Algebra 2: Sequnence and Inequalities

Erzhuo Wang

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Youth STEM Academy

Definition

The sequence $a_1, \ldots, a_n, \ldots$ is arithemetic if there is a constant d such that $a_n - a_{n-1} = d$ for $n = 2, 3, 4, \ldots$.(算术序列/等差数列)
The sequence $b_1, b_2 \ldots, b_n$ is geometric if there is a constant $r \neq 0$ such that $\frac{b_n}{b_{n-1}} = r$.(等比数列)

Example

$a_n = 3n + 5$ is a arithemetic sequence and $b_n = (-1)^n$ is a geometric Sequnence

Partial Sum(部分和公式)

Theorem (Partial Sum of a_n and b_n)

If $a_1, a_2, a_3, \cdots, a_n, \cdots$ is an arithmetic sequence with the common difference d, then $a_n = a_1 + (n-1)d$, where a_n is the n-th term

$$S_n = \frac{(a_1 + a_n)\,n}{2}$$

, where s_n is the partial sum of its first n terms.

If $a_1, a_2, a_3, \cdots, a_n, \cdots$ is a geometric sequence with the common ratio r, then $a_n = a_1 r^{n-1}$, where a_n is the n-th term

$$s_n = \frac{a_1\,(1 - r^n)}{1 - r}$$

, where s_n is the partial sum of its first n terms.

Exercise

Question

The sum of the odd positive integers from 1 to n is 9409, What is n?
从 1 到 n 的奇数正整数的和是 9409,n 是多少?

Question (AMC 8,2016-19)

The sum of 25 consecutive even integers is 10000. What is the largest of these 25 consecutive integers?
25 个连续偶数的和是 10000, 求其中最大的那个.

Question

Four numbers are written in a row. The average of the first two is 21 , the average of the middle two is 26 , and the average of the last two is 30 . What is the average of the first and last of the numbers?
(A) 24 (B) 25 (C) 26 (D) 27 (E) 28

Exercise

Question (AMC 8, 2023-25)

Fifteen integers $a_1, a_2, a_3, \ldots, a_{15}$ are arranged in order on a number line. The integers are equally spaced and have the property that

$$1 \leq a_1 \leq 10, 13 \leq a_2 \leq 20, \text{ and } 241 \leq a_{15} \leq 250$$

Find the value of a_{14}?

$a_1, a_2, \ldots, a_{15}$ 为一个等差数列, 且这 15 个数都是整数, 满足如下不等式:

$$1 \leq a_1 \leq 10, 13 \leq a_2 \leq 20, \text{ and } 241 \leq a_{15} \leq 250$$

求 a_{14}.

Perfect Cubes

Question (AMC 8, 2018-25)

考虑数列 $a_n = n^3$, 请问有多少个 n 使得 $2^8 + 1 \leq a_n \leq 2^{18} + 1$.

Some Useful Results(一些求和技巧)

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - \frac{1}{n+1}$$

不等式的基本性质

- $a^2 \geq 0$
- 若 $a > b$, 且 $c > d$，则 $a + c > b + d$.
- 若 $a > b$, 且 $c > 0$, 则 $ac > bc$, 若 $a > b$, 且 $c < 0$, 则 $ac < bc$.

A/G Mean Inequality(算术/几何均值不等式)

Theorem

For $a, b \geq 0$,

$$\sqrt{ab} \leq \frac{a+b}{2}$$

Example

Find the greatest possible value of the product xy, where x, y are positive real numbers with $x + 2y = 12$.

Baskets

Steph scored 15 baskets out of 20 attempts in the first half of a game, and 10 baskets out of 10 attempts in the second half. Candace took 12 attempts in the first half and 18 attempts in the second. In each half, Steph scored a higher percentage of baskets than Candace. Surprisingly they ended with the same overall percentage of baskets scored. How many more baskets did Candace score in the second half than in the first?

	First Half	Second Half
Steph	$\frac{15}{20}$	$\frac{10}{10}$
Candace	$\frac{\square}{12}$	$\frac{\square}{18}$

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

Solution

Let x be the number of shots that Candace made in the first half, and let y be the number of shots Candace made in the second half. Since Candace and Steph took the same number of attempts, with an equal percentage of baskets scored, we have $x + y = 10 + 15 = 25$. In addition, we have the following inequalities:

$$\frac{x}{12} < \frac{15}{20} \Longrightarrow x < 9$$

and

$$\frac{y}{18} < \frac{10}{10} \Longrightarrow y < 18$$

Pairing this up with $x + y = 25$ we see the only possible solution is $(x, y) = (8, 17)$, for an answer of $17 - 8 = \textbf{(C)}9$.

Homework

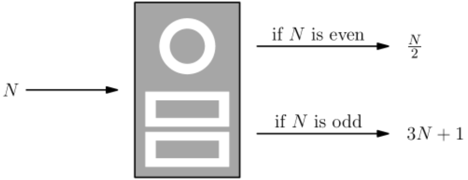
Question (AMC 8, 2023-22)

In a sequence of positive integers, each term after the second is the product of the previous two terms. The sixth term is 4000 . What is the first term? (A) 1 (B) 2 (C) 4 (D) 5 (E) 10

Question (AMC 8, 2023-20)

Two integers are inserted into the list $3, 3, 8, 11, 28$ to double its range. The mode(众数) and median(中位数) remain unchanged. What is the maximum possible sum of the two additional numbers?
(A) 56 (B) 57 (C) 58 (D) 60 (E) 61

When a positive integer N is fed into a machine, the output is a number calculated according to the rule shown below.



For example, starting with an input of $N = 7$, the machine will output $3 \cdot 7 + 1 = 22$. Then if the output is repeatedly inserted into the machine five more times, the final output is 26.

$$7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26$$

When the same 6-step process is applied to a different starting value of N, the final output is 1. What is the sum of all such integers N?

$$N \to \_\_ \to \_\_ \to \_\_ \to \_\_ \to \_\_ \to 1$$

$\textbf{(A)}\ 73 \qquad \textbf{(B)}\ 74 \qquad \textbf{(C)}\ 75 \qquad \textbf{(D)}\ 82 \qquad \textbf{(E)}\ 83$