Analysis

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Contents

Chapter 1

Foundation

1.1 Construction of Real Number

Definition 1.1.1 (ordered ring)**.** Thus, a ring(field) $R \neq 0$ with an order $<$ is called an ordered ring(field) if the following holds:

(1) $(R, <)$ is totally ordered

(2) $x < y \Rightarrow x + z < y + z, z \in R$

(3) $x, y > 0 \Rightarrow xy > 0$

Of course, an element $x \in R$ is called positive if $x > 0$ and negative if $x < 0$. We gather in the next proposition some simple properties of ordered fields.

Proposition 1.1.2. Let K be an ordered field, then for $x, y, a, b \in K$.

(1) $x > y \Leftrightarrow x - y > 0$.

(2) If $x > y$ and $a > b$, then $x + a > y + b$.

(3) If $a > 0$ and $x > y$, then $ax > ay$.

(4) If $x > 0$, then $-x < 0$. If $x < 0$, then $-x > 0$.

(5) Let $x > 0$. If $y > 0$, then $xy > 0$. If $y < 0$, then $xy < 0$.

(6) If $a < 0$ and $x > y$, then $ax < ay$.

(7) $x^2 > 0$ for all $x \neq 0$. In particular, $1 > 0$.

(8) If $x > 0$, then $x^{-1} > 0$.

(9) If $x > y > 0$, then $0 < x^{-1} < y^{-1}$ and $xy^{-1} > 1$.

Definition 1.1.3. K is a ordered field, K is said to be Archimedes if and only if for $x, y > 0$ there's $n \in \mathbb{Z}$ such that $nx > y$.

Example 1.1.4. $\mathbb{Q}$ is a Archimedes ordered field with original order.

Proposition 1.1.5. For an ordered field K, the absolute value function, $|\cdot|: K \to K$ and the sign function, $\operatorname{sign}(\cdot): K \to K$ are defined by

$$|x| := \begin{cases} x, & x > 0, \\ 0, & x = 0, \\ -x, & x < 0, \end{cases} \qquad \operatorname{sign} x := \begin{cases} 1, x > 0, \\ 0, x = 0, \\ -1, x < 0. \end{cases}$$

Let K be an ordered field and $x, y, a, \varepsilon \in K$ with $\varepsilon > 0$.

(1) $x = |x| \operatorname{sign}(x), |x| = x \operatorname{sign}(x)$.

(2) $|x| = |-x|, \quad x \leq |x|$.

(3) $|xy| = |x||y|$.

(4) $|x| \geq 0$ and $(|x| = 0 \Leftrightarrow x = 0)$.

(5) $|x - a| < \varepsilon \Leftrightarrow a - \varepsilon < x < a + \varepsilon$.

(6) $|x + y| \leq |x| + |y|$ (triangle inequality).

(7) $|x - y| \geq ||x| - |y||, \quad x, y \in K$

Definition 1.1.6. A ring homomorphism f between ordered field is said to be order-preserving if

$$a < b \Longleftrightarrow f(a) < f(b)$$

.

Definition 1.1.7. A sequence $r = (x_n)_{n \in \mathbb{Z}_{>0}}$ is a Cauchy sequence if for all $\epsilon \in \mathbb{Q} > 0$, there's $N > 0$ such that for all $m, n > N$, $|x_n - x_m| < \epsilon$.

Proposition 1.1.8. Cauchy sequence is bounded.

Definition 1.1.9. Let

$$\mathcal{R} = \left\{(x_n) \in \mathbb{Q}^{\mathbb{Z}_{>0}} : (x_n) \text{ is a Cauchy sequence}\right\}$$

and

$$\mathbf{c}_0 = \left\{(x_n) \in \mathbb{Q}^{\mathbb{Z}_{>0}} : \text{for all } \epsilon > 0, \text{ there's } N > 0 \text{ such that for all } n > N, |x_n| < \epsilon\right\}$$

It's clear that $\mathbf{c}_0 \subset \mathcal{R}$ is a maximal ideal of $\mathcal{R}$. Hence $\mathcal{R}/\mathbf{c}_0$ is a field and we denote it by $\mathbb{R}$. For convenience, we usually denote $(a_n) + \mathbf{c_0}$ by (a_n).

Definition 1.1.10. Now we define a order on $\mathbb{R}$, for $(a_n), (b_n)$ in $\mathbb{R}$, $(a_n) > (b_n)$ if there's $\epsilon > 0$, a sufficiently large integer N, such that $a_n - b_n > \epsilon$ for $n > N$. And denote this order by $<$. It's esay to check that '$<$' is well-defined and totally ordered.

Proposition 1.1.11. $(\mathbb{R}, <)$ is a Archimedes ordered field. And the embedding $l : \mathbb{Q} \to \mathbb{R}$ given by

$$r \mapsto (r, r, r, \dots)$$

is an order-preserving ring homomorphism.

Definition 1.1.12. For a sequence $(A_n) \in \mathbb{R}$, we say $A_n \to A$ if for all $\epsilon \in \mathbb{R} > 0$, there's $N > 0$ such that for all $n > N$, $|A_n - A| < \epsilon$. And we say (A_n) is a Cauchy sequence if for all $\epsilon \in \mathbb{R}_{>0}$, there's $N > 0$ such that for all $m, n > N$, $|x_n - x_m| < \epsilon$.

Proposition 1.1.13 (dense)**.** For all $a, b \in \mathbb{R}$, if $a < b$, there's $c \in \mathbb{Q}$ such that $a < l(c) < b$.

Proposition 1.1.14 (completeness)**.** (A_n) is a Cauchy sequence in $\mathbb{R}$ if and only if there's $A \in \mathbb{R}$ such that $A_n \to A$.

Proof: 'if' is obvious.

'only if': Take $x_n \in \mathbb{Q}$ such that:

$$A_n < l(x_n) < A_n + l(\frac{1}{n})$$

It's cleat that $a = (x_n) + \mathbf{c}_0 \in \mathbb{R}$.

Notice that $A_n \to a$, we have $\mathbb{R}$ is complete.

Now we identity $\mathbb{Q}$ with a subfield of $\mathbb{R}$ in the following content.

Proposition 1.1.15. (1) E is a non-empty subset of $\mathbb{R}$ and if E is lower-bounded, then E has a infimum; if E is upper-bounded, then E has a supremum.

(2) Every incresing bounded sequence $(x_n) \in \mathbb{R}$ has a limit.

(3) (Bolzano-Weierstress) Every bounded sequence has a convergent subsequnce.

(4) if

$$[a, b] \subset \bigcup_{i \in I} (a_i, b_i)$$

, then

$$[a, b] \subset \bigcup_{k \in J} (a_k, b_k)$$

for some finite subset J of I.

Proposition 1.1.16. $a > 0$, $n \in \mathbb{Z}_{>0}$, then there's unique $x \in \mathbb{R}_{>0}$ such that $x^n = a$. We denote the unique positive root by $\sqrt[n]{a}$. And for all $a \in \mathbb{R}$ and $r = \frac{p}{q} \in \mathbb{Q}$, define $a^r = \sqrt[q]{a^p}$. It's easy to check that $\sqrt[q]{a^p} = (\sqrt[q]{a})^p$.

Definition 1.1.17 (complex number)**.** Let $\mathbb{C} = \mathbb{R} \times \mathbb{R}$, define $(a, b) \cdot (c, d) = (ac - bd, bc + ad)$. Then $\mathbb{C}$ is a field under this operator and $\mathbb{R}$ is a subfield of $\mathbb{C}$.

1.2 Point-Set Topology

Munkres's Topology is a good reference for this section.

1.2.1 Definition

Definition 1.2.1. A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(1) $\varnothing$ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a topological space.

Definition 1.2.2. If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

(1) For each $x \in X$, there is at least one basis element B containing x.

(2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology $\boldsymbol{T}$ generated by $\mathcal{B}$ as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.

Definition 1.2.3. Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the subspace topology. With this topology, Y is called a subspace of X; its open sets consist of all intersections of open sets of X with Y.

Definition 1.2.4. X is Hausdorff if for any two elements $x \neq y$ in X, there's U, V open in X such that $x \in U, y \in V$ and $U \cap V = \varnothing$.

Definition 1.2.5 (convergence)**.**

Proposition 1.2.6. If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.

Example 1.2.7. Let X be a ordered set; assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

(1) All open intervals (a, b) in X.

(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.

(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X. The collection $\mathcal{B}$ is a basis for a topology on X, which is called the order topology.

Example 1.2.8. $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.

Proposition 1.2.9. Given a subset A of a topological space X, the interior of A is defined as the union of all open sets contained in A, and the closure of A is defined as the intersection of all closed sets containing A.

Let Y be a subspace of X; let A be a subset of Y; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.

Definition 1.2.10 (limit point)**.** If A is a subset of the topological space X and if x is a point of X, we say that x is a limit point of A if every neighborhood of x intersects A in some point other than x itself. Said differently, x is a limit point of A if it belongs to the closure of $A - \{x\}$. The point x may lie in A or not; for this definition it does not matter. Let A be a subset of the topological space X; let A' be the set of all limit points of A. Then

$$\bar{A} = A \cup A'.$$

Definition 1.2.11. X be a topological space, A is perfect if for all $a \in A$, a is a limit point.

Proposition 1.2.12. Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:

(1) f is continuous.(U open in X implies $f^{-1}(U)$ open in Y)

(2) For every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$.

(3) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

(4) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$. If the condition in (4) holds for the point x of X, we say that f is continuous at the point x.

Definition 1.2.13. Consider $(X_i)_{i \in I}$ be a family of topology spaces, then the sets of the form

$$\prod_{i \in I} U_i$$

$U_i = X_i$ for all but finite i, form a basis of $\prod_{i \in I} X_i$. We call it the topology induced by this product topology.

In language of category, product topology with projection $p_i : \prod_{i \in I} X_i \to X_i$ is the product object in the category of topological space.

Proposition 1.2.14. If each space X_α is Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in product topology.

Proposition 1.2.15. Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given the product topology, then

$$\prod \bar{A}_\alpha = \overline{\prod A_\alpha}$$

Theorem 1.2.16. Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J},$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

Definition 1.2.17 (locally closed)**.** A subset E of a topological space X is said to be locally closed if any of the following equivalent conditions are satisfied:

(1) E is the intersection of an open set and a closed set in X.

(2) For each point $x \in E$, there is a neighborhood U of x such that $E \cap U$ is closed in U.

(3) E is open in its closure $\bar{E}$.

(4) The set $\bar{E} \backslash E$ is closed in X.

Proof: (2) implies (1): For all $x \in E$, choose U_i open in X such that $E = \cap U_i = U_i \cap V_i$ for some V_i closed in X. Then consider

$$(\bigcup_{i \in I} U_i) \cap \bar{E}$$

For $x \in U_i \cap \bar{E}$, if $x \notin E$, then $x \in U_i \cap V_i^c$. Notice that

$$U_i \cap V_i^c \cap E = \varnothing$$

which contradicts to $x \in \bar{E}$.

(3) implies (4): If $\bar{E} \cap U = E$ for some U open in X, then $\bar{E} - E = \bar{E} \cap E^c = \bar{E} \cup U^c$.

Proposition 1.2.18. E is a locally closed subset in X, then E closed in the open subset $X - \bar{E} \backslash E$ and $X - \bar{E} \backslash E$ is the largest open subset contains E such that E is closed in it.

Proof: Since

$$X - \bar{E} \backslash E = (\bar{E})^c \cup E$$

and $((\bar{E})^c \cup E) \cap \bar{E} = E$, we have E closed in $X - \bar{E} \backslash E$.

In addition, if there's V open in X such that E closed in V,

$$E = \bar{E} \cap V$$

Hence, $V = (V \cap \bar{E}^c) \cup (V \cap \bar{E}) = E \cup (V \cap \bar{E}^c) \subset E \cup \bar{E}^c$.

1.2.2 Metric space

Definition 1.2.19. A metric on a set X is a function

$$d : X \times X \longrightarrow \mathbb{R}$$

having the following properties:

(1) $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(3) (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$, for all $x, y, z \in X$.

Given a metric d on X, the number $d(x, y)$ is often called the distance between x and y in the metric d. Given $\epsilon > 0$, consider the set

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

of all points y whose distance from x is less than ϵ. It is called the **$\boldsymbol{\epsilon}$-ball centered at $\boldsymbol{x}$**. Sometimes we omit the metric d from the notation and write this ball simply as $B(x, \epsilon)$, when no confusion will arise.

If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X, called the metric topology induced by d.

Example 1.2.20. $\mathbb{R}^n$ is a metric space with distance $d(x, y) = ||x - y||$

Theorem 1.2.21. Let X be a topological space; let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is metrizable.

Let $f : X \to Y$. If the function f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.

Theorem 1.2.22. Let $f_n : X \to Y$ be a sequence of functions from the set X to the metric space Y. Let d be the metric for Y. We say that the sequence (f_n) converges uniformly to the function $f : X \to Y$ if given $\epsilon > 0$, there exists an integer N such that

$$d(f_n(x), f(x)) < \epsilon$$

for all $n > N$ and all x in X.

Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to f, then f is continuous.

Proposition 1.2.23. If $f \in B(X)$, we define the uniform norm of f to be

$$\|f\|_u = \sup\{|f(x)| : x \in X\}.$$

The function $\rho(f, g) = \|f - g\|_u$ is easily seen to be a metric on $B(X)$, and convergence with respect to this metric is simply uniform convergence on $X.B(X)$ is obviously complete in the uniform metric: If $\{f_n\}$ is uniformly Cauchy, then $\{f_n(x)\}$ is Cauchy for each x, and if we set $f(x) = \lim_n f_n(x)$, it is easily verified that $\|f_n - f\|_u \to 0$.

If X is a topological space, $BC(X) = B(X) \cap C(X)$ is a closed subspace of $B(X)$ in the uniform metric; in particular, $BC(X)$ is complete.

1.2.3 Compactness

Definition 1.2.24. A collection $\mathcal{A}$ of subsets of a space X is said to cover X, or to be a covering of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an open covering of X if its elements are open subsets of X.

A space X is said to be compact if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Proposition 1.2.25. Every closed subspace of a compact space is compact. Every compact subspace of a Hausdorff space is closed.

Theorem 1.2.26. The image of a compact space under a continuous map is compact.

Corollary 1.2.27. X is a compact space, Y is a Hausdorff space, then continuous $f : X \to Y$ is closed.

Corollary 1.2.28. Let $f : X \to Y$ be a continuous bijection. X is a compact space, Y is a Hausdorff space, then f is homemorphism.

Lemma 1.2.29 (Lebesgue number lemma)**.** Let $\mathcal{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing it. The number δ is called a Lebesgue number for the covering $\mathcal{A}$.

Theorem 1.2.30. Let X be a metrizable space. Then the following are equivalent:

(1) X is compact.

(2) X is limit point compact(infinite subset has a limit point).

(3) X is sequentially compact(every sequnence has a convergent subsequnce).

Theorem 1.2.31 (Tychonoff theorem)**.** An arbitrary product of compact spaces is compact in the product topology.

Definition 1.2.32. A function f from the metric space (X, d_X) to the metric space (Y, d_Y) is said to be uniformly continuous if given $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points x_0, x_1 of X,

$$d_X(x_0, x_1) < \delta \Longrightarrow d_Y(f(x_0), f(x_1)) < \epsilon.$$

Theorem 1.2.33. Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous.

Proposition 1.2.34 (finite intersection)**.** A collection $\mathcal{C}$ of subsets of X is said to have the finite intersection property if for every finite subcollection

$$\{C_1, \ldots, C_n\}$$

of $\mathcal{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty.

Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty.

Definition 1.2.35 (locally compact)**.** A space X is said to be locally compact at $\boldsymbol{x}$ if there is some compact subspace C of X that contains a neighborhood of x. If X is locally compact at each of its points, X is said simply to be locally compact.

Definition 1.2.36 (one-point compactification)**.** Let X be a space. Then X is locally compact Hausdorff if and only if there exists a space Y satisfying the following conditions:

(1) X is a subspace of Y.

(2) The set $Y - X$ consists of a single point.

(3) Y is a compact Hausdorff space.

If Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X.

Proof: We only provide the form of the open sets in Y: U open in Y if and only if U open in X, or U is the complement of a compact subset in X.

Definition 1.2.37. If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is said to be a compactification of X. If $Y - X$ equals a single point, then Y is called the one-point compactification of X.

Proposition 1.2.38. Let X be a Hausdorff space. Then X is locally compact if and only if given x in X, and given a neighborhood U of x, there is a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subset U$.

Corollary 1.2.39. If X is an LCH space and $K \subset U \subset X$ where K is compact and U is open, there exists a precompact open V such that $K \subset V \subset \bar{V} \subset U$.

Proposition 1.2.40. Let X be locally compact Hausdorff; let A be a subspace of X. If A is closed in X or open in X, then A is locally compact.

Proposition 1.2.41. In a locally compact Hausdorff space E, a subset A is closed if and only if its intersection with every compact set is compact.

Proof: Let $A \subseteq E$ have the property that $A \cap K$ is closed in K for all compact $K \subseteq E$. We want to show that A is closed whenever E is locally compact Hausdorff, so we will show that $E - A$ is open.

Let $x \in E - A$, let K be a compact neighbourhood of x, and let $U \subseteq K$ be an open neighbourhood of x. Then $x \in U - K \cap A$ and $U - K \cap A$ is open in X. Hence $E - A$ is open in X.

Theorem 1.2.42 (Usysohn's Lemma, Locally Compact Version)**.** If X is an LCH space and $K \subset U \subset X$ where K is compact and U is open, there exists $f \in C(X, [0, 1])$ such that $f = 1$ on K and $f = 0$ outside a compact subset of U.

Definition 1.2.43. If X is a topological space and $f \in C(X)$, the support of f, denoted by $\mathrm{supp}(f)$, is the smallest closed set outside of which f vanishes, that is, the closure of $\{x : f(x) \neq 0\}$. If $\mathrm{supp}(f)$ is compact, we say that f is compactly supported, and we define

$$C_c(X) = \{f \in C(X) : \mathrm{supp}(f) \text{ is compact }\}.$$

Moreover, if $f \in C(X)$, we say that f vanishes at infinity if for every $\epsilon > 0$ the set $\{x : |f(x)| \geq \epsilon\}$ is compact, and we define

$$C_0(X) = \{f \in C(X) : f \text{ vanishes at infinity }\}.$$

Clearly $C_c(X) \subset C_0(X)$. Moreover, $C_0(X) \subset BC(X)$, because for $f \in C_0(X)$ the image of the set $\{x : |f(x)| \geq \epsilon\}$ is compact, and $|f| < \epsilon$ on its complement.

Proposition 1.2.44. If X is an LCH space, $C_0(X)$ is the closure of $C_c(X)$ in the uniform metric.

Proof: If $\{f_n\}$ is a sequence in $C_c(X)$ that converges uniformly to $f \in C(X)$, for each $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $\|f_n - f\|_u < \epsilon$. Then $|f(x)| < \epsilon$ if $x \notin \mathrm{supp}\,(f_n)$, so $f \in C_0(X)$. Conversely, if $f \in C_0(X)$, for $n \in \mathbb{N}$ let $K_n = \{x : |f(x)| \geq n^{-1}\}$. Then K_n is compact, so by Usysohn's Lemma, there exists $g_n \in C_c(X)$ with $0 \leq g_n \leq 1$ and $g_n = 1$ on K_n. Let $f_n = g_n f$. Then $f_n \in C_c(X)$ and $\|f_n - f\|_u \leq n^{-1}$, so $f_n \to f$ uniformly.

Proposition 1.2.45. If X is a σ-compact LCH space, there is a sequence $\{U_n\}$ of precompact open sets such that $\overline{U}_n \subset U_{n+1}$ for all n and $X = \bigcup_1^\infty U_n$.

1.2.4 Connectness

Definition 1.2.46. Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called the components (or the "connected components") of X.The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.

Definition 1.2.47. We define another equivalence relation on the space X by defining $x \sim y$ if there is a path in X from x to y. The equivalence classes are called the path components of X. The path components of X are path-connected disjoint subspaces of X whose union is X, such that each nonempty path-connected subspace of X intersects only one of them.

Definition 1.2.48. A space X is said to be locally connected at x if for every neighborhood U of x, there is a connected neighborhood V of x contained in U. If X is locally connected at each of its points, it is said simply to be locally connected. Similarly, a space X is said to be locally path connected at x if for every neighborhood U of x, there is a path-connected neighborhood V of x contained in U. If X is locally path connected at each of its points, then it is said to be locally path connected.

Proposition 1.2.49. (1) A space X is locally connected if and only if for every open set U of X, each component of U is open in X.

(2) A space X is locally path connected if and only if for every open set U of X, each path component of U is open in X.

(3) If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.

Proposition 1.2.50. The union of a collection of connected subspaces of X that have a point in common is connected.

Proposition 1.2.51. Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is also connected.

Proposition 1.2.52. The image of a connected space under a continuous map is connected.

Theorem 1.2.53 (Intermediate Value Theorem)**.** Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

Theorem 1.2.54 (Extreme value theorem)**.** Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.

Example 1.2.55. Let X be a connected open subset of $\mathbb{R}^n$, then any pair of points of X can be connected by a polygonal path in X.

1.2.5 Countability

Definition 1.2.56. A space X is said to have a countable basis at x if there is a countable collection B of neighborhoods of x such that each neighborhood of x contains at least one of the elements of B. A space that has a countable basis at each of its points is said to satisfy the first countability axiom, or to be first-countable.

Proposition 1.2.57. Let X be a topological space.

(1) Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is first-countable.

(2) Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is firstcountable.

Definition 1.2.58. If a space X has a countable basis for its topology, then X is said to satisfy the second countability axiom, or to be second-countable.

Definition 1.2.59. A subset A of a space X is said to be dense in X if $\bar{A} = X$.

Definition 1.2.60. Suppose that X has a countable basis. Then:

(1) Every open covering of X contains a countable subcollection covering X.(Lindelof space)

(2) There exists a countable subset of X that is dense in X.(separable)

Proposition 1.2.61. (1) Every metrizable space with a countable dense subset has a countable basis.

(2) Every metrizable Lindelöf space has a countable basis.

Proposition 1.2.62. Compact metric space is second-countable.

1.2.6 Separation

Definition 1.2.63. Suppose that one-point sets are closed in X. Then X is said to be regular if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively.

The space X is said to be normal if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.

Proposition 1.2.64. Let X be a topological space. Let one-point sets in X be closed.

(1) X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $\bar{V} \subset U$.

(2) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\bar{V} \subset U$.

Proposition 1.2.65. (1) Every metrizable space is normal.

(2) Every compact Hausdorff space is normal.

Theorem 1.2.66 (Usysohn's lemma)**.** Let X be a normal space; let A and B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map

$$f : X \longrightarrow [a, b]$$

such that $f(x) = a$ for every x in A, and $f(x) = b$ for every x in B.

Theorem 1.2.67 (Tietze extension theorem)**.** Let X be a normal space; let A be a closed subspace of X.

(1) Any continuous map of A into the closed interval $[a, b]$ of $\mathbb{R}$ may be extended to a continuous map of all of X into $[a, b]$.

(2) Any continuous map of A into $\mathbb{R}$ may be extended to a continuous map of all of X into $\mathbb{R}$.

1.2.7 Completeness

Definition 1.2.68. Let (X, d) be a metric space. A sequence (x_n) of points of X is said to be a Cauchy sequence in (X, d) if it has the property that given $\epsilon > 0$, there is an integer N such that

$$d(x_n, x_m) < \epsilon \quad \text{whenever } n, m \geq N.$$

The metric space (X, d) is said to be complete if every Cauchy sequence in X converges.

Theorem 1.2.69. A metric space (X, d) is compact if and only if it is complete and totally bounded.

Theorem 1.2.70 (extension theorem)**.** Suppose Y and Z are metric spaces, and Z is complete. Also suppose X is a dense subset of Y, and $f : X \to Z$ is uniformly continuous. Then f has a uniquely determined extension $\bar{f} : Y \to Z$ given by

$$\bar{f}(y) = \lim_{\substack{x \to y \\ x \in X}} f(x) \quad \text{for } y \in Y$$

and $\bar{f}$ is also uniformly continuous.

Definition 1.2.71. Let X be a metric space. If $h : X \to Y$ is an isometric imbedding of X into a complete metric space Y, such that $h(X)$ dense in Y. Then Y is called the completion of X. By extension theorem, the completion of X is uniquely determined up to an isometry.

Definition 1.2.72. A space X is said to be a Baire space if the following condition holds: Given any countable collection $\{A_n\}$ of closed sets of X each of which has empty interior in X, their union $\bigcup A_n$ also has empty interior in X.

Theorem 1.2.73 (Baire Category Theorem)**.** If X is a compact Hausdorff space or a complete metric space, then X is a Baire space.

Theorem 1.2.74. Any open subspace Y of a Baire space X is itself a Baire space.

Theorem 1.2.75. Let X be a space; let (Y, d) be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions such that $f_n(x) \to f(x)$ for all $x \in X$, where $f : X \to Y$. If X is a Baire space, the set of points at which f is continuous is dense in X.

1.3 Limit

1.3.1 Limit Superior and Limit Inferior

We work on $\bar{\mathbb{R}}$ in this subsection.

Definition 1.3.1. We call $a \in \bar{\mathbb{R}}$ a cluster point of (x_n) if every neighborhood of a contains infinitely many terms of the sequence.

Definition 1.3.2. *A* point a is a cluster point of a sequence (x_n) if and only if there is some subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of (x_n) which converges to a.

Definition 1.3.3. Let (x_n) be a sequence in $\mathbb{R}$. We can define two new sequences (y_n) and (z_n) by

$$y_n := \sup_{k\geq n} x_k := \sup\{x_k; k \geq n\}$$
$$z_n := \inf_{k\geq n} x_k := \inf\{x_k; k \geq n\}$$

Clearly (y_n) is a decreasing sequence and (z_n) is an increasing sequence in $\overline{\mathbb{R}}$. These sequences converge in $\overline{\mathbb{R}}$

$$\limsup_{n\to\infty} x_n := \overline{\lim_{n\to\infty}}\, x_n := \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right)$$

the limit superior, and

$$\liminf_{n\to\infty} x_n := \underline{\lim}_{n\to\infty}\, x_n := \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right)$$

the limit inferior.

Theorem 1.3.4. Any sequence (x_n) in $\mathbb{R}$ has a smallest cluster point x_* and a greatest cluster point x^* in $\overline{\mathbb{R}}$ and these satisfy

$$\liminf x_n = x_* \quad \text{and} \quad \limsup x_n = x^*$$

1.3.2 Series

In the following theorem, $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, $(E, |\cdot|)$ is a Banach space over $\mathbb{K}$ and (x_n) is a sequence in E.

Proposition 1.3.5. For a series $\sum x_k$ in a Banach space $(E, |\cdot|)$, the following are equivalent:

(1) $\sum x_k$ converges.

(2) For each $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$\left|\sum_{k=n+1}^{m} x_k\right| < \varepsilon, \quad m > n \geq N.$$

Proposition 1.3.6. Let $\sum x_k$ be a series in E and $\sum a_k$ a series in $\mathbb{R}^+$. Then the series $\sum a_k$ is called a majorant (or minorant) for $\sum x_k$ if there is some $K \in \mathbb{N}$ such that $|x_k| \leq a_k$ (or $a_k \leq |x_k|$) for all $k \geq K$. If a series in a Banach space has a convergent majorant, then it converges absolutely.

Proposition 1.3.7 (Abel)**.** Let $(a_n)_{n\in\mathbb{Z}}, (b_n)_{n\in\mathbb{Z}}$ be two sequneces in E, then

$$\sum_{M<n\leqslant M+N} a_n b_n = a_{M+N}B_{M+N} + \sum_{M<n\leqslant M+N-1} (a_n - a_{n+1}) B_n,$$

where $B_n = \sum_{M<k\leqslant n} b_k$.

If in particular $E = \mathbb{C}$ and (a_n) is a monotone sequence in $\mathbb{R}$, and

$$\sup_{M<n\leqslant M+N} |B_n| \leqslant \rho,$$

we have

$$\left|\sum_{M<n\leqslant M+N} a_n b_n\right| \leqslant \rho\left(|a_{M+1}| + 2\,|a_{M+N}|\right).$$

Example 1.3.8 (base g expansion)**.** Suppose that $g \geq 2$. Then every real number x has a base g expansion. This expansion is unique if expansions satisfying $x_k = g-1$ for almost all $k \in \mathbb{N}$ are excluded(for example, if $g = 10$, $0.999\ldots$ is excluded). Moreover, x is a rational number if and only if its base g expansion is periodic.

Theorem 1.3.9. Every rearrangement of an absolutely convergent series $\sum x_k$ is absolutely convergent and has the same value as $\sum x_k$.

Theorem 1.3.10. There is a bijection $\alpha : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$. If α is such a bijection, we call the series $\sum_n x_{\alpha(n)}$ an ordering of the double series $\sum x_{jk}$. If we fix $j \in \mathbb{N}$ (or $k \in \mathbb{N}$), then the series $\sum_k x_{jk}$ (or $\sum_j x_{jk}$) is called the j^{th} row series (or j^{th} column series) of $\sum x_{jk}$. If every row series (or column series) converges, then we can consider the series of row sums $\sum_j \left(\sum_{k=0}^{\infty} x_{jk}\right)$ (or the series of column sums $\sum_k \left(\sum_{j=0}^{\infty} x_{jk}\right)\Big)$. Finally we say that the double series $\sum x_{jk}$ is summable if

$$\sup_{n\in\mathbb{N}} \sum_{j,k=0}^{n} |x_{jk}| < \infty.$$

Let $\sum x_{jk}$ be a summable double series.

(1) Every ordering $\sum_n x_{\alpha(n)}$ of $\sum x_{jk}$ converges absolutely to a value $s \in E$ which is independent of α.

(2) The series of row sums $\sum_j \left(\sum_{k=0}^{\infty} x_{jk}\right)$ and column sums $\sum_k \left(\sum_{j=0}^{\infty} x_{jk}\right)$ converge absolutely, and

$$\sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty} x_{jk}\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty} x_{jk}\right) = s$$

Theorem 1.3.11. Suppose that the series $\sum x_j$ and $\sum y_k$ in $\mathbb{K}$ converge absolutely. Then the Cauchy product $\sum_n \sum_{k=0}^{n} x_k y_{n-k}$ of $\sum x_j$ and $\sum y_k$ converges absolutely, and

$$\left(\sum_{j=0}^{\infty} x_j\right)\left(\sum_{k=0}^{\infty} y_k\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} x_k y_{n-k}$$

Example 1.3.12. Let $k \in \mathbb{N}$ and $a \in \mathbb{C}$ be such that $|a| > 1$. Then

$$\lim_{n\to\infty} \frac{n^k}{a^n} = 0$$

that is, for $|a| > 1$ the function $n \mapsto a^n$ increases faster than any power function $n \mapsto n^k$.

Example 1.3.13. For all $a \in \mathbb{C}$,

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0$$

The factorial function $n \mapsto n\,!$ increases faster than the function $n \mapsto a^n$.

1.4 Functions of Single variable

1.4.1 Monotone Functions

Theorem 1.4.1. Suppose that $I \subseteq \mathbb{R}$ is a nonempty interval(connected subset of $\mathbb{R}$) and $f : I \to \mathbb{R}$ is continuous and strictly increasing (or strictly decreasing).

(1) $J := f(I)$ is an interval.

(2) $f : I \to J$ is bijective.

(3) $f^{-1} : J \to I$ is continuous and strictly increasing (or strictly decreasing).

Example 1.4.2. For each $n \in \mathbb{N}^\times$, the function

$$\mathbb{R}^+ \to \mathbb{R}^+, \quad x \mapsto \sqrt[n]{x}$$

is continuous and strictly increasing. In addition, $\lim_{x\to\infty} \sqrt[n]{x} = \infty$.

1.4.2 Power Series

Definition 1.4.3. Let

$$a := \sum a_k X^k := \sum_k a_k X^k$$

be a (formal) power series in one indeterminate with coefficients in $\mathbb{K}$. Then, for each $x \in \mathbb{K}$, $\sum a_k x^k$ is a series in $\mathbb{K}$. When this series converges we denote its value by $\underline{a}(x)$, the value of the (formal) power series at x. Set

$$\mathrm{dom}(\underline{a}) := \left\{ x \in \mathbb{K}; \sum a_k x^k \text{ converges in } \mathbb{K} \right\}$$

Then $\underline{a} : \mathrm{dom}(\underline{a}) \to \mathbb{K}$ is a well defined function:

$$\underline{a}(x) := \sum_{k=0}^{\infty} a_k x^k, \quad x \in \mathrm{dom}(\underline{a})$$

Note that $0 \in \mathrm{dom}(\underline{a})$ for any $a \in \mathbb{K}[[X]]$. The following examples show that each of the cases

$$\mathrm{dom}(\underline{a}) = \mathbb{K}, \quad \{0\} \subset \mathrm{dom}(\underline{a}) \subset \mathbb{K}, \quad \mathrm{dom}(\underline{a}) = \{0\}$$

is possible.

Proposition 1.4.4. For a power series $a = \sum a_k X^k$ with coefficients in $\mathbb{K}$ there is a unique $\rho := \rho_a \in [0, \infty]$ with the following properties:

(1) The series $\sum a_k x^k$ converges absolutely if $|x| < \rho$ and diverges if $|x| > \rho$.

(2) Hadamard's formula holds:

$$\rho_a = \frac{1}{\overline{\lim}_{k\to\infty} \sqrt[k]{|a_k|}}$$

The number $\rho_a \in [0, \infty]$ is called the radius of convergence of a, and

$$\rho_a \mathbb{B}_{\mathbb{K}} = \{x \in \mathbb{K}; |x| < \rho_a\}$$

is the disk of convergence of a.

Proposition 1.4.5. Let $a = \sum a_k X^k$ be a power series such that $\lim_{n\to\infty} |a_k/a_{k+1}|$ exists in $\overline{\mathbb{R}}$. Then the radius of convergence of a is given by the formula

$$\rho_a = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right|$$

Theorem 1.4.6. Let $a = \sum a_k X^k$ and $b = \sum b_k X^k$ be power series with radii of convergence ρ_a and ρ_b respectively. Set $\rho := \min(\rho_a, \rho_b)$. Then for all $x \in \mathbb{K}$ such that $|x| < \rho$ we have

$$\sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} (a_k + b_k)\, x^k$$

$$\left[\sum_{k=0}^{\infty} a_k x^k\right] \left[\sum_{k=0}^{\infty} b_k x^k\right] = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j}\right) x^k$$

Proposition 1.4.7. Let $\sum a_k X^k$ be a power series with positive radius of convergence ρ_a. If there is a null sequence (y_j) such that $0 < |y_j| < \rho_a$ and

$$\underline{a}\,(y_j) = \sum_{k=0}^{\infty} a_k y_j^k = 0, \quad j \in \mathbb{N}$$

then $a_k = 0$ for all $k \in \mathbb{N}$, that is, $a = 0 \in \mathbb{K}[[X]]$.

Proposition 1.4.8. Let $a = \sum a_k X^k$ be a power series with positive radius of convergence ρ_a. Then the function $\underline{a}$ represented by a is continuous on $\rho_a \mathbb{B}$.

Definition 1.4.9.

$$\exp : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

converges absolutely for all $z \in \mathbb{C}$.

Corollary 1.4.10.

$$\exp(x + y) = \exp(x) \exp(y)$$

for $x, y \in \mathbb{C}$

Definition 1.4.11.

$$\cos : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

and

$$\sin : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

are called the cosine and sine functions.

Definition 1.4.12. The sequence $((1+1/n)^n)$ converges and its limit

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

the Euler number, satisfies $2 < e \leq 3$. Moreover, we can show that

$$e = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!}$$

Proposition 1.4.13. As an application of this property of the exponential function, we determine the values of the exponential function for rational arguments. Namely,

$$\exp(r) = e^r, \quad r \in \mathbb{Q}$$

that is, for a rational number $r, \exp(r)$ is the r^{th} power of e.

Proposition 1.4.14 (Euler's formula)**.**

$$e^{iz} = \cos z + i\sin z, \quad z \in \mathbb{C}$$

Theorem 1.4.15. For all $z, w \in \mathbb{C}$ we have [3]

$$\begin{aligned}\cos(z \pm w) &= \cos z\cos w \mp \sin z\sin w\\ \sin(z \pm w) &= \sin z\cos w \pm \cos z\sin w\end{aligned}$$

$$\begin{aligned}\sin z - \sin w &= 2\cos\frac{z+w}{2}\sin\frac{z-w}{2}\\ \cos z - \cos w &= -2\sin\frac{z+w}{2}\sin\frac{z-w}{2}\end{aligned}$$

Proposition 1.4.16. Define $\exp_{\mathbb{R}} := e^z|_{\mathbb{R}}$.

(1) If $x < 0$, then $0 < e^x < 1$. If $x > 0$, then $1 < e^x < \infty$.

(2) $\exp_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}^+$is strictly increasing.

(3) For each $\alpha \in \mathbb{Q}$,

$$\lim_{x\to\infty}\frac{e^x}{x^\alpha} = \infty$$

that is, the exponential function increases faster than any power function.

(4) $\lim_{x\to-\infty} e^x = 0$.

Proposition 1.4.17. For all $a > 0$ and $r \in \mathbb{Q}$

$$a^r = e^{r\log a}.$$

Proposition 1.4.18. For all $\alpha > 0$,

$$\lim_{x\to\infty}\frac{\log x}{x^\alpha} = 0 \quad \text{and} \quad \lim_{x\to 0+} x^\alpha\log x = 0.$$

In particular, the logarithm function increases more slowly than any (arbitrarily small) positive power function.

Definition 1.4.19 (Definition of π)**.** Firstly, notice that

$$\left|e^{it}\right|^2 = e^{it}\overline{(e^{it})} = e^{it}e^{-it} = e^0 = 1, \quad t \in \mathbb{R},$$

Hence, define

$$\text{cis} : \mathbb{R} \to \mathbb{S}^1, \quad t \mapsto e^{it}$$

Now we show that cis is surjective:

Next step, we claim the set $M := \{t > 0; e^{it} = 1\}$ has a minimum element. First we show that M is nonempty. Since, there is some $t \in \mathbb{R}^\times$such that $e^{it} = -1$. Because

$$e^{-it} = \frac{1}{e^{it}} = \frac{1}{-1} = -1$$

we can suppose that $t > 0$. Then $e^{2it} = \left(e^{it}\right)^2 = (-1)^2 = 1$ and M is nonempty. Next we show that M is closed in $\mathbb{R}$. To prove this, choose a sequence (t_n) in M which converges to $t^* \in \mathbb{R}$. Since t_n is positive for all n, we have $t^* \geq 0$. In addition, the continuity of cis implies

$$e^{it^*} = \text{cis}\,(t^*) = \text{cis}\,(\lim t_n) = \lim \text{cis}\,(t_n) = 1$$

To prove that M is closed, it remains to show that t^* is positive. Suppose, to the contrary, that $t^* = 0$. Then there is some $m \in \mathbb{N}$ such that $t_m \in (0, 1)$. From Euler's formula we have $1 = e^{it_m} = \cos t_m + i \sin t_m$ and so $\sin t_m = 0$.

$$\sin t = t - \frac{t^3}{6} + (\frac{t^5}{5!} - \frac{t^7}{7!}) + \cdots \geq t - \frac{t^3}{6}$$

we get

$$\sin t \geq t\left(1 - t^2/6\right), \quad 0 < t < 1.$$

For t_m, this yields $0 = \sin t_m \geq t_m\,(1 - t_m^2/6) > 5t_m/6$, a contradiction. Thus M is closed. Since M is a nonempty closed set which is bounded below, it has minimum element.

The preceding lemma makes it possible to define a number π by

$$\pi := \frac{1}{2} \min\left\{t > 0; e^{it} = 1\right\}$$

Definition 1.4.20. The tangent and cotangent functions are defined by

$$\tan z := \frac{\sin z}{\cos z}, \quad z \in \mathbb{C}\backslash\left(\frac{\pi}{2} + \pi\mathbb{Z}\right), \quad \cot z := \frac{\cos z}{\sin z}, \quad z \in \mathbb{C}\backslash\pi\mathbb{Z}.$$

1.4.3 Differentiation in One Variable

Setting: $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, X be a subset of $\mathbb{K}$ and E be a normed vector space over $\mathbb{K}$. a is a limit point of X.

Definition 1.4.21. A function $f : X \to E$ is called differentiable at a if the limit

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists in E. When this occurs, $f'(a) \in E$ is called the derivative of f at a.

Proposition 1.4.22. (1) (linearity) Let $f, g : X \to E$ be differentiable at a and $\alpha, \beta \in \mathbb{K}$. Then the function $\alpha f + \beta g$ is also differentiable at a and

$$(\alpha f + \beta g)'(a) = \alpha f'(a) + \beta g'(a).$$

In other words, the set of functions which are differentiable at a forms a subspace V of E^X, and the function $V \to E, \quad f \mapsto f'(a)$ is linear.

(2) (product rule) Let $f, g : X \to \mathbb{K}$ be differentiable at a. Then the function $f \cdot g$ is also differentiable at a and

$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a).$$

The set of functions which are differentiable at a forms a subalgebra of $\mathbb{K}^X$.

(3) (quotient rule) Let $f, g : X \to \mathbb{K}$ be differentiable at a with $g(a) \neq 0$. Then the function f/g is also differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}$$

Proposition 1.4.23. Suppose that $f : X \to \mathbb{K}$ is differentiable at a, and $f(a)$ is a limit point of Y with $f(X) \subseteq Y \subseteq \mathbb{K}$. If $g : Y \to E$ is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

Proposition 1.4.24 (differentiability of inverse functions)**.** Let $f : X \to \mathbb{K}$ be injective and differentiable at a. Then, $f(a)$ is a limit point of $Y = f(X)$. Suppose that $f^{-1} : f(X) \to X$ is continuous at $b := f(a)$. Then f^{-1} is differentiable at b if and only if $f'(a)$ is nonzero. In this case,

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)}, \quad b = f(a)$$

Proof:

Setting: Let $X \subseteq \mathbb{K}$ be perfect.

Definition 1.4.25. $f : X \to E$ is called differentiable on X if f is differentiable at each point of X. The function

$$f' : X \to E, \quad x \mapsto f'(x)$$

is called the derivative of f. It is also denoted by $\dot{f}, \partial f, Df$ and df/dx.

If $f : X \to E$ is differentiable, then it is natural to ask whether the derivative f' is itself differentiable. When this occurs f is said to be twice differentiable and we call $\partial^2 f := f'' := \partial(\partial f)$ the second derivative of f. Repeating this process we can define further higher derivatives of f. Specifically, we set

$$\partial^0 f := f^{(0)} := f, \quad \partial^1 f(a) := f^{(1)}(a) := f'(a)$$
$$\partial^{n+1} f(a) := f^{(n+1)}(a) := \partial\left(\partial^n f\right)(a)$$

for all $n \in \mathbb{N}$. The element $\partial^n f(a) \in E$ is called the n^{th} derivative of f at a. The function f is called n-times differentiable on X if the n^{th} derivative exists at each $a \in X$. If f is n-times differentiable and the n^{th} derivative

$$\partial^n f : X \to E, \quad x \mapsto \partial^n f(x)$$

is continuous, then f is n-times continuously differentiable. The space of n-times continuously differentiable functions from X to E is denoted by $C^n(X, E)$. In particular, $C^0(X, E) = C(X, E)$ is the space of continuous E-valued functions on X. Finally

$$C^\infty(X, E) := \bigcap_{n \in \mathbb{N}} C^n(X, E)$$

If $E = \mathbb{K}$, we simply write $C^n(X, E)$ by $C^n(X)$.

Proposition 1.4.26. Let $X \subseteq \mathbb{K}$ be perfect, $k \in \mathbb{N}$ and $n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

(1) (linearity) For all $f, g \in C^k(X, E)$ and $\alpha, \beta \in \mathbb{K}$,

$$\alpha f + \beta g \in C^k(X, E) \quad \text{and} \quad \partial^k(\alpha f + \beta g) = \alpha \partial^k f + \beta \partial^k g.$$

Hence $C^n(X, E)$ is a subspace of $C(X, E)$ and the differentiation operator

$$\partial : C^{n+1}(X, E) \to C^n(X, E), \quad f \mapsto \partial f$$

is linear.

(2) (Leibniz' rule) Let $f, g \in C^k(X)$. Then $f \cdot g$ is in $C^k(X)$ and

$$\partial^k(fg) = \sum_{j=0}^{k} \binom{k}{j} (\partial^j f)\, \partial^{k-j} g$$

Hence $C^n(X)$ is a subalgebra of $\mathbb{K}^X$.

Theorem 1.4.27 (mean value theorem)**.** If $f \in C([a, b], \mathbb{R})$ is differentiable on (a, b), then there is some $\xi \in (a, b)$ such that

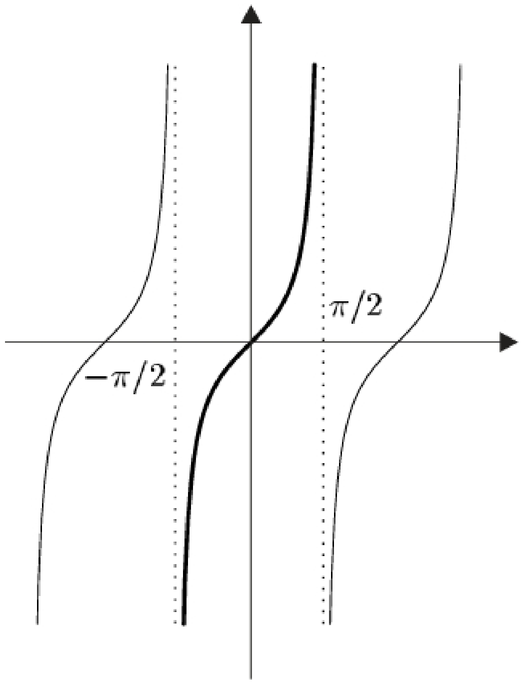
$$f(b) = f(a) + f'(\xi)(b - a).$$

Theorem 1.4.28 (mean value theorem, normed vector space version)**.** Suppose that E is a normed vector space and $f \in C([a, b], E)$ is differentiable on (a, b). Then

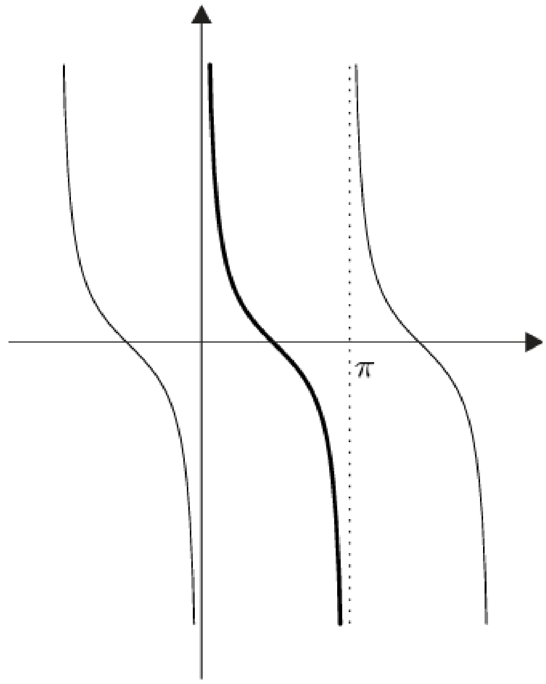
$$\|f(b) - f(a)\| \leq \sup_{t \in (a,b)} \|f'(t)\| \,(b - a).$$

Proposition 1.4.29. Suppose that I is a perfect interval and $f \in C(I, \mathbb{R})$ is differentiable on I°.

(1) f is increasing (or decreasing) if and only if $f'(x) \geq 0$ (or $f'(x) \leq 0$) for all $x \in I^\circ$.



Tangent



Cotangent

(2) If $f'(x) > 0$ (or $f'(x) < 0$) for all $x \in I^{\circ}$, then f is strictly increasing (or strictly decreasing).

Proposition 1.4.30. Suppose that I is a perfect interval and $f : I \to \mathbb{R}$ is differentiable with $f'(x) \neq 0$ for all $x \in I$.

(1) f is strictly monotone.

(2) $J := f(I)$ is a perfect interval.

(3) $f^{-1} : J \to \mathbb{R}$ is differentiable and $\left(f^{-1}\right)'(f(x)) = 1/f'(x)$ for all $x \in I$.

Definition 1.4.31. Define

$$\begin{aligned}
\arcsin &:= (\sin \mid (-\pi/2, \pi/2))^{-1} &&: (-1,1) \to (-\pi/2, \pi/2),\\
\arccos &:= (\cos \mid (0, \pi))^{-1} &&: (-1,1) \to (0, \pi),\\
\arctan &:= (\tan \mid (-\pi/2, \pi/2))^{-1} &&: \mathbb{R} \to (-\pi/2, \pi/2),\\
\operatorname{arccot} &:= (\cot \mid (0, \pi))^{-1} &&: \mathbb{R} \to (0, \pi).
\end{aligned}$$

To calculate the derivatives of the inverse trigonometric functions we use Theorem 2.8(iii). For the arcsine function this gives

$$\arcsin' x = \frac{1}{\sin' y} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}, \quad x \in (-1,1),$$

where we have set $y := \arcsin x$ and used $x = \sin y$. Similarly, for the arctangent function,

$$\arctan' x = \frac{1}{\tan' y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}, \quad x \in \mathbb{R},$$

where $y \in (-\pi/2, \pi/2)$ is determined by $x = \tan y$.

The derivatives of the arccosine and arccotangent functions can be calculated the same way and, summarizing, we have

$$\begin{aligned} \arcsin' x &= \frac{1}{\sqrt{1-x^2}}, \quad \arccos' x = \frac{-1}{\sqrt{1-x^2}}, \quad x \in (-1,1), \\ \arctan' x &= \frac{1}{1+x^2}, \quad \operatorname{arccot}' x = \frac{-1}{1+x^2}, \quad x \in \mathbb{R}. \end{aligned}$$

Setting: D convex perfect subset of $\mathbb{K}$, E be a Banach space, $f : D \to E$ be a function.

Theorem 1.4.32. For each $f \in C^n(D, E)$ and $a \in D$, there is a function $R_n(f, a) \in C(D, E)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(f,a)(x), \quad x \in D$$

The remainder function $R_n(f, a)$ satisfies

$$\|R_n(f,a)(x)\| \le \frac{1}{(n-1)!} \sup_{0<t<1} \left\| f^{(n)}(a + t(x-a)) - f^{(n)}(a) \right\| |x-a|^n$$

for all $x \in D$.

Definition 1.4.33. For $n \in \mathbb{N}, f \in C^n(D, E)$ and $a \in D$,

$$\mathcal{T}_n(f,a) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(X-a)^k$$

is a polynomial of degree $\le n$ with coefficients in E, the n^{th} Taylor polynomial of f at a, and

$$R_n(f,a) := f - \mathcal{T}_n(f,a)$$

is the n^{th} remainder function of f at a.

Now let $E := \mathbb{K}$ and $f \in C^\infty(D) := C^\infty(D, \mathbb{K})$. Then the formal expression

$$\mathcal{T}(f,a) := \sum_{k} \frac{f^{(k)}(a)}{k!}(X-a)^k$$

is called the Taylor series of f at a, and by the radius of convergence of $\mathcal{T}(f, a)$ we mean the radius of convergence of the power series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} X^k$$

Example 1.4.34 (a characterization of the exponential function)**.** Suppose that $a, b \in \mathbb{C}$, the function $f : \mathbb{C} \to \mathbb{C}$ is differentiable, and

$$f'(z) = bf(z), \quad z \in \mathbb{C}, \quad f(0) = a$$

Then $f(z) = ae^{bz}$ for all $z \in \mathbb{C}$.

Proof: Notice that $f \in C^\infty(\mathbb{C})$ and $f^{(k)} = b^k f$ for all $k \in \mathbb{N}$. If, in addition, $f(0) = a$, then

$$\sum_k \frac{f^{(k)}(0)}{k!} X^k = f(0) \sum_k \frac{b^k}{k!} X^k = a \sum_k \frac{b^k}{k!} X^k.$$

Since this power series has infinite radius of convergence, we have

$$\mathcal{T}(f,0)(z) = ae^{bz}, \quad z \in \mathbb{C}$$

To complete the proof, we need to prove that this Taylor series equals f on $\mathbb{C}$. Notice that

$$\begin{aligned} |R_n(f,0)(z)| &\le \sup_{0<t<1} \left| f^{(n)}(tz) - f^{(n)}(0) \right| \frac{|z|^n}{(n-1)!} = \frac{|b|^n |z|^n}{(n-1)!} \sup_{0<t<1} |f(tz) - a| \\ &\le M|bz| \frac{|bz|^{n-1}}{(n-1)!} \end{aligned}$$

where $M > 0$ has been chosen so that $|f(w) - a| \le M$ for all $w \in \overline{\mathbb{B}}(0, |z|)$.

Setting: $\mathbb{K} = \mathbb{R}$ and $E = \mathbb{R}$.

Theorem 1.4.35 (Schlömilch remainder formula)**.** Let I be a perfect interval, $a \in I$, $p > 0$ and $n \in \mathbb{N}$. Suppose that $f \in C^n(I, \mathbb{R})$ and $f^{(n+1)}$ exists on I°. Then, for each $x \in I \backslash \{a\}$, there is some $\xi := \xi(x) \in (\min\{x,a\}, \max\{x,a\})$ such that

$$R_n(f,a)(x) = \frac{f^{(n+1)}(\xi)}{pn!} \left(\frac{x-\xi}{x-a} \right)^{n-p+1} (x-a)^{n+1}.$$

In particular, take $p = n+1$ and $p = 1$ respectively, we obtain

$$R_n(f,a)(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1} \text{ (Lagrange)}$$

and

$$R_n(f,a)(x) = \frac{f^{(n+1)}(\xi)}{n!} \left(\frac{x-\xi}{x-a} \right)^n (x-a)^{n+1} \quad \text{(Cauchy)}$$

Example 1.4.36. Taylor series expansion for the general power function:

$$(1+x)^s = \sum_{k=0}^{\infty} \binom{s}{k} x^k, \quad x \in (-1,1)$$

Proof: Step 1: Notice that we only need to check the case when $s < 0$.

Step 2: Show that

$$\binom{s}{n} = \prod_{k=1}^{n} (1 - (s+1)/n) = \mathcal{O}(n)$$

.

Step 3: Show that, for each $x > -1$, there are some $\tau \in (0,1)$ and $\tau' \in (0,1)$ such that

$$(1+x)^s = \sum_{k=0}^{n} \binom{s}{k} x^k + \binom{s}{n+1} \frac{x^{n+1}}{(1+\tau x)^{n+1-s}} \text{ (Lagrange)}$$

and

$$(1+x)^s = \sum_{k=0}^{n} \binom{s}{k} x^k + \binom{s}{n+1} (n+1)(1+\tau' x)^{s-1} x (\frac{x - \tau' x}{1+\tau' x})^n \text{ (Cauchy)}$$

Step 4: For $-1 < x < 0$,

$$\frac{x - \tau' x}{1 + \tau' x} = 1 - \frac{x+1}{1+\tau' x} < -x < 1$$

1.4.4 Sequences of Functions

Setting: X is a set and E be a Banach space over $\mathbb{K}$.

Definition 1.4.37. An E-valued sequence of functions on X is simply a sequence (f_n) in E^X. If the choice of X and E is clear from the context (or irrelevant) we say simply that (f_n) is a sequence of functions.

The sequence of functions (f_n) converges pointwise to $f \in E^X$ if, for each $x \in X$, the sequence $(f_n(x))$ converges to $f(x)$ in E.

Definition 1.4.38. A sequence of functions (f_n) converges uniformly to f if, for each $\varepsilon > 0$, there is some $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon, \quad n \geq N, \quad x \in X.$$

Definition 1.4.39. Define $B(X, E)$ be the space of bounded functions. There's a natural norm

$$f \mapsto \sup_{x \in X} |f(x)|$$

on $B(X, E)$ and we denote it by $|| \cdot ||_\infty$. It's easy to check $(B(X, E), || \cdot ||_\infty)$ forms a Banach space.

If f_n and f are in $B(X, E)$, then (f_n) converges uniformly to f if and only if (f_n) converges to f in $B(X, E)$.

Proposition 1.4.40. The following are equivalent:

(1) The sequence of functions (f_n) converges uniformly.

(2) For each $\varepsilon > 0$, there is some $N := N(\varepsilon) \in \mathbb{N}$ such that

$$||f_n - f_m||_\infty < \varepsilon, \quad n, m \geq N.$$

Definition 1.4.41. Let (f_k) be an E-valued sequence of functions on X, that is, a sequence in E^X. Then

$$s_n := \sum_{k=0}^{n} f_k \in E^X, \quad n \in \mathbb{N}$$

and so we have a well defined sequence (s_n) in E^X.

The series $\sum f_k$ is called

$$\begin{aligned} \text{pointwise convergent } &:\Leftrightarrow \sum f_k(x) \text{ converges in } E \text{ for each } x \in X, \\ \text{absolutely convergent } &:\Leftrightarrow \sum |f_k(x)| < \infty \text{ for each } x \in X, \\ \text{uniformly convergent } &:\Leftrightarrow (s_n) \text{ converges uniformly}, \\ \text{norm convergent } &:\Leftrightarrow \sum ||f_k||_\infty < \infty. \end{aligned}$$

Remark 1.4.42. If f_k is norm convergent, then by Proposition 4.1.4, $f_k \mapsto f$ for some $f \in B(X, E)$. Hence, f_k is uniformly convergent and absolutely convergent.

Theorem 1.4.43 (Weierstrass majorant criterion)**.** Suppose that $f_k \in B(X,E)$ for all $k \in \mathbb{N}$. If there is a convergent series $\sum \alpha_k$ in $\mathbb{R}$ such that $\|f_k\|_\infty \leq \alpha_k$ for almost all $k \in \mathbb{N}$, then $\sum f_k$ is norm convergent. In particular, $\sum f_k$ converges absolutely and uniformly.

Theorem 1.4.44. Let $\sum a_k Y^k$ be a power series with positive radius of convergence ρ and $0 < r < \rho$. Then the series $\sum a_k Y^k$ is norm convergent on $r\overline{\mathbb{B}}_{\mathbb{K}}$. In particular, it converges absolutely and uniformly.

Setting: X metric space, E be a Banach space.

Definition 1.4.45. A sequence of functions (f_n) is called locally uniformly convergent if each $x \in X$ has a neighborhood U such that $(f_n \mid U)$ converges uniformly. A series of functions $\sum f_n$ is called locally uniformly convergent if the sequence of partial sums (s_n) converges locally uniformly.

Theorem 1.4.46. If a sequence of continuous functions (f_n) converges locally uniformly to f, then f is also continuous. In other words, locally uniform limits of continuous functions are continuous.

Theorem 1.4.47 (differentiability of the limits of sequences of functions)**.** Let X be an open subset of $\mathbb{K}$ and $f_n \in C^1(X,E)$ for all $n \in \mathbb{N}$. Suppose that there are $f, g \in E^X$ such that

(1) (f_n) converges pointwise to f, and

(2) (f_n') converges locally uniformly to g.

Then f is in $C^1(X,E)$, and $f' = g$. In addition, (f_n) converges locally uniformly to f.

Corollary 1.4.48. Suppose that $X \subseteq \mathbb{K}$ is open, and (f_n) is a sequence in $C^1(X,E)$ for which $\sum_n f_n$ converges pointwise and $\sum f_n'$ converges locally uniformly. Then the sum $\sum_{n=0}^\infty f_n$ is in $C^1(X,E)$ and

$$\left(\sum_{n=0}^\infty f_n\right)' = \sum_{n=0}^\infty f_n'$$

Setting: Let $a = \sum_k a_k X^k \in \mathbb{K}[\![X]\!]$ be a power series with radius of convergence $\rho = \rho_a > 0$, and $\underline{a}$ the function on $\rho\mathbb{B}_{\mathbb{K}}$ represented by a. When no misunderstanding is possible, we write $\mathbb{B}$ for $\mathbb{B}_{\mathbb{K}}$.

Theorem 1.4.49. Let $a = \sum_k a_k X^k$ be a power series. Then $\underline{a}$ is continuously differentiable on $\rho\mathbb{B}$. The 'termwise differentiated' series $\sum_{k\geq 1} k a_k X^{k-1}$ has radius of convergence ρ and

$$\underline{a}'(x) = \left(\sum_{k=0}^\infty a_k x^k\right)' = \sum_{k=1}^\infty k a_k x^{k-1}, \quad x \in \rho\mathbb{B}.$$

Corollary 1.4.50. If $a = \sum a_k X^k$ is a power series with positive radius of convergence ρ, then $\underline{a} \in C^\infty(\rho\mathbb{B}, \mathbb{K})$ and $a_k = \underline{a}^{(k)}(0)/k!$.

Definition 1.4.51. Let D be open in $\mathbb{K}$. A function $f : D \to \mathbb{K}$ is called analytic (on D) if, for each $x_0 \in D$, there is some $r = r(x_0) > 0$ such that $\mathbb{B}(x_0, r) \subseteq D$ and a power series $\sum_k a_k X^k$ with radius of convergence $\rho \geq r$, such that

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k, \quad x \in \mathbb{B}(x_0, r).$$

In this case, we say that $\sum_k a_k (X - x_0)^k$ is the power series expansion for f at x_0. The set of all analytic functions on D is denoted by $C^\omega(D, \mathbb{K})$, or by $C^\omega(D)$ if no misunderstanding is possible. Further, $f \in C^\omega(D)$ is called real (or complex) analytic if $\mathbb{K} = \mathbb{R}($ or $\mathbb{K} = \mathbb{C})$.

Proposition 1.4.52 (A power series represents an analytic function on its disk of convergence)**.** Suppose that $a = \sum a_k X^k$ is a power series with radius of convergence $\rho > 0$. Then $\underline{a} \in C^\omega(\rho\mathbb{B}, \mathbb{K})$ and

$$\underline{a}(x) = \mathcal{T}(\underline{a}, x_0)(x), \quad x_0 \in \rho\mathbb{B}, \quad x \in \mathbb{B}(x_0, \rho - |x_0|).$$

Definition 1.4.53. A nonempty open and connected subset of a metric space is called a domain.

Setting: Suppose that D is open in $\mathbb{K}$, E is a normed vector space and $f : D \to E$.

Definition 1.4.54. $F : D \to E$ is called an antiderivative of f if F is differentiable and $F' = f$.

Proposition 1.4.55. Let $D \subseteq \mathbb{K}$ be a domain and $f : D \to E$. If $F_1, F_2 \in E^D$ are antiderivatives of f, then $F_2 - F_1$ is constant. That is, antiderivatives are unique up to an additive constant.

Corollary 1.4.56. Let $E = \mathbb{K}$. If $f \in C^\omega(D, \mathbb{K})$ has an antiderivative F, then F is also analytic.

Proof: Let $x_0 \in D$. Then there is some $r > 0$ such that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k, \quad x \in \mathbb{B}(x_0, r) \subseteq D.$$

Then, there is some $a \in \mathbb{K}$ such that

$$F(x) = a + \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{(k+1)!} (x - x_0)^{k+1}, \quad x \in \mathbb{B}(x_0, r).$$

Them, F is analytic on $\mathbb{B}(x_0, r)$. Since analyticity is a local property, the claim follows.

Theorem 1.4.57. Let D be a domain in $\mathbb{K}$ and $f \in C^\omega(D, \mathbb{K})$. If the set of zeros of f has a limit point in D, then f is zero on D.

Example 1.4.58. Define $\log z = \log re^{i\theta} = \log r + i\theta$ for $\theta \in (-\pi, \pi)$. Then $\log z$ is a analytic function on $\mathbb{C} - (-\infty, 0]$.

Proof: It's easy to show that $(\log z)' = 1/z$. And notice that

$$\frac{1}{z} = \sum_{k=0}^{\infty} \frac{(-1)^k}{z_0^{k+1}} (z - z_0)^k, \quad z_0 \in \mathbb{C}^\times, \quad z \in \mathbb{B}_{\mathbb{C}}(z_0, |z_0|)$$

, we have $1/z$ is analytic in $\mathbb{C} - (-\infty, 0]$. By above Corollary, for some $c \in \mathbb{C}$,

$$\log z = c + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1) z_0^{k+1}} (z - z_0)^{k+1}, \quad z, z_0 \in \mathbb{C} \backslash (-\infty, 0], \quad |z - z_0| < |z_0|$$

In particular, for all $z \in \mathbb{B}_{\mathbb{C}}$,

$$\log(1+z) = \sum_{k=1}^{\infty} (-1)^{k-1} z^k / k$$

Example 1.4.59. For $z \in 1 + \mathbb{B}_{\mathbb{C}}$ and $\alpha \in \mathbb{C} - \mathbb{N}$, define $z^\alpha = e^{\alpha \log z}$, we have

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} z^k = (1+z)^\alpha, \quad z \in \mathbb{B}_{\mathbb{C}}$$

Proof: Let $a_k := \binom{\alpha}{k}$. Since $\alpha \notin \mathbb{N}$ we have $\lim |a_k / a_{k+1}| = \lim_k ((k+1)/|\alpha - k|) = 1$, the binomial series has radius of convergence 1 . Define $f(z) := \sum_{k=0}^{\infty} a_k z^k$ for all $z \in \mathbb{B}_{\mathbb{C}}$. We have

$$(1+z) f'(z) - \alpha f(z) = 0, \quad z \in \mathbb{B}_{\mathbb{C}},$$

from which follows

$$\left[(1+z)^{-\alpha} f(z)\right]' = (1+z)^{-\alpha-1} \left[(1+z) f'(z) - \alpha f(z)\right] = 0, \quad z \in \mathbb{B}_{\mathbb{C}}$$

Since $\mathbb{B}_{\mathbb{C}}$ is a domain, $(1+z)^{-\alpha} f(z) = c$ for some constant $c \in \mathbb{C}$. Since $f(0) = 1$, we have $c = 1$, and so $f(z) = (1+z)^\alpha$ for all $z \in \mathbb{B}_{\mathbb{C}}$.

Example 1.4.60 (The case $\alpha = 1/2$)**.** First we calculate the binomial coefficients:

$$\begin{aligned}
\binom{1/2}{k} &= \frac{1}{k!} \frac{1}{2} \left(\frac{1}{2} - 1\right) \cdot \cdots \cdot \left(\frac{1}{2} - k + 1\right) \\
&= \frac{(-1)^{k-1}}{k!} \frac{1 \cdot 3 \cdot \cdots \cdot (2k-3)}{2^k} \\
&= (-1)^{k-1} \frac{1 \cdot 3 \cdot \cdots \cdot (2k-3)}{2 \cdot 4 \cdot \cdots \cdot 2k}
\end{aligned}$$

for all $k \geq 2$. We get the series expansion

$$\sqrt{1+z} = 1 + \frac{z}{2} + \sum_{k=2}^{\infty} (-1)^{k-1} \frac{1 \cdot 3 \cdot \cdots \cdot (2k-3)}{2 \cdot 4 \cdot \cdots \cdot 2k} z^k, \quad z \in \mathbb{B}_{\mathbb{C}}$$

Example 1.4.61 (The case $\alpha = -1/2$)**.** Here we have

$$\binom{-1/2}{k} = (-1)^k \frac{1 \cdot 3 \cdot \cdots \cdot (2k-1)}{2 \cdot 4 \cdot \cdots \cdot 2k}, \quad k \geq 2$$

We get

$$\frac{1}{\sqrt{1+z}} = 1 - \frac{z}{2} + \sum_{k=2}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot \cdots \cdot (2k-1)}{2 \cdot 4 \cdot \cdots \cdot 2k} z^k, \quad z \in \mathbb{B}_{\mathbb{C}}.$$

If $|z| < 1$ then $|-z^2| < 1$ and so we can substitute $-z^2$ for z to get

$$\frac{1}{\sqrt{1-z^2}} = 1 + \frac{z^2}{2} + \sum_{k=2}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2k-1)}{2 \cdot 4 \cdot \cdots \cdot 2k} z^{2k}, \quad z \in \mathbb{B}_{\mathbb{C}}.$$

Example 1.4.62. The arcsine function is real analytic on $(-1, 1)$ and

$$\arcsin(x) = x + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2k-1)}{2 \cdot 4 \cdot \cdots \cdot 2k} \frac{x^{2k+1}}{2k+1}, \quad x \in (-1, 1).$$

1.5 Multivariable Differential Calculus

1.5.1 Differentiability

Setting: $E=(E,\|\cdot\|)$ and $F=(F,\|\cdot\|)$ are Banach spaces over the field $\mathbb{K}$, X open subset of E and $\mathcal{L}(E,F)$, the space of all bounded linear maps from E to F.

Definition 1.5.1. A function $f: X \rightarrow F$ is differentiable at $x_0 \in X$ if there is an $A_{x_0} \in \mathcal{L}(E,F)$ such that

$$\lim_{x \rightarrow x_0} \frac{f(x)-f\left(x_0\right)-A_{x_0}\left(x-x_0\right)}{\left\|x-x_0\right\|}=0.$$

Definition 1.5.2. Suppose $f: X \rightarrow F$ is differentiable at $x_0 \in X$. Then we denote by $\partial f\left(x_0\right)$ the linear operator $A_{x_0} \in \mathcal{L}(E,F)$ uniquely determined. This is called the derivative of f at x_0 and will also be written

$$D f\left(x_0\right) \text{ or } f^{\prime}\left(x_0\right).$$

Therefore $\partial f\left(x_0\right) \in \mathcal{L}(E,F)$, and

$$\lim_{x \rightarrow x_0} \frac{f(x)-f\left(x_0\right)-\partial f\left(x_0\right)\left(x-x_0\right)}{\left\|x-x_0\right\|}=0.$$

If $f: X \rightarrow F$ is differentiable at every point $x \in X$, we say f is differentiable and call the map

$$\partial f: X \rightarrow \mathcal{L}(E,F), \quad x \mapsto \partial f(x)$$

the derivative of f. Since $\mathcal{L}(E,F)$ is a Banach space, we can meaningfully speak of the continuity of the derivative. If ∂f is continuous, that is, $\partial f \in C(X, \mathcal{L}(E,F))$, we call f continuously differentiable. We set

$$C^1(X,F):=\{f: X \rightarrow F ; f \text{ is continuously differentiable }\}.$$

Definition 1.5.3 (Directional derivatives)**.** Suppose $f: X \rightarrow F, x_0 \in X$ and $v \in E \backslash\{0\}$. Because X is open, there is an $\varepsilon>0$ such that $x_0+t v \in X$ for $|t|<\varepsilon$. Therefore the function

$$(-\varepsilon, \varepsilon) \rightarrow F, \quad t \mapsto f\left(x_0+t v\right)$$

is well defined. When this function is differentiable at the point 0 , we call its derivative the directional derivative of f at the point x_0 in the direction v and denote it by $D_v f\left(x_0\right)$. Thus

$$D_v f\left(x_0\right)=\lim_{t \rightarrow 0} \frac{f\left(x_0+t v\right)-f\left(x_0\right)}{t}$$

Proposition 1.5.4. Suppose $f: X \rightarrow F$ is differentiable at $x_0 \in X$. Then $D_v f\left(x_0\right)$ exists for every $v \in E \backslash\{0\}$, and $D_v f\left(x_0\right)=\partial f\left(x_0\right) v$.

Example 1.5.5. We consider a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x,y):=\left\{\begin{array}{cl}
\dfrac{x^2 y}{x^2+y^2}, & (x,y) \neq(0,0) \\
0, & (x,y)=(0,0)
\end{array}\right.$$

For every $v=(\xi,\eta)\in\mathbb{R}^2\backslash\{(0,0)\}$, we have

$$f(tv)=\frac{t^3\xi^2\eta}{t^2\left(\xi^2+\eta^2\right)}=tf(v)$$

Thus

$$D_vf(0)=\lim_{t\to 0}f(tv)/t=f(v)$$

If f were differentiable at 0, $\partial f(0)v=D_vf(0)=f(v)$ for every $v\in\mathbb{R}^2\backslash\{(0,0)\}$. A contridiction!

Definition 1.5.6. Suppose X is open in $\mathbb{R}^n$ and $f=(f^1,\ldots,f^m):X\to\mathbb{R}^m$ is partially differentiable at x_0. We then call

$$\left[\partial_k f^j\left(x_0\right)\right]=\left[\begin{array}{ccc}\partial_1 f^1\left(x_0\right) & \cdots & \partial_n f^1\left(x_0\right)\\ \vdots & & \vdots\\ \partial_1 f^m\left(x_0\right) & \cdots & \partial_n f^m\left(x_0\right)\end{array}\right]$$

the Jacobi matrix of f at x_0.

Proposition 1.5.7. Suppose X is open in $\mathbb{R}^n$ and F is a Banach space. Then $f:X\to F$ is continuously differentiable if and only if f has continuous partial derivatives.

Corollary 1.5.8. Let X be open in $\mathbb{R}^n$. Then $f:X\to\mathbb{R}^m$ is continuously differentiable if and only if every coordinate function $f^j:X\to\mathbb{R}$ has continuous partial derivatives, and then

$$[\partial f(x)]=\left[\partial_k f^j(x)\right]\in\mathbb{R}^{m\times n}$$

Theorem 1.5.9 (Cauchy-Riemann Equation)**.** Suppose X is open at $\mathbb{C}$. For $f:X\to\mathbb{C}$, we set $u:=\operatorname{Re}f$ and $v:=\operatorname{Im}f$.

The function f is complex differentiable at $z_0=x_0+iy_0$ if and only if $F:=(u,v)$ is differentiable(as a function$f:X\subset\mathbb{R}^2\to\mathbb{R}^2$) at (x_0,y_0) and satisfies the Cauchy-Riemann equations

$$u_x=v_y,\quad u_y=-v_x$$

at (x_0,y_0). In that case,

$$f'\left(z_0\right)=u_x\left(x_0,y_0\right)+iv_x\left(x_0,y_0\right)$$

Proof: Suppose f is complex differentiable at z_0. We set

$$A:=\left[\begin{array}{cc}\alpha & -\beta\\ \beta & \alpha\end{array}\right]$$

where $\alpha:=\operatorname{Re}f'\left(z_0\right)$ and $\beta:=\operatorname{Im}f'\left(z_0\right)$. Then for $h=\xi+i\eta\longleftrightarrow(\xi,\eta)$, we have

$$\begin{aligned}&\lim_{(\xi,\eta)\to(0,0)}\frac{\left|F\left(x_0+\xi,y_0+\eta\right)-F\left(x_0,y_0\right)-A(\xi,\eta)\right|}{|(\xi,\eta)|}\\ &=\lim_{h\to 0}\left|\frac{f\left(z_0+h\right)-f\left(z_0\right)-f'\left(z_0\right)h}{h}\right|=0\end{aligned}$$

Therefore F is totally differentiable

$$\left[\partial F\left(x_0,y_0\right)\right]=\left[\begin{array}{cc}\partial_1 u\left(x_0,y_0\right) & \partial_2 u\left(x_0,y_0\right)\\ \partial_1 v\left(x_0,y_0\right) & \partial_2 v\left(x_0,y_0\right)\end{array}\right]=\left[\begin{array}{cc}\alpha & -\beta\\ \beta & \alpha\end{array}\right]$$

Theorem 1.5.10 (chain rule)**.** Suppose Y is open in F and G is a Banach space. Also suppose $f : X \to F$ is differentiable at x_0 and $g : Y \to G$ is differentiable at $y_0 := f(x_0)$ and that $f(X) \subset Y$. Then $g \circ f : X \to G$ is differentiable at x_0, and the derivative is given by

$$\partial(g \circ f)(x_0) = \partial g(f(x_0))\, \partial f(x_0).$$

Theorem 1.5.11 (mean value theorem in integral form)**.** In what follows, we use the notation $[\![x, y]\!]$ for the straight path $\{x + t(y - x); t \in [0, 1]\}$ between the points $x, y \in E$. Let $f \in C^1(X, F)$. Then we have

$$f(y) - f(x) = \int_0^1 \partial f(x + t(y - x))(y - x)dt$$

for $x, y \in X$ such that $[\![x, y]\!] \subset X$.

Proof: By Hahn-Banach Theorem, fundamental theorem of calculus and Proposition 2.3.12.

Corollary 1.5.12. X be a connect open subset of X, $f \in C^1(X, F)$, then if $\partial f = 0$, f is a constant.

1.5.2 Continuous Multilinear Map

Setting: In the following, $E_1, \ldots, E_m$ for $m \geq 2, E$, and F are Banach spaces over the field $\mathbb{K}$.

Definition 1.5.13. A map $\varphi : E_1 \times \cdots \times E_m \to F$ is multilinear or, equivalently, m-linear if for every $k \in \{1, \ldots, m\}$ and every choice of $x_j \in E_j$ for $j = 1, \ldots, m$ with $j \neq k$, the map

$$\varphi(x_1, \ldots, x_{k-1}, \cdot, x_{k+1}, \ldots, x_m) : E_k \to F$$

is linear.

Proposition 1.5.14. For the m-linear map $\varphi : E_1 \times \cdots \times E_m \to F$, these statements are equivalent:

(1) φ is continuous.

(2) φ is continuous at 0 .

(3) φ is bounded on bounded sets.

(4) There is an $\alpha \geq 0$ such that

$$\|\varphi(x_1, \ldots, x_m)\| \leq \alpha \|x_1\| \cdot \cdots \cdot \|x_m\| \quad \text{for } x_j \in E_j, \quad 1 \leq j \leq m$$

Theorem 1.5.15. Define

$$\|\varphi\| := \inf \{\alpha \geq 0; \|\varphi(x_1, \ldots, x_m)\| \leq \alpha \|x_1\| \cdot \cdots \cdot x_m\|, x_j \in E_j\}$$

for $\varphi \in \mathcal{L}(E_1, \ldots, E_m; F)$.

Then
$$\|\varphi\| = \sup\{\|\varphi(x_1,\ldots,x_m)\|\,;\|x_j\|\leq 1, 1\leq j\leq m\}$$
and
$$\mathcal{L}(E_1,\ldots,E_m;F) := (\mathcal{L}(E_1,\ldots,E_m;F), \|\cdot\|)$$
is a Banach space.

Theorem 1.5.16. The spaces $\mathcal{L}(E_1,\ldots,E_m;F)$ and $\mathcal{L}(E_1,\mathcal{L}(E_2,\ldots,\mathcal{L}(E_m,F)\ldots))$ are isometrically isomorphic.

Proof: We verify the statement for $m=2$. The general case obtains via a simple induction argument.

Firstly, for $T\in\mathcal{L}(E_1,\mathcal{L}(E_2,F))$ we set
$$\varphi_T(x_1,x_2) := (Tx_1)\,x_2 \quad \text{for } (x_1,x_2)\in E_1\times E_2$$
Then $\varphi_T : E_1\times E_2\to F$ is bilinear, and
$$\|\varphi_T(x_1,x_2)\| \leq \|T\|\,\|x_1\|\,\|x_2\| \quad \text{for } (x_1,x_2)\in E_1\times E_2.$$
Therefore φ_T belongs to $\mathcal{L}(E_1,E_2;F)$, and $\|\varphi_T\|\leq\|T\|$.

Secondly, suppose $\varphi\in\mathcal{L}(E_1,E_2;F)$. Then we set
$$T_\varphi(x_1)\,x_2 := \varphi(x_1,x_2) \quad \text{for } (x_1,x_2)\in E_1\times E_2$$
Because
$$\|T_\varphi(x_1)\,x_2\| = \|\varphi(x_1,x_2)\| \leq \|\varphi\|\,\|x_1\|\,\|x_2\| \quad \text{for } (x_1,x_2)\in E_1\times E_2$$
we get
$$T_\varphi(x_1)\in\mathcal{L}(E_2,F) \quad \text{for } \|T_\varphi(x_1)\|\leq\|\varphi\|\,\|x_1\|$$
for every $x_1\in E_1$. Therefore
$$T_\varphi := [x_1\mapsto T_\psi(x_1)]\in\mathcal{L}(E_1,\mathcal{L}(E_2,F)) \text{ and } \|T_\psi\|\leq\|\varphi\|$$

Altogether, we have proved that the maps
$$T\mapsto\varphi_{\mathrm{T}} : \mathcal{L}(E_1,\mathcal{L}(E_2,F))\to\mathcal{L}(E_1,E_2;F)$$
and
$$\varphi\mapsto T_\psi : \mathcal{L}(E_1,E_2;F)\to\mathcal{L}(E_1,\mathcal{L}(E_2,F))$$
are linear, bijective, isometry.

Proposition 1.5.17. Suppose $m\geq 2$ and $\varphi : E^m\to F$ is m-linear. We say φ is symmetric if
$$\varphi\left(x_{\sigma(1)},\ldots,x_{\sigma(m)}\right) = \varphi(x_1,\ldots,x_m)$$
for every $(x_1,\ldots,x_m)$ and every permutation σ of $\{1,\ldots,m\}$. We set
$$\mathcal{L}^m_{\text{sym}}(E,F) := \{\varphi\in\mathcal{L}^m(E,F);\varphi \text{ is symmetric }\}$$

$\mathcal{L}^m_{\text{sym}}(E,F)$ is a closed vector subspace of $\mathcal{L}^m(E,F)$ and is therefore itself a Banach space.

Proposition 1.5.18. $\mathcal{L}(E_1,\dots,E_m;F)$ is a vector subspace of $C^1(E_1\times\cdots\times E_m,F)$. And, for $\varphi\in\mathcal{L}(E_1,\dots,E_m;F)$ and $(x_1,\dots,x_m)\in E_1\times\cdots\times E_m$, we have

$$\partial\varphi(x_1,\dots,x_m)(h_1,\dots,h_m)=\sum_{j=1}^{m}\varphi(x_1,\dots,x_{j-1},h_j,x_{j+1},\dots,x_m)$$

for $(h_1,\dots,h_m)\in E_1\times\cdots\times E_m$.

Proof:

Definition 1.5.19. Suppose $f:X\to F$ and $x_0\in X$. We then set $\partial^0 f:=f$. Therefore $\partial^0 f(x_0)$ belongs to $F=\mathcal{L}^0(E,F)$. Suppose now that $m\in\mathbb{N}^\times$and $\partial^{m-1}f:X\to\mathcal{L}^{m-1}(E,F)$ is already defined. If

$$\partial^m f(x_0):=\partial\left(\partial^{m-1}f\right)(x_0)\in\mathcal{L}\left(E,\mathcal{L}^{m-1}(E,F)\right)=\mathcal{L}^m(E,F)$$

exists, we call $\partial^m f(x_0)$ the m-th derivative of f at x_0, we call

$$\partial^m f:X\to\mathcal{L}^m(E,F)$$

the m-th derivative of f. We set

$$C^m(X,F):=\{f:X\to F;f\text{ is }m\text{-times continuously differentiable }\}$$

and

$$C^\infty(X,F):=\bigcap_{m\in\mathbb{N}}C^m(X,F)$$

Proposition 1.5.20. For $f\in C^m(X,F)$ such that $m\geq 2$, we have $\partial^m f(x)\in\mathcal{L}^m_{\text{sym}}(E,F)$ for $x\in X$.

Proposition 1.5.21 (chain rule)**.** Suppose Y is open in F and G is a Banach space. Also suppose $m\in\mathbb{N}^\times$and $f\in C^m(X,F)$ with $f(X)\subset Y$ and $g\in C^m(Y,G)$. Then we have $g\circ f\in C^m(X,G)$.

Proposition 1.5.22. We consider first the case $E=\mathbb{R}^n$ with $n\geq 2$. For $q\in\mathbb{N}^\times$and indices $j_1,\dots,j_q\in\{1,\dots,n\}$, we call

$$\frac{\partial^q f(x)}{\partial x^{j_1}\partial x^{j_2}\cdots\partial x^{j_q}}:=\partial_{j_1}\partial_{j_2}\cdots\partial_{j_q}f(x)\qquad\text{for }x\in X$$

Suppose X is open in $\mathbb{R}^n$, $f:X\to F$, and $m\in\mathbb{N}^\times$. Then the following statements hold:

(1) f belongs to $C^m(X,F)$ if and only if f is m-times continuously partially differentiable.

(2) For $f\in C^m(X,F)$, we have

$$\frac{\partial^q f}{\partial x^{j_1}\cdots\partial x^{j_q}}=\frac{\partial^q f}{\partial x^{j_{\sigma(1)}}\cdots\partial x^{j_{\sigma(q)}}}\qquad\text{for }1\leq q\leq m$$

for every permutation $\sigma\in\mathrm{S}_q$, that is, the partial derivatives are independent of the order of differentiation.

Theorem 1.5.23 (Taylor's theorem)**.** Define

$$\partial^k f(x)[h]^k := \begin{cases} \partial^k f(x)[\underbrace{h,\ldots,h}_{k\text{-times}}], & 1 \le k \le q \\ f(x), & k = 0 \end{cases}$$

For $x \in X, h \in E$, and $f \in C^q(X,F)$. Suppose X is open in $E, q \in \mathbb{N}^\times$, and f belongs to $C^q(X,F)$. Then

$$f(x+h) = \sum_{k=0}^{q} \frac{1}{k!}\partial^k f(x)[h]^k + R_q(f,x;h)$$

for $x \in X$ and $h \in E$ such that $[\![x, x+h]\!] \subset X$. Here

$$R_q(f,x;h) := \int_0^1 \frac{(1-t)^{q-1}}{(q-1)!}\left[\partial^q f(x+th) - \partial^q f(x)\right][h]^q dt \in F$$

is the q-th order remainder of f at the point x.

1.5.3 Inverse maps and Implicit functions

Setting: E and F are Banach spaces over the field $\mathbb{K}$. $\mathrm{Lis}(E,F)$ be the set of bijective continous linear maps.(By Opeen Mapping Theorem, $A \in \mathrm{Lis}(E,F)$ implies $A^{-1} \in \mathrm{Lis}(F,E)$). $\mathbb{N} = \mathbb{Z}_{\ge 0}$, $\mathbb{N}^\times = \mathbb{Z}_{\ge 1}$.

Theorem 1.5.24 (inv is C^1)**.** Define $\mathrm{inv} : \mathrm{Lis}(E,F) \to \mathcal{L}(F,E), A \to A^{-1}$.

(1) $\mathrm{Lis}(E,F)$ is open in $\mathcal{L}(E,F)$.

(2) $\mathrm{inv} \in C^1(\mathrm{Lis}(E,F), \mathcal{L}(F,E))$ and

$$\partial \operatorname{Inv}(A)B = -A^{-1}BA^{-1} \quad \text{for } A \in \mathcal{L}\mathrm{is}(E,F) \text{ and } B \in \mathcal{L}(E,F)$$

Theorem 1.5.25 (inverse function)**.** Suppose X is open in E and $x_0 \in X$. Also suppose for $q \in \mathbb{N}^\times \cup \{\infty\}$ that $f \in C^q(X,F)$. Finally, suppose

$$\partial f(x_0) \in \mathcal{L}\mathrm{is}(E,F)$$

Then there is an open neighborhood U of x_0 in X and an open neighborhood V of $y_0 := f(x_0)$ with these properties:

(1) $f : U \to V$ is bijective.

(2) $f^{-1} \in C^q(V,E)$, and for every $x \in U$, we have

$$\partial f(x) \in \mathcal{L}\mathrm{is}(E,F) \quad \text{and} \quad \partial f^{-1}(f(x)) = [\partial f(x)]^{-1}$$

Definition 1.5.26. Suppose X is open in E, Y is open in F, and $q \in \mathbb{N} \cup \{\infty\}$. We call the map $f : X \to Y$ a C^q diffeomorphism from X to Y if it is bijective,

$$f \in C^q(X, F), \qquad \text{and} \qquad f^{-1} \in C^q(Y, E)$$

We may call a C^0 diffeomorphism a homeomorphism or a topological map. We set

$$\mathrm{Diff}^q(X, Y) := \{f : X \to Y; f \text{ is a } C^q \text{ diffeomorphism }\}$$

The map $g : X \to F$ is a locally C^q diffeomorphism if every $x_0 \in X$ has open neighborhoods U and V open neighborhood of $f(x_0)$ such that $g\,|_U$ belongs to $\mathrm{Diff}^q(U, V)$. We denote the set of all locally C^q diffeomorphisms from X to F by $\mathrm{Diff}^q_{\mathrm{loc}}(X, F)$.

Proposition 1.5.27. $f \in \mathrm{Diff}^q_{\mathrm{loc}}(X, F)$ for some $q \in \mathbb{N} \cup \{\infty\}$, then f is open.

Proof:

Proposition 1.5.28. Suppose X is open in $E, q \in \mathbb{N}^\times \cup \{\infty\}$, and $f \in C^q(X, F)$. Then $f \in \mathrm{Diff}^q_{\mathrm{loc}}(X, F) \Leftrightarrow \partial f(x) \in \mathrm{Lis}(E, F)$ for $x \in X$.

Setting: E_1, E_2 and F are Banach spaces over $\mathbb{K}$; $q \in \mathbb{N}^\times \cup \{\infty\}$. Suppose X_j is open in E_j for $j = 1, 2$, and $f : X_1 \times X_2 \to F$ is differentiable at (a, b). Then the functions $f(\cdot, b) : X_1 \to F$ and $f(a, \cdot) : X_2 \to F$ are also differentiable at a and b, respectively. We write $D_1 f(a, b)$ for the derivative of $f(\cdot, b)$ at a, and we write $D_2 f(a, b)$ for the derivative of $f(a, \cdot)$ at b.

Chapter 2

Measure

2.1 Measure Space

Definition 2.1.1 (algebra, σ-algebra)**.** Let X be a nonempty set. An algebra of sets on X is a nonempty collection $\mathcal{A}$ of subsets of X that is closed under finite unions and complements; in other words, if $E_1, \ldots, E_n \in \mathcal{A}$, then $\bigcup_1^n E_j \in \mathcal{A}$; and if $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$. A σ-algebra is an algebra that is closed under countable unions.

We observe that since $\bigcap_j E_j = \left(\bigcup_j E_j^c\right)^c$, algebras (resp. σ-algebras) are also closed under finite (resp. countable) intersections. Moreover, if $\mathcal{A}$ is an algebra, then $\varnothing \in \mathcal{A}$ and $X \in \mathcal{A}$, for if $E \in \mathcal{A}$ we have $\varnothing = E \cap E^c$ and $X = E \cup E^c$.

Definition 2.1.2. A countable intersection of open sets is called a G_δ set; a countable union of closed sets is called an F_σ set.

Definition 2.1.3. Let $\{X_\alpha\}_{\alpha \in A}$ be an indexed collection of nonempty sets, $X = \prod_{\alpha \in A} X_\alpha$, and $\pi_\alpha : X \to X_\alpha$ the coordinate maps. If $\mathcal{M}_\alpha$ is a σ-algebra on X_α for each α, the product σ-algebra on X is the σ-algebra generated by

$$\left\{\pi_\alpha^{-1}\left(E_\alpha\right) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\right\}.$$

We denote this σ-algebra by $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$. (If $A = \{1, \ldots, n\}$ we also write $\bigotimes_1^n \mathcal{M}_j$ or $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$.

Proposition 2.1.4. If A is countable, then $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is the σ-algebra generated by

$$\left\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\right\}$$

.

Proposition 2.1.5 (elementary family)**.** Define an elementary family to be a collection $\mathcal{E}$ of subsets of X such that

(1) $\varnothing \in \mathcal{E}$,

(2) If $E, F \in \mathcal{E}$ then $E \cap F \in \mathcal{E}$,

(3) If $E \in \mathcal{E}$ then E^c is a finite disjoint union of members of $\mathcal{E}$.

If $\mathcal{E}$ is an elementary family, the collection $\mathcal{A}$ of finite disjoint unions of members of $\mathcal{E}$ is an algebra.

Definition 2.1.6. Let X be a set equipped with a σ-algebra $\mathcal{M}$. A measure on $\mathcal{M}$ (or on $(X, \mathcal{M}\ )$, or simply on X if $\mathcal{M}$ is understood) is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

(1) $\mu(\varnothing) = 0$,

(2) if $\{E_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{M}$, then $\mu\left(\bigcup_1^\infty E_j\right) = \sum_1^\infty \mu\left(E_j\right)$.

If X is a set and $\mathcal{M} \subset \mathcal{P}(X)$ is a σ-algebra, $(X, \mathcal{M})$ is called a measurable space and the sets in $\mathcal{M}$ are called measurable sets. If μ is a measure on $(X, \mathcal{M})$, then $(X, \mathcal{M}, \mu)$ is called a measure space.

Definition 2.1.7. Let $(X, \mathcal{M}, \mu)$ be a measure space. Here is some standard terminology concerning the "size" of μ. If $\mu(X) < \infty$ (which implies that $\mu(E) < \infty$ for all $E \in \mathcal{M}$ since $\mu(X) = \mu(E) + \mu\left(E^c\right)), \mu$ is called finite. If $X = \bigcup_1^\infty E_j$ where $E_j \in \mathcal{M}$ and $\mu\left(E_j\right) < \infty$ for all j, μ is called σ-finite. More generally, if $E = \bigcup_1^\infty E_j$ where $E_j \in \mathcal{M}$ and $\mu\left(E_j\right) < \infty$ for all j, the set E is said to be σ-finite for μ.

If for each $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty, \mu$ is called semifinite.(σ-finite is semi-finte)

Example 2.1.8. Let X be any nonempty set, $\mathcal{M} = \mathcal{P}(X)$, and f any function from X to $[0, \infty]$. Then f determines a measure μ on $\mathcal{M}$ by the formula $\mu(E) = \sum_{x \in E} f(x)$.Two special cases are of particular significance: If $f(x) = 1$ for all x, μ is called counting measure; and if, for some $x_0 \in X, f$ is defined by $f\left(x_0\right) = 1$ and $f(x) = 0$ for $x \neq x_0$, μ is called the point mass or Dirac measure at x_0.

Proposition 2.1.9. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(1) (Monotonicity) If $E, F \in \mathcal{M}$ and $E \subset F$, then $\mu(E) \leq \mu(F)$.

(2) (Subadditivity) If $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu\left(\bigcup_1^\infty E_j\right) \leq \sum_1^\infty \mu\left(E_j\right)$.

(3) (Continuity from below) If $\{E_j\}_1^\infty \subset \mathcal{M}$ and $E_1 \subset E_2 \subset \cdots$, then $\mu\left(\bigcup_1^\infty E_j\right) = \lim_{j \to \infty} \mu\left(E_j\right)$.

(4) (Continuity from above) If $\{E_j\}_1^\infty \subset \mathcal{M}, E_1 \supset E_2 \supset \cdots$, and $\mu\left(E_1\right) < \infty$, then $\mu\left(\bigcap_1^\infty E_j\right) = \lim_{j \to \infty} \mu\left(E_j\right)$.

Definition 2.1.10. If $(X, \mathcal{M}, \mu)$ is a measure space, a set $E \in \mathcal{M}$ such that $\mu(E) = 0$ is called a null set. By subadditivity, any countable union of null sets is a null set, a fact which we shall use frequently. If a statement about points $x \in X$ is true except for x in some null set, we say that it is true almost everywhere (abbreviated a.e.), or for almost every x. (If more precision is needed, we shall speak of a μ-null set, or μ-almost everywhere).

Definition 2.1.11 (complete measure)**.** If $\mu(E) = 0$ and $F \subset E$, then $\mu(F) = 0$ by monotonicity provided that $F \in \mathcal{M}$, but in general it need not be true that $F \in \mathcal{M}$. A measure whose domain includes all subsets of null sets is called complete. Completeness can sometimes obviate annoying technical points, and it can always be achieved by enlarging the domain of μ, as follows.

Theorem 2.1.12. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}$. Then $\overline{\mathcal{M}}$ is a σ-algebra, and there is a unique extension $\bar{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

Definition 2.1.13 (outer measure)**.** The abstract generalization of the notion of outer area is as follows. An outer measure on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies

(1) $\mu^*(\varnothing) = 0$,

(2) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$,

(3) $\mu^*\left(\bigcup_1^\infty A_j\right) \leq \sum_1^\infty \mu^*\left(A_j\right)$.

Proposition 2.1.14. Let $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\varnothing \in \mathcal{E}, X \in \mathcal{E}$, and $\rho(\varnothing) = 0$. For any $A \subset X$, define

$$\mu^*(A) = \inf \left\{ \sum_1^\infty \mu\left(E_j\right) : E_j \in \mathcal{E} \text{ and } A \subset \bigcup_1^\infty E_j \right\}.$$

Then μ^* is an outer measure.

Proposition 2.1.15. If μ^* is an outer measure on X, a set $A \subset X$ is called $\boldsymbol{\mu}^{\star}$-measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*\left(E \cap A^c\right) \text{ for all } E \subset X.$$

Theorem 2.1.16 (Carathéodory's Theorem)**.** If μ^* is an outer measure on X, the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

Definition 2.1.17. If $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, a function $\mu_0 : \mathcal{A} \to [0, \infty]$ will be called a premeasure if

(1) $\mu_0(\varnothing) = 0$,

(2) if $\{A_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{A}$ such that $\bigcup_1^\infty A_j \in \mathcal{A}$, then $\mu_0\left(\bigcup_1^\infty A_j\right) = \sum_1^\infty \mu_0\left(A_j\right)$.

In particular, a premeasure is finitely additive since one can take $A_j = \varnothing$ for j large. The notions of finite and σ-finite premeasures are defined just as for measures.

Theorem 2.1.18. If μ_0 is a premeasure on $\mathcal{A} \subset \mathcal{P}(X)$, it induces an outer measure on X, namely,

$$\mu^*(E) = \inf\left\{\sum_1^\infty \mu_0\left(A_j\right) : A_j \in \mathcal{A}, E \subset \bigcup_1^\infty A_j\right\}.$$

then every set in $\mathcal{A}$ is μ^* measurable and $\mu^* \mid \mathcal{A} = \mu_0$.

Theorem 2.1.19. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, μ_0 a premeasure on $\mathcal{A}$, and $\mathcal{M}$ the σ-algebra generated by $\mathcal{A}$. There exists a measure μ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0 - namely, $\mu = \mu^* \mid \mathcal{M}$ where μ^* is given by Proposition 2.1.13. If μ_0 is σ-finite, then μ is the unique extension of μ_0 to a measure on $\mathcal{M}$ and the completion of μ is $\mu^*|M^*$ where M^* is the μ^*-measurable sets.

Example 2.1.20 (Lebesgue-Stieltjes measure)**.** Consider sets of the form $(a, b]$ or (a, ∞) or $\varnothing$, where $-\infty \leq a < b < \infty$. In this section we shall refer to such sets as h-intervals (h for "half-open"). Clearly the intersection of two h-intervals is an h-interval, and the complement of an h-interval is an h-interval or the disjoint union of two h-intervals. Hence the collection $\mathcal{A}$ of finite disjoint unions of h-intervals is an algebra. Notice tha the σ-algebra generated by $\mathcal{A}$ is $\mathcal{B}_{\mathbb{R}}$.

Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. If $(a_j, b_j]$ $(j = 1, \ldots, n)$ are disjoint h-intervals, let

$$\mu_0\left(\bigcup_1^n (a_j, b_j]\right) = \sum_1^n \left[F\left(b_j\right) - F\left(a_j\right)\right]$$

and let $\mu_0(\varnothing) = 0$. Then μ_0 is a premeasure on the algebra $\mathcal{A}$.

Example 2.1.21. If $F : \mathbb{R} \to \mathbb{R}$ is any increasing, right continuous function, there is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all a, b. If G is another such function, we have $\mu_F = \mu_G$ iff $F - G$ is constant. Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets and we define

$$F(x) = \begin{cases} \mu((0, x]) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu((-x, 0]) & \text{if } x < 0 \end{cases}$$

then F is increasing and right continuous, and $\mu = \mu_F$.

Example 2.1.22 (Lebesgue measure)**.** This is the complete measure μ_F associated to the function $F(x) = x$, for which the measure of an interval is simply its length. We shall denote it by m. The domain of m is called the class of Lebesgue measurable sets, and we shall denote it by $\mathcal{L}$. In this book, we refer to the restriction of m to $\mathcal{B}_{\mathbb{R}}$ as Lebesgue measure.

Proposition 2.1.23. If $E \in \mathcal{L}$, then $E + s \in \mathcal{L}$ and $rE \in \mathcal{L}$ for all $s, r \in \mathbb{R}$. Moreover, $m(E + s) = m(E)$ and $m(rE) = |r|m(E)$.

Definition 2.1.24. If X is any topological space, the σ-algebra generated by the family of open sets in X is called the Borel σ-algebra on X and is denoted by $\mathcal{B}_X$. Its members are called Borel sets. $\mathcal{B}_X$ thus includes open sets, closed sets, countable intersections of open sets, countable unions of closed sets, and so forth.

Proposition 2.1.25. Let $X_1, \ldots, X_n$ be topological spaces and let $X = \prod_1^n X_j$, equipped with the product topology. Then $\bigotimes_1^n \mathcal{B}_{X_j} \subset \mathcal{B}_X$. If the X_j 's are secound countable, then $\bigotimes_1^n \mathcal{B}_{X_j} = \mathcal{B}_X$

Proposition 2.1.26. X is a topological space, $Y \in B_X$ be a measurable set. Give Y the subspace topology from X, then B_Y equals to the σ-algebra $\{Y \cap E : E \in B_X\}$

2.2 Intergration, $\overline{\mathbb{R}}$-valued

Proposition 2.2.1. $f : X \to Y$ between two sets induces a mapping $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$, defined by $f^{-1}(E) = \{x \in X : f(x) \in E\}$, which preserves unions, intersections, and complements. Thus, if $\mathcal{N}$ is a σ-algebra on Y, $\{f^{-1}(E) : E \in \mathcal{N}\}$ is a σ-algebra on X. If $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces, a mapping $f : X \to Y$ is called $(\mathcal{M}, \mathcal{N})$-measurable, or just measurable when $\mathcal{M}$ and $\mathcal{N}$ are understood, if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

If $\mathcal{N}$ is generated by $\mathcal{E}$, then $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable iff $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

Proposition 2.2.2. If $(X, \mathcal{M})$ is a measurable space and $f : X \to \mathbb{R}$, the following are equivalent:

(1) f is $\mathcal{M}$-measurable.

(2) $f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

(3) $f^{-1}([a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

(4) $f^{-1}((-\infty, a)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

(5) $f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

Proposition 2.2.3. A function $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable iff $\operatorname{Re} f$ and $\operatorname{Im} f$ are $\mathcal{M}$-measurable.

Definition 2.2.4. It is sometimes convenient to consider functions with values in the extended real number system $\overline{\mathbb{R}} = [\infty, \infty]$(with order topology). It is easily verified that $\mathcal{B}_{\overline{\mathbb{R}}}$ is generated by the rays $(a, \infty]$ or $[-\infty, a)(a \in \mathbb{R})$, and we define $f : X \to \overline{\mathbb{R}}$ to be $\mathcal{M}$-measurable if it is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable. And we always define $0 \cdot \infty$ to be 0.

Proposition 2.2.5. If $f, g : X \to \mathbb{C}$ are $\mathcal{M}$-measurable, then so are $f + g$ and fg.

Proposition 2.2.6. If $\{f_j\}$ is a sequence of $\overline{\mathbb{R}}$-valued measurable functions on $(X, \mathcal{M})$, then the functions

$$g_1(x) = \sup_j f_j(x), \quad g_3(x) = \overline{\lim}_{j\to\infty} f_j(x),$$
$$g_2(x) = \inf_j f_j(x), \quad g_4(x) = \underline{\lim}_{j\to\infty} f_j(x)$$

are all measurable.

Corollary 2.2.7. If $f, g : X \to \overline{\mathbb{R}}$ are measurable, then so are $\max(f, g)$ and $\min(f, g)$.

If $\{f_j\}$ is a sequence of complex-valued measurable functions and $f(x) = \lim_{j\to\infty} f_j(x)$ exists for all x, then f is measurable.

Definition 2.2.8 (simple function)**.** Suppose that $(X, \mathcal{M})$ is a measurable space. If $E \subset X$, the characteristic function χ_E of E (sometimes called the indicator function of E and denoted by 1_E) is defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Equivalently, $f : X \to \mathbb{C}$ is simple iff f is measurable and the range of f is a finite subset of $\mathbb{C}$. Indeed, we have

$$f = \sum_1^n z_j \chi_{E_j}, \text{ where } E_j = f^{-1}(\{z_j\}) \text{ and range } (f) = \{z_1, \ldots, z_n\}.$$

Theorem 2.2.9. Let $(X, \mathcal{M})$ be a measurable space. If $f : X \to [0, \infty]$ is measurable, there is a sequence $\{\phi_n\}$ of simple functions such that $0 \le \phi_1 \le \phi_2 \le \cdots \le f, \phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on any set on which f is bounded.

If $f : X \to \mathbb{C}$ is measurable, there is a sequence $\{\phi_n\}$ of simple functions such that $0 \le |\phi_1| \le |\phi_2| \le \cdots \le |f|, \phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on any set on which f is bounded.

Definition 2.2.10. The following implications are valid iff the measure μ is complete:

(1) If f is measurable and $f = g$ μ-a.e., then g is measurable.

(2) If f_n is measurable for $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e., then f is measurable.

Proposition 2.2.11. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(X, \overline{\mathcal{M}}, \bar{\mu})$ be its completion. If f is an $\overline{\mathcal{M}}$-measurable function on X, there is an $\mathcal{M}$-measurable function g such that $f = g\bar{\mu}$-almost everywhere.

Definition 2.2.12. In this section we fix a measure space $(X, \mathcal{M}, \mu)$, and we define

$$L^+ = \text{the space of all measurable functions from } X \text{ to } [0, \infty].$$

If ϕ is a simple function in L^+ with standard representation $\phi = \sum_1^n a_j \chi_{E_j}$, we define the integral of ϕ with respect to μ by

$$\int \phi d\mu = \sum_1^n a_j \mu(E_j)$$

Proposition 2.2.13. Let ϕ and ψ be simple functions in L^+.

(1) If $c \ge 0, \int c\phi = c \int \phi$.

(2) $\int (\phi + \psi) = \int \phi + \int \psi$.

(3) If $\phi \le \psi$, then $\int \phi \le \int \psi$.

Definition 2.2.14. We now extend the integral to all functions $f \in L^+$ by defining

$$\int f d\mu = \sup \left\{ \int \phi d\mu : 0 \le \phi \le f, \phi \text{ simple} \right\}.$$

Theorem 2.2.15. If $\{f_n\}$ is a sequence in L^+such that $f_j \le f_{j+1}$ for all j, and

$$f = \lim_{n \to \infty} f_n \left(= \sup_n f_n \right)$$

, then $\int f = \lim_{n \to \infty} \int f_n$.

Corollary 2.2.16. If $\{f_n\}$ is a finite or infinite sequence in $L^+$and $f = \sum_n f_n$, then $\int f = \sum_n \int f_n$.

Proposition 2.2.17. If $f \in L^+$, then $\int f = 0$ iff $f = 0$ a.e.

Lemma 2.2.18 (Fatou's lemma,)**.** If $\{f_n\}$ is any sequence in L^+, then

$$\int (\liminf f_n) \leq \liminf \int f_n.$$

Proposition 2.2.19. The two definitions of $\int f$ agree when f is simple, as the family of simple functions over which the supremum is taken includes f itself and

$$\int f \leq \int g \text{ whenever } f \leq g, \text{ and } \int cf = c \int f \text{ for all } c \in [0, \infty).$$

Definition 2.2.20. If $f^+$and $f^-$are the positive and negative parts of f and at least one of $\int f^+$and $\int f^-$is finite, we define

$$\int f = \int f^+ - \int f^-.$$

We shall be mainly concerned with the case where $\int f^+$and $\int f^-$are both finite; we then say that f is integrable. Since $|f| = f^+ + f^-$, it is clear that f is integrable iff $\int |f| < \infty$

Next, if f is a complex-valued measurable function, we say that f is integrable if $\int |f| < \infty$. More generally, if $E \in \mathcal{M}$, f is integrable on E if $\int_E |f| < \infty$. Since $|f| \leq |\operatorname{Re} f| + |\operatorname{Im} f| \leq 2|f|$, f is integrable iff $\operatorname{Re} f$ and $\operatorname{Im} f$ are both integrable, and in this case we define

$$\int f = \int \operatorname{Re} f + i \int \operatorname{Im} f.$$

It follows easily that the space of complex-valued integrable functions is a complex vector space and that the integral is a complex-linear functional on it. We denote this space - provisionally - by $L^1(\mu)$ (or $L^1(X, \mu)$, or $L^1(X)$, or simply L^1, depending on the context).

Proposition 2.2.21. If $f \in L^1$, then $\left|\int f\right| \leq \int |f|$.

Proposition 2.2.22. (1) If $f \in L^1$, then $\{x : f(x) \neq 0\}$ is σ-finite.

(2) If $f, g \in L^1$, then $\int_E f = \int_E g$ for all $E \in \mathcal{M}$ iff $\int |f - g| = 0$ iff $f = g$ a.e.

Theorem 2.2.23 (Dominated Convergence Theorem)**.** Let $\{f_n\}$ be a sequence in L^1 such that

(1) $f_n \to f$

(2) there exists a nonnegative $g \in L^1$ such that $|f_n| \leq g$ for all n. Then $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n$.

Proof: By Fatou's lemma.

Theorem 2.2.24. Suppose that $\{f_j\}$ is a sequence in L^1 such that $\sum_1^\infty \int |f_j| < \infty$. Then $\sum_1^\infty f_j$ converges a.e. to a function in L^1, and

$$\int \sum_1^\infty f_j = \sum_1^\infty \int f_j$$

.

Theorem 2.2.25. If $f \in L^1(\mu)$ and $\epsilon > 0$, there is an integrable simple function $\phi = \sum a_j \chi_{E_j}$ such that $\int |f - \phi| d\mu < \epsilon$. (That is, the integrable simple functions are dense in L^1 in the L^1 metric.)

If μ is a Borel measure on $\mathbb{R}$, the sets E_j in the definition of ϕ can be taken to be finite unions of open intervals; moreover, there is a continuous function g that vanishes outside a bounded interval such that $\int |f - g| d\mu < \epsilon$.

Theorem 2.2.26. Suppose U is open in $\mathbb{R}^n$, or $U \subset \mathbb{C}$ is perfect and convex, and suppose $f : X \times U \to \mathbb{C}$ satisfies

(1) $f(\cdot, y) \in L^1(X, \mu)$ for every $y \in U$;

(2) $f(x, \cdot) \in C^1(U, \mathbb{C})$ for every $x \in X$;

(3) there exists $g \in L^1(X, \mu, \mathbb{R})$ such that

$$\left|\frac{\partial}{\partial y^j} f(x,y)\right| \le g(x) \qquad \text{for } (x,y) \in X \times U \text{ and } 1 \le j \le n$$

Then

$$F : U \to \mathbb{C}, \quad y \mapsto \int_X f(x,y)\mu(dx)$$

is continuously differentiable and

$$\partial_j F(y) = \int_X \frac{\partial}{\partial y^j} f(x,y)\mu(dx) \qquad \text{for } y \in U \text{ and } 1 \le j \le n$$

Definition 2.2.27. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. We have already discussed the product σ-algebra $\mathcal{M} \otimes \mathcal{N}$ on $X \times Y$; we now construct a measure on $\mathcal{M} \otimes \mathcal{N}$ that is, in an obvious sense, the product of μ and ν.

To begin with, we define a (measurable) rectangle to be a set of the form $A \times B$ where $A \in \mathcal{M}$ and $B \in \mathcal{N}$. Clearly

$$(A \times B) \cap (E \times F) = (A \cap E) \times (B \cap F), \quad (A \times B)^c = (X \times B^c) \cup (A^c \times B).$$

Therefore, by Proposition 2.1.5, the collection $\mathcal{A}$ of finite disjoint unions of rectangles is an algebra, and of course the σ-algebra it generates is $\mathcal{M} \otimes \mathcal{N}$.

If we integrate with respect to x

$$\begin{aligned}\mu(A)\chi_B(y) = \int \chi_A(x)\chi_B(y) d\mu(x) &= \sum \int \chi_{A_j}(x)\chi_{B_j}(y) d\mu(x) \\ &= \sum \mu(A_j)\, \chi_{B_j}(y).\end{aligned}$$

In the same way, integration in y then yields

$$\mu(A)\nu(B)=\sum\mu\left(A_j\right)\nu\left(B_j\right).$$

It follows that if $E\in\mathcal{A}$ is the disjoint union of rectangles $A_1\times B_1,\dots,A_n\times B_n$, and we set

$$\pi(E)=\sum_1^n\mu\left(A_j\right)\nu\left(E_j\right)$$

then π is well defined on $\mathcal{A}$ (since any two representations of E as a finite disjoint union of rectangles have a common refinement), and π is a premeasure on $\mathcal{A}$. Therefore, π generates an outer measure on $X\times Y$ whose restriction to $\mathcal{M}\times\mathcal{N}$ is a measure that extends π. We call this measure the product of μ and ν and denote it by $\mu\times\nu$. Moreover, if μ and ν are σ-finite - say, $X=\bigcup_1^\infty A_j$ and $Y=\bigcup_1^\infty B_k$ with $\mu\left(A_j\right)<\infty$ and $\nu\left(B_k\right)<\infty$ - then $X\times Y=\bigcup_{j,k}A_j\times B_k$, and $\mu\times\nu\left(A_j\times B_k\right)<\infty$, so $\mu\times\nu$ is also σ-finite.Then $\mu\times\nu$ is the unique measure on $\mathcal{M}\otimes\mathcal{N}$ such that $\mu\times\nu(A\times B)=\mu(A)\nu(B)$ for all rectangles $A\times B$.

The same construction works for any finite number of factors. That is, suppose $(X_j,\mathcal{M}_j,\mu_j)$ are measure spaces for $j=1,\dots,n$. If we define a rectangle to be a set of the form $A_1\times\cdots\times A_n$ with $A_j\in\mathcal{M}_j$, then the collection $\mathcal{A}$ of finite disjoint unions of rectangles is an algebra, and the same procedure as above produces a measure $\mu_1\times\cdots\times\mu_n$ on $\mathcal{M}_1\otimes\cdots\otimes\mathcal{M}_n$ such that

$$\mu_1\times\cdots\times\mu_n\left(A_1\times\cdots\times A_n\right)=\prod_1^n\mu_j\left(A_j\right).$$

Moreover, if the μ_j 's are σ-finite so that the extension from $\mathcal{A}$ to $\bigotimes_1'\mathcal{M}_j$ is uniquely determined.

Proposition 2.2.28. If $(X_j,\mathcal{M}_j)$ is a measurable space for $j=1,2,3$, then $\bigotimes_1^3\mathcal{M}_j=(\mathcal{M}_1\otimes\mathcal{M}_2)\otimes\mathcal{M}_3$. Moreover, if μ_j is a σ-finite measure on $(X_j,\mathcal{M}_j)$, then $\mu_1\times\mu_2\times\mu_3=(\mu_1\times\mu_2)\times\mu_3$.

Proof: Consider $\{A\subset M_1\otimes M_2 : A\times E_3\in M_1\otimes M_2\otimes M_3\}$ for some $E_3\in M_3$ is a σ-algebra.

Definition 2.2.29. We return to the case of two measure spaces $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$. If $E\subset X\times Y$, for $x\in X$ and $y\in Y$ we define the x-section E_x and the y-section E^y of E by

$$E_x=\{y\in Y:(x,y)\in E\},\quad E^y=\{x\in X:(x,y)\in E\}.$$

Also, if f is a function on $X\times Y$ we define the x-section f_x and the y-section f^y of f by

$$f_x(y)=f^y(x)=f(x,y).$$

Proposition 2.2.30. If $E\in\mathcal{M}\otimes\mathcal{N}$, then $E_x\in\mathcal{N}$ for all $x\in X$ and $E^y\in\mathcal{M}$ for all $y\in Y$.

If f is $\mathcal{M}\otimes\mathcal{N}$-measurable, then f_x is $\mathcal{N}$-measurable for all $x\in X$ and f^y is $\mathcal{M}$-measurable for all $y\in Y$.

Theorem 2.2.31. Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$, then the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable on X and Y, respectively, and

$$\mu \times \nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y).$$

Theorem 2.2.32 (The Fubini-Tonelli Theorem)**.** Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ finite measure spaces.

(1) (Tonelli) If $f \in L^+(X \times Y)$, then the functions $g(x) = \int f_x d\nu$ and $h(y) = \int f^y d\mu$ are in $L^+(X)$ and $L^+(Y)$, respectively, and

$$\begin{aligned} \int f d(\mu \times \nu) &= \int \left[\int f(x,y) d\nu(y) \right] d\mu(x) \\ &= \int \left[\int f(x,y) d\mu(x) \right] d\nu(y). \end{aligned}$$

(2) (Fubini) If $f \in L^1(\mu \times \nu)$, then $f_x \in L^1(\nu)$ for a.e. $x \in X$, $f^y \in L^1(\mu)$ for a.e. $y \in Y$, Define

$$g(x) = \begin{cases} \int f_x & \text{if } f_x \in L^1(\nu) \\ 0 & \text{otherwise} \end{cases}$$

and

$$h(y) = \begin{cases} \int f^y & \text{if } f_x \in L^1(\nu) \\ 0 & \text{otherwise} \end{cases}$$

, we have $g(x) \in L^1(\mu)$, $h(y) \in L^1(\nu)$ and $\int g(x) \mathrm{d}\mu = \int h(y) \mathrm{d}\mu = \int f \mathrm{d}(\mu \times \nu)$.

2.3 Bochner-Lebesgue integral

Setting: $(X, \mathcal{A}, \mu)$ a measure space and $E = (E, |\cdot|)$ a Banach space over $\mathbb{C}$.

Definition 2.3.1 (simple functions). We say $f \in E^X$ is μ-simple [1] if $f(X)$ is finite, $f^{-1}(e) \in \mathcal{A}$ for every $e \in E$, and $\mu\left(f^{-1}(E\backslash\{0\})\right) < \infty$. We denote by $\mathcal{S}(X, \mu, E)$ the set of all μ-simple functions.

Definition 2.3.2 (simple function). For $\varphi \in \sum_{j=0}^{m} e_j \chi_{A_j} \in \mathcal{S}(X, \mu, E)$, we define the integral of φ over X with respect to the measure μ as the sum

$$\int_X \varphi d\mu := \int \varphi d\mu := \sum_{j=0}^{m} e_j \mu\left(A_j\right)$$

If A is a μ-measurable set, we define the integral of φ over A with respect to the measure μ as

$$\int_A \varphi d\mu := \int_X \chi_A \varphi d\mu.$$

Definition 2.3.3. Let

$$\|\varphi\|_1 := \int_X |\varphi| d\mu \quad \text{ for } \varphi \in \mathcal{S}(X, \mu, E).$$

Then $\|\cdot\|_1$ is a semi-norm on $\mathcal{S}(X, \mu, E)$.

Definition 2.3.4. A function $f \in E^X$ is said to be μ-measurable if there is a sequence (f_j) in $\mathcal{S}(X, \mu, E)$ such that $f_j \to f$ μ-almost everywhere as $j \to \infty$. We set

$$\mathcal{L}_0(X, \mu, E) := \left\{f \in E^X : f \text{ is } \mu\text{-measurable}\right\}$$

Definition 2.3.5. A function $f \in E^X$ is said to be $\mathcal{A}$-measurable if the inverse images of open sets of E under f are measurable, that is, if $f^{-1}\left(\mathcal{T}_E\right) \subset \mathcal{A}$, where $\mathcal{T}_E$ is the norm topology on E. If there is a μ-null set N such that $f\left(N^c\right)$ is separable, we say f is μ-almost separable valued.

Proposition 2.3.6 (μ-measurable criterion). Function in E^X is μ-measurable if and only if it is $\mathcal{A}$-measurable and μ-almost separable valued.

Proposition 2.3.7. Suppose (f_j) is a sequence in $\mathcal{L}_0(X, \mu, E)$ and $f \in E^X$. If (f_j) converges μ-almost everywhere to f, then f is μ-measurable.

Definition 2.3.8 (integrable). A function $f \in E^X$ is called μ-integrable if f is a μ-a.e. limit of some $\mathcal{L}_1$-Cauchy sequence (φ_j) in $\mathcal{S}(X, \mu, E)$. We denote the set of E-valued, μ-integrable functions of X by $\mathcal{L}_1(X, \mu, E)$.

Definition 2.3.9 (Bochner Lebesgue integral). Let (φ_j) and (ψ_j) be Cauchy sequences in $\mathcal{S}(X, \mu, E)$ converging μ-a.e. to the same function. The sequences $\left(\int_X \varphi_j d\mu\right)$ and $\left(\int_X \psi_j d\mu\right)$ converge in E, and

$$\lim_j \int_X \varphi_j d\mu = \lim_j \int_X \psi_j d\mu$$

We define the integral of integrable functions in a natural way, extending the integral of simple functions. Suppose $f \in \mathcal{L}_1(X,\mu,E)$. Then there is an $\mathcal{L}_1$-Cauchy sequence (φ_j) in $\mathcal{S}(X,\mu,E)$ such that $\varphi_j \to f$ μ-a.e. Then, the quantity

$$\int_X f d\mu := \lim_j \int_X \varphi_j d\mu$$

exists in E, and is independent of the sequence (φ_j). This is called the Bochner Lebesgue integral of f.

Proposition 2.3.10. For $f \in \mathcal{L}_1(X,\mu,E)$,

$$\|f\|_1 = 0 \Leftrightarrow f = 0 \quad \mu\text{-a.e.}$$

In the following content, we only consider the quotient space of $\mathcal{L}^1$(up to a null-set) space and still use the same notation.

Proposition 2.3.11 (complete)**.** For $f \in \mathcal{L}_1(X,\mu,E)$, let $\|f\|_1 := \int_X |f| d\mu$. Then $\|\cdot\|_1$ is a norm on $\mathcal{L}_1(X,\mu,E)$, called the $\mathcal{L}_1$-norm. We have $\mathcal{S}(X,\mu,E)$ is dense in $\mathcal{L}_1(X,\mu,E)$. and the space $\mathcal{L}_1(X,\mu,E)$ is complete.

Proposition 2.3.12 (integral commutes with linear operator)**.** (1) $\int_X \cdot d\mu : \mathcal{L}_1(X,\mu,E) \to E$ is linear and continuous, and

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu = \|f\|_1.$$

(2) Suppose F is a Banach space and $T \in \mathcal{L}(E,F)$. Then

$$Tf \in \mathcal{L}_1(X,\mu,F) \quad \text{and} \quad T\int_X f d\mu = \int_X T f d\mu$$

for $f \in \mathcal{L}_1(X,\mu,E)$.

Proposition 2.3.13. For $f \in \mathcal{L}_0(X,\mu,E)$, the following are equivalent:

(1) $f \in \mathcal{L}_1(X,\mu,E)$;

(2) $|f| \in \mathcal{L}_1(X,\mu,\mathbb{R})$;

(3) $\int_X |f| d\mu < \infty$.

Proposition 2.3.14. Suppose $f \in \mathcal{L}_0(X,\mu,E)$ and $g \in \mathcal{L}_1(X,\mu,\mathbb{R})$ satisfy $|f| \leq g$. Then f belongs to $\mathcal{L}_1(X,\mu,E)$.

Proposition 2.3.15 (differentiability of parametrized integrals)**.** Suppose U is open in $\mathbb{R}^n$, or $U \subset \mathbb{C}$ is perfect and convex, and suppose $f : X \times U \to E$ satisfies

(1) $f(\cdot,y) \in \mathcal{L}_1(X,\mu,E)$ for every $y \in U$;

(2) $f(x,\cdot) \in C^1(U,E)$ for

(3) there exists $g \in \mathcal{L}_1(X, \mu, \mathbb{R})$ such that

$$\left|\frac{\partial}{\partial y^j} f(x, y)\right| \leq g(x) \quad \text{for } (x, y) \in X \times U \text{ and } 1 \leq j \leq n$$

Then

$$F : U \to E, \quad y \mapsto \int_X f(x, y)\mu(dx)$$

is continuously differentiable and

$$\partial_j F(y) = \int_X \frac{\partial}{\partial y^j} f(x, y)\mu(dx) \quad \text{for } y \in U \text{ and } 1 \leq j \leq n$$

Setting: X be a σ-compact LCH space with Radon measure μ, E be a Banach space over $\mathbb{C}$.

Proposition 2.3.16. $C(X, E)$ is a vector subspace of $\mathcal{L}_0(X, \mu, E)$ and $C_c(X, E)$ is a vector subspace of $\mathcal{L}_1(X, \mu, E)$.

Proof: The second statement is a trivial corollary of the first statement. Hence, it suffices to show $C(X, E) \subset \mathcal{L}_0(X, \mu, E)$. By Proposition 2.3.6, it suffices to show $f(X)$ is separable. Take $f \in C(X, E)$ and let (X_j) be a sequence of compact sets in X such that $X = \bigcup_j X_j$. Then, $f(X_j)$ being a compact subset of E. By Proposition 1.2.62, $f(X_j)$ is second-countable, hense separable for all j. Therefore $f(X) = \bigcup_j f(X_j)$ is also separable.

2.4 Lebesgue Measure

Definition 2.4.1. Lebesgue measure m^n on $\mathbb{R}^n$ is the n-fold product of Lebesgue measure on $\mathbb{R}$ with itself.

Proposition 2.4.2. Lebesgue measure is translation-invariant. More precisely, for $a \in \mathbb{R}^n$ define $\tau_a : \mathbb{R}^n \to \mathbb{R}^n$ by $\tau_a(x) = x + a$.

(1) If $E \in \mathcal{L}^n$, then $\tau_a(E) \in \mathcal{L}^n$ and $m\left(\tau_a(E)\right) = m(E)$.

(2) If $f : \mathbb{R}^n \to \mathbb{C}$ is Borel measurable, then so is $f \circ \tau_a$. Moreover, if either $f \geq 0$ or $f \in L^1(m)$, then $\int (f \circ \tau_a)\, dm = \int f dm$.

Theorem 2.4.3. Suppose $T \in GL(n, \mathbb{R})$.

(1) If f is a Borel measurable function on $\mathbb{R}^n$, so is $f \circ T$. If $f \geq 0$ or $f \in L^1(m)$, then

$$\int f(x)dx = |\det T| \int f \circ T(x)dx.$$

(2) If $E \in \mathcal{L}^n$, then $T(E) \in \mathcal{L}^n$ and $m(T(E)) = |\det T| m(E)$.

Theorem 2.4.4 (Change of Variables)**.** Let $G = (g_1, \ldots, g_n)$ be a map from an open set $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$ whose components g_j are of class C^1. G is called a C^1 diffeomorphism if G is injective and $D_x G$ is invertible for all $x \in \Omega$. In this case, the inverse function theorem guarantees that $G(\Omega)$ is open and $G^{-1} : G(\Omega) \to \Omega$ is also a C^1 diffeomorphism and that $D_x\left(G^{-1}\right) = \left[D_{G^{-1}(x)} G\right]^{-1}$ for all $x \in G(\Omega)$.

Suppose that Ω is an open set in $\mathbb{R}^n$ and $G : \Omega \to \mathbb{R}^n$ is a C^1 diffeomorphism.

(1) If f is a Lebesgue measurable function on $G(\Omega)$, then $f \circ G$ is Lebesgue measurable on Ω. If $f \geq 0$ or $f \in L^1(G(\Omega), m)$, then

$$\int_{G(\Omega)} f(x)dx = \int_\Omega f \circ G(x) \left|\det D_x G\right| dx.$$

(2) If $E \subset \Omega$ and $E \in \mathcal{L}^n$, then $G(E) \in \mathcal{L}^n$ and $m(G(E)) = \int_E \left|\det D_x G\right| dx$.

2.5 Signed Measure and Complex Measure

Definition 2.5.1. Let $(X, \mathcal{M})$ be a measurable space. A signed measure on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that

(1) $\nu(\varnothing) = 0$

(2) ν assumes at most one of the values $\pm\infty$;

(3) if $\{E_j\}$ is a sequence of disjoint sets in $\mathcal{M}$, then $\nu\left(\bigcup_1^\infty E_j\right) = \sum_1^\infty \nu\left(E_j\right)$, where the latter sum converges absolutely if $\nu\left(\bigcup_1^\infty E_j\right)$ is finite.

Definition 2.5.2. If ν is a signed measure on $(X, \mathcal{M})$, a set $E \in \mathcal{M}$ is called positive (resp. negative, null) for ν if $\nu(F) \geq 0$ (resp. $\nu(F) \leq 0, \nu(F) = 0$) for all $F \in \mathcal{M}$ such that $F \subset E$.

Definition 2.5.3 (mutually singular)**.** Two signed measures μ and ν on $(X, \mathcal{M})$ are mutually singular, or that ν is singular with respect to μ, if there exist $E, F \in \mathcal{M}$ such that $E \cap F = \varnothing, E \cup F = X, E$ is null for μ, and F is null for ν.We express this relationship symbolically with the perpendicularity sign:

$$\mu \perp \nu.$$

Theorem 2.5.4 (Jordan Decomposition Theorem)**.** If ν is a signed measure, there exist unique positive measures $\nu^+$and ν^-such that $\nu = \nu^+ - \nu^-$and $\nu^+ \perp \nu^-$.

Moreover, if ν omits $-\infty$, μ^- is finite and if ν omits ∞, ν^+ is finite.

Remark 2.5.5. The measures $\nu^+$and $\nu^-$are called the positive and negative variations of ν, and $\nu = \nu^+ - \nu^-$is called the Jordan decomposition of ν. Furthermore, we define the total variation of ν to be the measure $|\nu|$ defined by

$$|\nu| = \nu^+ + \nu^-.$$

Definition 2.5.6. Integration with respect to a signed measure ν is defined in the obvious way: We set

$$L^1(\nu) = L^1\left(\nu^+\right) \cap L^1\left(\nu^-\right)$$
$$\int f d\nu = \int f d\nu^+ - \int f d\nu^- \quad \left(f \in L^1(\nu)\right).$$

One more piece of terminology: a signed measure ν is called finite (resp. σ-finite) if $|\nu|$ is finite (resp. σ-finite).

Proposition 2.5.7. *$E \in \mathcal{M}$ is ν-null iff $|\nu|(E) = 0$, and $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$*

Definition 2.5.8. Suppose that ν is a signed measure and μ is a positive measure on $(X, \mathcal{M})$. We say that ν is absolutely continuous with respect to μ and write

$$\nu \ll \mu$$

if $\nu(E) = 0$ for every $E \in \mathcal{M}$ for which $\mu(E) = 0$. Absolute continuity is in a sense the

Proposition 2.5.9. $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$

Proposition 2.5.10. If $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$

Proposition 2.5.11. Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{M})$. Then $\nu \ll \mu$ iff for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$.

Proof: If there's $\varepsilon > 0$ such that for all $n > 0$, there's $E_n \in M$ such that $\mu(E_n) < 2^{-n}$ and $\nu(E) \geq \varepsilon$. Consider the set

$$\limsup E_n = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k$$

Corollary 2.5.12. If $f \in L^1(\mu)$, for every $\epsilon > 0$ there exists $\delta > 0$ such that $\left|\int_E f d\mu\right| < \epsilon$ whenever $\mu(E) < \delta$.

Definition 2.5.13. A complex measure on a measurable space $(X, \mathcal{M})$ is a map $\nu : \mathcal{M} \to \mathbb{C}$ such that

(1) $\nu(\varnothing) = 0$;

(2) if $\{E_j\}$ is a sequence of disjoint sets in $\mathcal{M}$, then $\nu\left(\bigcup_1^\infty E_j\right) = \sum_1^\infty \nu\left(E_j\right)$, where the series converges absolutely.

Example 2.5.14. If μ is a positive measure and$f \in L^1(\mu)$, then $f \mathrm{d}\mu$ is a complex measure.

If ν is a complex measure, we shall write ν_r and ν_i for the real and imaginary parts of ν. Thus ν_r and ν_i are signed measures that do not assume the values $\pm\infty$; hence they are finite, and so the range of ν is a bounded subset of $\mathbb{C}$.

The notions we have developed for signed measures generalize easily to complex measures. For example, we define $L^1(\nu)$ to be $L^1\left(\nu_r\right) \cap L^1\left(\nu_i\right)$, and for $f \in L^1(\nu)$, we set $\int f d\nu = \int f d\nu_r + i \int f d\nu_i$. If ν and μ are complex measures, we say that $\nu \perp \mu$ if $\nu_a \perp \mu_b$ for $a, b = r, i$, and if λ is a positive measure, we say that $\nu \ll \lambda$ if $\nu_r \ll \lambda$ and $\nu_i \ll \lambda$.

Theorem 2.5.15 (Lebesgue-Radon-Nikodym Theorem)**.** If ν is a complex measure and μ is a σ-finite positive measure on $(X, \mathcal{M})$, there exist a complex measure λ and an $f \in L^1(\mu)$ such that $\lambda \perp \mu$ and $d\nu = d\lambda + f d\mu$. If also $\lambda' \perp \mu$ and $d\nu = d\lambda' + f' d\mu$, then $\lambda = \lambda'$ and $f = f'$ μ-a.e.

Definition 2.5.16 (total variation of complex measure)**.** If ν is a complex measure and ν_r and ν_i be the real part and imaginary part of ν. Take a σ-finite positive measure μ on X, for example $|\nu_r| + |\nu_i|$, such that $\nu \ll \mu$. By Lebesgue-Radon-Nikodym Theorem, $\nu = f d\mu$ for some $f \in L^1(\mu)$. Define total variation of ν by $|f| d\mu$. This definition is independent of the choice of f and μ.

Proposition 2.5.17. Let ν be a complex measure on $(X, \mathcal{M})$.

(1) $|\nu(E)| \leq |\nu|(E)$ for all $E \in \mathcal{M}$.

(2) $\nu \ll |\nu|$

(3) $L^1(\nu) = L^1(|\nu|)$, and if $f \in L^1(\nu)$, then $\left|\int f d\nu\right| \leq \int |f| d|\nu|$.

Definition 2.5.18. A measurable function $f : \mathbb{R}^n \to \mathbb{C}$ is called locally integrable (with respect to Borel measure) if $\int_K |f(x)|dx < \infty$ for every bounded measurable set $K \subset \mathbb{R}^n$.

We denote the space of locally integrable functions by L^1_{loc} . If $f \in L^1_{\text{loc}}$, $x \in \mathbb{R}^n$, and $r > 0$, we define $A_r f(x)$ to be the average value of f on $B(r, x)$:

$$A_r f(x) = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y) dy$$

Theorem 2.5.19 (The Lebesgue Differentiation Theorem)**.** If $f \in L^1_{\text{loc}}$, then

$$\lim_{r \to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - f(x)| dy = 0 \text{ for a.e. } x \in \mathbb{R}^n$$

Definition 2.5.20. A family $\{E_r\}_{r>0}$ of Borel subsets of $\mathbb{R}^n$ is said to shrink nicely to $x \in \mathbb{R}^n$ if $E_r \subset B(r, x)$ for each r; and there is a constant $\alpha > 0$, independent of r, such that $m(E_r) > \alpha m(B(r, x))$.

Theorem 2.5.21. Let ν be a regular complex Borel measure on $\mathbb{R}^n$, and let $d\nu = d\lambda + f dm$ be its Lebesgue-Radon-Nikodym representation. Then for m-almost every $x \in \mathbb{R}^n$,

$$\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

where E_r shrinks nicely to 0.

2.6 Function of Bounded Variation

In this section, m denotes the Lebesgue measure on $\mathbb{R}$.

Definition 2.6.1. If $F : \mathbb{R} \to \mathbb{C}$ and $x \in \mathbb{R}$, we define

$$T_F(x) = \sup \left\{ \sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x \right\}.$$

T_F is called the total variation function of F.

T_F is an increasing function with values in $[0, \infty]$. If $T_F(\infty) = \lim_{x \to \infty} T_F(x)$ is finite, we say that F is of bounded variation on $\mathbb{R}$, and we denote the space of all such F by BV.

Proposition 2.6.2. We observe that the sums in the definition of T_F are made bigger if the additional subdivision points x_j are added. Hence, if $a < b$, the definition of $T_F(b)$ is unaffected if we assume that a is always one of the subdivision points. It follows that

$$T_F(b) = T_F(a) + \sup \left\{ \sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, a = x_0 < \cdots < x_n = b \right\}$$

Definition 2.6.3. Define $BV([a,b])$ to be the set of all functions on $[a,b]$ whose total variation

$$\sup\left\{\sum_{1}^{n}|F(x_j)-F(x_{j-1})| : n\in\mathbb{N}, a=x_0<\cdots<x_n=b\right\}$$

is finite.

Remark 2.6.4. If $F\in BV$, the restriction of F to $[a,b]$ is in $BV([a,b])$ for all a,b. Indeed, its total variation on $[a,b]$ is nothing but $T_F(b)-T_F(a)$. Conversely, if $F\in BV([a,b])$ and we set $F(x)=F(a)$ for $x<a$ and $F(x)=F(b)$ for $x>b$, then $F\in BV$. By this device the results that we shall prove for BV can also be applied to $BV([a,b])$.

Proposition 2.6.5. (1) If $F:\mathbb{R}\to\mathbb{R}$ is bounded and increasing, then $F\in BV$.

(2) If $F,G\in BV$ and $a,b\in\mathbb{C}$, then $aF+bG\in BV$.

(3) If $F\in BV$ is real-valued, then T_F+F and T_F-F are increasing.

(4) $F\in BV$ iff $\operatorname{Re}F\in BV$ and $\operatorname{Im}F\in BV$.

(5) If $F:\mathbb{R}\to\mathbb{R}$, then $F\in BV$ iff F is the difference of two bounded increasing functions; for $F\in BV$ these functions may be taken to be $\frac{1}{2}(T_F+F)$ and $\frac{1}{2}(T_F-F)$.

(6) If $F\in BV$ and $G(x)=F(x+)$, then $G(x)$ is right continous and F' and G' exist and are equal a.e.

(7) If F is differentiable on $\mathbb{R}$ and F' is bounded, then $F\in BV([a,b])$ for $-\infty<a<b<\infty$ (by the mean value theorem).

Remark 2.6.6. For $F\in BV$, denote $M\in\mathcal{B}_{\mathbb{R}}$ such that $m(M^c)=0$ and F is differentiable on M. Then,

$$f(x)=\begin{cases}F'(x), & x\in M\\ 0, & x\in M^c\end{cases}$$

is Lebesgue measurable and we still denote it by $F'(x)$.

Proposition 2.6.7. Define the normalized bounded variation function space to be

$$NBV=\{F\in BV : F \text{ is right continuous and } F(-\infty)=0\}$$

If $F\in BV$, then $T_F(-\infty)=0$. If F is also right continuous, then so is T_F. If $F\in NBV$, T_F is also in NBV.

Proposition 2.6.8. If μ is a complex Borel measure on $\mathbb{R}$ and $F(x)=\mu((-\infty,x])$, then $F\in NBV$. Conversely, if $F\in NBV$, there is a unique complex Borel measure μ_F such that $F(x)=\mu_F((-\infty,x])$. Moreover, $|\mu_F|=\mu_{T_F}$.

Proposition 2.6.9. A function $F : \mathbb{R} \to \mathbb{C}$ is called absolutely continuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any finite set of disjoint intervals $(a_1, b_1), \ldots, (a_N, b_N)$,

$$\sum_1^N (b_j - a_j) < \delta \Longrightarrow \sum_1^N |F(b_j) - F(a_j)| < \epsilon.$$

More generally, F is said to be absolutely continuous on $[a, b]$ if this condition is satisfied whenever the intervals (a_j, b_j) all lie in $[a, b]$. Clearly, if F is absolutely continuous, then F is uniformly continuous.

Example 2.6.10. If F is everywhere differentiable and F' is bounded, then F is absolutely continuous, for $|F(b_j) - F(a_j)| \leq (\max |F'|)(b_j - a_j)$ by the mean value theorem.

Proposition 2.6.11. If $F \in NBV$, then F is absolutely continuous iff $\mu_F \ll m$.

Theorem 2.6.12. If $F \in NBV$, then F is differentiable almost m-everywhere. We have $F'(x) \in L^1(m)$.

Moreover, $\mu_F \perp m$ iff $F' = 0$ a.e., and $\mu_F \ll m$ iff $F(x) = \int_{-\infty}^x F'(t)dt$.

Proof: Since total variation of μ_F is regular, the theorem follows from Theorem 2.5.21.

Theorem 2.6.13. If $f \in L^1(m)$, then the function $F(x) = \int_{-\infty}^x f(t)dt$ is in NBV and is absolutely continuous, and $f = F'$ a.e. Conversely, if $F \in NBV$ is absolutely continuous, then $F' \in L^1(m)$ and $F(x) = \int_{-\infty}^x F'(t)dt$.

Theorem 2.6.14. If $-\infty < a < b < \infty$ and $F : [a, b] \to \mathbb{C}$, the following are equivalent:

(1) F is absolutely continuous on $[a, b]$.

(2) $F(x) - F(a) = \int_a^x f(t)dt$ for some $f \in L^1([a, b], m)$.

(3) F is differentiable a.e. on $[a, b]$, $F' \in L^1([a, b], m)$, and $F(x) - F(a) = \int_a^x F'(t)dt$.

Theorem 2.6.15 (integrate by part)**.** If F and G are in NBV and at least one of them is continuous, then for $-\infty < a < b < \infty$,

$$\int_{(a,b]} F dG + \int_{(a,b]} G dF = F(b)G(b) - F(a)G(a).$$

2.7 Radon measure

Definition 2.7.1. Let μ be a Borel measure on X and E a Borel subset of X. The measure μ is called outer regular on E if

$$\mu(E) = \inf\{\mu(U) : U \supset E, U \text{ open }\}$$

and inner regular on E if

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact }\}.$$

If μ is outer and inner regular on all Borel sets, μ is called regular.

Definition 2.7.2. A Radon measure on X is a Borel measure that is finite on all compact sets, outer regular on all Borel sets, and inner regular on all open sets.

Definition 2.7.3. A complex measure is regular if its total variation is regular.

Proposition 2.7.4. Every σ-finite Radon measure is regular.

Proposition 2.7.5. In C_2 LCH space X, every open subset is σ-compact.

Proof: Since in C_2 space, every open subspace is still C_2 hence Lindelöf.

By Proposition 1.2.38, for all $x \in U$, there's V_x open and precompact such that $x \in V_x \subset \overline{V_x} \subset U$. Take a countable subcovering of $\{V_x\}$ indexed by J, we have

$$\bigcup_{x \in J} \overline{V_x} = U$$

.

Proposition 2.7.6. Let X be a σ-compact(for example, when X is second countable) LCH space. Then every Borel measure on X that is finite on compact sets is regular and hence Radon.

Proposition 2.7.7. If μ is a Radon measure on $X, C_c(X)$ is dense in $L^p(\mu)$ for $1 \leq p < \infty$

Theorem 2.7.8 (The Riesz Representation Theorem)**.** If U is open in X and $f \in C_c(X)$, we shall write

$$f \prec U$$

to mean that $0 \leq f \leq 1$ and $\operatorname{supp}(f) \subset U$.

If I is a positive linear functional on $C_c(X)$, there is a unique Radon measure μ on X such that $I(f) = \int f d\mu$ for all $f \in C_c(X)$. Moreover, μ satisfies

$$\mu(U) = \sup \{I(f) : f \in C_c(X), f \prec U\} \text{ for all open } U \subset X$$

and $\mu(K) = \inf \{I(f) : f \in C_c(X), f \geq \chi_K\}$ for all compact $K \subset X$.

Corollary 2.7.9. There's one-to-one correspondence between bounded positive linear functional $C_c(X)$ and finite Radon meausre on X. Moreover, since $C_c(X)$ is dense subset of Banach space $C_0(X)$, by Theorem 1.2.70, every bounded positive linear functional on $C_c(X)$ can be extended to $C_0(X)$ continuously.

Proof: If I is a bounded positive linear functional, by Riesz Representation Theorem,

$$\mu(X) = \sup \left\{ \int f d\mu : f \in C_c(X), 0 \leq f \leq 1 \right\} < \infty$$

Proposition 2.7.10. If μ is a σ-finite Radon measure on X and $A \in \mathcal{B}_X$, the Borel measure μ_A defined by $\mu_A(E) = \mu(E \cap A)$ is a Radon measure.

Proposition 2.7.11. Suppose that μ is a Radon measure on X. If $\phi \in L^1(\mu)$ and $\phi \geq 0$, then $\nu(E) = \int_E \phi d\mu$ is a Radon measure.

Proposition 2.7.12. Suppose that μ is a Radon measure on X and $\phi \in C(X,(0,\infty))$. Let $\nu(E) = \int_E \phi d\mu$, and let ν' be the Radon measure associated to the functional $f \mapsto \int f\phi d\mu$ on $C_c(X)$, then $\nu = \nu'$, and hence ν is a Radon measure.

Definition 2.7.13. A complex measure is Radon if its real and imaginary parts are difference of finite Radon measure.

Definition 2.7.14. $M(X)$ is the space of all the complex Radon measures and for $\mu \in M(X)$, define

$$||\mu|| = |\mu|(X)$$

Then,$||\cdot||$ is a norm on vector space $M(X)$.

Theorem 2.7.15. Let X be an LCH space, and for $\mu \in M(X)$, $I_\mu : f \in C_0(X) \mapsto \int f \mathrm{d}\mu$ is a bounded linear functional on $C_0(X)$. Then $\mu \mapsto I_\mu$ is an bijective isometry between $M(X)$ the space of complex Radon measure and space of bounded linear functional on $C_0(X)$.

Proposition 2.7.16. Suppose X, Y are LCH spaces.

(1) $\mathcal{B}_X \otimes \mathcal{B}_Y \subset \mathcal{B}_{X\times Y}$.

(2) If X and Y are second countable, then $\mathcal{B}_X \otimes \mathcal{B}_Y = \mathcal{B}_{X\times Y}$.

(3) If X and Y are second countable and μ and ν are Radon measures on X and Y, then $\mu\times\nu$ is a Radon measure on $X\times Y$.

(4) If $E \in \mathcal{B}_{X\times Y}$, then $E_x \in \mathcal{B}_Y$ for all $x \in X$ and $E^y \in \mathcal{B}_X$ for all $y \in Y$.

(5) If $f : X \times Y \to \mathbb{C}$ is $\mathcal{B}_{X\times Y}$-measurable, then f_x is $\mathcal{B}_Y$-measurable for all $x \in X$ and f^y is $\mathcal{B}_X$-measurable for all $y \in Y$.

Definition 2.7.17 (Radon product)**.** Every $f \in C_c(X \times Y)$ is $\mathcal{B}_X \otimes \mathcal{B}_Y$-measurable. Moreover, if μ and ν are σ-finite Radon measures on X and Y, then $C_c(X\times Y) \subset L^1(\mu\times\nu)$, and

$$\int fd(\mu\times\nu) = \iint fd\mu d\nu = \iint fd\nu d\mu \quad (f \in C_c(X\times Y)).$$

The formula $I(f) = \int fd(\mu\times\nu)$ defines a positive linear functional on $C_c(X\times Y)$, so it determines a Radon measure on $X\times Y$ by the Riesz representation theorem. We call this measure the Radon product of μ and ν and denote it by $\mu\widehat{\times}\nu$.

Proposition 2.7.18. Suppose that μ and ν are σ-finite Radon measures on X and Y. If $E \in \mathcal{B}_{X\times Y}$, then the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are Borel measurable on X and Y, and

$$\mu\widehat{\times}\nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y).$$

Moreover, the restriction of $\mu\hat{\times}\nu$ to $\mathcal{B}_X \otimes \mathcal{B}_Y$ is $\mu\times\nu$.

Theorem 2.7.19. Suppose that, for each $\alpha \in A$, μ_α is a σ-finite Radon measure on the compact Hausdorff space X_α such that $\mu_\alpha(X_\alpha) = 1$. Then there is a unique Radon measure μ on $X = \prod_{\alpha \in A} X_\alpha$ such that for any $\alpha_1, \ldots, \alpha_n \in A$ and any Borel set E in $\prod_1^n X_{\alpha_j}$,

$$\mu\left(\pi^{-1}_{(\alpha_1,\ldots,\alpha_n)}(E)\right) = \left(\mu_{\alpha_1} \widehat{\times} \cdots \widehat{\times} \mu_{\alpha_n}\right)(E).$$

Chapter 3

Complex Analysis

3.1 The Fundamental Theorem of Line Integrals

Setting: $E = (E, |\cdot|)$ is a Banach space over the field $\mathbb{K}$ and $I = [a, b]$ is a compact interval.

Definition 3.1.1. Suppose $f : I \to E$ and $Z = (t_0, \ldots, t_n)$ is a partition of I. Then

$$L_Z(f) := \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})|$$

is the length of the piecewise straight path $(f(t_0), \ldots, f(t_n))$ in E, and

$$\operatorname{Var}(f, I) := \sup \{L_Z(f); Z = (t_0, \ldots, t_n) \text{ is a partition of } I\}$$

is called the total variation (or simply the variation) of f over I. We say f is of bounded variation if $\operatorname{Var}(f, I) < \infty$.

Proposition 3.1.2. For $f : [a, b] \to E$ and $c \in [a, b]$, we have

$$\operatorname{Var}(f, [a, b]) = \operatorname{Var}(f, [a, c]) + \operatorname{Var}(f, [c, b])$$

Setting: $E = (E, |\cdot|)$ is a Banach space over the field $\mathbb{K}$ and $I = [a, b]$ is a compact interval, $\gamma \in C(I, E)$ as a continuous path in E.

Definition 3.1.3. We call $\operatorname{Var}(\gamma, I)$ the length (or arc length) of γ and write it as $L(\gamma)$. If $L(\gamma) < \infty$, that is, if γ has a finite length, we say γ is rectifiable.

Proposition 3.1.4. Suppose $\gamma \in C^1(I, E)$. Then γ is rectifiable, and we have

$$L(\gamma) = \int_a^b |\dot{\gamma}(t)| dt$$

Proof: If $\mathcal{Z} = (t_0, \ldots, t_n)$ is partition of $[a, b]$, then

$$\begin{aligned}\sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})| &= \sum_{j=1}^{n} \left| \int_{t_{j-1}}^{t_j} \dot{\gamma}(t) dt \right| \\ &\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} |\dot{\gamma}(t)| dt = \int_a^b |\dot{\gamma}(t)| dt\end{aligned}$$

Therefore

$$L(\gamma)=\operatorname{Var}(\gamma,[a,b]) \leq \int_a^b |\dot{\gamma}(t)| dt$$

Suppose now $s_0 \in [a,b)$. For every $s \in (s_0,b)$ that

$$\operatorname{Var}(\gamma,[a,s])-\operatorname{Var}\left(\gamma,\left[a,s_0\right]\right)=\operatorname{Var}\left(\gamma,\left[s_0,s\right]\right)$$

and

$$\left|\gamma(s)-\gamma\left(s_0\right)\right| \leq \operatorname{Var}\left(\gamma,\left[s_0,s\right]\right) \leq \int_{s_0}^s |\dot{\gamma}(t)| dt$$

Thus because $s_0<s$, we have

$$\left|\frac{\gamma(s)-\gamma\left(s_0\right)}{s-s_0}\right| \leq \frac{\operatorname{Var}(\gamma,[a,s])-\operatorname{Var}\left(\gamma,\left[a,s_0\right]\right)}{s-s_0} \leq \frac{1}{s-s_0} \int_{s_0}^s |\dot{\gamma}(t)| dt$$

Hence,

$$\left|\dot{\gamma}\left(s_0\right)\right|=\lim_{s \rightarrow s_0}\left|\frac{\gamma(s)-\gamma\left(s_0\right)}{s-s_0}\right| \leq \lim_{s \rightarrow s_0}\left[\frac{1}{s-s_0} \int_{s_0}^s |\dot{\gamma}(t)| dt\right]=\left|\dot{\gamma}\left(s_0\right)\right|$$

This implies

$$\frac{d}{ds} \operatorname{Var}(\gamma,[a,s])=|\dot{\gamma}(s)| \quad \text{ for } s \in [a,b]$$

Corollary 3.1.5. For $\gamma=(\gamma_1,\ldots,\gamma_n) \in C^1\left(I,\mathbb{R}^n\right)$, we have

$$L(\gamma)=\int_a^b \sqrt{\left(\dot{\gamma}_1(t)\right)^2+\cdots+\left(\dot{\gamma}_n(t)\right)^2} dt$$

Setting: Suppose J_1 and J_2 are intervals, $q \in \mathbb{N} \cup \{\infty\}$.

Definition 3.1.6 (reparametrization)**.** The map $\varphi: J_1 \rightarrow J_2$ is said to be an (orientation-preserving) C^q change of parameters if φ strictly increasing, bijective, C^q map such that $\varphi^{-1} \in C^q$.

If $\gamma_j \in C^q\left(J_j,E\right)$ for $j=1,2$, then γ_1 is said to be an (orientation-preserving) C^q reparametrization of γ_2 if there is a C^q change of parameters φ such that $\gamma_1=\gamma_2 \circ \varphi$.

Proposition 3.1.7. For $q \geq 1$ or $q=\infty$, a map $\varphi: J_1 \rightarrow J_2$ is a C^q change of parameters if and only if φ belongs to $C^q\left(J_1,J_2\right)$, is surjective, and satisfies $\dot{\varphi}(t)>0$ for $t \in J_1$.

Proof: If φ is a C^q change of parameters, by mean value theorem, $\varphi^{\prime}(t)>0$ for all $t \in J_1$.

If φ belongs to $C^q\left(J_1,J_2\right)$, is surjective, and satisfies $\dot{\varphi}(t)>0$ for $t \in J_1$, by mean value theorem, φ is injective and strictly increasing. Since φ is strictly increasing and continous, φ^{-1} is continous. By Proposition 1.4.24, φ^{-1} is differentiable and $(\varphi^{-1})^{\prime}$ is continous. That is, $(\varphi^{-1})^{\prime} \in C^1$. This shows the case when $q=1$. For $q>1$, notice that $(\varphi^{-1})^{\prime} \circ \varphi \in C^{q-1}$, it suffice to show $\varphi^{-1} \in C^{q-1}$ which follows from induction.

Proposition 3.1.8 (length is invariant under reparametrization)**.** Let I_1 and I_2 be compact intervals, and suppose $\gamma_1 \in C\left(I_1,E\right)$ is a continuous reparametrization of $\gamma_2 \in C\left(I_2,E\right)$. Then

$$\operatorname{Var}\left(\gamma_1,I_1\right)=\operatorname{Var}\left(\gamma_2,I_2\right)$$

Setting: $q \geq 1$ or $q = \infty$, E be a Banach space over $\mathbb{K}$.

Definition 3.1.9. A C^q path is a C^q map from a compact interval to E.

Definition 3.1.10. On the set of all C^q paths in E, we define the relation $\sim$ by

$$\gamma_1 \sim \gamma_2 :\Leftrightarrow \gamma_1 \text{ is a } C^q \text{ reparametrization of } \gamma_2$$

and define C^q curve to be the equivalent class of C^q paths under $\sim$.

Definition 3.1.11. We say a C^q curve $[\gamma]$ of C^q curve is regular if $\dot{\gamma}(t) \neq 0$ for $t \in \operatorname{dom}(\gamma)$. We say a C^q curve is a plane curve if $E = \mathbb{R}^2$.

Definition 3.1.12. Suppose $Z = (\alpha_0, \ldots, \alpha_m)$ for $m \in \mathbb{N}^\times$ is a partition of a compact interval I and $q \in \mathbb{N}^\times \cup \{\infty\}$.

A continuous path $\gamma \in C(I, E)$ is a piecewise C^q path in E if $\gamma_j := \gamma \mid [\alpha_{j-1}, \alpha_j] \in C^q\left([\alpha_{j-1}, \alpha_j], E\right)$ for $j = 1, \ldots, m$. A path $\eta \in C(J, E)$ that is piecewise C^q on the partition $Z' = (\beta_0, \ldots, \beta_m)$ of J is called a C^q reparametrization of γ if there is a C^q change of parameters $\varphi_j \in \operatorname{Diff}^q\left([\alpha_{j-1}, \alpha_j], [\beta_{j-1}, \beta_j]\right)$ such that $\gamma_j := \eta_j \circ \varphi_j$ for $j = 1, \ldots, m$. On the set of all piecewise C^q paths in E, we define $\sim$ through

$$\gamma \sim \eta :\Leftrightarrow \eta \text{ is a reparametrization of } \gamma$$

Define piecewise C^q curve to be the equivalent class of piecewise C^q paths under $\sim$.

Proposition 3.1.13. Suppose Γ is a curve in E with parametrization $\gamma \in C(I, E)$ that is piecewise C^q on the partition $\mathfrak{Z} = (\alpha_0, \ldots, \alpha_m)$. Define the length (or the arc length) of Γ through

$$L(\Gamma) := \operatorname{Var}(\gamma, I).$$

Show $L(\Gamma)$ is well defined and

$$L(\Gamma) = \sum_{j=1}^{m} L\left(\Gamma_j\right) = \sum_{j=1}^{m} \int_{\alpha_{j-1}}^{\alpha_j} |\dot{\gamma}_j(t)|\, dt$$

Definition 3.1.14 (oriented area)**.** If $\Gamma = [\gamma]$ is a plane closed piecewise C^q curve and $(\alpha_0, \ldots, \alpha_m)$ is a partition for γ,

$$A(\Gamma) := \frac{1}{2} \sum_{j=1}^{m} \int_{\alpha_{j-1}}^{\alpha_j} \det\left[\gamma_j(t), \dot{\gamma}_j(t)\right] dt$$

is the oriented area contained in Γ.

Proposition 3.1.15. Let $-\infty < \alpha < \beta < \infty$, and suppose $f \in C^1([\alpha, \beta], \mathbb{R})$ satisfies $f(\alpha) = f(\beta) = 0$. Let $a := \alpha$ and $b := 2\beta - \alpha$, and define $\gamma : [a, b] \to \mathbb{R}^2$ by

$$\gamma(t) := \begin{cases} (\alpha + \beta - t, f(\alpha + \beta - t)), & t \in [a, \beta] \\ (\alpha - \beta + t, 0), & t \in [\beta, b] \end{cases}$$

Show that $A(\Gamma) = \int_\alpha^\beta f(t) dt$.

Setting: X open in $\mathbb{R}^n$, $q \in \mathbb{N} \cup \{\infty\}$.

Definition 3.1.16 (C^q vector field)**.** A map $\boldsymbol{v} : X \to TX$ in which $\boldsymbol{v}(p) \in T_pX$ for $p \in X$ is a vector field on X.

A C^q vector field is a vector field $X \to TX$ whose representation under standard basis $\{\partial/\partial x_1, \dots \partial/\partial x_n\}$ belongs to $C^q(X, \mathbb{R}^n)$.

We denote the set of C^q vector field on X by $\mathcal{V}_{(q)}(X)$

Definition 3.1.17 (C^q 1-form)**.** A map $\boldsymbol{v} : X \to T^*X$ in which $\boldsymbol{v}(p) \in (T_pX)^*$ for $p \in X$ is a 1-form(Pfaff form) on X.

A C^q 1-form is a 1-form whose representation under standard basis $\{dx^1, \dots dx^n\}$ belongs to $C^q(X, \mathbb{R}^n)$. We denote the set of C^q 1-form on X by $\Omega_{(q)}(X)$

Example 3.1.18. For $f \in C^{q+1}(X)$,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx^i \in \Omega_{(q)}(X)$$

Definition 3.1.19 (exact form)**.** Suppose $\alpha \in \Omega_{(q)}(X)$, if there is a $f \in C^{q+1}(X)$ such that $df = \alpha$, we say α is exact, and f is an antiderivative of α.

Proposition 3.1.20. Suppose X is a domain and $\alpha \in \Omega_{(0)}(X)$ is exact. If f and g are antiderivatives of α, then $f - g$ is constant.

Proposition 3.1.21. Suppose $\alpha = \sum_{j=1}^{n} a_j dx^j \in \Omega_{(1)}(X)$ is exact. Then it satisfies integrability conditions

$$\partial_k a_j = \partial_j a_k \quad \text{for } 1 \leq j, k \leq n$$

We call a continous differentiable 1-form closed if it satisfies integrability conditions.

Example 3.1.22. Suppose $X := \mathbb{R}^n \backslash \{0\}$ and the components of the Pfaff form

$$\alpha = \sum_{j=1}^{n} a_j dx^j \in \Omega_{(\infty)}(X)$$

have the representation $a_j(x) := x^j \varphi(|x|)$ for $x = (x^1, \dots, x^n)$ and $1 \leq j \leq n$, where $\varphi \in C^\infty((0, \infty), \mathbb{R})$. Then α is exact. An antiderivative f is given by $f(x) := \Phi(|x|)$ with

$$\Phi(r) := \int_{r_0}^{r} t\varphi(t)dt \quad \text{for } r > 0$$

where r_0 is a strictly positive number.

Setting: X open in $\mathbb{R}^n$, $q \in \mathbb{Z}_{>0} \cup \{\infty\}$.

Theorem 3.1.23 (the Poincaré lemma)**.** Suppose X is star shaped and $q \geq 1$. When $\alpha \in \Omega_{(q)}(X)$ is closed, α is exact.

Proof: Suppose X is star shaped with respect to 0 and $\alpha = \sum_{j=1}^{n} a_j dx^j$. Because, for $x \in X$, the segment $[\![0, x]\!]$ lies in X,

$$f(x) := \sum_{k=1}^{n} \int_0^1 a_k(tx) x^k dt \quad \text{for } x \in X$$

is well defined. Since α satisfies integrability conditions, we have

$$\begin{aligned}
\partial_j f(x) &= \int_0^1 a_j(tx)dt + \int_0^1 t\left(\sum_{k=1}^{n} x^k \partial_j a_k(tx)\right) dt \\
&= \int_0^1 a_j(tx)dt + \int_0^1 t\left(\sum_{k=1}^{n} x^k \partial_k a_j(tx)\right) dt \\
&= \int_0^1 a_j(tx)dt + \int_0^1 t d(a_j(tx)) \\
&= a_j(x)
\end{aligned}$$

Hence, α is exact.

Setting: X be a open subset of $\mathbb{R}^n$. I be a compact interval. $\gamma \in C^1(I, X)$ and $\alpha \in \Omega_{(0)}(X)$.

Definition 3.1.24. If $\alpha = \sum_{j=1}^{n} a_j dx^j$, define

$$\int_\gamma \alpha := \sum_{j=1}^{n} \int_I (a_j \circ \gamma)\, \dot{\gamma}^j dt.$$

Notice that for all $\gamma_1 \in [\gamma]$(equivalent C^1 path), we have

$$\int_\gamma \alpha = \int_{\gamma_1} \alpha$$

Proposition 3.1.25. Let Γ be the circle parametrized by $\gamma : [0, 2\pi] \to \mathbb{R}^2, t \mapsto R(\cos t, \sin t)$. Then

$$\int_\Gamma X dy - Y dx = 2\pi R^2$$

Definition 3.1.26. Suppose γ is piecewise C^q path in X or a sum of the C^q paths γ_j. The curve $\Gamma = [\gamma]$ parametrized by γ is said to be a piecewise C^q curve in X, and we write $\Gamma := \Gamma_1 + \cdots + \Gamma_m$, where $\Gamma_j := [\gamma_j]$. Given a piecewise C^1 curve $\Gamma = \Gamma_1 + \cdots + \Gamma_m$ in X and $\alpha \in \Omega_{(0)}(X)$ we define the line integral of α along Γ by

$$\int_\Gamma \alpha := \sum_{j=1}^{m} \int_{\Gamma_j} \alpha$$

Definition 3.1.27. Suppose $I = [a, b]$ and $\gamma \in C(I, X)$. Then

$$\gamma^- : I \to X, \quad t \mapsto \gamma(a + b - t)$$

is the path inverse to γ, and $-\Gamma := [\gamma^-]$ is the curve inverse to $\Gamma := [\gamma]$.

Theorem 3.1.28 (The fundamental theorem of line integrals)**.** $X \subset \mathbb{R}^n$ is a domain and $\alpha \in \Omega_{(0)}(X)$. Then these statements are equivalent:

(1) α is exact;

(2) $\int_\Gamma \alpha = 0$ for every closed piecewise C^1 curve in X.

Proof: (1) implies (2): trivial

(2) implies (1): Suppose $x_0 \in X$. According to 1.2.55, there is for every $x \in X$ a continuous, piecewise straight path in X that leads to x from x_0. Thus there is for every $x \in X$ a piecewise C^1 curve Γ_x in X with initial point x_0 and final point x. We set

$$f : X \to \mathbb{R}, \quad x \mapsto \int_{\Gamma_x} \alpha.$$

(2) guarantees that f is well-defined. Suppose now $h \in \mathbb{R}^+$with $\overline{\mathbb{B}}(x,h) \subset X$ and $\Pi_j := [\pi_j]$ with

$$\pi_j : [0,1] \to X, \quad t \mapsto x + the_j \quad \text{ for } j = 1, \dots, n,$$

we have

$$f\left(x + he_j\right) - f(x) = \int_{\Pi_j} \alpha = ha_j(x) + h(\int_0^1 a_j(x + the_j) - a_j(x))dt$$

Hence, $\partial_j f(x) = a_j(x)$.

Corollary 3.1.29. Suppose X is open in $\mathbb{R}^n$ and star shaped, and let $x_0 \in X$. Also suppose $q \in \mathbb{N}^\times \cup \{\infty\}$ and $\alpha \in \Omega_{(q)}(X)$ is closed. Let

$$f(x) := \int_{\Gamma_x} \alpha \quad \text{ for } x \in X$$

where Γ_x is a piecewise C^1 curve in X with initial point x_0 and final point x. This function satisfies $f \in C^{q+1}(X)$ and $df = \alpha$.

Theorem 3.1.30 (homotopic invariant)**.** Suppose $\alpha \in \Omega_{(1)}(X)$ is closed. Let γ_0 and γ_1 be homotopic piecewise C^1 loops in X. Then $\int_{\gamma_0} \alpha = \int_{\gamma_1} \alpha$.

Proof: Suppose $H \in C(I \times [0,1], X)$ is a homotopy from γ_0 to γ_1. Since the image of H is compact and X^c is closed, there's an $\varepsilon > 0$ such that

$$|H(t,s) - y| \geq \varepsilon \quad \text{ for } (t,s) \in I \times [0,1] \text{ and } y \in X^c.$$

Since H is uniformly continuous. Hence there is a $\delta > 0$ such that

$$|H(t,s) - H(\tau,\sigma)| < \varepsilon \quad \text{ for } |t - \tau| < \delta, \quad |s - \sigma| < \delta$$

Now we choose a partition $(t_0, \dots, t_m)$ of I and a partition $(s_0, \dots, s_\ell)$ of $[0,1]$, both with mesh $< \delta$. Letting $A_{j,k} := H\left(t_j, s_k\right)$, we set

$$\widetilde{\gamma}_k(t) := A_{j-1,k} + \frac{t - t_{j-1}}{t_j - t_{j-1}}\left(A_{j,k} - A_{j-1,k}\right) \quad \text{ for } t_{j-1} \leq t \leq t_j$$

where

$$1 \le j \le m, \text{ and } 0 \le k \le \ell$$

Clearly every $\widetilde{\gamma}_k$ is a piecewise C^1 loop in X. The choice of δ shows that we can apply the Poincaré lemma in the convex neighborhood $\mathbb{B}\left(A_{j-1,k-1}, \varepsilon\right)$ of the points $A_{j-1,k-1}$. Thus we get that

$$\int_{\partial V_{j,k}} \alpha = 0 \quad \text{for } 1 \le j \le m \text{ and } 1 \le k \le \ell$$

where $\partial V_{j,k}$ denotes the closed piecewise straight curve from $A_{j-1,k-1}$ to $A_{j,k-1}$ to $A_{j,k}$ to $A_{j-1,k}$ and back to $A_{j-1,k-1}$. Therefore

$$\int_{\widetilde{\gamma}_{k-1}} \alpha = \int_{\widetilde{\gamma}_k} \alpha \quad \text{for } 1 \le k \le \ell$$

because the integral cancels itself over the "connecting pieces" between $\widetilde{\gamma}_{k-1}$ and $\widetilde{\gamma}_k$. Likewise, using the Poincaré lemma we conclude that $\int_{\widetilde{\gamma}_0} \alpha = \int_{\gamma_0} \alpha$ and $\int_{\widetilde{\gamma}_\ell} \alpha = \int_{\gamma_\ell} \alpha$, as the claim requires.

Corollary 3.1.31. Suppose X is open in $\mathbb{R}^n$ and simply connected. If $\alpha \in \Omega_{(1)}(X)$ is closed, α is exact.

3.2 Holomorphic functions

Setting: U open in $\mathbb{C}$, $f \in C^1(U, \mathbb{C})$. $u(x,y), v(x,y) \in C^1(U, \mathbb{R})$ be the real and imaginary part of f.

Definition 3.2.1. We say f is holomorphic if $f \in C^1(U, \mathbb{C})$.

Definition 3.2.2. Suppose $I \subset \mathbb{R}$ is a compact interval, and suppose Γ is a piecewise C^1 curve in U parametrized by

$$I \to U, \quad t \mapsto z(t) = x(t) + iy(t)$$

Then

$$\int_\Gamma f dz := \int_\Gamma f(z) dz := \int_\Gamma u dx - v dy + i \int_\Gamma u dy + v dx$$

is the complex line integral of f along Γ.

Proposition 3.2.3. Suppose Γ is a piecewise C^1 curve parametrized by $I \to U$, $t \mapsto z(t)$. Then

(1) $\int_\Gamma f(z) dz = \int_I f(z(t)) \dot{z}(t) dt$;

(2) $\left| \int_\Gamma f(z) dz \right| \le \max_{z \in \Gamma} |f(z)| L(\Gamma)$.

Proof:

Theorem 3.2.4 (the Cauchy integral theorem)**.** Suppose U is simply connected and f is holomorphic. Then, for every closed piecewise C^1 curve Γ in U,

$$\int_\Gamma f dz = 0$$

Proof: Since the 1-forms $\alpha_1 := udx - vdy$ and $\alpha_2 := udy + vdx$ are both closed. Because U is simply connected, it follows from Corollary 3.1.31 that α_1 and α_2 are exact. Now by fundamental theorem of line integrals

$$\int_\Gamma f dz = \int_\Gamma \alpha_1 + i \int_\Gamma \alpha_2 = 0$$

for every closed, piecewise C^1 curve Γ in U.

Proposition 3.2.5. Suppose f is holomorphic and γ_1 and γ_2 are homotopic piecewise C^1 loops in U. Then

$$\int_{\gamma_1} f dz = \int_{\gamma_2} f dz$$

Proof: By the homotopic invariant property of the integration of closed 1-form.

Theorem 3.2.6 (the Cauchy integral formula)**.** Suppose f is holomorphic and $\overline{\mathbb{D}}(z_0, r) \subset U$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}(z_0,r)} \frac{f(\zeta)}{\zeta - z} d\zeta \quad \text{for } z \in \mathbb{D}(z_0, r)$$

Proof: Suppose $z \in \mathbb{D}(z_0, r)$ and $\varepsilon > 0$. Then there is an $\delta > 0$ such that $\overline{\mathbb{D}}(z, \delta) \subset U$ and

$$|f(\zeta) - f(z)| \le \varepsilon \quad \text{for } \zeta \in \overline{\mathbb{D}}(z, \delta)$$

We set $\Gamma_\delta := \partial \mathbb{D}(z, \delta)$ and $\Gamma := \partial \mathbb{D}(z_0, r)$.

Since Γ_δ and Γ are homotopic in $U - \{z\}$,

$$\int_\Gamma \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta = \int_{\Gamma_\delta} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta$$

Since

$$\begin{aligned} \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta &= \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \\ &= f(z) + \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \end{aligned}$$

with estimate

$$\left| \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \right| \le \frac{\varepsilon}{2\pi\delta} 2\pi\delta = \varepsilon,$$

we have

$$\left| \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta - f(z) \right| \le \varepsilon$$

Because $\varepsilon > 0$ was arbitrary, the claim is proved.

Theorem 3.2.7. A function f is holomorphic if and only if it is analytic. Therefore

$$C^1(U, \mathbb{C}) = C^\omega(U, \mathbb{C})$$

Proof: Suppose f is holomorphic. Let $z_0 \in U$ and $r > 0$ with $\overline{\mathbb{D}}(z_0, r) \subset U$. We choose $z \in \mathbb{D}(z_0, r)$ and set $r_0 := |z - z_0|$. Notice that

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^k = \sum_{k=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} (z - z_0)^k$$

Because Γ is compact, there is an $M \geq 0$ such that $|f(\zeta)| \leq M$ for $\zeta \in \Gamma$. It follows that

$$\left|\frac{f(\zeta)}{(\zeta - z_0)^{k+1}} (z - z_0)^k\right| \leq \frac{M}{r^{k+1}} r_0^k = \frac{M}{r} \left(\frac{r_0}{r}\right)^k \qquad \text{for } \zeta \in \Gamma$$

Let

$$a_k := \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta \qquad \text{for } k \in \mathbb{N}$$

By the Weierstrass majorant criterion and Cauchy integral formula

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)} d\zeta \\ &= \frac{1}{2\pi i} \int_\Gamma \sum_{k=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} (z - z_0)^k \, d\zeta \\ &= \sum_{k=0}^{\infty} \left[\frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta\right] (z - z_0)^k \\ &= \sum_{k=0}^{\infty} a_k (z - z_0)^k \end{aligned}$$

Corollary 3.2.8 (Cauchy's derivative formula)**.** Suppose f is holomorphic, $z \in U$, and $r > 0$ with $\overline{\mathbb{D}}(z, r) \subset U$. Then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial \mathbb{D}(z,r)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \qquad \text{for } n \in \mathbb{N}$$

Theorem 3.2.9 (Liouville)**.** Every bounded entire function is constant.

Proposition 3.2.10. Suppose (g_n) is a locally uniformly convergent sequence of holomorphic functions in U. Then $g := \lim g_n$ is holomorphic in U.

3.3 Meromorphic functions

Setting: Let $r_1 > r_0 \geq 0$, define

$$z_0 + \Omega(\rho_0, \rho_1) = \{z \in \mathbb{C}; \rho_0 < |z - z_0| < \rho_1\}$$

and

$$\mathbb{D}^\bullet(z_0, r) := z_0 + \Omega(0, r) = \{z \in \mathbb{C}; 0 < |z - z_0| < r\}.$$

Proposition 3.3.1. Suppose $f : \Omega(r_0, r_1) \to \mathbb{C}$ is holomorphic.

(1) For $r, s \in (r_0, r_1)$, we have

$$\int_{r\partial\mathbb{D}} f(z)dz = \int_{s\partial\mathbb{D}} f(z)dz$$

(2) Suppose $a \in \Omega(\rho_0, \rho_1)$ with $r_0 < \rho_0 < \rho_1 < r_1$. Then

$$f(a) = \frac{1}{2\pi i}\int_{\rho_1\partial\mathrm{D}} \frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\int_{\rho_0\partial\mathrm{D}} \frac{f(z)}{z-a}dz$$

Proof: (2):Suppose $g : \Omega \to \mathbb{C}$ is defined through

$$g(z) := \begin{cases} (f(z)-f(a))/(z-a), & z \in \Omega\backslash\{a\} \\ f'(a), & z = a \end{cases}$$

It's easy to check g is holomorphic at a with $g'(a) = f''(a)/2$. Therefore,

$$\int_{r\partial\mathbb{D}} g(z)dz = \int_{s\partial\mathbb{D}} g(z)dz$$

Theorem 3.3.2. Every function f holomorphic in $\Omega := \Omega(r_0, r_1)$ has a unique Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \quad \text{for } z \in \Omega$$

The Laurent series converges uniformly on every compact subset of Ω, and its coefficients are given by

$$c_n = \frac{1}{2\pi i}\int_{r\partial\mathbb{D}} \frac{f(\zeta)}{\zeta^{n+1}}d\zeta \quad \text{for } n \in \mathbb{Z} \text{ and } r_0 < r < r_1$$

Corollary 3.3.3. Suppose f is holomorphic in $\mathbb{D}^{\bullet}(z_0, r)$. Then f has a unique Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \quad \text{for } z \in \mathbb{D}^{\bullet}(z_0, r)$$

where

$$c_n := \frac{1}{2\pi i}\int_{\partial\mathbb{D}(z_0,\rho)} \frac{f(z)}{(z-z_0)^{n+1}}dz \quad \text{for } n \in \mathbb{Z} \text{ and } \rho \in (0, r)$$

Definition 3.3.4 (Removable singularities)**.** Suppose U is an open subset of $\mathbb{C}$ and z_0 is a point in U. Given a holomorphic function $f : U\backslash\{z_0\} \to \mathbb{C}$, the point z_0 is a removable singularity if f has a holomorphic extension $F : U \to \mathbb{C}$.

Theorem 3.3.5 (Riemann's removability theorem)**.** Suppose $f : U\backslash\{z_0\} \to \mathbb{C}$ is holomorphic. Then the point z_0 is a removable singularity of f if and only if f is bounded in a neighborhood of z_0.

Proof: Suppose $r > 0$ with $\overline{\mathbb{D}}(z_0, r) \subset U$ and f is bounded in $\overline{\mathbb{D}}(z_0, r)$ with a upper bounded M.

For all $n \leq -1$, we have a estimate for each c_n in Corollary3.3.3

$$|c_n| \leq M\rho^{-n}$$

Let $\rho \to 0$, we have $c_n = 0$ for all $n \leq -1$. It follows that

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \qquad \text{for } z \in \mathbb{D}^\bullet (z_0, r)$$

The function defined through

$$z \mapsto \sum_{n=0}^{\infty} c_n (z - z_0)^n \qquad \text{for } z \in \mathbb{D} (z_0, r)$$

is holomorphic on $\mathbb{D} (z_0, r)$ and agrees with f on $\mathbb{D}^\bullet (z_0, r)$. Therefore z_0 is a removable singularity of f.

Definition 3.3.6 (Isolated singularities)**.** Suppose $f : U \backslash \{z_0\} \to \mathbb{C}$ is holomorphic and $r > 0$ with $\overline{\mathbb{D}} (z_0, r) \subset U$. Further suppose

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \qquad \text{for } z \in \mathbb{D}^\bullet (z_0, r)$$

is the Laurent expansion of f in $\mathbb{D}^\bullet (z_0, r)$. If z_0 is a singularity of f, it is isolated if it is not removable.

Due to the proof of Riemann's removability theorem, this is the case if and only if the principal part of the Laurent expansion of f does not vanish identically. If z_0 is an isolated singularity of f, we say z_0 is a pole of f if there is an $m \in \mathbb{N}^\times$ such that $c_{-m} \neq 0$ and $c_{-n} = 0$ for $n > m$. In this case, m is the order of the pole. If infinitely many coefficients of the principal part of the Laurent series are different from zero, z_0 is an essential singularity of f. Finally, we define the residue of f at z_0 through

$$\operatorname{Res} (f, z_0) := c_{-1}$$

Proposition 3.3.7. Suppose that f has an isolated singularity at the point z_0. Then z_0 is a pole of f if and only if $|f(z)| \to \infty$ as $z \to z_0$.

Proof: Apply Riemann's removability theorem to $1/f$.

Definition 3.3.8 (meromorphic functions)**.** A function g is said to be meromorphic in U if there is a closed and discrete subset $P(g)$ of U such that g is holomorphic in $U \backslash P(g)$ and every $z \in P(g)$ is a pole of g. Then $P(g)$ is the set of poles of g.

Proposition 3.3.9. A function g is meromorphic in U, then $P(g)$ has no limit point in U and $P(g)$ is countable.

Example 3.3.10. The Laurent expansion of the cotangent in $\pi\mathbb{D}^\bullet$ reads

$$\cot z = \frac{1}{z} - 2 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{\pi^{2k}} z^{2k-1} \qquad \text{for } z \in \pi\mathbb{D}^\bullet.$$

Setting: Γ is a closed piecewise C^1 curve in $\mathbb{C}$

Definition 3.3.11. For $a \in \mathbb{C}\backslash\Gamma$,

$$w(\Gamma, a) := \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-a}$$

is called the winding number, the circulation number, or the index of Γ about a.

Proposition 3.3.12. Winding number of a piecewise C^1 curve is an integer.

Proof:

Lemma 3.3.13 (lifting of piecewise C^1 curve)**.** Suppose I is a compact interval and $\gamma : I \to \mathbb{C}^\times$has piecewise continuous derivatives. Then there is a continuous and piecewise (the partation for C^1 curve are the same) continuously differentiable function $\varphi : I \to \mathbb{C}$ such that $\exp \circ \varphi = \gamma$.

Suppose γ is a piecewise C^1 parametrization of Γ and $(t_0, \ldots, t_m)$ is a partition of the parameter interval I such that $\gamma \mid [t_{j-1}, t_j]$ is continuously differentiable for $1 \le j \le m$. Then, for $a \in \mathbb{C}\backslash\Gamma$, there is a $\varphi \in C(I)$ such that $\varphi \mid [t_{j-1}, t_j] \in C^1[t_{j-1}, t_j]$ for $1 \le j \le m$, and $\exp\varphi = \gamma - a$. From this it follows that $\dot{\gamma}(t) = \dot{\varphi}(t)(\gamma(t) - a)$ for $t_{j-1} \le t \le t_j$ and $1 \le j \le m$. Therefore we get

$$\int_\Gamma \frac{dz}{z-a} = \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{\dot{\gamma}(t)dt}{\gamma(t)-a} = \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \dot{\varphi}(t)dt = \varphi(t_m) - \varphi(t_0)$$

Proposition 3.3.14. The map $w(\Gamma, \cdot) : \mathbb{C}\backslash\Gamma \to \mathbb{Z}$ is constant on every connected component. If a belongs to the unbounded connected component of $\mathbb{C}\backslash\Gamma$, then $w(\Gamma, a) = 0$.

Proposition 3.3.15. Suppose f is meromorphic in U and $w(\Gamma, a) = 0$ for $a \in U^c$. Then $\{z \in P(f)\backslash\Gamma; w(\Gamma, z) \neq 0\}$ is a finite set.

Definition 3.3.16. A curve Γ in U is said to be null homologous in U if it is closed and piecewise continuously differentiable and if $w(\Gamma, a) = 0$ for $a \in U^c$.

Theorem 3.3.17 (homology version of the Cauchy integral theorem and formula)**.** Suppose U is open in $\mathbb{C}$ and f is holomorphic in U. Then, if the curve Γ is null homologous in U,

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta = w(\Gamma, z)f(z) \quad \text{for } z \in U\backslash\Gamma$$

and

$$\int_\Gamma f(z)dz = 0$$

In particular,

$$w(\Gamma, z)f^{(k)}(z) = \frac{k!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}}d\zeta \quad \text{for } z \in U\backslash\Gamma, \quad k \in \mathbb{N}$$

Theorem 3.3.18 (residue theorem)**.** Suppose U is open in $\mathbb{C}$ and f is meromorphic in U. Further suppose Γ is a curve in $U\backslash P(f)$ that is null homologous in U. Then

$$\int_\Gamma f(z)dz = 2\pi i \sum_{p \in P(f)} \operatorname{Res}(f, p)w(\Gamma, p)$$

where only finitely many terms in the sum are distinct from zero.

Proof: We know that $A := \{a \in P(f); w(\Gamma, a) \neq 0\}$ is a finite set. Thus the sum in has only finitely many terms distinct from zero. Suppose $A = \{a_0, \ldots, a_m\}$ and f_j is the principal part of the Laurent expansion of f at a_j for $0 \leq j \leq m$. Then f_j is holomorphic in $\mathbb{C}\backslash \{a_j\}$, and the singularities of $F := f - \sum_{j=0}^{m} f_j$ at $a_0, \ldots, a_m$ are removable. Therefore, by the Riemann removability theorem, F has a holomorphic continuation (also denoted by F) on

$$U_0 := A \cup (U \backslash P(f))$$

Because Γ lies in $U \backslash P(f)$ and is null homologous in U, Γ lies in U_0 and is null homologous there. Thus it follows from the generalized Cauchy integral theorem that $\int_\Gamma F dz = 0$, which implies

$$\int_\Gamma f dz = \sum_{j=0}^{m} \int_\Gamma f_j dz$$

Because a_j is a pole of f, there are $n_j \in \mathbb{N}^\times$and $c_{jk} \in \mathbb{C}$ for $1 \leq k \leq n_j$ and $0 \leq j \leq m$ such that

$$f_j(z) = \sum_{k=1}^{n_j} c_{jk} \left(z - a_j\right)^{-k} \qquad \text{for } 0 \leq j \leq m$$

It therefore follows from the expression of $w(\Gamma, z) f^{(k)}(z)$ in homology version of the Cauchy integral theorem and formula

$$\int_\Gamma f_j dz = \sum_{k=1}^{n_j} c_{jk} \int_\Gamma \frac{dz}{\left(z - a_j\right)^k} = 2\pi i c_{j1} w\left(\Gamma, a_j\right)$$

Corollary 3.3.19 (Argument Principle)**.** Suppose f is meromorphic in an open set containing $\overline{\mathbb{B}}(a, r)$. If f has no poles and never vanishes on $\partial \mathbb{B}(a, r)$, then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \text{number of zeros inside the circle} - \text{number of poles inside the circle}$$

where the zeros and poles are counted with their multiplicities.

Corollary 3.3.20 (Open mapping theorem)**.** If f is holomorphic and nonconstant in a region Ω, then f is open.

Corollary 3.3.21 (Rouché's theorem)**.** Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If

$$|f(z)| > |g(z)| \qquad \text{for all } z \in C$$

then f and $f + g$ have the same number of zeros inside the circle C.

Corollary 3.3.22 (Maximum modulus principle)**.** If f is a non-constant holomorphic function in a region Ω, then f cannot attain a maximum in Ω.

3.4 Conformal mapping

We fix some terminology that we shall use in the rest of this chapter. A bijective holomorphic function $f : U \to V$ is called a conformal map or biholomorphism. Given such a mapping f, we say that U and V are conformally equivalent or simply biholomorphic.

Proposition 3.4.1. If $f : U \to \mathbb{C}$ is holomorphic and injective, then $f'(z) \neq 0$ for all $z \in U$. In particular, by Proposition 1.4.24, the inverse of f defined on its range is holomorphic, and thus the inverse of a conformal map is also holomorphic.

Proof: We argue by contradiction, and suppose that $f'(z_0) = 0$ for some $z_0 \in U$. Then

$$f(z) - f(z_0) = a(z - z_0)^k + G(z) \quad \text{for all } z \text{ near } z_0$$

with $a \neq 0, k \geq 2$ and G vanishing to order $k+1$ at z_0. For a sufficiently small positive real number r such that

(1) $\overline{\mathbb{B}}(z_0, r) \subset U$,

(2) $f'(z)$ is non-vanishing on $\mathbb{B}(z_0, r) - \{z_0\}$

(3) $|a|r^k - r^{k+1}M > 0$ or equivalently $r < |a|/M$, where

$$M = \max_{z \in \partial\mathbb{B}(z_0, r)} \left| \frac{G(z)}{(z - z_0)^{k+1}} \right|$$

Take $0 \neq \omega \in \mathbb{C}$ such that $|\omega| < |a|r^k - r^{k+1}M$, write

$$f(z) - f(z_0) - w = F(z) + G(z), \quad \text{where } F(z) = a(z - z_0)^k - w.$$

Then in the circle $\partial\mathbb{B}(z_0, r)$,

$$|G(z)| < r^{k+1}M < |a|r^k ar^k - |\omega| \leq |F(z)|$$

Since $|\omega| < |a|r^k - r^{k+1}M$, there's at least two zero of $F(z)$ inside the circle $\partial\mathbb{B}(z_0, r)$, by Rouché's theorem, $f(z) - f(z_0) - w$ has at least two zero inside the circle $\partial\mathbb{B}(z_0, r)$. Since $f'(z)$ is non-vanishing on $\mathbb{B}(z_0, r) - \{z_0\}$, there's two distinct complex number z_1 and z_2 such that $f(z_1) = f(z_2)$. A contridiction!

Chapter 4

Functional Analysis

4.1 Foundation

Definition 4.1.1. Let K denote either $\mathbb{R}$ or $\mathbb{C}$, and let X be a vector space over K. A seminorm on X is a function $x \mapsto \|x\|$ from X to $[0, \infty)$ such that

(1) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (the triangle inequality),

(2) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and $\lambda \in K$.

The second property clearly implies that $\|0\| = 0$. A seminorm such that $\|x\| = 0$ only when $x = 0$ is called a norm, and a vector space equipped with a norm is called a normed vector space (or normed linear space).

Definition 4.1.2. Banach space is a complete normed vector space.

Definition 4.1.3 (quotient space)**.** A related construction is that of quotient spaces. If $\mathcal{M}$ is a vector subspace of the vector space X, it defines an equivalence relation on X as follows: $x \sim y$ iff $x - y \in \mathcal{M}$. The equivalence class of $x \in \mathcal{X}$ is denoted by $x + \mathcal{M}$, and the set of equivalence classes, or quotient space, is denoted by $X/\mathcal{M}.X/\mathcal{M}$ is a vector space with vector operations $(x + \mathcal{M}) + (y + \mathcal{M}) = (x + y) + \mathcal{M}$ and $\lambda(x + \mathcal{M}) = (\lambda x) + \mathcal{M}$. If $\mathcal{X}$ is a normed vector space and $\mathcal{M}$ is closed, $X/\mathcal{M}$ inherits a norm from X called the quotient norm, namely

$$\|x + \mathcal{M}\| = \inf_{y \in \mathcal{M}} \|x + y\|$$

Proposition 4.1.4. A normed vector space is complete if and only if every absolutely convergent series converges.

Proposition 4.1.5. A linear map $T : X \to Y$ between two normed vector spaces is called bounded if there exists $C \geq 0$ such that

$$\|Tx\| \leq C\|x\| \text{ for all } x \in \mathcal{X}$$

If X and Y are normed vector spaces and $T : X \to Y$ is a linear map, the following are equivalent:

(1) T is continuous.

(2) T is continuous at 0 .

(3) T is bounded.

Definition 4.1.6. If X and Y are normed vector spaces, we denote the space of all bounded linear maps from X to Y by $L(X, Y)$. It is easily verified that $L(X, Y)$ is a vector space and that the function $T \mapsto \|T\|$ defined by

$$\begin{aligned}
\|T\| &= \sup\{\|Tx\| : \|x\| = 1\} \\
&= \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0\right\} \\
&= \inf\{C : \|Tx\| \leq C\|x\| \text{ for all } x\}
\end{aligned}$$

is a norm on $L(X, Y)$, called the operator norm.

Proposition 4.1.7. If Y is complete, so is $L(X, Y)$.

Corollary 4.1.8. Let X be a vector space over K, where $K = \mathbb{R}$ or $\mathbb{C}$. A linear map from X to K is called a linear functional on X. If X is a normed vector space, the space $L(X, K)$ of bounded linear functionals on X is called the dual space of X. Then X^* is a Banach space with the operator norm.

Proposition 4.1.9. Let X be a vector space over $\mathbb{C}$. If f is a complex linear functional on X and $u = \operatorname{Re} f$, then u is a real linear functional, and $f(x) = u(x) - iu(ix)$ for all $x \in X$. Conversely, if u is a real linear functional on X and $f : X \to \mathbb{C}$ is defined by $f(x) = u(x) - iu(ix)$, then f is complex linear. In this case, if X is normed, we have $\|u\| = \|f\|$.

Definition 4.1.10. If X is a real vector space, a sublinear functional on X is a map $p : X \to \mathbb{R}$ such that

$$p(x + y) \leq p(x) + p(y) \text{ and } p(\lambda x) = \lambda p(x) \text{ for all } x, y \in X \text{ and } \lambda \geq 0$$

Theorem 4.1.11 (The Hahn-Banach Theorem)**.** Let X be a real vector space, p a sublinear functional on $\mathcal{X}, \mathcal{M}$ a subspace of $\mathcal{X}$, and f a linear functional on $\mathcal{M}$ such that $f(x) \leq p(x)$ for all $x \in \mathcal{M}$. Then there exists a linear functional F on $\mathcal{X}$ such that $F(x) \leq p(x)$ for all $x \in \mathcal{X}$ and $F \mid \mathcal{M} = f$.

Definition 4.1.12 (Complex Hahn-Banach Theorem)**.** Let X be a complex vector space, p a seminorm on $\mathcal{X}, \mathcal{M}$ a subspace of $\mathcal{X}$, and f a complex linear functional on $\mathcal{M}$ such that $|f(x)| \leq p(x)$ for $x \in \mathcal{M}$. Then there exists a complex linear functional F on X such that $|F(x)| \leq p(x)$ for all $x \in \mathcal{X}$ and $F|_{\mathcal{M}} = f$.

Corollary 4.1.13. Let X be a normed vector space.

(1) If $\mathcal{M}$ is a closed subspace of X and $x \in X \backslash \mathcal{M}$, there exists $f \in X^*$ such that $f(x) \neq 0$ and $f \mid \mathcal{M} = 0$. In fact, if $\delta = \inf_{y \in \mathcal{M}} \|x - y\|$, f can be taken to satisfy $\|f\| = 1$ and $f(x) = \delta$.

(2) If $x \neq 0 \in X$, there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.

(3) The bounded linear functionals on X separate points.

(4) If $x \in X$, define $\widehat{x} : X^* \to K$ by $\widehat{x}(f) = f(x)$. Then the map $x \mapsto \widehat{x}$ is a linear isometry from X into X^{**} (the dual of X^*).

Corollary 4.1.14. Let X, Y be normed vector spaces and $A : X \to Y$ be a bounded linear operator. Let $x^* \in X^*$. The following are equivalent.

(1) $x^* \in \operatorname{im}(A^*)$.

(2) There is a constant $c \geq 0$ such that $|\langle x^*, x\rangle| \leq c\|Ax\|_Y$ for all $x \in X$.

Proof: (2) implies (1): Suppose x^* satisfies (2) and define the map $\psi : \operatorname{im}(A) \to \mathbb{C}$ as follows. Given $y \in \operatorname{im}(A)$ choose $x \in X$ such that $y = Ax$ and define $\psi(y) := \langle x^*, x\rangle$. By (2), for all $x \in \operatorname{Ker} A$, $\langle x^*, x\rangle = 0$. Then, ψ is a well-defined continous linear functional. By Hahn-Banach Theorem, there exists a $y^* \in Y^*$ such that $y^*|_{\operatorname{im}(A)} = \psi$. It satisfies $x^* = \psi \circ A = y^* \circ A = A^* y^*$.

Corollary 4.1.15 (separation of convex sets)**.** Let X be a real normed vector space and $A, B \subset X$ be nonempty disjoint convex sets such that A is closed and B is compact. Then there exists a bounded linear functional $\Lambda : X \to \mathbb{R}$ such that

$$\inf_{x \in A} \Lambda(x) > \sup_{y \in B} \Lambda(y)$$

Theorem 4.1.16. Let X be a normed vector space and let $Y \subset X$ be a linear subspace. Then the following holds.

(1) The linear map

$$X^*/Y^\perp \to Y^* : [x^*] \mapsto x^*|_Y$$

is an isometric isomorphism.

(2) Assume Y is closed and let $\pi : X \to X/Y$ be the canonical projection, given by $\pi(x) := x + Y$ for $x \in X$. Then the linear map

$$(X/Y)^* \to Y^\perp : \Lambda \mapsto \Lambda \circ \pi$$

is an isometric isomorphism.

Theorem 4.1.17 (open mapping theorem)**.** Let X and Y be Banach spaces. If $T \in L(X, Y)$ is surjective, then T is open.

Corollary 4.1.18. Let X and Y be Banach spaces. If $T \in L(X, Y)$ is bijective, then $T^{-1} \in L(Y, X)$.

Definition 4.1.19. If X and Y are normed vector spaces and T is a linear map from X to Y, we define the graph of T to be

$$\Gamma(T) = \{(x, y) \in X \times Y : y = Tx\}$$

which is a subspace of $X \times Y$. We say that T is closed if $\Gamma(T)$ is a closed subspace of $X \times Y$. ClearlY, if T is continuous, then T is closed.

Theorem 4.1.20 (The Closed Graph Theorem)**.** If X and Y are Banach spaces and $T : X \to Y$ is a closed linear map, then T is bounded.

Theorem 4.1.21 (The Uniform Boundedness Principle)**.** Suppose that X and Y are normed vector spaces and $\mathcal{A}$ is a subset of $L(X, Y)$. If X is a Banach space and $\sup_{T\in\mathcal{A}} \|Tx\| < \infty$ for all $x \in X$, then $\sup_{T\in\mathcal{A}} \|T\| < \infty$.

4.2 Topological Vector Space

Definition 4.2.1. A topological vector space is a vector space X over the field $K(= \mathbb{R}$ or $\mathbb{C})$ which is endowed with a topology such that the maps $(x, y) \to x + y$ and $(\lambda, x) \to \lambda x$ are continuous from $X \times X$ and $K \times X$ to X.

Definition 4.2.2 (locally convex)**.** A topological vector space is called locally convex if there is a base for the topology consisting of convex sets (that is, sets A such that if $x, y \in A$ then $tx + (1 - t)y \in A$ for $0 < t < 1$). Most topological vector spaces that arise in practice are locally convex and Hausdorff.

Proposition 4.2.3. Let $\{p_\alpha\}_{\alpha\in A}$ be a family of seminorms on the vector space X. If $x \in X, \alpha \in A$, and $\epsilon > 0$, let

$$U_{x\alpha\epsilon} = \{y \in X : p_\alpha(y - x) < \epsilon\}$$

and let $\mathcal{T}$ be the topology generated by the sets $U_{x\alpha\epsilon}$.

(1) For each $x \in X$, the finite intersections of the sets $U_{x\alpha\epsilon}(\alpha \in A, \epsilon > 0)$ form a neighborhood base at x.

(2) $x_i \to x$ iff $p_\alpha (x_i - x) \to 0$ for all $\alpha \in A$.

(3) $(X, \mathcal{T})$ is a locally convex topological vector space.

Proposition 4.2.4. Suppose X and y are vector spaces with topologies defined, respectively, by the families $\{p_\alpha\}_{\alpha\in A}$ and $\{q_\beta\}_{\beta\in B}$ of seminorms, and $T : X \to Y$ is a linear map. Then T is continuous iff for each $\beta \in B$ there exist $\alpha_1, \ldots, \alpha_k \in A$ and $C > 0$ such that $q_\beta(Tx) \leq C \sum_1^k p_{\alpha_j}(x)$

Proposition 4.2.5. Let X be a vector space equipped with the topology defined by a family $\{p_\alpha\}_{\alpha\in A}$ of seminorms. X is Hausdorff iff for each $x \neq 0$ there exists $\alpha \in A$ such that $p_\alpha(x) \neq 0$.

Proposition 4.2.6. A topological vector space X whose topology is defined by a countable family of seminorms such that X is Hausdorff is first-countable.

Definition 4.2.7. A topological vector space whose topology is defined by a countable family of seminorms is called a Fréchet space if it is Hausdorff and complete(every Cauchy sequence converges).

Definition 4.2.8 (weak topology)**.** X be a normed vector space, weak topology on X is the topological vector space induced by semi-norms: $x \mapsto |\langle x^*, x\rangle|, x^* \in X^*$.

Definition 4.2.9 (Strong Opertor Topology and Weak Opertor Topology)**.** Let X and Y be Banach spaces over $\mathbb{C}$. The topology on $L(X,Y)$ generated by semi-norms $T \mapsto ||Tx||$ (for all $x \in X$) is called the strong operator topology on $L(X,Y)$. The topology generated by the linear functionals $T \mapsto ||f(Tx)||(x \in X,\ f \in Y^*)$ is called the weak operator topology on $L(X,Y)$.

In addition, these topologies are best understood in terms of convergence: $T_\alpha \to T$ strongly iff $T_\alpha x \to Tx$ in the norm topology of Y for each $x \in X$, whereas $T_\alpha \to T$ weakly iff $T_\alpha x \to Tx$ in the weak topology of y for each $x \in X$.

Thus the strong operator topology is stronger than the weak operator topology but weaker than the norm topology on $L(X,Y)$.

Definition 4.2.10 (weak topology and weak star topology)**.** Let X be a normed vector space, X^* its dual space. The weak topology on X^* as defined above is the topology generated by X^{**}. The weak* topology on X^* as defined above is the topology generated by X as a subspace of X^{**}.

Therefore, weak* topology on X^* is simply the topology of pointwise convergence: $f_n \to f$ iff $f_n(x) \to f(x)$ for all $x \in X$. The weak* topology is even weaker than the weak topology on X^*.

4.3 Hilbert Space

Setting: all the vector space are over $\mathbb{C}$.

Definition 4.3.1. Let $\mathcal{H}$ be a complex vector space. An inner product (or scalar product) on $\mathcal{H}$ is a map $(x,y) \mapsto \langle x, y\rangle$ from $X \times X \to \mathbb{C}$ such that:

(1) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ for all $x, y, z \in \mathcal{H}$ and $a, b \in \mathbb{C}$.

(2) $\langle y, x\rangle = \overline{\langle x, y\rangle}$ for all $x, y \in \mathcal{H}$.

(3) $\langle x, x\rangle \in (0, \infty)$ for all nonzero $x \in X$.

A complex vector space with inner product is called a pre-Hilbert space. A Hilbert space is a vector space over $\mathbb{C}$ with a inner product such that the norm induced by this inner product is complete.

Proposition 4.3.2. If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$; that is, each $x \in \mathcal{H}$ can be expressed uniquelyas $x = y + z$ where $y \in \mathcal{M}$ and $z \in \mathcal{M}^{\perp}$. Moreover, y and z are the unique elements of $\mathcal{M}$ and $\mathcal{M}^{\perp}$ whose distance to x is minimal.

Theorem 4.3.3 (Riesz Representation Theoerm, Hilber Space Version)**.** If $f \in \mathcal{H}^*$, there is a unique $y \in \mathcal{H}$ such that $f(x) = \langle x, y \rangle$ for all $x \in X$.

Definition 4.3.4 (unitary map)**.** If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, a unitary map from $\mathcal{H}_1$ to $\mathcal{H}_2$ is an invertible linear map $U : \mathcal{H}_1 \to \mathcal{H}_2$ that preserves inner products:

$$\langle Ux, Uy \rangle_2 = \langle x, y \rangle_1 \text{ for all } x, y \in \mathcal{H}_1$$

Example 4.3.5. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $L^2(\mu)$ be the set of all measurable functions $f : X \to \mathbb{C}$ such that $\int |f|^2 d\mu < \infty$ (where, as usual, we identify two functions that are equal a.e.). From the inequality $ab \leq \frac{1}{2}\left(a^2 + b^2\right)$, valid for all $a, b \geq 0$, we see that if $f, g \in L^2(\mu)$ then $|f\bar{g}| \leq \frac{1}{2}\left(|f|^2 + |g|^2\right)$, so that $f\bar{g} \in L^1(\mu)$. It follows easily that the formula

$$\langle f, g \rangle = \int f\bar{g} d\mu$$

defines an inner product on $L^2(\mu)$.

Definition 4.3.6 (direct sum of Hilbert space)**.**

Definition 4.3.7 (adjoint operator)**.** Suppose that $\mathcal{H}$ is a Hilbert space and $T \in L(\mathcal{H}, \mathcal{H})$.

(1) There is a unique $T^* \in L(\mathcal{H}, \mathcal{H})$, called the adjoint of T, such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x, y \in \mathcal{H}$.

(2) $\|T^*\| = \|T\|, \|T^*T\| = \|T\|^2, (aS + bT)^* = \bar{a}S^* + \bar{b}T^*, (ST)^* = T^*S^*$, and $T^{**} = T$.

(3) Let $\mathcal{R}$ and $\mathcal{N}$ denote range and nullspace; then $\mathcal{R}(T)^{\perp} = \mathcal{N}(T^*)$ and $\mathcal{N}(T)^{\perp} = \overline{\mathcal{R}(T^*)}$.

(4) T is unitary iff T is invertible and $T^{-1} = T^*$.

4.4 Fredholm Theory

Definition 4.4.1 (Dual Operator)**.** $A : X \to Y$ be a bounded linear operator. The dual operator of A is the linear operator

$$A^* : Y^* \to X^*$$

defined by

$$A^* y^* := y^* \circ A : X \to \mathbb{C} \qquad \text{for } y^* \in Y^*.$$

Proposition 4.4.2. For every bounded linear functional $y^* : Y \to \mathbb{C}$, the bounded linear functional $A^* y^* : X \to \mathbb{C}$ is the composition of the bounded linear operator $A : X \to Y$ with y^*, i.e.

$$\langle A^* y^*, x \rangle = \langle y^*, Ax \rangle$$

for all $x \in X$.

Proposition 4.4.3. Let X and Y be normed vector spaces over $\mathbb{C}$ and let

$$A : X \to Y$$

be a bounded linear operator. Then the dual operator

$$A^* : Y^* \to X^*$$

is bounded and

$$\|A^*\| = \|A\|.$$

Proposition 4.4.4 (Weak* Closure of a Subspace)**.** Let X be a normed vector space and let $E \subset X^*$ be a linear subspace of its dual space. Then the following holds.

(1) The linear subspace $\left({}^{\perp}E\right)^{\perp}$ is the weak* closure of E.

(2) E is weak* closed if and only if $E = \left({}^{\perp}E\right)^{\perp}$.

(3) E is weak* dense in X^* if and only if ${}^{\perp}E = \{0\}$.

Proof: It suffices to show (1). Firstly, $\left({}^{\perp}E\right)^{\perp}$ is a weak star closed subset contains E. Hence, it suffice to show $\overline{E} \supset \left({}^{\perp}E\right)^{\perp}$. For all $f \in \left({}^{\perp}E\right)^{\perp}$, if there's a neighborhood $U_{x,\epsilon}$ of f given by $x \in X$ and $\epsilon > 0$ such that $U_{x,\epsilon} \cap E = \varnothing$, then $x \notin {}^{\perp}E$. Since ${}^{\perp}E$ is closed, by Hahn-Banach, there's $g \in \left({}^{\perp}E\right)^{\perp}$ such that $g(x) = f(x)$. A contradiction!

Corollary 4.4.5. If X is reflexive, E is weak star dense in X^* if and only if E is dense in X^*.

Proof: If E is not dense, by Hahn-Banach, there's $f \in X^*$ and $x \in X$ such that $\langle x^*, x\rangle = 0$ for all $x^* \in E$ and $f(x) \neq 0$. A contradiction.

Corollary 4.4.6. Let X and Y be complex normed vector spaces and let $A : X \to Y$ be a bounded linear operator. Then the following holds.

(1) $\mathrm{im}(A)^{\perp} = \ker(A^*)$ and ${}^{\perp}\mathrm{im}(A^*) = \ker(A)$.

(2) A has a dense image if and only if A^* is injective.

(3) A is injective if and only if A^* has a weak* dense image.

Proof: (1) and (2) follow from Hahn-Banach and (3) follows from above proposition.

Theorem 4.4.7 (Closed Image Theorem)**.** Let X and Y be Banach spaces, let $A : X \to Y$ be a bounded linear operator, and let $A^* : Y^* \to X^*$ be its dual operator. Then the following are equivalent.

(1) $\mathrm{im}(A) = {}^{\perp}\ker(A^*)$.

(2) The image of A is a closed subspace of Y.

(3) There exists a constant $c > 0$ such that every $x \in X$ satisfies

$$\inf_{A\xi=0} \|x+\xi\|_X \leq c\|Ax\|_Y.$$

Here the infimum runs over all $\xi \in X$ that satisfy $A\xi = 0$.

(4) $\operatorname{im}(A^*) = \ker(A)^\perp$.

(5) The image of A^* is a weak* closed subspace of X^*.

(6) The image of A^* is a closed subspace of X^*.

(7) There exists a constant $c > 0$ such that every $y^* \in Y^*$ satisfies

$$\inf_{A^*\eta^*=0} \|y^*+\eta^*\|_{Y^*} \leq c\,\|A^*y^*\|_{X^*}$$

Here the infimum runs over all $\eta^* \in Y^*$ that satisfy $A^*\eta^* = 0$.

Proof: (1) implies (2): trivial

(2) implies (3): Since A is bounded, there's $C > 0$ such that $\|Ax\| \leq C\|x\|$. Then, for all $\xi \in \mathrm{Ker}A$, $\|Ax\| = \|A(x+\xi)\| \leq C\|x+\xi\|$. We have, the map

$$X/\mathrm{Ker}A \to \mathrm{im}A$$

is bijective bounded linear map between Banach space. By open mapping theorem, the inverse map is also continous.

(3) implies (4): The inclusion $\operatorname{im}(A^*) \subset \ker(A)^\perp$ follows directly from the definitions. To prove the converse inclusion, fix an element $x^* \in \ker(A)^\perp$ so that $\langle x^*, \xi\rangle = 0$ for all $\xi \in \ker(A)$. Then

$$|\langle x^*, x\rangle| = |\langle x^*, x+\xi\rangle| \leq \|x^*\|_{X^*}\,\|x+\xi\|_X$$

for all $x \in X$ and all $\xi \in \ker(A)$. Take the infimum over all $\xi \in \ker(A)$ and use the inequality in (3) to obtain the estimate

$$|\langle x^*, x\rangle| \leq \|x^*\|_{X^*} \inf_{A\xi=0} \|x+\xi\|_X \leq c\,\|x^*\|_{X^*}\,\|Ax\|_Y$$

for all $x \in X$. It follows Corollary 4.1.14 that $x^* \in \operatorname{im}(A^*)$.

(4) implies (5): by Proposition 4.4.4.

(5) implies (6): Since weak* topology is weaker than norm topology.

(6) implies (7): the same as (2) implies (3)

(7) implies (1): non-trivial...

Proposition 4.4.8. Let X and Y be Banach spaces and let $A : X \to Y$ be a bounded linear operator. Then the following holds.

(1) The operator A is surjective if and only if A^* is injective and has a closed image. Equivalently, there exists a constant $c > 0$ such that

$$\|y^*\|_{Y^*} \leq c\,\|A^*y^*\|_{X^*} \qquad \text{for all } y^* \in Y^*$$

(2) The operator A^* is surjective if and only if A is injective and has a closed image. Equivalently, there exists a constant $c > 0$ such that

$$\|x\|_X \le c\|Ax\|_Y \qquad \text{for all } x \in X$$

Proof:

Definition 4.4.9 (compact operator)**.** Let X and Y be Banach spaces and let $K : X \to Y$ be a bounded linear operator. Then the following are equivalent.

(1) If $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in X then the sequence $(Kx_n)_{n\in\mathbb{N}}$ has a Cauchy subsequence.

(2) If $S \subset X$ is a bounded set then the set $K(S) := \{Kx \mid x \in S\}$ has a compact closure.

(3) The set $\overline{\{Kx \mid x \in X, \|x\|_X \le 1\}}$ is a compact subset of Y.

Proof: (1) implies (2): In a metric space, compact if and only if sequence compact. Hence, take $x_n \in \overline{K(S)}$, there's $a_n \in K(S)$ such that $\|x_n - a_n\| \le 1/n$, take a subsequence of x_{n_k} such that $a_{n_k} \to a$. We have $x_{n_k} \to a$.

Thus assume K satisfies (i) and let $S \subset X$ be a bounded set. Then every sequence in $K(S)$ has a Cauchy subsequence by (i). Hence Corollary 1.1.8 asserts that $\overline{K(S)}$ is a compact subset of Y, because Y is complete.

(2) implies (3): trivial

(3) implies (1): Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence and choose $c > 0$ such that $\|x_n\|_{X_1} \le c$ for all $n \in \mathbb{N}$. Then $(c^{-1}Kx_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(c^{-1}Kx_{n_i})_{i\in\mathbb{N}}$ by (3). Hence $(Kx_{n_i})_{i\in\mathbb{N}}$ is the required Cauchy subsequence.

Definition 4.4.10. Let X and Y be Banach spaces. A bounded linear operator $K : X \to Y$ is said to be of finite rank if its image is a finite-dimensional subspace of Y.

$K : X \to Y$ is said to be completely continuous if the image of every weakly convergent sequence in X under K converges in the norm topology on Y.

Proposition 4.4.11. Let X and Y be Banach spaces. Then the following holds.

(1) Every compact operator $K : X \to Y$ is completely continuous.

(2) Assume X is reflexive. Then a bounded linear operator $K : X \to Y$ is compact if and only if it is completely continuous.

Proposition 4.4.12 (set of compact operators is a closed ideal)**.** Let X, Y, and Z be Banach spaces. Then the following holds.

(1) Let $A : X \to Y$ and $B : Y \to Z$ be bounded linear operators and assume that A is compact or B is compact. Then $BA : X \to Z$ is compact.

(2) Let $K_i : X \to Y$ be a sequence of compact operators that converges to a bounded linear operator $K : X \to Y$ in the norm topology. Then K is compact.

(3) Let $K : X \to Y$ be a bounded linear operator and let $K^* : Y^* \to X^*$ be its dual operator. Then K is compact if and only if K^* is compact.

Definition 4.4.13 (Fredholm Operator)**.** Let X and Y be Banach spaces and let $A : X \to Y$ be a bounded linear operator. A is called a Fredholm operator if it has a closed image and its kernel and cokernel are finite-dimensional. If A is a Fredholm operator the difference of the dimensions of its kernel and cokernel is called the Fredholm index of A and is denoted by

$$\operatorname{index}(A) := \dim\ker(A) - \dim\operatorname{coker}(A)$$

Proposition 4.4.14. Let X and Y be Banach spaces and let $A : X \to Y$ be a bounded linear operator with a finite-dimensional cokernel. Then the image of A is a closed subspace of Y.

Proof: Let $m := \dim\operatorname{coker}(A)$ and choose vectors $y_1, \ldots, y_m \in Y$ such that the equivalence classes

$$[y_i] := y_i + \operatorname{im}(A) \in Y/\operatorname{im}(A), \quad i = 1, \ldots, m$$

form a basis of the cokernel of A. Define

$$\tilde{X} := X \times \mathbb{R}^m, \quad \|(x, \lambda)\|_{\tilde{X}} := \|x\|_X + \|\lambda\|_{\mathbb{R}^m}$$

for $x \in X$ and $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$. Then $\tilde{X}$ is a Banach space. Define the linear operator $\widetilde{A} : \widetilde{X} \to Y$ by

$$\widetilde{A}(x, \lambda) := Ax + \sum_{i=1}^{m} \lambda_i y_i$$

Then $\widetilde{A}$ is a surjective bounded linear operator and

$$\ker(\widetilde{A}) = \{(x, \lambda) \in X \times \mathbb{R}^m \mid Ax = 0, \lambda = 0\} = \ker(A) \times \{0\}$$

Since $\widetilde{A}$ is surjective, it follows from Closed Image Theorem that there exists a constant $c > 0$ such that

$$\inf_{\xi \in \ker(A)} \|x + \xi\|_X + \|\lambda\|_{\mathbb{R}^m} \leq c \left\| Ax + \sum_{i=1}^{m} \lambda_i y_i \right\|_Y$$

for all $x \in X$ and all $\lambda \in \mathbb{R}^m$. Take $\lambda = 0$ to obtain the inequality

$$\inf_{\xi \in \ker(A)} \|x + \xi\|_X \leq c\|Ax\|_Y \quad \text{for all } x \in X$$

Thus A has a closed image by Closed Image Theorem.

Proposition 4.4.15. Let X and Y be Banach spaces and let $A \in \mathcal{L}(X, Y)$. Then the following holds.

(1) If A and A^* have closed images then

$$\dim\ker(A^*) = \dim\operatorname{coker}(A), \quad \dim\operatorname{coker}(A^*) = \dim\ker(A)$$

(define $\dim V = \infty$ if V is a infinite-dimensional vector space)

(2) A is a Fredholm operator if and only if A^* is a Fredholm operator.

(3) If A is a Fredholm operator then $\operatorname{index}(A^*) = -\operatorname{index}(A)$.

Proof: (1): Assume A and A^* have closed images. Then

$$\operatorname{im}(A^*) = \ker(A)^\perp, \quad \ker(A^*) = \operatorname{im}(A)^\perp$$

by Closed Image Theorem. It follows from Theorem 4.1.16 that

$$\begin{aligned}(\ker(A))^* &\cong X^*/\ker(A)^\perp = X^*/\operatorname{im}(A^*) = \operatorname{coker}(A^*)\\ (\operatorname{coker}(A))^* &= (Y/\operatorname{im}(A))^* \cong \operatorname{im}(A)^\perp = \ker(A^*)\end{aligned}$$

Proposition 4.4.16 (Fredholm and Compact Operators)**.** Let X and Y be Banach spaces and let $A : Y \to X$ be a bounded linear operator. Then the following are equivalent.

(1) A is a Fredholm operator.

(2) There exists a bounded linear operator $F : X \to Y$ such that the operators $\mathrm{id}_X - FA : X \to X$ and $\mathrm{id}_Y - AF : Y \to Y$ are compact.

Proposition 4.4.17 (composition of Fredholm operator)**.** Let X, Y, Z be Banach spaces and let $A : X \to Y$ and $B : Y \to Z$ be Fredholm operators. Then $BA : X \to Z$ is a Fredholm operator and

$$\operatorname{index}(BA) = \operatorname{index}(A) + \operatorname{index}(B)$$

Proposition 4.4.18 (Stability of the Fredholm Index)**.** Let X and Y be Banach spaces and let $D : X \to Y$ be a Fredholm operator.

(1) If $K : X \to Y$ is a compact operator then $D + K$ is a Fredholm operator and index $(D + K) = \operatorname{index}(D)$.

(2) There is a constant $\varepsilon > 0$ such that the following holds. If $P : X \to Y$ is a bounded linear operator such that $\|P\| < \varepsilon$ then $D + P$ is a Fredholm operator and index $(D + P) = \operatorname{index}(D)$.

4.5 Spectrum of Operator

Definition 4.5.1 (Spectrum)**.** Let X be a complex Banach space and let $A \in \mathcal{L}^c(X)$. The spectrum of A is the set

$$\begin{aligned} \sigma(A) :&= \{\lambda \in \mathbb{C} \mid \text{ the operator } \lambda \mathrm{id} - A \text{ is not bijective } \} \\ &= \mathrm{P}\sigma(A) \cup \mathrm{R}\sigma(A) \cup \mathrm{C}\sigma(A) \end{aligned}$$

Here $\mathrm{P}\sigma(A)$ is the point spectrum, $\mathrm{R}\sigma(A)$ is the residual spectrum, and $\mathrm{C}\sigma(A)$ is the continuous spectrum. These are defined by

$$\begin{aligned} \mathrm{P}\,\sigma(A) :=&\{\lambda \in \mathbb{C} \mid \text{ the operator } \lambda \mathrm{id} - A \text{ is not injective } \} \\ \mathrm{C}\,\sigma(A) :=&\{\lambda \in \mathbb{C} \mid \text{ the operator } \lambda \mathrm{id} - A \text{ is injective and its image is dense } \} \\ \mathrm{R}\,\sigma(A) :=&\{\lambda \in \mathbb{C} \mid \text{ the operator } \lambda \mathrm{id} - A \text{ is injective but its image is not dense } \} \end{aligned}$$

The resolvent set of A is the complement of the spectrum. It is denoted by $\rho(A) := \mathbb{C}\backslash\sigma(A) = \{\lambda \in \mathbb{C} \mid \text{the operator } \lambda \mathrm{id} - A \text{ is bijective } \}$.

Definition 4.5.2. A complex Banach algebra is a pair consisting of complex Banach space $(\mathcal{A}, \|\cdot\|)$ and a bilinear map $\mathcal{A} \times \mathcal{A} \to \mathcal{A} : (a, b) \mapsto ab$ (called the product) that is associative, i.e.

$$(ab)c = a(bc) \quad \text{ for all } a, b, c \in \mathcal{A}$$

and satisfies the inequality

$$\|ab\| \leq \|a\|\|b\| \quad \text{ for all } a, b \in \mathcal{A}.$$

It is called unital if there exists an element $\mathrm{id} \in \mathcal{A}\backslash\{0\}$ such that

$$\mathrm{id}a = a\mathrm{id} = a \quad \text{ for all } a \in \mathcal{A}$$

The unit id, if it exists, is uniquely determined by the product.

Proposition 4.5.3. Let $\mathcal{A}$ be a unital Banach algebra.

(1) For every $a \in \mathcal{A}$ the limit

$$r_a := \lim_{n\to\infty} \|a^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|a^n\|^{1/n} \leq \|a\|$$

exists. It is called the spectral radius of a.

(2) If $a \in \mathcal{A}$ satisfies $r_a < 1$ then the element $\mathrm{id} - a$ is invertible and

$$(\mathrm{id} - a)^{-1} = \sum_{n=0}^{\infty} a^n$$

(3) The group $\mathcal{G} \subset \mathcal{A}$ of invertible elements is an open subset of $\mathcal{A}$ and the map $\mathcal{G} \to \mathcal{G} : a \mapsto a^{-1}$ is continuous.

More precisely, if $a \in \mathcal{G}$ and $b \in \mathcal{A}$ satisfy $\|a-b\| < 1/\|a^{-1}\|$, then $b \in \mathcal{G}$. Moreover $b^{-1} = \sum_{n=0}^{\infty} (\mathrm{id} - a^{-1}b)^n a^{-1}$,

$$\left\|b^{-1} - a^{-1}\right\| \le \frac{\|a-b\| \left\|a^{-1}\right\|^2}{1 - \|a-b\| \left\|a^{-1}\right\|}$$

and

$$\left\|b^{-1}\right\| \le \frac{\left\|a^{-1}\right\|}{1 - \|a-b\| \left\|a^{-1}\right\|}.$$

Proof: (1):Let $a \in \mathcal{A}$, define

$$r := \inf_{n \in \mathbb{N}} \|a^n\|^{1/n} \ge 0$$

and fix a real number $\varepsilon > 0$. Choose $m \in \mathbb{N}$ such that

$$\|a^m\|^{1/m} < r + \varepsilon$$

Fix two integers $k \ge 0$ and $0 \le \ell \le m-1$ and let $n := km + \ell$. Then

$$\begin{aligned}
\|a^n\|^{1/n} &= \left\|a^{km} a^{\ell}\right\|^{1/n} \\
&\le \|a\|^{\ell/n} \|a^m\|^{k/n} \\
&\le \|a\|^{\ell/n} (r+\varepsilon)^{km/n} \\
&= \left(\frac{\|a\|}{r+\varepsilon}\right)^{\ell/n} (r+\varepsilon)
\end{aligned}$$

Then, there is an integer $n_0 \in \mathbb{N}$ such that

$$\|a^n\|^{1/n} < r + 2\varepsilon$$

(2): Let $a \in \mathcal{A}$ and assume $r_a < 1$. Choose a real number α such that

$$r_a < \alpha < 1$$

Then there exists an $n_0 \in \mathbb{N}$ such that

$$\|a^n\|^{1/n} \le \alpha$$

for every integer $n \ge n_0$. Hence $\|a^n\| \le \alpha^n$ for every integer $n \ge n_0$. This implies $\sum_{n=0}^{\infty} \|a^n\| < \infty$, so the sequence

$$b_n := \sum_{i=0}^{n} a^i$$

converges.

Proposition 4.5.4. Let $A : X \to X$ be a bounded complex linear operator on a complex Banach space X and denote by $A^* : X^* \to X^*$ the complex dual operator. Then the following holds.

(1) The spectrum $\sigma(A)$ is a compact subset of $\mathbb{C}$.

(2) $\sigma(A^*) = \sigma(A)$.

(3) The point, residual, and continuous spectra of A and A^* are related by

$$\begin{aligned} &\mathrm{P}\sigma(A^*) \subset \mathrm{P}\sigma(A) \cup \mathrm{R}\sigma(A), && \mathrm{P}\sigma(A) \subset \mathrm{P}\sigma(A^*) \cup \mathrm{R}\sigma(A^*), \\ &\mathrm{R}\sigma(A^*) \subset \mathrm{P}\sigma(A) \cup \mathrm{C}\sigma(A), && \mathrm{R}\sigma(A) \subset \mathrm{P}\,\sigma(A^*), \\ &\mathrm{C}\sigma(A^*) \subset \mathrm{C}\sigma(A), && \mathrm{C}\sigma(A) \subset \mathrm{R}\,\sigma(A^*) \cup \mathrm{C}\sigma(A^*). \end{aligned}$$

(4) If X is reflexive then $\mathrm{C}\sigma(A^*) = \mathrm{C}\sigma(A)$ and

$$\begin{aligned} &\mathrm{P}\sigma(A^*) \subset \mathrm{P}\sigma(A) \cup \mathrm{R}\sigma(A), \\ &\mathrm{P}\sigma(A) \subset \mathrm{P}\sigma(A^*) \cup \mathrm{R}\sigma(A^*), \\ &\mathrm{R}\sigma(A^*) \subset \mathrm{P}\sigma(A), \\ &\mathrm{R}\sigma(A) \subset \mathrm{P}\sigma(A^*). \end{aligned}$$

Proof: (1)(2) follows from Proposition 4.5.3.
(3): By Proposition 4.4.6.
(4): By Corollary 4.4.5.

Proposition 4.5.5 (Spectral Radius)**.** Let X be a complex Banach space and let $A \in \mathcal{L}^c(X)$. Then $\sigma(A) \neq \emptyset$ and

$$r_A := \lim_{n\to\infty} \|A^n\|^{1/n} = \sup_{\lambda\in\sigma(A)} |\lambda|.$$

Proof: Step 1: Show that

$$\sup_{\lambda\in\sigma(A)} |\lambda| \leq r_A.$$

Step 2: Define the set $\Omega \subset \mathbb{C}$ by

$$\Omega := \left\{ z \in \mathbb{C} \mid z = 0 \text{ or } z^{-1} \in \rho(A) \right\}$$

and define the map $R : \Omega \to \mathcal{L}^c(X)$ by $R(0) := 0$ and by

$$R(z) := \left(z^{-1}\mathrm{id} - A \right)^{-1} \qquad \text{for } z \in \Omega \backslash \{0\}$$

Show that $R(z) \in C^1(\Omega, \mathcal{L}^c(X))$. Firstly, by Theorem 1.5.24, $R(z)|_{\Omega-0} \in C^1(\Omega - 0, \mathcal{L}^c(X))$. And for all z such that $|z| < 1/r_A$, by Proposition 4.5.3, $z \in \Omega$. Hence, $\mathbb{B}(0, 1/r_A) \subset \Omega$. For all $z \in \mathbb{B}(0, 1/r_A)$, we have

$$\|z^k A^k\| \leq (|z| r_A)^k.$$

Hence,

$$\sum_{k=0}^{\infty} z^{k+1} A^k = z(\mathrm{id} - zA)^{-1} = R(z), z \in \mathbb{B}(0, 1/r_A).$$

By Corollary 1.4.48, the restriction of $R(z)$ onto $\mathbb{B}(0, 1/r_A)$ lies in $C^1(\mathbb{B}(0, 1/r_A), \mathcal{L}^c(X))$. Therefore, $R(z) \in C^1(\Omega, \mathcal{L}^c(X))$.

Step 3: Now let $r > \sup_{\lambda\in\sigma(A)} |\lambda|$, so the closed disc $\overline{\mathbb{B}}(0, r^{-1})$ is contained in Ω. Define a loop

$$\gamma(t) := \frac{e^{2\pi it}}{r}, \quad 0 \le t \le 1$$

which lies in Ω. Show that for all $n \ge 1$,

$$A^n = \frac{1}{2\pi i}\int_\gamma \frac{R(z)}{z^{n+2}}dz = \frac{1}{2\pi i}\int_0^1 \frac{\dot\gamma(t)R(\gamma(t))}{\gamma(t)^{n+2}}dt = \int_0^1 \frac{R(\gamma(t))}{\gamma(t)^{n+1}}dt.$$

Recall Proposition2.3.12 of Bochner-Lebesgue Integration and Hahn-Banach Theorem, it suffice to show that for all $x^* \in X^*$ and $x \in X$, we have

$$\left\langle x^*, A^{n-1}x\right\rangle = \frac{1}{2\pi i}\int_\gamma \frac{\langle x^*, R(z)x\rangle}{z^{n+1}}dz$$

Since γ is homotopic to another loop

$$\gamma_1 : [0,1] \to \Omega, t \mapsto \frac{e^{2\pi it}}{2r_A}$$

and $\langle x^*, R(z)x\rangle \in C^1(\Omega, \mathbb{C})$ is holomorphic, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{\langle x^*, R(z)x\rangle}{z^{n+1}}dz = \frac{1}{2\pi i}\int_{\gamma_1} \frac{\langle x^*, R(z)x\rangle}{z^{n+1}}dz$$

It follows from the expression of $R(z)$ in $z \in \mathbb{B}(0, 1/r_A)$ that

$$\frac{1}{2\pi i}\int_{\gamma_1} \frac{\langle x^*, R(z)x\rangle}{z^{n+1}}dz = \left\langle x^*, A^{n-1}x\right\rangle.$$

Step 4: Notice that

$$\begin{aligned}\|A^n\| &\le \int_0^1 \frac{\|R(\gamma(t))\|}{|\gamma(t)|^{n+1}}dt \\ &= r^{n+1}\int_0^1 \|R(\gamma(t))\|dt \\ &\le r^{n+1}\sup_{0\le t\le 1}\|R(\gamma(t))\| \\ &= r^{n+1}\sup_{|\lambda|=r}\left\|(\lambda\mathrm{id} - A)^{-1}\right\|\end{aligned}$$

we have

$$r_A = \lim_{n\to\infty}\|A^n\|^{1/n} \le r$$

Setting:

Chapter 5

Harmonic Analysis

5.1 Topological Group

Definition 5.1.1. A topological group is a group G with a topology such that the maps $(g,h) \mapsto gh$ from $G \times G$ (with the product topology) to G and $g \mapsto g^{-1}$ from G to G are continuous.

Theorem 5.1.2 (topology defined by neighborhood basis)**.** Let G be a topological group, and let $\mathcal{N}$ be a neighbourhood base for the identity element e of G. Then

(1) for all $N_1, N_2 \in \mathcal{N}$, there exists an $N' \in \mathcal{N}$ such that $e \in N' \subset N_1 \cap N_2$;

(2) all $a \in N \in \mathcal{N}$, there exists an $N' \in \mathcal{N}$ such that $N'a \subset N$;

(3) all $N \in \mathcal{N}$, there exists an $V \in \mathcal{N}$ such that $V^{-1}V \subset N$;

(4) all $N \in \mathcal{N}$ and all $g \in G$, there exists an $N' \in \mathcal{N}$ such that $g^{-1}N'g \subset N$;

Conversely, if G is a group and $\mathcal{N}$ is a nonempty set of subsets of G contain e satisfying $(1), (2), (3), (4)$, then there is a (unique) topology on G such that G is a topological group and $\mathcal{N}$ form a neighborhood base at e. Morover, if subsets in $\mathcal{N}$ are all subgroup of G, we only need (1) and (4)

Proposition 5.1.3. G is a topological group.

(1) If H is a subgroup of G, so is $\bar{H}$.

(2) Every open subgroup of G is also closed.

(3) If K_1, K_2 are compact subsets of G, so is K_1K_2.

(4) Every subgroup of G, endowed with the subspace topology, is a topological group.

(5) Let G_1 and G_2 be topological groups. The direct product $G_1 \times G_2$ endowed with the product topology and componentwise group operation is a topological group.

Definition 5.1.4. A homomorphism $G \to H$ between topological groups is a continous group homomorphism $\varphi : G \to H$.

Proposition 5.1.5. G, H are topological groups. $\varphi : G \to H$ is a group homomorphism, then φ is continous if and only if φ is continous at identity.

Definition 5.1.6. Let f be a function on a group G. We define left and right translates of f by $L_h f(g) = f\left(h^{-1} g\right)$ and $R_h f(g) = f(gh)$, respectively. If f is a continuous function from G to $\mathbb{R}$ or $\mathbb{C}$, then we say that f is left uniformly continuous if, for all $\epsilon > 0$, there exists a neighborhood V of the identity such that

$$\|L_h f - f\|_u < \epsilon \quad \forall h \in V$$

where $\|\|_u$ is the uniform, or supremum, norm. And right uniform continuity is defined similarly. Let $C_c(G)$ be the space of continuous functions on G with compact support.

Proposition 5.1.7. Let G be a topological group. Every function $f \in C_c(G)$ is both left and right uniformly continuous.

Proposition 5.1.8. Let G be a topological group. Then the following assertions are equivalent:

(1) G is T_1.

(2) G is Hausdorff.

(3) The identity e is closed in G.

(4) Every point of G is closed in G.

Definition 5.1.9. X is a topological space, G is a topological group. If a topological group action is a group $G \times S \to S$ which is also continuous. If in addition the action is transitive, we call it transitive topological group action.

Example 5.1.10. G is a topological group and H be a subgroup of G. Give G/H, the set of left cosets, quotient topology. Then the group action $\rho : G \times G/H \to G/H : (g, aH) \mapsto gaH$ is a transtive topological group action.

Proof: If U open in G/H, let

$$W = \bigcup_{u \in U} u$$

and $\varphi : G \times G \to G$ be the multiplication and $\pi : G \times G \to G \times G/H$ be the product of identity and projection, we have $\rho^{-1}(U) = \pi(\varphi^{-1}(W))$.

Proposition 5.1.11. Let G be a topological group and let H be a subgroup of G. Then the following assertions hold:

(1) The canonical projection $\rho : G \to G/H$ is an open map.

(2) The quotient space G/H is T_1 if and only if H is closed.

(3) The quotient space G/H is discrete if and only if H is open. Moreover, if G is compact, then H is open if and only if G/H is finite.

(4) If H is normal in G, then G/H is a topological group with respect to coset multiplication and the quotient topology.

Proposition 5.1.12. Let G be a Hausdorff topological group. Then:

(1) The product of a closed subset F and a compact subset K is closed.

(2) If H is a compact subgroup of G, then $\rho : G \to G/H$ is a closed map.

Proposition 5.1.13. Let $\{G_i\}_i \in I$ be a set of LCHG(locally compact Hausdorff) such that G_i is compact for all but finitely many $i \in I$. Then

$$\prod_{i \in I} G_i$$

is a LCHG.

Proposition 5.1.14 (LCHG subgroup)**.** Let G be a Hausdorff topological group. Then a subgroup H of G is a LCHG (in the subspace topology) if and only if H is closed. In particular, every discrete subgroup of G is closed.

Proposition 5.1.15 (LCHG quotient group)**.** If G is LCHG and H is a closed subgroup, then G/H is a locally compact and Hausdorff space.

Theorem 5.1.16. Inverse limit exists in category of topological group.

Proof:

Example 5.1.17 (completion of $\mathbb{Z}$)**.** Define

$$\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$$

Since $\widehat{\mathbb{Z}}$ is completion, by Chinese Remainder Theorem, and Tychonoff theorem

$$\widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$$

Hence

$$\widehat{\mathbb{Z}}^\times = \varprojlim (\mathbb{Z}/n\mathbb{Z})^\times \cong \prod_p \mathbb{Z}_p^\times$$

Definition 5.1.18 (pro-finite group)**.** A topological group is pro-finite if it is isomorphic to a inverse limit of finite discrete topological group.

Proposition 5.1.19. A pro-finite group is compact, Hausdorff and totally disconnected.

Proof: Let G be a pro-finite group and $G \cong \varprojlim G_i$, since G_i is compact for each $i \in I$, it suffice to show $\varprojlim G_i$ is closed in product of G_i and also totally disconnected (connected component is one-point set).

Given $(g_i)_{i\in I} \notin \varprojlim G_i$, then there will exist p_{ij} such that $p_{ij}(g_j) \neq g_i$. Define

$$U = \{g_i\} \times \{g_j\} \times \prod_{k\neq i,j} G_k$$

which is open in $\prod_i G_i$ since G_i 's are discrete. Then $(g_i) \in U$, but $U \cap \varprojlim G_i = \emptyset$, which means $\prod_i G_i - \lim G_i$ is open.

Given any two elements $(g_i)_i$ and $(h_i)_i$ in $\prod_i G_i$ such that $(g_i)_i \neq (h_i)_i$, then there will exist some $j, g_j \neq h_j$. Define open subsets $U_j = \{g_j\} \times \prod_{i\neq j} G_i$ and $V_j = (G_j - \{g_j\}) \times \prod_{i\neq j} G_i$. Then $(g_i)_i \in U_j$ and $(h_i)_i \in V_j$ but $U_j \cap V_j = \emptyset$. Hence any subspace containing more than one element of X is not connected.

Definition 5.1.20 (compact-open topology)**.** Let G be a locally compact Hausdorff abelian group(LCHA). We will write the group operation multiplicatively. Define $\hat{G}$(group of unitary characters) to be the set of all continuous homomorphisms of G into the circle group, $S^1 := \{z \in \mathbb{C} : |z| = 1\}$, of the complex numbers.

Sets of the form

$$W(K, V) = \{\chi \in \hat{G} : \chi(K) \subseteq V\}$$

where K is a compact subset of G and V is a neighborhood of the identity in S^1 satisfies the four conditions in Theorem 5.1.2. Hence, it induces a topological group structure of $\hat{G}$. We call it compact-open topology.

Proposition 5.1.21. G is discrete, then $\hat{G}$ is compact.

Proof: G is compact, then by Yychonoff's Theorem, $(S^1)^G$ with product topology is compact. And its compact subspace $\hat{G}$ with subspace topology is the same as $\hat{G}$ itself with compact-open topology.

Proposition 5.1.22. G is comact, then $\hat{G}$ is discrete.

Proposition 5.1.23. χ_n converges to χ in $\hat{G}$ if and only if for each compact set K in G, $\chi_n|_K$ converges uniformly to $\chi|_K$. If G is compact, then the compact open topology coincides the topology of uniform convergence. If G is finite, then the compact-open topology coincides with the topology of pointwise convergence.

Proposition 5.1.24. G is a LCHA, then $\hat{G}$ is also LCHA.

Proof: Consider universal covering map $\phi : \mathbb{R} \to \mathbb{S}^1, x \mapsto e^{2\pi ix}$, define $N(\varepsilon) = \phi((-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}))$.

Hausdorff: if $\chi_1 \neq \chi_2$, there's $g \in G$ such that $\chi_1(g) \neq \chi_2$. Then there's $g \in K \subset U$, where K compact and U open, such that $|\chi_1 - \chi_2| \geq \varepsilon$ in U. Consider a sufficiently small ε_0, we have $\chi_1 U(K, N(\varepsilon_0)) \cap \chi_2 U(K, N(\varepsilon_0)) = \varnothing$.

Locally compact: Show that for every compact neighborhood K of G,

$$W(K, \overline{N(1/4)})$$

is a compact subset of $\hat{G}$.

Proposition 5.1.25. For a LCHA G, $\hat{G}$ is also LCHA.

(1) $\hat{\mathbb{R}} \cong \mathbb{R}$ as topological group with isometric map

$$\xi \mapsto (x \mapsto e^{2\pi i x \xi})$$

(2) $\hat{S}^1 \cong \mathbb{Z}$ as topological group, with isometric map

$$n \mapsto (z \mapsto z^n)$$

(3) $\hat{\mathbb{Z}} \cong S^1$, with isometric map

$$\alpha \mapsto (n \mapsto \alpha^n)$$

(4) $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$, with isometric map

$$m \mapsto (k \mapsto e^{2\pi i k m/n})$$

Definition 5.1.26. A left Haar measure is a non-zero Radon measure on a LCHG such that it is left-invariant.

Proposition 5.1.27. Let G be a LCHG. Define

$$C_c^+(G) = \{f \in C_c(G) : f \geq 0 \text{ and } \|f\|_u > 0\}.$$

we have

(1) A Radon measure μ on G is a left Haar measure iff the measure $\tilde{\mu}$ defined by $\widetilde{\mu}(E) = \mu\left(E^{-1}\right)$ is a right Haar measure.

(2) A nonzero Radon measure μ on G is a left Haar measure iff $\int f d\mu = \int L_y f d\mu$ for all $f \in C_c^+$ and $y \in G$.

(3) If μ is a left Haar measure on G, then $\mu(U) > 0$ for every nonempty open $U \subset G$, and $\int f d\mu > 0$ for all $f \in C_c^+$.

(4) If μ is a left Haar measure on G, then $\mu(G) < \infty$ iff G is compact.

Proposition 5.1.28. Every LCHG group G possesses a left Haar measure and it is unique up to a constant.

Example 5.1.29 (Haar measure on $\mathbb{T}^n$.)**.** Define $\varphi : Q = [0,1)^n \to \mathbb{T}^n : x \mapsto x + \mathbb{Z}^n$ a bijection map. and notive that $\mu : E \in B_{\mathbb{T}^n} \mapsto m(\varphi^{-1}(E)$ is a left invariant Radon measure.

And by Risez Representation Theorem, we can show that the measure induced by the positive linear functional

$$f \in C_c(\mathbb{T}^n) \mapsto \int_Q f \circ \pi$$

is left invariant, hence also Haar measure on $\mathbb{T}^n$.

Theorem 5.1.30 (Pontrjagin Duality). G LCHA. Then the map $G \to \hat{\hat{G}} : g \mapsto (\chi \mapsto \chi(g))$ is an isomorphic between topological group.

Definition 5.1.31 (Fourier Transform). Let $f \in L^1(G)$. Then we define $\hat{f} : \hat{G} \to \mathbb{C}$, the Fourier transform of f, to be

$$\hat{f}(\chi) = \int_G f(y)\overline{\chi(y)}dy \text{ for } \chi \in \hat{G}$$

Moreover, The Fourier Transform of $f \in L^1(G)$ is a continous function vanishes at infty.($\in C_0(G)$).

Theorem 5.1.32 (The Fourier Inversion Theorem). Let $\mathfrak{B}(G)$ denote the set of functions $f \in L^1(G)$ such that f is continuous and $\hat{f} \in L^1(\hat{G})$. There exists a Haar measure $d\chi$ on $\hat{G}$ such that for all $f \in \mathfrak{B}(G)$,

$$f(-y) = \int_{\hat{G}} \hat{f}(\chi)\overline{\chi(y)}d\chi$$

That is, $\hat{\hat{f}}(y) = f(-y)$. In addition, the Fourier transform $f \mapsto \hat{f}$ identifies $\mathfrak{B}(G)$ with $\mathfrak{B}(\hat{G})$.

Theorem 5.1.33 (The Plancherel Theorem). Fix a Haar measure on $\hat{G}$ such that Fourier Inversion Theorem holds. Then the Fourier transform on $L^1(G)\cap L^2(G)$ extends uniquely to a unitary map from $L^2(G)$ to $L^2(\widehat{G})$.

Definition 5.1.34 (modular function). If μ is a left Haar measure on G and $x \in G$, the measure $\mu_x(E) = \mu(Ex)$ is again a left Haar measure, because of the commutativity of left and right translations. Hence, by there is a positive number $\Delta(x)$ such that $\mu_x = \Delta(x)\mu$. The function $\Delta : G \to (0, \infty)$ thus defined. It is called the modular function of G.

Proposition 5.1.35. Δ is a continuous homomorphism from G to the multiplicative group of positive real numbers. Moreover, if μ is a left Haar measure on G, for any $f \in L^1(\mu)$ and y in G we have

$$\int (R_y f)\, d\mu = \Delta\left(y^{-1}\right) \int f d\mu$$

Proposition 5.1.36. The left Haar measures on G are also right Haar measures precisely when Δ is identically 1 , in which case G is called unimodular.

(1) If $G/[G, G]$ is finite or G is compact, then G is unimodular.

(2) If H is a compact subgroup of G, then $\Delta_G|H = \Delta_H = 1$

Proposition 5.1.37. Let G be a LCHG, S a LCH space, $\rho : G \times S \to S$ a transitive G-action on S. Take $s_0 \in S$, define $\varphi : G \to S, g \mapsto gs_0$. Let H be the stabilizer at s_0, a closed subgroup of G. It induces a continous bijection $\Phi : G/H \to S$.

If G is σ-compact, Φ is a homemorphism.

Definition 5.1.38. G is a LCHG with left Haar measure dx, H is a closed subgroup of G with left Haar measure $d\xi, q : G \to G/H$ is the canonical quotient map $q(x) = xH$, and Δ_G and Δ_H are the modular functions of G and H. We define a map $P : C_c(G) \to C_c(G/H)$ by

$$Pf(xH) = \int_H f(x\xi)d\xi.$$

Theorem 5.1.39. Suppose G is a LCHG and H is a closed subgroup. There is a G-invariant Radon measure μ on G/H if and only if $\Delta_G|_H = \Delta_H$. In this case, μ is unique up to a constant factor, and if this factor is suitably chosen we have

$$\int_G f(x)dx = \int_{G/H} Pf d\mu = \int_{G/H}\int_H f(x\xi)d\xi d\mu \quad (f \in C_c(G)).$$

Proposition 5.1.40. G a LCHA. Suppose H is a closed subgroup of G. Then $H^\perp$ is a closed subgroup of $\widehat{G}$. We have

(1) $(H^\perp)^\perp) = H$

(2) Define $\Phi : (G/H)^\wedge \to H^\perp$ and $\Psi : \widehat{G}/H^\perp \to \widehat{H}$ by

$$\Phi(\eta) = \eta \circ q, \quad \Psi\left(\xi H^\perp\right) = \xi|_H,$$

where $q : G \to G/H$ is the canonical projection. Then Φ and Ψ are isomorphisms of topological groups.

Definition 5.1.41 (Restricted Direct Product)**.** Let $J = \{\nu\}$ be a set of indices for which we are given G_ν, a LCHG, and let J_∞ be a fixed finite subset of J such that for each $\nu \notin J_\infty$ we are given a compact open subgroup $H_\nu \leq G_\nu$. The restricted direct product of G_v with respect to H_v is defined by

$$G = \prod_{\nu\in J}{}' G_\nu = \{(x_\nu) : x_\nu \in G_\nu \text{ with } x_\nu \in H_\nu \text{ for all but finitely many } \nu\}$$

Definition 5.1.42 (topology on restricted direct product)**.** Notice that subsets

$$B = \left\{\prod N_\nu : N_\nu \text{ a neighborhood of } 1 \in G_\nu \text{ and } N_\nu = H_\nu \text{ for all but finitely many } \nu\right\}$$

of G induces a topological group structure by Theorem 5.1.2.

Moreover, for any $S \subseteq J$, which necessarily contains J_∞, define G_S by

$$G_S = \prod_{\nu\in S} G_\nu \times \prod_{\nu\notin S} H_\nu$$

G_S is a open subgroup of G and product topology on G_S is identical to the subspace topology induced by restricted direct topology defined above. .

Proposition 5.1.43. G itself is a LCHG.

Proposition 5.1.44. A subset Y of G has compact closure if and only if $Y \subseteq \prod K_\nu$, for some family of compact subsets $K_\nu \subseteq G_v$, such that $K_\nu = H_\nu$ for all but finitely many indices ν.

Proposition 5.1.45. There exists a topological embedding of $G_\nu \longrightarrow G$ given by

$$x \longmapsto (\ldots, 1, 1, x, 1, 1, \ldots)$$

where the x is in the ν th component. And image of G_ν is a closed subgroup of G.

Lemma 5.1.46. Let $\chi \in \mathrm{Hom}_{\mathrm{Cont}}(G, \mathbb{C}^\times)$ (quasi-characters). Then χ is trivial on all but finitely many H_ν. Therefore, for $y \in G, \chi(y_\nu) = 1$ for all but finitely many ν, and

$$\chi(y) = \prod_\nu \chi(y_\nu).$$

Lemma 5.1.47. For each ν let $\chi_\nu \in \mathrm{Hom}_{\mathrm{Cont}}\ (G_\nu, \mathbb{C}^\times)$and $\chi_\nu|_{H_\nu} = 1$ for all but finitely many indices ν. Then we have that $\chi = \prod_\nu \chi_\nu \in \mathrm{Hom}_{\mathrm{Cont}}\ (G, \mathbb{C}^\times)$.

Theorem 5.1.48. Let G be the restricted direct product of LCHA G_ν with respect to compact-open subgroups H_ν. As topological groups, we have that

$$\hat{G} \cong \prod{}' \hat{G}_\nu$$

where the restricted direct product on the right is taken with respect to subgroups defined by

$$K(G_\nu, H_\nu) = \left\{\chi_\nu \in \hat{G}_\nu : \chi_\nu|_{H_\nu} = 1\right\}$$

for $\nu \notin J_\infty$. This subgroup traditionally is denoted $H_\nu^\perp$.

Proof: We will begin by showing that $K(G_\nu, H_\nu)$ is a compact-open subgroup of $\hat{G}_\nu$. It is clear that $K(G_\nu, H_\nu)$ is a subgroup of G_ν. Let U be a neighborhood of 1 in $\mathbb{C}^\times$that contains no other subgroup besides the trivial subgroup. Consider the neighborhood of the trivial character on G_ν defined by

$$W(H_\nu, U) = \left\{\chi \in \hat{G}_\nu : \chi(H_\nu) \subseteq U\right\}$$

Since $\chi(H_\nu)$ is a subgroup of U, then $\chi(H_\nu) = \{1\}$, and hence

$$W(H_\nu, U) = K(G_\nu, H_\nu)$$

This shows that $K(G_\nu, H_\nu)$ is an open subgroup of $\hat{G}_\nu$. By Proposition 5.1.11 and 5.1.40, $K(G_\nu, H_\nu)$ is a compact open subgroup.

Now, we assume Haar measure on G_v are all σ-finite.

Definition 5.1.49 (Restricted Direct Integration)**.** Let dg_ν denote a left (right) Haar measure on G_ν normalized so that

$$\int_{H_\nu} dg_\nu = 1$$

for almost all $\nu \notin J_\infty$. Then there is a unique left (respectively, right) Haar measure dg on G such that for each finite set of indices S containing J_∞, the restriction of dg_s of dg to G_S(open subgroup of G) is precisely the product measure(infinite Radon product described in Analysis 2.7.19, hence also Haar measure on G_S). We will write $dg = \prod_\nu dg_\nu$ for this measure.

Proposition 5.1.50. Let $f \in L^1(G)$, for all $S \supset J_\infty$, we have $f|_{G_S} \in L^1(G_S)$. And if S_n be a sequence of subsets of J such that $S_n \supset J_\infty$ with $S_n \subset S_{n+1}$ and

$$\bigcup_{i=1}^{\infty} S_n = J,$$

then

$$\int_G f(g) = \lim_{n\to\infty} \int_{G_{S_n}} f(g_s)\, dg_S$$

Proposition 5.1.51. Let S_0 denote the finite set of indices containing both J_∞ and the set of indices for which $\mathrm{Vol}(H_\nu, dg_\nu) \neq 1$. Suppose that for each index ν, we are given a continuous and integrable function f_ν on G_ν, such that $f_\nu|_{H_\nu} = 1$ for all ν outside some finite set S_1. Then for $g = (g_\nu) \in G$ we can define the function

$$f(g) = \prod_\nu f_\nu(g_\nu)$$

The function f is well-defined and continuous on G. Furthermore, if S is any finite set of indices including S_0 and S_1, then we have $f|_{G_S} \in L^1(G_S)$ and

$$\int_{G_S} f(g) dg_S = \prod_{\nu\in S} \left(\int_{G_\nu} f_\nu(g_\nu)\, dg_\nu \right)$$

Furthermore, if

$$\prod_\nu \left(\int_{G_\nu} |f_\nu(g_\nu)|\, dg_\nu \right) < \infty$$

then $f \in L^1(G)$ and

$$\int_G f(g) dg = \prod_\nu \left(\int_{G_\nu} f_\nu(g_\nu)\, dg_\nu \right)$$

Now we assume G_v are all abelian group.

Proposition 5.1.52. Let $f_\nu \in L^1(G) \cap C(G)$ and of f_ν being a characteristic function of H_ν for all but finite many ν. Then $f \in L^1(G)$ and the Fourier transform of f is given by

$$\hat{f}(g) = \prod_\nu \hat{f}_\nu(g_\nu)$$

Moreover, if we additionally assume $f_\nu \in \mathfrak{B}(G_\nu)$ for all ν, $f \in \mathfrak{B}(G)$.

Proof: The key point is to notice that

$$\hat{f}_\nu(\chi_\nu) = \mathrm{Vol}(H_\nu, dg_\nu)\, \mathbf{1}_{H_\nu^\perp}(\chi_\nu).$$

Now we need to define dual measure on $\hat{G}$ such that Fourier Inversion Theorem holds.

Theorem 5.1.53. The measure $d\chi = \prod_\nu d\chi_\nu$, where $d\chi_\nu = \widehat{dg_\nu}$, is dual the measure $dg = \prod_\nu dg_\nu$. Therefore,

$$f(g) = \int_{\hat{G}} \hat{f}(\chi)\chi(g) d\chi,$$

for all $f \in \mathfrak{B}(G)$.

Proof: Notice that

$$\hat{\hat{f}}_\nu\left(g_\nu\right)=\operatorname{Vol}\left(H_\nu, dg_\nu\right)\int_{\hat{G}_\nu}\mathbf{1}_{H_\nu^\perp}\left(\chi_\nu\right)\chi_\nu\left(g_\nu\right)d\chi_\nu=$$

$$\operatorname{Vol}\left(H_\nu, dg_\nu\right)\int_{H_\nu^\perp}\chi_\nu\left(g_\nu\right)d\chi_\nu=\operatorname{Vol}\left(H_\nu, dg_\nu\right)\operatorname{Vol}\left(H_\nu^\perp, d\chi_\nu\right)1_{\left(H_\nu^\perp\right)^\perp}$$

and $(H_\nu^\perp)^\perp = H_\nu$. We have $\operatorname{Vol}\left(H_\nu, dg_\nu\right)\operatorname{Vol}\left(H_\nu^\perp, d\chi_\nu\right)=1$

5.2 Unitary Representation of Topological Group

5.3 Fourier Transform

Definition 5.3.1 (Schwartz space)**.** The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consisting of those C^∞ functions which, together with all their derivatives, vanish at infinity faster than any power of $|x|$. More precisely, for any nonnegative integer N and any multi-index α we define

$$\|f\|_{(N,\alpha)}=\sup_{x\in\mathbb{R}^n}(1+|x|)^N\left|\partial^\alpha f(x)\right|$$

then

$$\mathcal{S}=\mathcal{S}(\mathbb{R}^n)=\left\{f\in C^\infty : \|f\|_{(N,\alpha)}<\infty \text{ for all } N,\alpha\right\}$$

Proposition 5.3.2. $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet Space and Fourier Transform is a linear bi-continous bijection between Schwartz space.

5.4 Pointwise Convergence of Fourier Series

Let f be a function $\mathbb{R}\to\mathbb{C}$ with period 1. Assume $f|_{[0,1]}\in L^1([0,1])$ and define

$$\widehat{f}(k)=\int_0^1 f(y)e^{-2\pi iky}dy$$

We denote by $S_m f$ the m th symmetric partial sum of the Fourier series of f :

$$S_m f(x)=\sum_{-m}^{m}\widehat{f}(k)e^{2\pi ikx}$$

From the definition of $\widehat{f}(k)$, we have

$$S_m f(x)=\sum_{-m}^{m}\int_0^1 f(y)e^{2\pi ik(x-y)}dy$$

where D_m is the m th Dirichlet kernel:

$$D_m(x)=\sum_{-m}^{m}e^{2\pi ikx}.$$

The terms in this sum form a geometric progression, so

$$D_m(x) = e^{-2\pi imx}\sum_0^{2m} e^{2\pi ikx} = e^{-2\pi imx}\frac{e^{2\pi(2m+1)x}-1}{e^{2\pi ix}-1}.$$

Multiplying top and bottom by $e^{-\pi ix}$ yields the standard closed formula for D_m :

$$D_m(x) = \frac{e^{(2m+1)\pi ix}-e^{-(2m+1)\pi ix}}{e^{\pi ix}-e^{-\pi ix}} = \frac{\sin(2m+1)\pi x}{\sin \pi x}$$

Theorem 5.4.1. If f is periodic on $\mathbb{R}$ with period 1 and is a bounded variation function on $\left[-\frac{1}{2},\frac{1}{2}\right]$. Then

$$\lim_{m\to\infty} S_m f(x) = \frac{1}{2}[f(x+)+f(x-)] \text{ for every } x.$$

In particular, $\lim_{m\to\infty} S_m f(x) = f(x)$ at every x at which f is continuous.

Example 5.4.2. Let us first consider a simple example: Let

$$\phi(x) = \frac{1}{2} - x - [x] \quad ([x] = \text{ greatest integer } \le x).$$

It is easy to check that $\widehat{\phi}(0) = 0$ and $\widehat{\phi}(k) = (2\pi ik)^{-1}$ for $k \neq 0$, so that

$$S_m\phi(x) = \sum_{0<|k|\le m}\frac{e^{2\pi ikx}}{2\pi ik} = \sum_1^m \frac{\sin 2\pi kx}{\pi k}.$$

Hence,

$$\sum_{k=1}^{\infty}\frac{\sin 2\pi kx}{\pi k} = \begin{cases} 1 & x\in\mathbb{Z} \\ \frac{1}{2}-x-[x] & x\notin\mathbb{Z}\end{cases}$$

Example 5.4.3. For $|t|\le\pi$, we have

$$|t| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos((2k+1)t)}{(2k+1)^2}$$

and the series converges uniformly on $\mathbb{R}$.

Example 5.4.4. For all $z\in\mathbb{C}-\mathbb{Z}$,

$$\cos(zt) = \frac{\sin(\pi z)}{\pi}\left(\frac{1}{z}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{z+n}+\frac{1}{z-n}\right)\cos(nt)\right)$$

In particular,

$$\pi\cot(\pi z) = \frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}+\frac{1}{z-n}\right).$$

Chapter 6

Differential Equation

6.1 Linear ODE

6.2 Initial Value Problem