Analysis

Erzhuo Wang

September 15, 2024

Contents

Chapter 1

Foundation

1.1 Construction of Real Number

Definition 1.1.1 (ordered ring)**.** Thus, a ring(field) $R \neq 0$ with an order $<$ is called an ordered ring(field) if the following holds:

(1) $(R, <)$ is totally ordered

(2) $x < y \Rightarrow x + z < y + z, z \in R$

(3) $x, y > 0 \Rightarrow xy > 0$

Of course, an element $x \in R$ is called positive if $x > 0$ and negative if $x < 0$. We gather in the next proposition some simple properties of ordered fields.

Proposition 1.1.2. Let K be an ordered field, then for $x, y, a, b \in K$.

(1) $x > y \Leftrightarrow x - y > 0$.

(2) If $x > y$ and $a > b$, then $x + a > y + b$.

(3) If $a > 0$ and $x > y$, then $ax > ay$.

(4) If $x > 0$, then $-x < 0$. If $x < 0$, then $-x > 0$.

(5) Let $x > 0$. If $y > 0$, then $xy > 0$. If $y < 0$, then $xy < 0$.

(6) If $a < 0$ and $x > y$, then $ax < ay$.

(7) $x^2 > 0$ for all $x \neq 0$. In particular, $1 > 0$.

(8) If $x > 0$, then $x^{-1} > 0$.

(9) If $x > y > 0$, then $0 < x^{-1} < y^{-1}$ and $xy^{-1} > 1$.

Definition 1.1.3. K is a ordered field, K is said to be Archimedes if and only if for $x, y > 0$ there's $n \in \mathbb{Z}$ such that $nx > y$.

Example 1.1.4. $\mathbb{Q}$ is a Archimedes ordered field with original order.

Proposition 1.1.5. For an ordered field K, the absolute value function, $|\cdot| : K \to K$ and the sign function, $\operatorname{sign}(\cdot) : K \to K$ are defined by

$$|x| := \begin{cases} x, & x > 0, \\ 0, & x = 0, \\ -x, & x < 0, \end{cases} \qquad \operatorname{sign} x := \begin{cases} 1, x > 0, \\ 0, x = 0, \\ -1, x < 0. \end{cases}$$

Let K be an ordered field and $x, y, a, \varepsilon \in K$ with $\varepsilon > 0$.

(1) $x = |x| \operatorname{sign}(x), |x| = x \operatorname{sign}(x)$.

(2) $|x| = |-x|, \quad x \leq |x|$.

(3) $|xy| = |x||y|$.

(4) $|x| \geq 0$ and $(|x| = 0 \Leftrightarrow x = 0)$.

(5) $|x - a| < \varepsilon \Leftrightarrow a - \varepsilon < x < a + \varepsilon$.

(6) $|x + y| \leq |x| + |y|$ (triangle inequality).

(7) $|x - y| \geq ||x| - |y||, \quad x, y \in K$

Definition 1.1.6. A ring homomorphism f between ordered field is said to be order-preserving if

$$a < b \Longleftrightarrow f(a) < f(b)$$

.

Definition 1.1.7. A sequence $r = (x_n)_{n \in \mathbb{Z}_{>0}}$ is a Cauchy sequence if for all $\epsilon \in \mathbb{Q} > 0$, there's $N > 0$ such that for all $m, n > N$, $|x_n - x_m| < \epsilon$.

Proposition 1.1.8. Cauchy sequence is bounded.

Definition 1.1.9. Let

$$\mathcal{R} = \left\{ (x_n) \in \mathbb{Q}^{\mathbb{Z}_{>0}} : (x_n) \text{ is a Cauchy sequence} \right\}$$

and

$$\mathbf{c}_0 = \left\{ (x_n) \in \mathbb{Q}^{\mathbb{Z}_{>0}} : \text{for all } \epsilon > 0, \text{ there's } N > 0 \text{ such that for all } n > N, |x_n| < \epsilon \right\}$$

It's clear that $\mathbf{c}_0 \subset \mathcal{R}$ is a maximal ideal of $\mathcal{R}$. Hence $\mathcal{R}/\mathbf{c}_0$ is a field and we denote it by $\mathbb{R}$. For convenience, we usually denote $(a_n) + \mathbf{c_0}$ by (a_n).

Definition 1.1.10. Now we define a order on $\mathbb{R}$, for $(a_n), (b_n)$ in $\mathbb{R}$, $(a_n) > (b_n)$ if there's $\epsilon > 0$, a sufficiently large integer N, such that $a_n - b_n > \epsilon$ for $n > N$. And denote this order by $<$. It's esay to check that '$<$' is well-defined and totally ordered.

Proposition 1.1.11. $(\mathbb{R}, <)$ is a Archimedes ordered field. And the embedding $l : \mathbb{Q} \to \mathbb{R}$ given by

$$r \mapsto (r, r, r, \dots)$$

is an order-preserving ring homomorphism.

Definition 1.1.12. For a sequence $(A_n) \in \mathbb{R}$, we say $A_n \to A$ if for all $\epsilon \in \mathbb{R} > 0$, there's $N > 0$ such that for all $n > N$, $|A_n - A| < \epsilon$. And we say (A_n) is a Cauchy sequence if for all $\epsilon \in \mathbb{R}_{>0}$, there's $N > 0$ such that for all $m, n > N$, $|x_n - x_m| < \epsilon$.

Proposition 1.1.13 (dense)**.** For all $a, b \in \mathbb{R}$, if $a < b$, there's $c \in \mathbb{Q}$ such that $a < l(c) < b$.

Proposition 1.1.14 (completeness)**.** (A_n) is a Cauchy sequence in $\mathbb{R}$ if and only if there's $A \in \mathbb{R}$ such that $A_n \to A$.

Proof: 'if' is obvious.

'only if': Take $x_n \in \mathbb{Q}$ such that:

$$A_n < l(x_n) < A_n + l(\frac{1}{n})$$

It's cleat that $a = (x_n) + \mathbf{c}_0 \in \mathbb{R}$.

Notice that $A_n \to a$, we have $\mathbb{R}$ is complete.

Now we identity $\mathbb{Q}$ with a subfield of $\mathbb{R}$ in the following content.

Proposition 1.1.15. (1) E is a non-empty subset of $\mathbb{R}$ and if E is lower-bounded, then E has a infimum; if E is upper-bounded, then E has a supremum.

(2) Every incresing bounded sequence $(x_n) \in \mathbb{R}$ has a limit.

(3) (Bolzano-Weierstress) Every bounded sequence has a convergent subsequnce.

(4) if

$$[a, b] \subset \bigcup_{i \in I} (a_i, b_i)$$

, then

$$[a, b] \subset \bigcup_{k \in J} (a_k, b_k)$$

for some finite subset J of I.

Proposition 1.1.16. $a > 0$, $n \in \mathbb{Z}_{>0}$, then there's unique $x \in \mathbb{R}_{>0}$ such that $x^n = a$. We denote the unique positive root by $\sqrt[n]{a}$. And for all $a \in \mathbb{R}$ and $r = \frac{p}{q} \in \mathbb{Q}$, define $a^r = \sqrt[q]{a^p}$. It's easy to check that $\sqrt[q]{a^p} = (\sqrt[q]{a})^p$.

Definition 1.1.17 (complex number)**.** Let $\mathbb{C} = \mathbb{R} \times \mathbb{R}$, define $(a, b) \cdot (c, d) = (ac - bd, bc + ad)$. Then $\mathbb{C}$ is a field under this operator and $\mathbb{R}$ is a subfield of $\mathbb{C}$.

1.2 Point-Set Topology

Munkres's Topology is a good reference for this section.

1.2.1 Definition

Definition 1.2.1. A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(1) $\varnothing$ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a topological space.

Definition 1.2.2. If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

(1) For each $x \in X$, there is at least one basis element B containing x.

(2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology $\boldsymbol{T}$ generated by $\mathcal{B}$ as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.

Definition 1.2.3. Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the subspace topology. With this topology, Y is called a subspace of X; its open sets consist of all intersections of open sets of X with Y.

Definition 1.2.4. X is Hausdorff if for any two elements $x \neq y$ in X, there's U, V open in X such that $x \in U, y \in V$ and $U \cap V = \varnothing$.

Definition 1.2.5 (convergence)**.**

Proposition 1.2.6. If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.

Example 1.2.7. Let X be a ordered set; assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

(1) All open intervals (a, b) in X.

(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.

(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X. The collection $\mathcal{B}$ is a basis for a topology on X, which is called the order topology.

Example 1.2.8. $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.

Proposition 1.2.9. Given a subset A of a topological space X, the interior of A is defined as the union of all open sets contained in A, and the closure of A is defined as the intersection of all closed sets containing A.

Let Y be a subspace of X; let A be a subset of Y; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.

Definition 1.2.10. If A is a subset of the topological space X and if x is a point of X, we say that x is a limit point of A if every neighborhood of x intersects A in some point other than x itself. Said differently, x is a limit point of A if it belongs to the closure of $A - \{x\}$. The point x may lie in A or not; for this definition it does not matter. Let A be a subset of the topological space X; let A' be the set of all limit points of A. Then

$$\bar{A} = A \cup A'.$$

Proposition 1.2.11. Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:

(1) f is continuous.(U open in X implies $f^{-1}(U)$ open in Y)

(2) For every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$.

(3) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

(4) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$. If the condition in (4) holds for the point x of X, we say that f is continuous at the point x.

Definition 1.2.12. Consider $(X_i)_{i\in I}$ be a family of topology spaces, then the sets of the form

$$\prod_{i\in I} U_i$$

$U_i = X_i$ for all but finite i, form a basis of $\prod_{i\in I} X_i$. We call it the topology induced by this product topology.

In language of category, product topology with projection $p_i : \prod_{i\in I} X_i \to X_i$ is the product object in the category of topological space.

Proposition 1.2.13. If each space X_α is Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in product topology.

Proposition 1.2.14. Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given the product topology, then

$$\prod \bar{A}_\alpha = \overline{\prod A_\alpha}$$

Theorem 1.2.15. Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J},$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

1.2.2 Metric space

Definition 1.2.16. A metric on a set X is a function

$$d : X \times X \longrightarrow \mathbb{R}$$

having the following properties:

(1) $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(3) (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$, for all $x, y, z \in X$.

Given a metric d on X, the number $d(x, y)$ is often called the distance between x and y in the metric d. Given $\epsilon > 0$, consider the set

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

of all points y whose distance from x is less than ϵ. It is called the **$\boldsymbol{\epsilon}$-ball centered at $\boldsymbol{x}$**. Sometimes we omit the metric d from the notation and write this ball simply as $B(x, \epsilon)$, when no confusion will arise.

If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X, called the metric topology induced by d.

Example 1.2.17. $\mathbb{R}^n$ is a metric space with distance $d(x, y) = ||x - y||$

Theorem 1.2.18. Let X be a topological space; let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is metrizable.

Let $f : X \to Y$. If the function f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.

Theorem 1.2.19. Let $f_n : X \to Y$ be a sequence of functions from the set X to the metric space Y. Let d be the metric for Y. We say that the sequence (f_n) converges uniformly to the function $f : X \to Y$ if given $\epsilon > 0$, there exists an integer N such that

$$d(f_n(x), f(x)) < \epsilon$$

for all $n > N$ and all x in X.

Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to f, then f is continuous.

Proposition 1.2.20. If $f \in B(X)$, we define the uniform norm of f to be

$$\|f\|_u = \sup\{|f(x)| : x \in X\}.$$

The function $\rho(f, g) = \|f - g\|_u$ is easily seen to be a metric on $B(X)$, and convergence with respect to this metric is simply uniform convergence on $X.B(X)$ is obviously complete in the uniform metric: If $\{f_n\}$ is uniformly Cauchy, then $\{f_n(x)\}$ is Cauchy for each x, and if we set $f(x) = \lim_n f_n(x)$, it is easily verified that $\|f_n - f\|_u \to 0$.

If X is a topological space, $BC(X) = B(X) \cap C(X)$ is a closed subspace of $B(X)$ in the uniform metric; in particular, $BC(X)$ is complete.

1.2.3 Compactness

Definition 1.2.21. A collection $\mathcal{A}$ of subsets of a space X is said to cover X, or to be a covering of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an open covering of X if its elements are open subsets of X.

A space X is said to be compact if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Proposition 1.2.22. Every closed subspace of a compact space is compact. Every compact subspace of a Hausdorff space is closed.

Theorem 1.2.23. The image of a compact space under a continuous map is compact.

Corollary 1.2.24. X is a compact space, Y is a Hausdorff space, then continuous $f : X \to Y$ is closed.

Corollary 1.2.25. Let $f : X \to Y$ be a continuous bijection. X is a compact space, Y is a Hausdorff space, then f is homemorphism.

Lemma 1.2.26 (Lebesgue number lemma)**.** Let $\mathcal{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing it. The number δ is called a Lebesgue number for the covering $\mathcal{A}$.

Theorem 1.2.27. Let X be a metrizable space. Then the following are equivalent:

(1) X is compact.

(2) X is limit point compact(infinite subset has a limit point).

(3) X is sequentially compact(every sequnence has a convergent subsequnce).

Theorem 1.2.28 (Tychonoff theorem)**.** An arbitrary product of compact spaces is compact in the product topology.

Definition 1.2.29. A function f from the metric space (X, d_X) to the metric space (Y, d_Y) is said to be uniformly continuous if given $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points x_0, x_1 of X,

$$d_X(x_0, x_1) < \delta \Longrightarrow d_Y(f(x_0), f(x_1)) < \epsilon.$$

Theorem 1.2.30. Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous.

Proposition 1.2.31 (finite intersection)**.** A collection $\mathcal{C}$ of subsets of X is said to have the finite intersection property if for every finite subcollection

$$\{C_1, \ldots, C_n\}$$

of $\mathcal{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty.

Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty.

Definition 1.2.32 (locally compact)**.** A space X is said to be locally compact at $\boldsymbol{x}$ if there is some compact subspace C of X that contains a neighborhood of x. If X is locally compact at each of its points, X is said simply to be locally compact.

Definition 1.2.33 (one-point compactification)**.** Let X be a space. Then X is locally compact Hausdorff if and only if there exists a space Y satisfying the following conditions:

(1) X is a subspace of Y.

(2) The set $Y - X$ consists of a single point.

(3) Y is a compact Hausdorff space.

If Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X.

Proof: We only provide the form of the open sets in Y: U open in Y if and only if U open in X, or U is the complement of a compact subset in X.

Definition 1.2.34. If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is said to be a compactification of X. If $Y - X$ equals a single point, then Y is called the one-point compactification of X.

Proposition 1.2.35. Let X be a Hausdorff space. Then X is locally compact if and only if given x in X, and given a neighborhood U of x, there is a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subset U$.

Corollary 1.2.36. If X is an LCH space and $K \subset U \subset X$ where K is compact and U is open, there exists a precompact open V such that $K \subset V \subset \bar{V} \subset U$.

Proposition 1.2.37. Let X be locally compact Hausdorff; let A be a subspace of X. If A is closed in X or open in X, then A is locally compact.

Proposition 1.2.38. In a locally compact Hausdorff space E, a subset A is closed if and only if its intersection with every compact set is compact

Proof: Let $A \subseteq E$ have the property that $A \cap K$ is closed in K for all compact $K \subseteq E$. We want to show that A is closed whenever E is locally compact Hausdorff, so we will show that $E - A$ is open.

Let $x \in E - A$, let K be a compact neighbourhood of x, and let $U \subseteq K$ be an open neighbourhood of x. Then $x \in U - K \cap A$ and $U - K \cap A$ is open in X. Hence $E - A$ is open in X.

Theorem 1.2.39 (Usysohn's Lemma, Locally Compact Version)**.** If X is an LCH space and $K \subset U \subset X$ where K is compact and U is open, there exists $f \in C(X, [0, 1])$ such that $f = 1$ on K and $f = 0$ outside a compact subset of U.

Definition 1.2.40. If X is a topological space and $f \in C(X)$, the support of f, denoted by $\mathrm{supp}(f)$, is the smallest closed set outside of which f vanishes, that is, the closure of $\{x : f(x) \neq 0\}$. If $\mathrm{supp}(f)$ is compact, we say that f is compactly supported, and we define

$$C_c(X) = \{f \in C(X) : \mathrm{supp}(f) \text{ is compact } \}.$$

Moreover, if $f \in C(X)$, we say that f vanishes at infinity if for every $\epsilon > 0$ the set $\{x : |f(x)| \geq \epsilon\}$ is compact, and we define

$$C_0(X) = \{f \in C(X) : f \text{ vanishes at infinity } \}.$$

Clearly $C_c(X) \subset C_0(X)$. Moreover, $C_0(X) \subset BC(X)$, because for $f \in C_0(X)$ the image of the set $\{x : |f(x)| \geq \epsilon\}$ is compact, and $|f| < \epsilon$ on its complement.

Proposition 1.2.41. If X is an LCH space, $C_0(X)$ is the closure of $C_c(X)$ in the uniform metric.

Proof: If $\{f_n\}$ is a sequence in $C_c(X)$ that converges uniformly to $f \in C(X)$, for each $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $\|f_n - f\|_u < \epsilon$. Then $|f(x)| < \epsilon$ if $x \notin \mathrm{supp}(f_n)$, so $f \in C_0(X)$. Conversely, if $f \in C_0(X)$, for $n \in \mathbb{N}$ let $K_n = \{x : |f(x)| \geq n^{-1}\}$. Then K_n is compact, so by Usysohn's Lemma, there exists $g_n \in C_c(X)$ with $0 \leq g_n \leq 1$ and $g_n = 1$ on K_n. Let $f_n = g_n f$. Then $f_n \in C_c(X)$ and $\|f_n - f\|_u \leq n^{-1}$, so $f_n \to f$ uniformly.

1.2.4 Connectness

Definition 1.2.42. Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called the components (or the "connected components") of X.The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.

Definition 1.2.43. We define another equivalence relation on the space X by defining $x \sim y$ if there is a path in X from x to y. The equivalence classes are called the path components of X. The path components of X are path-connected disjoint subspaces of X whose union is X, such that each nonempty path-connected subspace of X intersects only one of them.

Definition 1.2.44. A space X is said to be locally connected at x if for every neighborhood U of x, there is a connected neighborhood V of x contained in U. If X is locally connected at each of its points, it is said simply to be locally connected. Similarly, a space X is said to be locally path connected at x if for every neighborhood U of x, there is a path-connected neighborhood V of x contained in U. If X is locally path connected at each of its points, then it is said to be locally path connected.

Proposition 1.2.45. (1) A space X is locally connected if and only if for every open set U of X, each component of U is open in X.

(2) A space X is locally path connected if and only if for every open set U of X, each path component of U is open in X.

(3) If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.

Proposition 1.2.46. The union of a collection of connected subspaces of X that have a point in common is connected.

Proposition 1.2.47. Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is also connected.

Proposition 1.2.48. The image of a connected space under a continuous map is connected.

Theorem 1.2.49 (Intermediate Value Theorem)**.** Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

Theorem 1.2.50 (Extreme value theorem)**.** Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.

1.2.5 Countability

Definition 1.2.51. A space X is said to have a countable basis at x if there is a countable collection B of neighborhoods of x such that each neighborhood of x contains at least one of the elements of B. A space that has a countable basis at each of its points is said to satisfy the first countability axiom, or to be first-countable.

Proposition 1.2.52. Let X be a topological space.

(1) Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is first-countable.

(2) Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is firstcountable.

Definition 1.2.53. If a space X has a countable basis for its topology, then X is said to satisfy the second countability axiom, or to be second-countable.

Definition 1.2.54. A subset A of a space X is said to be dense in X if $\bar{A} = X$.

Definition 1.2.55. Suppose that X has a countable basis. Then:

(1) Every open covering of X contains a countable subcollection covering X.(Lindelof space)

(2) There exists a countable subset of X that is dense in X.(separable)

Proposition 1.2.56. (1) Every metrizable space with a countable dense subset has a countable basis.

(2) Every metrizable Lindelöf space has a countable basis.

1.2.6 Separation

Definition 1.2.57. Suppose that one-point sets are closed in X. Then X is said to be regular if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively.

The space X is said to be normal if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.

Proposition 1.2.58. Let X be a topological space. Let one-point sets in X be closed.

(1) X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $\bar{V} \subset U$.

(2) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\bar{V} \subset U$.

Proposition 1.2.59. (1) Every metrizable space is normal.

(2) Every compact Hausdorff space is normal.

Theorem 1.2.60 (Usysohn's lemma)**.** Let X be a normal space; let A and B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map

$$f : X \longrightarrow [a, b]$$

such that $f(x) = a$ for every x in A, and $f(x) = b$ for every x in B.

Theorem 1.2.61 (Tietze extension theorem)**.** Let X be a normal space; let A be a closed subspace of X.

(1) Any continuous map of A into the closed interval $[a, b]$ of $\mathbb{R}$ may be extended to a continuous map of all of X into $[a, b]$.

(2) Any continuous map of A into $\mathbb{R}$ may be extended to a continuous map of all of X into $\mathbb{R}$.

1.2.7 Completeness

Definition 1.2.62. Let (X, d) be a metric space. A sequence (x_n) of points of X is said to be a Cauchy sequence in (X, d) if it has the property that given $\epsilon > 0$, there is an integer N such that

$$d(x_n, x_m) < \epsilon \quad \text{whenever } n, m \geq N.$$

The metric space (X, d) is said to be complete if every Cauchy sequence in X converges.

Theorem 1.2.63. A metric space (X, d) is compact if and only if it is complete and totally bounded.

Theorem 1.2.64 (extension theorem)**.** Suppose Y and Z are metric spaces, and Z is complete. Also suppose X is a dense subset of Y, and $f : X \to Z$ is uniformly continuous. Then f has a uniquely determined extension $\bar{f} : Y \to Z$ given by

$$\bar{f}(y) = \lim_{\substack{x \to y \\ x \in X}} f(x) \quad \text{for } y \in Y$$

and $\bar{f}$ is also uniformly continuous.

Definition 1.2.65. Let X be a metric space. If $h : X \to Y$ is an isometric imbedding of X into a complete metric space Y, such that $h(X)$ dense in Y. Then Y is called the completion of X. By extension theorem, the completion of X is uniquely determined up to an isometry.

Definition 1.2.66. A space X is said to be a Baire space if the following condition holds: Given any countable collection $\{A_n\}$ of closed sets of X each of which has empty interior in X, their union $\bigcup A_n$ also has empty interior in X.

Theorem 1.2.67 (Baire Category Theorem)**.** If X is a compact Hausdorff space or a complete metric space, then X is a Baire space.

Theorem 1.2.68. Any open subspace Y of a Baire space X is itself a Baire space.

Theorem 1.2.69. Let X be a space; let (Y, d) be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions such that $f_n(x) \to f(x)$ for all $x \in X$, where $f : X \to Y$. If X is a Baire space, the set of points at which f is continuous is dense in X.

1.3 Limit

1.3.1 Limit Superior and Limit Inferior

Definition 1.3.1. Let (x_n) be a sequence in $\mathbb{R}$. We can define two new sequences (y_n) and (z_n) by

$$y_n := \sup_{k \geq n} x_k := \sup\{x_k; k \geq n\}$$
$$z_n := \inf_{k \geq n} x_k := \inf\{x_k; k \geq n\}$$

Clearly (y_n) is a decreasing sequence and (z_n) is an increasing sequence in $\overline{\mathbb{R}}$. These sequences converge in $\overline{\mathbb{R}}$

$$\limsup_{n\to\infty} x_n := \overline{\lim_{n\to\infty}}\, x_n := \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right)$$

the limit superior, and

$$\liminf_{n\to\infty} x_n := \underline{\lim_{n\to\infty}}\, x_n := \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right)$$

the limit inferior.

Theorem 1.3.2. Any sequence (x_n) in $\mathbb{R}$ has a smallest cluster point x_* and a greatest cluster point x^* in $\overline{\mathbb{R}}$ and these satisfy

$$\liminf x_n = x_* \quad \text{and} \quad \limsup x_n = x^*$$

1.3.2 Series

In the following theorem, $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, $(E, |\cdot|)$ is a Banach space over $\mathbb{K}$ and (x_n) is a sequence in E.

Proposition 1.3.3. For a series $\sum x_k$ in a Banach space $(E, |\cdot|)$, the following are equivalent:

(1) $\sum x_k$ converges.

(2) For each $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$\left|\sum_{k=n+1}^{m} x_k\right| < \varepsilon, \quad m > n \geq N.$$

Proposition 1.3.4. Let $\sum x_k$ be a series in E and $\sum a_k$ a series in $\mathbb{R}^+$. Then the series $\sum a_k$ is called a majorant (or minorant) for $\sum x_k$ if there is some $K \in \mathbb{N}$ such that $|x_k| \leq a_k$ (or $a_k \leq |x_k|$) for all $k \geq K$. If a series in a Banach space has a convergent majorant, then it converges absolutely.

Proposition 1.3.5 (Abel)**.** Let $(a_n)_{n\in\mathbb{Z}}, (b_n)_{n\in\mathbb{Z}}$ be two sequneces in E, then

$$\sum_{M<n\leqslant M+N} a_n b_n = a_{M+N} B_{M+N} + \sum_{M<n\leqslant M+N-1} (a_n - a_{n+1}) B_n,$$

where $B_n = \sum_{M<k\leqslant n} b_k$.

If in particular $E = \mathbb{C}$ and (a_n) is a monotone sequence in $\mathbb{R}$, and

$$\sup_{M<n\leqslant M+N} |B_n| \leqslant \rho,$$

we have

$$\left| \sum_{M<n\leqslant M+N} a_n b_n \right| \leqslant \rho \left(|a_{M+1}| + 2\, |a_{M+N}| \right).$$

Example 1.3.6 (base g expansion)**.** Suppose that $g \geq 2$. Then every real number x has a base g expansion. This expansion is unique if expansions satisfying $x_k = g - 1$ for almost all $k \in \mathbb{N}$ are excluded(for example, if $g = 10$, $0.999\ldots$ is excluded). Moreover, x is a rational number if and only if its base g expansion is periodic.

Definition 1.3.7.

$$\exp : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

converges absolutely for all $z \in \mathbb{C}$.

Theorem 1.3.8. Every rearrangement of an absolutely convergent series $\sum x_k$ is absolutely convergent and has the same value as $\sum x_k$.

Theorem 1.3.9. There is a bijection $\alpha : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. If α is such a bijection, we call the series $\sum_n x_{\alpha(n)}$ an ordering of the double series $\sum x_{jk}$. If we fix $j \in \mathbb{N}$ (or $k \in \mathbb{N}$), then the series $\sum_k x_{jk}$ (or $\sum_j x_{jk}$) is called the j^{th} row series (or j^{th} column series) of $\sum x_{jk}$. If every row series (or column series) converges, then we can consider the series of row sums $\sum_j \left(\sum_{k=0}^{\infty} x_{jk}\right)$ (or the series of column sums $\sum_k \left(\sum_{j=0}^{\infty} x_{jk}\right)$). Finally we say that the double series $\sum x_{jk}$ is summable if

$$\sup_{n\in\mathbb{N}} \sum_{j,k=0}^{n} |x_{jk}| < \infty.$$

Let $\sum x_{jk}$ be a summable double series.

(1) Every ordering $\sum_n x_{\alpha(n)}$ of $\sum x_{jk}$ converges absolutely to a value $s \in E$ which is independent of α.

(2) The series of row sums $\sum_j \left(\sum_{k=0}^{\infty} x_{jk}\right)$ and column sums $\sum_k \left(\sum_{j=0}^{\infty} x_{jk}\right)$ converge absolutely, and

$$\sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} x_{jk} \right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} x_{jk} \right) = s$$

Theorem 1.3.10. Suppose that the series $\sum x_j$ and $\sum y_k$ in $\mathbb{K}$ converge absolutely. Then the Cauchy product $\sum_n \sum_{k=0}^{n} x_k y_{n-k}$ of $\sum x_j$ and $\sum y_k$ converges absolutely, and

$$\left(\sum_{j=0}^{\infty} x_j \right) \left(\sum_{k=0}^{\infty} y_k \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} x_k y_{n-k}$$

Corollary 1.3.11.

$$\exp(x+y) = \exp(x)\exp(y)$$

for $x, y \in \mathbb{C}$

1.3.3 Some Important Limits

Example 1.3.12. Let $k \in \mathbb{N}$ and $a \in \mathbb{C}$ be such that $|a| > 1$. Then

$$\lim_{n\to\infty} \frac{n^k}{a^n} = 0$$

that is, for $|a| > 1$ the function $n \mapsto a^n$ increases faster than any power function $n \mapsto n^k$.

Example 1.3.13. For all $a \in \mathbb{C}$,

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0$$

The factorial function $n \mapsto n\,!$ increases faster than the function ${}^2n \mapsto a^n$.

Definition 1.3.14. The sequence $((1 + 1/n)^n)$ converges and its limit

$$e := \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

the Euler number, satisfies $2 < e \leq 3$. Morover, we can show that

$$e = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!}$$

Proposition 1.3.15. As an application of this property of the exponential function, we determine the values of the exponential function for rational arguments. Namely,

$$\exp(r) = e^r, \quad r \in \mathbb{Q}$$

that is, for a rational number $r, \exp(r)$ is the r^{th} power of e.

1.3.4 Power Series

Definition 1.3.16. Let

$$a := \sum a_k X^k := \sum_k a_k X^k$$

be a (formal) power series in one indeterminate with coefficients in $\mathbb{K}$. Then, for each $x \in \mathbb{K}, \sum a_k x^k$ is a series in $\mathbb{K}$. When this series converges we denote its value by $\underline{a}(x)$, the value of the (formal) power series at x. Set

$$\operatorname{dom}(\underline{a}) := \left\{x \in \mathbb{K}; \sum a_k x^k \text{ converges in } \mathbb{K}\right\}$$

Then $\underline{a} : \operatorname{dom}(\underline{a}) \to \mathbb{K}$ is a well defined function:

$$\underline{a}(x) := \sum_{k=0}^{\infty} a_k x^k, \quad x \in \operatorname{dom}(\underline{a})$$

Note that $0 \in \operatorname{dom}(\underline{a})$ for any $a \in \mathbb{K}[[X]]$. The following examples show that each of the cases

$$\operatorname{dom}(\underline{a}) = \mathbb{K}, \quad \{0\} \subset \operatorname{dom}(\underline{a}) \subset \mathbb{K}, \quad \operatorname{dom}(\underline{a}) = \{0\}$$

is possible.

Proposition 1.3.17. For a power series $a = \sum a_k X^k$ with coefficients in $\mathbb{K}$ there is a unique $\rho := \rho_a \in [0, \infty]$ with the following properties:

(1) The series $\sum a_k x^k$ converges absolutely if $|x| < \rho$ and diverges if $|x| > \rho$.

(2) Hadamard's formula holds:
$$\rho_a = \frac{1}{\overline{\lim}_{k\to\infty} \sqrt[k]{|a_k|}}$$
The number ${}^1\rho_a \in [0, \infty]$ is called the radius of convergence of a, and
$$\rho_a \mathbb{B}_{\mathbb{K}} = \{x \in \mathbb{K}; |x| < \rho_a\}$$
is the disk of convergence of a.

Proposition 1.3.18. Let $a = \sum a_k X^k$ be a power series such that $\lim |a_k / a_{k+1}|$ exists in $\overline{\mathbb{R}}$. Then the radius of convergence of a is given by the formula
$$\rho_a = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right|$$

Theorem 1.3.19. Let $a = \sum a_k X^k$ and $b = \sum b_k X^k$ be power series with radii of convergence ρ_a and ρ_b respectively. Set $\rho := \min(\rho_a, \rho_b)$. Then for all $x \in \mathbb{K}$ such that $|x| < \rho$ we have
$$\sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$
$$\left[\sum_{k=0}^{\infty} a_k x^k\right] \left[\sum_{k=0}^{\infty} b_k x^k\right] = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j}\right) x^k$$

Proposition 1.3.20. Let $\sum a_k X^k$ be a power series with positive radius of convergence ρ_a. If there is a null sequence (y_j) such that $0 < |y_j| < \rho_a$ and
$$\underline{a}(y_j) = \sum_{k=0}^{\infty} a_k y_j^k = 0, \quad j \in \mathbb{N}$$
then $a_k = 0$ for all $k \in \mathbb{N}$, that is, $a = 0 \in \mathbb{K}[[X]]$.

1.3.5 Expoential and Related Functions

1.4 Functions of Single variable

1.5 Several Variables functions

Chapter 2

Measure

2.1 Measure Space

Definition 2.1.1. Let X be a nonempty set. An algebra of sets on X is a nonempty collection $\mathcal{A}$ of subsets of X that is closed under finite unions and complements; in other words, if $E_1, \ldots, E_n \in \mathcal{A}$, then $\bigcup_1^n E_j \in \mathcal{A}$; and if $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$. A σ-algebra is an algebra that is closed under countable unions.

We observe that since $\bigcap_j E_j = \left(\bigcup_j E_j^c\right)^c$, algebras (resp. σ-algebras) are also closed under finite (resp. countable) intersections. Moreover, if $\mathcal{A}$ is an algebra, then $\varnothing \in \mathcal{A}$ and $X \in \mathcal{A}$, for if $E \in \mathcal{A}$ we have $\varnothing = E \cap E^c$ and $X = E \cup E^c$.

Definition 2.1.2. If X is any topological space, the σ-algebra generated by the family of open sets in X is called the Borel σ-algebra on X and is denoted by $\mathcal{B}_X$. Its members are called Borel sets. $\mathcal{B}_X$ thus includes open sets, closed sets, countable intersections of open sets, countable unions of closed sets, and so forth.

There is a standard terminology for the levels in this hierarchy. A countable intersection of open sets is called a G_δ set; a countable union of closed sets is called an F_σ set.

Definition 2.1.3. Let $\{X_\alpha\}_{\alpha\in A}$ be an indexed collection of nonempty sets, $X = \prod_{\alpha\in A} X_\alpha$, and $\pi_\alpha : X \to X_\alpha$ the coordinate maps. If $\mathcal{M}_\alpha$ is a σ-algebra on X_α for each α, the product σ-algebra on X is the σ-algebra generated by

$$\left\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\right\}.$$

We denote this σ-algebra by $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha$. (If $A = \{1, \ldots, n\}$ we also write $\bigotimes_1^n \mathcal{M}_j$ or $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$.

Proposition 2.1.4. If A is countable, then $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha$ is the σ-algebra generated by

$$\left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\right\}$$

.

Proposition 2.1.5. Let $X_1, \ldots, X_n$ be topological spaces and let $X = \prod_1^n X_j$, equipped with the product topology. Then $\bigotimes_1^n \mathcal{B}_{X_j} \subset \mathcal{B}_X$. If the X_j 's are secound countable, then $\bigotimes_1^n \mathcal{B}_{X_j} = \mathcal{B}_X$

Proposition 2.1.6. Define an elementary family to be a collection $\mathcal{E}$ of subsets of X such that

(1) $\varnothing \in \mathcal{E}$,

(2) If $E, F \in \mathcal{E}$ then $E \cap F \in \mathcal{E}$,

(3) If $E \in \mathcal{E}$ then E^c is a finite disjoint union of members of $\mathcal{E}$.

If $\mathcal{E}$ is an elementary family, the collection $\mathcal{A}$ of finite disjoint unions of members of $\mathcal{E}$ is an algebra.

Proposition 2.1.7. X is a topological space, $Y \in B_X$ be a measurable set. Give Y the subspace topology from X, then B_Y equals to the σ-algebra $\{Y \cap E : E \in B_X\}$

Definition 2.1.8. Let X be a set equipped with a σ-algebra $\mathcal{M}$. A measure on $\mathcal{M}$ (or on $(X, \mathcal{M}\ )$, or simply on X if $\mathcal{M}$ is understood) is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

(1) $\mu(\varnothing) = 0$,

(2) if $\{E_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{M}$, then $\mu\left(\bigcup_1^\infty E_j\right) = \sum_1^\infty \mu\left(E_j\right)$.

If X is a set and $\mathcal{M} \subset \mathcal{P}(X)$ is a σ-algebra, $(X, \mathcal{M})$ is called a measurable space and the sets in $\mathcal{M}$ are called measurable sets. If μ is a measure on $(X, \mathcal{M})$, then $(X, \mathcal{M}, \mu)$ is called a measure space.

Definition 2.1.9. Let $(X, \mathcal{M}, \mu)$ be a measure space. Here is some standard terminology concerning the "size" of μ. If $\mu(X) < \infty$ (which implies that $\mu(E) < \infty$ for all $E \in \mathcal{M}$ since $\mu(X) = \mu(E) + \mu\left(E^c\right))$, μ is called finite. If $X = \bigcup_1^\infty E_j$ where $E_j \in \mathcal{M}$ and $\mu\left(E_j\right) < \infty$ for all j, μ is called σ-finite. More generally, if $E = \bigcup_1^\infty E_j$ where $E_j \in \mathcal{M}$ and $\mu\left(E_j\right) < \infty$ for all j, the set E is said to be σ-finite for μ.

If for each $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty, \mu$ is called semifinite.(σ-finite is semi-finte)

Example 2.1.10. Let X be any nonempty set, $\mathcal{M} = \mathcal{P}(X)$, and f any function from X to $[0, \infty]$. Then f determines a measure μ on $\mathcal{M}$ by the formula $\mu(E) = \sum_{x \in E} f(x)$.Two special cases are of particular significance: If $f(x) = 1$ for all x, μ is called counting measure; and if, for some $x_0 \in X, f$ is defined by $f\left(x_0\right) = 1$ and $f(x) = 0$ for $x \neq x_0$, μ is called the point mass or Dirac measure at x_0.

Proposition 2.1.11. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(1) (Monotonicity) If $E, F \in \mathcal{M}$ and $E \subset F$, then $\mu(E) \leq \mu(F)$.

(2) (Subadditivity) If $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu\left(\bigcup_1^\infty E_j\right) \leq \sum_1^\infty \mu\left(E_j\right)$.

(3) (Continuity from below) If $\{E_j\}_1^\infty \subset \mathcal{M}$ and $E_1 \subset E_2 \subset \cdots$, then $\mu\left(\bigcup_1^\infty E_j\right) = \lim_{j\to\infty} \mu\left(E_j\right)$.

(4) (Continuity from above) If $\{E_j\}_1^\infty \subset \mathcal{M}$, $E_1 \supset E_2 \supset \cdots$, and $\mu\left(E_1\right) < \infty$, then $\mu\left(\bigcap_1^\infty E_j\right) = \lim_{j\to\infty} \mu\left(E_j\right)$.

Definition 2.1.12. If $(X, \mathcal{M}, \mu)$ is a measure space, a set $E \in \mathcal{M}$ such that $\mu(E) = 0$ is called a null set. By subadditivity, any countable union of null sets is a null set, a fact which we shall use frequently. If a statement about points $x \in X$ is true except for x in some null set, we say that it is true almost everywhere (abbreviated a.e.), or for almost every x. (If more precision is needed, we shall speak of a μ-null set, or μ-almost everywhere).

Definition 2.1.13. If $\mu(E) = 0$ and $F \subset E$, then $\mu(F) = 0$ by monotonicity provided that $F \in \mathcal{M}$, but in general it need not be true that $F \in \mathcal{M}$. A measure whose domain includes all subsets of null sets is called complete. Completeness can sometimes obviate annoying technical points, and it can always be achieved by enlarging the domain of μ, as follows.

Theorem 2.1.14. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M}$ and $F \subset N$ for some $N \in \mathcal{N}\}$. Then $\overline{\mathcal{M}}$ is a σ-algebra, and there is a unique extension $\bar{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

Definition 2.1.15 (outer measure)**.** The abstract generalization of the notion of outer area is as follows. An outer measure on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies

(1) $\mu^*(\varnothing) = 0$,

(2) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$,

(3) $\mu^*\left(\bigcup_1^\infty A_j\right) \leq \sum_1^\infty \mu^*\left(A_j\right)$.

Proposition 2.1.16. Let $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\varnothing \in \mathcal{E}, X \in \mathcal{E}$, and $\rho(\varnothing) = 0$. For any $A \subset X$, define

$$\mu^*(A) = \inf\left\{\sum_1^\infty \mu\left(E_j\right) : E_j \in \mathcal{E} \text{ and } A \subset \bigcup_1^\infty E_j\right\}.$$

Then μ^* is an outer measure.

Proposition 2.1.17. If μ^* is an outer measure on X, a set $A \subset X$ is called $\boldsymbol{\mu}^\star$-measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*\left(E \cap A^c\right) \text{ for all } E \subset X.$$

Theorem 2.1.18 (Carathéodory's Theorem)**.** If μ^* is an outer measure on X, the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

Definition 2.1.19. If $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, a function $\mu_0 : \mathcal{A} \to [0, \infty]$ will be called a premeasure if

(1) $\mu_0(\varnothing) = 0$,

(2) if $\{A_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{A}$ such that $\bigcup_1^\infty A_j \in \mathcal{A}$, then $\mu_0\left(\bigcup_1^\infty A_j\right) = \sum_1^\infty \mu_0\left(A_j\right)$.

In particular, a premeasure is finitely additive since one can take $A_j = \varnothing$ for j large. The notions of finite and σ-finite premeasures are defined just as for measures.

Theorem 2.1.20. If μ_0 is a premeasure on $\mathcal{A} \subset \mathcal{P}(X)$, it induces an outer measure on X, namely,

$$\mu^*(E) = \inf\left\{\sum_1^\infty \mu_0\left(A_j\right) : A_j \in \mathcal{A}, E \subset \bigcup_1^\infty A_j\right\}.$$

then every set in $\mathcal{A}$ is μ^* measurable and $\mu^* \mid \mathcal{A} = \mu_0$.

Theorem 2.1.21. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, μ_0 a premeasure on $\mathcal{A}$, and $\mathcal{M}$ the σ-algebra generated by $\mathcal{A}$. There exists a measure μ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0 - namely, $\mu = \mu^* \mid \mathcal{M}$ where μ^* is given by Proposition 2.1.15. If μ_0 is σ-finite, then μ is the unique extension of μ_0 to a measure on $\mathcal{M}$ and the completion of μ is $\mu^*|M^*$ where M^* is the μ^*-measurable sets.

Example 2.1.22 (Lebesgue-Stieltjes measure)**.** Consider sets of the form $(a, b]$ or (a, ∞) or $\varnothing$, where $-\infty \leq a < b < \infty$. In this section we shall refer to such sets as h-intervals (h for "half-open"). Clearly the intersection of two h-intervals is an h-interval, and the complement of an h-interval is an h-interval or the disjoint union of two h-intervals. Hence the collection $\mathcal{A}$ of finite disjoint unions of h-intervals is an algebra. Notice tha the σ-algebra generated by $\mathcal{A}$ is $\mathcal{B}_{\mathbb{R}}$.

Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. If $(a_j, b_j]$ $(j = 1, \ldots, n)$ are disjoint h-intervals, let

$$\mu_0\left(\bigcup_1^n (a_j, b_j]\right) = \sum_1^n \left[F\left(b_j\right) - F\left(a_j\right)\right]$$

and let $\mu_0(\varnothing) = 0$. Then μ_0 is a premeasure on the algebra $\mathcal{A}$.

Example 2.1.23. If $F : \mathbb{R} \to \mathbb{R}$ is any increasing, right continuous function, there is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all a, b. If G is another such function, we have $\mu_F = \mu_G$ iff $F - G$ is constant. Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets and we define

$$F(x) = \begin{cases} \mu((0, x]) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu((-x, 0]) & \text{if } x < 0 \end{cases}$$

then F is increasing and right continuous, and $\mu = \mu_F$.

Example 2.1.24 (Lebesgue measure)**.** This is the complete measure μ_F associated to the function $F(x) = x$, for which the measure of an interval is simply its length. We shall denote it by m. The domain of m is called the class of Lebesgue measurable sets, and we shall denote it by $\mathcal{L}$. We shall also refer to the restriction of m to $\mathcal{B}_{\mathbb{R}}$ as Lebesgue measure.

Proposition 2.1.25. If $E \in \mathcal{L}$, then $E + s \in \mathcal{L}$ and $rE \in \mathcal{L}$ for all $s, r \in \mathbb{R}$. Moreover, $m(E + s) = m(E)$ and $m(rE) = |r|m(E)$.

2.2 Intergration

Proposition 2.2.1. $f : X \to Y$ between two sets induces a mapping $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$, defined by $f^{-1}(E) = \{x \in X : f(x) \in E\}$, which preserves unions, intersections, and complements. Thus, if $\mathcal{N}$ is a σ-algebra on Y, $\{f^{-1}(E) : E \in \mathcal{N}\}$ is a σ-algebra on X. If $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces, a mapping $f : X \to Y$ is called $(\mathcal{M}, \mathcal{N})$-measurable, or just measurable when $\mathcal{M}$ and $\mathcal{N}$ are understood, if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

If $\mathcal{N}$ is generated by $\mathcal{E}$, then $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable iff $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

Definition 2.2.2. If $(X, \mathcal{M})$ is a measurable space, a real- or complex-valued function f on X will be called $\mathcal{M}$-measurable, or just measurable, if it is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ or $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ measurable. $\mathcal{B}_{\mathbb{R}}$ or $\mathcal{B}_{\mathbb{C}}$ is always understood as the σ-algebra on the range space unless otherwise specified. In particular, $f : \mathbb{R} \to \mathbb{C}$ is Lebesgue (resp. Borel) measurable if is $(\mathcal{L}, \mathcal{B}_{\mathbb{C}})$ (resp. $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{C}})$) measurable;

Proposition 2.2.3. If $(X, \mathcal{M})$ is a measurable space and $f : X \to \mathbb{R}$, the following are equivalent:

(1) f is $\mathcal{M}$-measurable.

(2) $f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

(3) $f^{-1}([a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

(4) $f^{-1}((-\infty, a)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

(5) $f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

Corollary 2.2.4. $f : X \to Y$ is continuous, then f is (B_X, B_Y)-measurable.

Proposition 2.2.5. A function $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable iff $\operatorname{Re} f$ and $\operatorname{Im} f$ are $\mathcal{M}$-measurable.

Definition 2.2.6. It is sometimes convenient to consider functions with values in the extended real number system $\overline{\mathbb{R}} = [\infty, \infty]$(with order topology). It is easily verified that $\mathcal{B}_{\overline{\mathbb{R}}}$ is generated by the rays $(a, \infty]$ or $[-\infty, a)(a \in \mathbb{R})$, and we define $f : X \to \overline{\mathbb{R}}$ to be $\mathcal{M}$-measurable if it is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable. And we always define $0 \cdot \infty$ to be 0.

Proposition 2.2.7. If $f, g : X \to \mathbb{C}$ are $\mathcal{M}$-measurable, then so are $f+g$ and fg.

Proposition 2.2.8. If $\{f_j\}$ is a sequence of $\overline{\mathbb{R}}$-valued measurable functions on $(X, \mathcal{M})$, then the functions

$$\begin{array}{ll} g_1(x) = \sup_j f_j(x), & g_3(x) = \overline{\lim}_{j\to\infty} f_j(x), \\ g_2(x) = \inf_j f_j(x), & g_4(x) = \underline{\lim}_{j\to\infty} f_j(x) \end{array}$$

are all measurable.

Corollary 2.2.9. If $f, g : X \to \overline{\mathbb{R}}$ are measurable, then so are $\max(f, g)$ and $\min(f, g)$.

If $\{f_j\}$ is a sequence of complex-valued measurable functions and $f(x) = \lim_{j\to\infty} f_j(x)$ exists for all x, then f is measurable.

Definition 2.2.10. We now discuss the functions that are the building blocks for the theory of integration. Suppose that $(X, \mathcal{M})$ is a measurable space. If $E \subset X$, the characteristic function χ_E of E (sometimes called the indicator function of E and denoted by 1_E) is defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

It is easily checked that χ_E is measurable iff $E \in \mathcal{M}$. A simple function on X is a finite linear combination, with complex coefficients, of characteristic functions of sets in $\mathcal{M}$. (We do not allow simple functions to assume the values $\pm\infty$.) Equivalently, $f : X \to \mathbb{C}$ is simple iff f is measurable and the range of f is a finite subset of $\mathbb{C}$. Indeed, we have

$$f = \sum_1^n z_j \chi_{E_j}, \text{ where } E_j = f^{-1}(\{z_j\}) \text{ and range } (f) = \{z_1, \ldots, z_n\}.$$

We call this the standard representation of f. It exhibits f as a linear combination, with distinct coefficients, of characteristic functions of disjoint sets whose union is X. Note: One of the coefficients z_j may well be 0 , but the term $z_j \chi_{E_j}$ is still to be envisioned as part of the standard representation, as the set E_j may have a role to play when f interacts with other functions.

Theorem 2.2.11. Let $(X, \mathcal{M})$ be a measurable space. If $f : X \to [0, \infty]$ is measurable, there is a sequence $\{\phi_n\}$ of simple functions such that $0 \le \phi_1 \le \phi_2 \le \cdots \le f, \phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on any set on which f is bounded.

If $f : X \to \mathbb{C}$ is measurable, there is a sequence $\{\phi_n\}$ of simple functions such that $0 \le |\phi_1| \le |\phi_2| \le \cdots \le |f|, \phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on any set on which f is bounded.

Definition 2.2.12. The following implications are valid iff the measure μ is complete:

(1) If f is measurable and $f = g$ μ-a.e., then g is measurable.

(2) If f_n is measurable for $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e., then f is measurable.

Proposition 2.2.13. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(X, \overline{\mathcal{M}}, \bar{\mu})$ be its completion. If f is an $\overline{\mathcal{M}}$-measurable function on X, there is an $\mathcal{M}$-measurable function g such that $f = g\bar{\mu}$-almost everywhere.

Definition 2.2.14. In this section we fix a measure space $(X, \mathcal{M}, \mu)$, and we define

$$L^+ = \text{the space of all measurable functions from } X \text{ to } [0, \infty].$$

If ϕ is a simple function in $L^+$with standard representation $\phi = \sum_1^n a_j \chi_{E_j}$, we define the integral of ϕ with respect to μ by

$$\int \phi d\mu = \sum_1^n a_j \mu(E_j)$$

Proposition 2.2.15. Let ϕ and ψ be simple functions in L^+.

(1) If $c \geq 0, \int c\phi = c \int \phi$.

(2) $\int (\phi + \psi) = \int \phi + \int \psi$.

(3) If $\phi \leq \psi$, then $\int \phi \leq \int \psi$.

Definition 2.2.16. We now extend the integral to all functions $f \in L^+$by defining

$$\int f d\mu = \sup \left\{ \int \phi d\mu : 0 \leq \phi \leq f, \phi \text{ simple} \right\}.$$

Theorem 2.2.17. If $\{f_n\}$ is a sequence in L^+such that $f_j \leq f_{j+1}$ for all j, and $f = \lim_{n\to\infty} f_n \,(= \sup_n f_n)$, then $\int f = \lim_{n\to\infty} \int f_n$.

Corollary 2.2.18. If $\{f_n\}$ is a finite or infinite sequence in $L^+$and $f = \sum_n f_n$, then $\int f = \sum_n \int f_n$.

Proposition 2.2.19. If $f \in L^+$, then $\int f = 0$ iff $f = 0$ a.e.

Lemma 2.2.20 (Fatou's lemma,)**.** If $\{f_n\}$ is any sequence in L^+, then

$$\int (\liminf f_n) \leq \liminf \int f_n.$$

Proposition 2.2.21. The two definitions of $\int f$ agree when f is simple, as the family of simple functions over which the supremum is taken includes f itself and

$$\int f \leq \int g \text{ whenever } f \leq g, \text{ and } \int cf = c \int f \text{ for all } c \in [0, \infty).$$

Definition 2.2.22. If $f^+$and $f^-$are the positive and negative parts of f and at least one of $\int f^+$and $\int f^-$is finite, we define

$$\int f = \int f^+ - \int f^-.$$

We shall be mainly concerned with the case where $\int f^+$and $\int f^-$are both finite; we then say that f is integrable. Since $|f| = f^+ + f^-$, it is clear that f is integrable iff $\int |f| < \infty$

Next, if f is a complex-valued measurable function, we say that f is integrable if $\int |f| < \infty$. More generally, if $E \in \mathcal{M}$, f is integrable on E if $\int_E |f| < \infty$. Since $|f| \leq |\operatorname{Re} f| + |\operatorname{Im} f| \leq 2|f|$, f is integrable iff $\operatorname{Re} f$ and $\operatorname{Im} f$ are both integrable, and in this case we define

$$\int f = \int \operatorname{Re} f + i \int \operatorname{Im} f.$$

It follows easily that the space of complex-valued integrable functions is a complex vector space and that the integral is a complex-linear functional on it. We denote this space - provisionally - by $L^1(\mu)$ (or $L^1(X, \mu)$, or $L^1(X)$, or simply L^1, depending on the context).

Proposition 2.2.23. If $f \in L^1$, then $\left|\int f\right| \leq \int |f|$.

Proposition 2.2.24. (1) If $f \in L^1$, then $\{x : f(x) \neq 0\}$ is σ-finite.

(2) If $f, g \in L^1$, then $\int_E f = \int_E g$ for all $E \in \mathcal{M}$ iff $\int |f - g| = 0$ iff $f = g$ a.e.

Remark 2.2.25. (X, M, μ) is a measurable space. Take $E \in M$ and $(E, M \cap E, \mu|_E)$ is also measurable space. If $f \in L^1(E)$, then

$$f' = \begin{cases} f & x \in E \\ 0 & x \in E^c \end{cases}$$

is a function in $L^1(X)$ and $\int_X f' = \int_E f$.

Remark 2.2.26. (X, M, μ) is a measurable space. (X, M', τ) is another measurable space such that $M' \supset M$ and $\tau|M = \mu$. Then if $f \in L^1(X, M)$, $f \in L^1(X, M')$ and values of integration of f on both measurable spaces are the same. This follows from Theorem 2.2.11 and Monotone Convergence Theorem.

Theorem 2.2.27 (Dominated Convergence Theorem)**.** Let $\{f_n\}$ be a sequence in L^1 such that (a) $f_n \to f$, and (b) there exists a nonnegative $g \in L^1$ such that $|f_n| \leq g$ for all n. Then $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n$.

Proof: By Fatou's lemma.

Theorem 2.2.28. Suppose that $\{f_j\}$ is a sequence in L^1 such that $\sum_1^\infty \int |f_j| < \infty$. Then $\sum_1^\infty f_j$ converges a.e. to a function in L^1, and $\int \sum_1^\infty f_j = \sum_1^\infty \int f_j$.

Theorem 2.2.29. If $f \in L^1(\mu)$ and $\epsilon > 0$, there is an integrable simple function $\phi = \sum a_j \chi_{E_j}$ such that $\int |f - \phi| d\mu < \epsilon$. (That is, the integrable simple functions are dense in L^1 in the L^1 metric.) If μ is a Lebesgue measure on $\mathbb{R}$, the sets E_j in the definition of ϕ can be taken to be finite unions of open intervals; moreover, there is a continuous function g that vanishes outside a bounded interval such that $\int |f - g| d\mu < \epsilon$.

Definition 2.2.30. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. We have already discussed the product σ-algebra $\mathcal{M} \otimes \mathcal{N}$ on $X \times Y$; we now construct a measure on $\mathcal{M} \otimes \mathcal{N}$ that is, in an obvious sense, the product of μ and ν.

To begin with, we define a (measurable) rectangle to be a set of the form $A \times B$ where $A \in \mathcal{M}$ and $B \in \mathcal{N}$. Clearly

$$(A \times B) \cap (E \times F) = (A \cap E) \times (B \cap F), \quad (A \times B)^c = (X \times B^c) \cup (A^c \times B).$$

Therefore, by Proposition 2.1.6, the collection $\mathcal{A}$ of finite disjoint unions of rectangles is an algebra, and of course the σ-algebra it generates is $\mathcal{M} \otimes \mathcal{N}$.

If we integrate with respect to x

$$\begin{aligned}\mu(A)\chi_B(y) = \int \chi_A(x)\chi_B(y)d\mu(x) &= \sum \int \chi_{A_j}(x)\chi_{B_j}(y)d\mu(x) \\ &= \sum \mu(A_j)\chi_{B_j}(y).\end{aligned}$$

In the same way, integration in y then yields

$$\mu(A)\nu(B) = \sum \mu(A_j)\nu(B_j).$$

It follows that if $E \in \mathcal{A}$ is the disjoint union of rectangles $A_1 \times B_1, \ldots, A_n \times B_n$, and we set

$$\pi(E) = \sum_1^n \mu(A_j)\nu(E_j)$$

then π is well defined on $\mathcal{A}$ (since any two representations of E as a finite disjoint union of rectangles have a common refinement), and π is a premeasure on $\mathcal{A}$. Therefore, π generates an outer measure on $X \times Y$ whose restriction to $\mathcal{M} \times \mathcal{N}$ is a measure that extends π. We call this measure the product of μ and ν and denote it by $\mu \times \nu$. Moreover, if μ and ν are σ-finite - say, $X = \bigcup_1^\infty A_j$ and $Y = \bigcup_1^\infty B_k$ with $\mu(A_j) < \infty$ and $\nu(B_k) < \infty$ - then $X \times Y = \bigcup_{j,k} A_j \times B_k$, and $\mu \times \nu(A_j \times B_k) < \infty$, so $\mu \times \nu$ is also σ-finite.Then $\mu \times \nu$ is the unique measure on $\mathcal{M} \otimes \mathcal{N}$ such that $\mu \times \nu(A \times B) = \mu(A)\nu(B)$ for all rectangles $A \times B$.

The same construction works for any finite number of factors. That is, suppose $(X_j, \mathcal{M}_j, \mu_j)$ are measure spaces for $j = 1, \ldots, n$. If we define a rectangle to be a set of the form $A_1 \times \cdots \times A_n$ with $A_j \in \mathcal{M}_j$, then the collection $\mathcal{A}$ of finite disjoint unions of rectangles is an algebra, and the same procedure as above produces a measure $\mu_1 \times \cdots \times \mu_n$ on $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ such that

$$\mu_1 \times \cdots \times \mu_n (A_1 \times \cdots \times A_n) = \prod_1^n \mu_j(A_j).$$

Moreover, if the μ_j 's are σ-finite so that the extension from $\mathcal{A}$ to $\bigotimes_1' \mathcal{M}_j$ is uniquely determined.

Proposition 2.2.31. If $(X_j, \mathcal{M}_j)$ is a measurable space for $j = 1, 2, 3$, then $\bigotimes_1^3 \mathcal{M}_j = (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$. Moreover, if μ_j is a σ-finite measure on $(X_j, \mathcal{M}_j)$, then $\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$.

Proof: Consider $\{A \subset M_1 \otimes M_2 : A \times E_3 \in M_1 \otimes M_2 \otimes M_3\}$ for some $E_3 \in M_3$ is a σ-algebra.

Definition 2.2.32. We return to the case of two measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$. If $E \subset X \times Y$, for $x \in X$ and $y \in Y$ we define the x-section E_x and the y-section E^y of E by

$$E_x = \{y \in Y : (x, y) \in E\}, \quad E^y = \{x \in X : (x, y) \in E\}.$$

Also, if f is a function on $X \times Y$ we define the x-section f_x and the y-section f^y of f by

$$f_x(y) = f^y(x) = f(x, y).$$

Proposition 2.2.33. If $E \in \mathcal{M} \otimes \mathcal{N}$, then $E_x \in \mathcal{N}$ for all $x \in X$ and $E^y \in \mathcal{M}$ for all $y \in Y$.

If f is $\mathcal{M} \otimes \mathcal{N}$-measurable, then f_x is $\mathcal{N}$-measurable for all $x \in X$ and f^y is $\mathcal{M}$-measurable for all $y \in Y$.

Theorem 2.2.34. Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$, then the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable on X and Y, respectively, and

$$\mu \times \nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y).$$

Theorem 2.2.35 (The Fubini-Tonelli Theorem)**.** Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ finite measure spaces.

(1) (Tonelli) If $f \in L^+(X \times Y)$, then the functions $g(x) = \int f_x d\nu$ and $h(y) = \int f^y d\mu$ are in $L^+(X)$ and $L^+(Y)$, respectively, and

$$\begin{aligned} \int f d(\mu \times \nu) &= \int \left[\int f(x, y) d\nu(y) \right] d\mu(x) \\ &= \int \left[\int f(x, y) d\mu(x) \right] d\nu(y). \end{aligned}$$

(2) (Fubini) If $f \in L^1(\mu \times \nu)$, then $f_x \in L^1(\nu)$ for a.e. $x \in X$, $f^y \in L^1(\mu)$ for a.e. $y \in Y$, Define

$$g(x) = \begin{cases} \int f_x & \text{if } f_x \in L^1(\nu) \\ 0 & \text{otherwise} \end{cases}$$

$$h(y) = \begin{cases} \int f^y & \text{if } f_x \in L^1(\nu) \\ 0 & \text{otherwise} \end{cases}$$

, we have $g(x) \in L^1(\mu)$, $h(y) \in L^1(\nu)$ and $\int g(x)\mathrm{d}\mu = \int h(y)\mathrm{d}\mu = \int f\mathrm{d}(\mu \times \nu)$.

Definition 2.2.36. Lebesgue measure m^n on $\mathbb{R}^n$ is the completion of the n-fold product of Lebesgue measure on $\mathbb{R}$ with itself, that is, the completion of $m \times \cdots \times m$ on $\mathcal{B}_{\mathbb{R}} \otimes \cdots \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^n}$.

Proposition 2.2.37. Lebesgue measure is translation-invariant. More precisely, for $a \in \mathbb{R}^n$ define $\tau_a : \mathbb{R}^n \to \mathbb{R}^n$ by $\tau_a(x) = x + a$.

(1) If $E \in \mathcal{L}^n$, then $\tau_a(E) \in \mathcal{L}^n$ and $m\left(\tau_a(E)\right) = m(E)$.

(2) If $f : \mathbb{R}^n \to \mathbb{C}$ is Lebesgue measurable, then so is $f \circ \tau_a$. Moreover, if either $f \geq 0$ or $f \in L^1(m)$, then $\int \left(f \circ \tau_a\right) dm = \int f dm$.

Theorem 2.2.38. Suppose $T \in GL(n, \mathbb{R})$.

(1) If f is a Lebesgue measurable function on $\mathbb{R}^n$, so is $f \circ T$. If $f \geq 0$ or $f \in L^1(m)$, then

$$\int f(x)dx = |\det T| \int f \circ T(x)dx.$$

(2) If $E \in \mathcal{L}^n$, then $T(E) \in \mathcal{L}^n$ and $m(T(E)) = |\det T| m(E)$.

Theorem 2.2.39 (Change of Variables)**.** Let $G = (g_1, \ldots, g_n)$ be a map from an open set $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$ whose components g_j are of class C^1. G is called a C^1 diffeomorphism if G is injective and $D_x G$ is invertible for all $x \in \Omega$. In this case, the inverse function theorem guarantees that $G(\Omega)$ is open and $G^{-1} : G(\Omega) \to \Omega$ is also a C^1 diffeomorphism and that $D_x\left(G^{-1}\right) = \left[D_{G^{-1}(x)} G\right]^{-1}$ for all $x \in G(\Omega)$.

Suppose that Ω is an open set in $\mathbb{R}^n$ and $G : \Omega \to \mathbb{R}^n$ is a C^1 diffeomorphism.

(1) If f is a Lebesgue measurable function on $G(\Omega)$, then $f \circ G$ is Lebesgue measurable on Ω. If $f \geq 0$ or $f \in L^1(G(\Omega), m)$, then

$$\int_{G(\Omega)} f(x)dx = \int_\Omega f \circ G(x) \left|\det D_x G\right| dx.$$

(2) If $E \subset \Omega$ and $E \in \mathcal{L}^n$, then $G(E) \in \mathcal{L}^n$ and $m(G(E)) = \int_E \left|\det D_x G\right| dx$.

2.3 Signed Measure and Complex Measure

Definition 2.3.1. Let $(X, \mathcal{M})$ be a measurable space. A signed measure on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that

(1) $\nu(\varnothing) = 0$

(2) ν assumes at most one of the values $\pm\infty$;

(3) if $\{E_j\}$ is a sequence of disjoint sets in $\mathcal{M}$, then $\nu\left(\bigcup_1^\infty E_j\right) = \sum_1^\infty \nu\left(E_j\right)$, where the latter sum converges absolutely if $\nu\left(\bigcup_1^\infty E_j\right)$ is finite.

Definition 2.3.2. If ν is a signed measure on $(X, \mathcal{M})$, a set $E \in \mathcal{M}$ is called positive (resp. negative, null) for ν if $\nu(F) \geq 0$ (resp. $\nu(F) \leq 0, \nu(F) = 0$) for all $F \in \mathcal{M}$ such that $F \subset E$.

Definition 2.3.3 (mutually singular)**.** Two signed measures μ and ν on $(X, \mathcal{M})$ are mutually singular, or that ν is singular with respect to μ, if there exist $E, F \in \mathcal{M}$ such that $E \cap F = \varnothing, E \cup F = X, E$ is null for μ, and F is null for ν.We express this relationship symbolically with the perpendicularity sign:

$$\mu \perp \nu.$$

Theorem 2.3.4 (Jordan Decomposition Theorem)**.** If ν is a signed measure, there exist unique positive measures $\nu^+$and ν^-such that $\nu = \nu^+ - \nu^-$and $\nu^+ \perp \nu^-$.

Moreover, if ν omits $-\infty$, μ^- is finite and if ν omits ∞, ν^+ is finite.

Remark 2.3.5. The measures $\nu^+$and $\nu^-$are called the positive and negative variations of ν, and $\nu = \nu^+ - \nu^-$is called the Jordan decomposition of ν. Furthermore, we define the total variation of ν to be the measure $|\nu|$ defined by

$$|\nu| = \nu^+ + \nu^-.$$

Definition 2.3.6. Integration with respect to a signed measure ν is defined in the obvious way: We set

$$L^1(\nu) = L^1\left(\nu^+\right) \cap L^1\left(\nu^-\right)$$
$$\int f d\nu = \int f d\nu^+ - \int f d\nu^- \quad \left(f \in L^1(\nu)\right).$$

One more piece of terminology: a signed measure ν is called finite (resp. σ-finite) if $|\nu|$ is finite (resp. σ-finite).

Proposition 2.3.7. $E \in \mathcal{M}$ is ν-null iff $|\nu|(E) = 0$, and $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$

Definition 2.3.8. Suppose that ν is a signed measure and μ is a positive measure on $(X, \mathcal{M})$. We say that ν is absolutely continuous with respect to μ and write

$$\nu \ll \mu$$

if $\nu(E) = 0$ for every $E \in \mathcal{M}$ for which $\mu(E) = 0$. Absolute continuity is in a sense the

Proposition 2.3.9. $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$

Proposition 2.3.10. If $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$

Proposition 2.3.11. Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{M})$. Then $\nu \ll \mu$ iff for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$.

Proof: If there's $\varepsilon > 0$ such that for all $n > 0$, there's $E_n \in M$ such that $\mu(E_n) < 2^{-n}$ and $\nu(E) \geq \varepsilon$. Consider the set

$$\limsup E_n = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k$$

Corollary 2.3.12. If $f \in L^1(\mu)$, for every $\epsilon > 0$ there exists $\delta > 0$ such that $\left|\int_E f d\mu\right| < \epsilon$ whenever $\mu(E) < \delta$.

Definition 2.3.13. A complex measure on a measurable space $(X, \mathcal{M})$ is a map $\nu : \mathcal{M} \to \mathbb{C}$ such that

(1) $\nu(\varnothing) = 0$;

(2) if $\{E_j\}$ is a sequence of disjoint sets in $\mathcal{M}$, then $\nu\left(\bigcup_1^\infty E_j\right)=\sum_1^\infty \nu\left(E_j\right)$, where the series converges absolutely.

Example 2.3.14. If μ is a positive measure and$f \in L^1(\mu)$, then $f \mathrm{d}\mu$ is a complex measure.

If ν is a complex measure, we shall write ν_r and ν_i for the real and imaginary parts of ν. Thus ν_r and ν_i are signed measures that do not assume the values $\pm\infty$; hence they are finite, and so the range of ν is a bounded subset of $\mathbb{C}$.

The notions we have developed for signed measures generalize easily to complex measures. For example, we define $L^1(\nu)$ to be $L^1\left(\nu_r\right) \cap L^1\left(\nu_i\right)$, and for $f \in L^1(\nu)$, we set $\int f d\nu=\int f d\nu_r+i \int f d\nu_i$. If ν and μ are complex measures, we say that $\nu \perp \mu$ if $\nu_a \perp \mu_b$ for $a, b=r, i$, and if λ is a positive measure, we say that $\nu \ll \lambda$ if $\nu_r \ll \lambda$ and $\nu_i \ll \lambda$.

Theorem 2.3.15 (Lebesgue-Radon-Nikodym Theorem)**.** If ν is a complex measure and μ is a σ-finite positive measure on $(X, \mathcal{M})$, there exist a complex measure λ and an $f \in L^1(\mu)$ such that $\lambda \perp \mu$ and $d\nu=d\lambda+f d\mu$. If also $\lambda^{\prime} \perp \mu$ and $d\nu=d\lambda^{\prime}+f^{\prime} d\mu$, then $\lambda=\lambda^{\prime}$ and $f=f^{\prime}\mu$-a.e.

Definition 2.3.16 (total variation of complex measure)**.** If ν is a complex measure and ν_r and ν_i be the real part and imaginary part of ν. Take a σ-finite positive measure μ on X, for example $|\nu_r|+|\nu_i|$, such that $\nu \ll \mu$. By Lebesgue-Radon-Nikodym Theorem, $\nu=f d\mu$ for some $f \in L^1(\mu)$. Define total variation of ν by $|f| d\mu$. This definition is independent of the choice of f and μ.

Proposition 2.3.17. Let ν be a complex measure on $(X, \mathcal{M})$.

(1) $|\nu(E)| \leq|\nu|(E)$ for all $E \in \mathcal{M}$.

(2) $\nu \ll|\nu|$

(3) $L^1(\nu)=L^1(|\nu|)$, and if $f \in L^1(\nu)$, then $\left|\int f d\nu\right| \leq \int|f| d|\nu|$.

Definition 2.3.18. A measurable function $f: \mathbb{R}^n \rightarrow \mathbb{C}$ is called locally integrable (with respect to Lebesgue measure) if $\int_K|f(x)| dx<\infty$ for every bounded measurable set $K \subset \mathbb{R}^n$.

We denote the space of locally integrable functions by L_{loc}^1 . If $f \in L_{\text{loc}}^1$, $x \in \mathbb{R}^n$, and $r>0$, we define $A_r f(x)$ to be the average value of f on $B(r, x)$:

$$
A_r f(x)=\frac{1}{m(B(r, x))} \int_{B(r, x)} f(y) dy
$$

Theorem 2.3.19 (The Lebesgue Differentiation Theorem)**.** Let us define the Lebesgue set L_f of f to be

$$
L_f=\left\{x: \lim_{r \rightarrow 0} \frac{1}{m(B(r, x))} \int_{B(r, x)}|f(y)-f(x)| dy=0\right\} .
$$

If $f \in L_{\text{loc}}^1$, then $m\left(\left(L_f\right)^c\right)=0$.

Definition 2.3.20. A family $\left\{E_r\right\}_{r>0}$ of Borel subsets of $\mathbb{R}^n$ is said to shrink nicely to $x \in \mathbb{R}^n$ if $E_r \subset B(r, x)$ for each r; and there is a constant $\alpha>0$, independent of r, such that $m\left(E_r\right)>\alpha m(B(r, x))$.

Theorem 2.3.21. Let ν be a regular complex Borel measure on $\mathbb{R}^n$, and let $d\nu = d\lambda + f dm$ be its Lebesgue-Radon-Nikodym representation. Thenfor m-almost every $x \in \mathbb{R}^n$,

$$\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

where E_r shrinks nicely to 0.

2.4 Function of bounded variation

Definition 2.4.1. If $F : \mathbb{R} \to \mathbb{C}$ and $x \in \mathbb{R}$, we define

$$T_F(x) = \sup \left\{ \sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x \right\}.$$

T_F is called the total variation function of F.

T_F is an increasing function with values in $[0, \infty]$. If $T_F(\infty) = \lim_{x \to \infty} T_F(x)$ is finite, we say that F is of bounded variation on $\mathbb{R}$, and we denote the space of all such F by BV.

Proposition 2.4.2. We observe that the sums in the definition of T_F are made bigger if the additional subdivision points x_j are added. Hence, if $a < b$, the definition of $T_F(b)$ is unaffected if we assume that a is always one of the subdivision points. It follows that

$$T_F(b) = T_F(a) + \sup \left\{ \sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, a = x_0 < \cdots < x_n = b \right\}$$

Definition 2.4.3. Define $BV([a, b])$ to be the set of all functions on $[a, b]$ whose total variation

$$\sup \left\{ \sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, a = x_0 < \cdots < x_n = b \right\}$$

is finite.

If $F \in BV$, the restriction of F to $[a, b]$ is in $BV([a, b])$ for all a, b; indeed, its total variation on $[a, b]$ is nothing but $T_F(b) - T_F(a)$. Conversely, if $F \in BV([a, b])$ and we set $F(x) = F(a)$ for $x < a$ and $F(x) = F(b)$ for $x > b$, then $F \in BV$. By this device the results that we shall prove for BV can also be applied to $BV([a, b])$.

Proposition 2.4.4. (1) If $F : \mathbb{R} \to \mathbb{R}$ is bounded and increasing, then $F \in BV$.

(2) If $F, G \in BV$ and $a, b \in \mathbb{C}$, then $aF + bG \in BV$.

(3) If F is differentiable on $\mathbb{R}$ and F' is bounded, then $F \in BV([a, b])$ for $-\infty < a < b < \infty$ (by the mean value theorem).

Proposition 2.4.5. Define the normalized bounded variation function space to be

$$NBV = \{F \in BV : F \text{ is right continuous and } F(-\infty) = 0\}$$

If $F \in BV$, then $T_F(-\infty) = 0$. If F is also right continuous, then so is T_F. If $F \in NBV$, T_F is also in NBV.

Proposition 2.4.6. If μ is a complex Borel measure on $\mathbb{R}$ and $F(x) = \mu((-\infty, x])$, then $F \in NBV$. Conversely, if $F \in NBV$, there is a unique complex Borel measure μ_F such that $F(x) = \mu_F((-\infty, x])$. Moreover, $|\mu_F| = \mu_{T_F}$.

Proposition 2.4.7. A function $F : \mathbb{R} \to \mathbb{C}$ is called absolutely continuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any finite set of disjoint intervals $(a_1, b_1), \ldots, (a_N, b_N)$,

$$\sum_1^N (b_j - a_j) < \delta \Longrightarrow \sum_1^N |F(b_j) - F(a_j)| < \epsilon.$$

More generally, F is said to be absolutely continuous on $[a, b]$ if this condition is satisfied whenever the intervals (a_j, b_j) all lie in $[a, b]$. Clearly, if F is absolutely continuous, then F is uniformly continuous (take $N = 1$ in (3.31)). On the other hand, if F is everywhere differentiable and F' is bounded, then F is absolutely continuous, for $|F(b_j) - F(a_j)| \leq (\max |F'|)(b_j - a_j)$ by the mean value theorem.

If $F \in NBV$, then F is absolutely continuous iff $\mu_F \ll m$.

Theorem 2.4.8. If $F \in NBV$, then F is differentiable almost m-everywhere(In this section we only consider Borel measure). Take a set $M \subset \mathbb{R}$ such that F is differentiable on M and its complement is Borel zero measure set. We have

$$F'(x) = \lim_{n\to\infty} \frac{F(x + \frac{1}{n}) - F(x)}{\frac{1}{n}} \cdot \chi_M \in L^1(m)$$

Moreover, $\mu_F \perp m$ iff $F' = 0$ a.e., and $\mu_F \ll m$ iff $F(x) = \int_{-\infty}^x F'(t)dt$.

Proof: Since total variation of μ_F is regular, the theorem follows from Theorem 2.3.21.

Theorem 2.4.9. If $f \in L^1(m)$, then the function $F(x) = \int_{-\infty}^x f(t)dt$ is in NBV and is absolutely continuous, and $f = F'$ a.e. Conversely, if $F \in NBV$ is absolutely continuous, then $F' \in L^1(m)$ and $F(x) = \int_{-\infty}^x F'(t)dt$.

Theorem 2.4.10. If $-\infty < a < b < \infty$ and $F : [a, b] \to \mathbb{C}$, the following are equivalent:

(1) F is absolutely continuous on $[a, b]$.

(2) $F(x) - F(a) = \int_a^x f(t)dt$ for some $f \in L^1([a, b], m)$.

(3) F is differentiable a.e. on $[a, b]$, $F' \in L^1([a, b], m)$, and $F(x) - F(a) = \int_a^x F'(t)dt$.

Theorem 2.4.11 (integrate by part)**.** If F and G are in NBV and at least one of them is continuous, then for $-\infty < a < b < \infty$,

$$\int_{(a,b]} FdG + \int_{(a,b]} GdF = F(b)G(b) - F(a)G(a).$$

2.5 L^p Space

2.6 Radon measure

Definition 2.6.1. Let μ be a Borel measure on X and E a Borel subset of X. The measure μ is called outer regular on E if

$$\mu(E) = \inf\{\mu(U) : U \supset E, U \text{ open }\}$$

and inner regular on E if

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact }\}.$$

If μ is outer and inner regular on all Borel sets, μ is called regular.

A Radon measure on X is a Borel measure that is finite on all compact sets, outer regular on all Borel sets, and inner regular on all open sets.

Definition 2.6.2. A complex measure is regular if its total variation is regular.

Proposition 2.6.3. Every σ-finite Radon measure is regular.

Lemma 2.6.4. In C_2 LCH space X, every open subset is σ-compact.

Proof: Since in C_2 space, every open subspace is still C_2 hence Lindelöf.

By Proposition 1.2.35, for all $x \in U$, there's V_x open and precompact such that $x \in V_x \subset \bar{V_x} \subset U$. Take a countable subcovering of $\{V_x\}$ indexed by J, we have

$$\bigcup_{x \in J} \bar{V_x} = U$$

.

Proposition 2.6.5. Let X be a C_2 LCH space. Then every Borel measure on X that is finite on compact sets is regular and hence Radon.

Proposition 2.6.6. If μ is a Radon measure on $X, C_c(X)$ is dense in $L^p(\mu)$ for $1 \leq p < \infty$

Theorem 2.6.7 (The Riesz Representation Theorem)**.** If U is open in X and $f \in C_c(X)$, we shall write

$$f \prec U$$

to mean that $0 \leq f \leq 1$ and $\operatorname{supp}(f) \subset U$.

If I is a positive linear functional on $C_c(X)$, there is a unique Radon measure μ on X such that $I(f) = \int f d\mu$ for all $f \in C_c(X)$. Moreover, μ satisfies

$$\mu(U) = \sup\left\{I(f) : f \in C_c(X), f \prec U\right\} \text{ for all open } U \subset X$$

and $\mu(K) = \inf\left\{I(f) : f \in C_c(X), f \geq \chi_K\right\}$ for all compact $K \subset X$.

Corollary 2.6.8. There's one-to-one correspondence between bounded positive linear functional $C_c(X)$ and finite Radon meausre on X. Moreover, since $C_c(X)$ is dense subset of Banach space $C_0(X)$, by Theorem 1.2.64, every bounded positive linear functional on $C_c(X)$ can be extended to $C_0(X)$ continuously.

Proof: If I is a bounded positive linear functional, by Riesz Representation Theorem,

$$\mu(X) = \sup\left\{\int f d\mu : f \in C_c(X), 0 \le f \le 1\right\} < \infty$$

Proposition 2.6.9. If μ is a σ-finite Radon measure on X and $A \in \mathcal{B}_X$, the Borel measure μ_A defined by $\mu_A(E) = \mu(E \cap A)$ is a Radon measure.

Proposition 2.6.10. Suppose that μ is a Radon measure on X. If $\phi \in L^1(\mu)$ and $\phi \ge 0$, then $\nu(E) = \int_E \phi d\mu$ is a Radon measure.

Proposition 2.6.11. Suppose that μ is a Radon measure on X and $\phi \in C(X, (0, \infty))$. Let $\nu(E) = \int_E \phi d\mu$, and let ν' be the Radon measure associated to the functional $f \mapsto \int f\phi d\mu$ on $C_c(X)$, then $\nu = \nu'$, and hence ν is a Radon measure.

Definition 2.6.12. A complex measure is Radon if its real and imaginary parts are difference of finite Radon measure.

Definition 2.6.13. $M(X)$ is the space of all the complex Radon measures and for $\mu \in M(X)$, define

$$||\mu|| = |\mu|(X)$$

Then,$||\cdot||$ is a norm on vector space $M(X)$.

Theorem 2.6.14. Let X be an LCH space, and for $\mu \in M(X)$, $I_\mu : f \in C_0(X) \mapsto \int f \mathrm{d}\mu$ is a bounded linear functional on $C_0(X)$. Then $\mu \mapsto I_\mu$ is an bijective isometry between $M(X)$ the space of complex Radon measure and space of bounded linear functional on $C_0(X)$.

Proposition 2.6.15. Suppose X, Y are LCH spaces.

(1) $\mathcal{B}_X \otimes \mathcal{B}_Y \subset \mathcal{B}_{X\times Y}$.

(2) If X and Y are second countable, then $\mathcal{B}_X \otimes \mathcal{B}_Y = \mathcal{B}_{X\times Y}$.

(3) If X and Y are second countable and μ and ν are Radon measures on X and Y, then $\mu \times \nu$ is a Radon measure on $X \times Y$.

(4) If $E \in \mathcal{B}_{X\times Y}$, then $E_x \in \mathcal{B}_Y$ for all $x \in X$ and $E^y \in \mathcal{B}_X$ for all $y \in Y$.

(5) If $f : X \times Y \to \mathbb{C}$ is $\mathcal{B}_{X\times Y}$-measurable, then f_x is $\mathcal{B}_Y$-measurable for all $x \in X$ and f^y is $\mathcal{B}_X$-measurable for all $y \in Y$.

Definition 2.6.16 (Radon product)**.** Every $f \in C_c(X \times Y)$ is $\mathcal{B}_X \otimes \mathcal{B}_Y$-measurable. Moreover, if μ and ν are σ-finite Radon measures on X and Y, then $C_c(X \times Y) \subset L^1(\mu \times \nu)$, and

$$\int f d(\mu \times \nu) = \iint f d\mu d\nu = \iint f d\nu d\mu \quad (f \in C_c(X \times Y)).$$

The formula $I(f) = \int f d(\mu \times \nu)$ defines a positive linear functional on $C_c(X \times Y)$, so it determines a Radon measure on $X \times Y$ by the Riesz representation theorem. We call this measure the Radon product of μ and ν and denote it by $\mu \widehat{\times} \nu$.

Proposition 2.6.17. Suppose that μ and ν are σ-finite Radon measures on X and Y. If $E \in \mathcal{B}_{X \times Y}$, then the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are Borel measurable on X and Y, and

$$\mu \widehat{\times} \nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y).$$

Moreover, the restriction of $\mu \hat{\times} \nu$ to $\mathcal{B}_X \otimes \mathcal{B}_Y$ is $\mu \times \nu$.

Theorem 2.6.18. Suppose that, for each $\alpha \in A, \mu_\alpha$ is a Radon measure on the compact Hausdorff space X_α such that $\mu_\alpha(X_\alpha) = 1$. Then there is a unique Radon measure μ on $X = \prod_{\alpha \in A} X_\alpha$ such that for any $\alpha_1, \ldots, \alpha_n \in A$ and any Borel set E in $\prod_1^n X_{\alpha_j}$,

$$\mu\left(\pi^{-1}_{(\alpha_1, \ldots, \alpha_n)}(E)\right) = \left(\mu_{\alpha_1} \widehat{\times} \cdots \widehat{\times} \mu_{\alpha_n}\right)(E).$$

Chapter 3

Complex Analysis

3.1 Line Integration

Theorem 3.1.1 (Open mapping Theorem)**.** If f is a holomorphic function and non-constant in a connected open set $\Omega \subset \mathbb{C}$, then f is open.

Proposition 3.1.2. U is an open subset of $\mathbb{C}$, $f : U \to \mathbb{C}$ is a injective holomorphic map, then $f'(z) \neq 0$ for all $z \in U$. By Open Mapping Theorem, the image of f is still open in $\mathbb{C}$, we denote it by V. Then $f : U \to V$ is a holomorphic bijective function. f^{-1} is also holomorphic and $(f^{-1})'(z) = \frac{1}{f(z)}$.

Proposition 3.1.3. f holomorphic, $f(a) \neq 0$, then f is local biholomorphic at a.

Proof: By inverse function theorem and Proposition 3.1.3.

3.2 Multiple Variables

Chapter 4

Functional Analysis

4.1 Foundation

Definition 4.1.1. Let K denote either $\mathbb{R}$ or $\mathbb{C}$, and let X be a vector space over K. A seminorm on X is a function $x \mapsto \|x\|$ from X to $[0, \infty)$ such that

(1) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (the triangle inequality),

(2) $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$ and $\lambda \in K$.

The second property clearly implies that $\|0\| = 0$. A seminorm such that $\|x\| = 0$ only when $x = 0$ is called a norm, and a vector space equipped with a norm is called a normed vector space (or normed linear space).

Definition 4.1.2. Banach space is a complete normed vector space.

Definition 4.1.3 (quotient space)**.** A related construction is that of quotient spaces. If $\mathcal{M}$ is a vector subspace of the vector space X, it defines an equivalence relation on X as follows: $x \sim y$ iff $x - y \in \mathcal{M}$. The equivalence class of $x \in \mathcal{X}$ is denoted by $x + \mathcal{M}$, and the set of equivalence classes, or quotient space, is denoted by $X/\mathcal{M}.X/\mathcal{M}$ is a vector space with vector operations $(x + \mathcal{M}) + (y + \mathcal{M}) = (x + y) + \mathcal{M}$ and $\lambda(x + \mathcal{M}) = (\lambda x) + \mathcal{M}$. If $\mathcal{X}$ is a normed vector space and $\mathcal{M}$ is closed, $X/\mathcal{M}$ inherits a norm from X called the quotient norm, namely

$$\|x + \mathcal{M}\| = \inf_{y \in \mathcal{M}} \|x + y\|$$

Proposition 4.1.4. A normed vector space is complete if and only if every absolutely convergent series converges.

Proposition 4.1.5. A linear map $T : X \to y$ between two normed vector spaces is called bounded if there exists $C \geq 0$ such that

$$\|Tx\| \leq C\|x\| \text{ for all } x \in \mathcal{X}$$

If X and y are normed vector spaces and $T : X \to y$ is a linear map, the following are equivalent:

(1) T is continuous.

(2) T is continuous at 0 .

(3) T is bounded.

Definition 4.1.6. If X and Y are normed vector spaces, we denote the space of all bounded linear maps from X to Y by $L(X,Y)$. It is easily verified that $L(X,Y)$ is a vector space and that the function $T \mapsto \|T\|$ defined by

$$\begin{aligned}\|T\| &= \sup\{\|Tx\| : \|x\| = 1\} \\ &= \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0\right\} \\ &= \inf\{C : \|Tx\| \leq C\|x\| \text{ for all } x\}\end{aligned}$$

is a norm on $L(X,Y)$, called the operator norm.

Proposition 4.1.7. If Y is complete, so is $L(X,Y)$.

Corollary 4.1.8. Let X be a vector space over K, where $K = \mathbb{R}$ or $\mathbb{C}$. A linear map from X to K is called a linear functional on X. If X is a normed vector space, the space $L(X,K)$ of bounded linear functionals on X is called the dual space of X. Then X^* is a Banach space with the operator norm.

Proposition 4.1.9. Let X be a vector space over $\mathbb{C}$. If f is a complex linear functional on X and $u = \operatorname{Re} f$, then u is a real linear functional, and $f(x) = u(x) - iu(ix)$ for all $x \in X$. Conversely, if u is a real linear functional on X and $f : X \to \mathbb{C}$ is defined by $f(x) = u(x) - iu(ix)$, then f is complex linear. In this case, if X is normed, we have $\|u\| = \|f\|$.

Definition 4.1.10. If X is a real vector space, a sublinear functional on X is a map $p : X \to \mathbb{R}$ such that

$$p(x+y) \leq p(x) + p(y) \text{ and } p(\lambda x) = \lambda p(x) \text{ for all } x, y \in X \text{ and } \lambda \geq 0$$

Theorem 4.1.11 (The Hahn-Banach Theorem)**.** Let X be a real vector space, p a sublinear functional on $\mathcal{X}, \mathcal{M}$ a subspace of $\mathcal{X}$, and f a linear functional on $\mathcal{M}$ such that $f(x) \leq p(x)$ for all $x \in \mathcal{M}$. Then there exists a linear functional F on $\mathcal{X}$ such that $F(x) \leq p(x)$ for all $x \in \mathcal{X}$ and $F \mid \mathcal{M} = f$.

Definition 4.1.12 (complex Hahn-Banach Theorem)**.** Let X be a complex vector space, p a seminorm on $\mathcal{X}, \mathcal{M}$ a subspace of $\mathcal{X}$, and f a complex linear functional on $\mathcal{M}$ such that $|f(x)| \leq p(x)$ for $x \in \mathcal{M}$. Then there exists a complex linear functional F on X such that $|F(x)| \leq p(x)$ for all $x \in \mathcal{X}$ and $F \mid \mathcal{M} = f$.

Corollary 4.1.13. Let X be a normed vector space.

(1) If $\mathcal{M}$ is a closed subspace of X and $x \in X \backslash \mathcal{M}$, there exists $f \in X^*$ such that $f(x) \neq 0$ and $f \mid \mathcal{M} = 0$. In fact, if $\delta = \inf_{y \in \mathcal{M}} \|x - y\|$, f can be taken to satisfy $\|f\| = 1$ and $f(x) = \delta$.

(2) If $x \neq 0 \in X$, there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.

(3) The bounded linear functionals on X separate points.

(4) If $x \in X$, define $\widehat{x} : X^* \to \mathbb{C}$ by $\widehat{x}(f) = f(x)$. Then the map $x \mapsto \widehat{x}$ is a linear isometry from X into X^{**} (the dual of X^*).

Theorem 4.1.14 (open mapping theorem)**.** Let X and Y be Banach spaces. If $T \in L(X, y)$ is surjective, then T is open.

Now we assume all the Banach spaces are over $\mathbb{C}$.

Definition 4.1.15. If X and Y are normed vector spaces and T is a linear map from X to Y, we define the graph of T to be

$$\Gamma(T) = \{(x, y) \in X \times Y : y = Tx\}$$

which is a subspace of $X \times Y$. We say that T is closed if $\Gamma(T)$ is a closed subspace of $X \times Y$. ClearlY, if T is continuous, then T is closed.

Theorem 4.1.16 (The Closed Graph Theorem)**.** If X and Y are Banach spaces and $T : X \to Y$ is a closed linear map, then T is bounded.

Theorem 4.1.17 (The Uniform Boundedness Principle)**.** Suppose that X and Y are normed vector spaces and $\mathcal{A}$ is a subset of $L(X, Y)$. If X is a Banach space and $\sup_{T \in \mathcal{A}} \|Tx\| < \infty$ for all $x \in X$, then $\sup_{T \in \mathcal{A}} \|T\| < \infty$.

4.2 Topological vector space

Definition 4.2.1. A topological vector space is a vector space X over the field $K(= \mathbb{R}$ or $\mathbb{C})$ which is endowed with a topology such that the maps $(x, y) \to x + y$ and $(\lambda, x) \to \lambda x$ are continuous from $X \times X$ and $K \times X$ to X. A topological vector space is called locally convex if there is a base for the topology consisting of convex sets (that is, sets A such that if $x, y \in A$ then $tx + (1-t)y \in A$ for $0 < t < 1$). Most topological vector spaces that arise in practice are locally convex and Hausdorff.

Proposition 4.2.2. Let $\{p_\alpha\}_{\alpha \in A}$ be a family of seminorms on the vector space X. If $x \in X, \alpha \in A$, and $\epsilon > 0$, let

$$U_{x\alpha\epsilon} = \{y \in X : p_\alpha(y - x) < \epsilon\}$$

and let $\mathcal{T}$ be the topology generated by the sets $U_{x\alpha\epsilon}$.

(1) For each $x \in X$, the finite intersections of the sets $U_{x\alpha\epsilon}(\alpha \in A, \epsilon > 0)$ form a neighborhood base at x.

(2) $x_i \to x$ iff $p_\alpha(x_i - x) \to 0$ for all $\alpha \in A$.

(3) $(X, \mathcal{T})$ is a locally convex topological vector space.

Proposition 4.2.3. Suppose X and y are vector spaces with topologies defined, respectively, by the families $\{p_\alpha\}_{\alpha\in A}$ and $\{q_\beta\}_{\beta\in B}$ of seminorms, and $T : X \to y$ is a linear map. Then T is continuous iff for each $\beta \in B$ there exist $\alpha_1, \ldots, \alpha_k \in A$ and $C > 0$ such that $q_\beta(Tx) \leq C\sum_1^k p_{\alpha_j}(x)$

Proposition 4.2.4. Let X be a vector space equipped with the topology defined by a family $\{p_\alpha\}_{\alpha\in A}$ of seminorms. X is Hausdorff iff for each $x \neq 0$ there exists $\alpha \in A$ such that $p_\alpha(x) \neq 0$.

Definition 4.2.5. A topological vector space whose topology is defined by a countable family of seminorms is called a Fréchet space if it is Hausdorff and complete.(every Cauchy sequence converges.)

4.3 Hilbert Space

Definition 4.3.1. Let $\mathcal{H}$ be a complex vector space. An inner product (or scalar product) on $\mathcal{H}$ is a map $(x, y) \mapsto \langle x, y\rangle$ from $X \times X \to \mathbb{C}$ such that:

(1) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ for all $x, y, z \in \mathcal{H}$ and $a, b \in \mathbb{C}$.

(2) $\langle y, x\rangle = \overline{\langle x, y\rangle}$ for all $x, y \in \mathcal{H}$.

(3) $\langle x, x\rangle \in (0, \infty)$ for all nonzero $x \in X$.

A Hilbert space is a vector space over $\mathbb{C}$ with a inner product such that the norm induced by this inner product is complete.

Proposition 4.3.2. If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$; that is, each $x \in \mathcal{H}$ can be expressed uniquelyas $x = y + z$ where $y \in \mathcal{M}$ and $z \in \mathcal{M}^\perp$. Moreover, y and z are the unique elements of $\mathcal{M}$ and $\mathcal{M}^\perp$ whose distance to x is minimal.

Theorem 4.3.3. If $f \in \mathcal{H}^*$, there is a unique $y \in \mathcal{H}$ such that $f(x) = \langle x, y\rangle$ for all $x \in X$.

Definition 4.3.4. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$, a unitary map from $\mathcal{H}_1$ to $\mathcal{H}_2$ is an invertible linear map $U : \mathcal{H}_1 \to \mathcal{H}_2$ that preserves inner products:

$$\langle Ux, Uy\rangle_2 = \langle x, y\rangle_1 \text{ for all } x, y \in \mathcal{H}_1$$

4.4 Spectrum of Opertor

Chapter 5

Harmonic Analysis

5.1 Fourier Transform

Definition 5.1.1 (Schwartz space)**.** The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consisting of those C^∞ functions which, together with all their derivatives, vanish at infinity faster than any power of $|x|$. More precisely, for any nonnegative integer N and any multi-index α we define

$$\|f\|_{(N,\alpha)} = \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha f(x)|$$

then

$$\mathcal{S} = \mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty : \|f\|_{(N,\alpha)} < \infty \text{ for all } N, \alpha \right\}$$

Proposition 5.1.2. $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet Space and Fourier Transform is a linear bi-continous bijection between Schwartz space.