Question 1. *assume* $n \geq 2$ *is positive integer,conside* $n \times n$ *matrix* $X = (a_{ij}), a_{ij} \in \{0, 1\}$

1. *there exists* X *such that* $\det X = n - 1$
2. $2 \leq n \leq 4$,*then* $\det X \leq n - 1$
3. $n \geq 2023$,*there exists* X *such that* $\det X > n^{\frac{n}{4}}$

Proof: (1)

Lemma 1. *let*

$$A_n = \begin{bmatrix} x & y & y & \dots & y \\ z & x & y & & \vdots \\ z & z & x & & \\ \vdots & & & \ddots & y \\ z & \dots & & z & x \end{bmatrix}$$

then

$$D_n = \det A_n = \begin{cases} \dfrac{(x-z)^n y - (x-y)^n z}{y - z} & y \neq z \\ (x + (n-1)y)(x-y)^{n-1} & y = z \end{cases}$$

By above lemma,take $x = 0, y = z = 1$, we have $D_n = (n-1)(-1)^{n-1}$,and we can exchange the row of A_n to make its determinant to be positive.Hence there's a matrix X whose elements in $\{0, 1\}$ such that $\det X = n - 1$.

Proof of the lemma:We only need to deal with the second case.Notice that

$$\begin{vmatrix} x & y & y & \dots & y \\ y & x & y & & \vdots \\ y & y & x & & \\ \vdots & & & \ddots & y \\ y & \dots & & y & x \end{vmatrix} = (x + (n-1)y) \begin{vmatrix} 1 & y & y & \dots & y \\ 1 & x & y & & \vdots \\ 1 & y & x & & \\ \vdots & & & \ddots & y \\ 1 & \dots & & y & x \end{vmatrix}$$

$$= (x + (n-1)y) \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & x-y & 0 & & \vdots \\ 1 & 0 & x-y & & \\ \vdots & & & \ddots & 0 \\ 1 & \dots & & 0 & x-y \end{vmatrix} = (x + (n-1)y)(x-y)^{n-1}$$

(2) let $f(n)$ be the maximal determinant of X,it suffices to show $f(2) = 1, f(3) = 2, f(4) = 3$. let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the 2×2 matrix who reaches to the maximal determinant. $f(2) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ac - bd \leq 1 - 0 = 1$. In (1) we have proved that $f(n) \geq n - 1$.Hence $f(2) = 1$. let $\begin{bmatrix} a & b & c \\ d & e & f \\ p & q & r \end{bmatrix}$ be the 3×3 matrix who reaches to the maximal determinant.We may assume one of a, b, c to be zero(c for example) since if $a = b = c = 1$, we can minius the first row by the second row or third row without changing the determinant.Hence, $f(3) = a \begin{vmatrix} e & f \\ q & r \end{vmatrix} - b \begin{vmatrix} d & f \\ p & r \end{vmatrix} \leq 2$,which means $f(3) = 2$

For $n=4$, let $g(n)$ be the maximal determinant of matrix of order n whose elements $\in \{-1,1\}$

Lemma 2 ([?],Theorem 2)**.**

$$g(n)=2^{n-1}f(n-1)$$

Lemma 3 (Hadamard's inequality)**.**

$$g(n)\le n^{\frac{n}{2}}$$

let $G=AA^T$,$\lambda_i, i=1,2\dots,n$ be its non-negative real eigenvalues

$$|g(n)|^{\frac{2}{n}}=|\det A|^{\frac{2}{n}}=(\det G)^{\frac{1}{n}}=(\prod_{i=1}^{n}\lambda_i)^{\frac{1}{n}}\le\frac{\sum_{i=1}^{n}\lambda_i}{n}=n$$

Hence,

$$g(n)\le n^{\frac{n}{2}}$$

By lemma 2 and 3,

$$f(4)=\frac{g(5)}{2^4}\le\frac{5^{2.5}}{16}\le 3.5$$

Since $f(4)\in\mathbb{Z}$,we have $f(4)\le 3$,hence $f(4)=3$ by (1)

(3) It suffices to show that $f(n)>n^{\frac{n}{4}}$ for $n\ge 2023$. By lemma 2, we need to show that $g(n)>2^n n^{\frac{n}{4}}$ for $n\ge 2023$, i.e

$$\log g(n)>\frac{n}{4}\log n+n\log 2$$

Lemma 4 ([?],Corollary of Theorem 2)**.**

$$g(n)>n^{\frac{n}{2}(1-\frac{\log\frac{4}{3}}{\log n})}$$

Hence by lemma 4,

$$\log g(n)>(\frac{n}{2}-\frac{n}{2}\frac{\log\frac{4}{3}}{\log n})\log n=\frac{n}{2}\log n-\frac{n}{2}\log\frac{4}{3}\ge\frac{n}{4}\log n+n\log 2$$

when $n\ge 2023$

Question 2. *1. Is there non-zero real numbers such that:*

$$\lim_{n\to\infty} ||(\sqrt{2}+1)^n s|| = 0$$

2. Is there non-zero real numbers such that:

$$\lim_{n\to\infty} ||(\sqrt{2}+3)^n s|| = 0$$

Proof: (1) take $s = 1$,we show that

$$\lim_{n\to\infty} ||(\sqrt{2}+1)^n|| = 0$$

Lemma 5.

$$a_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n \in \mathbb{Z}$$
$$b_n = \sqrt{2}((1+\sqrt{2})^n - (1-\sqrt{2})^n) \in \mathbb{Z}$$

we prove this lemma by induction,when $n = 1, a_1 = 2 \in \mathbb{Z}, b_1 = 4 \in \mathbb{Z}$,and notice that:

$$\begin{aligned}
a_n &= (1+\sqrt{2})^n + (1-\sqrt{2})^n \\
&= ((1+\sqrt{2})^{n-1} + (1-\sqrt{2})^{n-1})(1+\sqrt{2}+1-\sqrt{2}) - (1-\sqrt{2})(1+\sqrt{2})^{n-1} - (1+\sqrt{2})(1-\sqrt{2})^{n-1} \\
&= 2a_{n-1} - (a_{n-1} + b_{n-1}) \in \mathbb{Z} \\
b_n &= \sqrt{2}((1+\sqrt{2})^n - (1-\sqrt{2})^n) \\
&= 4((1+\sqrt{2})^{n-1} + \cdots + (1-\sqrt{2})^{n-1}) \\
&= 4((1+\sqrt{2})^{n-1} + (1-\sqrt{2})^{n-1} + (1+\sqrt{2})^{n-2}(1-\sqrt{2}) + (1-\sqrt{2})^{n-2}(1+\sqrt{2}) + \dots) \\
&= 4(a_{n-1} - a_{n-3} + \dots) \in \mathbb{Z}
\end{aligned}$$

Hence $a_n, b_n \in \mathbb{Z}$ for all n. Notice that

$$||(\sqrt{2}+1)^n|| = ||(1+\sqrt{2})^n + (1-\sqrt{2})^n - (1-\sqrt{2})^n|| = ||-(1-\sqrt{2})^n|| = ||1-(1-\sqrt{2})^n||$$

when n sufficiently large,we have

$$||(\sqrt{2}+1)^n|| = ||1-(1-\sqrt{2})^n|| = 1-(1-\sqrt{2})^n$$

Hence,

$$\lim_{n\to\infty} ||(\sqrt{2}+1)^n|| = 0$$

(2) there's no such $s \in \mathbb{R}$. If there's $s \neq 0$,such that

$$\lim_{n\to\infty} ||(\sqrt{2}+3)^n s|| = 0$$

let $\{x_n\}$ to be the positive integer sequence such that $||(\sqrt{2}+3)^n s|| = |(\sqrt{2}+3)^n s - x_n|$,then

$$\lim_{n\to\infty} |(\sqrt{2}+3)^n s - x_n| = 0$$

Also we have

$$s = \lim_{n\to\infty} \frac{x_n}{(3+\sqrt{2})^n}$$

we assume for all $n > N$,$|(\sqrt{2}+3)^n s - x_n| < \dfrac{1}{1000}$ let $\{a_n\},\{b_n\}$ to be the postive integer sequence such $a_n + b_n\sqrt{2} = (\sqrt{2}+3)^n$, Notice that $(3+\sqrt{2})^{n+1} = 3a_n + b_n + (a_n + 3b_n)\sqrt{2}$, we have

$$a_{n+1} = 3a_n + 2b_n, b_{n+1} = a_n + 3b_n$$

Hence

$$a_{n+2} = 3a_{n+1} + 2b_{n+1} = 3a_{n+1} + 2a_n + 6b_n = 6a_{n+1} - 7a_n$$

In the same approach,

$$b_{n+2} = 6b_{n+1} - 7b_n$$

When $n > N$

$$\begin{aligned} &|(\sqrt{2}+3)^{n+2}s - 6x_{n+1} + 7x_n| = |(a_{n+2} + \sqrt{2}b_{n+2})s - 6x_{n+1} + 7x_n| \\ &\leq 6|(a_{n+1} + b_{n+1}\sqrt{2})s - x_{n+1}| + 7|(a_n + b_n\sqrt{2})s - x_n| \leq \frac{6+7}{1000} < \frac{1}{2} \end{aligned}$$

so we can get

$$x_{n+2} = 6x_n - 7x_{n-1}$$

Hence for sufficiently large n,

$$x_n = A(3+\sqrt{2})^n + B(3-\sqrt{2})^n$$

Since

$$s = \lim_{n\to\infty} \frac{x_n}{(3+\sqrt{2})^n}$$

we have $A = s$, by the definition of $\{x_n\}$

$$\lim_{n\to\infty} B(3-\sqrt{2})^n = 0$$

we have $B = 0$.Hence $s(3+\sqrt{2})^n \in \mathbb{Z}$ for all n sufficiently large, we assume that

$$s = \frac{t}{(3+\sqrt{2})^m}, t \in \mathbb{Z} \text{ and m sufficiently large}$$

we have

$$s(3+\sqrt{2})^{2m} = t(3+\sqrt{2})^m \in \mathbb{Z}$$

A contradiction!

Question 3. *1. prove the existence of* m_N

2. $\lim_{n\to\infty} \frac{m_N}{N}$ *when* $p = 1$.

3. $\lim_{n\to\infty} \frac{m_N}{N}$ *when* $p \in (0, 1)$.

(a) denote the probability of finding the best candidate by $P(m, N)$,first we fix N and let $f(m) = P(m, N)$. Notice that

$$\begin{aligned} f(m) =& \frac{p}{N} + \frac{p}{N}(\frac{m+1-p}{m+1}) + \frac{p}{N}(\frac{m+1-p}{m+1})(\frac{m+2-p}{m+2}) + \dots \\ &+ \frac{p}{N}(\frac{m+1-p}{m+1})(\frac{m+2-p}{m+2})\dots(\frac{N-1-p}{N-1}) \end{aligned}$$

Since there's some of $1 \le m \le N$ such that $f(m)$ reaches to maximal,we prove the existence of m_N.

(b) we consider $f(m) - f(m+1)$

$$f(m)-f(m+1) = \frac{p}{N}(1 - \frac{p}{m+1} - \frac{p}{m+1}\frac{m+2-p}{m+2} - \dots - \frac{p}{m+1}\frac{m+2-p}{m+2}\dots\frac{N-1-p}{N-1}) \quad (1)$$

Notice the solution of the following equation(not necessarily be unique)

$$\begin{aligned} f(m) - f(m+1) \ge 0 \\ f(m-1) - f(m) \le 0 \end{aligned}$$

can be m_N, i.e. $f(m)$ reaches to maximal at the solution of above equation. By (1), above equation is equivalent to

$$\begin{aligned} &\frac{p}{m+1} + \frac{p}{m+1}\frac{m+2-p}{m+2} + \dots + \frac{p}{m+1}\frac{m+2-p}{m+2}\dots\frac{N-1-p}{N-1} \le 1 \\ &\frac{p}{m} + \frac{p}{m}\frac{m+1-p}{m+1} + \dots + \frac{p}{m}\frac{m+1-p}{m+1}\dots\frac{N-1-p}{N-1} \ge 1 \end{aligned}$$

Since

$$\begin{aligned} 1 &> \frac{p}{m+1} + \frac{p}{m+1}\frac{m+2-p}{m+2} + \dots + \frac{p}{m+1}\frac{m+2-p}{m+2}\dots\frac{N-1-p}{N-1} \\ &\ge \frac{p}{m+1} + \frac{p}{m+2} + \dots + \frac{p}{N-1} \\ &\ge p\int_{m+1}^{N} \frac{1}{x}\mathrm{d}x \\ &\ge p\log\frac{N}{m+1} \end{aligned}$$

we have

$$\frac{m_N}{N} \ge e^{-1/p} - \frac{1}{N}$$

On the other side,

$$\begin{aligned} 1 &\le \frac{p}{m} + \frac{p}{m}\frac{m+1-p}{m+1} + \dots + \frac{p}{m}\frac{m+1-p}{m+1}\dots\frac{N-1-p}{N-1} \\ &= (\frac{1}{m} + \frac{1}{m+1} + \dots + \frac{1}{N-1}) \\ &= \int_{m-1}^{N-1} \frac{1}{x}\mathrm{d}x = \log\frac{N-1}{m-1} \end{aligned}$$

Hence,

$$e^{-1} - \frac{1}{N} \leq \frac{m_N}{N} \leq e^{-1}$$

so we have

$$\lim_{n\to\infty} \frac{m_N}{N} = \frac{1}{e}$$

(c) Still conside

$$\frac{p}{m+1} + \frac{p}{m+1}\frac{m+2-p}{m+2} + \cdots + \frac{p}{m+1}\frac{m+2-p}{m+2}\cdots\frac{N-1-p}{N-1} \leq 1$$
$$\frac{p}{m} + \frac{p}{m}\frac{m+1-p}{m+1} + \cdots + \frac{p}{m}\frac{m+1-p}{m+1}\cdots\frac{N-1-p}{N-1} \geq 1$$

by Bernoulli inequality

$$\begin{aligned}
1 &\geq \frac{p}{m+1} + \frac{p}{m+1}\frac{m+2-p}{m+2} + \cdots + \frac{p}{m+1}\frac{m+2-p}{m+2}\cdots\frac{N-1-p}{N-1} \\
&\geq \frac{p}{m+1} + \frac{p}{m+1}(\frac{m+1}{m+2})^p + \cdots + \frac{p}{m+1}(\frac{m+1}{m+2})^p\cdots(\frac{N-2}{N-1})^p \\
&= \frac{p}{m+1}(\frac{m+1}{m+1})^p + \frac{p}{m+1}(\frac{m+1}{m+2})^p + \cdots + \frac{p}{m+1}(\frac{m+1}{N-1})^p \\
&\geq p\frac{1}{(m+1)^{1-p}}(\int_{m+1}^{N} \frac{1}{x^p}\mathrm{d}x) \\
&= p\frac{1}{(m+1)^{1-p}}\frac{N^{1-p} - (m+1)^{1-p}}{1-p}
\end{aligned}$$

Hence

$$\frac{m_N}{N} \geq p^{\frac{1}{1-p}} - \frac{1}{N} \tag{2}$$

by Bernoulli inequality,

$$\frac{k+1-p}{k+1} = \frac{1}{1+\dfrac{p}{k+1-p}} \leq \frac{1}{(1+\dfrac{1}{k+1-p})^p} = (\frac{k+1-p}{k+2-p})^p$$

Hence on the other side,we have

$$\begin{aligned}
1 &\leq \frac{p}{m} + \frac{p}{m}\frac{m+1-p}{m+1} + \cdots + \frac{p}{m}\frac{m+1-p}{m+1}\cdots\frac{N-1-p}{N-1} \\
&\leq \frac{p}{m} + \frac{p}{m}(\frac{m+1-p}{m+2-p})^p + \cdots + \frac{p}{m}(\frac{m+1-p}{m+2-p})^p\cdots(\frac{N-1-p}{N-p})^p \\
&= \frac{p}{m} + \frac{p}{m}(\frac{m+1-p}{m+2-p})^p + \cdots + \frac{p}{m}(\frac{m+1-p}{N-p})^p \\
&= \frac{p}{m}(m+1-p)^p((\frac{1}{m+1-p}))^p + \cdots + (\frac{1}{N-p})^p) \\
&\leq \frac{p}{m+1-p}(m+1-p)^p(\int_{m+1-p}^{N+1-p} \frac{1}{x^p}\mathrm{d}x) \\
&\leq \frac{p}{(m+1-p)^{1-p}}(\frac{(N+1-p)^{1-p} - (m+1-p)^{1-p}}{1-p})
\end{aligned}$$

Hence

$$\frac{m_N}{N} \leq p^{\frac{1}{1-p}} + \frac{p^{\frac{1}{1-p}}(1-p) + (p-1)}{N} \tag{3}$$

by (2) and (3) we have

$$\overline{\lim_{n\to\infty}} \frac{m_N}{N} \leq p^{\frac{1}{1-p}}$$
$$\underline{\lim_{n\to\infty}} \frac{m_N}{N} \geq p^{\frac{1}{1-p}}$$

Hence,

$$\lim_{n\to\infty} \frac{m_N}{N} = p^{\frac{1}{1-p}}$$

Question 4. *There's infinitely many Galois totally real field of degree d.*

Proof:

Lemma 6. *Subfield of totally real field is totally real.*

Proof of the lemma: Let K be a totally real field and L be a subfield of K.Since every embedding from L to $\mathbb{C}$ can be extended to K, L is naturally be a totally real field.

Consider infinitely many prime number p such that $p \equiv 1 (\bmod 2d)$. $\mathbb{Q}(\cos(2\pi/p))$ is a totally real subfield of degree $\frac{p-1}{2}$ of p-th cyclotomic field. Since d divides $\frac{p-1}{2}$, which is the order of the Galois group of $\mathbb{Q}(\cos(2\pi/p))$(circle group of order $\frac{p-1}{2}$), by Fundmental theorem in Galois theory, there's a Galois subfield L_p of $\mathbb{Q}(\cos(2\pi/p))$ which is a totaly real field by lemma 6, such that $[L_p : \mathbb{Q}] = d$.

It suffice to show that $L_p \neq L_{p'}$ if $p \neq p'$. If $L_p = L_{p'}$ and $p \neq p'$, $L_p \subset \mathbb{Q}(\zeta_p) \cap (\zeta_p') = \mathbb{Q}$ which is a contradiction.