AMC Introduction

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Youth STEM Academy

Basic Information

The American Mathematics Competitions(AMC) is the preeminent math competition for students K-12. Today, over 300,000 students in 50 states and over 30 countries take the AMC to bolster their confidence and passion for math.

Your Mathematical Journey: AMC to IMO

- Begin your adventure with the AMC 8 for an introductory challenge or the AMC 10/12 for a chance to advance further in competitive mathematics.
- American Invitational Mathematics Examination (AIME): Excel in the AMC 10/12 to qualify for the AIME, a more challenging examination that tests your problem-solving prowess and mathematical ingenuity.

Your Mathematical Journey: AMC to IMO

- USA Mathematical Olympiad and USA Junior Mathematical Olympiad (USAMO/USAJMO): Achieve high scores in the AIME and AMC 10/12 to earn an invitation to the USAMO or USAJMO, where the nation's top mathematical talents compete in proof-based problems.
- Mathematical Olympiad Program (MOP): Outstanding performance in the USAMO or USAJMO could lead to participation in the MOP, an intensive training camp designed to prepare the best young mathematicians for international competition.

Rules

	AMC 8	AMC 10	AMC 12
Grade	≤ 8	≤ 10	≤ 12
Form	25 questions	25 questions	25 questions
Time	40 minutes	75 minutes	75 minutes
Total Score	25	150	150
Scoring Criteria	(1,0,0)	(6,1.5,0)	(6,1.5,0)

- (x, y, z) means x points for 1 correct answer, y points for 1 blank answer and z points for 1 wrong answer.

规则

	AMC 8	AMC 10	AMC 12
年级	≤ 8	≤ 10	≤ 12
形式	25 道题	25 道题	25 道题
事件	40 分钟	75 分钟	75 分钟
总分	25	150	150
判分方法	(1,0,0)	(6,1.5,0)	(6,1.5,0)

- (x, y, z) 为对一个题 x 分, 不填 y 分, 答错 z 分.

Range for AMC 8

- Algebra: integer, rational number, real number, linear equation, inequality, number sequence, factorial, ratio.
- Combination: basic probability, Combinatorial number $\binom{n}{k}$
- Geometry: triangle, circle, rectangle, area, angle
- Number Theory: odd and even number, divisor, GCD(greatest common divisor)

AMC 8 考察范围

- 代数: 整数, 有理数, 实数, 线性方程, 不等式, 数列, 阶乘, 比率.
- 组合: 概率, 组合数
- 几何: 三角形, 圆形, 矩形, 面积, 角度
- 数列: 奇数偶数, 整除, 最大公约数, 唯一因子分解

Algebra(代数)

- 完全平方公式 $(x+y)^2 = x^2 + 2xy + y^2$
- 平方差公式 $(x+y)(x-y) = x^2 - y^2$
- 线性方程求解

$$x + y = 3$$
$$x - y = 1$$

- 等差数列求和 $(1+3+5+7+9+\cdots+101$ 是多少?)
- 等差数列求项数 $(1, 4, 7, 11, \ldots, 100$ 共多少项?)
- 开根号 $\sqrt{9} =?$
- 阶乘 $(5! =?)$

Number Theory(数论)

- 整除 (如何判断一个数能不能被 3 整除)
- 最大公约数, 最小公倍数 (24 和 16 最小公倍数是多少)
- 唯一因子分解, 勒让德公式
- 数论函数: 除数个数函数, 除数和函数 $\sigma(n), \tau(n)$

Combinnation(组合)

- 组合数 $\binom{n}{k}$
- 古典概型
- 条件概率 $\mathbb{P}(A|B)$

Geometry(几何)

- 图形的面积, 周长
- 勾股定理 ($a^2 + b^2 = c^2$)
- 三角形相似, 等腰直角三角形,30, 60, 90 度的三角形
- 直线的方程, 圆的方程

Question

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有多少个奇三位数, 其个位十位百位各不相同?

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Proof: If n is odd, we have 5 choices $1, 3, 5, 7, 9$ for its unit digits,

Case 1: tens digit $= 0$, there are $5 \times 1 \times 8$ numbers.

Case 2: tens digit $\neq 0$, there are $5 \times 8 \times 7$ numbers.

Handshakes

Question

Seven women and five men attend a party. At this party each man shakes hands with each other person once. Each woman shakes hands only with men. How many handshakes took place at the party?
七女五男参加一个聚会。在这个聚会中, 每个男人会和其他人都握一次手, 每个女人只和男人握手, 请问这个聚会上会有多少次握手?

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Proof: Firstly, let all the people shake hands with each other, then we get $\binom{12}{2} = 66$ times handshakes. The total amount equals to 66 minis the amount of handshakes between woman, that is $66 - \binom{7}{2} = 45$.

Solve this Euqation!

Question (AMC 8, 2019-20)

How many different real numbers x satisfy the equation

有多少不同的实数 x 满足这个方程

$$(x^2 - 5)^2 = 16$$

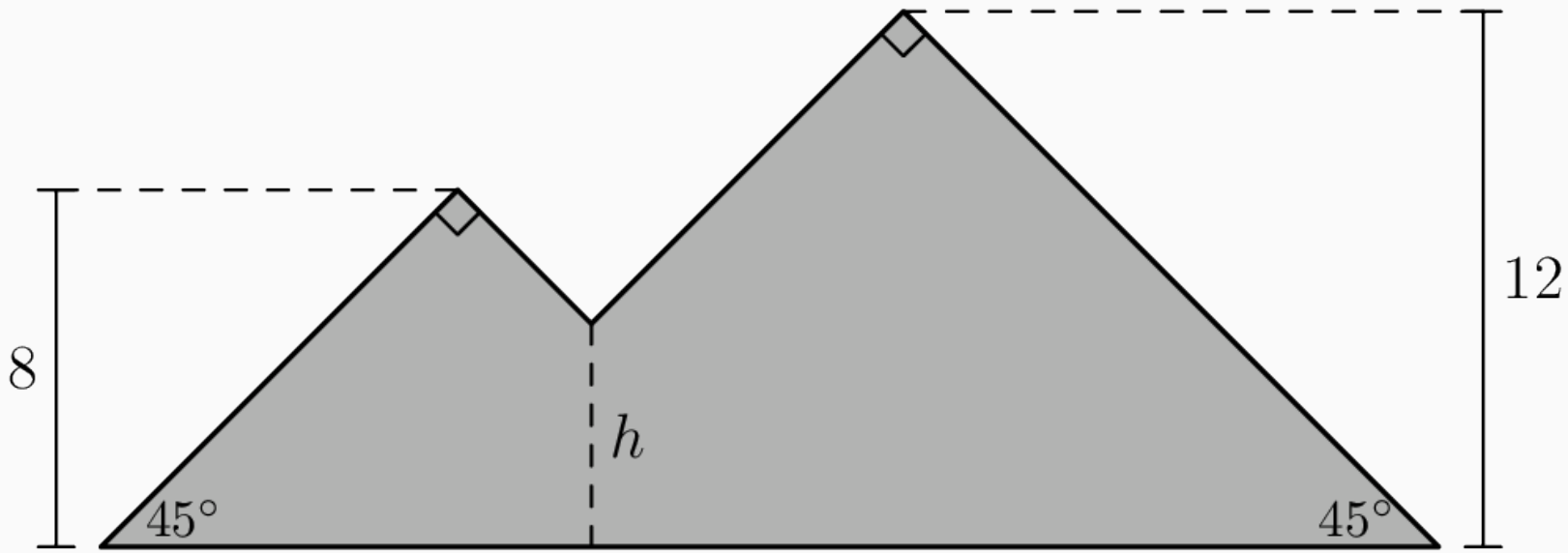
Double Factorial(AMC 8, 2022-17)

If n is an even positive integer, the double factorial notation n !! represents the product of all the even integers from 2 to n. For example, $8!! = 2 \cdot 4 \cdot 6 \cdot 8$. What is the units digit of the following sum?

$$2!! + 4!! + 6!! + \cdots + 2018!! + 2020!! + 2022!!$$

Two Mountains(AMC 8, 2022-24)

如下图, 阴影部分面积为 183, 求 h.



双阶乘 (AMC 8, 2022-17)

如果 n 是一个偶数, $n!!$ 定义为从 2 到 n 所有偶数的乘积. 例如, $8!! = 2 \cdot 4 \cdot 6 \cdot 8$. 请问如下求和的个位数?

$$2!! + 4!! + 6!! + \cdots + 2018!! + 2020!! + 2022!!$$

Solution

Notice that once $n > 8$, the units digit of n !! will be 0 because there will be a factor of 10 . Thus, we only need to calculate the units digit of

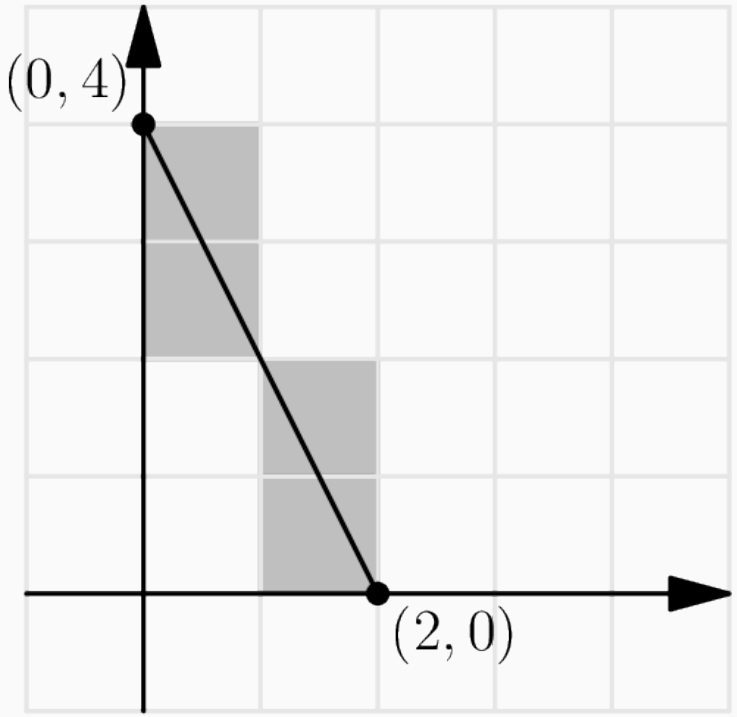
$$2!! + 4!! + 6!! + 8!! = 2 + 8 + 48 + 48 \times 8 = 442$$

Segment Coloring(AMC 8, 2024-23)

在平面直角坐标系上, 画一条从 $(2000, 3000)$ 到 $(5000, 8000)$ 的线段, 这条线段经过了多少个 1×1 的小方格 (必须进入该方格的内部).

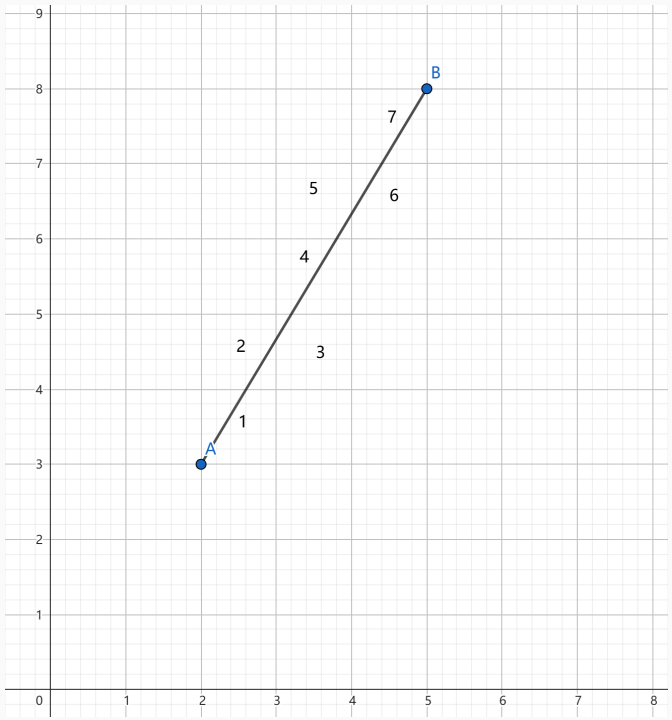
Segment Coloring(AMC 8, 2024-23)

Rodrigo has a very large sheet of graph paper. First he draws a line segment connecting point $(0, 4)$ to point $(2, 0)$ and colors the 4 cells whose interiors intersect the segment, as shown below. Next Rodrigo draws a line segment connecting point $(2000, 3000)$ to point $(5000, 8000)$. How many cells will he color this time?



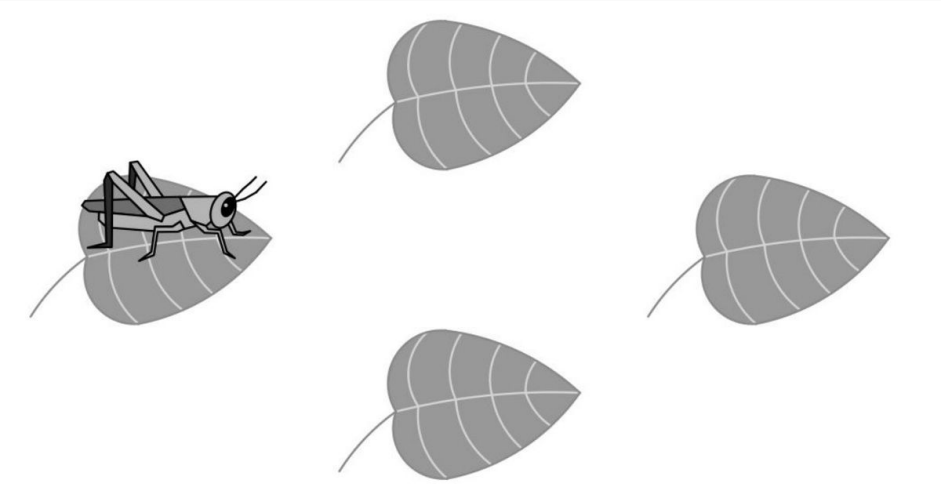
Solution

Draw a line in the lattice which from $(2, 3)$ to $(5, 8)$, notice that the line crossed 7 blocks in this pattern. Such a pattern is repeated 1000 times between $(2000, 3000)$ and $(5000, 8000)$, then the answer is 7000.



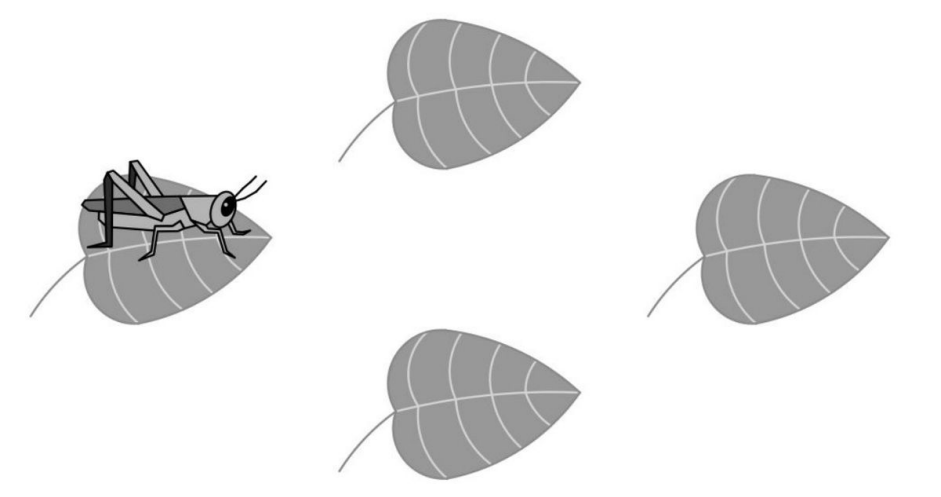
Jumping Cricket(AMC 8, 2022-25)

A cricket randomly hops between 4 leaves, on each turn hopping to one of the other 3 leaves with equal probability. After 4 hops what is the probability that the cricket has returned to the leaf where it started?



跳跃的蟋蟀 (AMC 8, 2022-25)

一个蟋蟀随意的在四片叶子上跳跃, 每次等概率得跳向其他三片叶子之一, 求蟋蟀跳四次后返回最初这片叶子的概率.



Solution

Let p_n be the probability that the cricket returns to the leaf after n times jumping. q_n be the probability that the cricket doesn't retrun to the leaf. We have the recursion equations

$$p_n + q_n = 1$$

and

$$p_n = \frac{1}{3} q_{n-1}$$

Hence

$$p_1 = 0, p_2 = \frac{1}{3}, p_3 = \frac{2}{9}, p_4 = \frac{7}{27}$$