Algebraic Geometry

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Contents

Chapter 1

Theory of Scheme

1.1 Sheaf Theory

Definition 1.1.1 (presheaf)**.** Let (Ouv_X) be the category whose objects are the open sets of X and, for two open sets $U, V \subseteq X, \mathrm{Hom}(U, V)$ is empty if $U \nsubseteq V$, and consists of the inclusion map $U \to V$ if $U \subseteq V$ (composition of morphisms being the composition of the inclusion maps). A presheaf is a contravariant functor $\mathscr{F}$ from the category (Ouv_X) to the category of category $\mathcal{C}$(such as the category of abelian groups, the category of rings, the category of R-modules, or the category of R-algebras)

Definition 1.1.2. Let $\mathscr{F}$ be a presheaf on a topological space X, let U be an open set in X and let $\mathscr{U} = (U_i)_{i \in I}$ be an open covering of U. We define maps (depending on $\mathscr{U}$)

$$\rho : \mathscr{F}(U) \to \prod_{i \in I} \mathscr{F}(U_i), \quad s \mapsto \left(s_{|U_i}\right)_i$$

$$\sigma : \prod_{i \in I} \mathscr{F}(U_i) \to \prod_{(i,j) \in I \times I} \mathscr{F}(U_i \cap U_j), \quad (s_i)_i \mapsto \left(s_{i|U_i \cap U_j}\right)_{(i,j)},$$

$$\sigma' : \prod_{i \in I} \mathscr{F}(U_i) \to \prod_{(i,j) \in I \times I} \mathscr{F}(U_i \cap U_j), \quad (s_i)_i \mapsto \left(s_{j|U_i \cap U_j}\right)_{(i,j)}.$$

The presheaf $\mathscr{F}$ is called a sheaf, if it satisfies for all U and all coverings (U_i) as above the following condition:

$$\mathscr{F}(U) \xrightarrow{\rho} \prod_{i \in I} \mathscr{F}(U_i) \underset{\sigma'}{\overset{\sigma}{\rightrightarrows}} \prod_{(i,j) \in I \times I} \mathscr{F}(U_i \cap U_j)$$

is exact. This means that the map ρ is injective and that its image is the set of elements $(s_i)_{i \in I} \in \prod_{i \in I} \mathscr{F}(U_i)$ such that $\sigma((s_i)_i) = \sigma'((s_i)_i)$.

In other words, a presheaf $\mathscr{F}$ is a sheaf if and only if for all open sets U in X and every open covering $U = \bigcup_i U_i$ the following two conditions hold:

(1) (Sh1) Let $s, s' \in \mathscr{F}(U)$ with $s_{|U_i} = s'_{|U_i}$ for all i. Then $s = s'$.

(2) (Sh2) Given $s_i \in \mathscr{F}(U_i)$ for all i such that $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ for all i, j. Then there exists an $s \in \mathscr{F}(U)$ such that $s_{|U_i} = s_i$ (note that s is unique by (Sh1)).

Definition 1.1.3 (restriction of sheaf)**.** If $\mathscr{F}$ is a presheaf on a topological space X and U is an open subspace of X, we obtain a presheaf $\mathscr{F}|_U$ on U by setting $\mathscr{F}|_U(V) = \mathscr{F}(V)$ for every open subset V in U. If $\mathscr{F}$ is a sheaf, $\mathscr{F}|_U$ is a sheaf on U. We call $\mathscr{F}|_U$ the restriction of $\mathscr{F}$ to U.

Definition 1.1.4. The inductive limit

$$\mathscr{F}_x := \varinjlim_{U \ni x} \mathscr{F}(U)$$

is called the stalk of $\mathscr{F}$ in x. In other words, $\mathscr{F}_x$ is the set of equivalence classes of pairs (U, s), where U is an open neighborhood of x and $s \in \mathscr{F}(U)$. Here two such pairs (U_1, s_1) and (U_2, s_2) are equivalent, if there exists an open neighborhood V of x with $V \subseteq U_1 \cap U_2$ such that $s_{1|V} = s_{2|V}$. For each open neighborhood U of x we have a canonical map

$$\mathscr{F}(U) \to \mathscr{F}_x, \quad s \mapsto s_x$$

which sends $s \in \mathscr{F}(U)$ to the class of (U, s) in $\mathscr{F}_x$. We call s_x the germ of s in x. If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves on X, we have an induced map

$$\mathscr{F}_x \to \mathscr{G}_x$$

of the stalks in x by Proposition 3.1.29. We obtain a functor $\mathscr{F} \mapsto \mathscr{F}_x$ from the category of presheaves on X to the category of sets.

If $\mathscr{F}$ is a presheaf with values in $\mathcal{C}$, where $\mathcal{C}$ is the category of abelian groups, of rings, or any category in which filtered inductive limits exist, then the stalk $\mathscr{F}_x$ is an object in $\mathcal{C}$ and we obtain a functor $\mathscr{F} \mapsto \mathscr{F}_x$ from the category of presheaves on X with values in $\mathcal{C}$ to the category $\mathcal{C}$.

Proposition 1.1.5. Let X be a topological space, $\mathscr{F}$ and $\mathscr{G}$ presheaves on X, and let $\varphi, \psi :$ $\mathscr{F} \to \mathscr{G}$ be two morphisms of presheaves.

(1) The induced maps on stalks $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ are injective for all $x \in X$ if $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all open subsets $U \subseteq X$.

(2) Assume that $\mathscr{F}$ is a sheaf. Then the induced maps on stalks $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ are injective for all $x \in X$ if and only if $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all open subsets $U \subseteq X$.

(3) If $\mathscr{F}$ and $\mathscr{G}$ are both sheaves, the maps φ_x are bijective for all $x \in X$ if and only if φ_U is bijective for all open subsets $U \subseteq X$.

(4) If $\mathscr{F}$ and $\mathscr{G}$ are both sheaves, the morphisms φ and ψ are equal if and only if $\varphi_x = \psi_x$ for all $x \in X$.

Proof: For $U \subseteq X$ open consider the map

$$\mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}_x, \quad s \mapsto (s_x)_{x \in U}$$

We claim that this map is injective if $\mathscr{F}$ is a sheaf. Indeed let $s, t \in \mathscr{F}(U)$ such that $s_x = t_x$ for all $x \in U$. Then for all $x \in U$ there exists an open neighborhood $V_x \subseteq U$ of x such that $s_{|V_x} = t_{|V_x}$. Clearly, $U = \bigcup_{x \in U} V_x$ and therefore $s = t$ by sheaf condition (Sh1). Using the commutative diagram

$$\begin{array}{ccc} \mathscr{F}(U) & \longrightarrow & \prod \mathscr{F}_x \\ \downarrow{\scriptstyle \varphi_U} & & \downarrow{\scriptstyle \prod \varphi_x} \\ \mathscr{G}(U) & \longrightarrow & \prod \mathscr{G}_x \end{array}$$

and Proposition 3.1.30, (1) and (3) hold.

(2): By proposition 3.1.30, it suffice to show the bijectivity of φ_x for all $x \in U$ implies the surjectivity of φ_U. Let $t \in \mathscr{G}(U)$. For all $x \in U$ we choose an open neighborhood U^x of x in U and $s^x \in \mathscr{F}(U^x)$ such that $(\varphi_{U^x}(s^x))_x = t_x$. Then there exists an open neighborhood $V^x \subseteq U^x$ of x with $\varphi_{V^x}\left({s^x}_{|V^x}\right) = t_{|V^x}$. Then $(V^x)_{x \in U}$ is an open covering of U and for $x, y \in U$

$$\varphi_{V^x \cap V^y}\left({s^x}_{|V^x \cap V^y}\right) = t_{|V^x \cap V^y} = \varphi_{V^x \cap V^y}\left(s^y \mid V^x \cap V^y\right).$$

As we already know that $\varphi_{V^x \cap V^y}$ is injective, this shows $s^x \left|V^x \cap V^y = s^y\right| V^x \cap V^y$ and the sheaf condition (Sh2) ensures that we find $s \in \mathscr{F}(U)$ such that $s_{|V^x} = {s^x}_{|V^x}$ for all $x \in U$. Clearly, we have $\varphi_U(s)_x = t_x$ for all $x \in U$ and hence $\varphi_U(s) = t$.

Definition 1.1.6. A morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of sheaves injective (resp. surjective, resp. bijective) if $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is injective (resp. surjective, resp. bijective) for all $x \in X$.

Remark 1.1.7. If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, φ is surjective if and only if for all open subsets $U \subseteq X$ and every $t \in \mathscr{G}(U)$ there exist an open covering $U = \bigcup_i U_i$ (depending on t) and sections $s_i \in \mathscr{F}(U_i)$ such that $\varphi_{U_i}(s_i) = t_{|U_i}$, i.e., locally we can find a preimage of t. But the surjectivity of φ does not imply that $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is surjective for all open sets U of X

Definition 1.1.8. If $\mathscr{F}, \mathscr{G}$ are (pre-)sheaves on X such that $\mathscr{F}(U) \subseteq \mathscr{G}(U)$ for all $U \subseteq X$ open, and such that the following diagram commute

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\subset} & \mathscr{G}(U) \\ {\scriptstyle \operatorname{res}_U^V}\uparrow & & \uparrow{\scriptstyle \operatorname{res}_U^V} \\ \mathscr{F}(V) & \xrightarrow[\subset]{} & \mathscr{G}(V) \end{array}$$

we call $\mathscr{F}$ sub(pre-)sheaf of $\mathscr{G}$.

Definition 1.1.9 (sheafification)**.** Let $\mathscr{F}$ be a presheaf on a topological space X. Then there exists a pair $\left(\tilde{\mathscr{F}}, \iota_{\mathscr{F}}\right)$, where $\tilde{\mathscr{F}}$ is a sheaf on X and $\iota_{\mathscr{F}} : \mathscr{F} \to \tilde{\mathscr{F}}$ is a morphism of presheaves, such that the following holds: If $\mathscr{G}$ is a sheaf on X and $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, then there exists a unique morphism of sheaves $\tilde{\varphi} : \tilde{\mathscr{F}} \to \mathscr{G}$ with $\tilde{\varphi} \circ \iota_{\mathscr{F}} = \varphi$. And the following properties hold:

(1) For all $x \in X$ the map on stalks $\iota_{\mathscr{F},x} : \mathscr{F}_x \to \tilde{\mathscr{F}}_x$ is bijective.

(2) For every presheaf $\mathscr{G}$ on X and every morphism of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$ there exists a unique morphism $\tilde{\varphi} : \tilde{\mathscr{F}} \to \tilde{\mathscr{G}}$ making the diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\iota_{\mathscr{F}}"] \arrow[d, "\varphi"'] & \tilde{\mathscr{F}} \arrow[d, "\tilde{\varphi}"] \\ \mathscr{G} \arrow[r, "\iota_{\mathscr{G}}"'] & \tilde{\mathscr{G}} \end{tikzcd}$$

commutative.

In particular, $\mathscr{F} \mapsto \tilde{\mathscr{F}}$ is a functor from the category of presheaves on X to the category of sheaves on X.

Proof: For $U \subseteq X$ open, elements of $\tilde{\mathscr{F}}(U)$ are by definition families of elements in the stalks of $\mathscr{F}$ which locally give rise to sections of $\mathscr{F}$. More precisely, we define

$$\tilde{\mathscr{F}}(U) := \Big\{ (s_x) \in \prod_{x \in U} \mathscr{F}_x : \forall x \in U, \exists \text{ an open neighborhood } W \subseteq U \text{ of } x, \\ \text{and } t \in \mathscr{F}(W) \text{ s.t. } \forall w \in W : s_w = t_w \Big\}.$$

For $U \subseteq V$ the restriction map $\tilde{\mathscr{F}}(V) \to \tilde{\mathscr{F}}(U)$ is induced by the natural projection $\prod_{x\in V} \mathscr{F}_x \to \prod_{x\in U} \mathscr{F}_x$. Then it is easy to check that $\tilde{\mathscr{F}}$ is a sheaf.

For $U \subseteq X$ open, we define $\iota_{\mathscr{F},U} : \mathscr{F}(U) \to \tilde{\mathscr{F}}(U)$ by $s \mapsto (s_x)_{x\in U}$. The definition of $\tilde{\mathscr{F}}$ shows that $\iota_{\mathscr{F},x} : \mathscr{F}_x \to \tilde{\mathscr{F}}_x$ is bijective.

Now let $\mathscr{G}$ be a presheaf on X and let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism. Sending $(s_x)_x \in \tilde{\mathscr{F}}(U)$ to $(\varphi_x(s_x))_x \in \tilde{\mathscr{G}}(U)$ defines a morphism $\tilde{\mathscr{F}} \to \tilde{\mathscr{G}}$. By Proposition 1.1.5, this is the unique morphism making the diagram commutative.

If we assume in addition that $\mathscr{G}$ is a sheaf, then the morphism of sheaves $\iota_{\mathscr{G}} : \mathscr{G} \to \tilde{\mathscr{G}}$, which is bijective on stalks, is an isomorphism by Proposition 1.1.5(3). Composing the morphism $\tilde{\mathscr{F}} \to \tilde{\mathscr{G}}$ with $\iota_{\mathscr{G}}^{-1}$, we obtain the morphism $\tilde{\varphi} : \tilde{\mathscr{F}} \to \mathscr{G}$. Finally, the uniqueness of $\left(\tilde{\mathscr{F}}, \iota_{\mathscr{F}}\right)$ is a formal consequence.

Remark 1.1.10.

Definition 1.1.11 (direct image)**.** Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathscr{F}$ be a presheaf on X. We define a presheaf $f_*\mathscr{F}$ on Y by

$$(f_*\mathscr{F})(V) = \mathscr{F}\left(f^{-1}(V)\right)$$

the restriction maps given by the restriction maps for $\mathscr{F}$. We call $f_*\mathscr{F}$ the direct image of $\mathscr{F}$ under f.

Whenever $\varphi : \mathscr{F}_1 \to \mathscr{F}_2$ is a morphism of presheaves, the family of maps $f_*(\varphi)_V := \varphi_{f^{-1}(V)}$ for $V \subseteq Y$ open is a morphism $f_*(\varphi) : f_*\mathscr{F}_1 \to f_*\mathscr{F}_2$. Therefore f_* is a functor from the category of presheaves on X to the category of presheaves on Y.

Proposition 1.1.12. (1) If $\mathscr{F}$ is a sheaf on X, $f_*\mathscr{F}$ is a sheaf on Y. Therefore f_* also defines a functor $f_* : (\mathrm{Sh}(X)) \to (\mathrm{Sh}(Y))$.

(2) If $g : Y \to Z$ is a second continuous map, there exists an identity $g_* (f_*\mathscr{F}) = (g \circ f)_*\mathscr{F}$ which is functorial in $\mathscr{F}$.

Definition 1.1.13 (inverse image)**.** Let $f : X \to Y$ be a continuous map and let $\mathscr{G}$ be a presheaf on Y. Define a presheaf on X by

$$U \mapsto \varinjlim_{V \supseteq f(U)} \mathscr{G}(V),$$

the restriction maps being induced by the restriction maps of $\mathscr{G}$ and the universal property of direct limit:

$$\begin{tikzcd}
\varinjlim_{V \supseteq f(U)} \mathscr{G}(V) \arrow[rr, dashed, "\mathrm{res}^U_W"'] & & \varinjlim_{V \supseteq f(W)} \mathscr{G}(V) \\
& \mathscr{G}(V_1) \arrow[ul] \arrow[ur] \arrow[d] & \\
& \mathscr{G}(V_2) \arrow[uul] \arrow[uur] &
\end{tikzcd}$$

We denote this presheaf by $f^+\mathscr{G}$. Let $f^{-1}\mathscr{G}$ be the sheafification of $f^+\mathscr{G}$. We call $f^{-1}\mathscr{G}$ the inverse image of $\mathscr{G}$ under f.

Proposition 1.1.14. f^{-1} is a functor from categroy of presheaf on Y to categroy of sheaf on X.

Proof: If $\varphi : \mathscr{G}_1 \to \mathscr{G}_2$ is a morphism of presheaf on Y, then $f^{-1}\varphi : f^{-1}\mathscr{G}_1 \to f^{-1}\mathscr{G}_1$ is induced by universal property of direct limit and Proposition 1.1.9.

Proposition 1.1.15 (stalks of inverse image)**.** Notice that

$$\left(f^{-1}\mathscr{G}\right)_x \cong \left(f^{+}\mathscr{G}\right)_x = \varinjlim_{x \in U} \left(f^{+}\mathscr{G}\right)(U)$$

Since f is continous, by uniqueness of direct limit,

$$\varinjlim_{x \in U} \varinjlim_{f(U) \subset V} \mathscr{G}(V) \cong \varinjlim_{f(x) \in V} \mathscr{G}(V)$$

Proof:

$$\varinjlim_{x \in U} \varinjlim_{f(U) \subset V} \mathscr{G}(V) \cong \varinjlim_{f(x) \in V} \mathscr{G}(V)$$

is given by $[[s]], s \in \mathscr{G}(V) \to [s], s \in \mathscr{G}(V)$ since f is continous.

Proposition 1.1.16. Now let $g : Y \to Z$ be a second continuous map and let $\mathscr{H}$ be a presheaf on Z. By the definition of f^+ and g^+, $f^+ (g^+\mathscr{H}) \cong (g \circ f)^+\mathscr{H}$. By taking sheafification,

$$f^{-1}\left(g^{+}\mathscr{H}\right) \cong (g \circ f)^{-1}\mathscr{H}$$

Since there's natural morphism of sheaves $f^{-1}g^{+}\mathscr{H} \to f^{-1}(g^{-1})\mathscr{H}$ and the morphism at stalks are isomorphism, we have

$$f^{-1}\left(g^{-1}\mathscr{H}\right) \cong f^{-1}\left(g^{+}\mathscr{H}\right) \cong (g \circ f)^{-1}\mathscr{H},$$

Theorem 1.1.17 (adjoint pair (f^{-1}, f_*))**.** Let $f : X \to Y$ be a continuous map, let $\mathscr{F}$ be a sheaf on X and let $\mathscr{G}$ be a presheaf on Y. Then there is a bijection

$$\begin{gathered}\operatorname{Hom}_{(\mathrm{Sh}(X))}\left(f^{-1}\mathscr{G}, \mathscr{F}\right) \leftrightarrow \operatorname{Hom}_{(\mathrm{PreSh}(Y))}\left(\mathscr{G}, f_*\mathscr{F}\right), \\ \varphi \to \varphi^b, \\ \psi^\sharp \leftarrow \psi\end{gathered}$$

and (f^{-1}, f_*) is an adjoint pair between $\mathrm{PreSh}(Y)$ and $\mathrm{Sh}(X)$.

Proof: Let $\varphi : f^{-1}\mathscr{G} \to \mathscr{F}$ be a morphism of sheaves on X, and let $V \subseteq Y$ be open. Since $f\left(f^{-1}(V)\right) \subseteq V$, we have a map $\mathscr{G}(V) \to f^+\mathscr{G}\left(f^{-1}(V)\right)$, and we define φ^b_V as the composition

$$\mathscr{G}(V) \to f^+\mathscr{G}\left(f^{-1}(V)\right) \longrightarrow f^{-1}\mathscr{G}\left(f^{-1}(V)\right) \xrightarrow{\varphi_{f^{-1}(V)}} \mathscr{F}\left(f^{-1}(V)\right) = f_*\mathscr{F}(V).$$

Conversely, let $\psi : \mathscr{G} \to f_*\mathscr{F}$ be a morphism of presheaves on Y. To define the morphism $\psi^\sharp$ it suffices to define a morphism of presheaves $f^+\mathscr{G} \to \mathscr{F}$, which we call again $\psi^\sharp$. Let U be open in X, and $s \in f^+\mathscr{G}(U)$. If V is some open neighborhood of $f(U), U$ is contained in $f^{-1}(V)$. Let V be such a neighborhood such that there exists $s_V \in \mathscr{G}(V)$ representing s. Then $\psi_V\left(s_V\right) \in f_*\mathscr{F}(V) = \mathscr{F}\left(f^{-1}(V)\right)$. Let $\psi^\sharp_U(s) \in \mathscr{F}(U)$ be the restriction of the section $\psi_V\left(s_V\right)$ to U.

Proposition 1.1.18. Let $f : X \to Y$ be a continuous map, let $\mathscr{F}$ be a sheaf on X and let $\mathscr{G}$ be a presheaf on Y, and a morphism of presheaves $\psi : \mathscr{G} \to f_*\mathscr{F}$. Then for each $x \in X$, the map

$$\psi^\sharp_x : \mathscr{G}_{f(x)} \cong \left(f^{-1}\mathscr{G}\right)_x \longrightarrow \mathscr{F}_x$$

induced by $\psi^\sharp : f^{-1}\mathscr{G} \to \mathscr{F}$ on stalks can be described in terms of ψ as follows: For every open neighborhood $V \subseteq Y$ of $f(x)$, we have maps

$$\mathscr{G}(V) \xrightarrow{\psi_V} \mathscr{F}\left(f^{-1}(V)\right) \longrightarrow \mathscr{F}_x,$$

and taking the inductive limit over all V we obtain the map $\psi^\sharp_x : \mathscr{G}_{f(x)} \to \mathscr{F}_x$.

Proposition 1.1.19. X be a topological space, then category of sheaves on X is an abelian category.

Proof: Cokernel exists: If $\mathcal{F}, \mathcal{G}$ are sheaves and $\varphi : \mathcal{F} \to \mathcal{G}$ is a sheaf map, then coker φ exists.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[ddrr, "0"'] & \mathcal{G} \arrow[rr, "\iota\circ\pi"] \arrow[dr, "\pi"] \arrow[ddr, "u"'] & & (\mathcal{G}/\mathrm{im}\mathcal{F})^+ \arrow[ddl, dotted, "\theta"] \\
& & \mathcal{G}/\mathrm{im}\mathcal{F} \arrow[ur, "\iota"] \arrow[d, dotted] & \\
& & X &
\end{tikzcd}$$

Here $+$ denotes the sheafification and θ is induced by universal property of sheafification.

Ab2:

$$\begin{tikzcd}
 & & \mathcal{G}/\mathrm{im}\mathcal{F} \arrow[dr] & \\
\mathrm{Ker}\varphi \arrow[r, "u"] & \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[r, "v"] \arrow[u, "\pi"] & (\mathcal{G}/\mathrm{im}\mathcal{F})^+ = \mathrm{coker}\varphi \\
 & \mathcal{F}/\mathrm{Ker}\varphi \arrow[r, "\simeq"] \arrow[d] & \mathrm{im}\mathcal{F} \arrow[u] \arrow[d] & \\
 & (\mathcal{F}/\mathrm{Ker}\varphi)^+ \arrow[r, "\simeq"] & (\mathrm{im}\mathcal{F})^+ \arrow[uu, dashed, bend right, "l"'] &
\end{tikzcd}$$

By the construction of cokernel, $(\mathcal{F}/\mathrm{Ker}\varphi)^+$ is the cokernel of u. Since kernel of v contains $\mathrm{im}\mathcal{F}$, by universal property of sheafification, l_U is injective for all U open in X and the image of l lie in the kernel of v. Now it suffice to show $l : (\mathrm{im}\mathcal{F})^+ \to \ker(v)$ is isomorphism on stalk. Notice that morphisms on stalk is clearly injective and for some $[g] \in \ker(v)_x$, where $g \in \mathscr{G}(U)$, since $\pi_x([g]) = 0$(By Proposition 1.1.9), there's $V \subset U$ such that $\pi_V(g|_V) = 0$. Hence, $g|_V \in \mathrm{im}(\mathscr{F})(V)$ which implies l_x is surjective. Hence, $(\mathrm{im}\mathcal{F})^+$ is the kernel of v.

Proposition 1.1.20. $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves, then coker $\varphi = 0$ if and only if φ_x be surjective for all $x \in X$.

Proof: coker$= 0$ implies φ_x is surjective for all x: By above diagram, if coker $\varphi = 0$, we have $l : (\mathrm{im}\mathcal{F})^+ \to \mathcal{G}$ is an isomorphism of sheaves. Hence, the map $\mathrm{im}\mathcal{F} \to \mathcal{G}$ is surjective on stalks. Hence, it suffice to check $\mathcal{F} \to \mathcal{F}/\mathrm{Ker}\varphi$ is surjective on stalk, which is obvious.

φ_x is surjective for all $x$implies coker$= 0$: it suffice to show $(\mathrm{coker}\varphi)_x$ for all $x \in X$. Since φ_x is surjective, $l : (\mathrm{im}\mathcal{F})^+ \to \mathcal{G}$ is an isomorphism of sheaves. Hence, the kernel of v is $\mathcal{G}$. Then v_x is surjective and $= 0$ for all $x \in X$.

Proposition 1.1.21. Let X be a topological space and $i : Z \to X$ the inclusion of a subspace Z. Let $\mathscr{F}$ be a sheaf on Z. Show the following properties for the stalks $i_*(\mathscr{F})_x$.

(1) For all $x \notin \bar{Z}, i_*(\mathscr{F})_x$ is a singleton (i.e., a set consisting of one element).

(2) For all $x \in Z, i_*(\mathscr{F})_x = \mathscr{F}_x$.

(3) If $Z = \{x\}$ and $\mathscr{F}$ is a constant sheaf on Z with value E, then $i_*(\mathscr{F})$ is called skyscraper sheaf in x with value E.

Theorem 1.1.22. X be a topological space and Z be a closed subset of X with $i : Z \to X$ be the embedding, $\mathscr{G}$ is a sheaf on X supported on Z(That is, Supp$\mathscr{G} \subset Z$), then $i^{-1}\mathscr{G}$ is a sheaf on Z. On the other hand, if $\mathscr{F}$ is a sheaf on Z, by Proposition 1.1.15, $i_*\mathscr{F}$ is a sheaf supported on Z.

$$\{\text{sheaf on } X \text{ suppoted on } Z\} \underset{i_*}{\overset{i^{-1}}{\rightleftarrows}} \{\text{sheaf on } Z\}$$

$$\mathscr{F} \longrightarrow i^{-1}\mathscr{F}$$

$$i_*\mathscr{G} \longleftarrow \mathscr{G}$$

Moreover, for a sheaf $\mathscr{F}$ supported on Z, the identify map $i^{-1}\mathscr{F} \to i^{-1}\mathscr{F}$ induces φ a natural isomorphism of sheaves

$$\mathscr{F} \to i_*i^{-1}\mathscr{F}$$

And, for a sheaf $\mathscr{G}$ on Z, the identify map $i_*\mathscr{G} \to i_*\mathscr{G}$ induces φ a natural isomorphism of sheaves

$$i^{-1}i_*\mathscr{G} \to \mathscr{G}$$

Proof: Since for all $x \in Z$

$$\varphi_x : \mathscr{F}_x \to (i_*i^{-1}\mathscr{F})_x \simeq (i^{-1}\mathscr{F})_x \simeq \mathscr{F}_x$$

is an identity map and for all $x \notin Z$, $\mathscr{F}_x = 0 = (i_*i^{-1}\mathscr{F})_x = 0$ by Proposition 1.1.15, we have $\mathscr{F} \simeq i_*i^{-1}\mathscr{F}$

1.2 Ringed Space

Definition 1.2.1. A ringed space is a pair $(X, \mathscr{O}_X)$, where X is a topological space and where $\mathscr{O}_X$ is a sheaf of (commutative) rings on X.

If $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are ringed spaces, we define a morphism of ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ as a pair (f, f^b), where $f : X \to Y$ is a continuous map and where $f^b : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is a homomorphism of sheaves of rings on Y.

Definition 1.2.2. If A is a local ring, we denote by $\mathfrak{m}_A$ its maximal ideal and by $\kappa(A) = A/\mathfrak{m}_A$ its residue field. A homomorphism of local rings $\varphi : A \to B$ is called local, if $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$.

A morphism $(f, f^b) : X \to Y$ of ringed spaces induces morphisms on the stalks as follows. Let $x \in X$. Let $f^\sharp : f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ be the morphism corresponding to f^b by adjointness. Using the identification $(f^{-1}\mathscr{O}_Y)_x = \mathscr{O}_{Y,f(x)}$, we get

$$f_x^\sharp : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$$

Definition 1.2.3. A locally ringed space is a ringed space $(X, \mathscr{O}_X)$ such that for all $x \in X$ the stalk $\mathscr{O}_{X,x}$ is a local ring.

A morphism of locally ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces (f, f^b) such that for all $x \in X$ the induced homomorphism on stalks

$$f_x^\sharp : \left(f^{-1}\mathscr{O}_Y\right)_x = \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$$

is a local ring homomorphism.

Definition 1.2.4. Let $(X, \mathscr{O}_X)$ be a locally ringed space and $x \in X$. We call the stalk $\mathscr{O}_{X,x}$ the local ring of X in x, denote by $\mathfrak{m}_x$ the maximal ideal of $\mathscr{O}_{X,x}$, and by $\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$ the residue field. If U is an open neighborhood of x and $f \in \mathscr{O}_X(U)$, we denote by $f(x) \in \kappa(x)$ the image of f under the canonical homomorphisms $\mathscr{O}_X(U) \to \mathscr{O}_{X,x} \to \kappa(x)$.

Definition 1.2.5 (sheaf of ring on Spec(A))**.** For each prime ideal $\mathfrak{p} \subseteq A$, let $A_\mathfrak{p}$ be the localization of A at $\mathfrak{p}$. For an open set $U \subseteq \operatorname{Spec} A$, we define $\mathcal{O}(U)$ to be the set of functions

$$s : U \to \coprod_{\mathfrak{p} \in U} A_p$$

, such that $s(\mathfrak{p}) \in A_p$ for each $\mathfrak{p}$, and such that s is locally a quotient of elements of A : to be precise, we require that for each $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$, contained in U, and elements $a, f \in A$, such that for each $\mathfrak{q} \in V, f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = a/f$ in $A_\mathfrak{q}$.

Proposition 1.2.6. Let A be a ring, and $(\operatorname{Spec} A, \mathcal{O})$ its spectrum.

(1) For any $\mathfrak{p} \in \operatorname{Spec} A$, the stalk $\mathcal{O}_\mathfrak{p}$ of the sheaf $\mathcal{O}$ is isomorphic to the local ring $A_\mathfrak{p}$.

(2) For any element $f \in A$, the ring $\mathcal{O}(D(f))$ is isomorphic to the localized ring A_f.

(3) In particular, $\Gamma(\operatorname{Spec} A, \mathcal{O}) \cong A$.

Proof: (1):First we define a homomorphism from $\mathcal{O}_\mathfrak{p}$ to $A_\mathfrak{p}$ by sending any local section s in a neighborhood of $\mathfrak{p}$ to its value $s(\mathfrak{p}) \in A_\mathfrak{p}$. This gives a well-defined homomorphism φ from $\mathcal{O}_\mathfrak{p}$ to $A_\mathfrak{p}$. The map φ is surjective, because any element of $A_\mathfrak{p}$ can be represented as a quotient a/f, with $a, f \in A, f \notin \mathfrak{p}$. Then $D(f)$ will be an open neighborhood of $\mathfrak{p}$, and a/f defines a section of $\mathcal{O}$ over $D(f)$ whose value at $\mathfrak{p}$ is the given element. To show that φ is injective, let U be a neighborhood of $\mathfrak{p}$, and let $s, t \in \mathcal{O}(U)$ be elements having the same value $s(\mathfrak{p}) = t(\mathfrak{p})$ at $\mathfrak{p}$. By shrinking U if necessary, we may assume that $s = a/f$, and $t = b/g$ on U, where $a, b, f, g \in A$, and $f, g \notin \mathfrak{p}$. Since a/f and b/g have the same image in $A_\mathfrak{p}$, it follows from the definition of localization that there is an $h \notin \mathfrak{p}$ such that $h(ga - fb) = 0$ in A. Therefore $a/f = b/g$ in every local ring $A_\mathfrak{q}$ such that $f, g, h \notin \mathfrak{q}$. But the set of such $\mathfrak{q}$ is the open set $D(f) \cap D(g) \cap D(h)$, which contains $\mathfrak{p}$.

(2): We define a homomorphism $\psi : A_f \to \mathcal{O}(D(f))$ by sending a/f^n to the section $s \in \mathcal{O}(D(f))$ which assigns to each $\mathfrak{p}$ the image of a/f^n in $A_\mathfrak{p}$.

Corollary 1.2.7. $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec}(A)})$ is a locally ringed space.

Proposition 1.2.8. A, B are commutative rings,

(1) If $\varphi : A \to B$ is a homomorphism of rings, then φ induces a natural morphism of locally ringed spaces

$$(f, f^b) : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$$

where

$$\begin{aligned} f_U^b :&\mathcal{O}_{\mathrm{Spec}(A)}(U) \to f_*\mathcal{O}_{\mathrm{Spec}(A)}(U) \\ &(s : U \to \coprod_{\mathfrak{p}\in U} A_{\mathfrak{p}}) \mapsto (s' : f^{-1}(U) \to U \to \coprod_{\mathfrak{p}\in U} A_{\mathfrak{p}} \to \coprod_{\mathfrak{q}\in f^{-1}(U)} B_{\mathfrak{q}}) \end{aligned}$$

(2) If A and B are rings, then any morphism of locally ringed spaces from $\operatorname{Spec} B$ to $\operatorname{Spec} A$ is induced by a homomorphism of rings $\varphi : A \to B$ as in (1).

Proof: (1): Assume $\mathfrak{p} \in \operatorname{Spec}(B)$ and $\varphi^{-1}(\mathfrak{p}) = \mathfrak{q}$. Then the ring homomorphism

$$\varphi_{\mathfrak{p}} : A_{\mathfrak{q}} \to B_{\mathfrak{p}}$$

induced by universal property of localization is a local ring homomorphism.

(2): Conversely, suppose given a morphism of locally ringed spaces $\left(f, f^{\#}\right)$ from $\operatorname{Spec} B$ to $\operatorname{Spec} A$. Taking global sections, $f^{\#}$ induces a homomorphism of rings $\varphi : \Gamma\left(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}\right) \to \Gamma\left(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}\right)$. These rings are A and B, respectively, so we have a homomorphism $\varphi : A \to B$. For any $\mathfrak{p} \in \operatorname{Spec} B$, we have an induced local homomorphism on the stalks(universal property of direct limit), $\mathcal{O}_{\operatorname{Spec} A, f(p)} \to \mathcal{O}_{\operatorname{Spec} B, p}$ or $A_{f(p)} \to B_p$, which must be compatible with the map φ on global sections. In other words, we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A_{f(\mathfrak{p})} \arrow[r, "f^{\#}_{\mathfrak{p}}"'] & B_{\mathfrak{p}} \end{tikzcd}$$

Since $f^{\#}$ is a local homomorphism, it follows that $\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$, which shows that f coincides with the map $\operatorname{Spec} B \to \operatorname{Spec} A$ induced by φ.

By universal property of localization, $\varphi_{\mathfrak{p}} = f_{\mathfrak{p}}^{\#}$. Then by Theorem 1.1.5(3), $(f, f^{\#})$ is induced by φ.

Corollary 1.2.9.

Definition 1.2.10. A locally ringed space $(X, \mathscr{O}_X)$ is called affine scheme, if there exists a ring A such that $(X, \mathscr{O}_X)$ is isomorphic to $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$.

Definition 1.2.11. *A* scheme is a locally ringed space $(X, \mathscr{O}_X)$ which admits an open covering $X = \bigcup_{i\in I} U_i$ such that all locally ringed spaces $\left(U_i, \mathscr{O}_X|_{U_i}\right)$ are affine schemes. A morphism of schemes is a morphism of locally ringed spaces.

Definition 1.2.12 (principal oepn subschmems of an affine scheme)**.** Let $X = \operatorname{Spec} A$ be an affine scheme. For $f \in A$ let $j : \operatorname{Spec} A_f \to \operatorname{Spec} A$ be the morphism of affine schemes that corresponds to the canonical homomorphism $A \to A_f$. Then j induces a homeomorphism of $\operatorname{Spec} A_f$ onto $D(f)$. Moreover, for all $x \in D(f)$, $j_x^{\sharp}$ is the canonical isomorphism $A_{\mathfrak{p}_x} \xrightarrow{\sim} (A_f)_{\mathfrak{p}_x}$ by Algebra Theorem 2.5.26. Hence we see that $\left(j, j^{\sharp}\right)$ induces an isomorphism of the affine scheme $\operatorname{Spec} A_f$ with the locally ringed space $\left(D(f), \mathscr{O}_{X|D(f)}\right)$.

Definition 1.2.13 (closed subschmems of affine schemes)**.** Let $X = \operatorname{Spec} A$ be an affine scheme. For an ideal $\mathfrak{a}$ of A let $i : \operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$ be the morphism of affine schemes that corresponds to the canonical homomorphism $A \to A/\mathfrak{a}$. Then i induces a homeomorphism of $\operatorname{Spec} A/\mathfrak{a}$ onto the closed subset $V(\mathfrak{a})$ of $\operatorname{Spec} A$. Moreover, for all $x \in V(\mathfrak{a})$ the morphism i_x^b is the canonical surjective homomorphism $A_{\mathfrak{p}_x} \to (A/\mathfrak{a})_{\overline{\mathfrak{p}_x}}$ where $\overline{\mathfrak{p}}_x$ is the image of $\mathfrak{p}_x$ in $A/\mathfrak{a}$.

1.3 Scheme: Basic Propositions

Definition 1.3.1. Let S be a fixed scheme. The category (Sch/S) of schemes over S (or of S-schemes) is the category whose objects are the morphisms $X \to S$ of schemes, and whose morphisms $\operatorname{Hom}(X \to S, Y \to S)$ are the morphisms $X \to Y$ of schemes with the property that

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

commutes.

Proposition 1.3.2 (open subscheme)**.** (1) Let X be a scheme, and $U \subseteq X$ an open subset. Then the locally ringed space $\left(U, \mathscr{O}_{X|U}\right)$ is a scheme. We call U an open subscheme of X. If U is an affine scheme, then U is called an affine open subscheme.

(2) Let X be a scheme. The affine open subschemes are a basis of the topology.

(3) There's a canonical morphism between scheme $(U, \mathscr{O}_X|_U)$ and $(X, \mathscr{O}_X)$.

(4): $(f, f^b) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of scheme and $f(X) \subset U$ for some open subset of Y, then there's a natrual morphism $(X, \mathscr{O}_X) \to (U, \mathscr{O}_Y|_U)$ making the following diagram ccommute

$$\begin{tikzcd} (X, \mathscr{O}_X) \arrow[r] \arrow[dr] & (Y, \mathscr{O}_Y) \\ & (U, \mathscr{O}_Y|_U) \arrow[u] \end{tikzcd}$$

Proof: (3): For all the V open in X, the restriction maps

$$\Gamma(V, \mathscr{O}_X) \to \Gamma(V \cap U, \mathscr{O}_X|_U)$$

induce a morphism $j^b : \mathscr{O}_X \to j_*(\mathscr{O}_X|_U)$ of sheaves.

Hence, there's a canonical morphism $(U, \mathscr{O}_X|_U) \to (X, \mathscr{O}_X)$ of scheme.

Lemma 1.3.3 (Nike's Trick)**.** Let X be a scheme, and let U, V be affine open subschemes of X. Then there exists for all $x \in U \cap V$ an open subscheme $W \subseteq U \cap V$ with $W \ni x$ such that W is principal open in U as well as in V.

Proof: We may assume $x \in V \subset U$ and U, V are all open affine, hence

$$(j, j^b) : (V, \mathscr{O}_X|_V) \to (U, \mathscr{O}_X|_U)$$

is a morphism of scheme.

$$
\begin{array}{ccccc}
(V, \mathscr{O}_X|_V) & \xrightarrow{\simeq} & (\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A}) & & B \\
\downarrow j & & \downarrow \varphi & & \uparrow \phi \\
(U, \mathscr{O}_X|_U) & \xrightarrow{\simeq} & (\operatorname{Spec} B, \mathscr{O}_{\operatorname{Spec} B}) & & A
\end{array}
$$

Take $f \in B$ such that the principal open subset $D(f)$ satisfies $x \in D(f) \subset V \subset U$, then

$$D(f) = j^{-1}(D(f)) = \varphi^{-1}(D(f)) = D(\phi(f))$$

Lemma 1.3.4 (Gluing of morphisms)**.** Let X, Y be schemes. If $X = \bigcup_i U_i$ is an open covering, then a family of morphisms $\varphi_i : (U_i, \mathscr{O}_X|_{U_i}) \to (Y, \mathscr{O}_Y)$ glues to a morphism (f, f^b) : $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ if and only if the morphisms coincide on intersections $U_i \cap U_j$, and the resulting morphism $X \to Y$ is uniquely determined.

Proof: Firstly, define

$$f : X \to Y, x \mapsto \varphi_i(x) \text{ if } x \in U_i$$

For some V open in Y, we can obtain φ_V by the following diagram:

$$
\begin{array}{ccc}
\mathscr{O}_Y(V) & \overset{\varphi_V}{\dashrightarrow} & f_*\mathscr{O}_X(V) = \mathscr{O}_X(f^{-1}(V)) \\
\downarrow \text{id} & & \text{glue} \uparrow \\
\mathscr{O}_Y(V) & \xrightarrow{(\varphi_i)_V} & (\varphi_i)_* \, \mathscr{O}_X|_{U_i}(V) = \mathscr{O}_X(U_i \cap f^{-1}(V))
\end{array}
$$

Example 1.3.5 (zero section)**.** Consider $\mathbb{A}_R^{n+1} = \operatorname{Spec}(R[T_0, \ldots, T_n])$, define

$$\mathbb{A}_R^{n+1} - \{0\} = \bigcup_{i=0}^{n} D(T_i)$$

be an open subscheme of $\mathbb{A}_R^{n+1}$. Since there's natural morphism p_i given by

$$p_i : D(T_i) = \operatorname{Spec} R\left[T_0, \ldots, T_n, T_i^{-1}\right] \to D_+(X_i) = \operatorname{Spec} R\left[\frac{X_0}{X_i}, \ldots, \widehat{\frac{X_i}{X_i}}, \ldots, \frac{X_n}{X_i}\right]$$

, by gluing of morphisms of scheme, there's a natural morphism

$$p : \mathbb{A}_R^{n+1} - \{0\} \to \mathbb{P}_R^n$$

Example 1.3.6. Consider $X = \operatorname{Spec}(\mathbb{R}[x, y]) - \{0\}$ and $p : X \to \mathbb{P}_{\mathbb{R}}^1$. We have, for $(\alpha, \beta) \neq (0, 0)$,

$$p((x - \alpha, y - \beta)) = (\alpha y - \beta x)$$

Example 1.3.7. Let A be an R-algebra, let $f : \operatorname{Spec} A \to \mathbb{A}_R^n$ be an R-morphism, and denote the corresponding R-algebra homomorphism by $\varphi : R[T_1, \ldots, T_n] \to A$. Set $a_i = \varphi(T_i) \in A$. Then f factors through $\mathbb{A}_R^n - \{0\}$ if and only if for all $\mathfrak{p} \in \operatorname{Spec}(A)$,

$$f(\mathfrak{p}) \in \bigcup_{i=0}^{n} D(T_i)$$

Equivalently, there's no such prime ideal $\mathfrak{p} \subset A$ such that $\varphi^{-1}(\mathfrak{p}) \supset (T_1, \ldots, T_n)$. Since $\varphi^{-1}(\mathfrak{p}) \supset (T_1, \ldots, T_n)$ if and only if $\mathfrak{p} \supset (\varphi(T_1), \ldots, \varphi(T_n))$, we have

$$\operatorname{Hom}_R(\operatorname{Spec}(A), \mathbb{A}_R^n - \{0\}) = \left\{\varphi \in \operatorname{Hom}_{(R-\mathrm{Alg})}(R[x_1, \ldots, x_n], A) : (\varphi(x_1), \ldots, \varphi(x_n)) = (1)\right\}$$

Example 1.3.8 (group scheme)**.** Set $X = \operatorname{Spec} R\left[U, U^{-1}\right] = R[U,T]/(UT-1)$. Then we obtain for every R-scheme T

$$\operatorname{Hom}_R(T, X) = \operatorname{Hom}_{(R-\mathrm{Alg})}\left(R\left[U, U^{-1}\right], \Gamma\left(T, \mathscr{O}_T\right)\right) = \Gamma\left(T, \mathscr{O}_T\right)^{\times}.$$

Proposition 1.3.9. Let $(X, \mathscr{O}_X)$ be a scheme, $Y = \operatorname{Spec} A$ an affine scheme. Then the natural map

$$\operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}\left(A, \Gamma\left(X, \mathscr{O}_X\right)\right), \quad \left(f, f^b\right) \mapsto f_Y^b,$$

is a functorial bijection. Here the set on the left side denotes the set of morphisms $X \to Y$ of scheme, and the set on the right side denotes the set of ring homomorphisms $A \to \Gamma\left(X, \mathscr{O}_X\right)$.

Proof:

$$\begin{tikzcd}
\operatorname{Hom}(X,Y) \arrow[r, dashed] \arrow[d, shift left, "\mathrm{res}"] & \operatorname{Hom}(A, \Gamma(X, \mathscr{O}_X)) \arrow[d, shift left, "\mathrm{res}"] \\
\operatorname{Hom}(U_i, Y) \arrow[r, "\simeq"] \arrow[u, shift left, "\mathrm{glue}"] & \operatorname{Hom}(A, \Gamma(U_i, \mathscr{O}_X)) \arrow[u, shift left, "\mathrm{glue}"]
\end{tikzcd}$$

Injective: For $f : X \to Y$, define $f_i : U_i \to X \to Y$ a morphism of scheme. It's easy to check the follow diagram commutes

$$\begin{tikzcd}
A \arrow[r, "f_Y^b"] \arrow[dr, "(f_i)_Y^b"'] & \Gamma(X, \mathscr{O}_X) \arrow[d, "j_X^b"] \\
 & \Gamma(U_i, \mathscr{O}_X)
\end{tikzcd}$$

Hence, $(f, f^b) = (g, g^b)$ iff $(f_i, f_i^b) = (g_i, g_i^b)$ iff $(f_i)_Y^b = (g_i)_Y^b$ iff $f_Y^b = g_Y^b$

Surjective:

$$\begin{tikzcd}
U_i \arrow[r, "f_i"] & Y \\
V \arrow[u, "l_i"] \arrow[r, "l_j"'] & U_j \arrow[u, "f_j"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma(U_i, \mathscr{O}_X) \arrow[d] & A \arrow[l, "\tilde{f}_i"'] \arrow[d, "\tilde{f}_j"'] \\
\Gamma(V, \mathscr{O}_X) & \Gamma(U_j, \mathscr{O}_X) \arrow[l]
\end{tikzcd}$$

Take $\tilde{f} \in \operatorname{Hom}(A, \Gamma(X, \mathscr{O}_X))$, and define $\tilde{f}_i : \tilde{f} \circ \operatorname{Res}_{U_i}^X$ and f_i be the corresponding morphisms with respect to category equivalence(commutative rings and affine schemes). Consider above diagram, V is an affine open subset of $U_i \cap U_j$. Since opposite category of commutative rings is equivalent to category of affine scheme, the fact that the right diagram commutes implies the left diagram commute.

Proposition 1.3.10. Let $(X, \mathscr{O}_X)$ be a k-scheme, A be a k-algebra and $Y = \operatorname{Spec} A$ an affine scheme over k. Then the natural map

$$\operatorname{Hom}_{\operatorname{Spec}(k)}(X, Y) \longrightarrow \operatorname{Hom}_k\left(A, \Gamma\left(X, \mathscr{O}_X\right)\right), \quad \left(f, f^b\right) \mapsto f_Y^b,$$

is a functorial bijection. Here the set on the left side denotes the set of morphisms $X \to Y$ of k-scheme, and the set on the right side denotes the set of k-algebra homomorphisms $A \to \Gamma(X, \mathscr{O}_X)$.

Proposition 1.3.11. Let X be a scheme. Let $x \in X$, and let $U \subseteq X$ be an affine open neighborhood of x, say $U = \operatorname{Spec} A$. Denote by $\mathfrak{p} \subset A$ the prime ideal of A corresponding to x. Then $\mathscr{O}_{X,x} = \mathscr{O}_{U,x} = A_{\mathfrak{p}}$, and the natural homomorphism $A \to A_{\mathfrak{p}}$ gives us a morphism

$$j_x : \operatorname{Spec} \mathscr{O}_{X,x} = \operatorname{Spec} A_{\mathfrak{p}} \to \operatorname{Spec} A = U \subseteq X$$

of schemes. This morphism is independent of the choice of U.

Proof: Assume V is an open affine subset of U with $x \in V$, $V = \operatorname{Spec}(B)$ and $x = \mathfrak{q}$. Then, it suffices to show j_x induced by V and j_x induced by U identifies. Consider the following commutative diagram

$$\begin{tikzcd}
 & & & & X \\
\operatorname{Spec}\mathcal{O}_{X,x} \arrow[r, "\simeq"] & \operatorname{Spec}A_{\mathfrak{p}} \arrow[r] & \operatorname{Spec}A \arrow[r, "\simeq"] & U \arrow[ur] & \\
\operatorname{Spec}\mathcal{O}_{X,x} \arrow[r, "\simeq"] \arrow[u, "\mathrm{id}"'] & \operatorname{Spec}B_{\mathfrak{q}} \arrow[r] \arrow[u] & \operatorname{Spec}B \arrow[r, "\simeq"] \arrow[u, "\varphi"'] & V \arrow[u] \arrow[uur] &
\end{tikzcd}$$

where the morphism $\operatorname{Spec}(B_{\mathfrak{q}}) \to \operatorname{Spec}(A_{\mathfrak{p}})$ is induced both by universal propoty of localization and the morphism of sheaves $\mathscr{O}_{\operatorname{Spec}(A)} \to \varphi_* \mathscr{O}_{\operatorname{Spec}(B)}$.

Proposition 1.3.12. The image of the canonical map $j_x : \operatorname{Spec}\mathcal{O}_{X,x} \to X$ is

$$Z = \left\{ y \in X : x \in \overline{\{y\}} \right\} = \bigcap_{x \in W, W \text{ open in } X} W$$

Proof: Trivial.

Proposition 1.3.13. Let $\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$ be the residue class field of x in X. We obtain a morphism of schemes

$$i_x : \operatorname{Spec} \kappa(x) \longrightarrow \operatorname{Spec} \mathscr{O}_{X,x} \longrightarrow X$$

called canonical. The image point of the unique point in $\operatorname{Spec} \kappa(x)$ is x. Notice that the map $\mathscr{O}_{X,x} \to \kappa(x)$ induced by considering the stalk of i_x is exactly the projective map.

Now let K be any field, let $f : \operatorname{Spec} K \to X$ be a morphism, and let $x \in X$ be the image point of the unique point p of $\operatorname{Spec} K$. Since f is a morphism of locally ringed spaces, f induces a local homomorphism $\mathscr{O}_{X,x} \to K = \mathscr{O}_{\operatorname{Spec} K, p}$, and hence a homomorphism $\iota : \kappa(x) \to K$ between the residue class fields.

Then, the morphism f factors as $f = i_x \circ (\operatorname{Spec} \iota) : \operatorname{Spec} K \to \operatorname{Spec} \kappa(x) \to X$ since we have a commutative diagram in stalks of those sheaves:

$$\begin{tikzcd}
K & \mathscr{O}_{X,x} \arrow[l] \arrow[dl] \\
\kappa(x) \arrow[u] &
\end{tikzcd}$$

The above construction gives rise to a bijection

$$\operatorname{Hom}(\operatorname{Spec} K, X) \longrightarrow \{(x, \iota); x \in X, \iota : \kappa(x) \to K\}$$

This is because, we can map an element $(x, \iota : \kappa(x) \to K)$ of the right hand side to the morphism

$$\operatorname{Spec} K \xrightarrow{\operatorname{Spec} \iota} \operatorname{Spec} \kappa(x) \xrightarrow{i_x} X,$$

and these two maps are inverse to each other.

Proposition 1.3.14. Assume $(X, \mathscr{O}_X) \to \operatorname{Spec}(k)$ be a k-scheme, then this map induces a local ring homomorphism

$$k \to \mathscr{O}_{X,x}$$

which induecs a field extension

$$k \to \kappa(x)$$

Hence there's natural k-scheme structure on $\operatorname{Spec}(\kappa(x))$. Moreover, above natural morphism i_x becomes a k-scheme morphism:

$$\begin{tikzcd}
(X, \mathscr{O}_X) \arrow[dr] & \\
\operatorname{Spec}\kappa(x) \arrow[u, "i_x"'] \arrow[r] & \operatorname{Spec}k \\
\operatorname{Spec}K \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Hence, if $k \to K$ be a field extension, there's a bijection

$$\operatorname{Hom}_k(\operatorname{Spec} K, X) \longrightarrow \{(x, \iota) : x \in X, \iota : \kappa(x) \to K \quad k\text{-algebra homomorphism}\}$$

And for an arbitrary k-scheme, define $X(K) = \operatorname{Hom}_k(\operatorname{Spec} K, X)$ to be its K-points.

Definition 1.3.15 (Structure sheaf on $\operatorname{Proj}S$)**.** Let S be a graded ring, we will define a sheaf of rings $\mathscr{O}$ on $\operatorname{Proj} S$. For each $\mathfrak{p} \in \operatorname{Proj} S$, we consider the ring $S_{(\mathfrak{p})}$ of elements of degree zero in the localized ring $T^{-1}S$, where T is the multiplicative system consisting of all homogeneous elements of S which are not in $\mathfrak{p}$. For any open subset $U \subseteq \operatorname{Proj} S$, we define $\mathscr{O}(U)$ to be the set of functions $s : U \to \coprod S_{(\mathfrak{p})}$ such that for each $\mathfrak{p} \in U, s(\mathfrak{p}) \in S_{(\mathfrak{p})}$, and such that s is locally a quotient of elements of S : for each $\mathfrak{p} \in U$, there exists a neighborhood V of $\mathfrak{p}$ in U, and homogeneous elements a, f in S, of the same degree, such that for all $\mathfrak{q} \in V, f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = a/f$ in $S_{(\mathfrak{q})}$. Now it is clear that $\mathscr{O}$ is a presheaf of rings, with the natural restrictions, and it is also clear from the local nature of the definition that $\mathscr{O}$ is a sheaf.

Proposition 1.3.16. Let S be a graded ring.

(1) For any $\mathfrak{p} \in \operatorname{Proj} S$, the stalk $\mathscr{O}_{\mathfrak{p}}$ is isomorphic to the local ring $S_{(\mathfrak{p})}$.

(2) For any homogeneous element $f \in S_+$, let $D_+(f) = \{\mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p}\}$. Then $D_+(f)$ is open in Proj S. Furthermore, these open sets cover Proj S, and for each such open set, we have an isomorphism of locally ringed spaces

$$\left(D_+(f), \mathscr{O}|_{D_+(f)}\right) \cong \operatorname{Spec} S_{(f)}$$

where

$$S_{(f)} = \{a/f^n \in S_f : a \text{ homogeneous and } \deg(a) = n\deg(f), n \geq 0\}$$

In particular, the global section of S is S_0.

Proof: (1): $S_{(\mathfrak{p})}$ is a local ring: The unique maximal ideal of $S_{(\mathfrak{p})}$ is of the form

$$\{a/f : a \in \mathfrak{p}, f \notin \mathfrak{p}, \deg a = \deg f\}$$

(2): Define

$$\varphi : D_+(f) \to \operatorname{Spec}(S_{(f)}), \mathfrak{a} \mapsto \mathfrak{a}S_f \cap S_{(f)}$$

φ is injective: If $\mathfrak{a}S_f \cap S_{(f)} = \mathfrak{b}S_f \cap S_{(f)}$, for some homogeneous element $s \in \mathfrak{a}$, there's $b \in \mathfrak{b}$ such that

$$\frac{s^n}{f^m} = \frac{b}{f^t}$$

for some integer n, m, t. Hence, $s^n \in \mathfrak{b}$ which implies $s \in \mathfrak{b}$.

φ is surjective: P be a prime ideal of $S_{(f)}$, define

$$\mathfrak{p} = \{s \in S : s/f^n \in P \text{ for some } n \geq 0\}$$

Then $\varphi(\mathfrak{p}) = P$.

Isomorphism on stalk: For $\mathfrak{p} \in D_+(f)$, there's a natural ring homomorphism

$$S_{(f)} \to S_{(\mathfrak{p})}, a/f^n \mapsto a/f^n$$

and by universal property of localization, it induces a ring homomorphism

$$\varphi_{\mathfrak{p}} : (S_{(f)})_{\varphi(\mathfrak{p})} \to S_{(\mathfrak{p})}$$

Acturally, $\varphi_{\mathfrak{p}}$ is an isomorphism: injective is easy to check, and for some $a/g \in S_{\mathfrak{p}}$, notice that

$$\frac{a}{g} = \frac{ag^{\deg f - 1}}{f^{\deg g}} \frac{f^{\deg g}}{g^{\deg f}}$$

Hence, $\varphi_{\mathfrak{p}}$ is surjective.

Isomorphism $\varphi_{\mathfrak{p}}$ induces a isomorphism of sheaves

$$\varphi^b : \mathscr{O}_{\operatorname{Spec} S_{(f)}} \simeq \varphi_*(\mathscr{O}_{\operatorname{Proj} S}|_{D+(f)})$$

Proposition 1.3.17 (morphisms between projective spectrum)**.** Let S be a graded ring.

(1) Let $\varphi : S \to T$ be a graded homomorphism of graded rings (preserving degrees). Let $U = \{\mathfrak{p} \in \operatorname{Proj} T \mid \mathfrak{p} \not\supseteq \varphi(S_+)\}$. Show that U is an open subset of Proj T, and show that φ determines a natural morphism $f : U \to \operatorname{Proj} S$.

(2) The morphism f can be an isomorphism even when φ is not. For example, suppose that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geqslant d_0$, where d_0 is an integer. Then show that $U = \operatorname{Proj} T$ and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is an isomorphism.

(3) Let $\varphi : S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Then, the open set U of is equal to Proj T, and the morphism f :Proj $T \to$ Proj S is a closed immersion.

(4) If $I \subseteq S$ is a homogeneous ideal, take $T = S/I$ and let Y be the closed subscheme of $X = \operatorname{Proj} S$ defined by the closed immersion Proj $S/I \to X$. Show that different homogeneous ideals can give rise to the same closed subscheme. For example, let d_0 be an integer, and let $I' = \bigoplus_{d \geqslant d_0} I_d$. Show that I and I' determine the same closed subscheme.

Proof: (1): Since graded homomorphism preserves order, $\mathfrak{p} \in U \mapsto \varphi^{-1}(\mathfrak{p})$ is a well-define map from U to Proj S. Notice that

$$U = \bigcup_{g \in \varphi(S_+)} D_+(g),$$

U is a open subset of Proj T. And the morphism of presheaves f^b is induced by the natural local ring homomorphism

$$\varphi_{\mathfrak{p}} : S_{(f(\mathfrak{p}))} \to T_{(\mathfrak{p})}$$

And it's easy to check f together with f^b forms a morphism of scheme $(f, f^b) : U \to$ Proj S.

(2): For $U = \text{Proj}T$, assume $\mathfrak{p} \supset \varphi(S_+)$ and $\mathfrak{p} \not\supseteq T_+$, there's $a \in T_r$ with $r \geq 1$ such that $a \notin \mathfrak{p}$. Consider the element a^k for k sufficiently large. Next step, we are going to show $f : \text{Proj}T \to \text{Proj}S$ is an isomorphism.

Since, φ_d are isomorphic for all $d \geq d_0$,

$$\{\{D_+(t_i)\} : t_i \in T_+, \deg t_i \geq d_0\}$$

be a open covering of Proj T. Put $s_i = \varphi^{-1}(t_i)$, we also have

$$\{\{D_+(s_i)\} : s_i \in S_+, \deg s_i \geq d_0\}$$

be a open covering of Proj S

$f_i = f|_{D_+(t_i)} \to D_+(s_i)$ is a morphism of affine schemes (as $D_+(t_i) \simeq \operatorname{Spec} T_{(t_i)}$ and $D_+(s_i) \simeq \operatorname{Spec} S_{(s_i)}\big)$ corresponding to the ring homomorphism $\varphi_i : S_{(s_i)} \to T_{(t_i)}$ induced by φ. But φ_i is an isomorphism since s_i has degree at least d_0, and φ_d is an isomorphism for all $d \geq d_0$. Hence, f is surjective.

To show f is injective, take $\mathfrak{p}_1, \mathfrak{p}_2 \in$ Proj T with $f(\mathfrak{p}_1) = f(\mathfrak{P}_2)$. We have $\mathfrak{p}_1 \cap T_d = \mathfrak{p}_2 \cap T_d$ for all $d \geq d_0$. If $t_r \in \mathfrak{p}_1 \cap T_r$, take $s \notin \mathfrak{p}_2$, we have $s^k t_r \in \mathfrak{p}_2$. It implies $t_r \in \mathfrak{p}_2 \cap T_r$.

(3):Since $\varphi : S \to T$ is surjective, f is injective and

$$\varphi_{\mathfrak{p}} : S_{(f(\mathfrak{p}))} \to T_{(\mathfrak{p})}$$

is surjective.

Then, it suffice to check $f(\mathrm{Proj}T)$ is a closed subset. Notice that $\mathrm{Ker}\varphi$ be a homogenous ideal and for all $\mathfrak{p} \in \mathrm{Proj}(T)$, we have

$$f(\mathfrak{p}) \subset \mathrm{Ker}\varphi$$

Hence, $f(\mathrm{Proj}(T)) \subset V(\mathrm{Ker}\varphi)$. On the other hand, Since φ is surjective, $T \simeq S/\mathrm{Ker}\varphi$ as graded ring. Hence, it's easy to show $V(\mathrm{Ker}\varphi) \subset f(\mathrm{Proj}(T))$.

(4): By (2).

Proposition 1.3.18. A gluing datum of schemes consists of the following data:

(1) an index set I,

(2) for all $i \in I$ a scheme U_i,

(3) for all $i, j \in I$ an open subset $U_{ij} \subseteq U_i$ (we consider U_{ij} as open subscheme of U_i),

(4) for all $i, j \in I$ an isomorphism $\varphi_{ji} : U_{ij} \to U_{ji}$ of schemes, such that $U_{ii} = U_i$ for all $i \in I$ and the cocycle condition holds: $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ on $U_{ij} \cap U_{ik}, i, j, k \in I$.

Remark 1.3.19. In the cocycle condition we implicitly assume that in particular $\varphi_{ji}\left(U_{ij} \cap U_{ik}\right) \subseteq U_{jk}$, such that the composition is meaningful.

For $i = j = k$, the cocycle condition implies that $\varphi_{ii} = \mathrm{id}_{U_i}$ and for $i = k$ that $\varphi_{ij}^{-1} = \varphi_{ji}$.

Moreover, φ_{ji} is an isomorphism $U_{ij} \cap U_{ik} \to U_{ji} \cap U_{jk}$. This is because, consider the cocycle conditions $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ and $\varphi_{ki} \circ \varphi_{ij} = \varphi_{kj}$. We obtain two natural morphisms $\varphi_{ji} : U_{ij} \cap U_{ik} \to U_{ji} \cap U_{jk}$ and $\varphi_{ij} : U_{ji} \cap U_{jk} \to U_{ij} \cap U_{ik}$. Then, the claim follows from the fact $\varphi_{ji}^{-1} = \varphi_{ij}$.

Proposition 1.3.20. Let $\left((U_i)_{i\in I}, (U_{ij})_{i,j\in I}, (\varphi_{ij})_{i,j\in I}\right)$ be a gluing datum of schemes. Then there exists a scheme X together with morphisms $\psi_i : U_i \to X$, such that

(1) for all i the map ψ_i is a isomorphism of U_i onto the open subscheme $\psi_i(U_i)$ of X.

$$\begin{tikzcd} & X \\ U_i \arrow[ur] \arrow[r, "\psi_i"'] & \psi_i(U_i) \arrow[u, "l"'] \end{tikzcd}$$

(2) $\psi_j \circ \varphi_{ji} = \psi_i$ on U_{ij} for all i, j,

(3) $X = \bigcup_i \psi_i\left(U_i\right)$,

(4) $\psi_i\left(U_i\right) \cap \psi_j\left(U_j\right) = \psi_i\left(U_{ij}\right) = \psi_j\left(U_{ji}\right)$ for all $i, j \in I$.

Furthermore, X together with the ψ_i is uniquely determined up to unique isomorphism.

Proof: Underlying topological space: To define the underlying topological space of X, we start with the disjoint union $\coprod_{i\in I} U_i$ of the (underlying topological spaces of the) U_i and define an equivalence relation $\sim$ on it as follows: points $x_i \in U_i, x_j \in U_j, i, j \in I$, are equivalent, if and only if $x_i \in U_{ij}$, $x_j \in U_{ji}$ and $x_j = \varphi_{ji}(x_i)$. The cocycle condition implies that $\sim$ is in fact an equivalence relation. As a set, define X to be the set of equivalence classes,

$$X := \coprod_{i\in I} U_i / \sim .$$

The natural maps $\psi_i : U_i \to X$ are injective and we have $\psi_i(U_{ij}) = \psi_i(U_i) \cap \psi_j(U_j)$ for all $i, j \in I$. We equip X with the quotient topology, i. e. with the finest topology such that all ψ_i are continuous. That means that a subset $U \subseteq X$ is open if and only if for all i the preimage $\psi_i^{-1}(U)$ is open in U_i. In particular, the $\psi_i(U_i)$ and the $\psi_i(U_{ij}) = \psi_i(U_i) \cap \psi_j(U_j)$ are open in X.

Structure sheaf: Define for W open in X,

$$\mathscr{O}_X(W) = \left\{ (s_i)_{i\in I} : s_i \in \mathscr{O}_{U_i}(W \cap U_i), \varphi_{ji}(s_i|_{W\cap U_{ij}}) = s_j|_{W\cap U_{ji}} \right\}$$

where $W \cap U_i$ is actually $\psi_i^{-1}(W)$.

morphism of sheaves ψ_i^b:

$$\psi_i^b : (\psi_i)_* \mathscr{O}_X \to \mathscr{O}_{U_i}, (s_i)_{i\in I} \mapsto s_i$$

Example 1.3.21 (line with double origin)**.** We denote the line with double origin by X. It is obtained by gluing Spec$k[u]$ and Spec$k[t]$ along the isomorphism $D(u) \simeq \mathrm{Spec}(k[u, 1/u]) \simeq \mathrm{Spec}(k[t, 1/t]) = D(t)$. Notice taht $(X, \mathscr{O}_X)$ is affine if the morphism $(f, f^b) : (X, \mathscr{O}_X) \to \mathrm{Spec}(\Gamma(X, \mathscr{O}_X))$ induced by $\mathrm{id} : \Gamma(X, \mathscr{O}_X) \to \Gamma(X, \mathscr{O}_X)$ is an isomorphism.

An element of $\Gamma(X, \mathscr{O}_X)$ is the same as giving two polynomials $\sum_n f_n u^n$ and $\sum_m g_m t^m$ such that $\sum_n f_n u^n = \sum_m g_m u^m$ in $k[u, 1/u]$. Note that this just means that $f_n = g_n$ for all n. Hence $\Gamma(X, \mathcal{O}_X)$ is isomorphic to $k[u]$. If X is affine, then we have isomorphisms

$$(X, \mathcal{O}_X) \xrightarrow{(f, f^b)} \mathrm{Spec}\,\Gamma(X, \mathscr{O}_X) \to \mathrm{Spec}\, k[u]$$

Now consider the vanishing set $V(u)$ of X where $V(f)$ for some f in global section consists of all those points $x \in X$ such that $f_x = 0$ modulo $\mathfrak{m}_x$. and u denotes the global section $u = v$ of $\Gamma(X, \mathscr{O}_X)$.

Note that $V(u)$ contains at least two points, the two origins of X. But $V(u)$ in Spec $k[u]$ consists of only one point. Hence line with double origin is not affine.

Example 1.3.22 (projective space)**.** R is a ring and $S = R[x_0, \ldots, x_n]$ be a graded ring. Consider the scheme $\mathbb{P}^n_R = \mathrm{Proj}S$. For $f = x_i, i = 1, \ldots n$, we have

$$S_{(f)} = \{a/x_i^n \in R[x_0, \ldots, x_n]_{x_i} : a \in R[x_0, \ldots, x_n]_n\} = R\left[\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right]$$

and for $U_i = D_+(f)$,

$$(U_i, \mathscr{O}_{\mathbb{P}^n_R}|_{U_i}) = \left(D_+(f), \mathscr{O}|_{D_+(f)}\right) \simeq \mathrm{Spec}R\left[\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right]$$

We define a gluing datum with index set $\{0, \ldots, n\}$ as follows: For $0 \leq i, j \leq n$ let $U_{ij} = D_{U_i}\left(\frac{X_j}{X_i}\right) \subseteq U_i$ if $i \neq j$, and $U_{ii} = U_i$. Further, let $\varphi_{ii} = \mathrm{id}_{U_i}$ and for $i \neq j$ let

$$\varphi_{ji} : U_{ij} \to U_{ji}$$

be the isomorphism defined by the equality

$$R\left[\frac{X_0}{X_i}, \ldots, \widehat{\frac{X_i}{X_i}}, \ldots, \frac{X_n}{X_i}\right]_{\frac{X_j}{X_i}} \longleftarrow R\left[\frac{X_0}{X_j}, \ldots, \widehat{\frac{X_j}{X_j}}, \ldots, \frac{X_n}{X_j}\right]_{\frac{X_i}{X_j}},$$

(as subrings of $R\left[X_0, \ldots, X_n, X_0^{-1}, \ldots, X_n^{-1}\right]$) of the affine schemes U_{ij} and U_{ji}.

Corollary 1.3.23. If $R = k$ be a ring, the global section of $\mathbb{P}_k^n$ is k. Hence, $\mathbb{P}_k^n$ is not affine.

Example 1.3.24 (structure of $\mathbb{P}_{\mathbb{R}}^1$)**.** For U_x, U_y, there are $\mathbb{R}$-scheme isomorphisms

$$(U_x, \mathscr{O}_{\mathbb{P}_{\mathbb{R}}^1}\big|_{U_x}) \simeq \mathrm{Spec}\mathbb{R}[y]$$

and

$$(U_y, \mathscr{O}_{\mathbb{P}_{\mathbb{R}}^1}\big|_{U_y}) \simeq \mathrm{Spec}\mathbb{R}[x]$$

Hence,

$$\mathbb{P}_{\mathbb{R}}^1 = \{(x - ay) : a \in \mathbb{R}\} \bigcup \{(y - ax) : a \in \mathbb{R}\} \bigcup \left\{(ax^2 + bxy + cy^2) : b^2 - 4ac < 0\right\}$$

Definition 1.3.25. (1) A scheme is called connected, if the underlying topological space is connected.

(2) A scheme is called quasi-compact, if the underlying topological space is quasi-compact, i. e., if every open covering admits a finite subcovering.

(3) A scheme is called irreducible, if the underlying topological space is irreducible, i. e., if it is non-empty and not equal to the union of two proper closed subsets.

(4) A morphism $f : X \to Y$ of schemes is called injective, surjective or bijective, respectively, if the continuous map $X \to Y$ of the underlying topological spaces has this property.

(5) f is called open, closed, or a homeomorphism, respectively, if the underlying continuous map has this property.

(6) f is called dominant if $f(X)$ is a dense subspace of Y.

(7) A scheme X is called locally noetherian, if X admits an affine open cover $X = \bigcup U_i$, such that all the affine coordinate rings $\Gamma(U_i, \mathscr{O}_X)$ are noetherian. If in addition X is quasi-compact, X is called noetherian.

(8) A scheme X is called reduced, if all local rings $\mathscr{O}_{X,x}, x \in X$, are reduced rings.

(9) An integral scheme is a scheme which is reduced and irreducible.

Proposition 1.3.26. Let $X = \operatorname{Spec} A$ be an affine scheme. Then X is noetherian if and only if A is a noetherian ring.

Proof: By Nike's Trick, $\operatorname{Spec} A$ can be covered by affine open subschemes of the form $D(f_i), f_i \in A, i = 1, \dots, n$, such that all A_{f_i} are noetherian rings.

If I is an ideal of A, $I_{f_i} = IA_{f_i}$ is finitely generated ideal in A_{f_i}. By Algebra 2.5.29, I is finitely generated ideal in A.

Proposition 1.3.27. X is any noetherian scheme, the underlying topological space of X is noetherian

Proof: Since spectrum of a noetherian ring is a noetherian topological space. Then this proposition follows from the fact that a topological space covered by finite many noetherian subspace is notherian.

Proposition 1.3.28. Let X be a (locally) noetherian scheme and $U \subseteq X$ an open subscheme. Then U is (locally) noetherian.

Proof: In a noetherian topological space, every open subset is quasi-compact.

Proposition 1.3.29. Let X be a scheme. The mapping

$$\begin{aligned} X &\longrightarrow \{Z \subseteq X; Z \text{ closed, irreducible }\} \\ x &\mapsto \overline{\{x\}} \end{aligned}$$

is a bijection, i. e. every irreducible closed subset contains a unique generic point.

Proof: Step 1: If Z is a closed irreducible subset of X and U is an affine open subset of X, $Z \cap U$ is irreducible. This is because, for W_1, W_2 be open subets of X and $Y_j = W_j \cap Z \cap U_i \neq \varnothing, j = 1, 2$, since Z is irreducible, the intersection of Y_1 and Y_2 is non-empty. Hence, $Z \cap U_i$ is irreducible.

Step 2: Since Z is closed, $\overline{Z \cap U} \subset Z$. Since Z is irreducible, $Z \cap U$ is a dense subset of Z. Then $\overline{Z \cap U} \cap Z = Z$.

Step 3: Since $Z \cap U$ is a irreducible closed subset of U, there's $x \in Z \cap U$ such that $\overline{\{x\}} \supset \overline{\{x\}} \cap U = Z \cap U$. Hence, $\{x\} \supset \overline{Z \cap U} = Z$.

Step 4: To show the uniqueness of x, consider $x, y \in X$ such that $\overline{\{x\}} = \overline{\{y\}} = Z$ and U be an open affine subset of X with $U \cap Z \neq \varnothing$. Since there's $z \in \overline{\{x\}} \cap U$, we have $x \in U$. Similarly, $y \in U$. Then, $\overline{\{x\}} \cap U = \overline{\{y\}} \cap U = Z \cap U$, by the uniqueness of affine case, $x = y$.

Proposition 1.3.30. Let X be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathscr{O}_x$.

If $U = \operatorname{Spec} B$ is an open affine subscheme of X, and if $\bar{f} \in B = \Gamma(U, \mathcal{O}_X|_U)$ is the restriction of f, show that $U \cap X_f = D(\bar{f})$. Conclude that X_f is an open subset of X.

Proof: For $\mathfrak{p} \in \operatorname{Spec}(B)$, $\mathfrak{p} \in D(\bar{f})$ iff $\bar{f} \notin \mathfrak{p}$ iff $\bar{f}$ viewed as an element in $A_{\mathfrak{p}}$ does not lie in $\mathfrak{p}A_{\mathfrak{p}}$.

Proposition 1.3.31. (1) A scheme X is reduced if and only if for every open subset $U \subseteq X$ the ring $\Gamma(U, \mathscr{O}_X)$ is reduced.

(2) A non-empty scheme X is integral if and only if for every open subset $\emptyset \neq U \subseteq X$ the ring $\Gamma(U, \mathscr{O}_X)$ is an integral domain.

(3) If X is an integral scheme, then for all $x \in X$ the local ring $\mathscr{O}_{X,x}$ is an integral domain.

(4) An affine scheme $X = \operatorname{Spec} A$ is integral if and only if A is a domain.

(5) Let X be an integral scheme, and let $\eta \in X$ be its generic point. Then the local ring $\mathscr{O}_{X,\eta}$ is a field.

Proof: (1): Trivial.

(2): Let X be integral. Because all open subschemes of X are integral, too, it is enough to show that $\Gamma(X, \mathscr{O}_X)$ is a domain. Take $f, g \in \Gamma(X, \mathscr{O}_X)$ such that $fg = 0$. Then $\varnothing = X_f \cap X_g$ since $f_x g_x \in \mathfrak{m}_x$ for all $x \in X$. By the irreducibility we get $X_f = \varnothing$ or $X_g = \varnothing$. Assume $X_f = \varnothing$. We want to show that f must then be 0. We can check this locally on X, so we may assume that X is affine. Then f lies in the intersection of all prime ideals, i. e. in the nil-radical of the affine coordinate ring of X. Since X is reduced, by (1) the nil-radical is the zero ideal.

If conversely all $\Gamma(U, \mathscr{O}_X)$ are integral domains, then by (1) X is reduced. If there existed non-empty affine open subsets $U_1, U_2 \subseteq X$ with empty intersection, then the sheaf axioms imply that

$$\Gamma(U_1 \cup U_2, \mathscr{O}_X) = \Gamma(U_1, \mathscr{O}_X) \times \Gamma(U_2, \mathscr{O}_X)$$

But the product on the right hand side obviously contains zero divisors.

(3): Trivial.

(4): A is integral domain, then it has a unique minimal prime ideal. Hence, $\operatorname{Spec}(A)$ is irreducible. Since $A_{\mathfrak{p}}$ is a subring of $\operatorname{Frac}(A)$, $\operatorname{Spec}(A)$ is reduced scheme. Hence, $\operatorname{Spec}(A)$ is an integral scheme. By (2), if $\operatorname{Spec}(A)$ is an integer scheme, A is an integral domain.

(5): If η is a generic point of X, for all affine open subscheme U such that $\eta \in U$, η is a generic point of $U = \operatorname{Spec}(A)$. That is, η corresponds to (0) in A. Then, $\mathscr{O}_{X,x} \simeq A_{(0)} = \operatorname{Frac}(A)$ is a field.

Definition 1.3.32. Let X be an integral scheme, and let $\eta \in X$ be its generic point. Then the local ring $\mathscr{O}_{X,\eta}$ is a field, which is called the function field of X and denoted by $K(X)$.

Proposition 1.3.33. X be an integral scheme with generic point η.

(1) Let $U \subseteq V \subseteq X$ be non-empty open subsets. Then the maps

$$\Gamma(V, \mathscr{O}_X) \xrightarrow{\operatorname{res}^V_U} \Gamma(U, \mathscr{O}_X) \xrightarrow{f \mapsto f_\eta} K(X)$$

(2) For all $x \in X$, there's a canonical injective map $\mathscr{O}_{X,x} \to \mathscr{O}_{X,\eta}$ gievn by $[s] \mapsto [s]$ and under this map, $\operatorname{Frac}(\mathscr{O}_{X,x}) = \mathscr{O}_{X,\eta}$.

(3) For every non-empty open subset $U \subseteq X$ and for every covering $U = \bigcup_i U_i$ by non-empty open subsets U_i we have

$$\Gamma(U, \mathscr{O}_X) = \bigcap_i \Gamma(U_i, \mathscr{O}_X) = \bigcap_{x \in U} \mathscr{O}_{X,x},$$

where the intersection takes place in $K(X)$.

Proof: (1): It suffice to show the map $f \mapsto f_\eta$ is injective. Since $f_\eta = 0$ is equivalent to $f|_W = 0$ for all W open affine subscheme of U, we may assume U is an affine open subscheme. Consider the following commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{X,\eta} \arrow[r, "\simeq"] & \mathscr{O}_{\operatorname{Spec}A,(0)} \arrow[r, "\simeq"] & \operatorname{Frac}(A) \\
\mathscr{O}_X(U) \arrow[u] \arrow[r, "\simeq"] & \mathscr{O}_{\operatorname{Spec}A}(\operatorname{Spec}A) \arrow[u] \arrow[r, "\simeq"] & A \arrow[u]
\end{tikzcd}$$

Since $A \to \operatorname{Frac}(A)$ is injective, we have $f = 0$.

(2): By (1) and the following diagram

$$\begin{tikzcd}
\mathscr{O}_{X,\eta} \arrow[r, "\simeq"] & \mathscr{O}_{\operatorname{Spec}A,(0)} \arrow[r, "\simeq"] & \operatorname{Frac}(A) \\
\mathscr{O}_{X,x} \arrow[u] \arrow[r, "\simeq"] & \mathscr{O}_{\operatorname{Spec}A,\mathfrak{p}} \arrow[u] \arrow[r, "\simeq"] & A_{\mathfrak{p}} \arrow[u]
\end{tikzcd}$$

(3): Consider the following commutative diagram

$$
\begin{tikzcd}
\Gamma(U, \mathscr{O}_X) \arrow[r] & \Gamma(U_i, \mathscr{O}_X) \arrow[r] \arrow[dr] & \mathscr{O}_{X,\eta} \\
 & & \Gamma(U_i \cap U_j, \mathscr{O}_X) \arrow[u]
\end{tikzcd}
$$

and notice that $\mathscr{O}_X$ is a sheaf.

Notice we define locally finite type and finite type k-scheme. The morphisms below are all in the category (Sch/k).

Definition 1.3.34. Let k be a field, and let $X \to \operatorname{Spec} k$ be a k-scheme. We call X a k-scheme locally of finite type or say that X is locally of finite type over k, if there is an affine open cover $X = \bigcup_{i \in I} U_i$ such that for all i, there's a k-algebra A_i such that

$$(U_i, \mathscr{O}_X|_{U_i}) \simeq (\operatorname{Spec}(A_i), \mathscr{O}_{\operatorname{Spec}(A_i)})$$

as k-scheme. We say that X is of finite type over k if X is locally of finite type and quasi-compact.

Proposition 1.3.35. Every (locally) finite type k-scheme is (locally) noetherian.

Proposition 1.3.36. Let X be a k-scheme locally of finite type and let $U \subseteq X$ be an open affine subset. Then the k-algebra $\Gamma(U, \mathscr{O}_X)$ is a finitely generated k-algebra.

Proof: Let $B = \Gamma(U, \mathscr{O}_X)$. Since the localization of a finitely generated k-algebra with respect to a single element is again finitely generated, we see, by Nike's Trick, that we can cover U by finitely many(since spectrum of a ring is compact) principal open subsets $D(f_i), f_1, \ldots, f_n \in B$, such that all localizations B_{f_i} are finitely generated k-algebras. The claim now follows from Algebra Proposition 2.5.30

Proposition 1.3.37. Let k be a field, let X be a k-scheme locally of finite type, and let $x \in X$. Then the following assertions are equivalent.

(1) The point $x \in X$ is closed.

(2) The field extension $k \hookrightarrow \kappa(x)$ is finite.

(3) The field extension $k \hookrightarrow \kappa(x)$ is algebraic.

Proof: (1) implies (2): Take U with $x \in U$ and there's k-scheme

$$(U, \mathscr{O}_X|_U) \simeq (\operatorname{Spec}(A), \mathscr{O}_{\operatorname{Spec}(A)})$$

where A be a finitely generated k-algebra and x corresponds to a maximal ideal $\mathfrak{m}$ of A. Consider the follow commutative diagram

$$\begin{tikzcd}
\kappa(x) \arrow[rr, "\simeq"] & & A/\mathfrak{m} \\
\mathscr{O}_{X,x} \arrow[u] \arrow[rr, "\simeq"] & & A_{\mathfrak{m}} \arrow[u] \\
\Gamma(U, \mathscr{O}_X) \arrow[u] \arrow[rr, "\simeq"] & & A \arrow[u] \\
& k \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Since $A/\mathfrak{m}$ is a field and finite generated k-algebra, by Algebra 2.8.2, $\kappa(x)$ is a finte extension of k.

(3) implies (1): Again take U with $x \in U$ and there's k-scheme

$$(U, \mathscr{O}_X|_U) \simeq (\operatorname{Spec}(A), \mathscr{O}_{\operatorname{Spec}(A)})$$

where A be a finitely generated k-algebra and x corresponds to a prime ideal $\mathfrak{p}$ of A.

$$\begin{tikzcd}
\kappa(x) \arrow[rr, "\simeq"] & & A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \arrow[r, "\simeq"] & \mathrm{Frac} A/\mathfrak{p} \\
\mathscr{O}_{X,x} \arrow[u] \arrow[rr, "\simeq"] & & A_{\mathfrak{p}} \arrow[u] & A/\mathfrak{p} \arrow[u] \\
\Gamma(U, \mathscr{O}_X) \arrow[u] \arrow[rr, "\simeq"] & & A \arrow[u] \arrow[ur] & \\
& k \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

Since $\kappa(x)$ is algebraic over k, $A/\mathfrak{p}$ is integral over k. Hence $\mathfrak{p}$ is a closed point in U. Consider all such U, we have x is closed in X.

Corollary 1.3.38. Let k be algebraically closed and let X be a k-scheme locally of finite type. Then

$$\{x \in X; x \text{ closed }\} = \{x \in X; k = \kappa(x)\} = \operatorname{Hom}_k(\operatorname{Spec} k, X),$$

Proof: Field extension $k \to \kappa(x)$ is an isomorphism if and only if there's k-algebra homomorphism $\kappa(x) \to k$. And if there's k-algebra homomorphism $\kappa(x) \to k$, it is obviously unique.

Example 1.3.39. $\mathbb{P}^n_k$ is an integral, finite type scheme over k.

Proof: reduced: $\mathbb{P}^n_k$ is reduced since for all $x \in \mathbb{P}^n_k$, we may find $i \in \{0, \ldots, n\}$ such that $x \in U_i = D_+(x_i)$. Then $\mathscr{O}_{\mathbb{P}^n_k, x}$ is a localization of a polynomial ring at a prime ideal, hence reduced.

irreducible: $D_+(f) \cap D_+(g) = D_+(fg)$ and notice that for all $h \in k[x_0, \ldots, x_n]_+$, $D_+(h)$ is non-empty.

locally finite type: trivial

quasi-compact: $\mathbb{P}^n_k$ is a finite union of compact open subset U_i.

Example 1.3.40. For $X = \mathrm{Spec}(\mathbb{Q}[x,y]/(x^n+y^n-1))$ be a $\mathbb{Q}$-scheme, then

$$X(\mathbb{R}) = \mathrm{Hom}_{\mathrm{Spec}(\mathbb{Q})}(X, \mathrm{Spec}(\mathbb{R})) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[x,y]/(x^n+y^n-1), \mathbb{R}) = \left\{(x,y) \in \mathbb{R}^2 : x^n+y^n=1\right\}$$

and

$$X(\mathbb{Q}) = \mathrm{Hom}_{\mathrm{Spec}(\mathbb{Q})}(X, \mathrm{Spec}(\mathbb{Q})) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[x,y]/(x^n+y^n-1), \mathbb{Q}) = \left\{(x,y) \in \mathbb{Q}^2 : x^n+y^n=1\right\}$$

Moreover, since closed point corresponds to the maximal ideal of $\mathbb{Q}[x,y]/(x^n+y^n-1)$ and $X(\mathbb{Q})$ be those maximal ideals $\mathfrak{m}$ of $\mathbb{Q}[x,y]$ which contain x^n+y^n-1 and have a $\mathbb{Q}$-algebra isomorphism $\mathbb{Q}[x,y]/\mathfrak{m} \to \mathbb{Q}$. Therefore, $\mathfrak{m}$ is of the form $(x-x_0, y-y_0)$ where (x_0,y_0) be a solution of $x^n+y^n=1$.

1.4 Immersions

Definition 1.4.1. A morphism $j : Y \to X$ of schemes is called an open immersion, if the underlying continuous map is a homeomorphism of Y with an open subset U of X, and the sheaf homomorphism $\mathscr{O}_X \to j_*\mathscr{O}_Y$ induces an isomorphism $\mathscr{O}_{X|U} \cong (j_*\mathscr{O}_Y)_{|U}$ (of sheaves on U).

Remark 1.4.2. There's a natural one-to-one correspondence between open immersion and open subscheme.

Definition 1.4.3. Given a scheme $(X, \mathscr{O}_X)$, we call a subsheaf $\mathscr{J} \subseteq \mathscr{O}_X$ a sheaf of ideals, if for every open subset $U \subseteq X$ the sections $\Gamma(U, \mathscr{J})$ are an ideal in $\Gamma(U, \mathscr{O}_X)$. The quotient sheaf $\mathscr{O}_X/\mathscr{J}$ is defined as the sheafification of the presheaf $U \mapsto \mathscr{O}_X(U)/\mathscr{J}(U)$. It is a sheaf of rings. The canonical projection $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{J}$ is surjective.

Definition 1.4.4. Let X be a scheme.

(1) A closed subscheme of X is given by a closed subset $Z \subseteq X$ with inclusion map $i : Z \to X$ and an ideal sheaf $\mathscr{J} \subseteq \mathscr{O}_X$ such that $Z = \left\{x \in X : (\mathscr{O}_X/\mathscr{J})_x \neq 0\right\}$ and $(Z, i^{-1}\mathscr{O}_X/\mathscr{J})$ is a scheme.

(2) A morphism $i : (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$ of schemes is called a closed immersion, if the underlying continuous map is a homeomorphism between Z and a closed subset of X, and the sheaf homomorphism $i^b : \mathscr{O}_X \to i_*\mathscr{O}_Z$ is surjective.

Proposition 1.4.5. *X be a scheme and Z be a closed subscheme associated to ideal sheaf $\mathscr{J}$. Then, the morphism of ringed space $(Z, i^{-1}\mathscr{O}_X/\mathscr{J}) \to (X, \mathscr{O}_X)$ induced by the natural projection $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{J}$ and the isomorphism $\mathscr{O}_X/\mathscr{J} \to i_*i^{-1}\mathscr{O}_X/\mathscr{J}$ is a morphism of locally ringed space and closed immersion.*

Proof: Step 1: The stalk of the morphism of sheaves $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{J}$ is a local ring homomorphism.

It's clear for the case when $x \notin Z$, since $(\mathscr{O}_X/\mathscr{J})_x = 0$. For $x \in Z$, since the stalk of the presheaf $U \mapsto \mathscr{O}_X(U)/\mathscr{J}(U)$ at x is $\mathscr{O}_{X,x}/\mathscr{J}_x$ where $J_x \neq \mathscr{O}_{X,x}$. And notice that the projection $\mathscr{O}_{X,x} \to \mathscr{O}_{X,x}/\mathscr{J}_x$ is a local ring homomorphism.

Step 2: $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{J}$ is surjective.

By taking stalks, it suffice to show $\mathscr{O}_X(U) \to \mathscr{O}_X(U)/\mathscr{J}(U)$ is surjective for all U open in X.

Proposition 1.4.6. If $i : (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$ is a closed immersion, consider the kernel of the morphism of sheaves $\varphi : \mathscr{O}_X \to i_*\mathscr{O}_Z$. It's clear that $\mathrm{Ker}\varphi$ is an ideal sheaf. By Proposition 1.1.9, the natural morphism

$$\mathscr{O}_X/\mathrm{Ker}\varphi \to i_*\mathscr{O}_Z$$

is an isomorphism of sheaves.

Moreover, since $(Z, \mathscr{O}_Z)$ is a scheme and Z is closed in X, $\mathrm{Supp}(i_*\mathscr{O}_Z) = Z$. Hence the support of $\mathscr{O}_X/\mathrm{Ker}\varphi$ is Z. Then by Proposition 1.1.22, a closed immersion induces a closed subscheme.

Theorem 1.4.7 (closed subscheme of affine scheme)**.** Let $X = \mathrm{Spec}\, A$ be an affine scheme. $i : (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$ be a closed immersion. Then the global section map

$$\varphi : A \to \Gamma(Z, \mathscr{O}_Z)$$

induces a commutative diagram of scheme:

$$\begin{tikzcd}
\mathrm{Spec}A & Z \arrow[l, "i"'] \arrow[dl, "\psi"] \\
\mathrm{Spec}A/\ker\varphi \arrow[u, "\pi"] &
\end{tikzcd}$$

Then ψ is an isomorphism of scheme.

Proof: Since i is an closed immersion, ψ is closed and injective. Hence, ψ is also a closed immersion. To prove ψ is surjective, it suffices to show the following lemma:

Lemma 1.4.8. Let $X = \mathrm{Spec}\, A$ be an affine scheme. $i : (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$ be a closed immersion such that the induced map on global section $\varphi : A \to \Gamma(Z, \mathscr{O}_Z)$ is injective. Then, i is surjective.

Proof of the lemma: Assume $X - Z$ is non-empty. Let $s \in A$ with $\varnothing \neq D(s) = X_s \subset X - Z$. Then $Z \subset X - X_s$. Hence, $Z \subset Z \cap (X - X_s)$. For all $x \in Z$, we have the following commutative diagram

$$\begin{tikzcd}
\Gamma(Z, \mathscr{O}_Z) \arrow[d, leftrightarrow] & A \arrow[l] \arrow[d] \\
\mathscr{O}_{Z,x} = i_*(\mathscr{O}_Z)_x & \mathscr{O}_{X,x} \arrow[l]
\end{tikzcd}$$

Hence, $Z \subset Z \cap (X - X_s) \subset Z - Z_{\varphi(s)}$. If $U \subseteq Z$ is open, such that $\big(U, \mathscr{O}_{Z|U}\big) \simeq \mathrm{Spec}(B)$ is affine. By Proposition 1.3.30, $U \subset \mathrm{Spec}(B) - D(\varphi(s)|_U)$. Hence, $\varphi(s)|_U$ is nilpotent. Moreover, since Z can be covered by finite many affine open subscheme, there's some sufficiently large N such that $\varphi(s)^N$ is nilpotent. Hence, $s^N = 0$. It contradicts to $\varnothing \neq X_s$. $\square$

To show that ψ is an isomorphism of scheme. We still need the following lemma

Lemma 1.4.9. Let $X = \operatorname{Spec} A$ be an affine scheme. $(i, i^b) : (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$ be a closed immersion such that the induced map on global section $\varphi : A \to \Gamma(Z, \mathscr{O}_Z)$ is injective. Then, i^b is injective.

Proof of the lemma: For $x \in X, \mathscr{O}_{X,x} = A_{\mathfrak{p}_x}$, and we see that it is enough to show that every element of $\operatorname{Ker}(\mathscr{O}_{X,x} \to \mathscr{O}_{Z,x})$ of the form $g/1$ is 0 in $\mathscr{O}_{X,x}$. Given g, we cover $Z = U \cup \bigcup_{i \in I} U_i$ by finitely many open subsets U, U_i, such that: (1) The schemes $\left(U, \mathscr{O}_{Z|U}\right)$ and $\left(U_i, \mathscr{O}_{Z|U_i}\right)$ for all i are affine. (2) We have $x \in U$ and $\varphi(g)_{|U} = 0$.

Choose $s \in A$ with $x \in D(s) \subseteq U$. If we can show that $\varphi\left(s^N g\right) = 0$ for some N, then $s^N g = 0$ because φ is injective, and it follows that $g/1 = 0$ in $\mathscr{O}_{X,x}$, as desired, since s is a unit in $\mathscr{O}_{X,x}$. Since $\varphi(g)_{|U} = 0$ by assumption, we have $\varphi(sg)_{|U} = 0$. Now I is finite, so we can search a suitable N for each U_i separately. Because

$$D_{U_i}\left(\varphi(s)_{|U_i}\right) = Z_{\varphi(s)} \cap U_i \subset D(s) \cap U_i$$

, we obtain $\varphi(g)_{|D_{U_i}\left(\varphi(s)_{|U_i}\right)} = 0$. In other words, the image of $\varphi(g)$ in the localization $\Gamma\left(U_i, \mathscr{O}_Z\right)_{\varphi(s)|_{U_i}}$ is 0. □

Definition 1.4.10 (immersion)**.** (1) Let X be a scheme. A subscheme of X is a scheme $(Y, \mathscr{O}_Y)$, such that $Y \subseteq X$ is a locally closed subset, and such that Y is a closed subscheme of the open subscheme $U \subseteq X$, where U is the largest open subset of X which contains Y and in which Y is closed. We then have a natural morphism of schemes $Y \to X$.

(2) An immersion $i : Y \to X$ is a morphism of schemes whose underlying continuous map is a homeomorphism of Y onto a locally closed subset of X, and such that for all $y \in Y$ the ring homomorphism $i_y^\sharp : \mathscr{O}_{X,i(y)} \to \mathscr{O}_{Y,y}$ between the local rings is surjective.

Definition 1.4.11. Let k be a field. (1) A k-scheme X is called projective, if there exist $n \geq 0$ and a closed immersion $X \hookrightarrow \mathbb{P}_k^n$.

(2) A k-scheme X is called quasi-projective, if there exist $n \geq 0$ and an immersion $X \hookrightarrow \mathbb{P}_k^n$.

Definition 1.4.12. Consider $X = \operatorname{Proj}\mathbb{C}[x, y, z]/(zy^2 - (4x^3 - a_2xz^2 - a_3z^3))$ with $a_2^3 - 27a_2^2 \neq 0$ as a $\mathbb{C}$- scheme. Firstly, the natural morphism $X \to \mathbb{P}_{\mathbb{C}}^2$ is a closed immersion. Moreover, consider the $\mathbb{C}$-points $X(\mathbb{C})$ of X. We have

$$X(\mathbb{C}) = \{\infty\} \cup \left\{(x, y) \in \mathbb{C}^2 : y^2 = 4x^3 - a_2x - a_3\right\}$$

where ∞ represents the point (x, z) in $\operatorname{Proj} \mathbb{C}[x, y, z]/(zy^2 - (4x^3 - a_2xz^2 - a_3z^3))$.

Now we show that X is an integral, projective , $\mathbb{C}$-finite type scheme.

irreducible: show that $D_+(z) \cap D_+(f) \neq \varnothing$ for all $D_+(f) \neq \varnothing$.

reduced: It suffice to show

$$\operatorname{Spec}(\mathbb{C}[x, y]/(y^2 - (4x^3 - a_2x - a_3)))$$

is integral and

$$\operatorname{Spec}(\mathbb{C}[x, z]/(z - (x^3 - a_2xz^2 - a_3z^3)))$$

is integral.

$\mathbb{C}$-finite type: trivial

1.5 Fibered Products