We denote the equalizer of f and g by $\mathrm{Eq}(f, g)_S$ or simply $\mathrm{Eq}(f, g)$ and call the morphism $i : \mathrm{Eq}(f, g) \to X$ the canonical morphism.

Remark 3.2.11. By Yoneda's Lemma, $g \circ i = f \circ i$.

Proposition 3.2.12.

3.3 Abelian Category

Definition 3.3.1 (additive category)**.** A category $\mathcal{C}$ is additive if

(1) $\mathrm{Hom}(A, B)$ is an (additive) abelian group for every $A, B \in \mathrm{obj}(\mathcal{C})$,

(2) composition map
$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$$
is $\mathbb{Z}$-bilinear.

(3) $\mathcal{C}$ has a zero object (a zero object is an object that is both initial and terminal),

(4) $\mathcal{C}$ has finite products and finite coproducts: for all objects A, B in $\mathcal{C}$, both $A \sqcap B$ and $A \sqcup B$ exist in $\mathrm{obj}(\mathcal{C})$.

Definition 3.3.2 (Additive Functor)**.** If $\mathcal{C}$ and $\mathcal{D}$ are additive categories, a functor $T : \mathcal{C} \to \mathcal{D}$ (of either variance) is additive if, for all A, B and all $f, g \in \mathrm{Hom}(A, B)$, we have
$$T(f + g) = Tf + Tg;$$
that is, the function $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(TA, TB)$, given by $f \mapsto Tf$, is a homomorphism of abelian groups.

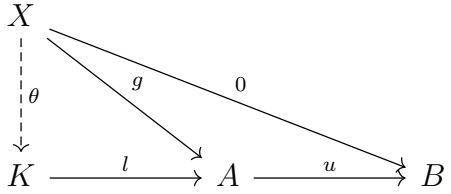
Proposition 3.3.3. Additive Functor preserve zero object.

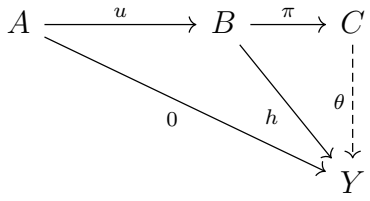
Proposition 3.3.4. If $\mathcal{C}$ and $\mathcal{D}$ are additive categories and $T : \mathcal{C} \to \mathcal{D}$ is an additive functor of either variance, then $T(A \oplus B) \cong T(A) \oplus T(B)$ for all $A, B \in \mathrm{obj}(\mathcal{C})$.

Definition 3.3.5. A morphism $u : B \to C$ in a category $\mathcal{C}$ is a monomorphism (or is monic) if u can be canceled from the left; that is, for all objects A and all morphisms $f, g : A \to B$, we have that $uf = ug$ implies $f = g$.
$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{u} C$$

Definition 3.3.6. A morphism $v : B \to C$ in a category $\mathcal{C}$ is an epimorphism (or is epic) if v can be canceled from the right; that is, for all objects D and all morphisms $h, k : C \to D$, we have that $hv = kv$ implies $h = k$.
$$B \xrightarrow{v} C \underset{k}{\overset{h}{\rightrightarrows}} D$$

Definition 3.3.7 (kernel,cokernel)**.** If $u : A \to B$ is a morphism in an additive category $\mathcal{A}$, then its kernel $\ker u$ is a morphism $i : K \to A$ that satisfies the following universal mapping property: $u \circ l = 0$ and, for every $g : X \to A$ with $ug = 0$, there exists a unique $\theta : X \to K$ with $i\theta = g$.





There is a dual definition for cokernel (the morphism π in the diagram).

Proposition 3.3.8. Let $u : A \to B$ be a morphism in an additive category $\mathcal{A}$.

(1) If $\ker u$ exists, then u is monic if and only if $\ker u = 0$.

(2) Dually, if coker u exists, then u is epic if and only if coker $u = 0$.

Proof: We refer to the diagrams in the definitions of kernel and cokernel. Let ker u be $\iota : K \to A$, and assume that $\iota = 0$. If $g : X \to A$ satisfies $ug = 0$, then the universal property of kernel provides a morphism $\theta : X \to K$ with $g = \iota\theta = 0$ (because $\iota = 0$). Hence, u is monic. Conversely, if u is monic, consider

$$K \underset{0}{\overset{\iota}{\rightrightarrows}} A \xrightarrow{u} B.$$

Since $u\iota = 0 = u0$, we have $\iota = 0$. The proof for epimorphisms and cokers is dual.

Proposition 3.3.9. Every kernel is monomorphism, every cokernel is epimorphism.

Definition 3.3.10. Let $\mathcal{A}$ be an additive category admitting kernels and cokernels and let $f : A \to B$ be a morphism. We define the coimage and image of f to be $\mathrm{coim}(f) = \mathrm{coker}(g), \mathrm{im}(f) = \ker(h)$, where $g : \ker(f) \to A$ and $h : B \to \mathrm{coker}(f)$ are the canonical morphisms.

In the above situation, every morphism $f : A \to B$ factors uniquely into

$$A \twoheadrightarrow \mathrm{coim}(f) \to \mathrm{im}(f) \hookrightarrow B.$$

as the following diagram

$$\begin{array}{ccccccc} \mathrm{Ker}f & \xrightarrow{g} & A & \xrightarrow{f} & B & \xrightarrow{h} & \mathrm{coker}f \\ & & \downarrow & & \uparrow & & \\ & & \mathrm{coker}g & \xrightarrow{\tilde{f}} & \mathrm{Ker}h & & \end{array}$$

Definition 3.3.11. A category $\mathcal{C}$ is an abelian category if it is an additive category such that

(1) every morphism has a kernel and a cokernel (AB1)

(2) For each morphism $f : A \to B$, the morphism $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism.(AB2)

Proposition 3.3.12. The following properties follow from (AB2):

(1) If a morphism is both a monomorphism and an epimorphism, then it is an isomorphism.

(2) Every monomorphism is the kernel of its cokernel.

(3) Every epimorphism is the cokernel of its kernel.

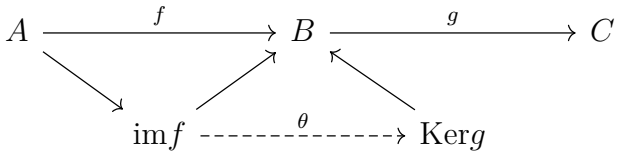
(4) Every morphism $f : A \to B$ can be decomposed into

$$A \xrightarrow{g} \mathrm{im}(f) \xrightarrow{h} B,$$

where g is an epimorphism and h is a monomorphism.

Example 3.3.13. Category of abelian topological group is not an abelian category, but it is an additive category. For example, consider the identity map between $\mathbb{R}$ with discrete topology and euclidean topology.

Definition 3.3.14. We say that a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in an abelian category is exact at B if $gf = 0$ and the morphism $\mathrm{im}(f) \to \ker(g)$ is an isomorphism.



Definition 3.3.15. Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories. An additive functor $F : \mathcal{C} \to \mathcal{D}$ is left exact (resp. right exact) if for any exact sequence $0 \to X' \to X \to X''$ (resp. $X' \to X \to X'' \to 0$) the sequence $0 \to F(X') \to F(X) \to F(X'')$ (resp. $F(X') \to F(X) \to F(X'') \to 0$) is exact. The functor F is exact if it is right exact and left exact. A functor F is exact if and only if for all exact sequences $X \xrightarrow{u} Y \xrightarrow{v} Z$ the sequence

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z)$$

is exact.

Proposition 3.3.16. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Then the following conditions are equivalent: (1) F is left exact. (2) For every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}, 0 \to FX \to FY \to FZ$ is an exact sequence in $\mathcal{B}$.

Proposition 3.3.17. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Then the following conditions are equivalent:

(1) F is exact.

(2) For every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}, 0 \to FX \to FY \to FZ \to 0$ is a short exact sequence in $\mathcal{B}$.

(3) F is left exact and preserves epimorphisms.

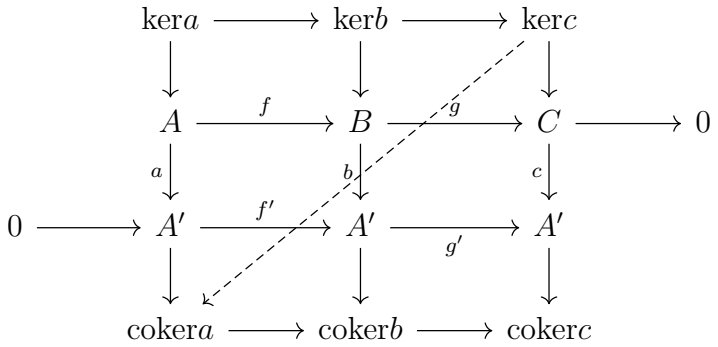
(4) F is right exact and preserves monomorphisms.

Definition 3.3.18. An object P in an abelian category $\mathcal{A}$ is projective if, for every epic $g : B \to C$ and every $f : P \to C$, there exists $h : P \to B$ with $f = gh$.

An object E in an abelian category $\mathcal{A}$ is injective if, for every monic $g : A \to B$ and every $f : A \to E$, there exists $h : B \to E$ with $f = hg$.

An abelian category $\mathcal{A}$ has enough injectives if, for every $A \in \mathrm{obj}(\mathcal{A})$, there exist an injective E and a monic $A \to E$. Dually, $\mathcal{A}$ has enough projectives if, for every $A \in \mathrm{obj}(\mathcal{A})$, there exist a projective P and an epic $P \to A$.

Theorem 3.3.19 (snake lemma)**.** In an abelian category , consider a commutative diagram:



where the rows are exact sequences and 0 is the zero object. Then there is an exact sequence relating the kernels and cokernels of a, b, and c.

$$\ker a \longrightarrow \ker b \longrightarrow \ker c \xrightarrow{d} \mathrm{coker}\, a \longrightarrow \mathrm{coker}\, b \longrightarrow \mathrm{coker}\, c$$

where d is a homomorphism, known as the connecting homomorphism. Furthermore, if the morphism f is a monomorphism, then so is the morphism $\ker a \longrightarrow \ker b$, and if g' is an epimorphism, then so is $\mathrm{coker}\, b \longrightarrow \mathrm{coker}\, c$.

3.4 Derived Functor

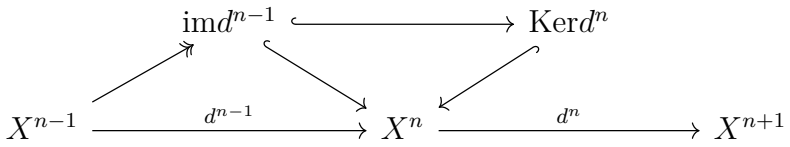
$\mathcal{A}$ be an abelian category.

Definition 3.4.1 (cochain complex)**.** A (cochain) complex in $\mathcal{A}$ consists of $X = (X^n, d_X^n)_{n\in\mathbb{Z}}$, where X^n is an object of $\mathcal{A}, d_X^n : X^n \to X^{n+1}$ is a morphism of $\mathcal{A}$ (called differential) such that for any $n, d_X^{n+1} d_X^n = 0$. The index n in X^n is called the degree. A (cochain) morphism of complexes $X \to Y$ is a collection of morphisms $(f^n)_{n\in\mathbb{Z}}$ of morphisms $f^n : X^n \to Y^n$ in $\mathcal{A}$ such that $d_Y^n f^n = f^{n+1} d_X^n$. We let $C(\mathcal{A})$ denote the category of complexes in $\mathcal{A}$.

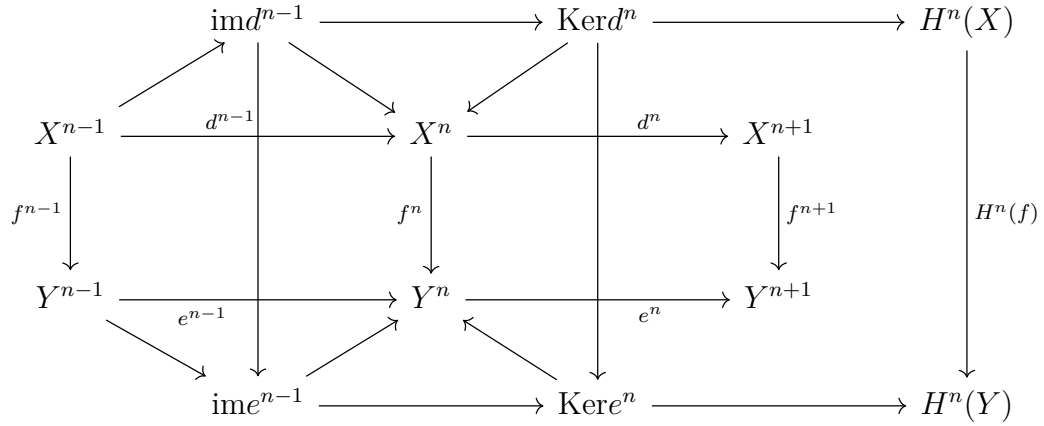
Definition 3.4.2. Let X be a complex in $\mathcal{A}$. We define

$$\begin{aligned} Z^n X &= \ker\left(d_X^n : X^n \to X^{n+1}\right), \\ B^n X &= \operatorname{im}\left(d_X^{n-1} : X^{n-1} \to X^n\right), \\ H^n X &= \operatorname{coker}\left(B^n X \hookrightarrow Z^n X\right), \end{aligned}$$

and call them the cocycle, coboundary, cohomology objects, of degree n.



Definition 3.4.3. Let X and Y be cochain complexes in $\mathcal{A}$ and $f : X \to Y$ be a morphism, we can induce a morphism $H^n(f) : H^n(X) \to H^n(Y)$ by the following diagram:



Definition 3.4.4. A complex X is said to be acyclic if $H^n X = 0$ for all n. A morphism of complexes $X \to Y$ is called a quasi-isomorphism if $H^n f : H^n X \to H^n Y$ is an isomorphism for all n.

Theorem 3.4.5 (long exact sequence cohomology)**.** Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short

exact sequence of complexes. Then we have a long exact sequence

$$\begin{array}{ccccccc} H^n(A) & \xrightarrow{H^n(f)} & H^n(B) & \xrightarrow{H^n(g)} & H^n(C) & & \\ & \searrow^{\delta_n} & & & & & \\ H^{n+1}(A) & \xrightarrow{H^{n+1}(f)} & H^{n+1}(B) & \xrightarrow{H^{n+1}(g)} & H^{n+1}(C) & & \\ & \searrow^{\delta_{n+1}} & & & & & \\ H^{n+2}(A) & & & & & & \end{array}$$

where δ_n are called connecting morphisms.

Theorem 3.4.6. Given a commutative diagram in category of cochain complex with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

then for every $n \in \mathbb{Z}$, there is a commutative diagram

$$\begin{array}{ccccccccccc} H^n(A) & \xrightarrow{H^n(f)} & H^n(B) & \xrightarrow{H^n(g)} & H^n(C) & \xrightarrow{\delta_n} & H^{n+1}(A) & \xrightarrow{H^{n+1}(f)} & H^{n+1}(B) & \xrightarrow{H^{n+1}(g)} & H^{n+1}(C) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^n(A') & \longrightarrow & H^n(B') & \longrightarrow & H^n(C') & \xrightarrow[\delta'_n]{} & H^{n+1}(A') & \longrightarrow & H^{n+1}(B') & \longrightarrow & H^{n+1}(C') \end{array}$$

Definition 3.4.7. Let X be an object of $\mathcal{A}$. A left resolution of X is an exact sequence

$$\cdots \to P^{-n} \to \cdots \to P^0 \to X \to 0$$

in $\mathcal{A}$. It is called a projective resolution if each P^i is projective.

Dually, a right resolution of X is an exact sequence

$$0 \to X \to I^0 \to \cdots \to I^n \to \cdots$$

in $\mathcal{A}$. It is called an injective resolution if each I^i is injective.

Definition 3.4.8 (Abelian Category with enough injectives and projectives)**.**

Proposition 3.4.9. Let X an object in an Abelian category $\mathcal{A}$ with enough injectives. Then X has an injective resolution.

Proof: Consider the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{u} & I^0 & \longrightarrow & I^1 \\ & & & & & \searrow \quad \nearrow & \\ & & & & & \operatorname{coker}(u) & \end{array}$$

in which I^1 is an injective object such that $\operatorname{coker}(u) \to I^1$ is monomorphism. Then it's easy to check $0 \to X \to I^0 \to I^1$ is an exact sequence.

Definition 3.4.10. Let $\mathcal{A}$ be an additive category. Let X and Y be complexes in $\mathcal{A}$. We let

$$\operatorname{Ht}(X, Y) = \prod_n \operatorname{Hom}_{\mathcal{A}}\left(X^n, Y^{n-1}\right)$$

denote the abelian group of families of morphisms $h = \left(h^n : X^n \to Y^{n-1}\right)_{n\in\mathbb{Z}}$. Given h, consider $f^n = d_Y^{n-1}h^n + h^{n+1}d_X^n : X^n \to Y^n$. We have

$$d_Y^n f^n = d_Y^{n-1}d_Y^n h^n + d_Y^n h^{n+1}d_X^n = d_Y^n h^{n+1}d_X^n = d_Y^n h^{n+1}d_X^n + h^{n+2}d_X^{n+1}d_X^n = f^{n+1}d_X^n.$$

Thus we get a morphism of complexes $f : X \to Y$. We get a homomorphism of abelian groups

$$\operatorname{Ht}(X, Y) \to \operatorname{Hom}_{C(\mathcal{A})}(X, Y).$$

We say that a morphism of complexes $f : X \to Y$ is nullhomotopic if there exists $h \in \operatorname{Ht}(X, Y)$ such that $f^n = d_Y^{n-1}h^n + h^{n+1}d_X^n$. We say that two morphisms of complexes $f, g : X \to Y$ are homotopic if $f - g$ is null-homotopic.

Let $f : X \to Y, g : Y \to Z$ be morphisms of complexes in $\mathcal{A}$. If f or g is null-homotopic, then gf is null-homotopic.

Proposition 3.4.11. If $f, g : X \to Y$ are homotopic, then $H^n f = H^n g : H^n X \to H^n Y$.

Proposition 3.4.12. Suppose we are given injective resolutions of objects A, B in $\mathcal{A}$ and a morphism $f : A \to B$,

$$\begin{aligned} I: &\quad 0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \\ J: &\quad 0 \longrightarrow B \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots \end{aligned}$$

there are $f_i, i \geq 0$ making the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \\ 0 & \longrightarrow & B & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & \cdots \end{array}$$

and f_i is unique up to homotopy.

Definition 3.4.13. Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be left exact an additive covariant functor. Suppose we have an injective resolution of A

$$I: \quad 0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

This gives rise to a cochain complex of objects of $\mathcal{B}$

$$TI: \quad 0 \longrightarrow T\left(I^0\right) \longrightarrow T\left(I^1\right) \longrightarrow \cdots \longrightarrow T\left(I^n\right) \longrightarrow \cdots$$

We define $R^n T(A) = H^n(TI)$ for $n \geq 0$ and call it n-th right derived functors of T. And Proposition 3.4.12 gives us that induced morphism of the functor.

Proposition 3.4.14. R^0T and T are naturally isomorphic.

Theorem 3.4.15 (long exact cohomology sequence in right derived functor)**.** Let $\mathcal{A}$ be an abelian category with enough injectives, $\mathcal{B}$ an abelian category, and let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor. Suppose we have an exact sequence

$$0 \longrightarrow A' \xrightarrow{\varphi} A \xrightarrow{\psi} A'' \longrightarrow 0$$

Then there exist canonical connecting morphisms $\omega^n : R^nT(A'') \longrightarrow R^{n+1}T(A')$ for $n \geq 0$ with the property that the following sequence is exact

$$\begin{aligned} 0 \longrightarrow R^0T(A') \longrightarrow R^0T(A) \longrightarrow R^0T(A'') \xrightarrow{\omega^0} R^1T(A') \longrightarrow \cdots \\ \cdots \longrightarrow R^nT(A'') \xrightarrow{\omega^n} R^{n+1}T(A') \longrightarrow R^{n+1}T(A) \longrightarrow R^{n+1}T(A'') \longrightarrow \cdots \end{aligned}$$

Remark 3.4.16. Left derived functor can be obtained if we replace the injective resolution by projective resolution and all the theorem with respect to left derived functor and right derived functor are similar.

3.5 Ext and Tor

Definition 3.5.1. Consider the opposite category of left R-module, $\mathrm{Hom}(\square, D)$ is a left exact functor. Denote the n-th right derived functor of $\mathrm{Hom}(\square, D)$ by $\mathrm{Ext}^n_R(\square, D)$

Definition 3.5.2. Consider category of left R-module, $\mathrm{Hom}(D, \square)$ is a left exact functor. Denote the n-th right derived functor of $\mathrm{Hom}(D, \square)$ by $\mathrm{ext}^n_R(D, \square)$

Proposition 3.5.3.

$$\mathrm{ext}^n_R(A, B) \simeq \mathrm{Ext}^n_R(A, B)$$

Hence we won't use the notation ext anymore.

Long exact cohomology sequence has the following obvious corollary:

Corollary 3.5.4. For an R-module Q the following are equivalent:

(1) Q is injective,

(2) $\mathrm{Ext}^1_R(A, Q) = 0$ for all R-modules A, and

(3) $\mathrm{Ext}^n_R(A, Q) = 0$ for all R-modules A and all $n \geq 1$.

Corollary 3.5.5. For an R-module P the following are equivalent:

(1) P is projective,

(2) $\mathrm{Ext}^1_R(P, B) = 0$ for all R-modules B, and

(3) $\mathrm{Ext}^n_R(P, B) = 0$ for all R-modules B and all $n \geq 1$.

Proposition 3.5.6. Show that $\mathrm{Ext}^n_R(A_1 \oplus A_2, B) \cong \mathrm{Ext}^n_R(A_1, B) \oplus \mathrm{Ext}^n_R(A_2, B)$ for all $n \geq 0$

Example 3.5.7. Let $R = \mathbb{Z}$ and let $A = \mathbb{Z}/m\mathbb{Z}$ for some $m \geq 2$. By the proposition we have $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D)$, and it follows that $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong mD$, where ${}_mD = \{d \in D \mid md = 0\}$ are the elements of D that have order dividing m. For the higher cohomology groups, we use the simple projective resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

for A given by multiplication by m on $\mathbb{Z}$. Taking homomorphisms into a fixed $\mathbb{Z}$ module D gives the cochain complex

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, D) \xrightarrow{m} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, D) \longrightarrow 0 \longrightarrow \cdots .$$

We have $D \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, D)$and under this isomorphism we have $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong D/mD$ for any abelian group D. It follows immediately from the definition and the cochain complex above that $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) = 0$ for all $n \geq 2$ and any abelian group D, which we summarize as

$$\begin{aligned}
&\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong {}_mD \\
&\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong D/mD \\
&\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) = 0, \qquad \text{for all } n \geq 2
\end{aligned}$$

Example 3.5.8. Suppose A is a torsion abelian group. Then we have $\mathrm{Ext}^0(A, \mathbb{Z}) \cong \mathrm{Hom}(A, \mathbb{Z}) = 0$ since $\mathbb{Z}$ is torsion free. The sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ gives an injective resolution of $\mathbb{Z}$. Applying Hom $(A, \ldots)$ gives the cochain complex

$$0 \longrightarrow \mathrm{Hom}(A, \mathbb{Z}) \longrightarrow \mathrm{Hom}(A, \mathbb{Q}) \longrightarrow \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0 \longrightarrow \cdots$$

and since $\mathbb{Q}$ is also torsion free, this shows that

$$\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}).$$

The group Hom $(A, \mathbb{Q}/\mathbb{Z})$ is called the Pontriagin dual group to A. If A is a finite abelian group the Pontriagin dual of A is isomorphic to A. In particular, $\mathrm{Ext}^1(A, \mathbb{Z}) \cong A$ is nonzero for all nonzero finite abelian groups A.

Example 3.5.9. Suppose $R = \mathbb{Z}$ and A and B are $\mathbb{Z}$-modules, i.e., are abelian groups. The group B can be embedded in an injective $\mathbb{Z}$-module Q_0 and the quotient, Q_1, of Q_0 by the image of B is again injective. Hence we have an injective resolution

$$0 \longrightarrow B \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow 0$$

of B. Applying $\mathrm{Hom}_{\mathbb{Z}}(A, \square)$ to this sequence gives the cochain complex

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, Q_0) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, Q_1) \longrightarrow 0 \longrightarrow \cdots$$

from which it follows immediately that

$$\mathrm{Ext}^n_{\mathbb{Z}}(A, B) = 0$$

for all abelian groups A and B and all $n \geq 2$.

Definition 3.5.10. D is a right R-module, then $D \otimes_R \square$ is a right exact functor from category of left R-module to abelian group. We denote the n-th left derived functor of $D \otimes_R \square$ by $\mathrm{Tor}_R^n(D, \square)$

Definition 3.5.11. If D is a left R-module, $\square \otimes_R D$ is a right exact functor from category of right R-module to abelian group, we denote its the n-the left derived functor by $\mathrm{tor}_n^R(\square, D)$

Theorem 3.5.12. $\mathrm{tor}_n^R(A, B) = \mathrm{Tor}_n^R(A, B)$

Proposition 3.5.13. For a right R-module D the following are equivalent:

(1) D is a flat R-module,

(2) $\mathrm{Tor}_1^R(D, B) = 0$ for all left R-modules B, and

(3) $\mathrm{Tor}_n^R(D, B) = 0$ for all left R-modules B and all $n \geq 1$.

Example 3.5.14. Let $R = \mathbb{Z}$ and let $B = \mathbb{Z}/m\mathbb{Z}$ for some $m \geq 2$. $\mathrm{Tor}_0^{\mathbb{Z}}(D, \mathbb{Z}/m\mathbb{Z})$ is isomorphic to $D \otimes \mathbb{Z}/m\mathbb{Z}$, so we have $\mathrm{Tor}_0^{\mathbb{Z}}(D, \mathbb{Z}/m\mathbb{Z}) \cong D/mD$.

For the higher groups we apply $D \otimes_{\mathbb{Z}} \square$ to the projective resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

of B and use the isomorphisms $D \otimes \mathbb{Z} \cong D$ and $D \otimes \mathbb{Z}/m\mathbb{Z} \cong D/mD$. This gives

$$\cdots \longrightarrow 0 \longrightarrow D \xrightarrow{m} D \longrightarrow 0.$$

It follows that $\mathrm{Tor}_1^{\mathbb{Z}}(D, \mathbb{Z}/m\mathbb{Z}) \cong {}_mD$ is the subgroup of D annihilated by m and that $\mathrm{Tor}_n^{\mathbb{Z}}(D, \mathbb{Z}/m\mathbb{Z}) = 0$ for all $n \geq 2$, which we summarize as

$$\begin{aligned} &\mathrm{Tor}_0(D, \mathbb{Z}/m\mathbb{Z}) \cong D/mD \\ &\mathrm{Tor}_1(D, \mathbb{Z}/m\mathbb{Z}) \cong {}_mD \\ &\mathrm{Tor}_n(D, \mathbb{Z}/m\mathbb{Z}) = 0, \qquad \text{for all } n \geq 2 \end{aligned}$$

3.6 Group Cohomology

Assume G is be finite group.

Definition 3.6.1. Consider a left exact functor $\square^G$: category of left $\mathbb{Z}[G]$-module to abelian group, with $A^G = \{a \in A : ga = a \quad \forall g \in G\}$ and a right exact functor $\square_G$: category of left $\mathbb{Z}[G]$-module to abelian group with $A_G = A/I_G A$ where I_G is the left ideal of $\mathbb{Z}[G]$ generated by $\sigma - 1$.

Proposition 3.6.2. $\square^G = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \square)$ and $\square_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} \square$.

Definition 3.6.3. Let A be a G-module and let $N_G : A \to A$ be the G-module endomorphism $a \mapsto N_G a$. We then have $I_G A \subseteq \ker N_G$ and $\mathrm{im}\, N_G \subseteq A^G$, thus N_G induces a morphism $N_G : A_G \to A^G$ of trivial G-modules.

Definition 3.6.4. Let $N_G = \sum_{\sigma \in G} \sigma$. Define Tate cohomology group:

$$H^n(G,A) = \begin{cases} \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A) & \text{if } n \geq 1 \\ A^G / N_G A & \text{if } n = 0 \\ {}_{N_G}A / I_G A & \text{if } n = -1 \\ \mathrm{Tor}^{\mathbb{Z}[\mathbb{G}]}_{-n-1}(\mathbb{Z}, A) & \text{if } n \leq -2 \end{cases}$$

where $N_G A = \{N_G a : a \in A\}$ and ${}_{N_G}A = \{a \in A : N_G a = a\}$.

Theorem 3.6.5 (standard projective resolution of $\mathbb{Z}$)**.** Let G be a finite group. The standard resolution (projection resolution in right $\mathbb{Z}[G]$-module category and injective resolution in the opposite category of left $\mathbb{Z}[G]$-module category) of $\mathbb{Z}$ by G-modules is the exact sequence of (both left and right) G-module homomorphisms

$$\cdots \longrightarrow \mathbb{Z}\left[G^{n+1}\right] \xrightarrow{d_n} \mathbb{Z}\left[G^n\right] \longrightarrow \cdots \xrightarrow{d_1} \mathbb{Z}[G] \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0,$$

where the boundary maps d_n are defined by

$$d_n\left(g_0, \ldots, g_n\right) := \sum_{i=0}^{n} (-1)^i \left(g_0, \ldots, \hat{g}_i, \ldots, g_n\right)$$

and extended $\mathbb{Z}$-linearly (the notation $\hat{g}_i$ means omit g_i from the tuple). The map d_0 sends each $g \in G$ to 1 , and extends to the map $\sum_g a_g g \mapsto \sum_g a_g$, which is also known as the augmentation map and may be denoted ε.

Proposition 3.6.6. Let A be a G-module. For every $n \geq 0$ we have an isomorphism of abelian groups

$$\Phi^n : \mathrm{Hom}_{\mathbb{Z}[G]}\left(\mathbb{Z}\left[G^{n+1}\right], A\right) \xrightarrow{\sim} C^n(G,A)$$

that sends $\varphi : \mathbb{Z}\left[G^{n+1}\right] \to A$ to the function $f : G^n \to A$ defined by

$$f\left(g_1, \ldots, g_n\right) := \varphi\left(1, g_1, g_1 g_2, \ldots, g_1 g_2 \cdots g_n\right).$$

And under this isomorphism, we have cochain complex of abelian group

$$0 \longrightarrow C^0(G,A) \xrightarrow{d^0} C^1(G,A) \xrightarrow{d^1} C^2(G,A) \longrightarrow \cdots$$

where

$$\begin{aligned} d^n(f)\left(g_0, \ldots, g_n\right) :=& g_0 f\left(g_1, \ldots, g_n\right) - f\left(g_0 g_1, g_2, \ldots, g_n\right) + f\left(g_0, g_1 g_2, \ldots, g_n\right) \\ & \cdots + (-1)^n f\left(g_0, \ldots, g_{n-2}, g_{n-1} g_n\right) + (-1)^{n+1} f\left(g_0, \ldots, g_{n-1}\right). \end{aligned}$$

for $n \geq 1$ and $d_0(a)(g) := ga - a$ for $n = 0$.

The group $C^n(G,A)$ contains subgroups of n-cocycles and n-coboundaries defined by

$$Z^n(G,A) := \ker d^n \quad \text{and} \quad B^n(G,A) := \mathrm{im}\, d^{n-1},$$

and we have $H^n(G,A) = Z^n(G,A)/B^n(G,A)$ for all $n \geq 1$

Corollary 3.6.7 (cohomology group of degree 1)**.** For the group $H^1(G, A)$, the 1-cocycles are the functions $x : G \to A$ with $\partial_2 x = 0$, thus satisfying the property

$$x(\sigma\tau) = \sigma x(\tau) + x(\sigma) \quad \text{for } \sigma, \tau \in G.$$

Because this relation is similar to the one of being a homomorphism, the 1-cocycles are often also called crossed homomorphisms. The 1-coboundaries are obviously the functions

$$x(\sigma) = \sigma a - a, \quad \sigma \in G,$$

with fixed $a \in A = A_0$ (i.e., $x = \partial_1 a$). If the group G acts trivially (i.e., as the identity) on A, then obviously $Z^1(G, A) = \mathrm{Hom}(G, A)$ and $B^1(G, A) = 0$; therefore

$$H^1(G, A) = \mathrm{Hom}(G, A)$$

Definition 3.6.8 (induced G-module by abelian group)**.** Let G be a group and let A be an abelian group. The abelian group

$$\mathrm{Ind}^G(A) := \mathbb{Z}[G] \otimes_{\mathbb{Z}} A$$

with G-action defined by $g(z \otimes a) = (gz) \otimes a$ is the induced left G-module associated to A.

Definition 3.6.9 (coinduced G-module by abelian group)**.** Let G be a group and let A be an abelian group. The abelian group

$$\mathrm{CoInd}^G(A) := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$$

with G-action defined by $(g\varphi)(z) := \varphi(zg)$ is the coinduced left G-module associated to A.

Theorem 3.6.10. Let G be a finite group and A an abelian group. The G-modules $\mathrm{Ind}^G(A)$ and $\mathrm{CoInd}^G(A)$ are isomorphic.

Proof: The canonical G-module isomorphism given by

$$\begin{aligned} \alpha : \mathrm{CoInd}^G(A) &\xrightarrow{\sim} \mathrm{Ind}^G(A) \\ \varphi &\mapsto \sum_{g \in G} g^{-1} \otimes \varphi(g) \\ \left(g^{-1} \mapsto a\right) &\longleftarrow g \otimes a \end{aligned}$$

Theorem 3.6.11 (long exact cohomology sequence for Tate cohomology)**.** If

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

is an exact sequence of G-modules and G-homomorphisms, then there exists a canonical homomorphism

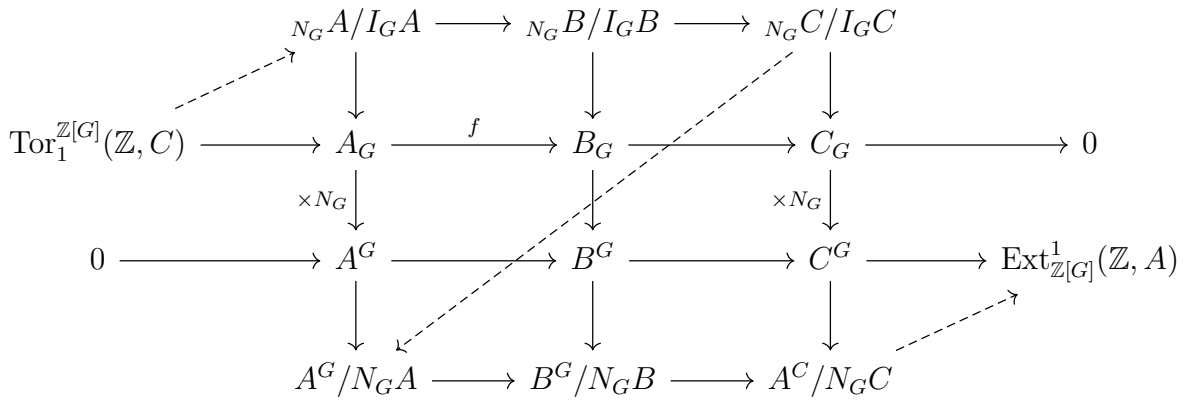
$$\delta_q : H^q(G, C) \longrightarrow H^{q+1}(G, A).$$

The map δ_q is called the connecting homomorphism or also the δ homomorphism. And the induced infinite sequence

$$\cdots \longrightarrow H^q(G, A) \xrightarrow{i_q} H^q(G, B) \xrightarrow{i_q} H^q(G, C) \xrightarrow{\delta_q} H^{q+1}(G, A) \longrightarrow \cdots$$

is also exact. It is called the long exact cohomology sequence.

Proof: By the following commute diagram and snake lemma:



Theorem 3.6.12. Let G be a group and A an abelian group. Then $\left(\mathrm{CoInd}^G(A)\right)^G \simeq A$ and $H^n\left(G, \mathrm{CoInd}^G(A)\right) = 0$ for all $n \geq 1$.

Proof: For all $n \geq 1$ we have an isomorphisms of abelian groups

$$\begin{aligned} \alpha : \mathrm{Hom}_{\mathbb{Z}[G]}\left(\mathbb{Z}\left[G^n\right], \mathrm{CoInd}^G(A)\right) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}\left[G^n\right], A\right) \\ \varphi &\mapsto (z \mapsto \varphi(z)(1)) \\ (z \mapsto (y \mapsto \phi(yz))) &\longleftarrow\!\shortmid \phi \end{aligned}$$

Since

$$\cdots \longrightarrow \mathbb{Z}\left[G^{n+1}\right] \xrightarrow{d_n} \mathbb{Z}\left[G^n\right] \longrightarrow \cdots \xrightarrow{d_1} \mathbb{Z}[G] \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0,$$

is an injective resolution of $\mathbb{Z}$ in the opposite category of abelian groups, we have for all $n \geq 1$, $H^n(G, \mathrm{CoInd}^G(A)) = \mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}, A) = 0$ and $\left(\mathrm{CoInd}^G(A)\right)^G = \mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}, A) = A$

Theorem 3.6.13. Let G be a group and A an abelian group. Then $\left(\mathrm{Ind}^G(A)\right)_G \simeq A$ and $H^{-n-1}\left(G, \mathrm{Ind}^G(A)\right) = 0$ for all $n \geq 1$.

Proof: Viewing $\mathbb{Z}\left[G^n\right]$ as a right $\mathbb{Z}[G]$-module and $\mathbb{Z}[G]$ as a left $\mathbb{Z}[G]$-module, for all $n \geq 1$, we have G-isomorphism

$$\mathbb{Z}\left[G^n\right] \otimes_{\mathbb{Z}[G]} \left(\mathbb{Z}[G] \otimes_{\mathbb{Z}} A\right) \simeq \left(\mathbb{Z}\left[G^n\right] \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]\right) \otimes_{\mathbb{Z}} A \simeq \mathbb{Z}\left[G^n\right] \otimes_{\mathbb{Z}} A$$

Hence $H^{-n-1}(G, A) = \mathrm{Tor}^{\mathbb{Z}}_n(\mathbb{Z}, A) = 0$ for all $n \geq 1$ and $\left(\mathrm{Ind}^G(A)\right)_G = \mathrm{Tor}^{\mathbb{Z}}_0(\mathbb{Z}, A) \simeq A$.

Proposition 3.6.14. Let G be a finite group and let B be an induced or co-induced G-module associated to some abelian group A. Then $H^n(G, B) = 0$ for all $n \in \mathbb{Z}$.

Proof: If $n \leq -2$ or $n \geq 1$, it follows from Theorem 3.6.13 and 3.6.12. , we have $H^{-1}(G, B) = H^0(G, 0)$.

By Proposition 3.6.10, it suffices to check the case when $B = \mathbb{Z}[G] \otimes_{\mathbb{Z}} A$ and $n = 0, 1$.

And by the definition of $H^{-1}(G, \square)$ and $H^{-0}(G, \square)$ and Theorem 2.1.22, we can explicitly write down the form of elements in ${}_{N_G}B$ and I_GB.

Lemma 3.6.15. For a G-module A, we newly define a G-actions on $\mathrm{Ind}^G(A) := \mathbb{Z}[G] \otimes_{\mathbb{Z}} A$ by $g(z \otimes a) := gz \otimes ga$ and on $\mathrm{CoInd}^G(A) := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ by $g\varphi := (z \mapsto g\varphi\left(g^{-1}z\right))$. Let A be a G-module and let A° denote the corresponding abelian group. We have G-module isomorphisms

$$\begin{aligned} \Phi : \mathrm{Ind}^G(A) &\xrightarrow{\sim} \mathrm{Ind}^G\left(A^\circ\right), \quad \Phi(g \otimes a) := g \otimes g^{-1}a \\ \Psi : \mathrm{CoInd}^G(A) &\xrightarrow{\sim} \mathrm{CoInd}^G\left(A^\circ\right), \quad \Psi(\varphi) := (g \mapsto g\varphi\left(g^{-1}\right)) \end{aligned}$$

Lemma 3.6.16. Let A be a G-module. The map $\pi : \mathrm{Ind}^G(A) \to A$ defined by $z \otimes a \mapsto \varepsilon(z)a$ is a surjective morphism of G-modules. By Proposition 2.1.23, it's trivial that

$$0 \to I_G \otimes_{\mathbb{Z}} A \to \mathrm{Ind}^G(A) = \mathbb{Z}[G] \otimes_{\mathbb{Z}} A \xrightarrow{\pi} A \to 0$$

is an exact sequence of G-modules.

The map $\iota : A \to \mathrm{CoInd}^G(A)$ defined by $a \mapsto (z \mapsto \varepsilon(z)a)$ for $g \in G$ is an injective morphism of G-modules with cokernel isomorphic to $\mathrm{Hom}_{\mathbb{Z}}\left(I_G, A\right)$ (by restricting the map $\varphi : \mathbb{Z}[G] \to A$ to I_G).

Theorem 3.6.17 (dimension shifting)**.** Let A be a G-module and let H be any subgroup of G. Then for all $n \in \mathbb{Z}$ we have

$$H^{n+1}(H, A) \simeq H^n\left(H, \mathrm{Hom}_{\mathbb{Z}}\left(I_G, A\right)\right) \quad \text{and} \quad H^{n-1}(H, A) \simeq H^n\left(H, I_G \otimes_{\mathbb{Z}} A\right).$$

Proof: By long exact Tate cohomology sequence and above two lemma.

Theorem 3.6.18. Let G be a finite group and let A be a G-module. The Tate cohomology groups $H^n(G, A)$ are all torsion groups of exponent dividing $\#G$.

Proof: By dimension shifting, it suffices to prove the case $n = 0$.

For any $a \in A^G$ we have $N_G a = (\#G)a$, thus every element of $H^0(G, A) = A^G/N_G A$ has order dividing $\#G$.

Corollary 3.6.19. Let G be a finite group and let A be a G-module. If multiplication by $\#G$ induces an isomorphism $A \to A$, then $H^n(G, A) = 0$ for all $n \in \mathbb{Z}$.

Proof: Consider the long exact sequence in Tate cohomology corresponding to the short exact sequence $0 \to 0 \to A \to A \to 0$). Above statement implies that multiplication by $\#G$ induces the zero morphism $H^n(G, A) \to H^n(G, A)$ for all $n \in \mathbb{Z}$, which is an isomorphism if and only if $\hat{H}^n(G, A) = 0$.

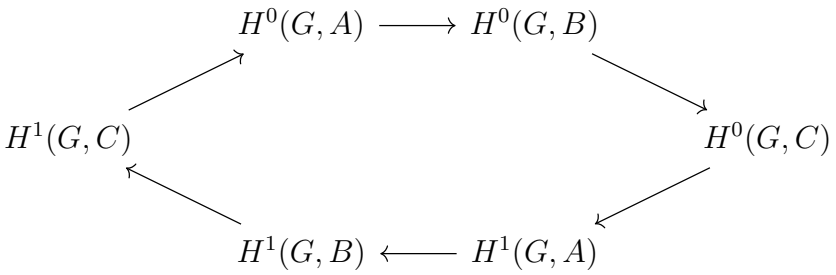
Proposition 3.6.20 (cohomology of finite cyclic group)**.** Let $G = \langle g \rangle$ be a finite cyclic group and let A be a G-module.

(1) For all $n \in \mathbb{Z}$, we have $H^{2n+1}(G, A) = H^1(G, A), H^{2n}(G, A) = H^0(G, A)$.

(2) If

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

is an exact sequence of G-modules, we have exact hexagon



(3) If $H^0(G,A)$ and $H^{-1}(G,A)$ are finite, define Herbrand quotient $h(A) = \#H^0(G,A)/\#H^{-1}(G,A) \in \mathbb{Q}$.

(4) For any exact sequence of G-modules

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0,$$

if any two of $h(A), h(B), h(C)$ are defined then so is the third and $h(B) = h(A)h(C)$.

(5) Let A and B be G-modules. If $h(A)$ and $h(B)$ are defined then so is $h(A \oplus B) = h(A)h(B)$.

(6) If A is an induced, coinduced, or $\#A < \infty$, then $h(A) = 1$.

(7) Let A be a trivial G-module that is a finitely generated abelian group. Then $h(A) = (\#G)^r$, where r is the rank of A.

Proof: (2): By Tate Cohomology sequence.

(4): By (2).

(5): Apply (4) to the split exact sequence $0 \to A \to A \oplus B \to B \to 0$.

(6): If A is finite and G generated by g, then the exact sequence

$$0 \longrightarrow A^G \longrightarrow A \xrightarrow{g-1} A \longrightarrow A_G \longrightarrow 0$$

implies $\#A^G = \#\ker(g-1) = \#\operatorname{coker}(g-1) = \#A_G$. Consdier the morphism $N_G : A_G \to A^G, x \mapsto N_G x$, we have

$$\#H^{-1}(G,A) = \#\ker N_G = \#A_G/\#\mathrm{im}N_G = \#A^G/\#\mathrm{im}N_G = \#\operatorname{coker} N_G = \#H^0(G,A)$$

so $h(A) = 1$.

(7): We have $A/A_{\mathrm{tor}} \simeq \mathbb{Z}^r$, where $\mathbb{Z}$ is a trivial G-module. Then $\mathbb{Z}_G = \mathbb{Z} = \mathbb{Z}^G$, and $N_G : \mathbb{Z}_G \to \mathbb{Z}^G$ is multiplication by $\#G$, so $h(\mathbb{Z}) = \#\mathrm{coker}N_G/\#\ker N_G = \#G$.

Theorem 3.6.21 (Shapiro's Lemma)**.** Let H be a subgroup of G and let A be an H-module. Let $\mathrm{Ind}_H^G(A) = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A$ and $\mathrm{CoInd}_H^G = \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], A)$. For all $n \geq 1$ or $n \leq -2$, we have canonical isomorphisms

$$H^n\left(G, \mathrm{CoInd}_H^G(A)\right) \simeq H^n(H,A)$$

and

$$H^n\left(G, \mathrm{Ind}_H^G(A)\right) \simeq H^n(H,A)$$

Chapter 4

Representation Theory

4.1 Definition

Definition 4.1.1. G is a finite group, a representation of G over a field F is a group homomorphism $\rho : G \to \mathrm{GL}(V)$ where V is a vector space over F.

Proposition 4.1.2. If F is a field and G is a finite group, there's a one-to-one correspondence:

$$\{F[G]\text{-module}\} \longleftrightarrow \{\text{representation of } G\text{over F}\}\,.$$

Definition 4.1.3. Two representations of G are equivalent (or similar) if the FG-modules affording them are isomorphic modules.

Proposition 4.1.4. Let R be a ring and let M be a nonzero R-module.

(1) The module M is said to be irreducible (or simple) if its only submodules are 0 and M; otherwise M is called reducible.

(2) The module M is said to be indecomposable if M cannot be written as $M_1 \oplus M_2$ for any nonzero submodules M_1 and M_2; otherwise M is called decomposable.

(3) The module M is said to be completely reducible(or semisimple) if it is a direct sum of irreducible submodules.

(4) A representation is called irreducible, reducible, indecomposable, decomposable or completely reducible according to whether the $F[G]$-module affording it has the corresponding property.

(5) If M is a completely reducible R-module, any direct summand of M is called a constituent of M (i.e., N is a constituent of M if there is a submodule N' of M such that $M = N \oplus N'$).

(6) Let R be a ring. A left ideal of R which is simple as a left R-module is said to be minimal.

Remark 4.1.5. By the definition of irreducible representation, a finite group has no infinite-dimention irreducible representation.

Definition 4.1.6 (simple ring)**.** A simple ring is a ring has no proper, nonzero 2-sided ideals.

Theorem 4.1.7 (Maschke's theorem)**.** Let G be a finite group and let F be a field whose characteristic does not divide $|G|$. If V is any $F[G]$-module and U is any submodule of V, then V has a submodule W such that $V = U \oplus W$

Proof: The idea of the proof of Maschke's Theorem is to produce an $F[G]$-module homomorphism

$$\pi : V \to U$$

which is a projection onto U, i.e., which satisfies the following two properties:

(1) $\pi(u) = u \quad$ for all $u \in U$

(2) $\pi(\pi(v)) = \pi(v) \quad$ for all $v \in V$

Then we have $V = \mathrm{Ker}\pi \oplus U$.

Let W_0 be subspace of V such that $V = U \oplus W_0$. π_0 be the projection onto U. For each $g \in G$ define

$$g\pi_0 g^{-1} : V \to U \quad \text{by} \quad g\pi_0 g^{-1}(v) = g \cdot \pi_0 \left(g^{-1} \cdot v\right), \quad \text{for all } v \in V$$

Define

$$\pi = \frac{1}{n} \sum_{g \in G} g\pi_0 g^{-1}$$

Then π is a $F[G]$-module homomorphism and satisfies above propositions.

Corollary 4.1.8. chap $F \nmid |G|$, then every representation of finite group G of finite degree over F is completely reducible.

Definition 4.1.9. R is a ring.

(1) A nonzero element e in a ring R is called an idempotent if $e^2 = e$.

(2) Idempotents e_1 and e_2 are said to be orthogonal if $e_1 e_2 = e_2 e_1 = 0$.

(3) An idempotent e is said to be primitive if it cannot be written as a sum of two (commuting) orthogonal idempotents.

(4) The idempotent e is called a primitive central idempotent if $e \in Z(R)$ and e cannot be written as a sum of two orthogonal idempotents in the ring $Z(R)$.

Lemma 4.1.10. Any quotient of a semisimple module is semisimple.

Proof: Let M be a completely reducible module, say $M = \bigoplus_{i \in I} M_i$ with each M_i irreducible, and let $\varphi : M \to N$ be a surjective homomorphism. Then N is the sum of the images of the submodules $M_i \subset M$:

$$N = \sum_{i \in I} \varphi\left(M_i\right)$$

It's clear that $\varphi(M_i)$ is 0 or $\varphi(M_i) \cong M_i$. By Zorn's lemma(take a maximal subset such that it's a direct sum), there's $J \subset I$ such that

$$N = \bigoplus_{i \in J} \varphi(M_i)$$

Hence N is semisimple

Lemma 4.1.11. Let R be an arbitrary nonzero ring.

(1) If M and N are simple R-modules and $\varphi : M \to N$ is a nonzero R-module homomorphism, then φ is an isomorphism.

(2) (Schur's Lemma) If M is a simple R-module, then $\mathrm{Hom}_R(M, M)$ is a division ring.

Proof: Notice that

(a) $E_{ij}A$ is the matrix whose i^{th} row equals the j^{th} row of A and all other rows are zero.

(b) AE_{ij} is the matrix whose j^{th} column equals the i^{th} column of A and all other columns are zero.

(c) $E_{pq}AE_{rs}$ is the matrix whose p, s entry is a_{qr} and all other entries are zero.

Hence (1): By (c).
(2): By $AE_{ij} = E_{ij}A$.
(3): trivial.
(4): L_i is simple by (a). Direct sum is obvious. Let M be any simple R-module. Since $Im = m$ for all $m \in M$ and since $I = \sum_{i=1}^{n} e_i$, there exists some i and some $m \in M$ such that $e_i m \neq 0$. For this i and m the map $re_i \mapsto re_i m$ is a nonzero R-module homomorphism from the simple R-module Re_i to the simple module M. By Schur's Lemma, it is an isomorphism. Also, the map $r \mapsto rE_{i1}$ gives $Re_i \cong Re_1$.

Lemma 4.1.12. Let Δ be a division ring, let $n \in \mathbb{Z}^+$, let R be the ring of all $n \times n$ matrices with entries from Δ and let I be the identity matrix (= the 1 of R).

(1) The only two-sided ideals of R are 0 and R.

(2) The center of R consists of the scalar matrices αI, where α is in the center of Δ : $Z(R) = \{\alpha I \mid \alpha \in Z(\Delta)\}$, and this is a field isomorphic to $Z(\Delta)$. In particular, if Δ is a field, the center of R is the subring of all scalar matrices. The only central idempotent in R is I (in particular, I is primitive).

(3) Let e_i be the matrix with a 1 in position i, i and zeros elsewhere. Then $e_1, \ldots, e_n$ are orthogonal primitive idempotents and $\sum_{i=1}^{n} e_i = I$.

(4) $L_i = Re_i$ is the left ideal consisting of arbitrary entries in column i and zeros in all other columns. L_i is a simple left R-module. Every simple left R-module is isomorphic to L_1 (in particular, all L_i are isomorphic R-modules) and as a left R-module we have $R = L_1 \oplus \cdots \oplus L_n$.

Lemma 4.1.13. Let $R = R_1 \times R_2 \times \cdots \times R_r$, where R_i is the ring of $n_i \times n_i$ matrices over the division ring Δ_i, for $i = 1, 2, \ldots, r$.

(1) Let z_i be the r-tuple with the identity of R_i in position i and zero in all other positions. Then $R_i = z_i R$ and for any $a \in R_i, z_i a = a$ and $z_j a = 0$ for all $j \neq i$. The elements $z_1, \ldots, z_r$ are all of the primitive central idempotents of R. They are pairwise orthogonal and $\sum_{i=1}^{r} z_i = 1$.

(2) Let N be any left R-module and let $z_i N = \{z_i x \mid x \in N\}, 1 \leq i \leq r$. Then $z_i N$ is a left R-submodule of N, each $z_i N$ is an R_i-module on which R_j acts trivially for all $j \neq i$, and

$$N = z_1 N \oplus z_2 N \oplus \cdots \oplus z_r N.$$

(3) Let M_i be the unique simple R_i-module in Lemma 4.1.12. We may consider M_i as an R-module by letting R_j act trivially for all $j \neq i$. Then $M_1, \ldots, M_r$ are pairwise nonisomorphic simple R-modules and any simple R-module is isomorphic to one of $M_1, \ldots, M_r$. Explicitly, the R-module M_i is isomorphic to the simple left ideal $\left(0, \ldots, 0, L^{(i)}, 0, \ldots, 0\right)$ of all elements of R whose i^{th} component, $L^{(i)}$, consists of matrices with arbitrary entries in the first column and zeros elsewhere.

Proof: (3): we show $M_1, , \ldots, M_r$ are pairwise non-isomorphic:Suppose $i \neq j$ and suppose $\varphi : M_i \to M_j$ is an R-module isomorphism. If $s_i \in M_i$ then $s_i = z_i s_i$ so

$$\varphi(s_i) = \varphi(z_i s_i) = z_i \varphi(s_i) = 0,$$

since $\varphi(s_i) \in M_j$ and z_i acts trivially on M_j. This contradicts the fact that φ is an isomorphism and proves that $M_1, \ldots, M_r$ are pairwise nonisomorphic simple R-modules.

Definition 4.1.14 (semisimple ring)**.** Let R be a nonzero ring with I (not necessarily commutative). Then the following are equivalent:

(1) every R-module is projective

(2) every R-module is injective

(3) every R-module is completely reducible

(4) the ring R considered as a left R-module is a direct sum:

$$R = L_1 \oplus L_2 \oplus \cdots \oplus L_n,$$

where each L_i is a simple module (i.e., a simple left ideal) with $L_i = Re_i$, for some $e_i \in R$ with (i) $e_i e_j = 0$ if $i \neq j$ (ii) $e_i^2 = e_i$ for all i (iii) $\sum_{i=1}^{n} e_i = 1$

(5) as rings, R is isomorphic to a direct product of matrix rings over division rings, i.e., $R = R_1 \times R_2 \times \cdots \times R_r$ where R_j is isomorphic to the ring of all $n_j \times n_j$ matrices with entries in a division ring Δ_j, $j = 1, 2, \ldots, r$. The integer r, the integers n_j, and the division rings Δ_j (up to isomorphism) are uniquely determined by R.

Proof: (1) $\Leftrightarrow$ (2): By Definition 2.1.33 and Definition 2.1.29.

(3) $\Rightarrow$ (2): Let N be a R-module, Q is a submodule of N. Consider all the submodules of N such that its intersection with Q is 0. Take a maximal element M of the set. If $M+Q \neq N$. Consider

$$N = \bigoplus_{i \in I} M_i$$

where M_i is irreducible submodule of N. Then if $M + Q \not\supseteq M_i$, since M_i is irreducible, $(M+Q) \cap M_i = \varnothing$. A contradiction!

(4) $\Rightarrow$ (3): Since R itself is a semisimple R-module, the direct sum of R is also semisimple. Hence by Lemma 4.1.10, we have every R-module is semisimple.

(5) $\Rightarrow$ (4): By Lemma 4.1.12.

(2) $\Rightarrow$ (5):

Step 1: A is a ring, then the ring homomorphism $\varphi : A^{\text{opp}} \to \text{Hom}_A(A, A)$ given by $a \mapsto (x \mapsto xa)$ is an isomorphism.

Step 2: Let A be any ring with 1 , let L be any left A-module and let L^n be the direct sum of n copies of L with itself. Then the ring homomorphism $\varphi : \text{Hom}_A(L^n, L^n) \to M_n(D)$, where $D = \text{Hom}_A(L, L)$ given by

$$\varphi \in \text{Hom}_A(L^n, L^n) \mapsto (\varphi_{ij})$$

where $\varphi_{ij}(a) = j^{\text{th}}$ component of $\varphi(0, \ldots, a, \ldots, 0)$.

Step 3: Use Schur's lemma to how that if L is a simple A-module, then $\text{Hom}_A(L^n, L^n)$ is isomorphic to a matrix ring over a division ring.

Step 4: Let S be a simple ring(i.e., has no proper, nonzero 2-sided ideals) with 1 satisfying D.C.C. on left ideals, then there's a minimal left ideal in S. Let L be a minimal left ideal in S. Show that $S \cong L^n$ as left S-modules, where $L^n = L \oplus \cdots \oplus L$ with n factors.

By simplicity of S, $LS = S$, so $1 = l_1 s_1 + \cdots + l_n s_n$ for some $l_i \in L$ and $s_i \in S$ with n minimal. Then the map $(x_1, \ldots, x_n) \mapsto x_1 s_1 + \cdots + x_n s_n$ is a surjective homomorphism of left S-modules. If $y_1 s_1 + \cdots + y_n s_n = 0$ and $y_1 \neq 0$ in which $y_i \in L$. Since L is a minimal left ideal, $Sy_1 = L$. Take $sy_1 = l_1$ Then $1 = l_1 s_1 + \ldots l_n s_n - s(y_1 s_1 + \cdots + y_n s_n)$, a contradiction!

Step 5: Let Δ be a division ring, and $n \in \mathbb{N}$. Then $M_n(\Delta^{op}) \cong M_n(\Delta)^{op}$. The isomorphic map is given by $A \mapsto A^T$.

Step 6: Show that (2) implies R has the strict descending chain condition (D.C.C.) on left ideals

If $I_1 \supset \ldots I_n \supset \ldots$ be a decending chain of left ideals. Then $R = I_1 \oplus J_1 = J_1 \oplus J_2 \oplus I_2 = \ldots$. Let

$$J = \bigoplus_{i=1}^{\infty} I_i$$

, then $R = J \oplus K$. Consider $1 = j_1 + \ldots j_s + k$. Then

$$R = \bigoplus_{i=1}^{s} I_i \oplus K$$

A contradiction!

Step 7: Show that $R \cong R_1 \times R_2 \times \cdots \times R_r$ where R_j a simple ring with identity satisfying D.C.C.

By Zorn's Lemma and Step 6, for all 2-sided ideal J, there's a minimal 2-sided ideal contained in J. Take R_1 be a minimal 2-sided ideal of R. There's a left ideal R_2 such that $R = R_1 \oplus R_2$. We can check that R_2 is also a 2-sided ideal of R. Hence we can write R as a direct sum of finite many 2-sided minial ideal. If $R = R_1 \oplus \dots R_r$, then we have

$$R \cong R_1 \times \cdots \times R_r$$

where R_j are simple ring with 1 satisfying D.C.C.

Step 8: (local uniqueness) Suppose $S = M_n(\Delta) \cong M_{n'}(\Delta')$ as rings, where Δ and Δ' are division rings. Then $\Delta \cong \Delta'$ and $n = n'$.

By Step 6, (6) $\Rightarrow$ (2) and Lemma 4.1.12, S is a simple ring satisfying D.C.C. Then let J be a minimal left ideal, we have $S = M_n(\Delta) \cong M_m(\mathrm{Hom}_S(L, L)^{\mathrm{opp}})$ for some m. Then $\Delta \cong \mathrm{Hom}_S(L, L)^{\mathrm{opp}}$. By Lemma 4.1.12, $\Delta \cong \Delta'$. $n = n'$ follows from the fact dimensions of S over Δ and Δ' are equal.

Step 9: (global uniqueness) $W_1 \times \dots W_{r'} \cong R_1 \times R_2 \times \cdots \times R_r$ where R_i and W_j are simple rings. Then $r = r'$ and $R_i \cong W_i$ for some order.

Hint: Show that $(1, 0, \dots, 0) \mapsto (1, 0, \dots, 0)$.(reset the order if it's necessary).

Lemma 4.1.15. If Δ is a division ring that is a finite dimensional vector space over an algebraically closed field F and $F \cdot 1 \subseteq Z(\Delta)$, then $\Delta = F \cdot 1$.

Proof: For all $a \in \Delta$, consider $F \cdot 1 \subset F[a] \subset \Delta$. Then $F[a]$ is an integral domain. Since $[\Delta : F] < \infty$, $F[a] \cong F[x]/(m(x))$ is a field. Since every algebraically closed field have no nontrivial algebraic extension, $F[a] = F$.

Theorem 4.1.16. Let G be a finite group. There's $\mathbb{C}$-algebra isomorphism:

$$\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$$

where $n_1, \dots, n_r$ and r are uniquely determined. In particular, G is isomorphic to a finite subgroup of $\mathrm{GL}_{n_1}(\mathbb{C}) \times \dots \mathrm{GL}_{n_k}(\mathbb{C})$

Proof: By Maschke's Theorem, every $\mathbb{C}[G]$-module is injective, by equivalent definition of semisimple ring, as rings, $R = \mathbb{C}[G]$ is isomorphic to a direct product of matrix rings over division rings, i.e., $R = R_1 \times R_2 \times \cdots \times R_r$ where R_j is isomorphic to the ring of all $n_j \times n_j$ matrices with entries in a division ring Δ_j, $j = 1, 2, \dots, r$. The integer r, the integers n_j, and the division rings Δ_j (up to isomorphism) are uniquely determined by R.

Let φ be the isomorphic map, then φ induce a ring homomorphism $\varphi_i : \mathbb{C} \to \Delta_i$ such that $\varphi_i(\mathbb{C}) \subset Z(\Delta_i)$. Then by Lemma 4.1.15, as $\mathbb{C}$-algebra,

$$\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$$

where $n_1, \dots, n_r$ and r are uniquely determined.

Theorem 4.1.17. $\mathbb{C}[G]$ has exactly r distinct isomorphism types of irreducible modules and these have complex dimensions $n_1, n_2, \ldots, n_r$ given by Lemma 4.1.13.

Proposition 4.1.18. $\sum_{i=1}^{r} n_i^2 = |G|$

Proposition 4.1.19. r equals the number of conjugacy classes in G and the dimension of $Z(\mathbb{C}G)$.

Proof: Let $\mathcal{K}_1, \ldots, \mathcal{K}_s$ be the distinct conjugacy classes of G (recall that these partition G). For each conjugacy class $\mathcal{K}_i$ of G let

$$X_i = \sum_{g \in \mathcal{K}_i} g \quad \in \mathbb{C}G.$$

It's clear that $X_i \in Z(\mathbb{C}G)$.

We show the X_i 's form a basis of $Z(\mathbb{C}G)$, which will prove $s = \dim_{\mathbb{C}} Z(\mathbb{C}G) = r$. Since the X_i 's are linearly independent it remains to show they span $Z(\mathbb{C}G)$. Let $X = \sum_{g \in G} \alpha_g g$ be an arbitrary element of $Z(\mathbb{C}G)$. Since $h^{-1}Xh = X$,

$$\sum_{g \in G} \alpha_g h^{-1} g h = \sum_{g \in G} \alpha_g g.$$

Since the elements of G form a basis of $\mathbb{C}G$ the coefficients of g in the above two sums are equal:

$$\alpha_{hgh^{-1}} = \alpha_g.$$

Since h was arbitrary, every element in the same conjugacy class of a fixed group element g has the same coefficient in X, hence X can be written as a linear combination of the X_i 's.

4.2 Character

All representations considered are assumed to be finite dimensional.

Definition 4.2.1. A class function is any function from G into F which is constant on the conjugacy classes of G, i.e., $f : G \to F$ such that $f(g^{-1}xg) = f(x)$ for all $g, x \in G$.

Definition 4.2.2. If φ is a representation of G afforded by the FG-module V, the character of φ is the function

$$\chi : G \to F \quad \text{defined by} \quad \chi(g) = \operatorname{tr} \varphi(g),$$

Proposition 4.2.3. (1) Equivalent representations have the same character.

(2) the character of a representation is a class function.

(3) χ is the character of $\varphi : G \to \mathrm{GL}(V)$. Then $\chi(1)$ is the degree of φ.

Proposition 4.2.4. V_1 and V_2 are $F[G]$-module, χ and ψ are their character respectively. Then character of $V_1 \oplus V_2$ is $\chi + \psi$.

Example 4.2.5. Consider $\mathbb{C}[G]$ itself as a $\mathbb{C}[G]$-module, we call this representation the regular representation of G. By Lemma 4.1.12 and Lemma 4.1.13,

$$\chi_\rho(g) = \sum_{i=1}^{r} \chi_i(1)\chi_i(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases}$$

Lemma 4.2.6. V_1 and V_2 are finite-dimensional k-vector space, $A : V_1 \to V_1$ and $B : V_2 \to V_2$ are linear transforms. Then for the linear transform $A \otimes B : V_1 \otimes V_2 \to V_1 \otimes_k V_2$, we have $\det(A \otimes B) = \det(A)^{\dim V_2} \det(B)^{\dim V_1}$ and $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B)$.

Proposition 4.2.7. $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_1 : G \to \mathrm{GL}(V_2)$ are representations of G. Then the representation $\rho_1 \otimes \rho_2 : G \to \mathrm{GL}(V_1 \otimes_k V_2)$ defined by

$$g \mapsto \rho_1(g) \otimes \rho_2(g)$$

is called tensor product of ρ_1 and ρ_2. And by lemma 4.2.6 we have $\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1}\chi_{\rho_2}$

Proposition 4.2.8. (1) Two representations are equivalent if and only if they have the same character.(linearly independent of characters)

(2) characters of irreducible representation form a basis of class function.

Proof: Let $M_1, \ldots, M_r$ be the distinct irreducible $C[G]$-module defined by Lemma 4.1.12 and Lemma 4.1.13, Let $z_1, z_2, \ldots, z_r$ be the primitive central idempotents of $\mathbb{C}[G]$. Let χ_i be the character of M_i. Notice that if $j \neq i$ then $z_j M_i = 0$, i.e., z_j acts as the zero matrix on M_j, hence $\chi_j(z_i) = 0$, and z_i acts as the identity on M_i, hence $\chi_i(z_i) = n_i$. Hence $\chi_1, \ldots, \chi_r$ are linearly independent as class functions on G. By Proposition 4.1.19, $\chi_1, \ldots, \chi_r$ form a basis of class functions on G.

Corollary 4.2.9. Let G be a finite group. There's $\mathbb{C}$-algebra isomorphism:

$$\varphi : \mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$$

Then representation $\rho_1, \ldots, \rho_r$ given by $\varphi(g) = (\rho_1(g), \ldots, \rho_r(g))$ is isomorphic to the irreducible representation defined by Lemma 4.1.12 and Lemma 4.1.13.

Proof: Check the character.

Definition 4.2.10. For class functions θ and ψ define

$$(\theta, \psi) = \frac{1}{|G|} \sum_{g \in G} \theta(g)\overline{\psi(g)}$$

(where the bar denotes complex conjugation). One easily checks that $(,)$ is Hermitian: for $\alpha, \beta \in \mathbb{C}$

(1) $(\alpha\theta_1 + \beta\theta_2, \psi) = \alpha(\theta_1, \psi) + \beta(\theta_2, \psi)$,

(2) $(\theta, \alpha\psi_1 + \beta\psi_2) = \bar{\alpha}(\theta, \psi_1) + \bar{\beta}(\theta, \psi_2)$, and

(3) $(\theta, \psi) = (\psi, \theta)$.

Proposition 4.2.11. Consider all the distinct irreducible representation $\rho_1, \ldots, \rho_r$ with characters $\chi_1, ;\chi_r$. There's a $\mathbb{C}$-algebra homomorphism

$$\varphi : \mathbb{C}[G] \to M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$$

induced by $\varphi(g) = (\rho_1(g), \ldots, \rho_r(g))$. This homomorphism is an isomorphism.

Proof: Let $\tau = \sum_{g\in G} \alpha_g g$, $\varphi(\tau) = 0$. If $\alpha_{g_0} \neq 0$, then

$$\chi_i(g_0^{-1}\tau) = \sum_{g\in G} \alpha_g \chi_i(g_0^{-1}g) = 0$$

Hence

$$\sum_{i=1}^{r} \chi_i(1) \sum_{g\in G} \alpha_g \chi_i(g_0^{-1}g) = 0$$

So we have $\alpha_{g_0} = 0$, a contradiction!

Corollary 4.2.12. Let $z_1, \ldots, z_r$ be the orthogonal primitive central idempotents in $\mathbb{C}[G]$ such that $\varphi(z_i) = (O, \ldots, I_{n_i}, \ldots, O)$. Then

$$z_i = \frac{\chi_i(1)}{|G|} \sum_{g\in G} \chi_i\left(g^{-1}\right) g$$

Proof: Let

$$z_i = \sum_{g\in G} \alpha_g g$$

If ρ is the character of regular representation of G, then

$$\sum_{j=1}^{r} \chi_j(1)\chi_j\left(z_i g^{-1}\right) = \rho\left(z_i g^{-1}\right) = \alpha_g |G|.$$

It's esay to check that $\chi_j(z_i g^{-1}) = \chi_i(g^{-1})\delta_{ij}$, then

$$z_i = \sum_{g\in G} \frac{1}{|G|} \sum_{j=1}^{r} \chi_j(1)\chi_j(g^{-1})\delta_{ij} g = \frac{\chi_i(1)}{|G|} \sum_{g\in G} \chi_i\left(g^{-1}\right) g$$

Lemma 4.2.13. If ψ is any character of G then $\psi(x)$ is a sum of roots of 1 in $\mathbb{C}$ and $\psi\left(x^{-1}\right) = \overline{\psi(x)}$ for all $x \in G$.

Proof: Let $n = |G|$ Notice that $\psi(1) = \psi(g^n) = \psi(g)^n$, then $\psi(g)$ can be diagonalized.

Theorem 4.2.14 (The First Orthogonality Relation for Group Characters)**.** Let G be a finite group and let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G over $\mathbb{C}$. Then

$$(\chi_i, \chi_j) = \delta_{ij}$$

Proof: The orthonormality of the irreducible characters will follow directly from the orthogonality of the central primitive idempotents via the following calculation:

$$\begin{aligned} z_i\delta_{ij} &= z_i z_j \\ &= \frac{\chi_i(1)}{|G|}\frac{\chi_j(1)}{|G|}\sum_{g,h\in G}\chi_i\left(g^{-1}\right)\chi_j\left(h^{-1}\right)gh \\ &= \frac{\chi_i(1)}{|G|}\frac{\chi_j(1)}{|G|}\sum_{y\in G}\left[\sum_{x\in G}\chi_i\left(xy^{-1}\right)\chi_j\left(x^{-1}\right)\right]y \end{aligned}$$

By the expression of the coefficient of z_i, we have

$$\delta_{ij}\frac{\chi_i(1)}{|G|}\chi_i\left(g^{-1}\right) = \frac{\chi_i(1)\chi_j(1)}{|G|^2}\sum_{x\in G}\chi_i\left(xg^{-1}\right)\chi_j\left(x^{-1}\right).$$

Simplifying (and replacing g by g^{-1}) gives

$$\delta_{ij}\frac{\chi_i(g)}{\chi_j(1)} = \frac{1}{|G|}\sum_{x\in G}\chi_i(xg)\chi_j\left(x^{-1}\right) \qquad \text{for all } g\in G$$

Taking $g=1$, we have

$$\delta_{ij} = \frac{1}{|G|}\sum_{x\in G}\chi_i(x)\chi_j\left(x^{-1}\right)$$

Then by Lemma 4.2.13, we get the finial result.

Theorem 4.2.15 (The Second Orthogonality Relation for Group Characters)**.** Denote the r conjugate class by $C_1,\ldots,C_r$ and denote the set of element in G commutes with x by $C_G(x)$

$$\sum_{i=1}^{r}\chi_i(x)\overline{\chi_i(y)} = \begin{cases} |C_G(x)| & \text{if } x \text{ and } y \text{ are conjugate in } G \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Take x_i to be the element of C_i, notice that

$$|G|\delta_{jk} = |G|(\chi_j,\chi_k) = \sum_{i=1}^{r}|C_i|\chi_j(x_i)\overline{\chi_k(x_i)}$$

Let $W=\operatorname{diag}\{|C_1|,\ldots,|C_r|\}$, $X=(\chi_i(x_j))$. Then $WXX^H=\operatorname{diag}\{|G|,\ldots,|G|\}$,

4.3 Induced Representation

Definition 4.3.1. Let H be a subgroup of the finite group G and let V be an FH-module affording the representation φ of H. The FG-module $FG\otimes_{FH}V$ is called the induced module of V and the representation of G it affords is called the induced representation of φ. If ψ is the character of φ then the character of the induced representation is called the induced character and is denoted by $\operatorname{Ind}_H^G(\psi)$.

Proposition 4.3.2. Let H be a subgroup of the finite group G and let $g_1, \ldots, g_m$ be representatives for the distinct left cosets of H in G. Let V be an FH-module affording the matrix representation φ of H of degree n under the bais $v_1, \ldots, v_n$ The FG-module $W = FG \otimes_{FH} V$ has dimension nm over F and there is a basis of W such that W affords the matrix representation under the basis

$$g_1 \otimes v_1, g_1 \otimes v_2, \ldots, g_1 \otimes v_n, g_2 \otimes v_1, \ldots, g_2 \otimes v_n, \ldots\ldots, g_m \otimes v_n$$

, Φ defined for each $g \in G$ by

$$\Phi(g) = \begin{pmatrix} \varphi\left(g_1^{-1} g g_1\right) & \cdots & \varphi\left(g_1^{-1} g g_m\right) \\ \vdots & \vdots & \vdots \\ \varphi\left(g_m^{-1} g g_1\right) & \cdots & \varphi\left(g_m^{-1} g g_m\right) \end{pmatrix}$$

where each $\varphi\left(g_i^{-1} g g_j\right)$ is an $n \times n$ block appearing in the i, j block position of $\Phi(g)$, and where $\varphi\left(g_i^{-1} g g_j\right)$ is defined to be the zero block whenever $g_i^{-1} g g_j \notin H$.

Corollary 4.3.3. If ψ is the character afforded by V then the induced character is given by

$$\mathrm{Ind}_H^G(\psi)(g) = \sum_{i=1}^{m} \psi\left(g_i^{-1} g g_i\right)$$

where $\psi\left(g_i^{-1} g g_i\right)$ is defined to be 0 if $g_i^{-1} g g_i \notin H$, and

$\mathrm{Ind}_H^G(\psi)(g) = 0$ if g is not conjugate in G to some element of H. In particular, if H is a normal subgroup of G then $\mathrm{Ind}_H^G(\psi)$ is zero on all elements of $G - H$.

Theorem 4.3.4. Let G be a group, let H be a subgroup of G and let ψ and ψ' be characters of H.

(1) (Induction of characters is additive) $\mathrm{Ind}_H^G(\psi + \psi') = \mathrm{Ind}_H^G(\psi) + \mathrm{Ind}_H^G(\psi')$.

(2) (Induction of characters is transitive) If $H \leq K \leq G$ then

$$\mathrm{Ind}_K^G\left(\mathrm{Ind}_H^K(\psi)\right) \cong \mathrm{Ind}_H^G(\psi)$$

Proposition 4.3.5 (Frobenius reciprocity)**.** Let G be a group, let H be a subgroup of G and let ψ and φ be characters of H and G. Then

$$\langle \psi, \mathrm{Res}\, \varphi \rangle_{\mathrm{H}} = \langle \mathrm{Ind}\, \psi, \varphi \rangle_{\mathrm{G}}$$

Proof: Take V be a $\mathbb{C}[G]$-module and W be a $\mathbb{C}[H]$-module such that ψ is the character of W and φ is the character of V. Let $\tilde{V}$ be the $\mathbb{C}[H]$-module induced by V.

By Theorem 2.1.18,

$$\langle \psi, \mathrm{Res}\, \varphi \rangle_{\mathrm{H}} = \dim_{\mathbb{C}} \mathrm{Hom}(W, \tilde{V}) = \dim_{\mathbb{C}} \mathrm{Hom}(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W, V) = \langle \mathrm{Ind}\, \psi, \varphi \rangle_{\mathrm{G}}$$

Proposition 4.3.6. G is a group, H and K are the subgroup of G. Let $\{s_1, \ldots, s_r\}$ be the representation elements of double coset decomposition of G by H and K. i.e.

$$G = Ks_1H \cup \ldots Ks_rH$$

where $Ks_iH \cap Ks_jH = \varnothing$ if $i \neq j$. (V, p) be a representation of H and $\mathrm{Ind}_H^G(\rho)$ be the induced representation. The restriction $\mathrm{Res}_K\left(\mathrm{Ind}_H^G(\rho)\right)$ of the G-representation $\mathrm{Ind}_H^G(\rho)$ to a K-representation decomposes as a direct sum

$$\bigoplus_{i=1}^{r} \mathrm{Ind}_{s_i^{-1}Hs_i\cap K}^{K}\left(\rho^{s_i}\right)$$

$\rho^{s_i}(x) := \rho\left(s_i x s_i^{-1}\right)$ for x belonging to the subgroup $s_i^{-1}Hs_i \cap K$ that depends only on equivalence class of s_i and not on its chosen representative s.

Proof: Notice that there's $\mathbb{C}[K]$-module isomorphism

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \cong \bigoplus_{i=1}^{r} \mathbb{C}[K]s_i\mathbb{C}[H] \otimes_{\mathbb{C}[H]} V \cong \bigoplus_{i=1}^{r} \mathbb{C}[K] \otimes_{\mathbb{C}[K\cap s_iHs_i^{-1}]} \tilde{V}$$

where $\tilde{V}$ is a representation of $K \cap s_iHs_i^{-1}$ defined by $(s_ihs_i^{-1}) \cdot v = hv$ and in this equation, the isomorphisc map between $\mathbb{C}[K]\otimes_{\mathbb{C}[K\cap s_iHs_i^{-1}]}\tilde{V}$ and $\mathbb{C}[K]s_i\mathbb{C}[H]\otimes_{\mathbb{C}[H]}V$ is induced by bi-additive maps

$$\begin{aligned} \mathbb{C}[K] \times \tilde{V} &\to \mathbb{C}[K]s_i\mathbb{C}[H] \otimes_{\mathbb{C}[H]} V \\ (k, v) &\mapsto ks_i \otimes v \end{aligned}$$

and

$$\begin{aligned} \mathbb{C}[K]s_i\mathbb{C}[H] \times V &\to \mathbb{C}[K] \otimes_{\mathbb{C}[K\cap s_iHs_i^{-1}]} \tilde{V} \\ (ks_ih, v) &\mapsto k \otimes hv \end{aligned}$$

Corollary 4.3.7 (Mackey's irreducibility criterion)**.** With all the notations in above Proposition, in order to make the induced representation $\mathrm{Ind}_H^G \rho$ be irreducible, it is necessary and sufficient that the following two conditions be satisfied:

(1) ρ is irreducible.

(2) For each $s \in G - H$ the two representations ρ^s and $\mathrm{Res}_s(\rho)$ of H_s are disjoint.

Proof: Let φ be the character of ρ, by Frobenius reciprocity,

$$\langle \mathrm{Ind}_H^G(\varphi), \mathrm{Ind}_H^G(\varphi)\rangle = \langle \varphi, \mathrm{Res}(\mathrm{Ind}(\varphi))\rangle$$

Let χ_i be the character of ρ^{s_i}, then by Frobenius reciprocity and Proposition 4.3.6

$$\langle \varphi, \mathrm{Res}(\mathrm{Ind}(\varphi))\rangle = \sum_{i=1}^{r} \langle \mathrm{Res}_{s_i^{-1}Hs_i\cap H}(\varphi), \chi_i\rangle$$

Corollary 4.3.8. Suppose H is normal in G. In order that $\mathrm{Ind}_H^G(\rho)$ be irreducible, it is necessary and sufficient that ρ be irreducible and not isomorphic to any of its conjugates ρ^s for $s \notin H$.

Definition 4.3.9. Let G be a finite group and let $\chi_1, \ldots, \chi_h$ be its distinct irreducible characters. A class function on G is a character if and only if it is a linear combination of the χ_i' 's with non-negative integer coefficients. We will denote by $R^+(G)$ the set of these functions, and by $R(G)$ the group generated by $R^+(G)$, i.e., the set of differences of two characters. We have

$$R(\mathrm{G}) = \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_h.$$

An element of $R(G)$ is called a virtual character. Since the product of two characters is a character, $R(G)$ is a subring of the ring $F(G)$ of class functions on G with complex values. Since the χ_i form a basis of $F(G)$, by Theorem 2.1.22, we have $\mathbb{C}$-algebra isomorphism

$$F(G) \cong \mathbb{C} \otimes_{\mathbb{Z}} R(G)$$

Theorem 4.3.10.

Theorem 4.3.11 (Artin)**.**

Reference: Matsumura, Atiyah, Gortz 1, GTM73, GTM42, Dummit, GTM52, Miline:Galois Theory, Red book, Hu Yong quan:Galois Theory