Analysis

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Contents

Chapter 1

Foundation

1.1 Construction of Real Number

Definition 1.1.1 (ordered ring)**.** Thus, a ring(field) $R \neq 0$ with an order $<$ is called an ordered ring(field) if the following holds:

(1) $(R, <)$ is totally ordered

(2) $x < y \Rightarrow x + z < y + z, z \in R$

(3) $x, y > 0 \Rightarrow xy > 0$

Of course, an element $x \in R$ is called positive if $x > 0$ and negative if $x < 0$. We gather in the next proposition some simple properties of ordered fields.

Proposition 1.1.2. Let K be an ordered field, then for $x, y, a, b \in K$.

(1) $x > y \Leftrightarrow x - y > 0$.

(2) If $x > y$ and $a > b$, then $x + a > y + b$.

(3) If $a > 0$ and $x > y$, then $ax > ay$.

(4) If $x > 0$, then $-x < 0$. If $x < 0$, then $-x > 0$.

(5) Let $x > 0$. If $y > 0$, then $xy > 0$. If $y < 0$, then $xy < 0$.

(6) If $a < 0$ and $x > y$, then $ax < ay$.

(7) $x^2 > 0$ for all $x \neq 0$. In particular, $1 > 0$.

(8) If $x > 0$, then $x^{-1} > 0$.

(9) If $x > y > 0$, then $0 < x^{-1} < y^{-1}$ and $xy^{-1} > 1$.

Definition 1.1.3. K is a ordered field, K is said to be Archimedes if and only if for $x, y > 0$ there's $n \in \mathbb{Z}$ such that $nx > y$.

Example 1.1.4. $\mathbb{Q}$ is a Archimedes ordered field with original order.

Proposition 1.1.5. For an ordered field K, the absolute value function, $|\cdot|: K \to K$ and the sign function, $\operatorname{sign}(\cdot): K \to K$ are defined by

$$|x| := \begin{cases} x, & x > 0, \\ 0, & x = 0, \\ -x, & x < 0, \end{cases} \qquad \operatorname{sign} x := \begin{cases} 1, x > 0, \\ 0, x = 0, \\ -1, x < 0. \end{cases}$$

Let K be an ordered field and $x, y, a, \varepsilon \in K$ with $\varepsilon > 0$.

(1) $x = |x| \operatorname{sign}(x), |x| = x \operatorname{sign}(x)$.

(2) $|x| = |-x|, \quad x \leq |x|$.

(3) $|xy| = |x||y|$.

(4) $|x| \geq 0$ and $(|x| = 0 \Leftrightarrow x = 0)$.

(5) $|x - a| < \varepsilon \Leftrightarrow a - \varepsilon < x < a + \varepsilon$.

(6) $|x + y| \leq |x| + |y|$ (triangle inequality).

(7) $|x - y| \geq ||x| - |y||, \quad x, y \in K$

Definition 1.1.6. A ring homomorphism f between ordered field is said to be order-preserving if

$$a < b \Longleftrightarrow f(a) < f(b)$$

.

Definition 1.1.7. A sequence $r = (x_n)_{n \in \mathbb{Z}_{>0}}$ is a Cauchy sequence if for all $\epsilon \in \mathbb{Q} > 0$, there's $N > 0$ such that for all $m, n > N$, $|x_n - x_m| < \epsilon$.

Proposition 1.1.8. Cauchy sequence is bounded.

Definition 1.1.9. Let

$$\mathcal{R} = \left\{ (x_n) \in \mathbb{Q}^{\mathbb{Z}_{>0}} : (x_n) \text{ is a Cauchy sequence} \right\}$$

and

$$\mathbf{c}_0 = \left\{ (x_n) \in \mathbb{Q}^{\mathbb{Z}_{>0}} : \text{for all } \epsilon > 0, \text{ there's } N > 0 \text{ such that for all } n > N, |x_n| < \epsilon \right\}$$

It's clear that $\mathbf{c}_0 \subset \mathcal{R}$ is a maximal ideal of $\mathcal{R}$. Hence $\mathcal{R}/\mathbf{c}_0$ is a field and we denote it by $\mathbb{R}$. For convenience, we usually denote $(a_n) + \mathbf{c_0}$ by (a_n).

Definition 1.1.10. Now we define a order on $\mathbb{R}$, for $(a_n), (b_n)$ in $\mathbb{R}$, $(a_n) > (b_n)$ if there's $\epsilon > 0$, a sufficiently large integer N, such that $a_n - b_n > \epsilon$ for $n > N$. And denote this order by $<$. It's esay to check that '$<$' is well-defined and totally ordered.

Proposition 1.1.11. $(\mathbb{R}, <)$ is a Archimedes ordered field. And the embedding $l : \mathbb{Q} \to \mathbb{R}$ given by

$$r \mapsto (r, r, r, \dots)$$

is an order-preserving ring homomorphism.

Definition 1.1.12. For a sequence $(A_n) \in \mathbb{R}$, we say $A_n \to A$ if for all $\epsilon \in \mathbb{R} > 0$, there's $N > 0$ such that for all $n > N$, $|A_n - A| < \epsilon$. And we say (A_n) is a Cauchy sequence if for all $\epsilon \in \mathbb{R}_{>0}$, there's $N > 0$ such that for all $m, n > N$, $|x_n - x_m| < \epsilon$.

Proposition 1.1.13 (dense)**.** For all $a, b \in \mathbb{R}$, if $a < b$, there's $c \in \mathbb{Q}$ such that $a < l(c) < b$.

Proposition 1.1.14 (completeness)**.** (A_n) is a Cauchy sequence in $\mathbb{R}$ if and only if there's $A \in \mathbb{R}$ such that $A_n \to A$.

Proof: 'if' is obvious.

'only if': Take $x_n \in \mathbb{Q}$ such that:

$$A_n < l(x_n) < A_n + l(\frac{1}{n})$$

It's cleat that $a = (x_n) + \mathbf{c}_0 \in \mathbb{R}$.

Notice that $A_n \to a$, we have $\mathbb{R}$ is complete.

Now we identity $\mathbb{Q}$ with a subfield of $\mathbb{R}$ in the following content.

Proposition 1.1.15. (1) E is a non-empty subset of $\mathbb{R}$ and if E is lower-bounded, then E has a infimum; if E is upper-bounded, then E has a supremum.

(2) Every incresing bounded sequence $(x_n) \in \mathbb{R}$ has a limit.

(3) (Bolzano-Weierstress) Every bounded sequence has a convergent subsequnce.

(4) if

$$[a, b] \subset \bigcup_{i \in I} (a_i, b_i)$$

, then

$$[a, b] \subset \bigcup_{k \in J} (a_k, b_k)$$

for some finite subset J of I.

Proposition 1.1.16. $a > 0$, $n \in \mathbb{Z}_{>0}$, then there's unique $x \in \mathbb{R}_{>0}$ such that $x^n = a$. We denote the unique positive root by $\sqrt[n]{a}$. And for all $a \in \mathbb{R}$ and $r = \frac{p}{q} \in \mathbb{Q}$, define $a^r = \sqrt[q]{a^p}$. It's easy to check that $\sqrt[q]{a^p} = (\sqrt[q]{a})^p$.

Proof: To prove the existence of a solution, we can, without loss of generality, assume that $n \geq 2$ and $a \neq 1$. We only prove with the case $a > 1$. Then we have

$$x^n > a^n > a > 0 \quad \text{for all } x > a.$$

Now set $A := \{x \geq 0 : x^n \leq a\}$. Then $0 \in A$ and $x \leq a$ for all $x \in A$. Thus $s := \sup(A)$ is a well defined real number such that $s \geq 0$. We will prove that $s^n = a$ holds by showing that $s^n \neq a$ leads to a contradiction. Suppose first that $s^n < a$ so that $a - s^n > 0$.

$$b := \sum_{k=0}^{n-1} \begin{pmatrix} n \\ k \end{pmatrix} s^k > 0$$

implies that there is some $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < (a - s^n)/b$. By making ε smaller if needed, we can further suppose that $\varepsilon \leq 1$. Then $\varepsilon^k \leq \varepsilon$ for all $k \in \mathbb{Z}_{>0}$, and, using the binomial theorem, we have

$$(s+\varepsilon)^n = s^n + \sum_{k=0}^{n-1} \begin{pmatrix} n \\ k \end{pmatrix} s^k \varepsilon^{n-k} \leq s^n + \left(\sum_{k=0}^{n-1} \begin{pmatrix} n \\ k \end{pmatrix} s^k \right) \varepsilon < a.$$

This shows that $s + \varepsilon \in A$, a contradiction of $\sup(A) = s < s + \varepsilon$. Therefore $s^n < a$ cannot be true. Now suppose that $s^n > a$. Then, in particular, $s > 0$ and

$$b := \sum^{*} \begin{pmatrix} n \\ 2j-1 \end{pmatrix} s^{2j-1} > 0,$$

where the symbol $\sum^{*}$ means that we sum over all indices $j \in \mathbb{Z}_{>0}$ such that $2j \leq n$. Then there is some $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < (s^n - a)/b$ and $\varepsilon \leq \min\{1, s\}$. Thus we have

$$\begin{aligned}
(s-\varepsilon)^n &= s^n + \sum_{k=0}^{n-1} (-1)^{n-k} \begin{pmatrix} n \\ k \end{pmatrix} s^k \varepsilon^{n-k} \\
&\geq s^n - \sum^{*} \begin{pmatrix} n \\ 2j-1 \end{pmatrix} s^{2j-1} \varepsilon^{n-2j+1} \geq s^n - \varepsilon \sum^{*} \begin{pmatrix} n \\ 2j-1 \end{pmatrix} s^{2j-1} \\
&> a
\end{aligned}$$

1.2 Point-Set Topology

Munkres's Topology is a good reference for this section.

1.2.1 Definition

Definition 1.2.1. A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(1) $\varnothing$ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a topological space.

Definition 1.2.2. If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

(1) For each $x \in X$, there is at least one basis element B containing x.

(2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology $\boldsymbol{T}$ generated by $\mathcal{B}$ as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.

Definition 1.2.3. Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the subspace topology. With this topology, Y is called a subspace of X; its open sets consist of all intersections of open sets of X with Y.

Definition 1.2.4. X is Hausdorff if for any two elements $x \neq y$ in X, there's U, V open in X such that $x \in U, y \in V$ and $U \cap V = \varnothing$.

Definition 1.2.5 (convergence)**.**

Proposition 1.2.6. If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.

Example 1.2.7. Let X be a ordered set; assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

(1) All open intervals (a, b) in X.

(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.

(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X. The collection $\mathcal{B}$ is a basis for a topology on X, which is called the order topology.

Proposition 1.2.8. Given a subset A of a topological space X, the interior of A is defined as the union of all open sets contained in A, and the closure of A is defined as the intersection of all closed sets containing A.

Let Y be a subspace of X; let A be a subset of Y; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.

Definition 1.2.9. If A is a subset of the topological space X and if x is a point of X, we say that x is a limit point of A if every neighborhood of x intersects A in some point other than x itself. Said differently, x is a limit point of A if it belongs to the closure of $A - \{x\}$. The point x may lie in A or not; for this definition it does not matter. Let A be a subset of the topological space X; let A' be the set of all limit points of A. Then

$$\bar{A} = A \cup A'.$$

Proposition 1.2.10. Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:

(1) f is continuous.(U open in X implies $f^{-1}(U)$ open in Y)

(2) For every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$.

(3) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

(4) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$. If the condition in (4) holds for the point x of X, we say that f is continuous at the point x.

Definition 1.2.11. Consider $(X_i)_{i\in I}$ be a family of topology spaces, then the sets of the form

$$\prod_{i\in I} U_i$$

$U_i = X_i$ for all but finite i, form a basis of $\prod_{i\in I} X_i$. We call it the topology induced by this product topology.

In language of category, product topology with projection $p_i : \prod_{i\in I} X_i \to X_i$ is the product object in the category of topological space.

Proposition 1.2.12. If each space X_α is Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in product topology.

Proposition 1.2.13. Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given the product topology, then

$$\prod \bar{A}_\alpha = \overline{\prod A_\alpha}$$

Theorem 1.2.14. Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J},$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

1.2.2 Metric space

Definition 1.2.15. A metric on a set X is a function

$$d : X \times X \longrightarrow \mathbb{R}$$

having the following properties:

(1) $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(3) (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$, for all $x, y, z \in X$.

Given a metric d on X, the number $d(x, y)$ is often called the distance between x and y in the metric d. Given $\epsilon > 0$, consider the set

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

of all points y whose distance from x is less than ϵ. It is called the **$\boldsymbol{\epsilon}$-ball centered at $\boldsymbol{x}$**. Sometimes we omit the metric d from the notation and write this ball simply as $B(x, \epsilon)$, when no confusion will arise.

If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X, called the metric topology induced by d.

Example 1.2.16. $\mathbb{R}^n$ is a metric space with distance $d(x, y) = ||x - y||$

Theorem 1.2.17. Let X be a topological space; let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is metrizable.

Let $f : X \to Y$. If the function f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.

Theorem 1.2.18. Let $f_n : X \to Y$ be a sequence of functions from the set X to the metric space Y. Let d be the metric for Y. We say that the sequence (f_n) converges uniformly to the function $f : X \to Y$ if given $\epsilon > 0$, there exists an integer N such that

$$d(f_n(x), f(x)) < \epsilon$$

for all $n > N$ and all x in X.

Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to f, then f is continuous.

1.2.3 Compactness

Definition 1.2.19. A collection $\mathcal{A}$ of subsets of a space X is said to cover X, or to be a covering of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an open covering of X if its elements are open subsets of X.

A space X is said to be compact if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Proposition 1.2.20. Every closed subspace of a compact space is compact. Every compact subspace of a Hausdorff space is closed.

Theorem 1.2.21. The image of a compact space under a continuous map is compact.

Corollary 1.2.22. X is a compact space, Y is a Hausdorff space, then continuous $f : X \to Y$ is closed.

Corollary 1.2.23. Let $f : X \to Y$ be a continuous bijection. X is a compact space, Y is a Hausdorff space, then f is homemorphism.

Lemma 1.2.24 (Lebesgue number lemma)**.** Let $\mathcal{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing it. The number δ is called a Lebesgue number for the covering $\mathcal{A}$.

Theorem 1.2.25. Let X be a metrizable space. Then the following are equivalent:

(1) X is compact.

(2) X is limit point compact(infinite subset has a limit point).

(3) X is sequentially compact(every sequnence has a convergent subsequnce).

Theorem 1.2.26 (Tychonoff theorem)**.** An arbitrary product of compact spaces is compact in the product topology.

Definition 1.2.27. A function f from the metric space (X, d_X) to the metric space (Y, d_Y) is said to be uniformly continuous if given $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points x_0, x_1 of X,

$$d_X(x_0, x_1) < \delta \Longrightarrow d_Y(f(x_0), f(x_1)) < \epsilon.$$

Theorem 1.2.28. Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous.

Definition 1.2.29 (one-point compactification)**.**

1.2.4 Connectness

Definition 1.2.30. Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called the components (or the "connected components") of X.The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.

Definition 1.2.31. We define another equivalence relation on the space X by defining $x \sim y$ if there is a path in X from x to y. The equivalence classes are called the path components of X. The path components of X are path-connected disjoint subspaces of X whose union is X, such that each nonempty path-connected subspace of X intersects only one of them.

Definition 1.2.32. A space X is said to be locally connected at $\boldsymbol{x}$ if for every neighborhood U of x, there is a connected neighborhood V of x contained in U. If X is locally connected at each of its points, it is said simply to be locally connected. Similarly, a space X is said to be locally path connected at $\boldsymbol{x}$ if for every neighborhood U of x, there is a path-connected neighborhood V of x contained in U. If X is locally path connected at each of its points, then it is said to be locally path connected.

Proposition 1.2.33. (1) A space X is locally connected if and only if for every open set U of X, each component of U is open in X.

(2) A space X is locally path connected if and only if for every open set U of X, each path component of U is open in X.

(3) If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.

Proposition 1.2.34. The union of a collection of connected subspaces of X that have a point in common is connected.

Proposition 1.2.35. Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is also connected.

Proposition 1.2.36. The image of a connected space under a continuous map is connected.

Theorem 1.2.37 (Intermediate Value Theorem)**.** Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

Theorem 1.2.38 (Extreme value theorem)**.** Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.

1.2.5 Completeness

Definition 1.2.39. Let (X, d) be a metric space. A sequence (x_n) of points of X is said to be a Cauchy sequence in (X, d) if it has the property that given $\epsilon > 0$, there is an integer N such that

$$d(x_n, x_m) < \epsilon \quad \text{whenever } n, m \geq N.$$

The metric space (X, d) is said to be complete if every Cauchy sequence in X converges.

Theorem 1.2.40. A metric space (X, d) is compact if and only if it is complete and totally bounded.

Theorem 1.2.41 (extension theorem)**.** Suppose Y and Z are metric spaces, and Z is complete. Also suppose X is a dense subset of Y, and $f : X \to Z$ is uniformly continuous. Then f has a uniquely determined extension $\bar{f} : Y \to Z$ given by

$$\bar{f}(y) = \lim_{\substack{x \to y \\ x \in X}} f(x) \quad \text{for } y \in Y$$

and $\bar{f}$ is also uniformly continuous.

Definition 1.2.42. Let X be a metric space. If $h : X \to Y$ is an isometric imbedding of X into a complete metric space Y, such that $h(X)$ dense in Y. Then Y is called the completion of X. By extension theorem, the completion of X is uniquely determined up to an isometry.

Definition 1.2.43. A space X is said to be a Baire space if the following condition holds: Given any countable collection $\{A_n\}$ of closed sets of X each of which has empty interior in X, their union $\bigcup A_n$ also has empty interior in X.

Theorem 1.2.44 (Baire Category Theorem)**.** If X is a compact Hausdorff space or a complete metric space, then X is a Baire space.

Theorem 1.2.45. Any open subspace Y of a Baire space X is itself a Baire space.

Theorem 1.2.46. Let X be a space; let (Y, d) be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions such that $f_n(x) \to f(x)$ for all $x \in X$, where $f : X \to Y$. If X is a Baire space, the set of points at which f is continuous is dense in X.

1.2.6 Separation axiom

1.3 Limit

1.4 Series

In the following theorem, $(E, |\cdot|)$ is a complex Banach space and (x_n) is a sequence in E.

Proposition 1.4.1. For a series $\sum x_k$ in a Banach space $(E, |\cdot|)$, the following are equivalent:

(1) $\sum x_k$ converges.

(2) For each $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$\left|\sum_{k=n+1}^{m} x_k\right| < \varepsilon, \quad m > n \geq N.$$

Proposition 1.4.2. A normed vector space is complete if and only if every absolutely convergent series converges.

Proposition 1.4.3. Let $\sum x_k$ be a series in complex Banach space E and $\sum a_k$ a series in $\mathbb{R}^+$. Then the series $\sum a_k$ is called a majorant (or minorant) for $\sum x_k$ if there is some $K \in \mathbb{N}$ such that $|x_k| \leq a_k$ (or $a_k \leq |x_k|$) for all $k \geq K$. If a series in a Banach space has a convergent majorant, then it converges absolutely.

Proposition 1.4.4. Let $(a_n)_{n\in\mathbb{Z}}, (b_n)_{n\in\mathbb{Z}}$ be two sequneces in E, then

$$\sum_{M<n\leqslant M+N} a_n b_n = a_{M+N} B_{M+N} + \sum_{M<n\leqslant M+N-1} (a_n - a_{n+1}) B_n,$$

where $B_n = \sum_{M<k\leqslant n} b_k$.

If in particular $E = \mathbb{C}$ and (a_n) is a monotone sequence of $\mathbb{R}$, and

$$\sup_{M<n\leqslant M+N} |B_n| \leqslant \rho,$$

then

$$\left|\sum_{M<n\leqslant M+N} a_n b_n\right| \leqslant \rho\left(|a_{M+1}| + 2|a_{M+N}|\right).$$

Corollary 1.4.5. (1) (Dirichlet's Rule)

(2) (Leibniz's Rule)

Theorem 1.4.6. Let $\sum x_k$ be a series in E and

$$\alpha := \overline{\lim} \sqrt[k]{|x_k|}.$$

Then the following hold: $\sum x_k$ converges absolutely if $\alpha < 1$. $\sum x_k$ diverges if $\alpha > 1$. For $\alpha = 1$, both convergence and divergence of $\sum x_k$ are possible.

Example 1.4.7. (1) $m \geq 2 \in \mathbb{R}$, $\sum n^{-m}$ converges.

(2) For any $z \in \mathbb{C}$ such that $|z| < 1$, the series $\sum z^k$ converges absolutely.

(3)

$$\exp : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

converges absolutely for all $z \in \mathbb{C}$.

Theorem 1.4.8. Every rearrangement of an absolutely convergent series $\sum x_k$ is absolutely convergent and has the same value as $\sum x_k$.

Theorem 1.4.9. There is a bijection $\alpha : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. If α is such a bijection, we call the series $\sum_n x_{\alpha(n)}$ an ordering of the double series $\sum x_{jk}$. If we fix $j \in \mathbb{N}$ (or $k \in \mathbb{N}$), then the series $\sum_k x_{jk}$ (or $\sum_j x_{jk}$) is called the j^{th} row series (or j^{th} column series) of $\sum x_{jk}$. If every row series (or column series) converges, then we can consider the series of row sums $\sum_j \left(\sum_{k=0}^{\infty} x_{jk}\right)$ (or the series of column sums $\sum_k \left(\sum_{j=0}^{\infty} x_{jk}\right)\Big)$. Finally we say that the double series $\sum x_{jk}$ is summable if

$$\sup_{n \in \mathbb{N}} \sum_{j,k=0}^{n} |x_{jk}| < \infty.$$

Let $\sum x_{jk}$ be a summable double series.

(1) Every ordering $\sum_n x_{\alpha(n)}$ of $\sum x_{jk}$ converges absolutely to a value $s \in E$ which is independent of α.

(2) The series of row sums $\sum_j \left(\sum_{k=0}^{\infty} x_{jk}\right)$ and column sums $\sum_k \left(\sum_{j=0}^{\infty} x_{jk}\right)$ converge absolutely, and

$$\sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} x_{jk} \right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} x_{jk} \right) = s$$

Theorem 1.4.10. Suppose that the series $\sum x_j$ and $\sum y_k$ in $\mathbb{C}$ converge absolutely. Then the Cauchy product $\sum_n \sum_{k=0}^{n} x_k y_{n-k}$ of $\sum x_j$ and $\sum y_k$ converges absolutely, and

$$\left(\sum_{j=0}^{\infty} x_j \right) \left(\sum_{k=0}^{\infty} y_k \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} x_k y_{n-k}$$

Corollary 1.4.11.

$$\exp(x+y) = \exp(x) \exp(y)$$

for $x, y \in \mathbb{C}$

1.5 Functions of single variable

1.6 Several Variables functions

Chapter 2

Measure

Chapter 3

Complex Analysis

3.1 Single Variable

Theorem 3.1.1 (Open mapping Theorem)**.** If f is a holomorphic function and non-constant in a connected open set $\Omega \subset \mathbb{C}$, then f is open.

Proposition 3.1.2. U is an open subset of $\mathbb{C}$, $f : U \to \mathbb{C}$ is a injective holomorphic map, then $f'(z) \neq 0$ for all $z \in U$. By Open Mapping Theorem, the image of f is still open in $\mathbb{C}$, we denote it by V. Then $f : U \to V$ is a holomorphic bijective function. f^{-1} is also holomorphic and $(f^{-1})'(z) = \frac{1}{f(z)}$.

Proposition 3.1.3. f holomorphic, $f(a) \neq 0$, then f is local biholomorphic at a.

3.2 Multiple Variables

Chapter 4

Functional Analysis

4.1 Foundation

4.2 Spectrum of Opertor

4.3 Banach Algebra

Chapter 5

Differential Equation

Chapter 6

Harmonicl Analysis

6.1 Abstact Harmonic Analysis