Number Theory 3: Modular Arithmetic

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Modulus(同余)

Definition

Let n be a natural number. Two integers a and b are said to be equal modulo n(模 n 同余) if and only if n divides $a - b$. We write

$$a \equiv b(\mathrm{mod}\, n)$$

Example

For example, $7 \equiv 1(\mathbf{mod}3)$, and $11 \equiv 2(\mathbf{mod}3)$.$2016 \equiv 0(\mathbf{mod}7)$.

Proposition

For any positive integer n, if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then

$$a \pm c \equiv b \pm d \pmod{n}$$
$$ac \equiv bd \pmod{n}$$
$$a^k \equiv b^k \pmod{n}, \text{ for any positive integer } k$$

Proposition

If $am \equiv bm \pmod{n}$ and $\gcd(m, n) = 1$, then $a \equiv b \pmod{n}$.

Theorem

For any integer n, $n^2 \bmod 3$ or 4 can only be 0 or 1 .
For any integer n, $n^2 \bmod 8$ can only be 0 or 1 or 4 .

Proof:

$$(4k+1)^2 \equiv 1 \pmod 4, (4k+2)^2 \equiv 0 \pmod 4$$

$$(4k+3)^2 \equiv 1 \pmod 4, (4k)^2 \equiv 0 \pmod 4$$

$$(1)^2 \equiv 1 \pmod 8, (2)^2 \equiv 4 \pmod 8, (3)^2 \equiv 1 \pmod 8, (4)^2 \equiv 0 \pmod 8$$

$$(5)^2 \equiv 1 \pmod 8, (6)^2 \equiv 4 \pmod 8, (7)^2 \equiv 1 \pmod 8, (0)^2 \equiv 0 \pmod 8$$

Rational roots of polynomial equation(有理根的判定)

Lemma

p is a prime, a, b are integers, $p|ab$, then either $p|a$ or $p|b$.

Theorem

If $a_n, a_{n-1}, \ldots, a_0$ are integers, and if p and q are relatively prime integers where p/q is a solution of

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = 0,$$

then, $p \mid a_0$ and $q \mid a_n$.

Divisibility Rules

If n is a natural number with $m+1$ digits a_m, a_{m-1}, a_{m-1},
$n = a_m \times 10^m + a_{m-1} \times 10^{m-1} + \cdots + a_1 \times 10 + a_0$, then

- $2|n$ if and only if $2 \mid a_0$.
- $3|n$ if and only if $3 \mid (a_0 + a_1 + \cdots + a_m)$.
- $5|n$ if and only if $5 \mid a_0$.
- $9|n$ if and only if $9 \mid (a_0 + a_1 + \cdots + a_m)$.
- $11|n$ if and only if $11 \mid (a_0 - a_1 + a_2 - \cdots + (-1)^m a_m)$.

Question (难)

$19! = 1216T510040M832H00$, 则 $T + M + H =$

Exercise

- If $2^{100} = 5m + k$, where k and m are integers and $0 \leq k \leq 4$, then k is
  2^{100} 除以 5 的余数是多少?
- What is the units digit of 3^{2013} ?
  3^{2013} 的个位数是多少?
- Consider the sequence $a_1 = 1, a_2 = 13, \ldots$, where each term a_n is obtained from the previous term a_{n-1} by appending the n^{th} odd number. So $a_3 = 135, a_4 = 1357$. Find the number m so that a_m is the 30^{th} multiple of 9 in the sequence.

Exercise

- Consider the sum $1^2 + 2^2 + \cdots + 2012^2$. What is its last digit?
  $1^2 + 2^2 + \cdots + 2012^2$ 个位数是多少?
- Consider the 2700 digit number $N = 100101102\ldots999$ obtained by listing all the three digit numbers in order. What is the remainder when N is divided by 11 ?
  $N = 100101102\ldots999$ 除以 11 的余数是多少?

Question (AMC 8, 2020-19)

A number is called flippy if its digits alternate between two distinct digits. For example, 2020 and 37373 are flippy, but 3883 and 123123 are not. How many five-digit flippy numbers are divisible by 15?
如果一个数的各位数在两个不同的数字之间交替, 那么这个数就被称为翻转数. 例如，2020 和 37373 是翻转数, 而 3883 和 123123 不是翻转的。有多少个五位翻转数能被 15 整除?

Question

What is the tens digit of $2015^{2016} - 2017$?
$2015^{2016} - 2017$ 的十位数是什么?