Combination 2: Probability

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Introduction

这节课我们来处理与概率有关的问题, 这类的问题的一个特点是没有固定的思路, 需要灵活运用知识点。

但是一些与概率有关的经典问题和思想我们需要进行讨论, 如条件概率的思想以及二项分布问题。

Definition of Probability

如果样本空间的总量有限, 且每个样本空间的子事件概率相同, 则某个事件发生的概率可以理解为

$$\frac{\text{该事件发生的情况总数}}{\text{总事件个数}}$$

Question

Two different numbers are randomly selected from $-2, -1, 0, 3, 4, 5$ and multiplied together. What is the probablity that the product is 0?
从 $-2, -1, 0, 3, 4, 5$ 中随机选择两个不同的数字并相乘, 为 0 的概率是多少?

如果子事件发生概率不同, 某个事件的概率应该为该事件的所有子事件的概率分别相加。

Binomial Distribution(二项分布)

A typical problem about binomial distribution is the coin toss. Consider a coin of uneven texture with a probability of $p \in (0, 1)$ for heads and a probability of $1 - p$ for tails. Find the probability of k heads after flipping n times.

与二项分布有关的问题中最经典的问题为抛硬币问, 考虑一枚质地不均匀的硬币, 每次抛该硬币得到正面的概率为 $p \in (0, 1)$, 反面的概率为 $1 - p$. 求抛 n 次后 k 次正面的概率.

Binomial Distribution(二项分布)

Theorem

Probability of k heads after flipping n times is $\binom{n}{k}p^k(1-p)^{n-k}$

Example

A fair coin is tossed ten times. Find the probability of getting six heads and four tails.

质地均匀的硬币投十次, 求反面出现四次的概率.

Special Dice (AMC 8, 2018-18)

Question

The faces of a "special dice" are numbered 1, 2, 3, 5, 7, and 8. When two "special dice" are tossed, what is the probability that their sum will be an even number?

一种特殊的骰子面上的数字为 1, 2, 3, 5, 7, 8, 投两个这样的筛子, 得到的数字加起来为偶数的概率是多少?

Question

Replace "two special dice are tossed" in above Question by "three special dice are tossed".

将上题投两个改成投三个。

Conditional Probability(条件概率)

We denote the probability that event A occurs under the condition B by $\mathbb{P}(A|B)$.

Theorem (Conditional probability of A given B)

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Example

Flip a fair coin three times. Find the probability that the first flip comes out heads given that there aree exactly 2 heads flipped.
抛一枚质地均匀的硬币两次, 求在有共两次正面的条件下第一次为正面的概率.

Theorem (Law of total probability(全概率公式))

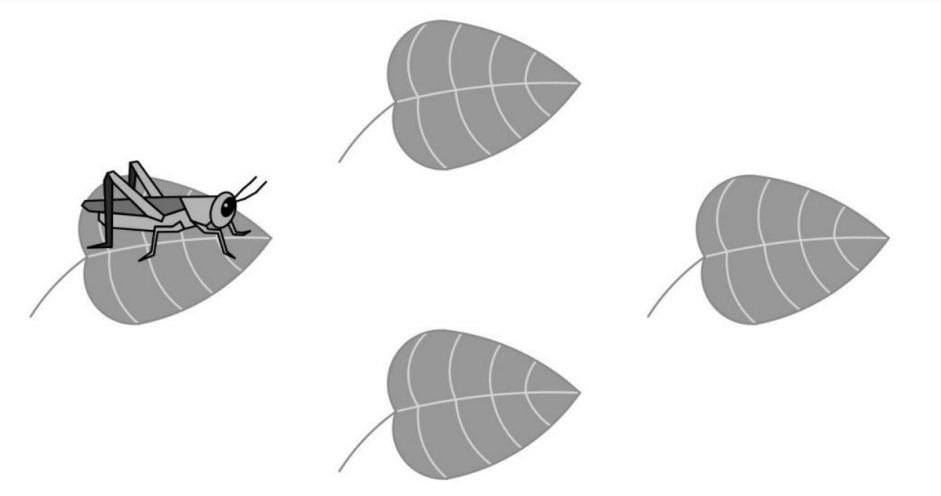
$$\mathbb{P}(A) = \mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|B^C)\mathbb{P}(B^C)$$

Example

用全概率公式证明: 任意抛 n 次均匀的硬币, 正面朝上的次数是偶数 (或奇数) 的概率为 1/2.

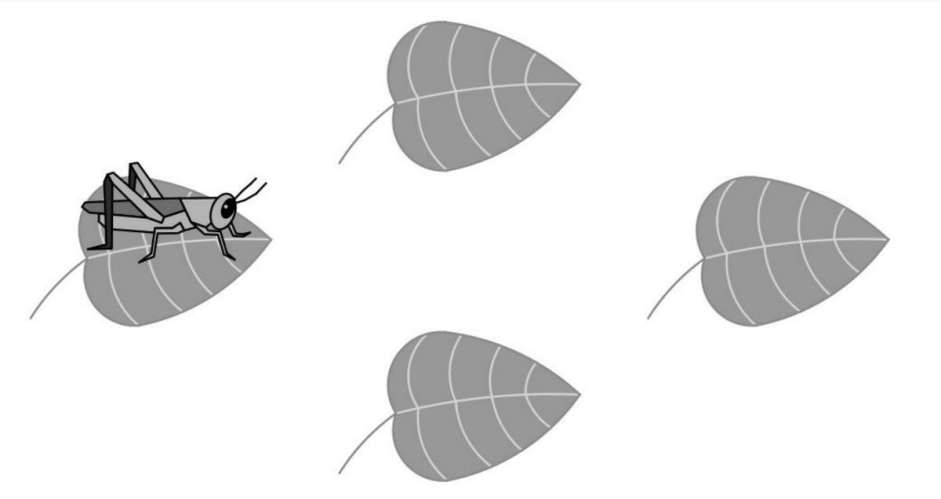
Jumping Cricket(AMC 8, 2022-25)

A cricket randomly hops between 4 leaves, on each turn hopping to one of the other 3 leaves with equal probability. After 4 hops what is the probability that the cricket has returned to the leaf where it started?



跳跃的蟋蟀 (AMC 8, 2022-25)

一个蟋蟀随意的在四片叶子上跳跃, 每次等概率得跳向其他三片叶子之一, 求蟋蟀跳四次后返回最初这片叶子的概率.



Solution

Assume the cricket start from the leaf A. Let $\mathbb{P}(A_n)$ be the probability that the cricket returns to the leaf after n times jumping. $\mathbb{P}(B_n)$ be the probability that the cricket doesn't retrun to the leaf after n times jumping. By Law of total probability,

$$\begin{aligned}\mathbb{P}(A_n) &= \mathbb{P}(A_n|B_{n-1})\mathbb{P}(B_{n-1}) + \mathbb{P}(A_n|B_{n-1}^c)\mathbb{P}(B_{n-1}^c) = \frac{1}{3}\mathbb{P}(B_{n-1}) \\ &= \frac{1}{3}(1 - \mathbb{P}(A_{n-1}))\end{aligned}$$

Since $\mathbb{P}(A_1) = 0$, it's easy to calculate $\mathbb{P}(A_4)$.

Seats Distributing(分配座位)

Question

A small airplane has 4 rows of seats with 3 seats in each row. Eight passengers have boarded the plane and are distributed randomly among the seats. A married couple is next to board. What is the probability there will be 2 adjacent seats in the same row for the couple?

一辆小飞机排布有四行三列座位, 8 个乘客随机地坐在这 12 个座位上, 求这八个人坐完后还可以剩下一对同一行的相邻座位的概率?

Solution

Proof: There are $\binom{12}{8} = 495$ total ways to distribute the passengers. It suffices to count the number of passenger arrangements such that the couple cannot sit anywhere. Consider the partitions of 8 among the rows of 3 seats. We have third pattern of partition:
$(3, 3, 1, 1), (3, 2, 2, 1), (2, 2, 2, 2)$

For the first partition, there are 6 ways to permute the partition. Now the rows with exactly 1 passenger must be in the middle, so this case generates 6 cases.

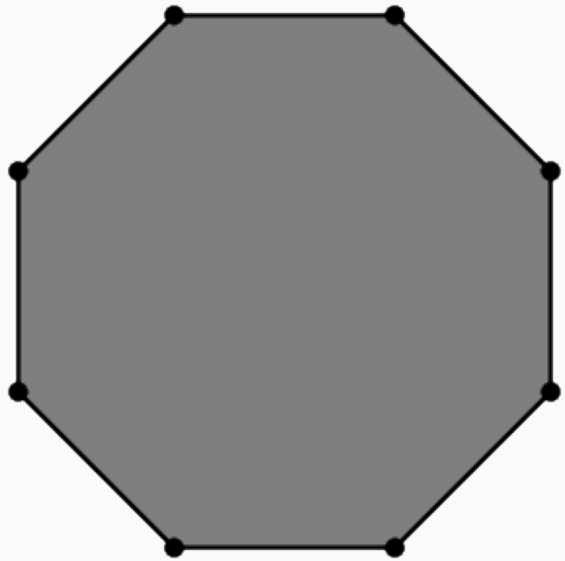
For the second partition, there are 12 ways to permute the partition. For rows with 2 passengers, there are $\binom{3}{2} = 3$ ways to arrange them in the row so that the couple cannot sit there. The row with 1 passenger must be in the middle. We obtain $12 \cdot 3^2 = 108$ cases.

For the third partition, rows with 2 passengers can be arranged in 3 ways, so we obtain $3^4 = 81$ cases.

Collectively, we obtain a grand total of $6 + 108 + 81 = 195$ cases. The final probability is $\frac{20}{33}$.

Octagon(AMC8 2018-23)

From a regular octagon, a triangle is formed by connecting three randomly chosen vertices of the octagon. What is the probability that at least one of the sides of the triangle is also a side of the octagon?



(A) $\frac{2}{7}$ (B) $\frac{5}{42}$ (C) $\frac{11}{14}$ (D) $\frac{5}{7}$ (E) $\frac{6}{7}$

Exercise

Question (AMC8 2017-20)

An integer x satisfying $1000 \leq x \leq 9999$, is chosen at random. What is the probability that it is an odd integer whose digits are all distinct?
(A) $\frac{14}{75}$ (B) $\frac{56}{225}$ (C) $\frac{107}{400}$ (D) $\frac{7}{25}$ (E) $\frac{9}{25}$

Question (AMC 8 2016-21)

A top hat contains 3 red chips and 2 green chips. Chips are drawn randomly, one at a time without replacement, until all 3 of the reds are drawn or until both green chips are drawn. What is the probability that the 3 reds are drawn?
(A) $\frac{3}{10}$ (B) $\frac{2}{5}$ (C) $\frac{1}{2}$ (D) $\frac{3}{5}$ (E) $\frac{7}{10}$

Homework

Question

Coin A is flipped three times and coin B is flipped four times.

(1): Find the probability that both of these coins have two heads.

(2): Find the probability that these two coins have the same number of heads.

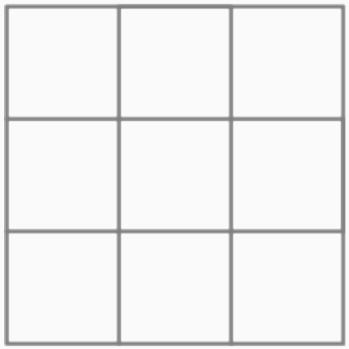
抛质地均匀的硬币 A 共 3 次, B 共 4 次,

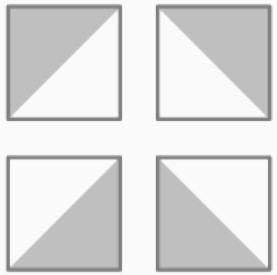
(1): 求这两个硬币正面次数均为两次的概率的概率.

(2): 求这两个硬币正面次数相同的概率.

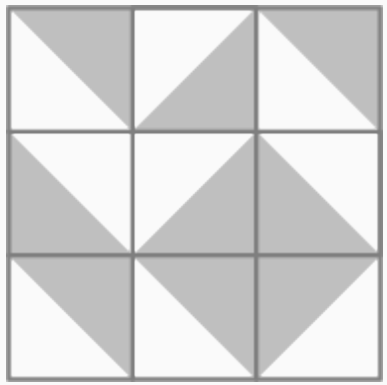
Homework

Each square in a 3×3 grid is randomly filled with one of the 4 gray and white tiles shown below on the right.





What is the probability that the tiling will contain a large gray diamond in one of the smaller 2×2 grids? Below is an example of such tiling.



(A) 1/2024 (B) 1/256 (C) 1/64 (D) 1/16 (E) 1/4

Question

在海滩上,50 个人戴着太阳镜,35 个人戴着帽子。有些人既戴太阳镜又戴帽子。如果随机选择一个戴帽子的人, 这个人也戴太阳镜的概率是 $\frac{2}{5}$. 如果随机选择一个戴太阳镜的人, 这个人也戴帽子的概率是多少?

Question

A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

一个盒子里正好有 5 个筹码,3 个红牌,2 个白牌。每次随机取出一张筹码, 直到所有红牌或白牌都被取出为止。最后一张牌是白色的概率是多少?