Algebraic Geometry

Erzhuo Wang

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Contents

Chapter 1

Theory of Scheme

1.1 Sheaf Theory

Definition 1.1.1 (presheaf)**.** Let (Ouv_X) be the category whose objects are the open sets of X and, for two open sets $U, V \subseteq X, \mathrm{Hom}(U, V)$ is empty if $U \nsubseteq V$, and consists of the inclusion map $U \to V$ if $U \subseteq V$ (composition of morphisms being the composition of the inclusion maps). A presheaf is a contravariant functor $\mathscr{F}$ from the category (Ouv_X) to the category of category $\mathcal{C}$(such as the category of abelian groups, the category of rings, the category of R-modules, or the category of R-algebras)

Definition 1.1.2. Let $\mathscr{F}$ be a presheaf on a topological space X, let U be an open set in X and let $\mathscr{U} = (U_i)_{i\in I}$ be an open covering of U. We define maps (depending on $\mathscr{U}$)

$$\rho : \mathscr{F}(U) \to \prod_{i\in I} \mathscr{F}(U_i), \quad s \mapsto \left(s_{|U_i}\right)_i$$

$$\sigma : \prod_{i\in I} \mathscr{F}(U_i) \to \prod_{(i,j)\in I\times I} \mathscr{F}(U_i \cap U_j), \quad (s_i)_i \mapsto \left(s_{i|U_i\cap U_j}\right)_{(i,j)},$$

$$\sigma' : \prod_{i\in I} \mathscr{F}(U_i) \to \prod_{(i,j)\in I\times I} \mathscr{F}(U_i \cap U_j), \quad (s_i)_i \mapsto \left(s_{j|U_i\cap U_j}\right)_{(i,j)}.$$

The presheaf $\mathscr{F}$ is called a sheaf, if it satisfies for all U and all coverings (U_i) as above the following condition:

$$\mathscr{F}(U) \xrightarrow{\rho} \prod_{i\in I} \mathscr{F}(U_i) \underset{\sigma'}{\overset{\sigma}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathscr{F}(U_i \cap U_j)$$

is exact. This means that the map ρ is injective and that its image is the set of elements $(s_i)_{i\in I} \in \prod_{i\in I} \mathscr{F}(U_i)$ such that $\sigma((s_i)_i) = \sigma'((s_i)_i)$.

In other words, a presheaf $\mathscr{F}$ is a sheaf if and only if for all open sets U in X and every open covering $U = \bigcup_i U_i$ the following two conditions hold:

(1) (Sh1) Let $s, s' \in \mathscr{F}(U)$ with $s_{|U_i} = s'_{|U_i}$ for all i. Then $s = s'$.

(2) (Sh2) Given $s_i \in \mathscr{F}(U_i)$ for all i such that $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$ for all i, j. Then there exists an $s \in \mathscr{F}(U)$ such that $s_{|U_i} = s_i$ (note that s is unique by (Sh1)).

Definition 1.1.3 (restriction of sheaf)**.** If $\mathscr{F}$ is a presheaf on a topological space X and U is an open subspace of X, we obtain a presheaf $\mathscr{F}|_U$ on U by setting $\mathscr{F}|_U(V) = \mathscr{F}(V)$ for every open subset V in U. If $\mathscr{F}$ is a sheaf, $\mathscr{F}|_U$ is a sheaf on U. We call $\mathscr{F}|_U$ the restriction of $\mathscr{F}$ to U.

Definition 1.1.4. The inductive limit

$$\mathscr{F}_x := \varinjlim_{U \ni x} \mathscr{F}(U)$$

is called the stalk of $\mathscr{F}$ in x. In other words, $\mathscr{F}_x$ is the set of equivalence classes of pairs (U, s), where U is an open neighborhood of x and $s \in \mathscr{F}(U)$. Here two such pairs (U_1, s_1) and (U_2, s_2) are equivalent, if there exists an open neighborhood V of x with $V \subseteq U_1 \cap U_2$ such that $s_{1|V} = s_{2|V}$. For each open neighborhood U of x we have a canonical map

$$\mathscr{F}(U) \to \mathscr{F}_x, \quad s \mapsto s_x$$

which sends $s \in \mathscr{F}(U)$ to the class of (U, s) in $\mathscr{F}_x$. We call s_x the germ of s in x. If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves on X, we have an induced map

$$\mathscr{F}_x \to \mathscr{G}_x$$

of the stalks in x by Proposition 3.1.30. We obtain a functor $\mathscr{F} \mapsto \mathscr{F}_x$ from the category of presheaves on X to the category of sets.

If $\mathscr{F}$ is a presheaf with values in $\mathcal{C}$, where $\mathcal{C}$ is the category of abelian groups, of rings, or any category in which filtered inductive limits exist, then the stalk $\mathscr{F}_x$ is an object in $\mathcal{C}$ and we obtain a functor $\mathscr{F} \mapsto \mathscr{F}_x$ from the category of presheaves on X with values in $\mathcal{C}$ to the category $\mathcal{C}$.

Proposition 1.1.5. Let X be a topological space, $\mathscr{F}$ and $\mathscr{G}$ presheaves on X, and let $\varphi, \psi : \mathscr{F} \to \mathscr{G}$ be two morphisms of presheaves.

(1) The induced maps on stalks $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ are injective for all $x \in X$ if $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all open subsets $U \subseteq X$.

(2) Assume that $\mathscr{F}$ is a sheaf. Then the induced maps on stalks $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ are injective for all $x \in X$ if and only if $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all open subsets $U \subseteq X$.

(3) If $\mathscr{F}$ and $\mathscr{G}$ are both sheaves, the maps φ_x are bijective for all $x \in X$ if and only if φ_U is bijective for all open subsets $U \subseteq X$.

(4) If $\mathscr{F}$ and $\mathscr{G}$ are both sheaves, the morphisms φ and ψ are equal if and only if $\varphi_x = \psi_x$ for all $x \in X$.

Proof: For $U \subseteq X$ open consider the map

$$\mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}_x, \quad s \mapsto (s_x)_{x \in U}$$

We claim that this map is injective if $\mathscr{F}$ is a sheaf. Indeed let $s, t \in \mathscr{F}(U)$ such that $s_x = t_x$ for all $x \in U$. Then for all $x \in U$ there exists an open neighborhood $V_x \subseteq U$ of x such that $s_{|V_x} = t_{|V_x}$. Clearly, $U = \bigcup_{x\in U} V_x$ and therefore $s = t$ by sheaf condition (Sh1). Using the commutative diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] \arrow[d, "\varphi_U"] & \prod \mathscr{F}_x \arrow[d, "\prod \varphi_x"] \\
\mathscr{G}(U) \arrow[r] & \prod \mathscr{G}_x
\end{tikzcd}$$

and Proposition 3.1.31, (1) and (3) hold.

(2): By proposition 3.1.31, it suffice to show the bijectivity of φ_x for all $x \in U$ implies the surjectivity of φ_U. Let $t \in \mathscr{G}(U)$. For all $x \in U$ we choose an open neighborhood U^x of x in U and $s^x \in \mathscr{F}(U^x)$ such that $(\varphi_{U^x}(s^x))_x = t_x$. Then there exists an open neighborhood $V^x \subseteq U^x$ of x with $\varphi_{V^x}\left(s^x{}_{|V^x}\right) = t_{|V^x}$. Then $(V^x)_{x\in U}$ is an open covering of U and for $x, y \in U$

$$\varphi_{V^x\cap V^y}\left(s^x{}_{|V^x\cap V^y}\right) = t_{|V^x\cap V^y} = \varphi_{V^x\cap V^y}\left(s^y \mid V^x \cap V^y\right).$$

As we already know that $\varphi_{V^x\cap V^y}$ is injective, this shows $s^x\,|V^x \cap V^y = s^y|\,V^x \cap V^y$ and the sheaf condition (Sh2) ensures that we find $s \in \mathscr{F}(U)$ such that $s_{|V^x} = s^x{}_{|V^x}$ for all $x \in U$. Clearly, we have $\varphi_U(s)_x = t_x$ for all $x \in U$ and hence $\varphi_U(s) = t$.

Definition 1.1.6. A morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of sheaves injective (resp. surjective, resp. bijective) if $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is injective (resp. surjective, resp. bijective) for all $x \in X$.

Remark 1.1.7. If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, φ is surjective if and only if for all open subsets $U \subseteq X$ and every $t \in \mathscr{G}(U)$ there exist an open covering $U = \bigcup_i U_i$ (depending on t) and sections $s_i \in \mathscr{F}(U_i)$ such that $\varphi_{U_i}(s_i) = t_{|U_i}$, i.e., locally we can find a preimage of t. But the surjectivity of φ does not imply that $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is surjective for all open sets U of X

Definition 1.1.8. If $\mathscr{F}, \mathscr{G}$ are (pre-)sheaves on X such that $\mathscr{F}(U) \subseteq \mathscr{G}(U)$ for all $U \subseteq X$ open, and such that the following diagram commute

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\subset"] & \mathscr{G}(U) \\
\mathscr{F}(V) \arrow[r, "\subset"'] \arrow[u, "\mathrm{res}_U^V"] & \mathscr{G}(V) \arrow[u, "\mathrm{res}_U^V"']
\end{tikzcd}$$

we call $\mathscr{F}$ sub(pre-)sheaf of $\mathscr{G}$.

Definition 1.1.9 (sheafification)**.** Let $\mathscr{F}$ be a presheaf on a topological space X. Then there exists a pair $\left(\tilde{\mathscr{F}}, \iota_{\mathscr{F}}\right)$, where $\tilde{\mathscr{F}}$ is a sheaf on X and $\iota_{\mathscr{F}} : \mathscr{F} \to \tilde{\mathscr{F}}$ is a morphism of presheaves, such that the following holds: If $\mathscr{G}$ is a sheaf on X and $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, then there exists a unique morphism of sheaves $\tilde{\varphi} : \tilde{\mathscr{F}} \to \mathscr{G}$ with $\tilde{\varphi} \circ \iota_{\mathscr{F}} = \varphi$. And the following properties hold:

(1) For all $x \in X$ the map on stalks $\iota_{\mathscr{F},x} : \mathscr{F}_x \to \tilde{\mathscr{F}}_x$ is bijective.

(2) For every presheaf $\mathscr{F}, \mathscr{G}$ on X and every morphism of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$ there exists a unique morphism $\tilde{\varphi} : \tilde{\mathscr{F}} \to \tilde{\mathscr{G}}$ making the diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\iota_{\mathscr{F}}"] \arrow[d, "\varphi"'] & \tilde{\mathscr{F}} \arrow[d, "\tilde{\varphi}"] \\ \mathscr{G} \arrow[r, "\iota_{\mathscr{G}}"'] & \tilde{\mathscr{G}} \end{tikzcd}$$

commutative.

In particular, $\mathscr{F} \mapsto \tilde{\mathscr{F}}$ is a functor from the category of presheaves on X to the category of sheaves on X.

Proof: For $U \subseteq X$ open, elements of $\tilde{\mathscr{F}}(U)$ are by definition families of elements in the stalks of $\mathscr{F}$ which locally give rise to sections of $\mathscr{F}$. More precisely, we define

$$\tilde{\mathscr{F}}(U) := \Big\{ (s_x) \in \prod_{x \in U} \mathscr{F}_x : \forall x \in U, \exists \text{ an open neighborhood } W \subseteq U \text{ of } x, \\ \text{and } t \in \mathscr{F}(W) \text{ s.t. } \forall w \in W : s_w = t_w \Big\}.$$

For $U \subseteq V$ the restriction map $\tilde{\mathscr{F}}(V) \to \tilde{\mathscr{F}}(U)$ is induced by the natural projection $\prod_{x\in V} \mathscr{F}_x \to \prod_{x \in U} \mathscr{F}_x$. Then it is easy to check that $\tilde{\mathscr{F}}$ is a sheaf.

For $U \subseteq X$ open, we define $\iota_{\mathscr{F},U} : \mathscr{F}(U) \to \tilde{\mathscr{F}}(U)$ by $s \mapsto (s_x)_{x \in U}$. The definition of $\tilde{\mathscr{F}}$ shows that $\iota_{\mathscr{F},x} : \mathscr{F}_x \to \tilde{\mathscr{F}}_x$ is bijective.

Now let $\mathscr{G}$ be a presheaf on X and let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism. Sending $(s_x)_x \in \tilde{\mathscr{F}}(U)$ to $(\varphi_x(s_x))_x \in \tilde{\mathscr{G}}(U)$ defines a morphism $\tilde{\mathscr{F}} \to \tilde{\mathscr{G}}$. By Proposition 1.1.5, this is the unique morphism making the diagram commutative.

If we assume in addition that $\mathscr{G}$ is a sheaf, then the morphism of sheaves $\iota_{\mathscr{G}} : \mathscr{G} \to \tilde{\mathscr{G}}$, which is bijective on stalks, is an isomorphism by Proposition 1.1.5(3). Composing the morphism $\tilde{\mathscr{F}} \to \tilde{\mathscr{G}}$ with $\iota_{\mathscr{G}}^{-1}$, we obtain the morphism $\tilde{\varphi} : \tilde{\mathscr{F}} \to \mathscr{G}$. Finally, the uniqueness of $\left(\tilde{\mathscr{F}}, \iota_{\mathscr{F}}\right)$ is a formal consequence.

Remark 1.1.10. For every presheaf $\mathscr{F}, \mathscr{G}$ on X and every morphism of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$ there exists a unique morphism $\tilde{\varphi} : \tilde{\mathscr{F}} \to \tilde{\mathscr{G}}$ making the diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\iota_{\mathscr{F}}"] \arrow[d, "\varphi"'] & \tilde{\mathscr{F}} \arrow[d, "\tilde{\varphi}"] \\ \mathscr{G} \arrow[r, "\iota_{\mathscr{G}}"'] & \tilde{\mathscr{G}} \end{tikzcd}$$

commutative. If in addition $\mathscr{G}$ is a sheaf and φ_U is injective for all U open in X, we have $\iota_{\mathscr{F},U}$ is injective for all U open in X.

Definition 1.1.11 (direct image)**.** Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathscr{F}$ be a presheaf on X. We define a presheaf $f_*\mathscr{F}$ on Y by

$$(f_*\mathscr{F})(V) = \mathscr{F}\left(f^{-1}(V)\right)$$

the restriction maps given by the restriction maps for $\mathscr{F}$. We call $f_*\mathscr{F}$ the direct image of $\mathscr{F}$ under f.

Whenever $\varphi : \mathscr{F}_1 \to \mathscr{F}_2$ is a morphism of presheaves, the family of maps $f_*(\varphi)_V := \varphi_{f^{-1}(V)}$ for $V \subseteq Y$ open is a morphism $f_*(\varphi) : f_*\mathscr{F}_1 \to f_*\mathscr{F}_2$. Therefore f_* is a functor from the category of presheaves on X to the category of presheaves on Y.

Proposition 1.1.12. (1) If $\mathscr{F}$ is a sheaf on X, $f_*\mathscr{F}$ is a sheaf on Y. Therefore f_* also defines a functor $f_* : (\mathrm{Sh}(X)) \to (\mathrm{Sh}(Y))$.

(2) If $g : Y \to Z$ is a second continuous map, there exists an identity $g_*(f_*\mathscr{F}) = (g \circ f)_*\mathscr{F}$ which is functorial in $\mathscr{F}$.

Definition 1.1.13 (inverse image)**.** Let $f : X \to Y$ be a continuous map and let $\mathscr{G}$ be a presheaf on Y. Define a presheaf on X by

$$U \mapsto \varinjlim_{V \supseteq f(U)} \mathscr{G}(V),$$

the restriction maps being induced by the restriction maps of $\mathscr{G}$ and the universal property of direct limit:

$$\begin{tikzcd}
\varinjlim_{V\supseteq f(U)} \mathscr{G}(V) \arrow[rr, dashed, "\mathrm{res}^U_W"'] & & \varinjlim_{V\supseteq f(W)} \mathscr{G}(V) \\
& \mathscr{G}(V_1) \arrow[ul] \arrow[ur] \arrow[d] & \\
& \mathscr{G}(V_2) \arrow[uul] \arrow[uur] &
\end{tikzcd}$$

We denote this presheaf by $f^+\mathscr{G}$. Let $f^{-1}\mathscr{G}$ be the sheafification of $f^+\mathscr{G}$. We call $f^{-1}\mathscr{G}$ the inverse image of $\mathscr{G}$ under f.

Proposition 1.1.14. f^{-1} is a functor from categroy of presheaf on Y to categroy of sheaf on X.

Proof: If $\varphi : \mathscr{G}_1 \to \mathscr{G}_2$ is a morphism of presheaf on Y, then $f^{-1}\varphi : f^{-1}\mathscr{G}_1 \to f^{-1}\mathscr{G}_1$ is induced by universal property of direct limit and Proposition 1.1.9.

Proposition 1.1.15 (stalks of inverse image)**.** Notice that

$$\left(f^{-1}\mathscr{G}\right)_x \cong \left(f^+\mathscr{G}\right)_x = \varinjlim_{x \in U} \left(f^+\mathscr{G}\right)(U)$$

Since f is continous, by uniqueness of direct limit,

$$\varinjlim_{x\in U} \varinjlim_{f(U)\subset V} \mathscr{G}(V) \cong \varinjlim_{f(x)\in V} \mathscr{G}(V)$$

Proof:

$$\varinjlim_{x\in U} \varinjlim_{f(U)\subset V} \mathscr{G}(V) \cong \varinjlim_{f(x)\in V} \mathscr{G}(V)$$

is given by $[[s]], s \in \mathscr{G}(V) \to [s], s \in \mathscr{G}(V)$ since f is continous.

Proposition 1.1.16. Now let $g : Y \to Z$ be a second continuous map and let $\mathscr{H}$ be a presheaf on Z. By the definition of f^+ and g^+, $f^+ (g^+\mathscr{H}) \cong (g \circ f)^+\mathscr{H}$. By taking sheafification,

$$f^{-1}\left(g^+\mathscr{H}\right) \cong (g \circ f)^{-1}\mathscr{H}$$

Since there's natural morphism of sheaves $f^{-1}g^+\mathscr{H} \to f^{-1}(g^{-1})\mathscr{H}$ and the morphism at stalks are isomorphism, we have

$$f^{-1}\left(g^{-1}\mathscr{H}\right) \cong f^{-1}\left(g^+\mathscr{H}\right) \cong (g \circ f)^{-1}\mathscr{H},$$

Theorem 1.1.17 (adjoint pair (f^{-1}, f_*))**.** Let $f : X \to Y$ be a continuous map, let $\mathscr{F}$ be a sheaf on X and let $\mathscr{G}$ be a presheaf on Y. Then there is a bijection

$$\begin{gathered}\operatorname{Hom}_{(\operatorname{Sh}(X))}\left(f^{-1}\mathscr{G}, \mathscr{F}\right) \leftrightarrow \operatorname{Hom}_{(\operatorname{PreSh}(Y))}\left(\mathscr{G}, f_*\mathscr{F}\right), \\ \varphi \to \varphi^b, \\ \psi^\sharp \leftarrow \psi\end{gathered}$$

and (f^{-1}, f_*) is an adjoint pair between $\operatorname{PreSh}(Y)$ and $\operatorname{Sh}(X)$.

Proof: Let $\varphi : f^{-1}\mathscr{G} \to \mathscr{F}$ be a morphism of sheaves on X, and let $V \subseteq Y$ be open. Since $f\left(f^{-1}(V)\right) \subseteq V$, we have a map $\mathscr{G}(V) \to f^+\mathscr{G}\left(f^{-1}(V)\right)$, and we define φ_V^b as the composition

$$\mathscr{G}(V) \to f^+\mathscr{G}\left(f^{-1}(V)\right) \longrightarrow f^{-1}\mathscr{G}\left(f^{-1}(V)\right) \xrightarrow{\varphi_{f^{-1}(V)}} \mathscr{F}\left(f^{-1}(V)\right) = f_*\mathscr{F}(V).$$

Conversely, let $\psi : \mathscr{G} \to f_*\mathscr{F}$ be a morphism of presheaves on Y. To define the morphism $\psi^\sharp$ it suffices to define a morphism of presheaves $f^+\mathscr{G} \to \mathscr{F}$, which we call again $\psi^\sharp$. Let U be open in X, and $s \in f^+\mathscr{G}(U)$. If V is some open neighborhood of $f(U), U$ is contained in $f^{-1}(V)$. Let V be such a neighborhood such that there exists $s_V \in \mathscr{G}(V)$ representing s. Then $\psi_V\left(s_V\right) \in f_*\mathscr{F}(V) = \mathscr{F}\left(f^{-1}(V)\right)$. Let $\psi_U^\sharp(s) \in \mathscr{F}(U)$ be the restriction of the section $\psi_V\left(s_V\right)$ to U.

Proposition 1.1.18. Let $f : X \to Y$ be a continuous map, let $\mathscr{F}$ be a sheaf on X and let $\mathscr{G}$ be a presheaf on Y, and a morphism of presheaves $\psi : \mathscr{G} \to f_*\mathscr{F}$. Then for each $x \in X$, the map

$$\psi_x^\sharp : \mathscr{G}_{f(x)} \cong \left(f^{-1}\mathscr{G}\right)_x \longrightarrow \mathscr{F}_x$$

induced by $\psi^\sharp : f^{-1}\mathscr{G} \to \mathscr{F}$ on stalks can be described in terms of ψ as follows: For every open neighborhood $V \subseteq Y$ of $f(x)$, we have maps

$$\mathscr{G}(V) \xrightarrow{\psi_V} \mathscr{F}\left(f^{-1}(V)\right) \longrightarrow \mathscr{F}_x,$$

and taking the inductive limit over all V we obtain the map $\psi_x^\sharp : \mathscr{G}_{f(x)} \to \mathscr{F}_x$.

Proposition 1.1.19. X be a topological space, then category of sheaves on X is an abelian category.

Proof: Cokernel exists: If $\mathcal{F}, \mathcal{G}$ are sheaves and $\varphi : \mathcal{F} \to \mathcal{G}$ is a sheaf map, then coker φ exists.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[ddrr, "0"'] & \mathcal{G} \arrow[rr, "\iota\circ\pi"] \arrow[dr, "\pi"] \arrow[ddr, "u"] & & (\mathcal{G}/\mathrm{im}\mathcal{F})^+ \arrow[ddl, dotted, "\theta"] \\
& & \mathcal{G}/\mathrm{im}\mathcal{F} \arrow[ur, "\iota"] \arrow[d, dotted] & \\
& & X &
\end{tikzcd}$$

Here $+$ denotes the sheafification and θ is induced by universal property of sheafification.

Ab2:

$$\begin{tikzcd}
& & \mathcal{G}/\mathrm{im}\mathcal{F} \arrow[dr] & \\
\mathrm{Ker}\varphi \arrow[r, "u"] & \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[u, "\pi"] \arrow[r, "v"] & (\mathcal{G}/\mathrm{im}\mathcal{F})^+ = \mathrm{coker}\varphi \\
& \mathcal{F}/\mathrm{Ker}\varphi \arrow[r, "\simeq"] \arrow[d] & \mathrm{im}\mathcal{F} \arrow[u] \arrow[d] & \\
& (\mathcal{F}/\mathrm{Ker}\varphi)^+ \arrow[r, "\simeq"] & (\mathrm{im}\mathcal{F})^+ \arrow[uu, dashed, bend right, "l"'] &
\end{tikzcd}$$

By the construction of cokernel, $(\mathcal{F}/\mathrm{Ker}\varphi)^+$ is the cokernel of u. Since kernel of v contains $\mathrm{im}\mathcal{F}$, by universal property of sheafification, l_U is injective for all U open in X and the image of l lie in the kernel of v. Now it suffice to show $l : (\mathrm{im}\mathcal{F})^+ \to \ker(v)$ is isomorphism on stalk. Notice that morphisms on stalk is clearly injective and for some $[g] \in \ker(v)_x$, where $g \in \mathscr{G}(U)$, since $\pi_x([g]) = 0$(By Proposition 1.1.9), there's $V \subset U$ such that $\pi_V(g|_V) = 0$. Hence, $g|_V \in \mathrm{im}(\mathscr{F})(V)$ which implies l_x is surjective. Hence, $(\mathrm{im}\mathcal{F})^+$ is the kernel of v.

Proposition 1.1.20. $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves, then coker $\varphi = 0$ if and only if φ_x be surjective for all $x \in X$.

Proof: coker$= 0$ implies φ_x is surjective for all x: By above diagram, if coker $\varphi = 0$, we have $l : (\mathrm{im}\mathcal{F})^+ \to \mathcal{G}$ is an isomorphism of sheaves. Hence, the map $\mathrm{im}\mathcal{F} \to \mathcal{G}$ is surjective on stalks. Hence, it suffice to check $\mathcal{F} \to \mathcal{F}/\mathrm{Ker}\varphi$ is surjective on stalk, which is obvious.

φ_x is surjective for all $x$implies coker$= 0$: it suffice to show $(\mathrm{coker}\varphi)_x$ for all $x \in X$. Since φ_x is surjective, $l : (\mathrm{im}\mathcal{F})^+ \to \mathcal{G}$ is an isomorphism of sheaves. Hence, the kernel of v is $\mathcal{G}$. Then v_x is surjective and $= 0$ for all $x \in X$.

Proposition 1.1.21. Let X be a topological space and $i : Z \to X$ the inclusion of a subspace Z. Let $\mathscr{F}$ be a sheaf on Z. Show the following properties for the stalks $i_*(\mathscr{F})_x$.

(1) For all $x \notin \bar{Z}, i_*(\mathscr{F})_x$ is a singleton (i.e., a set consisting of one element).

(2) For all $x \in Z, i_*(\mathscr{F})_x = \mathscr{F}_x$.

(3) If $Z = \{x\}$ and $\mathscr{F}$ is a constant sheaf on Z with value E, then $i_*(\mathscr{F})$ is called skyscraper sheaf in x with value E.

Theorem 1.1.22. X be a topological space and Z be a closed subset of X with $i : Z \to X$ be the embedding, $\mathscr{G}$ is a sheaf on X supported on Z(That is, Supp$\mathscr{G} \subset Z$), then $i^{-1}\mathscr{G}$ is a sheaf on Z. On the other hand, if $\mathscr{F}$ is a sheaf on Z, by Proposition 1.1.15, $i_*\mathscr{F}$ is a sheaf supported on Z.

$$\begin{tikzcd} \{\text{sheaf on } X \text{ suppoted on } Z\} \arrow[r, bend left, "i^{-1}"] & \{\text{sheaf on } Z\} \arrow[l, bend left, "i_*"] \end{tikzcd}$$

$$\begin{tikzcd} \mathscr{F} \arrow[r] & i^{-1}\mathscr{F} \\ i_*\mathscr{G} & \mathscr{G} \arrow[l] \end{tikzcd}$$

Moreover, for a sheaf $\mathscr{F}$ supported on Z, the identify map $i^{-1}\mathscr{F} \to i^{-1}\mathscr{F}$ induces φ a natural isomorphism of sheaves

$$\mathscr{F} \to i_*i^{-1}\mathscr{F}$$

And, for a sheaf $\mathscr{G}$ on Z, the identify map $i_*\mathscr{G} \to i_*\mathscr{G}$ induces φ a natural isomorphism of sheaves

$$i^{-1}i_*\mathscr{G} \to \mathscr{G}$$

Proof: Since for all $x \in Z$

$$\varphi_x : \mathscr{F}_x \to (i_*i^{-1}\mathscr{F})_x \simeq (i^{-1}\mathscr{F})_x \simeq \mathscr{F}_x$$

is an identity map and for all $x \notin Z$, $\mathscr{F}_x = 0 = (i_*i^{-1}\mathscr{F})_x = 0$ by Proposition 1.1.15, we have $\mathscr{F} \simeq i_*i^{-1}\mathscr{F}$

1.2 Ringed Space

Definition 1.2.1. A ringed space is a pair $(X, \mathscr{O}_X)$, where X is a topological space and where $\mathscr{O}_X$ is a sheaf of (commutative) rings on X.

If $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are ringed spaces, we define a morphism of ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ as a pair $\left(f, f^b\right)$, where $f : X \to Y$ is a continuous map and where $f^b : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is a homomorphism of sheaves of rings on Y.

Definition 1.2.2. If A is a local ring, we denote by $\mathfrak{m}_A$ its maximal ideal and by $\kappa(A) = A/\mathfrak{m}_A$ its residue field. A homomorphism of local rings $\varphi : A \to B$ is called local, if $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$.

A morphism $\left(f, f^b\right) : X \to Y$ of ringed spaces induces morphisms on the stalks as follows. Let $x \in X$. Let $f^\sharp : f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ be the morphism corresponding to f^b by adjointness. Using the identification $(f^{-1}\mathscr{O}_Y)_x = \mathscr{O}_{Y,f(x)}$, we get

$$f_x^\sharp : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$$

Definition 1.2.3. A locally ringed space is a ringed space $(X, \mathscr{O}_X)$ such that for all $x \in X$ the stalk $\mathscr{O}_{X,x}$ is a local ring.

A morphism of locally ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces $\left(f, f^b\right)$ such that for all $x \in X$ the induced homomorphism on stalks

$$f_x^\sharp : \left(f^{-1}\mathscr{O}_Y\right)_x = \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$$

is a local ring homomorphism.

Definition 1.2.4. Let $(X, \mathscr{O}_X)$ be a locally ringed space and $x \in X$. We call the stalk $\mathscr{O}_{X,x}$ the local ring of X in x, denote by $\mathfrak{m}_x$ the maximal ideal of $\mathscr{O}_{X,x}$, and by $\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$ the residue field. If U is an open neighborhood of x and $f \in \mathscr{O}_X(U)$, we denote by $f(x) \in \kappa(x)$ the image of f under the canonical homomorphisms $\mathscr{O}_X(U) \to \mathscr{O}_{X,x} \to \kappa(x)$.

Definition 1.2.5 (sheaf of ring on $\operatorname{Spec}(A)$)**.** For each prime ideal $\mathfrak{p} \subseteq A$, let $A_\mathfrak{p}$ be the localization of A at $\mathfrak{p}$. For an open set $U \subseteq \operatorname{Spec} A$, we define $\mathcal{O}(U)$ to be the set of functions

$$s : U \to \coprod_{\mathfrak{p} \in U} A_p$$

, such that $s(\mathfrak{p}) \in A_p$ for each $\mathfrak{p}$, and such that s is locally a quotient of elements of A : to be precise, we require that for each $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$, contained in U, and elements $a, f \in A$, such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = a/f$ in $A_\mathfrak{q}$.

Proposition 1.2.6. Let A be a ring, and $(\operatorname{Spec} A, \mathcal{O})$ its spectrum.

(1) For any $\mathfrak{p} \in \operatorname{Spec} A$, the stalk $\mathcal{O}_\mathfrak{p}$ of the sheaf $\mathcal{O}$ is isomorphic to the local ring $A_\mathfrak{p}$.

(2) For any element $f \in A$, the ring $\mathcal{O}(D(f))$ is isomorphic to the localized ring A_f.

(3) In particular, $\Gamma(\operatorname{Spec} A, \mathcal{O}) \cong A$.

Proof: (1):First we define a homomorphism from $\mathcal{O}_\mathfrak{p}$ to $A_\mathfrak{p}$ by sending any local section s in a neighborhood of $\mathfrak{p}$ to its value $s(\mathfrak{p}) \in A_\mathfrak{p}$. This gives a well-defined homomorphism φ from $\mathcal{O}_\mathfrak{p}$ to $A_\mathfrak{p}$. The map φ is surjective, because any element of $A_\mathfrak{p}$ can be represented as a quotient a/f, with $a, f \in A, f \notin \mathfrak{p}$. Then $D(f)$ will be an open neighborhood of $\mathfrak{p}$, and a/f defines a section of $\mathcal{O}$ over $D(f)$ whose value at $\mathfrak{p}$ is the given element. To show that φ is injective, let U be a neighborhood of $\mathfrak{p}$, and let $s, t \in \mathcal{O}(U)$ be elements having the same value $s(\mathfrak{p}) = t(\mathfrak{p})$ at $\mathfrak{p}$. By shrinking U if necessary, we may assume that $s = a/f$, and $t = b/g$ on U, where $a, b, f, g \in A$, and $f, g \notin \mathfrak{p}$. Since a/f and b/g have the same image in $A_\mathfrak{p}$, it follows from the definition of localization that there is an $h \notin \mathfrak{p}$ such that $h(ga - fb) = 0$ in A. Therefore $a/f = b/g$ in every local ring $A_\mathfrak{q}$ such that $f, g, h \notin \mathfrak{q}$. But the set of such $\mathfrak{q}$ is the open set $D(f) \cap D(g) \cap D(h)$, which contains $\mathfrak{p}$.

(2): We define a homomorphism $\psi : A_f \to \mathcal{O}(D(f))$ by sending a/f^n to the section $s \in \mathcal{O}(D(f))$ which assigns to each $\mathfrak{p}$ the image of a/f^n in $A_\mathfrak{p}$.

Corollary 1.2.7. $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec}(A)})$ is a locally ringed space.

Proposition 1.2.8. *A, B are commutative rings,*

(1) If $\varphi : A \to B$ is a homomorphism of rings, then φ induces a natural morphism of locally ringed spaces

$$\left(f, f^b\right) : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$$

where

$$\begin{aligned} f_U^b : & \mathcal{O}_{\operatorname{Spec}(A)}(U) \to f_* \mathcal{O}_{\operatorname{Spec}(A)}(U) \\ & (s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}) \mapsto (s' : f^{-1}(U) \to U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \to \coprod_{\mathfrak{q} \in f^{-1}(U)} B_{\mathfrak{q}}) \end{aligned}$$

(2) If A and B are rings, then any morphism of locally ringed spaces from $\operatorname{Spec} B$ to $\operatorname{Spec} A$ is induced by a homomorphism of rings $\varphi : A \to B$ as in (1).

Proof: (1): Assume $\mathfrak{p} \in \operatorname{Spec}(B)$ and $\varphi^{-1}(\mathfrak{p}) = \mathfrak{q}$. Then the ring homomorphism

$$\varphi_{\mathfrak{p}} : A_{\mathfrak{q}} \to B_{\mathfrak{p}}$$

induced by universal property of localization is a local ring homomorphism.

(2): Conversely, suppose given a morphism of locally ringed spaces $\left(f, f^{\#}\right)$ from $\operatorname{Spec} B$ to $\operatorname{Spec} A$. Taking global sections, $f^{\#}$ induces a homomorphism of rings $\varphi : \Gamma\left(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}\right) \to \Gamma\left(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}\right)$. These rings are A and B, respectively, so we have a homomorphism $\varphi : A \to B$. For any $\mathfrak{p} \in \operatorname{Spec} B$, we have an induced local homomorphism on the stalks(universal property of direct limit), $\mathcal{O}_{\operatorname{Spec} A, f(p)} \to \mathcal{O}_{\operatorname{Spec} B, p}$ or $A_{f(p)} \to B_p$, which must be compatible with the map φ on global sections. In other words, we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A_{f(\mathfrak{p})} \arrow[r, "f_{\mathfrak{p}}^{\#}"'] & B_{\mathfrak{p}} \end{tikzcd}$$

Since $f^{\#}$ is a local homomorphism, it follows that $\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$, which shows that f coincides with the map $\operatorname{Spec} B \to \operatorname{Spec} A$ induced by φ.

By universal property of localization, $\varphi_{\mathfrak{p}} = f_{\mathfrak{p}}^{\#}$. Then by Theorem 1.1.5(3), $(f, f^{\#})$ is induced by φ.

Corollary 1.2.9.

Definition 1.2.10. A locally ringed space $(X, \mathscr{O}_X)$ is called affine scheme, if there exists a ring A such that $(X, \mathscr{O}_X)$ is isomorphic to $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$.

Definition 1.2.11. *A* scheme is a locally ringed space $(X, \mathscr{O}_X)$ which admits an open covering $X = \bigcup_{i \in I} U_i$ such that all locally ringed spaces $\left(U_i, \mathscr{O}_X|_{U_i}\right)$ are affine schemes. A morphism of schemes is a morphism of locally ringed spaces.

Definition 1.2.12 (principal oepn subschmems of an affine scheme)**.** Let $X = \operatorname{Spec} A$ be an affine scheme. For $f \in A$ let $j : \operatorname{Spec} A_f \to \operatorname{Spec} A$ be the morphism of affine schemes that corresponds to the canonical homomorphism $A \to A_f$. Then j induces a homeomorphism of $\operatorname{Spec} A_f$ onto $D(f)$. Moreover, for all $x \in D(f)$, $j_x^\sharp$ is the canonical isomorphism $A_{\mathfrak{p}_x} \xrightarrow{\sim} (A_f)_{\mathfrak{p}_x}$ by Algebra Theorem 2.4.26. Hence we see that $(j, j^\sharp)$ induces an isomorphism of the affine scheme $\operatorname{Spec} A_f$ with the locally ringed space $(D(f), \mathscr{O}_{X|D(f)})$.

Definition 1.2.13 (closed subschmems of affine schemes)**.** Let $X = \operatorname{Spec} A$ be an affine scheme. For an ideal $\mathfrak{a}$ of A let $i : \operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$ be the morphism of affine schemes that corresponds to the canonical homomorphism $A \to A/\mathfrak{a}$. Then i induces a homeomorphism of $\operatorname{Spec} A/\mathfrak{a}$ onto the closed subset $V(\mathfrak{a})$ of $\operatorname{Spec} A$. Moreover, for all $x \in V(\mathfrak{a})$ the morphism i_x^b is the canonical surjective homomorphism $A_{\mathfrak{p}_x} \to (A/\mathfrak{a})_{\overline{\mathfrak{p}_x}}$ where $\overline{\mathfrak{p}}_x$ is the image of $\mathfrak{p}_x$ in $A/\mathfrak{a}$.

1.3 Basic Propositions

Definition 1.3.1. Let S be a fixed scheme. The category (Sch/S) of schemes over S (or of S-schemes) is the category whose objects are the morphisms $X \to S$ of schemes, and whose morphisms $\operatorname{Hom}(X \to S, Y \to S)$ are the morphisms $X \to Y$ of schemes with the property that

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

commutes.

Proposition 1.3.2 (open subscheme)**.** (1) Let X be a scheme, and $U \subseteq X$ an open subset. Then the locally ringed space $(U, \mathscr{O}_{X|U})$ is a scheme. We call U an open subscheme of X. If U is an affine scheme, then U is called an affine open subscheme.

(2) Let X be a scheme. The affine open subschemes are a basis of the topology.

(3) There's a canonical morphism between scheme $(U, \mathscr{O}_X|_U)$ and $(X, \mathscr{O}_X)$.

(4): $(f, f^b) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of scheme and $f(X) \subset U$ for some open subset of Y, then there's a natrual morphism $(X, \mathscr{O}_X) \to (U, \mathscr{O}_Y|_U)$ making the following diagram ccommute

$$\begin{tikzcd} (X, \mathscr{O}_X) \arrow[r] \arrow[dr] & (Y, \mathscr{O}_Y) \\ & (U, \mathscr{O}_Y|_U) \arrow[u] \end{tikzcd}$$

Proof: (3): For all the V open in X, the restriction maps

$$\Gamma(V, \mathscr{O}_X) \to \Gamma(V \cap U, \mathscr{O}_X|_U)$$

induce a morphism $j^b : \mathscr{O}_X \to j_*(\mathscr{O}_X|_U)$ of sheaves.

Hence, there's a canonical morphism $(U, \mathscr{O}_X|_U) \to (X, \mathscr{O}_X)$ of scheme.

Lemma 1.3.3 (Nike's Trick)**.** Let X be a scheme, and let U, V be affine open subschemes of X. Then there exists for all $x \in U \cap V$ an open subscheme $W \subseteq U \cap V$ with $W \ni x$ such that W is principal open in U as well as in V.

Proof: We may assume $x \in V \subset U$ and U, V are all open affine, hence

$$(j, j^b) : (V, \mathscr{O}_X|_V) \to (U, \mathscr{O}_X|_U)$$

is a morphism of scheme.

$$\begin{array}{ccccc} (V, \mathscr{O}_X|_V) & \xrightarrow{\simeq} & (\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A}) & & B \\ \big\downarrow j & & \big\downarrow \varphi & & \phi \big\uparrow \\ (U, \mathscr{O}_X|_U) & \xrightarrow{\simeq} & (\operatorname{Spec} B, \mathscr{O}_{\operatorname{Spec} B}) & & A \end{array}$$

Take $f \in B$ such that the principal open subset $D(f)$ satisfies $x \in D(f) \subset V \subset U$, then

$$D(f) = j^{-1}(D(f)) = \varphi^{-1}(D(f)) = D(\phi(f))$$

Lemma 1.3.4 (Gluing of morphisms)**.** Let X, Y be schemes. If $X = \bigcup_i U_i$ is an open covering, then a family of morphisms $\varphi_i : (U_i, \mathscr{O}_X|_{U_i}) \to (Y, \mathscr{O}_Y)$ glues to a morphism (f, f^b) : $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ if and only if the morphisms coincide on intersections $U_i \cap U_j$, and the resulting morphism $X \to Y$ is uniquely determined.

Proof: Firstly, define

$$f : X \to Y, x \mapsto \varphi_i(x) \text{ if } x \in U_i$$

For some V open in Y, we can obtain φ_V by the following diagram:

$$\begin{array}{ccc} \mathscr{O}_Y(V) & \overset{\varphi_V}{\dashrightarrow} & f_*\mathscr{O}_X(V) = \mathscr{O}_X(f^{-1}(V)) \\ \big\downarrow \text{id} & & \text{glue} \big\uparrow \\ \mathscr{O}_Y(V) & \xrightarrow{(\varphi_i)_V} & (\varphi_i)_* \mathscr{O}_X|_{U_i}(V) = \mathscr{O}_X(U_i \cap f^{-1}(V)) \end{array}$$

Example 1.3.5 (zero section)**.** Consider $\mathbb{A}_R^{n+1} = \operatorname{Spec}(R[T_0, \ldots, T_n])$, define

$$\mathbb{A}_R^{n+1} - \{0\} = \bigcup_{i=0}^{n} D(T_i)$$

be an open subscheme of $\mathbb{A}_R^{n+1}$. Since there's natural morphism p_i given by

$$p_i : D(T_i) = \operatorname{Spec} R\left[T_0, \ldots, T_n, T_i^{-1}\right] \to D_+(X_i) = \operatorname{Spec} R\left[\frac{X_0}{X_i}, \ldots, \frac{\widehat{X_i}}{X_i}, \ldots, \frac{X_n}{X_i}\right]$$

, by gluing of morphisms of scheme, there's a natural morphism

$$p : \mathbb{A}_R^{n+1} - \{0\} \to \mathbb{P}_R^n$$

Example 1.3.6. Consider $X = \operatorname{Spec}(\mathbb{R}[x, y]) - \{0\}$ and $p : X \to \mathbb{P}_{\mathbb{R}}^1$. For $(\alpha, \beta) \neq (0, 0)$,

$$p((x - \alpha, y - \beta)) = (\alpha y - \beta x)$$

Example 1.3.7. Let A be an R-algebra, let $f : \operatorname{Spec} A \to \mathbb{A}_R^n$ be an R-morphism, and denote the corresponding R-algebra homomorphism by $\varphi : R[T_1, \ldots, T_n] \to A$. Set $a_i = \varphi(T_i) \in A$. Then f factors through $\mathbb{A}_R^n - \{0\}$ if and only if for all $\mathfrak{p} \in \operatorname{Spec}(A)$,

$$f(\mathfrak{p}) \in \bigcup_{i=0}^{n} D(T_i)$$

Equivalently, there's no such prime ideal $\mathfrak{p} \subset A$ such that $\varphi^{-1}(\mathfrak{p}) \supset (T_1, \ldots, T_n)$. Since $\varphi^{-1}(\mathfrak{p}) \supset (T_1, \ldots, T_n)$ if and only if $\mathfrak{p} \supset (\varphi(T_1), \ldots, \varphi(T_n))$, we have

$$\operatorname{Hom}_R(\operatorname{Spec}(A), \mathbb{A}_R^n - \{0\}) = \left\{\varphi \in \operatorname{Hom}_{(R-\mathrm{Alg})}(R[x_1, \ldots, x_n], A) : (\varphi(x_1), \ldots, \varphi(x_n)) = (1)\right\}$$

Example 1.3.8 ($\mathbb{G}_m$)**.** Set $X = \operatorname{Spec} R[U, U^{-1}] = R[U, T]/(UT - 1)$. Then we obtain for every R-scheme T

$$\operatorname{Hom}_R(T, X) = \operatorname{Hom}_{(R-\mathrm{Alg})}\left(R\left[U, U^{-1}\right], \Gamma(T, \mathscr{O}_T)\right) = \Gamma(T, \mathscr{O}_T)^{\times}.$$

Proposition 1.3.9. Let $(X, \mathscr{O}_X)$ be a scheme, $Y = \operatorname{Spec} A$ an affine scheme. Then the natural map

$$\operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}(A, \Gamma(X, \mathscr{O}_X)), \quad \left(f, f^b\right) \mapsto f_Y^b,$$

is a functorial bijection. Here the set on the left side denotes the set of morphisms $X \to Y$ of scheme, and the set on the right side denotes the set of ring homomorphisms $A \to \Gamma(X, \mathscr{O}_X)$.

Proof:

$$\begin{tikzcd}
\operatorname{Hom}(X,Y) \arrow[r, dashed] \arrow[d, shift left, "\mathrm{res}"] & \operatorname{Hom}(A, \Gamma(X, \mathscr{O}_X)) \arrow[d, shift left, "\mathrm{res}"] \\
\operatorname{Hom}(U_i, Y) \arrow[r, "\simeq"] \arrow[u, shift left, "\mathrm{glue}"] & \operatorname{Hom}(A, \Gamma(U_i, \mathscr{O}_X)) \arrow[u, shift left, "\mathrm{glue}"]
\end{tikzcd}$$

Injective: For $f : X \to Y$, define $f_i : U_i \to X \to Y$ a morphism of scheme. It's easy to check the follow diagram commutes

$$\begin{tikzcd}
A \arrow[r, "f_Y^b"] \arrow[dr, "(f_i)_Y^b"'] & \Gamma(X, \mathscr{O}_X) \arrow[d, "j_X^b"] \\
 & \Gamma(U_i, \mathscr{O}_X)
\end{tikzcd}$$

Hence, $(f, f^b) = (g, g^b)$ iff $(f_i, f_i^b) = (g_i, g_i^b)$ iff $(f_i)_Y^b = (g_i)_Y^b$ iff $f_Y^b = g_Y^b$

Surjective:

$$\begin{tikzcd}
U_i \arrow[r, "f_i"] & Y & \Gamma(U_i, \mathscr{O}_X) \arrow[d] & A \arrow[l, "\tilde{f}_i"'] \arrow[d, "\tilde{f}_j"'] \\
V \arrow[u, "l_i"] \arrow[r, "l_j"'] & U_j \arrow[u, "f_j"'] & \Gamma(V, \mathscr{O}_X) & \Gamma(U_j, \mathscr{O}_X) \arrow[l]
\end{tikzcd}$$

Take $\tilde{f} \in \operatorname{Hom}(A, \Gamma(X, \mathscr{O}_X))$, and define $\tilde{f}_i : \tilde{f} \circ \operatorname{Res}_{U_i}^X$ and f_i be the corresponding morphisms with respect to category equivalence(commutative rings and affine schemes). Consider above diagram, V is an affine open subset of $U_i \cap U_j$. Since opposite category of commutative rings is equivalent to category of affine scheme, the fact that the right diagram commutes implies the left diagram commute.

Proposition 1.3.10. Let $(X, \mathscr{O}_X)$ be a k-scheme, A be a k-algebra and $Y = \operatorname{Spec} A$ an affine scheme over k. Then the natural map

$$\operatorname{Hom}_{\operatorname{Spec}(k)}(X, Y) \longrightarrow \operatorname{Hom}_k(A, \Gamma(X, \mathscr{O}_X)), \quad (f, f^b) \mapsto f^b_Y,$$

is a functorial bijection. Here the set on the left side denotes the set of morphisms $X \to Y$ of k-scheme, and the set on the right side denotes the set of k-algebra homomorphisms $A \to \Gamma(X, \mathscr{O}_X)$.

Proposition 1.3.11. Let X be a scheme. Let $x \in X$, and let $U \subseteq X$ be an affine open neighborhood of x, say $U = \operatorname{Spec} A$. Denote by $\mathfrak{p} \subset A$ the prime ideal of A corresponding to x. Then $\mathscr{O}_{X,x} = \mathscr{O}_{U,x} = A_{\mathfrak{p}}$, and the natural homomorphism $A \to A_{\mathfrak{p}}$ gives us a morphism

$$j_x : \operatorname{Spec} \mathscr{O}_{X,x} = \operatorname{Spec} A_{\mathfrak{p}} \to \operatorname{Spec} A = U \subseteq X$$

of schemes. This morphism is independent of the choice of U.

Proof: Assume V is an open affine subset of U with $x \in V$, $V = \operatorname{Spec}(B)$ and $x = \mathfrak{q}$. Then, it suffices to show j_x induced by V and j_x induced by U identifies. Consider the following commutative diagram

$$\begin{tikzcd}
 & & & & X \\
\operatorname{Spec}\mathcal{O}_{X,x} \arrow[r, "\simeq"] & \operatorname{Spec}A_{\mathfrak{p}} \arrow[r] & \operatorname{Spec}A \arrow[r, "\simeq"] & U \arrow[ur] & \\
\operatorname{Spec}\mathcal{O}_{X,x} \arrow[r, "\simeq"] \arrow[u, "\mathrm{id}"] & \operatorname{Spec}B_{\mathfrak{q}} \arrow[r] \arrow[u] & \operatorname{Spec}B \arrow[r, "\simeq"] \arrow[u, "\varphi"] & V \arrow[u] \arrow[uur] &
\end{tikzcd}$$

where the morphism $\operatorname{Spec}(B_{\mathfrak{q}}) \to \operatorname{Spec}(A_{\mathfrak{p}})$ is induced both by universal propoty of localization and the morphism of sheaves $\mathscr{O}_{\operatorname{Spec}(A)} \to \varphi_* \mathscr{O}_{\operatorname{Spec}(B)}$.

Proposition 1.3.12. The image of the canonical map $j_x : \operatorname{Spec}\mathcal{O}_{X,x} \to X$ is

$$Z = \left\{ y \in X : x \in \overline{\{y\}} \right\} = \bigcap_{x \in W, W \text{ open in } X} W$$

Proof: Trivial.

Proposition 1.3.13. Let $\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$ be the residue class field of x in X. We obtain a morphism of schemes

$$i_x : \operatorname{Spec} \kappa(x) \longrightarrow \operatorname{Spec} \mathscr{O}_{X,x} \longrightarrow X$$

called canonical. The image point of the unique point in $\operatorname{Spec} \kappa(x)$ is x. Notice that the map $\mathscr{O}_{X,x} \to \kappa(x)$ induced by considering the stalk of i_x is exactly the projective map.

Now let K be any field, let $f : \operatorname{Spec} K \to X$ be a morphism, and let $x \in X$ be the image point of the unique point p of $\operatorname{Spec} K$. Since f is a morphism of locally ringed spaces, f induces a local homomorphism $\mathscr{O}_{X,x} \to K = \mathscr{O}_{\operatorname{Spec} K,p}$, and hence a homomorphism $\iota : \kappa(x) \to K$ between the residue class fields.

Then, the morphism f factors as $f = i_x \circ (\operatorname{Spec} \iota) : \operatorname{Spec} K \to \operatorname{Spec} \kappa(x) \to X$ since we have a commutative diagram in stalks of those sheaves:

$$\begin{tikzcd} K & \mathscr{O}_{X,x} \arrow[l] \arrow[dl] \\ \kappa(x) \arrow[u] & \end{tikzcd}$$

The above construction gives rise to a bijection

$$\operatorname{Hom}(\operatorname{Spec} K, X) \longrightarrow \{(x,\iota); x \in X, \iota : \kappa(x) \to K\}$$

This is because, we can map an element $(x, \iota : \kappa(x) \to K)$ of the right hand side to the morphism

$$\operatorname{Spec} K \xrightarrow{\operatorname{Spec} \iota} \operatorname{Spec} \kappa(x) \xrightarrow{i_x} X,$$

and these two maps are inverse to each other.

Proposition 1.3.14. Assume $(X, \mathscr{O}_X) \to \operatorname{Spec}(k)$ be a k-scheme, then this map induces a local ring homomorphism

$$k \to \mathscr{O}_{X,x}$$

which induecs a field extension

$$k \to \kappa(x)$$

Hence there's natural k-scheme structure on $\operatorname{Spec}(\kappa(x))$. Moreover, above natural morphism i_x becomes a k-scheme morphism:

$$\begin{tikzcd} (X, \mathscr{O}_X) \arrow[dr] & \\ \operatorname{Spec}\kappa(x) \arrow[u, "i_x"] \arrow[r] & \operatorname{Spec}k \\ \operatorname{Spec}K \arrow[u] \arrow[ur] & \end{tikzcd}$$

Hence, if $k \to K$ be a field extension, there's a bijection

$$\operatorname{Hom}_k(\operatorname{Spec} K, X) \longrightarrow \{(x,\iota) : x \in X, \iota : \kappa(x) \to K \quad k\text{-algebra homomorphism}\}$$

And for an arbitrary k-scheme, define $X(K) = \operatorname{Hom}_k(\operatorname{Spec} K, X)$ to be its K-points.

Definition 1.3.15 (Structure sheaf on Proj$S$)**.** Let S be a graded ring, we will define a sheaf of rings $\mathscr{O}$ on $\operatorname{Proj} S$. For each $\mathfrak{p} \in \operatorname{Proj} S$, we consider the ring $S_{(\mathfrak{p})}$ of elements of degree zero in the localized ring $T^{-1}S$, where T is the multiplicative system consisting of all homogeneous elements of S which are not in $\mathfrak{p}$. For any open subset $U \subseteq \operatorname{Proj} S$, we define $\mathscr{O}(U)$ to be the set of functions $s : U \to \coprod S_{(\mathfrak{p})}$ such that for each $\mathfrak{p} \in U, s(\mathfrak{p}) \in S_{(\mathfrak{p})}$, and such that s is locally a quotient of elements of $S :$ for each $\mathfrak{p} \in U$, there exists a neighborhood V of $\mathfrak{p}$ in U, and homogeneous elements a, f in S, of the same degree, such that for all $\mathfrak{q} \in V, f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = a/f$ in $S_{(\mathfrak{q})}$. Now it is clear that $\mathscr{O}$ is a presheaf of rings, with the natural restrictions, and it is also clear from the local nature of the definition that $\mathscr{O}$ is a sheaf.

Proposition 1.3.16. Let S be a graded ring.

(1) For any $\mathfrak{p} \in \operatorname{Proj} S$, the stalk $\mathscr{O}_\mathfrak{p}$ is isomorphic to the local ring $S_{(\mathfrak{p})}$.

(2) For any homogeneous element $f \in S_+$, let $D_+(f) = \{\mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p}\}$. Then $D_+(f)$ is open in Proj S. Furthermore, these open sets cover Proj S, and for each such open set, we have an isomorphism of locally ringed spaces

$$\left(D_+(f), \mathscr{O}|_{D_+(f)}\right) \cong \operatorname{Spec} S_{(f)}$$

where

$$S_{(f)} = \{a/f^n \in S_f : a \text{ homogeneous and } \deg(a) = n\deg(f), n \geq 0\}$$

In particular, the global section of S is S_0.

Proof: (1): $S_{(\mathfrak{p})}$ is a local ring: The unique maximal ideal of $S_{(\mathfrak{p})}$ is of the form

$$\{a/f : a \in \mathfrak{p}, f \notin \mathfrak{p}, \deg a = \deg f\}$$

(2): Define

$$\varphi : D_+(f) \to \operatorname{Spec}(S_{(f)}), \mathfrak{a} \mapsto \mathfrak{a}S_f \cap S_{(f)}$$

φ is injective: If $\mathfrak{a}S_f \cap S_{(f)} = \mathfrak{b}S_f \cap S_{(f)}$, for some homogeneous element $s \in \mathfrak{a}$, there's $b \in \mathfrak{b}$ such that

$$\frac{s^n}{f^m} = \frac{b}{f^t}$$

for some integer n, m, t. Hence, $s^n \in \mathfrak{b}$ which implies $s \in \mathfrak{b}$.

φ is surjective: P be a prime ideal of $S_{(f)}$, define

$$\mathfrak{p} = \{s \in S : s/f^n \in P \text{ for some } n \geq 0\}$$

Then $\varphi(\mathfrak{p}) = P$.

Isomorphism on stalk: For $\mathfrak{p} \in D_+(f)$, there's a natural ring homomorphism

$$S_{(f)} \to S_{(\mathfrak{p})}, a/f^n \mapsto a/f^n$$

and by universal property of localization, it induces a ring homomorphism

$$\varphi_\mathfrak{p} : (S_{(f)})_{\varphi(\mathfrak{p})} \to S_{(\mathfrak{p})}$$

Acturally, $\varphi_\mathfrak{p}$ is an isomorphism: injective is easy to check, and for some $a/g \in S_\mathfrak{p}$, notice that

$$\frac{a}{g} = \frac{ag^{\deg f - 1}}{f^{\deg g}} \frac{f^{\deg g}}{g^{\deg f}}$$

Hence, $\varphi_\mathfrak{p}$ is surjective.

Isomorphism $\varphi_\mathfrak{p}$ induces a isomorphism of sheaves

$$\varphi^b : \mathscr{O}_{\operatorname{Spec} S_{(f)}} \simeq \varphi_*(\mathscr{O}_{\operatorname{Proj} S}|_{D^+(f)})$$

Proposition 1.3.17 (morphisms between projective spectrum)**.** Let S be a graded ring.

(1) Let $\varphi : S \to T$ be a graded homomorphism of graded rings (preserving degrees). Let $U = \{\mathfrak{p} \in \operatorname{Proj} T \mid \mathfrak{p} \not\supseteq \varphi(S_+)\}$. Show that U is an open subset of Proj T, and show that φ determines a natural morphism $f : U \to \operatorname{Proj} S$.

(2) The morphism f can be an isomorphism even when φ is not. For example, suppose that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geqslant d_0$, where d_0 is an integer. Then show that $U = \operatorname{Proj} T$ and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is an isomorphism.

(3) Let $\varphi : S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Then, the open set U of is equal to Proj T, and the morphism f :Proj $T \to$ Proj S is a closed immersion.

(4) If $I \subseteq S$ is a homogeneous ideal, take $T = S/I$ and let Y be the closed subscheme of $X = \operatorname{Proj} S$ defined by the closed immersion Proj $S/I \to X$. Show that different homogeneous ideals can give rise to the same closed subscheme. For example, let d_0 be an integer, and let $I' = \bigoplus_{d \geqslant d_0} I_d$. Show that I and I' determine the same closed subscheme.

Proof: (1): Since graded homomorphism preserves order, $\mathfrak{p} \in U \mapsto \varphi^{-1}(\mathfrak{p})$ is a well-define map from U to Proj S. Notice that

$$U = \bigcup_{g \in \varphi(S_+)} D_+(g),$$

U is a open subset of Proj T. And the morphism of presheaves f^b is induced by the natural local ring homomorphism

$$\varphi_{\mathfrak{p}} : S_{(f(\mathfrak{p}))} \to T_{(\mathfrak{p})}$$

And it's easy to check f together with f^b forms a morphism of scheme $(f, f^b) : U \to$ Proj S.

(2): For $U = \operatorname{Proj} T$, assume $\mathfrak{p} \supset \varphi(S_+)$ and $\mathfrak{p} \not\supseteq T_+$, there's $a \in T_r$ with $r \geq 1$ such that $a \notin \mathfrak{p}$. Consider the element a^k for k sufficiently large. Next step, we are going to show $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is an isomorphism.

Since, φ_d are isomorphic for all $d \geq d_0$,

$$\{\{D_+(t_i)\} : t_i \in T_+, \deg t_i \geq d_0\}$$

be a open covering of Proj T. Put $s_i = \varphi^{-1}(t_i)$, we also have

$$\{\{D_+(s_i)\} : s_i \in S_+, \deg s_i \geq d_0\}$$

be a open covering of Proj S

$f_i = f|_{D_+(t_i)} \to D_+(s_i)$ is a morphism of affine schemes (as $D_+(t_i) \simeq \operatorname{Spec} T_{(t_i)}$ and $D_+(s_i) \simeq \operatorname{Spec} S_{(s_i)}$) corresponding to the ring homomorphism $\varphi_i : S_{(s_i)} \to T_{(t_i)}$ induced by φ. But φ_i is an isomorphism since s_i has degree at least d_0, and φ_d is an isomorphism for all $d \geq d_0$. Hence, f is surjective.

To show f is injective, take $\mathfrak{p}_1, \mathfrak{p}_2 \in$ Proj T with $f(\mathfrak{p}_1) = f(\mathfrak{P}_2)$. We have $\mathfrak{p}_1 \cap T_d = \mathfrak{p}_2 \cap T_d$ for all $d \geq d_0$. If $t_r \in \mathfrak{p}_1 \cap T_r$, take $s \notin \mathfrak{p}_2$, we have $s^k t_r \in \mathfrak{p}_2$. It implies $t_r \in \mathfrak{p}_2 \cap T_r$.

(3):Since $\varphi : S \to T$ is surjective, f is injective and

$$\varphi_{\mathfrak{p}} : S_{(f(\mathfrak{p}))} \to T_{(\mathfrak{p})}$$

is surjective.

Then, it suffice to check $f(\mathrm{Proj}T)$ is a closed subset. Notice that $\mathrm{Ker}\varphi$ be a homogenous ideal and for all $\mathfrak{p} \in \mathrm{Proj}(T)$, we have

$$f(\mathfrak{p}) \subset \mathrm{Ker}\varphi$$

Hence, $f(\mathrm{Proj}(T)) \subset V(\mathrm{Ker}\varphi)$. On the other hand, Since φ is surjective, $T \simeq S/\mathrm{Ker}\varphi$ as graded ring. Hence, it's easy to show $V(\mathrm{Ker}\varphi) \subset f(\mathrm{Proj}(T))$.

(4): By (2).

Example 1.3.18. Let $f_1, \dots, f_r \in k[X_0, \dots, X_n]$ be homogeneous polynomials and let $X = \mathrm{Proj}\, k[X_0, \dots .X_n]/(f_1, \dots, f_r)$. For every field extension $k \hookrightarrow K$ we have

$$X(K) = \{x = (x_0 : \dots : x_n) \in \mathbb{P}^n(K) : f_1(x) = \cdots = f_r(x) = 0\}$$

Definition 1.3.19 (Galois Actions)**.** Assume K/k be a Galois extension and $G = \mathrm{Gal}(K/k)$. Let X be a k-scheme, we obtain an action of G on $X(K)$ by composition of the morphism $x : \mathrm{Spec}\, K \to X$ with ${}^a\sigma : \mathrm{Spec}\, K \to \mathrm{Spec}\, K$ for $\sigma \in G$. Hence, by Proposition 1.3.14, the Galois group action on $X(K)$ by $\sigma \in G$ is actually a transform of k-algebra homeomorphism through composition

$$l \in \mathrm{Hom}_k(\kappa(x), K) \mapsto \sigma \circ l \in \mathrm{Hom}_k(\kappa(x), K).$$

Denote the K-points which is stable under a subgroup H of G by $X(K)^H$, we have

$$X(K)^H = X(K^H).$$

Proposition 1.3.20. k be a perfect field. Then $\bar{k}/k$ is a Galois extension. Denote $G = \mathrm{Gal}(\bar{k}/k)$. Let X be a k-scheme locally of finite type. There's a one-to-one correspondence between G-orbits of $X(\bar{k})$ and closed point of X.

Proof: Since the point in $X(\bar{k})$ is the pair (x, l), where $x \in X$ and l be a k-algebra homeomorphism from $\kappa(x)$ to $\bar{k}$. By Proposition 1.3.42, for all $(x, l) \in X(\bar{k})$, x is a closed point. Moreover, G-action doesn't change x, so $(x, l) \mapsto x$ be a map from G-orbits of $X(\bar{k})$ to closed point of X. By Algebra 1.4.15, $(x, l) \mapsto x$ is surjective. By Numebr Theory Theorem 2.2.3, $(x, l) \to x$ is injective.

Proposition 1.3.21. A gluing datum of schemes consists of the following data:

(1) an index set I,

(2) for all $i \in I$ a scheme U_i,

(3) for all $i, j \in I$ an open subset $U_{ij} \subseteq U_i$ (we consider U_{ij} as open subscheme of U_i),

(4) for all $i, j \in I$ an isomorphism $\varphi_{ji} : U_{ij} \to U_{ji}$ of schemes, such that $U_{ii} = U_i$ for all $i \in I$ and the cocycle condition holds: $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ on $U_{ij} \cap U_{ik}, i, j, k \in I$.

Remark 1.3.22. In the cocycle condition we implicitly assume that in particular $\varphi_{ji}\left(U_{ij} \cap U_{ik}\right) \subseteq U_{jk}$, such that the composition is meaningful.

For $i = j = k$, the cocycle condition implies that $\varphi_{ii} = \mathrm{id}_{U_i}$ and for $i = k$ that $\varphi_{ij}^{-1} = \varphi_{ji}$.

Moreover, φ_{ji} is an isomorphism $U_{ij} \cap U_{ik} \to U_{ji} \cap U_{jk}$. This is because, consider the cocycle conditions $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ and $\varphi_{ki} \circ \varphi_{ij} = \varphi_{kj}$. We obtain two natural morphisms $\varphi_{ji} : U_{ij} \cap U_{ik} \to U_{ji} \cap U_{jk}$ and $\varphi_{ij} : U_{ji} \cap U_{jk} \to U_{ij} \cap U_{ik}$. Then, the claim follows from the fact $\varphi_{ji}^{-1} = \varphi_{ij}$.

Proposition 1.3.23. Let $\left((U_i)_{i\in I}, (U_{ij})_{i,j\in I}, (\varphi_{ij})_{i,j\in I}\right)$ be a gluing datum of schemes. Then there exists a scheme X together with morphisms $\psi_i : U_i \to X$, such that

(1) for all i the map ψ_i is a isomorphism of U_i onto the open subscheme $\psi_i(U_i)$ of X.

$$\begin{tikzcd} & X \\ U_i \arrow[ur] \arrow[r, "\psi_i"'] & \psi_i(U_i) \arrow[u, "l"'] \end{tikzcd}$$

(2) $\psi_j \circ \varphi_{ji} = \psi_i$ on U_{ij} for all i, j,

(3) $X = \bigcup_i \psi_i\left(U_i\right)$,

(4) $\psi_i\left(U_i\right) \cap \psi_j\left(U_j\right) = \psi_i\left(U_{ij}\right) = \psi_j\left(U_{ji}\right)$ for all $i, j \in I$.

Furthermore, X together with the ψ_i is uniquely determined up to unique isomorphism.

Proof: Underlying topological space: To define the underlying topological space of X, we start with the disjoint union $\coprod_{i\in I} U_i$ of the (underlying topological spaces of the) U_i and define an equivalence relation $\sim$ on it as follows: points $x_i \in U_i, x_j \in U_j, i, j \in I$, are equivalent, if and only if $x_i \in U_{ij}$, $x_j \in U_{ji}$ and $x_j = \varphi_{ji}\left(x_i\right)$. The cocycle condition implies that $\sim$ is in fact an equivalence relation. As a set, define X to be the set of equivalence classes,

$$X := \coprod_{i\in I} U_i / \sim .$$

The natural maps $\psi_i : U_i \to X$ are injective and we have $\psi_i\left(U_{ij}\right) = \psi_i\left(U_i\right) \cap \psi_j\left(U_j\right)$ for all $i, j \in I$. We equip X with the quotient topology, i. e. with the finest topology such that all ψ_i are continuous. That means that a subset $U \subseteq X$ is open if and only if for all i the preimage $\psi_i^{-1}(U)$ is open in U_i. In particular, the $\psi_i\left(U_i\right)$ and the $\psi_i\left(U_{ij}\right) = \psi_i\left(U_i\right) \cap \psi_j\left(U_j\right)$ are open in X.

Structure sheaf: Define for W open in X,

$$\mathscr{O}_X(W) = \left\{(s_i)_{i\in I} : s_i \in \mathscr{O}_{U_i}(W \cap U_i), \varphi_{ji}(s_i|_{W\cap U_{ij}}) = s_j|_{W\cap U_{ji}}\right\}$$

where $W \cap U_i$ is actually $\psi_i^{-1}(W)$.

morphism of sheaves ψ_i^b:

$$\psi_i^b : (\psi_i)_* \mathscr{O}_X \to \mathscr{O}_{U_i}, (s_i)_{i \in I} \mapsto s_i$$

Example 1.3.24 (line with double origin)**.** We denote the line with double origin by X. It is obtained by gluing Spec$k[u]$ and Spec$k[t]$ along the isomorphism $D(u) \simeq \mathrm{Spec}(k[u, 1/u]) \simeq \mathrm{Spec}(k[t, 1/t]) = D(t)$. Notice taht $(X, \mathscr{O}_X)$ is affine if the morphism $(f, f^b) : (X, \mathscr{O}_X) \to \mathrm{Spec}(\Gamma(X, \mathscr{O}_X))$ induced by id : $\Gamma(X, \mathscr{O}_X) \to \Gamma(X, \mathscr{O}_X)$ is an isomorphism.

An element of $\Gamma(X, \mathscr{O}_X)$ is the same as giving two polynomials $\sum_n f_n u^n$ and $\sum_m g_m t^m$ such that $\sum_n f_n u^n = \sum_m g_m u^m$ in $k[u, 1/u]$. Note that this just means that $f_n = g_n$ for all n. Hence $\Gamma(X, \mathcal{O}_X)$ is isomorphic to $k[u]$. If X is affine, then we have isomorphisms

$$(X, \mathcal{O}_X) \xrightarrow{(f, f^b)} \mathrm{Spec}\,\Gamma(X, \mathscr{O}_X) \to \mathrm{Spec}\,k[u]$$

Now consider the vanishing set $V(u)$ of X where $V(f)$ for some f in global section consists of all those points $x \in X$ such that $f_x = 0$ modulo $\mathfrak{m}_x$. and u denotes the global section $u = v$ of $\Gamma(X, \mathscr{O}_X)$.

Note that $V(u)$ contains at least two points, the two origins of X. But $V(u)$ in Spec $k[u]$ consists of only one point. Hence line with double origin is not affine.

Example 1.3.25 (projective space)**.** R is a ring and $S = R[X_0, \ldots, X_n]$ be a graded ring. Consider the scheme $\mathbb{P}_R^n = \mathrm{Proj}S$. For $f = x_i, i = 1, \ldots n$, we have

$$S_{(f)} = \{a/X_i^n \in R[X_0, \ldots, X_n]_{X_i} : a \in R[X_0, \ldots, X_n]_n\} = R\left[\frac{X_0}{X_i}, \ldots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \ldots, \frac{X_n}{X_i}\right]$$

and for $U_i = D_+(f)$,

$$(U_i, \mathscr{O}_{\mathbb{P}_R^n}|_{U_i}) = \left(D_+(f), \mathscr{O}|_{D_+(f)}\right) \simeq \mathrm{Spec}R\left[\frac{X_0}{X_i}, \ldots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \ldots, \frac{X_n}{X_i}\right]$$

We define a gluing datum with index set $\{0, \ldots, n\}$ as follows: For $0 \le i, j \le n$ let $U_{ij} = D_{U_i}\left(\frac{X_j}{X_i}\right) \subseteq U_i$ if $i \ne j$, and $U_{ii} = U_i$. Further, let $\varphi_{ii} = \mathrm{id}_{U_i}$ and for $i \ne j$ let

$$\varphi_{ji} : U_{ij} \to U_{ji}$$

be the isomorphism defined by the equality

$$R\left[\frac{X_0}{X_i}, \ldots, \widehat{\frac{X_i}{X_i}}, \ldots, \frac{X_n}{X_i}\right]_{\frac{X_j}{X_i}} \longleftarrow R\left[\frac{X_0}{X_j}, \ldots, \widehat{\frac{X_j}{X_j}}, \ldots, \frac{X_n}{X_j}\right]_{\frac{X_i}{X_j}},$$

(as subrings of $R\left[X_0, \ldots, X_n, X_0^{-1}, \ldots, X_n^{-1}\right]$) of the affine schemes U_{ij} and U_{ji}.

Corollary 1.3.26. If $R = k$ be a ring, the global section of $\mathbb{P}_k^n$ is k. Hence, $\mathbb{P}_k^n$ is not affine.

Example 1.3.27 (structure of $\mathbb{P}^1_{\mathbb{R}}$)**.** For U_x, U_y, there are $\mathbb{R}$-scheme isomorphisms

$$(U_x,\, \mathscr{O}_{\mathbb{P}^1_{\mathbb{R}}}\Big|_{U_x}) \simeq \mathrm{Spec}\mathbb{R}[y]$$

and

$$(U_y,\, \mathscr{O}_{\mathbb{P}^1_{\mathbb{R}}}\Big|_{U_y}) \simeq \mathrm{Spec}\mathbb{R}[x]$$

Hence,

$$\mathbb{P}^1_{\mathbb{R}} = \{(x-ay) : a \in \mathbb{R}\} \bigcup \{(y-ax) : a \in \mathbb{R}\} \bigcup \left\{(ax^2+bxy+cy^2) : b^2-4ac<0\right\}$$

Definition 1.3.28. (1) A scheme is called connected, if the underlying topological space is connected.

(2) A scheme is called quasi-compact, if the underlying topological space is quasi-compact, i. e., if every open covering admits a finite subcovering.

(3) A scheme is called irreducible, if the underlying topological space is irreducible, i. e., if it is non-empty and not equal to the union of two proper closed subsets.

(4) A morphism $f : X \to Y$ of schemes is called injective, surjective or bijective, respectively, if the continuous map $X \to Y$ of the underlying topological spaces has this property.

(5) f is called open, closed, or a homeomorphism, respectively, if the underlying continuous map has this property.

(6) f is called dominant if $f(X)$ is a dense subspace of Y.

(7) A scheme X is called locally noetherian, if X admits an affine open cover $X = \bigcup U_i$, such that all the affine coordinate rings $\Gamma(U_i, \mathscr{O}_X)$ are noetherian. If in addition X is quasi-compact, X is called noetherian.

(8) A scheme X is called reduced, if all local rings $\mathscr{O}_{X,x}, x \in X$, are reduced rings.

(9) An integral scheme is a scheme which is reduced and irreducible.

Proposition 1.3.29. Let $X = \operatorname{Spec} A$ be an affine scheme. Then X is noetherian if and only if A is a noetherian ring.

Proof: By Nike's Trick, $\operatorname{Spec} A$ can be covered by affine open subschemes of the form $D(f_i), f_i \in A, i = 1, \ldots, n$, such that all A_{f_i} are noetherian rings.

If I is an ideal of A, $I_{f_i} = IA_{f_i}$ is finitely generated ideal in A_{f_i}. By Algebra 2.4.29, I is finitely generated ideal in A.

Proposition 1.3.30. X is any noetherian scheme, the underlying topological space of X is noetherian

Proof: Since spectrum of a noetherian ring is a noetherian topological space. Then this proposition follows from the fact that a topological space covered by finite many noetherian subspace is notherian.

Proposition 1.3.31. Let X be a (locally) noetherian scheme and $U \subseteq X$ an open subscheme. Then U is (locally) noetherian.

Proof: In a noetherian topological space, every open subset is quasi-compact.

Proposition 1.3.32. Let X be a scheme. The mapping

$$X \longrightarrow \{Z \subseteq X; Z \text{ closed, irreducible }\}$$
$$x \mapsto \overline{\{x\}}$$

is a bijection, i. e. every irreducible closed subset contains a unique generic point.

Proof: Step 1: If Z is a closed irreducible subset of X and U is an affine open subset of X, $Z \cap U$ is irreducible. This is because, for W_1, W_2 be open subets of X and $Y_j = W_j \cap Z \cap U_i \neq \varnothing, j = 1, 2$, since Z is irreducible, the intersection of Y_1 and Y_2 is non-empty. Hence, $Z \cap U_i$ is irreducible.

Step 2: Since Z is closed, $\overline{Z \cap U} \subset Z$. Since Z is irreducible, $Z \cap U$ is a dense subset of Z. Then $\overline{Z \cap U} \cap Z = Z$.

Step 3: Since $Z \cap U$ is a irreducible closed subset of U, there's $x \in Z \cap U$ such that $\overline{\{x\}} \supset \overline{\{x\}} \cap U = Z \cap U$. Hence, $\{x\} \supset \overline{Z \cap U} = Z$.

Step 4: To show the uniqueness of x, consider $x, y \in X$ such that $\overline{\{x\}} = \overline{\{y\}} = Z$ and U be an open affine subset of X with $U \cap Z \neq \varnothing$. Since there's $z \in \overline{\{x\}} \cap U$, we have $x \in U$. Similarly, $y \in U$. Then, $\overline{\{x\}} \cap U = \overline{\{y\}} \cap U = Z \cap U$, by the uniqueness of affine case, $x = y$.

Corollary 1.3.33. X be a scheme, Z be a irreducible closed subset with generic point η, then there's a one-to-one order preserving correspondence between prime ideal of $\mathscr{O}_{X,\eta}$ and irreducible closed subset of X contains Z.

Proof: If $\eta \in U = \operatorname{Spec}(A)$ be a affine open neighborhood of η. Then prime ideal of $\mathscr{O}_{X,\eta}$ corresponds to prime ideal of A which is contained in $\mathfrak{p}_\eta$.

Let $Z_1 = \overline{\{\theta_1\}}, Z_2 = \overline{\{\theta_2\}}$ be two irreducible closed subset of X containing Z. Then θ_1, θ_2 lie in U. Hence, $Z_1 \cap U = Z_2 \cap U$ implies $Z_1 = Z_2$.

Notice that for a irreducible closed subset V contains Z, $V \cap U$ correspondes to a prime ideal of A contained in $\mathfrak{p}_\eta$, and by above proposition, this map is injective. Next step we show the map is surjective.

If W be a irreducible closed subet of U with $\eta \in W$, then $\overline{W}$ is a irreducible closed subset of X containing Z and $W = \overline{W} \cap U$.

Proposition 1.3.34. Let X be a scheme, let $f \in \Gamma(X, \mathscr{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathscr{O}_x$.

If $U = \operatorname{Spec} B$ is an open affine subscheme of X, and if $\bar{f} \in B = \Gamma(U, \mathscr{O}_X|_U)$ is the restriction of f, show that $U \cap X_f = D(\bar{f})$. Conclude that X_f is an open subset of X.

Proof: For $\mathfrak{p} \in \operatorname{Spec}(B)$, $\mathfrak{p} \in D(\bar{f})$ iff $\bar{f} \notin \mathfrak{p}$ iff $\bar{f}$ viewed as an element in $A_\mathfrak{p}$ does not lie in $\mathfrak{p}A_\mathfrak{p}$.

Proposition 1.3.35. (1) A scheme X is reduced if and only if for every open subset $U \subseteq X$ the ring $\Gamma(U, \mathscr{O}_X)$ is reduced.

(2) A non-empty scheme X is integral if and only if for every open subset $\emptyset \neq U \subseteq X$ the ring $\Gamma(U, \mathscr{O}_X)$ is an integral domain.

(3) If X is an integral scheme, then for all $x \in X$ the local ring $\mathscr{O}_{X,x}$ is an integral domain.

(4) An affine scheme $X = \operatorname{Spec} A$ is integral if and only if A is a domain.

(5) Let X be an integral scheme, and let $\eta \in X$ be its generic point. Then the local ring $\mathscr{O}_{X,\eta}$ is a field.

Proof: (1): Trivial.

(2): Let X be integral. Because all open subschemes of X are integral, too, it is enough to show that $\Gamma(X, \mathscr{O}_X)$ is a domain. Take $f, g \in \Gamma(X, \mathscr{O}_X)$ such that $fg = 0$. Then $\varnothing = X_f \cap X_g$ since $f_x g_x \in \mathfrak{m}_x$ for all $x \in X$. By the irreducibility we get $X_f = \varnothing$ or $X_g = \varnothing$. Assume $X_f = \varnothing$. We want to show that f must then be 0. We can check this locally on X, so we may assume that X is affine. Then f lies in the intersection of all prime ideals, i. e. in the nil-radical of the affine coordinate ring of X. Since X is reduced, by (1) the nil-radical is the zero ideal.

If conversely all $\Gamma(U, \mathscr{O}_X)$ are integral domains, then by (1) X is reduced. If there existed non-empty affine open subsets $U_1, U_2 \subseteq X$ with empty intersection, then the sheaf axioms imply that

$$\Gamma(U_1 \cup U_2, \mathscr{O}_X) = \Gamma(U_1, \mathscr{O}_X) \times \Gamma(U_2, \mathscr{O}_X)$$

But the product on the right hand side obviously contains zero divisors.

(3): Trivial.

(4): A is integral domain, then it has a unique minimal prime ideal. Hence, $\mathrm{Spec}(A)$ is irreducible. Since $A_{\mathfrak{p}}$ is a subring of $\mathrm{Frac}(A)$, $\mathrm{Spec}(A)$ is reduced scheme. Hence, $\mathrm{Spec}(A)$ is an integral scheme. By (2), if $\mathrm{Spec}(A)$ is an integer scheme, A is an integral domain.

(5): If η is a generic point of X, for all affine open subscheme U such that $\eta \in U$, η is a generic point of $U = \mathrm{Spec}(A)$. That is, η corresponds to (0) in A. Then, $\mathscr{O}_{X,x} \simeq A_{(0)} = \mathrm{Frac}(A)$ is a field.

Definition 1.3.36. Let X be an integral scheme, and let $\eta \in X$ be its generic point. Then the local ring $\mathscr{O}_{X,\eta}$ is a field, which is called the function field of X and denoted by $K(X)$.

Proposition 1.3.37. X be an integral scheme with generic point η.

(1) Let $U \subseteq V \subseteq X$ be non-empty open subsets. Then the maps

$$\Gamma(V, \mathscr{O}_X) \xrightarrow{\mathrm{res}^V_U} \Gamma(U, \mathscr{O}_X) \xrightarrow{f \mapsto f_\eta} K(X)$$

(2) For all $x \in X$, there's a canonical injective map $\mathscr{O}_{X,x} \to \mathscr{O}_{X,\eta}$ gievn by $[s] \mapsto [s]$ and under this map, $\mathrm{Frac}(\mathscr{O}_{X,x}) = \mathscr{O}_{X,\eta}$.

(3) For every non-empty open subset $U \subseteq X$ and for every covering $U = \bigcup_i U_i$ by non-empty open subsets U_i we have

$$\Gamma(U, \mathscr{O}_X) = \bigcap_i \Gamma(U_i, \mathscr{O}_X) = \bigcap_{x \in U} \mathscr{O}_{X,x},$$

where the intersection takes place in $K(X)$.

Proof: (1): It suffice to show the map $f \mapsto f_\eta$ is injective. Since $f_\eta = 0$ is equivalent to $f|_W = 0$ for all W open affine subscheme of U, we may assume U is an affine open subscheme. Consider the following commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{X,\eta} \arrow[r, "\simeq"] & \mathscr{O}_{\mathrm{Spec}A,(0)} \arrow[r, "\simeq"] & \mathrm{Frac}(A) \\
\mathscr{O}_X(U) \arrow[r, "\simeq"] \arrow[u] & \mathscr{O}_{\mathrm{Spec}A}(\mathrm{Spec}A) \arrow[r, "\simeq"] \arrow[u] & A \arrow[u]
\end{tikzcd}$$

Since $A \to \mathrm{Frac}(A)$ is injective, we have $f = 0$.

(2): By (1) and the following diagram

$$\begin{tikzcd}
\mathscr{O}_{X,\eta} \arrow[r, "\simeq"] & \mathscr{O}_{\mathrm{Spec}A,(0)} \arrow[r, "\simeq"] & \mathrm{Frac}(A) \\
\mathscr{O}_{X,x} \arrow[r, "\simeq"] \arrow[u] & \mathscr{O}_{\mathrm{Spec}A,\mathfrak{p}} \arrow[r, "\simeq"] \arrow[u] & A_\mathfrak{p} \arrow[u]
\end{tikzcd}$$

(3): Consider the following commutative diagram

$$\begin{tikzcd}
\Gamma(U, \mathscr{O}_X) \arrow[r] & \Gamma(U_i, \mathscr{O}_X) \arrow[r] \arrow[dr] & \mathscr{O}_{X,\eta} \\
& & \Gamma(U_i \cap U_j, \mathscr{O}_X) \arrow[u]
\end{tikzcd}$$

and notice that $\mathscr{O}_X$ is a sheaf.

Notice we define locally finite type and finite type k-scheme. The morphisms below are all in the category (Sch/k).

Definition 1.3.38. Let k be a field, and let $X \to \operatorname{Spec} k$ be a k-scheme. We call X a k-scheme locally of finite type or say that X is locally of finite type over k, if there is an affine open cover $X = \bigcup_{i \in I} U_i$ such that for all i, there's a k-algebra A_i such that

$$(U_i, \mathscr{O}_X|_{U_i}) \simeq (\mathrm{Spec}(A_i), \mathscr{O}_{\mathrm{Spec}(A_i)})$$

as k-scheme. We say that X is of finite type over k if X is locally of finite type and quasi-compact.

Proposition 1.3.39. Every (locally) finite type k-scheme is (locally) noetherian.

Proposition 1.3.40. Let X be a locally noetherian scheme. Prove that the set of irreducible components of X is locally finite (every point of X has an open neighborhood which meets only finitely many irreducible components of X).

Proof: Take $x \in U = \mathrm{Spec}(A)$ with A noetherian. Assume $Z_i, i = 1, \ldots, n$ be irreducible components of U and $\overline{Z_i}$ is contained in an irreducible component V_i of X. Then $Z_i = \overline{Z_i} \cap U \subset V_i \cap U$. Since $V_i = \overline{\{\theta_i\}}$, $V_i \cap U$ is the closure of θ_i in U hence irreducible closed in U. Since Z_i is maximal, $Z_i = V_i \cap U$.

Take $Z = \overline{\{y\}}$ be irreducible component of X such that $x \in Z$, then $y \in U$. Hence, $\overline{\{y\}} \cap U \subset \overline{\{\theta_i\}} \cap U$ for some i, hence $\{y\} \subset \overline{\{y\}} \cap U \subset \overline{\{\theta_i\}}$. Therefore, we have $Z = V_i$.

Proposition 1.3.41. Let X be a k-scheme locally of finite type and let $U \subseteq X$ be an open affine subset. Then the k-algebra $\Gamma\left(U, \mathscr{O}_X\right)$ is a finitely generated k-algebra.

Proof: Let $B=\Gamma\left(U, \mathscr{O}_X\right)$. Since the localization of a finitely generated k-algebra with respect to a single element is again finitely generated, we see, by Nike's Trick, that we can cover U by finitely many(since spectrum of a ring is compact) principal open subsets $D\left(f_i\right), f_1, \ldots, f_n \in B$, such that all localizations B_{f_i} are finitely generated k-algebras. The claim now follows from Algebra Proposition 2.4.30

Proposition 1.3.42. Let k be a field, let X be a k-scheme locally of finite type, and let $x \in X$. Then the following assertions are equivalent.

(1) The point $x \in X$ is closed.

(2) The field extension $k \hookrightarrow \kappa(x)$ is finite.

(3) The field extension $k \hookrightarrow \kappa(x)$ is algebraic.

Proof: (1) implies (2): Take U with $x \in U$ and there's k-scheme

$$(U, \mathscr{O}_X|_U) \simeq (\operatorname{Spec}(A), \mathscr{O}_{\operatorname{Spec}(A)})$$

where A be a finitely generated k-algebra and x corresponds to a maximal ideal $\mathfrak{m}$ of A. Consider the follow commutative diagram

$$\begin{tikzcd}
\kappa(x) \arrow[rr, "\simeq"] & & A/\mathfrak{m} \\
\mathscr{O}_{X,x} \arrow[u] \arrow[rr, "\simeq"] & & A_{\mathfrak{m}} \arrow[u] \\
\Gamma(U, \mathscr{O}_X) \arrow[u] \arrow[rr, "\simeq"] & & A \arrow[u] \\
& k \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Since $A/\mathfrak{m}$ is a field and finite generated k-algebra, by Algebra 2.8.2, $\kappa(x)$ is a finte extension of k.

(3) implies (1): Again take U with $x \in U$ and there's k-scheme

$$(U, \mathscr{O}_X|_U) \simeq (\operatorname{Spec}(A), \mathscr{O}_{\operatorname{Spec}(A)})$$

where A be a finitely generated k-algebra and x corresponds to a prime ideal $\mathfrak{p}$ of A.

$$\begin{tikzcd}
\kappa(x) \arrow[rr, "\simeq"] & & A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \arrow[r, "\simeq"] & \operatorname{Frac} A/\mathfrak{p} \\
\mathscr{O}_{X,x} \arrow[u] \arrow[rr, "\simeq"] & & A_{\mathfrak{p}} \arrow[u] & A/\mathfrak{p} \arrow[u] \\
\Gamma(U, \mathscr{O}_X) \arrow[u] \arrow[rr, "\simeq"] & & A \arrow[u] \arrow[ur] & \\
& k \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

Since $\kappa(x)$ is algebraic over k, $A/\mathfrak{p}$ is integral over k. Hence $\mathfrak{p}$ is a closed point in U. Consider all such U, we have x is closed in X.

Corollary 1.3.43. Let k be algebraically closed and let X be a k-scheme locally of finite type. Then

$$\{x \in X; x \text{ closed }\} = \{x \in X; k = \kappa(x)\} = \mathrm{Hom}_k(\mathrm{Spec}\, k, X),$$

Proof: Field extension $k \to \kappa(x)$ is an isomorphism if and only if there's k-algebra homomorphism $\kappa(x) \to k$. And if there's k-algebra homomorphism $\kappa(x) \to k$, it is obviously unique.

Example 1.3.44. $\mathbb{P}^n_k$ is an integral, finite type scheme over k.

Proof: reduced: $\mathbb{P}^n_k$ is reduced since for all $x \in \mathbb{P}^n_k$, we may find $i \in \{0, \ldots, n\}$ such that $x \in U_i = D_+(x_i)$. Then $\mathscr{O}_{\mathbb{P}^n_k,x}$ is a localization of a polynomial ring at a prime ideal, hence reduced.

irreducible: $D_+(f) \cap D_+(g) = D_+(fg)$ and notice that for all $h \in k[x_0, \ldots, x_n]_+$, $D_+(h)$ is non-empty.

locally finite type: trivial

quasi-compact: $\mathbb{P}^n_k$ is a finite union of compact open subset U_i.

Example 1.3.45. For $X = \mathrm{Spec}(\mathbb{Q}[x,y]/(x^n + y^n - 1))$ be a $\mathbb{Q}$-scheme, then

$$X(\mathbb{R}) = \mathrm{Hom}_{\mathrm{Spec}(\mathbb{Q})}(X, \mathrm{Spec}(\mathbb{R})) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[x,y]/(x^n+y^n-1), \mathbb{R}) = \left\{(x,y) \in \mathbb{R}^2 : x^n + y^n = 1\right\}$$

and

$$X(\mathbb{Q}) = \mathrm{Hom}_{\mathrm{Spec}(\mathbb{Q})}(X, \mathrm{Spec}(\mathbb{Q})) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[x,y]/(x^n+y^n-1), \mathbb{Q}) = \left\{(x,y) \in \mathbb{Q}^2 : x^n + y^n = 1\right\}$$

Moreover, since closed point corresponds to the maximal ideal of $\mathbb{Q}[x,y]/(x^n + y^n - 1)$ and $X(\mathbb{Q})$ be those maximal ideals $\mathfrak{m}$ of $\mathbb{Q}[x,y]$ which contain $x^n + y^n - 1$ and have a $\mathbb{Q}$-algebra isomorphism $\mathbb{Q}[x,y]/\mathfrak{m} \to \mathbb{Q}$. Therefore, $\mathfrak{m}$ is of the form $(x - x_0, y - y_0)$ where (x_0, y_0) be a solution of $x^n + y^n = 1$.

1.4 Immersions

Definition 1.4.1. A morphism $j : Y \to X$ of schemes is called an open immersion, if the underlying continuous map is a homeomorphism of Y with an open subset U of X, and the sheaf homomorphism $\mathscr{O}_X \to j_*\mathscr{O}_Y$ induces an isomorphism $\mathscr{O}_{X|U} \cong (j_*\mathscr{O}_Y)_{|U}$ (of sheaves on U).

Remark 1.4.2. There's a natural one-to-one correspondence between open immersion and open subscheme.

Definition 1.4.3. Given a scheme $(X, \mathscr{O}_X)$, we call a subsheaf $\mathscr{J} \subseteq \mathscr{O}_X$ a sheaf of ideals, if for every open subset $U \subseteq X$ the sections $\Gamma(U, \mathscr{J})$ are an ideal in $\Gamma(U, \mathscr{O}_X)$. The quotient sheaf $\mathscr{O}_X/\mathscr{J}$ is defined as the sheafification of the presheaf $U \mapsto \mathscr{O}_X(U)/\mathscr{J}(U)$. It is a sheaf of rings. The canonical projection $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{J}$ is surjective.

Definition 1.4.4. Let X be a scheme.

(1) A closed subscheme of X is given by a closed subset $Z \subseteq X$ with inclusion map $i : Z \to X$ and an ideal sheaf $\mathscr{J} \subseteq \mathscr{O}_X$ such that $Z = \{x \in X : (\mathscr{O}_X/\mathscr{J})_x \neq 0\}$ and $(Z, i^{-1}\mathscr{O}_X/\mathscr{J})$ is a scheme.

(2) A morphism $i : (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$ of schemes is called a closed immersion, if the underlying continuous map is a homeomorphism between Z and a closed subset of X, and the sheaf homomorphism $i^b : \mathscr{O}_X \to i_*\mathscr{O}_Z$ is surjective.

Proposition 1.4.5. X be a scheme and Z be a closed subscheme associated to ideal sheaf $\mathscr{J}$. Then, the morphism of ringed space $(Z, i^{-1}\mathscr{O}_X/\mathscr{J}) \to (X, \mathscr{O}_X)$ induced by the natural projection $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{J}$ and the isomorphism $\mathscr{O}_X/\mathscr{J} \to i_*i^{-1}\mathscr{O}_X/\mathscr{J}$ is a morphism of locally ringed space and closed immersion.

Proof: Step 1: The stalk of the morphism of sheaves $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{J}$ is a local ring homomorphism.

It's clear for the case when $x \notin Z$, since $(\mathscr{O}_X/\mathscr{J})_x = 0$. For $x \in Z$, since the stalk of the presheaf $U \mapsto \mathscr{O}_X(U)/\mathscr{J}(U)$ at x is $\mathscr{O}_{X,x}/\mathscr{J}_x$ where $\mathscr{J}_x \neq \mathscr{O}_{X,x}$. And notice that the projection $\mathscr{O}_{X,x} \to \mathscr{O}_{X,x}/\mathscr{J}_x$ is a local ring homomorphism.

Step 2: $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{J}$ is surjective.

By taking stalks, it suffice to show $\mathscr{O}_X(U) \to \mathscr{O}_X(U)/\mathscr{J}(U)$ is surjective for all U open in X.

Proposition 1.4.6. If $i : (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$ is a closed immersion, consider the kernel of the morphism of sheaves $\varphi : \mathscr{O}_X \to i_*\mathscr{O}_Z$. It's clear that $\mathrm{Ker}\varphi$ is an ideal sheaf. By Proposition 1.1.9, the natural morphism

$$\mathscr{O}_X/\mathrm{Ker}\varphi \to i_*\mathscr{O}_Z$$

is an isomorphism of sheaves.

Moreover, since $(Z, \mathscr{O}_Z)$ is a scheme and Z is closed in X, $\mathrm{Supp}(i_*\mathscr{O}_Z) = Z$. Hence the support of $\mathscr{O}_X/\mathrm{Ker}\varphi$ is Z. Then by Proposition 1.1.22, a closed immersion induces a closed subscheme.

Theorem 1.4.7 (closed subscheme of affine scheme)**.** Let $X = \operatorname{Spec} A$ be an affine scheme. $i : (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$ be a closed immersion. Then the global section map

$$\varphi : A \to \Gamma(Z, \mathscr{O}_Z)$$

induces a commutative diagram of scheme:

$$\begin{tikzcd}
\mathrm{Spec}A & & Z \arrow[ll, "i"'] \arrow[dll, "\psi"] \\
\mathrm{Spec}A/\ker\varphi \arrow[u, "\pi"] & &
\end{tikzcd}$$

Then ψ is an isomorphism of scheme.

Proof: Since i is an closed immersion, ψ is closed and injective. Hence, ψ is also a closed immersion. To prove ψ is surjective, it suffices to show the following lemma:

Lemma 1.4.8. Let $X = \operatorname{Spec} A$ be an affine scheme. $i : (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$ be a closed immersion such that the induced map on global section $\varphi : A \to \Gamma(Z, \mathscr{O}_Z)$ is injective. Then, i is surjective.

Proof of the lemma: Assume $X - Z$ is non-empty. Let $s \in A$ with $\varnothing \neq D(s) = X_s \subset X - Z$. Then $Z \subset X - X_s$. Hence, $Z \subset Z \cap (X - X_s)$. For all $x \in Z$, we have the following commutative diagram

$$\begin{tikzcd}
\Gamma(Z, \mathscr{O}_Z) \arrow[d, leftrightarrow] & A \arrow[l] \arrow[d] \\
\mathscr{O}_{Z,x} = i_*(\mathscr{O}_Z)_x & \mathscr{O}_{X,x} \arrow[l]
\end{tikzcd}$$

Hence, $Z \subset Z \cap (X - X_s) \subset Z - Z_{\varphi(s)}$. If $U \subseteq Z$ is open, such that $\left(U, \mathscr{O}_{Z|U}\right) \simeq \operatorname{Spec}(B)$ is affine. By Proposition 1.3.34, $U \subset \operatorname{Spec}(B) - D(\varphi(s)|_U)$. Hence, $\varphi(s)|_U$ is nilpotent. Moreover, since Z can be covered by finite many affine open subscheme, there's some sufficiently large N such that $\varphi(s)^N$ is nilpotent. Hence, $s^N = 0$. It contradicts to $\varnothing \neq X_s$. □

To show that ψ is an isomorphism of scheme. We still need the following lemma

Lemma 1.4.9. Let $X = \operatorname{Spec} A$ be an affine scheme. $(i, i^b) : (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$ be a closed immersion such that the induced map on global section $\varphi : A \to \Gamma(Z, \mathscr{O}_Z)$ is injective. Then, i^b is injective.

Proof of the lemma: For $x \in X, \mathscr{O}_{X,x} = A_{\mathfrak{p}_x}$, and we see that it is enough to show that every element of $\operatorname{Ker}(\mathscr{O}_{X,x} \to \mathscr{O}_{Z,x})$ of the form $g/1$ is 0 in $\mathscr{O}_{X,x}$. Given g, we cover $Z = U \cup \bigcup_{i \in I} U_i$ by finitely many open subsets U, U_i, such that: (1) The schemes $\left(U, \mathscr{O}_{Z|U}\right)$ and $\left(U_i, \mathscr{O}_{Z|U_i}\right)$ for all i are affine. (2) We have $x \in U$ and $\varphi(g)_{|U} = 0$.

Choose $s \in A$ with $x \in D(s) \subseteq U$. If we can show that $\varphi\left(s^N g\right) = 0$ for some N, then $s^N g = 0$ because φ is injective, and it follows that $g/1 = 0$ in $\mathscr{O}_{X,x}$, as desired, since s is a unit in $\mathscr{O}_{X,x}$. Since $\varphi(g)_{|U} = 0$ by assumption, we have $\varphi(sg)_{|U} = 0$. Now I is finite, so we can search a suitable N for each U_i separately. Because

$$D_{U_i}\left(\varphi(s)_{|U_i}\right) = Z_{\varphi(s)} \cap U_i \subset D(s) \cap U_i$$

, we obtain $\varphi(g)_{|D_{U_i}\left(\varphi(s)_{|U_i}\right)} = 0$. In other words, the image of $\varphi(g)$ in the localization $\Gamma\left(U_i, \mathscr{O}_Z\right)_{\varphi(s)|_{U_i}}$ is 0. □

Definition 1.4.10 (immersion)**.** (1) Let X be a scheme. A subscheme of X is a scheme $(Y, \mathscr{O}_Y)$, such that $Y \subseteq X$ is a locally closed subset, and such that Y is a closed subscheme of the open subscheme $U \subseteq X$, where U is the largest open subset of X which contains Y and in which Y is closed. We then have a natural morphism of schemes $Y \to X$.

(2) An immersion $i : Y \to X$ is a morphism of schemes whose underlying continuous map is a homeomorphism of Y onto a locally closed subset of X, and such that for all $y \in Y$ the ring homomorphism $i_y^\sharp : \mathscr{O}_{X,i(y)} \to \mathscr{O}_{Y,y}$ between the local rings is surjective.

It's easy to check there's one-to-one correspondence between immersion and open subscheme.

Definition 1.4.11. Let k be a field.

(1) A k-scheme X is called projective, if there exist $n \geq 0$ and a closed immersion $X \hookrightarrow \mathbb{P}_k^n$.

(2) A k-scheme X is called quasi-projective, if there exist $n \geq 0$ and an immersion $X \hookrightarrow \mathbb{P}_k^n$.

Definition 1.4.12. We say that a proposition is local on the target if for every morphism $f : X \to Y$ of schemes and for every open covering $Y = \bigcup_{j \in J} V_j$ the morphism f possesses the proposition if and only if $f_{|f^{-1}(V_j)} : f^{-1}(V_j) \to V_j$ possesses the proposition for all $j \in J$.

Proposition 1.4.13. Open immersion, closed immersion, immersion are stable under base change and composition.

Proposition 1.4.14. Open immersion, closed immersion, immersion are local on target.

Proposition 1.4.15. Every affine k-scheme X of finite type is quasi-projective: Indeed, let $X = \operatorname{Spec} A$, where $A \cong k[T_1, \ldots, T_n]/\mathfrak{a}$. Therefore there exists a closed immersion $i : X \to \mathbb{A}_k^n$. Moreover, projective space $\mathbb{P}_k^n$ is covered by open subschemes which are isomorphic to $\mathbb{A}_k^n$. Hence, the composition $j \circ i$ is then an immersion $X \to \mathbb{P}_k^n$.

Example 1.4.16. Consider $X = \text{Proj}\mathbb{C}[x, y, z]/(zy^2 - (4x^3 - a_2xz^2 - a_3z^3))$ with $a_2^3 - 27a_2^2 \neq 0$ as a $\mathbb{C}$- scheme. Firstly, the natural morphism $X \to \mathbb{P}_{\mathbb{C}}^2$ is a closed immersion. Moreover, consider the $\mathbb{C}$-points $X(\mathbb{C})$ of X. We have

$$X(\mathbb{C}) = \{\infty\} \cup \left\{(x, y) \in \mathbb{C}^2 : y^2 = 4x^3 - a_2x - a_3\right\}$$

where ∞ represents the point (x, z) in Proj $\mathbb{C}[x, y, z]/(zy^2 - (4x^3 - a_2xz^2 - a_3z^3))$.

Now we show that X is an integral, projective , $\mathbb{C}$-finite type scheme.

irreducible: show that $D_+(z) \cap D_+(f) \neq \varnothing$ for all $D_+(f) \neq \varnothing$.

reduced: It suffice to show

$$\text{Spec}(\mathbb{C}[x, y]/(y^2 - (4x^3 - a_2x - a_3)))$$

is integral and

$$\text{Spec}(\mathbb{C}[x, z]/(z - (4x^3 - a_2xz^2 - a_3z^3)))$$

is integral.

$\mathbb{C}$-finite type: trivial.

Definition 1.4.17 (reduced subscheme of a scheme)**.** Let X be a scheme. Let $T \subset X$ be a closed subset. There exists a closed subscheme $Z \subset X$ with the following properties:

(1) the underlying topological space of Z is equal to T,

(2) Z is reduced.

If $T = X$, we usually denote the resulting closed subscheme by X_{red}.

Proof: Let $\mathcal{I} \subset \mathcal{O}_X$ be the sub presheaf defined by the rule

$$\mathcal{I}(U) = \{f \in \mathcal{O}_X(U) \mid f(t) = 0 \text{ for all } t \in T \cap U\}$$

Here we use $f(t)$ to indicate the image of f in the residue field $\kappa(t)$ of X at t. Because of the local nature of the condition it is clear that $\mathcal{I}$ is a sheaf of ideals. It's easy to check the stalk of $\mathscr{O}_X/\mathcal{I}$ at $x \notin Z$ vanishes. And for $x \in Z$, there's open subscheme $x \in U = \operatorname{Spec}(R)$ where x correspondes to prime ideal $\mathfrak{p}$.

Let I be the unique radical ideal correspondes to closed subset $Z \cap U$. It's easy to check

$$(\mathcal{O}_X/\mathcal{I})_x \simeq R_\mathfrak{p}/I_\mathfrak{p} = (R/I)_\mathfrak{p}$$

Hence, $(Z \cap U, i^{-1}(\mathscr{O}_X/\mathcal{I})|_{U \cap Z}) \simeq \operatorname{Spec}(A/I)$. So, $(Z, i^{-1}(\mathscr{O}_X/\mathcal{I}))$ is a reduced, closed subscheme of X with under lying topological space T.

Corollary 1.4.18. X be a scheme. Z be a closed, quasi-compact subset. Then there's a closed point in Z.

Proof: By Proposition 1.4.17, it suffice to show for a quasi-compact scheme X, there's a closed point in X. Assume X is covered by affine open subscheme $U_i, i = 1, \ldots, n$ and U_i is not contained in

$$\bigcup_{j \neq i} U_j$$

Since U_1 is affine, there's p closed in U_1. Notice that

$$U_1 \cap (\bigcup_{j \neq 1} U_j)^c = \varnothing$$

, we have p closed in X.

Proposition 1.4.19 (category of schemes to the category of reduced schemes)**.** For every morphism of schemes $f : X \to Y$ there exists a unique morphism of schemes $f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$ such that commutes, where i_X and i_Y are the canonical inclusion morphisms.

$$\begin{tikzcd} X_{\text{red}} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y_{\text{red}} \arrow[r] & Y \end{tikzcd}$$

If $g : Y \to Z$ is a second morphism of schemes, we have $(g \circ f)_{\text{red}} = g_{\text{red}} \circ f_{\text{red}}$.

1.5 Fibered Products

Proposition 1.5.1. Let S be a scheme and let X and Y be two S-schemes. Then the fiber product $X \times_S Y$ exists in the category of schemes.

Proposition 1.5.2. Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Let $(X \times_S Y, p, q)$ be the fibre product. Suppose that $U \subset S, V \subset X, W \subset Y$ are open subschemes such that $f(V) \subset U$ and $g(W) \subset U$. Then the canonical morphism $V \times_U W \to X \times_S Y$ is an open immersion which identifies $V \times_U W$ with $p^{-1}(V) \cap q^{-1}(W)$.

Corollary 1.5.3. Let k be a field and let X and Y be k-schemes (locally) of finite type. Then $X \times_k Y$ is (locally) of finite type over k.

Example 1.5.4. Let $A \leftarrow R \to B$ be homomorphisms of rings, let $S = \operatorname{Spec}(R)$, $X = \operatorname{Spec}(A)$, and $Y = \operatorname{Spec}(B)$. Set $Z = \operatorname{Spec}(A \otimes_R B)$ and let $p : Z \to X$ and $q : Z \to Y$ be the morphisms of schemes corresponding to the ring homomorphisms

$$\begin{aligned} \alpha : A \to A \otimes_R B, &\quad a \mapsto a \otimes 1 \\ \beta : B \to A \otimes_R B, &\quad b \mapsto 1 \otimes b \end{aligned}$$

Then (Z, p, q) is a fiber product of X and Y over S in the category of schemes.

Definition 1.5.5 (Relative Frobenius)**.** Let p be a prime number and let S be a scheme over $\mathbb{F}_p$. We denote by $\mathrm{Frob}_S : S \to S$ the absolute Frobenius of S : Frob_S is the identity on the underlying topological spaces and Frob_S^b is the map $x \mapsto x^p$ on $\Gamma(U, \mathscr{O}_S)$ for all open subsets U of S.

Now let $f : X \to S$ be an S-scheme. Note that Frob_X is in general not an S-morphism. Instead of the absolute Frobenius we therefore introduce a relative variant.

$$\begin{tikzcd}
X \arrow[dr, dashed, "F_{X/S}"] \arrow[drr, "\mathrm{Frob}_X"] \arrow[ddr, "f"'] & & \\
& X^{(p)} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& S \arrow[r, "\mathrm{Frob}_S"'] & S
\end{tikzcd}$$

Let $X^{(p)}$ be the fiber product of $S \xrightarrow{\mathrm{Frob}_S} S$ and $X \to S$, then $F_{X/S}$ is called relative Frobenius of X over S.

Example 1.5.6. Let $\mathbb{F}_q = \mathbb{F}_{p^n}$ be a finite field over $\mathbb{F}_p$. If $f = \sum a_\alpha x^\alpha \in \mathbb{F}_q[x_1, \ldots, x_n]$, define $f^{(p)} = \sum a_\alpha^p x^\alpha$. Assume X is a scheme, consider the following commutative diagram

$$\begin{tikzcd}
\Gamma(X, \mathscr{O}_X) & & \\
& \mathbb{F}_q[x_1, \ldots, x_n]/(f_j^{(p)}) \arrow[ul, dashed, "h"] & \mathbb{F}_q[x_1, \ldots, x_n]/(f_j) \arrow[l, "f^{(p)} \leftarrow f"'] \arrow[ull, "f"'] \\
& \mathbb{F}_q \arrow[u, "\mathrm{id}"] \arrow[uul, bend left, "g"] & \mathbb{F}_q \arrow[l, "x^p \leftarrow x"'] \arrow[u]
\end{tikzcd}$$

where h is defined by $h(\alpha_i x_i) = g(\alpha_i)f(x_i)$. This shows that $\mathrm{Spec}(\mathbb{F}_q[x_1, \dots, x_n]/(f_j^{(p)}))$ is the fiber product of $\mathrm{Spec}(\mathbb{F}_q) \xrightarrow{\text{Frob}} \mathrm{Spec}(\mathbb{F}_q)$ and $\mathrm{Spec}(\mathbb{F}_q[x_1, \dots, x_n]/(f_j)) \to \mathrm{Spec}(\mathbb{F}_q)$.

In particular, if $X = \mathrm{Spec}(\mathbb{F}_q[x_1, \dots, x_n]/(f_j))$, $f = \mathrm{Frob}_X$ and $g = \mathrm{id}$, then h is a $\mathbb{F}_p$-algebra homomorphism such that $h(x_i) = x_i^p$.

$$\begin{tikzcd}
\mathbb{F}_q[x_1, \dots, x_n]/(f_j) & & \\
& \mathbb{F}_q[x_1, \dots, x_n]/(f_j^{(p)}) \arrow[ul, dashed, "x_i \to x_i^p"'] & \mathbb{F}_q[x_1, \dots, x_n]/(f_j) \arrow[l, "f^{(p)} \leftarrow f"] \arrow[ull, "f \to f^p"'] \\
& \mathbb{F}_q \arrow[u, "\mathrm{id}"] \arrow[uul, bend left, "\mathrm{id}"] & \mathbb{F}_q \arrow[l, "x^p \leftarrow x"] \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

That is, although $f \to f^p$ can factor through two $\mathbb{F}_p$-algebra homomorphisms.

Example 1.5.7. Since fiber product exists in category of scheme, consider a morphism $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ and a $\mathbb{R}$-scheme Y, we have

$$\mathrm{Hom}_{\mathrm{Spec}(\mathbb{R})}(\mathrm{Spec}(\mathbb{C}), Y) = \mathrm{Hom}_{\mathrm{Spec}(\mathbb{C})}(\mathrm{Spec}(\mathbb{C}), Y \times_{\mathbb{R}} \mathrm{Spec}(\mathbb{C}))$$

Example 1.5.8. If field K is a extension of k, consider a morphism $\mathrm{Spec}(\mathbb{K}) \to \mathrm{Spec}(\daleth)$ and k-schemes X, Y, we have

$$\mathrm{Hom}_k(\mathrm{Spec}(K), X \times_k Y) = \mathrm{Hom}_k(\mathrm{Spec}(K), X) \times \mathrm{Hom}_k(\mathrm{Spec}(K), Y)$$

Proposition 1.5.9. Let S be a scheme, X and Y two S-schemes, and let $f : X' \to X$ be a morphism of S-schemes. Let g be the morphism induced by universal property of fiber product

$$\begin{tikzcd}
Z' = X' \times_S Y \arrow[r] \arrow[d] \arrow[dr, dashed, "g"] & Y \arrow[dr] & \\
X' \arrow[dr, "f"] & Z = X \times_S Y \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

Then all squares in the following diagram are cartesian

$$\begin{tikzcd}
Z' = X' \times_S Y \arrow[r, dashed, "g"] \arrow[d, "p'"] & Z = X \times_S Y \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d] \\
X' \arrow[r, "f"] & X \arrow[r] & S
\end{tikzcd}$$

In addition, assume that $f : X' \to X$ can be written as the composition of scheme morphisms which satisfy the following condition: each morphism is a homeomorphism onto its image and also satisfies one of the assumptions (1), (2):

(1) For each point $x' \in X'$, the homomorphism $f_{x'}^{\sharp} : \mathscr{O}_{X,f(x')} \to \mathscr{O}_{X',x'}$ is surjective, and there exists an open affine neighborhood V of $f(x')$ such that $f^{-1}(V)$ is quasi-compact.

(2) For each point $x' \in X'$, the homomorphism $f^\sharp_{x'} : \mathscr{O}_{X,f(x')} \to \mathscr{O}_{X',x'}$ is bijective.

Then, the morphism g is a homeomorphism of Z' onto

$$g\left(Z'\right) = p^{-1}\left(f\left(X'\right)\right)$$

Besides, for all $z' \in Z'$, consider following diagram

$$\begin{tikzcd}
\mathscr{O}_{Z',z'} & \mathscr{O}_{Z,g(z')} \arrow[l, "g^\sharp_{z'}"] \\
\mathscr{O}_{X',p'(z')} \arrow[u] & \mathscr{O}_{X,p(g(z'))} \arrow[l, "f^\sharp_{p'(z')}"] \arrow[u, "p^\sharp_{g(z')}"']
\end{tikzcd}$$

induced by the "left square" of above diagram. We have the homomorphism $g^\sharp_{z'}$, is surjective and its kernel is generated by the image of the kernel of $f^\sharp_{p'(z')}$ under $p^\sharp_{g(z')}$.

Example 1.5.10. The following f satisfying above assumption

(1) f is an immersion of schemes

(2) f is the canonical morphism $\operatorname{Spec} \mathscr{O}_{X,x} \to X$ for some point $x \in X$.

(3) f is the canonical morphism $\operatorname{Spec} \kappa(x) \to X$ for some point $x \in X$.

Proof:

Definition 1.5.11 (fibers of morphism)**.** Consider the natural morphism $\operatorname{Spec} \kappa(s) \to S$ and a morphism $f : X \to S$. We define $X_s = X \times_S \operatorname{Spec} \kappa(s)$ be the fiber of $f : X \to S$ in s.

$$\begin{tikzcd}
X \times_S \operatorname{Spec} \kappa(s) \arrow[r] \arrow[d] \arrow[dr, dashed, "g"] & X \arrow[dr, "\mathrm{id}"] & \\
\operatorname{Spec} \kappa(s) \arrow[dr, "0 \mapsto s"'] & X \arrow[r, "\mathrm{id}"] \arrow[d, "f"] & X \arrow[d] \\
& S \arrow[r] & S
\end{tikzcd}$$

By Proposition 1.5.9, the underlying topological space of X_s is $f^{-1}(s)$.

Example 1.5.12. Consider a integral k-scheme of finite type be

$$X = \operatorname{Spec} k[U,T,S]/(UT - S)$$

Let $f : X \to \mathbb{A}^1_k = \operatorname{Spec} k[S]$ be the natural morphism, then $\operatorname{Spec} A_s$ be the fiber of f in $(S - s)$ where

$$A_s = k[U,T,S]/(UT - S) \otimes_{k[S]} k[S]/(S - s) = k[U,T,S]/(UT - S, S - s) = k[U,T]/(UT - s)$$

Hence, $X_s(k) = \{(x, y) \in k^2 : xy = s\}$

Definition 1.5.13 (inverse image of Z under f)**.** Let $f : X \to Y$ be a morphism of schemes and let $i : Z \to Y$ be an immersion. Proposition 1.5.9 shows that the base change $i_{(X)} : Z \times_Y X \to X$ is surjective on stalks and a homeomorphism of $Z \times_Y X$ onto the locally closed subspace $f^{-1}(Z)$.

$$\begin{tikzcd}
Z \times_S Y \arrow[r] \arrow[d] \arrow[dr, dashed, "i_{(X)}"] & X \arrow[dr, "\mathrm{id}"] & \\
Z \arrow[dr, "i"'] & X \arrow[r] \arrow[d, "f"'] & X \arrow[d, "f"] \\
 & Y \arrow[r] & Y
\end{tikzcd}$$

Therefore $i_{(X)}$ is an immersion.

Proposition 1.5.14. In above definition, if Z is closed subscheme of Y, then $f^{-1}(Z)$ is closed which implies $i_{(X)}$ is a closed immersion. By the second result of Proposition **??**, if i is open immersion, $i_{(X)}$ is also open immersion.

Definition 1.5.15 (intersection of subscheme)**.** As a special case of the inverse image of a subscheme we can define the intersection of two subschemes: Let $i : Y \to X$ and $j : Z \to X$ be two subschemes.

$$\begin{tikzcd}
Z \times_X Y \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, dashed, "i_{(Z)}=p"] & Z \arrow[dr, "\mathrm{id}"] & \\
Y \arrow[dr, "i"'] & Z \arrow[r] \arrow[d, "j"'] & Z \arrow[d, "f"] \\
 & X \arrow[r] & X
\end{tikzcd}$$

Then the map $j \circ p$ is an immersion onto the locally closed subset $Y \cap X$.

Definition 1.5.16. For an arbitrary scheme S, define $\mathbb{A}^n_S = \mathbb{A}^n_{\mathbb{Z}} \times_{\mathbb{Z}} S$, $\mathbb{P}^n_S = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} S$.

Example 1.5.17. Suppose $I \subset A[x_1, \dots, x_m]$ and $J \subset A[y_1, \dots, y_n]$ are ideals.

$$A[x_1, \dots, x_m]/I \otimes_A A[y_1, \dots, y_n]/J \simeq A[x_1, \dots, x_m, y_1, \dots, y_n]/(I, J).$$

In particular, $\mathbb{A}^n_k \times_k \mathbb{A}^m_k = \mathbb{A}^{m+n}_k$

Proof: The bi-linear map

$$A[x_1, \dots, x_m]/I \times A[y_1, \dots, y_n]/J \to A[x_1, \dots, x_m, y_1, \dots, y_n]/(I, J)$$
$$(f + I, g + J) \to fg + (I, J)$$

induces an isomorphism

$$A[x_1, \dots, x_m]/I \otimes_A A[y_1, \dots, y_n]/J \simeq A[x_1, \dots, x_m, y_1, \dots, y_n]/(I, J).$$

Example 1.5.18. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \times \mathbb{C}$ as $\mathbb{R}$-algebra.

Proof:

$$\begin{aligned}
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\cong \mathbb{C} \otimes_{\mathbb{R}} \left(\mathbb{R}[x] / \left(x^2+1\right)\right) \\
&\cong (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[x]) / \left(x^2+1\right) \quad \text{Since } \otimes_{\mathbb{R}} \mathbb{C} \text{ is an exact functor} \\
&\cong \mathbb{C}[x] / \left(x^2+1\right) \\
&\cong \mathbb{C}[x] / ((x-i)(x+i)) \\
&\cong \mathbb{C}[x]/(x-i) \times \mathbb{C}[x]/(x+i) \quad \text{by the Chinese Remainder Theorem} \\
&\cong \mathbb{C} \times \mathbb{C}
\end{aligned}$$

Definition 1.5.19. Let (Grp) be the category of groups and $V : (\mathrm{Grp}) \to (\mathrm{Sets})$ the forgetful functor. Let S be a scheme and let G be an S-scheme. The following data for G are equivalent by Yoneda's lemma

(1) A factorization of the functor $h_G : (\mathrm{Sch}/S)^{\mathrm{opp}} \to (\mathrm{Sets})$ through the forgetful functor $V : (\mathrm{Grp}) \to (\mathrm{Sets})$.

(2) For all S-schemes T the structure of a group on $G_S(T)$ which is functorial in T (i.e., for all S-morphisms $T' \to T$ the associated map $G_S(T) \to G_S(T')$ is a homomorphism of groups).

Definition 1.5.20. A homomorphism of S-group schemes G and H is a morphism $G \to H$ of S-schemes such that for all S-schemes T the induced map $G(T) \to H(T)$ is a group homomorphism.

Example 1.5.21. $S = \operatorname{Spec} \mathbb{Z}$ and $G := \mathrm{GL}_n$ with $\mathrm{GL}_n(T) := \mathrm{GL}_n(\Gamma(T, \mathscr{O}_T))$, the group of invertible $(n \times n)$-matrices over $\Gamma(T, \mathscr{O}_T)$, for any scheme T and for a fixed integer $n \geq 1$. The underlying scheme of GL_n is $\operatorname{Spec} A$ with $A = \mathbb{Z}\left[(T_{ij})_{1 \leq i,j \leq n}\right]\left[\det^{-1}\right]$, where $\det := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) T_{1\sigma(1)} \cdots T_{n\sigma(n)}$ is the determinant of the matrix $(T_{ij})_{i,j}$. This group scheme is called the general linear group scheme. We call $\mathbb{G}_m := \mathrm{GL}_1$ the multiplicative group scheme.

Example 1.5.22. The additive group scheme $\mathbb{G}_{a,S}$ over S is defined by $\mathbb{G}_{a,S}(T) = \Gamma(T, \mathscr{O}_T)$ for every S-scheme T. Its underlying S-scheme is $\mathbb{A}^1_S$.

1.6 Dimension of Scheme over k

Even for noetherian schemes the notion of dimension is sometimes quite counter-intuitive. If one restricts oneself to the case of schemes of finite type over a field, then the theory of dimension works mostly as expected, and is a very useful invariant.

Proposition 1.6.1. Let X be a topological space.

(1) Let Y be a subspace of X. Then $\dim Y \leq \dim X$. If X is irreducible, $\dim X < \infty$, and $Y \subsetneq X$ is a proper closed subset, then $\dim Y < \dim X$.

(2) Let $X = \bigcup_\alpha U_\alpha$ be an open covering. Then

$$\dim X = \sup_\alpha \dim U_\alpha.$$

(3) Let I be the set of irreducible components of X. Then

$$\dim X = \sup_{Y \in I} \dim Y.$$

(4) Let X be a scheme. Then

$$\dim X = \sup_{x \in X} \dim \mathscr{O}_{X,x}$$

Example 1.6.2. $\dim \mathbb{A}^n_k = \dim \mathbb{P}^n_k = n$

Proposition 1.6.3. Let $i : Y \to X$ be a closed immersion of schemes, where X is integral. If $\dim X = \dim Y < \infty$, then i is an isomorphism.

Proof: By Proposition 1.6.1, i is a homeomorphism. Hence, we need to show the map on stalks $\mathscr{O}_{X,i(x)} \to \mathscr{O}_{Y,x}$ is injective. Since X is integral, take $i(\eta)$ be the generic point of X. It's easy to check the follow diagram commute

$$\begin{tikzcd}
\mathscr{O}_{X,i(x)} \arrow[r] \arrow[d, hook] & \mathscr{O}_{Y,x} \arrow[d] \\
\mathscr{O}_{X,i(\eta)} \arrow[r, hook] & \mathscr{O}_{Y,\eta}
\end{tikzcd}$$

Since $\mathscr{O}_{X,i(\eta)}$ is a field, $\mathscr{O}_{X,i(\eta)} \to \mathscr{O}_{Y,\eta}$ is injective. Hence, $\mathscr{O}_{X,i(x)} \to \mathscr{O}_{Y,x}$ is injective.

Proposition 1.6.4. Let X be an k-scheme locally of finite type with closed subset $Y = \overline{\{\theta\}}$. Then $\dim Y = \operatorname{trdeg}_k \kappa(\theta)$

Proof: By Definition 1.4.17, Algebra 2.9.12 and Algebra 2.9.13.

Proposition 1.6.5. Let X be an irreducible k-scheme locally of finite type with generic point η.

(1) $\dim X = \operatorname{trdeg}_k \kappa(\eta)$.

(2) $\dim U = \dim X$ for any non-empty open subscheme U of X.

(3) Let $x \in X$ be any closed point. Then $\dim \mathscr{O}_{X,x} = \dim X$.

(4) Let $f : Y \to X$ be a morphism of k-schemes of locally finite type such that $f(Y)$ contains the generic point η of X. Then $\dim Y \geq \dim X$.

If X is integral, then $\kappa(\eta)$ is simply the function field of X.

Proof:

(2): Notice that U is also an irreducible k-scheme locally of finite type, the information on stalks are the same. So we have $\dim U = \operatorname{trdeg}_k \kappa(\eta) = \dim X$.

(3):For all closed $x \in X$, we may find open subset U such that $U = \operatorname{Spec}(A)$ where A is an finitely generated k-algebra. Since U is irreducible, $\operatorname{nil}(A)$ is a prime ideal. Since x is closed, for some maximal ideal $\mathfrak{m}$,

$$\dim \mathscr{O}_{X,x} = \dim A_{\mathfrak{m}} = \dim(A/\operatorname{nil}(A))_{\mathfrak{m}} = \dim A/\operatorname{nil}(A) = \dim U = \dim X$$

where the third "=" follows from Algebra 2.9.13.

(4): By hypothesis there exists $\theta \in Y$ such that $f(\theta) = \eta$. Therefore f induces a k-embedding $\kappa(\eta) \hookrightarrow \kappa(\theta)$. Denote by Z the closure of θ.

$$\dim X = \operatorname{trdeg} \kappa(\eta) \leq \operatorname{trdeg} \kappa(\theta) \leq \dim Y.$$

Proposition 1.6.6. Let X be a non-empty k-scheme of finite type. The following are equivalent:

(1) $\dim X = 0$.

(2) The scheme X is affine, the k-vector space $\Gamma(X, \mathscr{O}_X)$ is finite-dimensional, and $\Gamma(X, \mathscr{O}_X) = \prod_x \mathscr{O}_{X,x}$.

(3) The underlying topological space of X is discrete.

(4) The underlying topological space of X has only finitely many points.

Proposition 1.6.7. Let $f : Y \to X$ be a morphism of k-schemes of locally finite type with finite fibers. Then $\dim Y \leq \dim X$.

Proof: Let Z be an irreducible component of Y with generic point θ and set $x := f(\theta)$. By Proposition 1.6.4 and Proposition 1.6.1 (2), we only need to show $\operatorname{trdeg}_k \kappa(\theta) \leq \dim X$.

Replacing X by an open affine neighborhood U of x and Y by an open affine neighborhood of θ in $f^{-1}(U)$ we may assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine. Then B is a k-algebra of finite type and in particular an A-algebra of finite type. The fiber $f^{-1}(x) = \operatorname{Spec}(B \otimes_A \kappa(x))$ is thus a $\kappa(x)$-scheme of finite type with only finitely many points.

Notice that the induced morphism on residue field of a immersion is an isomorphism, the residue field of $\operatorname{Spec}(B \otimes_A \kappa(x))$ at θ is the same as the residue field of $\operatorname{Spec}(B)$ at θ.

Since the point θ is closed in $f^{-1}(x)$ by Proposition 1.6.6 and therefore $\kappa(\theta)$ is a finite extension of $\kappa(x)$. This shows $\operatorname{trdeg}_k \kappa(\theta) = \operatorname{trdeg}_k \kappa(x) = \dim \overline{\{x\}} \leq \dim X$.

Proposition 1.6.8. Let X be a k-scheme locally of finite type and let $x \in X$ be a closed point. Then $\dim \mathscr{O}_{X,x} = \sup_Z \dim Z$, where Z runs through the (finitely many) irreducible components of X containing x.

Proof: Assume $I = \{Z_1, \dots, Z_r\} = \left\{\overline{\{\theta_1\}}, \dots, \overline{\{\theta_r\}}\right\}$.

Since X is locally finite type, there's some $x \in U$ open in X, $U = \mathrm{Spec}(A)$ where A be a finitely generated k-algebra.

If x corresponds to $\mathfrak{m}$, by Proposition 1.3.40, $\theta_i \in Z_i \cap U$ correspond to minimal prime ideal contained in $\mathfrak{m}$. Then by Proposition 2.9.13, for some i, $\dim A_{\mathfrak{m}} = \dim A/\mathfrak{p}_{\theta_i}$.

Then,

$$\dim \mathscr{O}_{X,x} = \dim A_{\mathfrak{m}} = \dim A/\mathfrak{p}_{\theta_i} = \mathrm{trdeg}_k \kappa(\theta_i) = \dim Z_i$$

Definition 1.6.9 (local dimension)**.** Let X be a topological space and $x \in X$. The dimension of X at x is

$$\dim_x X = \inf_U \dim U$$

Corollary 1.6.10. Let X be a scheme locally of finite type over a field and let I be the (finite) set of irreducible components of X containing x. Then $\dim_x X = \sup_{Z \in I} \dim Z$. If $x \in X$ is a closed point, then $\dim_x X = \dim \mathscr{O}_{X,x}$.

Proof: Assume $I = \{Z_1, \dots, Z_n\}$. By Proposition 1.3.40, there's a open subset $x \in U$ such that U only meets with fintie many irreducible components of X. Since irreducible components is closed, we may assume U only meets with some subset of I.

By definition of local dimension, we may in addition assume U is affine and $\dim U = \dim_x X$. Then by Proposition 1.3.41, $U = \mathrm{Spec}(A)$ with A finitely generated k-algebra. By Proposition 1.3.40 and ,

$$\dim U = \sup_{Z \in I} \dim(Z \cap U)$$

Notice that for all i, Z_i is irreducible, there's a reduced scheme structure on Z_i such that Z_i is locally finite type by Proposition 1.4.17. By Proposition 1.6.5, $Z_i \cap U \neq \varnothing$ implies $\dim Z_i \cap U = \dim Z_i$.

Proposition 1.6.11. Let X be a topological space.

(1) Let $Z \subseteq X$ be a closed irreducible subset. The codimension $\mathrm{codim}_X Z$ of Z in X is the supremum of the lengths of chains of irreducible closed subsets $Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_l$ such that $Z_l = Z$.

(2) Let $Z \subseteq X$ be a closed subset. We say that Z is equi-codimensional (of codimension r), if all irreducible components of Z have the same codimension in X (equal to r).

Proposition 1.6.12. For an arbitrary scheme X and a closed irreducible subset Z with generic point η we have

$$\mathrm{codim}_X Z = \dim \mathscr{O}_{X,\eta} = \inf_{z \in Z} \dim \mathscr{O}_{X,z}$$

Proof: By Proposition 1.3.33.

Definition 1.6.13. Let X be a scheme and let $Y \subseteq X$ be an arbitrary subset. Then

$$\operatorname{codim}_X(Y) := \inf_{y \in Y} \dim \mathscr{O}_{X,y}$$

is called the codimension of Y in X.

Proposition 1.6.14. Let X be a scheme. If Y is a closed subset of X, we find

$$\operatorname{codim}_X Y = \inf_Z \operatorname{codim}_X Z,$$

where Z runs through the set of irreducible components of Y.

Proof:

$$\operatorname{codim}_X Y = \inf_{Z \subset Y, Z \text{ irreducible closed}} \operatorname{codim}_X Z = \inf_{Z \subset Y, Z \text{ irreducible components}} \operatorname{codim}_X Z$$

Proposition 1.6.15. Let X be an irreducible scheme of finite type over a field k. Set $d := \dim X$.

(1) All maximal chains of closed irreducible subsets of X have the same length.

(2) For all closed subsets Y of X we have

$$\dim Y + \operatorname{codim}_X Y = \dim X$$

Proof: (1): If $Z_r \subsetneq \cdots \subsetneq Z_0$ is a maximal chain, then $Z_r = \{x\}$ for some closed point $x \in X$ by Proposition 1.4.18. Hence by Proposition 1.3.33, $r = \dim \mathscr{O}_{X,x}$. And by Proposition 1.6.5, $d = \dim \mathscr{O}_{X,x}$ which is independent of the choice of maximal chain.

(2): We first assume that Y is irreducible. Then $\dim Y + \operatorname{codim}_X Y$ is the supremum of the lengths of maximal chains of closed irreducible subsets of X having Y as a member. Thus the claim follows from (1). General case follows from above proposition.

1.7 Separated Morphisms

Proposition 1.7.1. Equalizer exists in category Sch/S.

Proof: Conisder $f, g : X \to Y$ be two S-morphisms and $h : T \to X$ be S-morphisms such that $f \circ h = g \circ h$.

By universal property of fiber product, there's unique S-morphism (f, g), making the following diagram commutes:

$$\begin{tikzcd}
X \arrow[dr, dashed, "{(f,g)}"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& Y \times_S Y \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}$$

Consider the following diagram

$$\begin{tikzcd}
T \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "f\circ h"'] & & & \\
& X \times_{Y\times_S Y} Y \arrow[r, "\pi_X"'] \arrow[d, "\pi_Y"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "{(f,g)}"'] & Y \\
& Y \arrow[r, "\Delta_{Y/S}"] & Y \times_S Y \arrow[ur, shift left, "p"] \arrow[ur, shift right, "q"'] &
\end{tikzcd}$$

It's easy to check $f \circ \pi_X = \pi_Y = g \circ \pi_X$. Moreover, $p \circ (f, g) \circ h = p \circ \Delta_{Y/S} \circ f \circ h$ and $q \circ (f, g) \circ h = q \circ \Delta_{Y/S} \circ f \circ h$ implies $(f, g) \circ h = \Delta_{Y/S} \circ f \circ h$. Hence, there's unique θ such that above diagram commutes.

Proposition 1.7.2. Let $S = \operatorname{Spec} R$ be an affine scheme, let $X = \operatorname{Spec} B \to S$ and $Y = \operatorname{Spec} A \to S$ be affine S-schemes and let $f : X \to Y$ be an S-morphism corresponding to an R-algebra morphism $\varphi : A \to B$. Then the diagonal morphism $\Delta_{X/S}$ and graph morphism Γ_f correspond to the following surjective ring homomorphisms.

$$\begin{aligned} \Delta_{B/R} : B \otimes_R B \to B, &\qquad b \otimes b' \mapsto bb', \\ \Gamma_\varphi : A \otimes_R B \to B, &\quad a \otimes b \mapsto \varphi(a)b. \end{aligned}$$

In particular $\Delta_{X/S}$ and Γ_f are closed immersions.

Proposition 1.7.3. Let S be a scheme, let X and Y be S-schemes, and let $f, g : X \to Y$ be morphisms of S-schemes. Then $\Delta_{X/S}, \Gamma_f$, and the canonical morphism $\mathrm{Eq}(f, g) \to X$ are immersions.

Proof: $\Delta_{X/S}$: Assume S is affine. By Proposition 1.5.2, Proposition 1.7.2, we may find $U_i \times_S U_i, i \in I$ open in $Y \times_S Y$ such that U_i are affine open subschemes of Y which cover the image of Y(notice that $U_i \times_S U_i$ may not cover $Y \times_S Y$). Then the diagonal morphism is locally a closed immersion, which implies the image of Y is locally closed.

Γ_f: the same as $\Delta_{X/S}$.

$\mathrm{Eq}(f, g)$: Since immersion is stable under base change, then it follows from the proof of existence of equalizer in category of schemes.

Lemma 1.7.4. Let $u : X \to S, v : Y \to S$ be S-objects, let $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ be the projections, and $f, g : X \to Y$ two S-morphisms.

$$\Delta_{X/S} = \Gamma_{\mathrm{id}_X}, \quad \Gamma_f = \left(\mathrm{can} : \mathrm{Eq}\left(X \times_S Y \underset{f \circ p}{\overset{q}{\rightrightarrows}} Y\right) \to X \times_S Y\right).$$

Proof:

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] & X \times_S Y \arrow[r, shift left, "f \circ p"] \arrow[r, shift right, "q"'] & Y \\ T \arrow[u, dashed, "p \circ h"] \arrow[ur, "h"'] & & \end{tikzcd}$$

Definition 1.7.5. A morphism of schemes $v : Y \to S$ is called separated if the following equivalent conditions are satisfied.

(1) The diagonal morphism $\Delta_{Y/S}$ is a closed immersion.

(2) For every S-scheme X and for any two S-morphisms $f, g : X \to Y$ the equalizer $\mathrm{Eq}(f, g) \subseteq X$ is a closed subscheme of X.

(3) For every S-scheme X and for any S-morphism $f : X \to Y$ its graph Γ_f is a closed immersion.

Proof: (1) implies (2): closed immersion stalbe under base change
(2) implies (3): By Lemma 1.7.4.
(3) implies (1): Take $X = Y$ and $f = \mathrm{id}$.

Proposition 1.7.6. These are basic examples of separated morphism.

(1) Every monomorphism of schemes (and in particular every immersion) is separated.

(2) The property of being separated is stable under composition, stable under base change, and local on the target.

Proof: (1): $f : X \to S$ be a monomorphism, then X is isomorphic to $X \times_S X$ under $\Delta_{X/S}$ since X also satisfies the universal property of fiber product.

Proposition 1.7.7. Let $S = \operatorname{Spec} R$ be an affine scheme and let X be an S-scheme. Then the following assertions are equivalent.

(1) X is separated.

(2) For every two open affine sets $U, V \subseteq X$ the intersection $U \cap V$ is affine and

$$\rho_{U,V} : \Gamma(U, \mathscr{O}_X) \otimes_R \Gamma(V, \mathscr{O}_X) \to \Gamma(U \cap V, \mathscr{O}_X), \quad (s, t) \mapsto s_{|U \cap V} \cdot t_{|U \cap V}$$

is surjective.

(3) There exists an open affine covering $X = \bigcup_i U_i$ such that $U_i \cap U_j$ is affine and $\rho_{U_i, U_j} : \Gamma(U_i, \mathscr{O}_X) \otimes_R \Gamma(U_j, \mathscr{O}_X) \to \Gamma(U_i \cap U_j, \mathscr{O}_X)$ is surjective for all i, j.

Example 1.7.8. k be a field, $\mathbb{P}_k^n$ is separated.

1.8 Quasi-coherent modules

Chapter 2

Algebraic Group