Geometry

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Contents

Chapter 1

Topology

1.1 Qutoient Map

Definition 1.1.1. Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is said to be a quotient map provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

Definition 1.1.2. If X is a space and A is a set and if $p : X \to A$ is a surjective map, then there exists exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map; it is called the quotient topology induced by p.

The topology $\mathcal{T}$ is of course defined by letting it consist of those subsets U of A such that $p^{-1}(U)$ is open in X. It is easy to check that $\mathcal{T}$ is a topology. The sets $\varnothing$ and A are open because $p^{-1}(\varnothing) = \varnothing$ and $p^{-1}(A) = X$. The other two conditions follow from the equations

$$p^{-1}\left(\bigcup_{\alpha \in J} U_\alpha\right) = \bigcup_{\alpha \in J} p^{-1}\left(U_\alpha\right)$$
$$p^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \bigcap_{i=1}^{n} p^{-1}\left(U_i\right).$$

Proposition 1.1.3. We say that a subset C of X is saturated with respect to the surjective map $p : X \to Y$ if C contains every set $p^{-1}(\{y\})$ that it intersects. Thus C is saturated if it equals the complete inverse image of a subset of Y. Then surjective map $p : X \to Y$ is quotient map if and only if it is continuous and maps saturated open(or closed) sets of X to open(closed) sets of Y.

Proposition 1.1.4. $p : X \to Y$ is a surjective continuous map that is either open or closed, then p is a quotient map.

Theorem 1.1.5 (universal proptery of quotient map)**.** Let $p : X \to Y$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$, for $y \in Y$. Then g induces a map $f : Y \to Z$ such that $f \circ p = g$. The induced map f is continuous if and only if g is continuous; f is a quotient map if and only if g is a quotient map.

$$\begin{tikzcd}
X \arrow[d, "p"'] \arrow[dr, "g"] & \\
Y \arrow[r, dashed, "f"] & Z
\end{tikzcd}$$

Definition 1.1.6. Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a quotient space of X.

Theorem 1.1.7. Let $g : X \to Z$ be a surjective continuous map. Let X^* be the following collection of subsets of X :

$$X^* = \left\{ g^{-1}(\{z\}) \mid z \in Z \right\}.$$

Give X^* the quotient topology.

(1) The map g induces a bijective continuous map $f : X^* \to Z$, which is a homeomorphism if and only if g is a quotient map.

$$\begin{tikzcd}
X \arrow[d, "p"] \arrow[dr, "g"] & \\
X^* \arrow[r, "f"] & Z
\end{tikzcd}$$

(2) If Z is Hausdorff, so is X^*.

1.2 Fundamental Group and Covering Space

Definition 1.2.1. If f and f' are continuous maps of the space X into the space Y, we say that f is homotopic to f' if there is a continuous map $F : X \times I \to Y$ such that

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = f'(x)$$

for each x. (Here $I = [0, 1]$.) The map F is called a homotopy between f and f'. If f is homotopic to f', we write $f \simeq f'$. If $f \simeq f'$ and f' is a constant map, we say that f is nulhomotopic.

Now we consider the special case in which f is a path in X. Recall that if $f : [0, 1] \to X$ is a continuous map such that $f(0) = x_0$ and $f(1) = x_1$, we say that f is a path in X from x_0 to x_1. We also say that x_0 is the initial point, and x_1 the final point, of the path f. In this chapter, we shall for convenience use the interval $I = [0, 1]$ as the domain for all paths.

If f and f' are two paths in X, there is a stronger relation between them than mere homotopy. It is defined as follows:

Two paths f and f', mapping the interval $I = [0, 1]$ into X, are said to be path homotopic if they have the same initial point x_0 and the same final point x_1, and if there is a continuous map $F : I \times I \to X$ such that

$$\begin{aligned} F(s, 0) &= f(s) \quad &\text{and} \quad F(s, 1) &= f'(s), \\ F(0, t) &= x_0 \quad &\text{and} \quad F(1, t) &= x_1, \end{aligned}$$

for each $s \in I$ and each $t \in I$. We call F a path homotopy between f and f'.

Proposition 1.2.2. The relations $\simeq$ and $\simeq p$ are equivalence relations.

Proof: Let us verify the properties of an equivalence relation. Given f, it is trivial that $f \simeq f$; the map $F(x, t) = f(x)$ is the required homotopy. If f is a path, F is a path homotopy.

Given $f \simeq f'$, we show that $f' \simeq f$. Let F be a homotopy between f and f'. Then $G(x, t) = F(x, 1 - t)$ is a homotopy between f' and f. If F is a path homotopy, so is G.

Suppose that $f \simeq f'$ and $f' \simeq f''$. We show that $f \simeq f''$. Let F be a homotopy between f and f', and let F' be a homotopy between f' and f''. Define $G : X \times I \to Y$ by the equation

$$G(x, t) = \begin{cases} F(x, 2t) & \text{for } t \in \left[0, \frac{1}{2}\right], \\ F'(x, 2t - 1) & \text{for } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

The map G is well defined, since if $t = \frac{1}{2}$, we have $F(x, 2t) = f'(x) = F'(x, 2t - 1)$. Because G is continuous on the two closed subsets $X \times \left[0, \frac{1}{2}\right]$ and $X \times \left[\frac{1}{2}, 1\right]$ of $X \times I$, it is continuous on all of $X \times I$, by the pasting lemma. Thus G is the required homotopy between f and f''.

Definition 1.2.3. If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, we define the product $f * g$ of f and g to be the path h given by the equations

$$h(s) = \begin{cases} f(2s) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ g(2s - 1) & \text{for } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

The function h is well-defined and continuous. It is a path in X from x_0 to x_2. We think of h as the path whose first half is the path f and whose second half is the path g.

The product operation on paths induces a well-defined operation on path-homotopy classes, defined by the equation

$$[f] * [g] = [f * g].$$

To verify this fact, let F be a path homotopy between f and f' and let G be a path homotopy between g and g'. Define

$$H(s,t) = \begin{cases} F(2s,t) & \text{for } s \in \left[0, \frac{1}{2}\right], \\ G(2s-1,t) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Because $F(1,t) = x_1 = G(0,t)$ for all t, the map H is well-defined. You can check that H is the required path homotopy between $f * g$ and $f' * g'$.

Example 1.2.4. let A be any convex subspace of $\mathbb{R}^n$, Let f and g be any two maps of a space X into A. It is easy to see that f and g are homotopic; the map

$$F(x,t) = (1-t)f(x) + tg(x)$$

is a homotopy between them. It is called a straight-line homotopy because it moves the point $f(x)$ to the point $g(x)$ along the straight-line segment joining them.

If f and g are paths from x_0 to x_1, then F will be a path homotopy.

Proposition 1.2.5. The operation $*$ has the following properties:

(1) (Associativity) If $[f] * ([g] * [h])$ is defined, so is $([f] * [g]) * [h]$, and they are equal.

(2) (Right and left identities) Given $x \in X$, let e_x denote the constant path $e_x : I \to X$ carrying all of I to the point x. If f is a path in X from x_0 to x_1, then

$$[f] * [e_{x_1}] = [f] \quad \text{and} \quad [e_{x_0}] * [f] = [f].$$

(3) (Inverse) Given the path f in X from x_0 to x_1, let $\bar{f}$ be the path defined by $\bar{f}(s) = f(1-s)$. It is called the reverse of f. Then

$$[f] * [\bar{f}] = [e_{x_0}] \quad \text{and} \quad [\bar{f}] * [f] = [e_{x_1}].$$

Proof: (1), (2) and (3) follow from the fact that if $k : X \to Y$ is a continuous map, and if F is a path homotopy in X between the paths f and f', then $k \circ F$ is a path homotopy in Y between the paths $k \circ f$ and $k \circ f'$.

Notice that $I = [0,1]$ is convex, we can construct path-homotopy between different paths in $I = [0,1]$ to prove (1), (2) and (3) respectively.

Definition 1.2.6. Let X be a space; let x_0 be a point of X. A path in X that begins and ends at x_0 is called a loop based at x_0. The set of path homotopy classes of loops based at x_0, with the operation $*$, is called the fundamental group of X relative to the base point x_0. It is denoted by $\pi_1(X, x_0)$.

Proposition 1.2.7. Let α be a path in X from x_0 to x_1. We define a map

$$\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

by the equation

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha].$$

The map $\hat{\alpha}$ is well-defined and a group isomorphism.

Remark 1.2.8. If X is path connected, all the groups $\pi_1(X, x)$ are isomorphic, so it is tempting to try to "identify" all these groups with one another and to speak simply of the fundamental group of the space X, without reference to base point. The difficulty with this approach is that there is no natural way of identifying $\pi_1(X, x_0)$ with $\pi_1(X, x_1)$; different paths α and β from x_0 to x_1 may give rise to different isomorphisms between these groups. For this reason, omitting the base point can lead to error.

Theorem 1.2.9. A space X is said to be simply connected if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial (one-element) group for some $x_0 \in X$, and hence for every $x_0 \in X$. We often express the fact that $\pi_1(X, x_0)$ is the trivial group by writing $\pi_1(X, x_0) = 0$.

In a simply connected space X, any two paths having the same initial and final points are path homotopic.

Proof: Let α and β be two paths from x_0 to x_1. Then $\alpha * \bar{\beta}$ is defined and is a loop on X based at x_0. Since X is simply connected, this loop is path homotopic to the constant loop at x_0. Then

$$[\alpha * \bar{\beta}] * [\beta] = [e_{x_0}] * [\beta],$$

from which it follows that $[\alpha] = [\beta]$.

Definition 1.2.10. Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map such that $h(x_0) = y_0$. Define

$$h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

by the equation

$$h_*([f]) = [h \circ f].$$

The map h_* is called the homomorphism induced by $\boldsymbol{h}$, relative to the base point x_0. The map h_* is well-defined, for if F is a path homotopy between the paths f and f', then $h \circ F$ is a path homotopy between the paths $h \circ f$ and $h \circ f'$. The fact that h_* is a homomorphism follows from the equation

$$(h \circ f) * (h \circ g) = h \circ (f * g).$$

The homomorphism h_* depends not only on the map $h : X \to Y$ but also on the choice of the base point x_0. (Once x_0 is chosen, y_0 is determined by h.) So some notational difficulty will arise if we want to consider several different base points for X. If x_0 and x_1 are two different points of X, we cannot use the same symbol h_* to stand for two different homomorphisms, one having domain $\pi_1(X, x_0)$ and the other having domain $\pi_1(X, x_1)$. Even if X is path connected, so these groups are isomorphic, they are still not the same group. In such a case, we shall use the notation

$$(h_{x_0})_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

for the first homomorphism and $(h_{x_1})_*$ for the second. If there is only one base point under consideration, we shall omit mention of the base point and denote the induced homomorphism merely by h_*.

Proposition 1.2.11. If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.

If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X with Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.

Remark 1.2.12. In the language of category, above statements give us a covariant functor from category of pointed space to category of group.

Example 1.2.13. A subset A of $\mathbb{R}^n$ is said to be star convex if for some point a_0 of A, all the line segments joining a_0 to other points of A lie in A. Show that if A is star convex, A is simply connected.

Definition 1.2.14. Let $p : E \to B$ be a continuous surjective map. The open set U of B is said to be evenly covered by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into slices.

Let $p : E \to B$ be continuous and surjective. If every point b of B has a neighborhood U that is evenly covered by p, then p is called a covering map, and E is said to be a covering space of B.

Example 1.2.15. Consider a lattice $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$, then then natural projection $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ is a covering map.

Theorem 1.2.16. It $p : E \to B$ and $p' : E' \to B'$ are covering maps, then

$$p \times p' : E \times E' \to B \times B'$$

is a covering map.

Definition 1.2.17. Let $p : E \to B$ be a map. If f is a continuous mapping of some space X into B, a lifting of f is a continuous map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Lemma 1.2.18 (lift of path)**.** Let $p : E \to B$ be a covering map, let $p(e_0) = b_0$. Any path $f : [0,1] \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f}$ in E beginning at e_0.

Proof: Cover B by open sets $(U_i)_{i\in I}$ each of which is evenly covered by p. Find a subdivision of $[0,1]$, say $s_0, \ldots, s_n$, such that for each i the set $f([s_i, s_{i+1}])$ lies in some open set U_i. (Here we use the Lebesgue number lemma.) We define the lifting $\tilde{f}$ step by step.

First, define $\tilde{f}(0) = e_0$. Then, supposing $\tilde{f}(s)$ is defined for $0 \le s \le s_i$, we define $\tilde{f}$ on $(s_i, s_{i+1}]$ as follows: The set $f([s_i, s_{i+1}])$ lies in some open set U_i that is evenly covered by p. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices; each set V_α is mapped homeomorphically onto U by p. Now $\tilde{f}(s_i)$ lies in only one of these sets, say in V_0. Define $\tilde{f}(s)$ for $s \in (s_i, s_{i+1}]$ by the equation

$$\tilde{f}(s) = (p \mid V_0)^{-1}(f(s)).$$

Uniqness: trivial.

Lemma 1.2.19 (lift of path homotopy)**.** Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let the map $F : I \times I \to B$ be continuous, with $F(0,0) = b_0$. There is a unique lifting of F to a continuous map

$$\tilde{F} : I \times I \to E$$

such that $\tilde{F}(0,0) = e_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.

Proof: The proof of existence and uniqness is similiar to the existence and uniqness of lift of path.

Now suppose that F is a path homotopy. We wish to show that F is a path homotopy. The map F carries the entire left edge $0 \times I$ of I^2 into a single point b_0 of B. Because $\tilde{F}$ is a lifting of F, it carries this edge into the set $p^{-1}(b_0)$. But this set has the discrete topology as a subspace of E. Since $0 \times I$ is connected and $\tilde{F}$ is continuous, $\tilde{F}(0 \times I)$ is connected and thus must equal a one-point set. Similarly, $\tilde{F}(1 \times I)$ must be a one-point set. Thus $\tilde{F}$ is a path homotopy.

Theorem 1.2.20. Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let f and g be two paths in B from b_0 to b_1; let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths in E beginning at e_0. If f and g are path homotopic, then $\tilde{f}$ and $\tilde{g}$ end at the same point of E and are path homotopic.

Proof: By Lemma 1.2.18 and Lemma 1.2.19.

Theorem 1.2.21. Let $p : E \to B$ be a covering map; let $b_0 \in B$. Choose e_0 so that $p(e_0) = b_0$. Given an element $[f]$ of $\pi_1(B, b_0)$, let $\tilde{f}$ be the lifting of f to a path in E that begins at e_0. Let $\phi([f])$ denote the end point $\tilde{f}(1)$ of $\tilde{f}$. Then ϕ is a well-defined set map

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0).$$

We call ϕ the lifting correspondence derived from the covering map p. It depends of course on the choice of the point e_0.

Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. If E is path connected, then the lifting correspondence

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

is surjective. If E is simply connected, it is bijective.

Proof: If E is path connected, then, given $e_1 \in p^{-1}(b_0)$, there is a path $\tilde{f}$ in E from e_0 to e_1. Then $f = p \circ \tilde{f}$ is a loop in B at b_0, and $\phi([f]) = e_1$ by definition.

Suppose E is simply connected. Let $[f]$ and $[g]$ be two elements of $\pi_1(B, b_0)$ such that $\phi([f]) = \phi([g])$. Let $\tilde{f}$ and $\tilde{g}$ be the liftings of f and g, respectively, to paths in E that begin at e_0; then $\tilde{f}(1) = \tilde{g}(1)$. Since E is simply connected, there is a path homotopy $\tilde{F}$ in E between $\tilde{f}$ and $\tilde{g}$. Then $p \circ \tilde{F}$ is a path homotopy in B between f and g.

Example 1.2.22. Fundamental group of $\mathbb{S}^1 \simeq \mathbb{Z}$

Proof: Let $p : \mathbb{R} \to S^1 : x \mapsto e^{2\pi i x}$, let $e_0 = 0$, and let $b_0 = p(e_0) = 1$. Then $p^{-1}(b_0)$ is the set $\mathbb{Z}$ of integers. Since $\mathbb{R}$ is simply connected, the lifting correspondence

$$\phi : \pi_1\left(S^1, b_0\right) \to \mathbb{Z}$$

is bijective.

Given $[f]$ and $[g]$ in $\pi_1(B, b_0)$, let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths on $\mathbb{R}$ beginning at 0 . Let $n = \tilde{f}(1)$ and $m = \tilde{g}(1)$; then $\phi([f]) = n$ and $\phi([g]) = m$, by definition. Let $\tilde{\tilde{g}}$ be the path

$$\tilde{\tilde{g}}(s) = n + \tilde{g}(s)$$

on $\mathbb{R}$. Because $p(n + x) = p(x)$ for all $x \in \mathbb{R}$, the path $\tilde{\tilde{g}}$ is a lifting of g; it begins at n. Then the product $\tilde{f} * \tilde{\tilde{g}}$ is defined, and it is the lifting of $f * g$ that begins at 0 , as you can check. The end point of this path is $\tilde{\tilde{g}}(1) = n + m$. Then by definition,

$$\phi([f] * [g]) = n + m = \phi([f]) + \phi([g]).$$

Definition 1.2.23. Let $p : E \to B$ and $p' : E' \to B$ be covering maps. They are said to be equivalent if there exists a homeomorphism $h : E \to E'$ such that $p = p' \circ h$. The homeomorphism h is called an equivalence of covering maps or an equivalence of covering spaces.

Proposition 1.2.24. Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$.

(1) The homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism.

(2) Let $H = p_*(\pi_1(E, e_0))$. The lifting correspondence ϕ induces an injective map

$$\Phi : \pi_1(B, b_0)/H \to p^{-1}(b_0)$$

of the collection of right cosets of H into $p^{-1}(b_0)$, which is bijective if E is path connected.

(3) If f is a loop in B based at b_0, then $[f] \in H$ if and only if f lifts to a loop in E based at e_0.

Proof: (1): Suppose $\tilde{h}$ is a loop in E at e_0, and $p_*([\tilde{h}])$ is the identity element. Let F be a path homotopy between $p \circ \tilde{h}$ and the constant loop. If $\tilde{F}$ is the lifting of F to E such that $\tilde{F}(0,0) = e_0$, then by Lemma 1.2.19, $\tilde{F}$ is a path homotopy between $\tilde{h}$ and the constant loop at e_0.

(2): Let $h \in \pi_1(B, b_0)$ and $\tilde{h}$ be the lift of h and f be an element in $\pi_1(E, e_0)$ then f is a lift of $p \circ f$. Notice that $p \circ (f * \tilde{h}) = (p \circ f) * (p \circ \tilde{h}) = (p \circ f) * h$, then $f * \tilde{h}$ is a lift of $(p \circ f) * h$. Hence Φ is well-defined. If E is path connected, then Φ is surjective by Theorem 1.2.21.

Injectivity of Φ means that $\phi([f]) = \phi([g])$ if and only if $[f] \in H * [g]$ which follows from the definition of H.

(3) Trivial.

Theorem 1.2.25. Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let $f : Y \to B$ be a continuous map, with $f(y_0) = b_0$. Suppose Y is path connected and locally path connected. The map f can be lifted to a map $\tilde{f} : Y \to E$ such that $\tilde{f}(y_0) = e_0$ if and only if

$$f_*\left(\pi_1\left(Y, y_0\right)\right) \subset p_*\left(\pi_1\left(E, e_0\right)\right).$$

Furthermore, if such a lifting exists, it is unique.

Proof: If the lifting $\tilde{f}$ exists, then

$$f_*\left(\pi_1\left(Y, Y_0\right)\right) = p_*\left(\tilde{f}_*\left(\pi_1\left(Y, y_0\right)\right)\right) \subset p_*\left(\pi_1\left(E, e_0\right)\right).$$

This proves the 'only if' part of the theorem. Now we prove that if $\tilde{f}$ exists, it is unique. Given $y_1 \in Y$, choose a path α in Y from y_0 to y_1. Take the path $\underset{\sim}{f} \circ \alpha$ in B and lift it to a path γ in E beginning at e_0. If a lifting $\tilde{f}$ of f exists, then $\tilde{f}(y_1)$ must equal the end point $\gamma(1)$ of γ, for $\tilde{f} \circ \alpha$ is a lifting of $f \circ \alpha$ that begins at e_0, and path liftings are unique.

Finally, we prove the "if" part of the theorem. The uniqueness part of the proof gives us a clue how to proceed. Given $y_1 \in Y$, choose a path α in Y from y_0 to y_1. Lift the path $f \circ \alpha$ to a path γ in E beginning at e_0, and define $\tilde{f}(y_1) = \gamma(1)$. Now we show that $\tilde{f}$ is well-defined and continuous.

Let α and β be two paths in Y from y_0 to y_1. We must show that if we lift $f \circ \alpha$ and $f \circ \beta$ to paths in E beginning at e_0, then these lifted paths end at the same point of E.

We lift $f \circ \alpha$ to a path γ in E beginning at e_0; then we lift $f \circ \bar{\beta}$ to a path δ in E beginning at the end point $\gamma(1)$ of γ. Then $\gamma * \delta$ is a lifting of the loop $f \circ (\alpha * \bar{\beta})$. Now by hypothesis,

$$f_*\left(\pi_1\left(Y, y_0\right)\right) \subset p_*\left(\pi_1\left(E, e_0\right)\right).$$

Hence $[f \circ (\alpha * \bar{\beta})]$ belongs to the image of p_*. Hence its lift $\gamma * \delta$ is a loop in E.

It follows that $\tilde{f}$ is well defined. For $\bar{\delta}$ is a lifting of $f \circ \beta$ that begins at e_0, and γ is a lifting of $f \circ \alpha$ that begins at e_0, and both liftings end at the same point of E.

To prove continuity of $\tilde{f}$ at the point y_1 of Y, we show that, given a neighborhood N of $\tilde{f}(y_1)$, there is a neighborhood W of y_1 such that $\tilde{f}(W) \subset N$. To begin, choose a neighborhood U of $f(y_1)$ that is evenly covered by p. Break $p^{-1}(U)$ up into slices, and let V_0 be the slice that contains the point $\tilde{f}(y_1)$. Replacing U by a smaller neighborhood of $f(y_1)$ if necessary, we can assume that $V_0 \subset N$. Let $p_0 : V_0 \to U$ be obtained by restricting p; then p_0 is a homeomorphism. Because f is continuous at y_1 and Y is locally path connected, we can find a path-connected neighborhood W of y_1 such that $f(W) \subset U$. We shall show that $\tilde{f}(W) \subset V_0$; then our result is proved.

Let α be a path begins at y_0 and ends at y_1. Given $y \in W$, choose a path β in W from y_1 to y. Since $\tilde{f}$ is well defined, $\tilde{f}(y)$ can be obtained by taking the path $\alpha * \beta$ from y_0 to y, lifting the path $f \circ (\alpha * \beta)$ to a path in E beginning at e_0, and letting $\tilde{f}(y)$ be the end point of this lifted path. Now let γ be a lifting of $f \circ \alpha$ that begins at e_0, ends at $\tilde{f}(y_1)$. Since the path $f \circ \beta$ lies in U, the path $\delta = p_0^{-1} \circ f \circ \beta$ is a lifting of it that begins at $\tilde{f}(y_1)$. Then $\gamma * \delta$ is a lifting of $f \circ (\alpha * \beta)$ that begins at e_0; it ends at the point $\delta(1)$ of V_0. Hence $\tilde{f}(W) \subset V_0$, as desired.

Theorem 1.2.26. Let $p : E \to B$ and $p' : E' \to B$ be covering maps; let $p(e_0) = p'(e_0') = b_0$. There is an equivalence $h : E \to E'$ such that $h(e_0) = e_0'$ if and only if the groups

$$H_0 = p_*(\pi_1(E, e_0)) \quad \text{and} \quad H_0' = p_*'(\pi_1(E', e_0'))$$

are equal. If h exists, it is unique.

1.3 Retraction and Deformation

Definition 1.3.1. If $A \subset X$, a retraction of X onto A is a continuous map $r : X \to A$ such that $r|_A$ is the identity map of A. If such a map r exists, we say that A is a retract of X.

Proposition 1.3.2. If A is a retract of X, then the homomorphism of fundamental groups induced by inclusion $j : A \to X$ is injective.

Proof: If $r : X \to A$ is a retraction, then the composite map $r \circ j$ equals the identity map of A. It follows that $r_* \circ j_*$ is the identity map of $\pi_1(A, a)$, so that j_* must be injective.

Corollary 1.3.3. There is no retraction of $B^2 = \{x \in \mathbb{R}^2 : ||x|| \leq 1\}$ onto $\mathbb{S}^1$.

Proof: By Example 1.2.22.

Theorem 1.3.4. Let $h : S^1 \to X$ be a continuous map. Then the following conditions are equivalent:

(1) h is nulhomotopic.

(2) h extends to a continuous map $k : B^2 \to X$.

(3) $(h_{x_0})_*$ is the trivial homomorphism of fundamental groups for all $x_0 \in \mathbb{S}^1$

Proof: (1) $\Rightarrow$ (2): Let $H : S^1 \times I \to X$ be a homotopy between h and a constant map. Let $\pi : S^1 \times I \to B^2$ be the map

$$\pi(x, t) = (1 - t)x.$$

Then π is continuous, closed and surjective, so it is a quotient map; it collapses $S^1 \times 1$ to the point $\mathbf{0}$ and is otherwise injective. Because H is constant on $S^1 \times 1$, it induces, via the quotient map π, a continuous map $k : B^2 \to X$ that is an extension of h.

(2) $\Rightarrow$ (3): If $j : S^1 \to B^2$ is the inclusion map, then h equals the composite $k \circ j$. Hence $h_* = k_* \circ j_*$. But

$$j_* : \pi_1\left(S^1, b_0\right) \to \pi_1\left(B^2, b_0\right)$$

is trivial because the fundamental group of B^2 is trivial. Therefore h_* is trivial.

(3) $\Rightarrow$ (1): Let $p : \mathbb{R} \to S^1$ be the standard covering map, and let $p_0 : I \to S^1$ be its restriction to the unit interval. Then $[p_0]$ generates $\pi_1\left(S^1, b_0\right)$ because p_0 is a loop in S^1 whose lift to $\mathbb{R}$ begins at 0 and ends at 1 .

Let $x_0 = h\left(b_0\right)$. Because h_* is trivial, the loop $f = h \circ p_0$ represents the identity element of $\pi_1\left(X, x_0\right)$. Therefore, there is a path homotopy F in X between f and the constant path at x_0. The map $p_0 \times \mathrm{id} : I \times I \to S^1 \times I$ is a quotient map, being continuous, closed, and surjective by Proposition **??**; it maps $0 \times t$ and $1 \times t$ to $b_0 \times t$ for each t, but is otherwise injective. The path homotopy F maps $0 \times I$ and $1 \times I$ and $I \times 1$ to the point x_0 of X, so it induces a continuous map $H : S^1 \times I \to X$ that is a homotopy between h and a constant map. See Figure 55.2.

Corollary 1.3.5. The inclusion map $j : \mathbb{S}^1 \to \mathbb{R}^2 - \{0\}$ and identity map $\mathbb{S}^1 \to \mathbb{S}^1$ are not nulhomotopic.

Theorem 1.3.6. Given a continuous map $v : \mathbb{S}^1 \to \mathbb{R}^2 - \{0\}$, there exists a point of S^1 where the vector field points directly inward and a point of S^1 where it points directly outward.

Proof: Let w be its restriction to S^1. Because the map w extends to a map of B^2 into $\mathbb{R}^2 - \mathbf{0}$, it is nulhomotopic.

On the other hand, w is homotopic to the inclusion map $j : S^1 \to \mathbb{R}^2 - \mathbf{0}$. Figure 55.3 illustrates the homotopy; one defines it formally by the equation

$$F(x,t) = tx + (1-t)w(x),$$

for $x \in S^1$. We must show that $F(x,t) \neq \mathbf{0}$. Clearly, $F(x,t) \neq \mathbf{0}$ for $t = 0$ and $t = 1$. If $F(x,t) = \mathbf{0}$ for some t with $0 < t < 1$, then $tx + (1-t)w(x) = 0$, so that $w(x)$ equals a negative scalar multiple of x. But this means that $w(x)$ points directly inward at x ! Hence F maps $S^1 \times I$ into $\mathbb{R}^2 - \mathbf{0}$, as desired. It follows that j is nulhomotopic, contradicting the preceding corollary. To show that v points directly outward at some point of S^1, we apply the result just proved to the vector field $(x, -v(x))$.

Theorem 1.3.7 (Brouwer fixed-point theorem for the disc)**.** If $f : B^2 \to B^2$ is continuous, then there exists a point $x \in B^2$ such that $f(x) = x$.

Proof: We proceed by contradiction. Suppose that $f(x) \neq x$ for every x in B^2. Then defining $v(x) = f(x) - x$ gives us a nonvanishing vector field $(x, v(x))$ on B^2. But the vector field v cannot point directly outward at any point x of S^1, for that would mean

$$f(x) - x = ax$$

for some positive real number a, so that $f(x) = (1+a)x$ would lie outside the unit ball B^2. We thus arrive at a contradiction.

Example 1.3.8 (Fundamental theorem of Algebra)**.** A polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

of degree $n > 0$ with complex coefficients has at least one complex root.

Proof: Step 1 : Let $f : \mathbb{S}^1 \to \mathbb{S}^1 : z \mapsto z^n$. Then by Theorem 1.2.21, the induced group homomorphism $f_* : \pi(\mathbb{S}^1, 1) \to \pi(\mathbb{S}^1, 1)$ is injective.

Step 2 : We show that if $g : \mathbb{S}^1 \to \mathbb{R}^2 - \mathbf{0}$ is the map $g(z) = z^n$, then g is not nulhomotopic.

Let $j : \mathbb{S}^1 \to \mathbb{R}^2 - \mathbf{0}$ be inclusion. Notice that $j \circ f = g$, then by Theorem 1.3.2, g_* is injective. Hence g is not nulhomotopic.

Step 3 : Now we prove a stronger case of the theorem. Given a polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

we assume that

$$|a_{n-1}| + \cdots + |a_1| + |a_0| < 1$$

and show that the equation has a root lying in the unit ball B^2. Notice that if we replace x by cx for a sufficiently large $c > 0$, we can obtain the original Fundemental Theorem of Algebra. Assume it has no such root. Then we can define a map $k : B^2 \to \mathbb{R}^2 - \mathbf{0}$ by the equation

$$k(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

Let h be the restriction of k to S^1. Because h extends to a map of the unit ball into $\mathbb{R}^2 - \mathbf{0}$, the map h is nulhomotopic.

On the other hand, we shall define a homotopy F between h and the map g of Step 2; since g is not nulhomotopic, we have a contradiction. We define $F : S^1 \times I \to \mathbb{R}^2 - \mathbf{0}$ by the equation

$$F(z,t) = z^n + t\left(a_{n-1}z^{n-1} + \cdots + a_0\right).$$

$F(z,t)$ never equals $\mathbf{0}$ because

$$\begin{aligned} |F(z,t)| &\geq |z^n| - \left|t\left(a_{n-1}z^{n-1} + \cdots + a_0\right)\right| \\ &\geq 1 - t\left(\left|a_{n-1}z^{n-1}\right| + \cdots + |a_0|\right) \\ &= 1 - t\left(|a_{n-1}| + \cdots + |a_0|\right) > 0. \end{aligned}$$

Definition 1.3.9. Let $f : X \to Y$ and $g : Y \to X$ be continuous maps. Suppose that the map $g \circ f : X \to X$ is homotopic to the identity map of X, and the map $f \circ g : Y \to Y$ is homotopic to the identity map of Y. Then the maps f and g are called homotopy equivalences, and each is said to be a homotopy inverse of the other.

Lemma 1.3.10. Let $h, k : (X, x_0) \to (Y, y_0)$ be continuous maps. If h and k are homotopic, and if the image of the base point x_0 of X remains fixed at y_0 during the homotopy, then the homomorphisms h_* and k_* are equal.

Proof: The proof is immediate. By assumption, there is a homotopy $H : X \times I \to Y$ between h and k such that $H(x_0, t) = y_0$ for all t. It follows that if f is a loop in X based at x_0, then the composite

$$I \times I \xrightarrow{f \times \mathrm{id}} X \times I \xrightarrow{H} Y$$

is a homotopy between $h \circ f$ and $k \circ f$; it is a path homotopy because f is a loop at x_0 and H maps $x_0 \times I$ to y_0.

Theorem 1.3.11. The inclusion map $j : S^n \to \mathbb{R}^{n+1} - \mathbf{0}$ induces an isomorphism of fundamental groups.

Proof: Let $X = \mathbb{R}^{n+1} - \mathbf{0}$; let $b_0 = (1, 0, \ldots, 0)$. Let $r : X \to S^n$ be the map $r(x) = x/\|x\|$. Then $r \circ j$ is the identity map of S^n, so that $r_* \circ j_*$ is the identity homomorphism of $\pi_1(S^n, b_0)$. Now consider the composite $j \circ r$, which maps X to itself;

$$X \xrightarrow{r} S^n \xrightarrow{j} X.$$

This map is not the identity map of X, but it is homotopic to the identity map. Indeed, the straight-line homotopy $H : X \times I \to X$, given by

$$H(x,t) = (1-t)x + tx/\|x\|,$$

is a homotopy between the identity map. It follows from the preceding lemma that the homomorphism $(j \circ r)_* = j_* \circ r_*$ is the identity homomorphism of $\pi_1(X, b_0)$.

Theorem 1.3.12. Let A be a subspace of X. We say that A is a deformation retract of X if the identity map of X is homotopic to a map that carries all of X into A, such that each point of A remains fixed during the homotopy. This means that there is a continuous map $H : X \times I \to X$ such that $H(x, 0) = x$ and $H(x, 1) \in A$ for all $x \in X$, and $H(a, t) = a$ for all $a \in A$. The homotopy H is called a deformation retraction of X onto A. The map $r : X \to A$ defined by the equation $r(x) = H(x, 1)$ is a retraction of X onto A, and H is a homotopy between the identity map of X and the map $j \circ r$, where $j : A \to X$ is inclusion.

Let A be a deformation retract of X; let $x_0 \in A$. Then the inclusion

$$j : (A, x_0) \to (X, x_0)$$

induces an isomorphism of fundamental groups.

Theorem 1.3.13. Let $h, k : X \to Y$ be continuous maps; let $h(x_0) = y_0$ and $k(x_0) = y_1$. If h and k are homotopic, there is a path α in Y from y_0 to y_1 such that $k_* = \hat{\alpha} \circ h_*$. Indeed, if $H : X \times I \to Y$ is the homotopy between h and k, then α is the path $\alpha(t) = H(x_0, t)$.

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "h_*"] \arrow[dr, "k_*"'] & \pi_1(Y, y_0) \arrow[d, "\hat{\alpha}"] \\
& \pi_1(Y, y_1)
\end{tikzcd}$$

Proof: Let $f : I \to X$ be a loop in X based at x_0. We must show that

$$k_*([f]) = \hat{\alpha}(h_*([f])).$$

This equation states that $[k \circ f] = [\bar{\alpha}] * [h \circ f] * [\alpha]$, or equivalently, that

$$[\alpha] * [k \circ f] = [h \circ f] * [\alpha].$$

This is the equation we shall verify. To begin, consider the loops f_0 and f_1 in the space $X \times I$ given by the equations

$$f_0(s) = (f(s), 0) \quad \text{and} \quad f_1(s) = (f(s), 1).$$

Consider also the path c in $X \times I$ given by the equation

$$c(t) = (x_0, t).$$

Then $H \circ f_0 = h \circ f$ and $H \circ f_1 = k \circ f$, while $H \circ c$ equals the path α. Let $F : I \times I \to X \times I$ be the map $F(s, t) = (f(s), t)$. Consider the following paths in $I \times I$, which run along the four edges of $I \times I$:

$$\begin{aligned} \beta_0(s) &= (s, 0) \quad \text{and} \quad \beta_1(s) = (s, 1), \\ \gamma_0(t) &= (0, t) \quad \text{and} \quad \gamma_1(t) = (1, t). \end{aligned}$$

Then $F \circ \beta_0 = f_0$ and $F \circ \beta_1 = f_1$, while $F \circ \gamma_0 = F \circ \gamma_1 = c$. The broken-line paths $\beta_0 * \gamma_1$ and $\gamma_0 * \beta_1$ are paths in $I \times I$ from $(0,0)$ to $(1,1)$; since $I \times I$ is convex, there is a path homotopy G between them. Then $F \circ G$ is a path homotopy in $X \times I$ between $f_0 * c$ and $c * f_1$. And $H \circ (F \circ G)$ is a path homotopy in Y between

$$(H \circ f_0) * (H \circ c) = (h \circ f) * \alpha \quad \text{and}$$
$$(H \circ c) * (H \circ f_1) = \alpha * (k \circ f),$$

Corollary 1.3.14. Let $h, k : X \to Y$ be homotopic continuous maps; let $h(x_0) = y_0$ and $k(x_0) = y_1$. If h_* is injective, or surjective, or trivial, so is k_*.

Corollary 1.3.15. Let $h : X \to Y$. If h is nulhomotopic, then h_* is the trivial homomorphism.

Corollary 1.3.16. Let $f : X \to Y$ be continuous; let $f(x_0) = y_0$. If f is a homotopy equivalence, then

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

is an isomorphism.

Proof: Let $f : X \to Y$ be continuous; let $f(x_0) = y_0$. If f is a homotopy equivalence, then

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

is an isomorphism.

Let $g : Y \to X$ be a homotopy inverse for f. Consider the maps

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1),$$

where $x_1 = g(y_0)$ and $y_1 = f(x_1)$. We have the corresponding induced homomorphisms:

$$\pi_1(X, x_0) \xrightarrow{(f_{x_0})_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(X, x_1) \xrightarrow{(f_{x_1})_*} \pi_1(Y, y_1)$$

Now

$$g \circ f : (X, x_0) \longrightarrow (X, x_1)$$

is by hypothesis homotopic to the identity map, so there is a path α in X such that

$$(g \circ f)_* = \hat{\alpha} \circ (i_X)_* = \hat{\alpha}.$$

It follows that $(g \circ f)_* = g_* \circ (f_{x_0})_*$ is an isomorphism. Hence g_* is surjective. Similarly, because $f \circ g$ is homotopic to the identity map i_Y, the homomorphism $(f \circ g)_* = (f_{x_1})_* \circ g_* = \hat{\beta}$ for some path β in Y. Hence g_* is injective.

Chapter 2

Differential Geometry

Chapter 3

Riemannian Geometry

Chapter 4

Riemann Surface and Complex Manifold

4.1 Riemann Surface

Let X be a connected, Hausdorff topological space, which is locally homeomorphism to a open subset of $\mathbb{C}$.

Definition 4.1.1. A complex chart on X is a homeomorphism $\varphi : U \to V$ of an open subset $U \subseteq X$ onto an open subset $V \subseteq \mathbb{C}$. A complex atlas on X is an open cover $\mathfrak{A} = \{(U_i, \phi_i)\}_{i \in I}$ of X by complex charts such that the transition maps

$$\varphi_i \circ \varphi_j^{-1}\big|_{\varphi_j(U_i \cap U_j)} : \varphi_j (U_i \cap U_j) \to \varphi_i (U_i \cap U_j)$$

Proposition 4.1.2. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas. If two charts (V, ψ) and (W, σ) are both compatible with the atlas $\{(U_\alpha, \phi_\alpha)\}$, then they are compatible with each other.

Proof: Let $p \in V \cap W$. We need to show that $\sigma \circ \psi^{-1}$ is holomorphic at $\psi(p)$. Since $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for $M, p \in U_\alpha$ for some α. Then p is in the triple intersection $V \cap W \cap U_\alpha$.

By the remark above, $\sigma \circ \psi^{-1} = (\sigma \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \psi^{-1})$ is holomorphic on $\psi (V \cap W \cap U_\alpha)$, hence at $\psi(p)$. Since p was an arbitrary point of $V \cap W$, this proves that $\sigma \circ \psi^{-1}$ is holomorphic on $\psi(V \cap W)$.

Definition 4.1.3 (Riemann Surface)**.** A complex structrue on X is a maximal atlas on X. We call X with a complex structrue on X a Riemann Surface.

Example 4.1.4 (complex plane)**.** The complex structure is defined by the atlas $\{\mathrm{id} : \mathbb{C} \to \mathbb{C}\}$.

Example 4.1.5. Let $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and we introduce the following topology. A subset of $\widehat{\mathbb{C}}$ is open if it is either an open subset of $\mathbb{C}$ or it is of the form $U \cup \{\infty\}$, where $U \subseteq \mathbb{C}$ is the complement of a compact subset of $\mathbb{C}$. With this topology $\widehat{\mathbb{C}}$ is a compact Hausdorff topological space, homeomorphic to the 2-sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ via the stereographic projection. Let $U_1 := \mathbb{C}$ and $U_2 := \mathbb{C}^* \cup \{\infty\}$. Let $\varphi_1 := \mathrm{id} : U_1 \to \mathbb{C}$ and let $\varphi_2 : U_2 \to \mathbb{C}$ be defined by $\varphi_2(z) = 1/z$ if $z \in \mathbb{C}^*$ and $\varphi_2(\infty) = 0$. Then φ_1, φ_2 are homeomorphisms.

Example 4.1.6. Let X be a Riemann surface. Let $Y \subseteq X$ be an open connected subset. Then Y is a Riemann surface in a natural way. An atlas is formed by all complex charts $\varphi : U \to V$ on X with $U \subseteq Y$.

Definition 4.1.7. Let X, Y be Riemann surfaces. A continuous map $f : X \to Y$ is called holomorphic if for every pair of charts $\varphi_1 : U_1 \to V_1$ on X and $\varphi_2 : U_2 \to V_2$ on Y with $f(U_1) \subseteq U_2$,

$$\varphi_2 \circ f \circ \varphi_1^{-1} : V_1 \to V_2$$

is holomorphic. A map $f : X \to Y$ is a biholomorphism if there is a holomorphic map $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Two Riemann surfaces are called isomorphic if there is a biholomorphism between them.

Theorem 4.1.8. Let X, Y be Riemann surfaces. Let $f_1, f_2 : X \to Y$ be holomorphic maps which coincide on a set $A \subseteq X$ with a limit point a in X. Then $f_1 = f_2$.

Theorem 4.1.9. Let X, Y be Riemann surfaces and let $f : X \to Y$ be a non-constant holomorphic map. Let $a \in X$ and $b = f(a)$. Then there is an integer $k \geq 1$ and charts $\varphi : U \to V$ on X and $\psi : U' \to V'$ on Y such that $a \in U, \varphi(a) = 0, b \in U', \psi(b) = 0, f(U) \subseteq U'$ and $\psi \circ f \circ \varphi^{-1} : V \to V' : z \mapsto z^k$

Theorem 4.1.10. Let $f : X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Then f is open.

Theorem 4.1.11. Let $f : X \to Y$ be an injective holomorphic map between Riemann surfaces. Then f is a biholomorphism from X to $f(X)$.

Proof: Since f is injective the multiplicity is always 1 , so the inverse map is holomorphic.

Definition 4.1.12. Let X be a Riemann surface and let Y be an open subset of X. A meromorphic function on Y is a holomorphic function $f : Y' \to \mathbb{C}$, where Y' is an open subset of Y such that $Y \backslash Y'$ contains only isolated points and

$$\lim_{x \to a} |f(x)| = \infty \quad \text{ for all } a \in Y \backslash Y'.$$

The points of $Y \backslash Y'$ are called the poles of f. The set of all meromorphic functions on Y is denoted by $\mathscr{M}(Y)$. It is easy to see that $\mathscr{M}(Y)$ is a $\mathbb{C}$-algebra.

Theorem 4.1.13. Let X be a Riemann surface and let $f \in \mathscr{M}(X)$. For each pole a of f define $f(a) := \infty$. The resulting map $f : X \to \widehat{\mathbb{C}}$ is holomorphic. Conversely, let $f : X \to \widehat{\mathbb{C}}$ be holomorphic. Then f is either identically equal to ∞ or $f^{-1}(\infty)$ consists of isolated points and $f : X \backslash f^{-1}(\infty) \to \mathbb{C}$ is meromorphic on X.

Definition 4.1.14. Let $p : Y \to X$ be a non-constant holomorphic map between Riemann surfaces. A point $y \in Y$ is called a branch point of p if there is no neighborhood of y on which p is injective, or equivalently, if $m_y(p) \geq 2$. We say that p is unbranched if it has no branch points.

Theorem 4.1.15. Let X, Y, Z be Riemann surfaces. Let $p : Y \to X$ be an unbranched holomorphic map and let $f : Z \to X$ be holomorphic. Then every lifting $g : Z \to Y$ of f is holomorphic.

Definition 4.1.16. Let X, Y, Z be topological spaces and let $p : Y \to X$ and $f : Z \to X$ be continuous maps. A lifting of f over p is a continuous map $g : Z \to Y$ such that $f = p \circ g$.

Proof: Notice that p is locally a biholomorphic map, so we have every lift g is holomorphic.

Chapter 5

Lie Group and Lie Algebra