

# Combination 3: Binomial Coefficients

---

Erzhuo Wang

September 20, 2024

Youth STEM Academy

这节课我们主要讲一些组合数的基本性质, 然后做一些组合方面的综合性题目来巩固之前学的内容。

# Binomial Theorem(二项式定理)

## Theorem (Binomial Theorem)

The expansion of  $(x + y)^n$  for positive integer  $n$  is

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0y^n$$

## Example

Show that  $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

# Properties of Binomial Coefficients

If  $k$  and  $n$  are two integers and  $0 \leq k \leq n$ , then

- $\binom{n}{k} = \binom{n}{n-k}$

- $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

- $k\binom{n}{k} = n\binom{n-1}{k-1}$

- 

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

- 

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n} = 0$$

# Example

## Question

What is the coefficient of  $x^7$  in the polynomial  $(x + 3)^{10}$ .

# Exercise

## Question

Two people flip a fair coin. One flipped it 5 times while the other 6 times. What is the probability that the second person flipped more tails than the first person?

两个人抛一枚均匀的硬币。一个人翻了 5 次,另一个人翻了 6 次。求第二个人抛到反面的次数比第一个人多的概率?

## Question

A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

一个盒子里正好有 5 个筹码,3 个红牌,2 个白牌。每次随机取出一张筹码,直到所有红牌或白牌都被取出为止。最后一张牌是白色的概率是多少?

# Exercise

## Question

Two different numbers are randomly selected from  $-2, -1, 0, 3, 4, 5$  and multiplied together. What is the probability that the product is 0?  
从  $-2, -1, 0, 3, 4, 5$  中随机选择两个不同的数字并相乘, 为 0 的概率是多少?

## Question

在海滩上, 50 个人戴着太阳镜, 35 个人戴着帽子。有些人既戴太阳镜又戴帽子。如果随机选择一个戴帽子的人, 这个人也戴太阳镜的概率是  $\frac{2}{5}$ 。如果随机选择一个戴太阳镜的人, 这个人也戴帽子的概率是多少?

### Question (AMC 8, 2019-25)

Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the three people has at least two apples?

爱丽丝有 24 个苹果。她有多少种方法可以与贝基和克里斯分享这些苹果, 使得这样三个人每人至少有两个苹果?

### Question (AMC8, 2020-23)

Five different awards are to be given to three students. Each student will receive at least one award. In how many different ways can the awards be distributed?

(A) 120 (B) 150 (C) 180 (D) 210 (E) 240



# Challenging Problem

## Question

What is the probability that any 50 person has the same birthday?  
(Assuming there are 365 days in a year)

任取 50 个人, 存在两人生日相同的概率是多少?(假设一年有 365 天)

## Question

A set of teams held a round-robin tournament in which every team played every other team exactly once. Every team won 10 games and lost 10 games. How many sets of three teams  $\{A, B, C\}$  were there in which  $A$  beat  $B$ ,  $B$  beat  $C$ ,  $C$  beat  $A$ .