

# Analysis

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# Chapter 1

## Foundation

### 1.1 Construction of Real Number

**Definition 1.1.1** (ordered ring). Thus, a ring(field)  $R \neq 0$  with an order  $<$  is called an ordered ring(field) if the following holds:

- (1)  $(R, <)$  is totally ordered
- (2)  $x < y \Rightarrow x + z < y + z, z \in R$
- (3)  $x, y > 0 \Rightarrow xy > 0$

Of course, an element  $x \in R$  is called positive if  $x > 0$  and negative if  $x < 0$ . We gather in the next proposition some simple properties of ordered fields.

**Proposition 1.1.2.** Let  $K$  be an ordered field, then for  $x, y, a, b \in K$ .

- (1)  $x > y \Leftrightarrow x - y > 0$ .
- (2) If  $x > y$  and  $a > b$ , then  $x + a > y + b$ .
- (3) If  $a > 0$  and  $x > y$ , then  $ax > ay$ .
- (4) If  $x > 0$ , then  $-x < 0$ . If  $x < 0$ , then  $-x > 0$ .
- (5) Let  $x > 0$ . If  $y > 0$ , then  $xy > 0$ . If  $y < 0$ , then  $xy < 0$ .
- (6) If  $a < 0$  and  $x > y$ , then  $ax < ay$ .
- (7)  $x^2 > 0$  for all  $x \neq 0$ . In particular,  $1 > 0$ .
- (8) If  $x > 0$ , then  $x^{-1} > 0$ .
- (9) If  $x > y > 0$ , then  $0 < x^{-1} < y^{-1}$  and  $xy^{-1} > 1$ .

**Definition 1.1.3.**  $K$  is an ordered field,  $K$  is said to be Archimedes if and only if for  $x, y > 0$  there's  $n \in \mathbb{Z}$  such that  $nx > y$ .

**Example 1.1.4.**  $\mathbb{Q}$  is a Archimedes ordered field with original order.

**Proposition 1.1.5.** For an ordered field  $K$ , the absolute value function,  $|\cdot| : K \rightarrow K$  and the sign function,  $\text{sign}(\cdot) : K \rightarrow K$  are defined by

$$|x| := \begin{cases} x, & x > 0, \\ 0, & x = 0, \\ -x, & x < 0, \end{cases} \quad \text{sign } x := \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

Let  $K$  be an ordered field and  $x, y, a, \varepsilon \in K$  with  $\varepsilon > 0$ .

- (1)  $x = |x| \text{sign}(x)$ ,  $|x| = x \text{sign}(x)$ .
- (2)  $|x| = |-x|$ ,  $x \leq |x|$ .
- (3)  $|xy| = |x||y|$ .
- (4)  $|x| \geq 0$  and  $(|x| = 0 \Leftrightarrow x = 0)$ .
- (5)  $|x - a| < \varepsilon \Leftrightarrow a - \varepsilon < x < a + \varepsilon$ .
- (6)  $|x + y| \leq |x| + |y|$  (triangle inequality).
- (7)  $|x - y| \geq ||x| - |y||$ ,  $x, y \in K$

**Definition 1.1.6.** A ring homomorphism  $f$  between ordered field is said to be order-preserving if

$$a < b \iff f(a) < f(b)$$

.

**Definition 1.1.7.** A sequence  $r = (x_n)_{n \in \mathbb{Z}_{>0}}$  is a Cauchy sequence if for all  $\epsilon \in \mathbb{Q} > 0$ , there's  $N > 0$  such that for all  $m, n > N$ ,  $|x_n - x_m| < \epsilon$ .

**Proposition 1.1.8.** Cauchy sequence is bounded.

**Definition 1.1.9.** Let

$$\mathcal{R} = \{(x_n) \in \mathbb{Q}^{\mathbb{Z}_{>0}} : (x_n) \text{ is a Cauchy sequence}\}$$

and

$$\mathbf{c}_0 = \{(x_n) \in \mathbb{Q}^{\mathbb{Z}_{>0}} : \text{for all } \epsilon > 0, \text{ there's } N > 0 \text{ such that for all } n > N, |x_n| < \epsilon\}$$

It's clear that  $\mathbf{c}_0 \subset \mathcal{R}$  is a maximal ideal of  $\mathcal{R}$ . Hence  $\mathcal{R}/\mathbf{c}_0$  is a field and we denote it by  $\mathbb{R}$ . For convenience, we usually denote  $(a_n) + \mathbf{c}_0$  by  $(a_n)$ .

**Definition 1.1.10.** Now we define a order on  $\mathbb{R}$ , for  $(a_n), (b_n)$  in  $\mathbb{R}$ ,  $(a_n) > (b_n)$  if there's  $\epsilon > 0$ , a sufficiently large integer  $N$ , such that  $a_n - b_n > \epsilon$  for  $n > N$ . And denote this order by  $<$ . It's easy to check that ' $<$ ' is well-defined and totally ordered.

**Proposition 1.1.11.**  $(\mathbb{R}, <)$  is a Archimedes ordered field. And the embedding  $l : \mathbb{Q} \rightarrow \mathbb{R}$  given by

$$r \mapsto (r, r, r, \dots)$$

is an order-preserving ring homomorphism.

**Definition 1.1.12.** For a sequence  $(A_n) \in \mathbb{R}$ , we say  $A_n \rightarrow A$  if for all  $\epsilon \in \mathbb{R} > 0$ , there's  $N > 0$  such that for all  $n > N$ ,  $|A_n - A| < \epsilon$ . And we say  $(A_n)$  is a Cauchy sequence if for all  $\epsilon \in \mathbb{R}_{>0}$ , there's  $N > 0$  such that for all  $m, n > N$ ,  $|x_n - x_m| < \epsilon$ .

**Proposition 1.1.13** (dense). For all  $a, b \in \mathbb{R}$ , if  $a < b$ , there's  $c \in \mathbb{Q}$  such that  $a < l(c) < b$ .

**Proposition 1.1.14** (completeness).  $(A_n)$  is a Cauchy sequence in  $\mathbb{R}$  if and only if there's  $A \in \mathbb{R}$  such that  $A_n \rightarrow A$ .

*Proof:* 'if' is obvious.

'only if': Take  $x_n \in \mathbb{Q}$  such that:

$$A_n < l(x_n) < A_n + l\left(\frac{1}{n}\right)$$

It's cleat that  $a = (x_n) + \mathbf{c}_0 \in \mathbb{R}$ .

Notice that  $A_n \rightarrow a$ , we have  $\mathbb{R}$  is complete.

Now we identity  $\mathbb{Q}$  with a subfield of  $\mathbb{R}$  in the following content.

**Proposition 1.1.15.** (1)  $E$  is a non-empty subset of  $\mathbb{R}$  and if  $E$  is lower-bounded, then  $E$  has a infimum; if  $E$  is upper-bounded, then  $E$  has a supremum.

(2) Every incresing bounded sequence  $(x_n) \in \mathbb{R}$  has a limit.

(3) (Bolzano-Weierstress) Every bounded sequence has a convergent subsequence.

(4) if

$$[a, b] \subset \bigcup_{i \in I} (a_i, b_i)$$

, then

$$[a, b] \subset \bigcup_{k \in J} (a_k, b_k)$$

for some finite subset  $J$  of  $I$ .

**Proposition 1.1.16.**  $a > 0$ ,  $n \in \mathbb{Z}_{>0}$ , then there's unique  $x \in \mathbb{R}_{>0}$  such that  $x^n = a$ . We denote the unique positive root by  $\sqrt[n]{a}$ . And for all  $a \in \mathbb{R}$  and  $r = \frac{p}{q} \in \mathbb{Q}$ , define  $a^r = \sqrt[q]{a^p}$ . It's easy to check that  $\sqrt[q]{a^p} = (\sqrt[q]{a})^p$ .

**Definition 1.1.17** (complex number). Let  $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ , define  $(a, b) \cdot (c, d) = (ac - bd, bc + ad)$ . Then  $\mathbb{C}$  is a field under this operator and  $\mathbb{R}$  is a subfield of  $\mathbb{C}$ .

## 1.2 Point-Set Topology

Munkres's Topology is a good reference for this section.

### 1.2.1 Definition

**Definition 1.2.1.** A topology on a set  $X$  is a collection  $\mathcal{T}$  of subsets of  $X$  having the following properties:

- (1)  $\emptyset$  and  $X$  are in  $\mathcal{T}$ .
- (2) The union of the elements of any subcollection of  $\mathcal{T}$  is in  $\mathcal{T}$ .
- (3) The intersection of the elements of any finite subcollection of  $\mathcal{T}$  is in  $\mathcal{T}$ .

A set  $X$  for which a topology  $\mathcal{T}$  has been specified is called a topological space.

**Definition 1.2.2.** If  $X$  is a set, a basis for a topology on  $X$  is a collection  $\mathcal{B}$  of subsets of  $X$  (called basis elements) such that

- (1) For each  $x \in X$ , there is at least one basis element  $B$  containing  $x$ .
- (2) If  $x$  belongs to the intersection of two basis elements  $B_1$  and  $B_2$ , then there is a basis element  $B_3$  containing  $x$  such that  $B_3 \subset B_1 \cap B_2$ .

If  $\mathcal{B}$  satisfies these two conditions, then we define the topology  $\mathcal{T}$  generated by  $\mathcal{B}$  as follows: A subset  $U$  of  $X$  is said to be open in  $X$  (that is, to be an element of  $\mathcal{T}$ ) if for each  $x \in U$ , there is a basis element  $B \in \mathcal{B}$  such that  $x \in B$  and  $B \subset U$ . Note that each basis element is itself an element of  $\mathcal{T}$ .

**Definition 1.2.3.** Let  $X$  be a topological space with topology  $\mathcal{T}$ . If  $Y$  is a subset of  $X$ , the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on  $Y$ , called the subspace topology. With this topology,  $Y$  is called a subspace of  $X$ ; its open sets consist of all intersections of open sets of  $X$  with  $Y$ .

**Definition 1.2.4.**  $X$  is Hausdorff if for any two elements  $x \neq y$  in  $X$ , there's  $U, V$  open in  $X$  such that  $x \in U, y \in V$  and  $U \cap V = \emptyset$ .

**Definition 1.2.5** (convergence).

**Proposition 1.2.6.** If  $X$  is a Hausdorff space, then a sequence of points of  $X$  converges to at most one point of  $X$ .

**Example 1.2.7.** Let  $X$  be a ordered set; assume  $X$  has more than one element. Let  $\mathcal{B}$  be the collection of all sets of the following types:

- (1) All open intervals  $(a, b)$  in  $X$ .



- (2) All intervals of the form  $[a_0, b)$ , where  $a_0$  is the smallest element (if any) of  $X$ .
- (3) All intervals of the form  $(a, b_0]$ , where  $b_0$  is the largest element (if any) of  $X$ . The collection  $\mathcal{B}$  is a basis for a topology on  $X$ , which is called the order topology.

**Example 1.2.8.**  $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ .

**Proposition 1.2.9.** Given a subset  $A$  of a topological space  $X$ , the interior of  $A$  is defined as the union of all open sets contained in  $A$ , and the closure of  $A$  is defined as the intersection of all closed sets containing  $A$ .

Let  $Y$  be a subspace of  $X$ ; let  $A$  be a subset of  $Y$ ; let  $\bar{A}$  denote the closure of  $A$  in  $X$ . Then the closure of  $A$  in  $Y$  equals  $\bar{A} \cap Y$ .

**Definition 1.2.10.** If  $A$  is a subset of the topological space  $X$  and if  $x$  is a point of  $X$ , we say that  $x$  is a limit point of  $A$  if every neighborhood of  $x$  intersects  $A$  in some point other than  $x$  itself. Said differently,  $x$  is a limit point of  $A$  if it belongs to the closure of  $A - \{x\}$ . The point  $x$  may lie in  $A$  or not; for this definition it does not matter. Let  $A$  be a subset of the topological space  $X$ ; let  $A'$  be the set of all limit points of  $A$ . Then

$$\bar{A} = A \cup A'.$$

**Proposition 1.2.11.** Let  $X$  and  $Y$  be topological spaces; let  $f : X \rightarrow Y$ . Then the following are equivalent:

- (1)  $f$  is continuous. ( $U$  open in  $X$  implies  $f^{-1}(U)$  open in  $Y$ )
- (2) For every subset  $A$  of  $X$ , one has  $f(\bar{A}) \subset \overline{f(A)}$ .
- (3) For every closed set  $B$  of  $Y$ , the set  $f^{-1}(B)$  is closed in  $X$ .
- (4) For each  $x \in X$  and each neighborhood  $V$  of  $f(x)$ , there is a neighborhood  $U$  of  $x$  such that  $f(U) \subset V$ . If the condition in (4) holds for the point  $x$  of  $X$ , we say that  $f$  is continuous at the point  $x$ .

**Definition 1.2.12.** Consider  $(X_i)_{i \in I}$  be a family of topology spaces, then the sets of the form

$$\prod_{i \in I} U_i$$

$U_i = X_i$  for all but finite  $i$ , form a basis of  $\prod_{i \in I} X_i$ . We call it the topology induced by this product topology.

In language of category, product topology with projection  $p_i : \prod_{i \in I} X_i \rightarrow X_i$  is the product object in the category of topological space.

**Proposition 1.2.13.** If each space  $X_\alpha$  is Hausdorff space, then  $\prod X_\alpha$  is a Hausdorff space in product topology.

**Proposition 1.2.14.** Let  $\{X_\alpha\}$  be an indexed family of spaces; let  $A_\alpha \subset X_\alpha$  for each  $\alpha$ . If  $\prod X_\alpha$  is given the product topology, then

$$\prod \bar{A}_\alpha = \overline{\prod A_\alpha}$$

**Theorem 1.2.15.** Let  $f : A \rightarrow \prod_{\alpha \in J} X_\alpha$  be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J},$$

where  $f_\alpha : A \rightarrow X_\alpha$  for each  $\alpha$ . Let  $\prod X_\alpha$  have the product topology. Then the function  $f$  is continuous if and only if each function  $f_\alpha$  is continuous.

**Definition 1.2.16** (locally closed). A subset  $E$  of a topological space  $X$  is said to be locally closed if any of the following equivalent conditions are satisfied:

- (1)  $E$  is the intersection of an open set and a closed set in  $X$ .
- (2) For each point  $x \in E$ , there is a neighborhood  $U$  of  $x$  such that  $E \cap U$  is closed in  $U$ .
- (3)  $E$  is open in its closure  $\bar{E}$ .
- (4) The set  $\bar{E} \setminus E$  is closed in  $X$ .

*Proof:* (2) implies (1): For all  $x \in E$ , choose  $U_i$  open in  $X$  such that  $E = \cap U_i = U_i \cap V_i$  for some  $V_i$  closed in  $X$ . Then consider

$$\left( \bigcup_{i \in I} U_i \right) \cap \bar{E}$$

For  $x \in U_i \cap \bar{E}$ , if  $x \notin E$ , then  $x \in U_i \cap V_i^c$ . Notice that

$$U_i \cap V_i^c \cap E = \emptyset$$

which contradicts to  $x \in \bar{E}$ .

(3) implies (4): If  $\bar{E} \cap U = E$  for some  $U$  open in  $X$ , then  $\bar{E} - E = \bar{E} \cap E^c = \bar{E} \cap U^c$ .

**Proposition 1.2.17.**  $E$  is a locally closed subset in  $X$ , then  $E$  closed in the open subset  $X - \bar{E} \setminus E$  and  $X - \bar{E} \setminus E$  is the largest open subset contains  $E$  such that  $E$  is closed in it.

*Proof:* Since

$$X - \bar{E} \setminus E = (\bar{E})^c \cup E$$

and  $((\bar{E})^c \cup E) \cap \bar{E} = E$ , we have  $E$  closed in  $X - \bar{E} \setminus E$ .

In addition, if there's  $V$  open in  $X$  such that  $E$  closed in  $V$ ,

$$E = \bar{E} \cap V$$

Hence,  $V = (V \cap \bar{E}^c) \cup (V \cap \bar{E}) = E \cup (V \cap \bar{E}^c) \subset E \cup \bar{E}^c$ .

### 1.2.2 Metric space

**Definition 1.2.18.** A metric on a set  $X$  is a function

$$d : X \times X \longrightarrow \mathbb{R}$$

having the following properties:

- (1)  $d(x, y) \geq 0$  for all  $x, y \in X$ ; equality holds if and only if  $x = y$ .
- (2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ .
- (3) (Triangle inequality)  $d(x, y) + d(y, z) \geq d(x, z)$ , for all  $x, y, z \in X$ .

Given a metric  $d$  on  $X$ , the number  $d(x, y)$  is often called the distance between  $x$  and  $y$  in the metric  $d$ . Given  $\epsilon > 0$ , consider the set

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

of all points  $y$  whose distance from  $x$  is less than  $\epsilon$ . It is called the  $\epsilon$ -ball centered at  $x$ . Sometimes we omit the metric  $d$  from the notation and write this ball simply as  $B(x, \epsilon)$ , when no confusion will arise.

If  $d$  is a metric on the set  $X$ , then the collection of all  $\epsilon$ -balls  $B_d(x, \epsilon)$ , for  $x \in X$  and  $\epsilon > 0$ , is a basis for a topology on  $X$ , called the metric topology induced by  $d$ .

**Example 1.2.19.**  $\mathbb{R}^n$  is a metric space with distance  $d(x, y) = \|x - y\|$

**Theorem 1.2.20.** Let  $X$  be a topological space; let  $A \subset X$ . If there is a sequence of points of  $A$  converging to  $x$ , then  $x \in \bar{A}$ ; the converse holds if  $X$  is metrizable.

Let  $f : X \rightarrow Y$ . If the function  $f$  is continuous, then for every convergent sequence  $x_n \rightarrow x$  in  $X$ , the sequence  $f(x_n)$  converges to  $f(x)$ . The converse holds if  $X$  is metrizable.

**Theorem 1.2.21.** Let  $f_n : X \rightarrow Y$  be a sequence of functions from the set  $X$  to the metric space  $Y$ . Let  $d$  be the metric for  $Y$ . We say that the sequence  $(f_n)$  converges uniformly to the function  $f : X \rightarrow Y$  if given  $\epsilon > 0$ , there exists an integer  $N$  such that

$$d(f_n(x), f(x)) < \epsilon$$

for all  $n > N$  and all  $x$  in  $X$ .

Let  $f_n : X \rightarrow Y$  be a sequence of continuous functions from the topological space  $X$  to the metric space  $Y$ . If  $(f_n)$  converges uniformly to  $f$ , then  $f$  is continuous.

**Proposition 1.2.22.** If  $f \in B(X)$ , we define the uniform norm of  $f$  to be

$$\|f\|_u = \sup\{|f(x)| : x \in X\}.$$

The function  $\rho(f, g) = \|f - g\|_u$  is easily seen to be a metric on  $B(X)$ , and convergence with respect to this metric is simply uniform convergence on  $X$ .  $B(X)$  is obviously complete in the uniform metric: If  $\{f_n\}$  is uniformly Cauchy, then  $\{f_n(x)\}$  is Cauchy for each  $x$ , and if we set  $f(x) = \lim_n f_n(x)$ , it is easily verified that  $\|f_n - f\|_u \rightarrow 0$ .

If  $X$  is a topological space,  $BC(X) = B(X) \cap C(X)$  is a closed subspace of  $B(X)$  in the uniform metric; in particular,  $BC(X)$  is complete.

### 1.2.3 Compactness

**Definition 1.2.23.** A collection  $\mathcal{A}$  of subsets of a space  $X$  is said to cover  $X$ , or to be a covering of  $X$ , if the union of the elements of  $\mathcal{A}$  is equal to  $X$ . It is called an open covering of  $X$  if its elements are open subsets of  $X$ .

A space  $X$  is said to be compact if every open covering  $\mathcal{A}$  of  $X$  contains a finite subcollection that also covers  $X$ .

**Proposition 1.2.24.** Every closed subspace of a compact space is compact. Every compact subspace of a Hausdorff space is closed.

**Theorem 1.2.25.** The image of a compact space under a continuous map is compact.

**Corollary 1.2.26.**  $X$  is a compact space,  $Y$  is a Hausdorff space, then continuous  $f : X \rightarrow Y$  is closed.

**Corollary 1.2.27.** Let  $f : X \rightarrow Y$  be a continuous bijection.  $X$  is a compact space,  $Y$  is a Hausdorff space, then  $f$  is homomorphism.

**Lemma 1.2.28** (Lebesgue number lemma). Let  $\mathcal{A}$  be an open covering of the metric space  $(X, d)$ . If  $X$  is compact, there is a  $\delta > 0$  such that for each subset of  $X$  having diameter less than  $\delta$ , there exists an element of  $\mathcal{A}$  containing it. The number  $\delta$  is called a Lebesgue number for the covering  $\mathcal{A}$ .

**Theorem 1.2.29.** Let  $X$  be a metrizable space. Then the following are equivalent:

- (1)  $X$  is compact.
- (2)  $X$  is limit point compact(infinite subset has a limit point).
- (3)  $X$  is sequentially compact(every sequence has a convergent subsequence).

**Theorem 1.2.30** (Tychonoff theorem). An arbitrary product of compact spaces is compact in the product topology.

**Definition 1.2.31.** A function  $f$  from the metric space  $(X, d_X)$  to the metric space  $(Y, d_Y)$  is said to be uniformly continuous if given  $\epsilon > 0$ , there is a  $\delta > 0$  such that for every pair of points  $x_0, x_1$  of  $X$ ,

$$d_X(x_0, x_1) < \delta \implies d_Y(f(x_0), f(x_1)) < \epsilon.$$

**Theorem 1.2.32.** Let  $f : X \rightarrow Y$  be a continuous map of the compact metric space  $(X, d_X)$  to the metric space  $(Y, d_Y)$ . Then  $f$  is uniformly continuous.

**Proposition 1.2.33** (finite intersection). A collection  $\mathcal{C}$  of subsets of  $X$  is said to have the finite intersection property if for every finite subcollection

$$\{C_1, \dots, C_n\}$$

of  $\mathcal{C}$ , the intersection  $C_1 \cap \dots \cap C_n$  is nonempty.

Let  $X$  be a topological space. Then  $X$  is compact if and only if for every collection  $\mathcal{C}$  of closed sets in  $X$  having the finite intersection property, the intersection  $\bigcap_{C \in \mathcal{C}} C$  of all the elements of  $\mathcal{C}$  is nonempty.

**Definition 1.2.34** (locally compact). A space  $X$  is said to be locally compact at  $x$  if there is some compact subspace  $C$  of  $X$  that contains a neighborhood of  $x$ . If  $X$  is locally compact at each of its points,  $X$  is said simply to be locally compact.

**Definition 1.2.35** (one-point compactification). Let  $X$  be a space. Then  $X$  is locally compact Hausdorff if and only if there exists a space  $Y$  satisfying the following conditions:

- (1)  $X$  is a subspace of  $Y$ .
- (2) The set  $Y - X$  consists of a single point.
- (3)  $Y$  is a compact Hausdorff space.

If  $Y$  and  $Y'$  are two spaces satisfying these conditions, then there is a homeomorphism of  $Y$  with  $Y'$  that equals the identity map on  $X$ .

*Proof:* We only provide the form of the open sets in  $Y$ :  $U$  open in  $Y$  if and only if  $U$  open in  $X$ , or  $U$  is the complement of a compact subset in  $X$ .

**Definition 1.2.36.** If  $Y$  is a compact Hausdorff space and  $X$  is a proper subspace of  $Y$  whose closure equals  $Y$ , then  $Y$  is said to be a compactification of  $X$ . If  $Y - X$  equals a single point, then  $Y$  is called the one-point compactification of  $X$ .

**Proposition 1.2.37.** Let  $X$  be a Hausdorff space. Then  $X$  is locally compact if and only if given  $x$  in  $X$ , and given a neighborhood  $U$  of  $x$ , there is a neighborhood  $V$  of  $x$  such that  $\bar{V}$  is compact and  $\bar{V} \subset U$ .

**Corollary 1.2.38.** If  $X$  is an LCH space and  $K \subset U \subset X$  where  $K$  is compact and  $U$  is open, there exists a precompact open  $V$  such that  $K \subset V \subset \bar{V} \subset U$ .

**Proposition 1.2.39.** Let  $X$  be locally compact Hausdorff; let  $A$  be a subspace of  $X$ . If  $A$  is closed in  $X$  or open in  $X$ , then  $A$  is locally compact.

**Proposition 1.2.40.** In a locally compact Hausdorff space  $E$ , a subset  $A$  is closed if and only if its intersection with every compact set is compact

*Proof:* Let  $A \subseteq E$  have the property that  $A \cap K$  is closed in  $K$  for all compact  $K \subseteq E$ . We want to show that  $A$  is closed whenever  $E$  is locally compact Hausdorff, so we will show that  $E - A$  is open.

Let  $x \in E - A$ , let  $K$  be a compact neighbourhood of  $x$ , and let  $U \subseteq K$  be an open neighbourhood of  $x$ . Then  $x \in U - K \cap A$  and  $U - K \cap A$  is open in  $X$ . Hence  $E - A$  is open in  $X$ .

**Theorem 1.2.41** (Urysohn's Lemma, Locally Compact Version). If  $X$  is an LCH space and  $K \subset U \subset X$  where  $K$  is compact and  $U$  is open, there exists  $f \in C(X, [0, 1])$  such that  $f = 1$  on  $K$  and  $f = 0$  outside a compact subset of  $U$ .

**Definition 1.2.42.** If  $X$  is a topological space and  $f \in C(X)$ , the support of  $f$ , denoted by  $\text{supp}(f)$ , is the smallest closed set outside of which  $f$  vanishes, that is, the closure of  $\{x : f(x) \neq 0\}$ . If  $\text{supp}(f)$  is compact, we say that  $f$  is compactly supported, and we define

$$C_c(X) = \{f \in C(X) : \text{supp}(f) \text{ is compact}\}.$$

Moreover, if  $f \in C(X)$ , we say that  $f$  vanishes at infinity if for every  $\epsilon > 0$  the set  $\{x : |f(x)| \geq \epsilon\}$  is compact, and we define

$$C_0(X) = \{f \in C(X) : f \text{ vanishes at infinity}\}.$$

Clearly  $C_c(X) \subset C_0(X)$ . Moreover,  $C_0(X) \subset BC(X)$ , because for  $f \in C_0(X)$  the image of the set  $\{x : |f(x)| \geq \epsilon\}$  is compact, and  $|f| < \epsilon$  on its complement.

**Proposition 1.2.43.** If  $X$  is an LCH space,  $C_0(X)$  is the closure of  $C_c(X)$  in the uniform metric.

*Proof:* If  $\{f_n\}$  is a sequence in  $C_c(X)$  that converges uniformly to  $f \in C(X)$ , for each  $\epsilon > 0$  there exists  $n \in \mathbb{N}$  such that  $\|f_n - f\|_u < \epsilon$ . Then  $|f(x)| < \epsilon$  if  $x \notin \text{supp}(f_n)$ , so  $f \in C_0(X)$ . Conversely, if  $f \in C_0(X)$ , for  $n \in \mathbb{N}$  let  $K_n = \{x : |f(x)| \geq n^{-1}\}$ . Then  $K_n$  is compact, so by Urysohn's Lemma, there exists  $g_n \in C_c(X)$  with  $0 \leq g_n \leq 1$  and  $g_n = 1$  on  $K_n$ . Let  $f_n = g_n f$ . Then  $f_n \in C_c(X)$  and  $\|f_n - f\|_u \leq n^{-1}$ , so  $f_n \rightarrow f$  uniformly.

### 1.2.4 Connectness

**Definition 1.2.44.** Given  $X$ , define an equivalence relation on  $X$  by setting  $x \sim y$  if there is a connected subspace of  $X$  containing both  $x$  and  $y$ . The equivalence classes are called the components (or the "connected components") of  $X$ . The components of  $X$  are connected disjoint subspaces of  $X$  whose union is  $X$ , such that each nonempty connected subspace of  $X$  intersects only one of them.

**Definition 1.2.45.** We define another equivalence relation on the space  $X$  by defining  $x \sim y$  if there is a path in  $X$  from  $x$  to  $y$ . The equivalence classes are called the path components of  $X$ . The path components of  $X$  are path-connected disjoint subspaces of  $X$  whose union is  $X$ , such that each nonempty path-connected subspace of  $X$  intersects only one of them.

**Definition 1.2.46.** A space  $X$  is said to be locally connected at  $x$  if for every neighborhood  $U$  of  $x$ , there is a connected neighborhood  $V$  of  $x$  contained in  $U$ . If  $X$  is locally connected at each of its points, it is said simply to be locally connected. Similarly, a space  $X$  is said to be locally path connected at  $x$  if for every neighborhood  $U$  of  $x$ , there is a path-connected neighborhood  $V$  of  $x$  contained in  $U$ . If  $X$  is locally path connected at each of its points, then it is said to be locally path connected.

**Proposition 1.2.47.** (1) A space  $X$  is locally connected if and only if for every open set  $U$  of  $X$ , each component of  $U$  is open in  $X$ .

- (2) A space  $X$  is locally path connected if and only if for every open set  $U$  of  $X$ , each path component of  $U$  is open in  $X$ .
- (3) If  $X$  is a topological space, each path component of  $X$  lies in a component of  $X$ . If  $X$  is locally path connected, then the components and the path components of  $X$  are the same.

**Proposition 1.2.48.** The union of a collection of connected subspaces of  $X$  that have a point in common is connected.

**Proposition 1.2.49.** Let  $A$  be a connected subspace of  $X$ . If  $A \subset B \subset \bar{A}$ , then  $B$  is also connected.

**Proposition 1.2.50.** The image of a connected space under a continuous map is connected.

**Theorem 1.2.51** (Intermediate Value Theorem). Let  $f : X \rightarrow Y$  be a continuous map, where  $X$  is a connected space and  $Y$  is an ordered set in the order topology. If  $a$  and  $b$  are two points of  $X$  and if  $r$  is a point of  $Y$  lying between  $f(a)$  and  $f(b)$ , then there exists a point  $c$  of  $X$  such that  $f(c) = r$ .

**Theorem 1.2.52** (Extreme value theorem). Let  $f : X \rightarrow Y$  be continuous, where  $Y$  is an ordered set in the order topology. If  $X$  is compact, then there exist points  $c$  and  $d$  in  $X$  such that  $f(c) \leq f(x) \leq f(d)$  for every  $x \in X$ .

### 1.2.5 Countability

**Definition 1.2.53.** A space  $X$  is said to have a countable basis at  $x$  if there is a countable collection  $B$  of neighborhoods of  $x$  such that each neighborhood of  $x$  contains at least one of the elements of  $B$ . A space that has a countable basis at each of its points is said to satisfy the first countability axiom, or to be first-countable.

**Proposition 1.2.54.** Let  $X$  be a topological space.

- (1) Let  $A$  be a subset of  $X$ . If there is a sequence of points of  $A$  converging to  $x$ , then  $x \in \bar{A}$ ; the converse holds if  $X$  is first-countable.
- (2) Let  $f : X \rightarrow Y$ . If  $f$  is continuous, then for every convergent sequence  $x_n \rightarrow x$  in  $X$ , the sequence  $f(x_n)$  converges to  $f(x)$ . The converse holds if  $X$  is first-countable.

**Definition 1.2.55.** If a space  $X$  has a countable basis for its topology, then  $X$  is said to satisfy the second countability axiom, or to be second-countable.

**Definition 1.2.56.** A subset  $A$  of a space  $X$  is said to be dense in  $X$  if  $\bar{A} = X$ .

**Definition 1.2.57.** Suppose that  $X$  has a countable basis. Then:

- (1) Every open covering of  $X$  contains a countable subcollection covering  $X$ . (Lindelof space)
- (2) There exists a countable subset of  $X$  that is dense in  $X$ . (separable)

**Proposition 1.2.58.** (1) Every metrizable space with a countable dense subset has a countable basis.

- (2) Every metrizable Lindelöf space has a countable basis.

### 1.2.6 Separation

**Definition 1.2.59.** Suppose that one-point sets are closed in  $X$ . Then  $X$  is said to be regular if for each pair consisting of a point  $x$  and a closed set  $B$  disjoint from  $x$ , there exist disjoint open sets containing  $x$  and  $B$ , respectively.

The space  $X$  is said to be normal if for each pair  $A, B$  of disjoint closed sets of  $X$ , there exist disjoint open sets containing  $A$  and  $B$ , respectively.

**Proposition 1.2.60.** Let  $X$  be a topological space. Let one-point sets in  $X$  be closed.

- (1)  $X$  is regular if and only if given a point  $x$  of  $X$  and a neighborhood  $U$  of  $x$ , there is a neighborhood  $V$  of  $x$  such that  $\bar{V} \subset U$ .
- (2)  $X$  is normal if and only if given a closed set  $A$  and an open set  $U$  containing  $A$ , there is an open set  $V$  containing  $A$  such that  $\bar{V} \subset U$ .

**Proposition 1.2.61.** (1) Every metrizable space is normal.

(2) Every compact Hausdorff space is normal.

**Theorem 1.2.62** (Urysohn's lemma). Let  $X$  be a normal space; let  $A$  and  $B$  be disjoint closed subsets of  $X$ . Let  $[a, b]$  be a closed interval in the real line. Then there exists a continuous map

$$f : X \longrightarrow [a, b]$$

such that  $f(x) = a$  for every  $x$  in  $A$ , and  $f(x) = b$  for every  $x$  in  $B$ .

**Theorem 1.2.63** (Tietze extension theorem). Let  $X$  be a normal space; let  $A$  be a closed subspace of  $X$ .

- (1) Any continuous map of  $A$  into the closed interval  $[a, b]$  of  $\mathbb{R}$  may be extended to a continuous map of all of  $X$  into  $[a, b]$ .
- (2) Any continuous map of  $A$  into  $\mathbb{R}$  may be extended to a continuous map of all of  $X$  into  $\mathbb{R}$ .

### 1.2.7 Completeness

**Definition 1.2.64.** Let  $(X, d)$  be a metric space. A sequence  $(x_n)$  of points of  $X$  is said to be a Cauchy sequence in  $(X, d)$  if it has the property that given  $\epsilon > 0$ , there is an integer  $N$  such that

$$d(x_n, x_m) < \epsilon \quad \text{whenever } n, m \geq N.$$

The metric space  $(X, d)$  is said to be complete if every Cauchy sequence in  $X$  converges.

**Theorem 1.2.65.** A metric space  $(X, d)$  is compact if and only if it is complete and totally bounded.



**Theorem 1.2.66** (extension theorem). Suppose  $Y$  and  $Z$  are metric spaces, and  $Z$  is complete. Also suppose  $X$  is a dense subset of  $Y$ , and  $f : X \rightarrow Z$  is uniformly continuous. Then  $f$  has a uniquely determined extension  $\bar{f} : Y \rightarrow Z$  given by

$$\bar{f}(y) = \lim_{\substack{x \rightarrow y \\ x \in X}} f(x) \quad \text{for } y \in Y$$

and  $\bar{f}$  is also uniformly continuous.

**Definition 1.2.67.** Let  $X$  be a metric space. If  $h : X \rightarrow Y$  is an isometric imbedding of  $X$  into a complete metric space  $Y$ , such that  $h(X)$  dense in  $Y$ . Then  $Y$  is called the completion of  $X$ . By extension theorem, the completion of  $X$  is uniquely determined up to an isometry.

**Definition 1.2.68.** A space  $X$  is said to be a Baire space if the following condition holds: Given any countable collection  $\{A_n\}$  of closed sets of  $X$  each of which has empty interior in  $X$ , their union  $\bigcup A_n$  also has empty interior in  $X$ .

**Theorem 1.2.69** (Baire Category Theorem). If  $X$  is a compact Hausdorff space or a complete metric space, then  $X$  is a Baire space.

**Theorem 1.2.70.** Any open subspace  $Y$  of a Baire space  $X$  is itself a Baire space.

**Theorem 1.2.71.** Let  $X$  be a space; let  $(Y, d)$  be a metric space. Let  $f_n : X \rightarrow Y$  be a sequence of continuous functions such that  $f_n(x) \rightarrow f(x)$  for all  $x \in X$ , where  $f : X \rightarrow Y$ . If  $X$  is a Baire space, the set of points at which  $f$  is continuous is dense in  $X$ .

## 1.3 Limit

### 1.3.1 Limit Superior and Limit Inferior

We work on  $\bar{\mathbb{R}}$  in this subsection.

**Definition 1.3.1.** We call  $a \in \bar{\mathbb{R}}$  a cluster point of  $(x_n)$  if every neighborhood of  $a$  contains infinitely many terms of the sequence.

**Definition 1.3.2.** A point  $a$  is a cluster point of a sequence  $(x_n)$  if and only if there is some subsequence  $(x_{n_k})_{k \in \mathbb{N}}$  of  $(x_n)$  which converges to  $a$ .

**Definition 1.3.3.** Let  $(x_n)$  be a sequence in  $\mathbb{R}$ . We can define two new sequences  $(y_n)$  and  $(z_n)$  by

$$y_n := \sup_{k \geq n} x_k := \sup \{x_k; k \geq n\}$$

$$z_n := \inf_{k \geq n} x_k := \inf \{x_k; k \geq n\}$$

Clearly  $(y_n)$  is a decreasing sequence and  $(z_n)$  is an increasing sequence in  $\bar{\mathbb{R}}$ . These sequences converge in  $\bar{\mathbb{R}}$

$$\limsup_{n \rightarrow \infty} x_n := \overline{\lim}_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} \left( \sup_{k \geq n} x_k \right)$$

the limit superior, and

$$\liminf_{n \rightarrow \infty} x_n := \underline{\lim}_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} x_k \right)$$

the limit inferior.

**Theorem 1.3.4.** Any sequence  $(x_n)$  in  $\mathbb{R}$  has a smallest cluster point  $x_*$  and a greatest cluster point  $x^*$  in  $\bar{\mathbb{R}}$  and these satisfy

$$\liminf x_n = x_* \quad \text{and} \quad \limsup x_n = x^*$$

### 1.3.2 Series

In the following theorem,  $\mathbb{K}$  is  $\mathbb{R}$  or  $\mathbb{C}$ ,  $(E, |\cdot|)$  is a Banach space over  $\mathbb{K}$  and  $(x_n)$  is a sequence in  $E$ .

**Proposition 1.3.5.** For a series  $\sum x_k$  in a Banach space  $(E, |\cdot|)$ , the following are equivalent:

- (1)  $\sum x_k$  converges.
- (2) For each  $\varepsilon > 0$  there is some  $N \in \mathbb{N}$  such that

$$\left| \sum_{k=n+1}^m x_k \right| < \varepsilon, \quad m > n \geq N.$$

**Proposition 1.3.6.** Let  $\sum x_k$  be a series in  $E$  and  $\sum a_k$  a series in  $\mathbb{R}^+$ . Then the series  $\sum a_k$  is called a majorant (or minorant) for  $\sum x_k$  if there is some  $K \in \mathbb{N}$  such that  $|x_k| \leq a_k$  (or  $a_k \leq |x_k|$ ) for all  $k \geq K$ . If a series in a Banach space has a convergent majorant, then it converges absolutely.

**Proposition 1.3.7** (Abel). Let  $(a_n)_{n \in \mathbb{Z}}, (b_n)_{n \in \mathbb{Z}}$  be two sequences in  $E$ , then

$$\sum_{M < n \leq M+N} a_n b_n = a_{M+N} B_{M+N} + \sum_{M < n \leq M+N-1} (a_n - a_{n+1}) B_n,$$

where  $B_n = \sum_{M < k \leq n} b_k$ .

If in particular  $E = \mathbb{C}$  and  $(a_n)$  is a monotone sequence in  $\mathbb{R}$ , and

$$\sup_{M < n \leq M+N} |B_n| \leq \rho,$$

we have

$$\left| \sum_{M < n \leq M+N} a_n b_n \right| \leq \rho (|a_{M+1}| + 2|a_{M+N}|).$$

**Example 1.3.8** (base  $g$  expansion). Suppose that  $g \geq 2$ . Then every real number  $x$  has a base  $g$  expansion. This expansion is unique if expansions satisfying  $x_k = g - 1$  for almost all  $k \in \mathbb{N}$  are excluded (for example, if  $g = 10$ ,  $0.999 \dots$  is excluded). Moreover,  $x$  is a rational number if and only if its base  $g$  expansion is periodic.

**Theorem 1.3.9.** Every rearrangement of an absolutely convergent series  $\sum x_k$  is absolutely convergent and has the same value as  $\sum x_k$ .

**Theorem 1.3.10.** There is a bijection  $\alpha : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ . If  $\alpha$  is such a bijection, we call the series  $\sum_n x_{\alpha(n)}$  an ordering of the double series  $\sum x_{jk}$ . If we fix  $j \in \mathbb{N}$  (or  $k \in \mathbb{N}$ ), then the series  $\sum_k x_{jk}$  (or  $\sum_j x_{jk}$ ) is called the  $j^{\text{th}}$  row series (or  $j^{\text{th}}$  column series) of  $\sum x_{jk}$ . If every row series (or column series) converges, then we can consider the series of row sums  $\sum_j (\sum_{k=0}^{\infty} x_{jk})$  (or the series of column sums  $\sum_k (\sum_{j=0}^{\infty} x_{jk})$ ). Finally we say that the double series  $\sum x_{jk}$  is summable if

$$\sup_{n \in \mathbb{N}} \sum_{j,k=0}^n |x_{jk}| < \infty.$$

Let  $\sum x_{jk}$  be a summable double series.

- (1) Every ordering  $\sum_n x_{\alpha(n)}$  of  $\sum x_{jk}$  converges absolutely to a value  $s \in E$  which is independent of  $\alpha$ .
- (2) The series of row sums  $\sum_j (\sum_{k=0}^{\infty} x_{jk})$  and column sums  $\sum_k (\sum_{j=0}^{\infty} x_{jk})$  converge absolutely, and

$$\sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} x_{jk} \right) = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{\infty} x_{jk} \right) = s$$

**Theorem 1.3.11.** Suppose that the series  $\sum x_j$  and  $\sum y_k$  in  $\mathbb{K}$  converge absolutely. Then the Cauchy product  $\sum_n \sum_{k=0}^n x_k y_{n-k}$  of  $\sum x_j$  and  $\sum y_k$  converges absolutely, and

$$\left( \sum_{j=0}^{\infty} x_j \right) \left( \sum_{k=0}^{\infty} y_k \right) = \sum_{n=0}^{\infty} \sum_{k=0}^n x_k y_{n-k}$$

**Example 1.3.12.** Let  $k \in \mathbb{N}$  and  $a \in \mathbb{C}$  be such that  $|a| > 1$ . Then

$$\lim_{n \rightarrow \infty} \frac{n^k}{a^n} = 0$$

that is, for  $|a| > 1$  the function  $n \mapsto a^n$  increases faster than any power function  $n \mapsto n^k$ .

**Example 1.3.13.** For all  $a \in \mathbb{C}$ ,

$$\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0$$

The factorial function  $n \mapsto n!$  increases faster than the function  $n \mapsto a^n$ .

## 1.4 Functions of Single variable

### 1.4.1 Monotone Functions

**Theorem 1.4.1.** Suppose that  $I \subseteq \mathbb{R}$  is a nonempty interval (connected subset of  $\mathbb{R}$ ) and  $f : I \rightarrow \mathbb{R}$  is continuous and strictly increasing (or strictly decreasing).

- (1)  $J := f(I)$  is an interval.
- (2)  $f : I \rightarrow J$  is bijective.
- (3)  $f^{-1} : J \rightarrow I$  is continuous and strictly increasing (or strictly decreasing).

**Example 1.4.2.** For each  $n \in \mathbb{N}^\times$ , the function

$$\mathbb{R}^+ \rightarrow \mathbb{R}^+, \quad x \mapsto \sqrt[n]{x}$$

is continuous and strictly increasing. In addition,  $\lim_{x \rightarrow \infty} \sqrt[n]{x} = \infty$ .

### 1.4.2 Power Series

**Definition 1.4.3.** Let

$$a := \sum a_k X^k := \sum_k a_k X^k$$

be a (formal) power series in one indeterminate with coefficients in  $\mathbb{K}$ . Then, for each  $x \in \mathbb{K}$ ,  $\sum a_k x^k$  is a series in  $\mathbb{K}$ . When this series converges we denote its value by  $\underline{a}(x)$ , the value of the (formal) power series at  $x$ . Set

$$\text{dom}(\underline{a}) := \left\{ x \in \mathbb{K}; \sum a_k x^k \text{ converges in } \mathbb{K} \right\}$$

Then  $\underline{a} : \text{dom}(\underline{a}) \rightarrow \mathbb{K}$  is a well defined function:

$$\underline{a}(x) := \sum_{k=0}^{\infty} a_k x^k, \quad x \in \text{dom}(\underline{a})$$

Note that  $0 \in \text{dom}(\underline{a})$  for any  $a \in \mathbb{K}[[X]]$ . The following examples show that each of the cases

$$\text{dom}(\underline{a}) = \mathbb{K}, \quad \{0\} \subset \text{dom}(\underline{a}) \subset \mathbb{K}, \quad \text{dom}(\underline{a}) = \{0\}$$

is possible.

**Proposition 1.4.4.** For a power series  $a = \sum a_k X^k$  with coefficients in  $\mathbb{K}$  there is a unique  $\rho := \rho_a \in [0, \infty]$  with the following properties:

- (1) The series  $\sum a_k x^k$  converges absolutely if  $|x| < \rho$  and diverges if  $|x| > \rho$ .

(2) Hadamard's formula holds:

$$\rho_a = \frac{1}{\overline{\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|}}}$$

The number  $\rho_a \in [0, \infty]$  is called the radius of convergence of  $a$ , and

$$\rho_a \mathbb{B}_{\mathbb{K}} = \{x \in \mathbb{K}; |x| < \rho_a\}$$

is the disk of convergence of  $a$ .

**Proposition 1.4.5.** Let  $a = \sum a_k X^k$  be a power series such that  $\lim_{n \rightarrow \infty} |a_k/a_{k+1}|$  exists in  $\overline{\mathbb{R}}$ . Then the radius of convergence of  $a$  is given by the formula

$$\rho_a = \lim_{k \rightarrow \infty} \left| \frac{a_k}{a_{k+1}} \right|$$

**Theorem 1.4.6.** Let  $a = \sum a_k X^k$  and  $b = \sum b_k X^k$  be power series with radii of convergence  $\rho_a$  and  $\rho_b$  respectively. Set  $\rho := \min(\rho_a, \rho_b)$ . Then for all  $x \in \mathbb{K}$  such that  $|x| < \rho$  we have

$$\begin{aligned} \sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} b_k x^k &= \sum_{k=0}^{\infty} (a_k + b_k) x^k \\ \left[ \sum_{k=0}^{\infty} a_k x^k \right] \left[ \sum_{k=0}^{\infty} b_k x^k \right] &= \sum_{k=0}^{\infty} \left( \sum_{j=0}^k a_j b_{k-j} \right) x^k \end{aligned}$$

**Proposition 1.4.7.** Let  $\sum a_k X^k$  be a power series with positive radius of convergence  $\rho_a$ . If there is a null sequence  $(y_j)$  such that  $0 < |y_j| < \rho_a$  and

$$\underline{a}(y_j) = \sum_{k=0}^{\infty} a_k y_j^k = 0, \quad j \in \mathbb{N}$$

then  $a_k = 0$  for all  $k \in \mathbb{N}$ , that is,  $a = 0 \in \mathbb{K}[[X]]$ .

**Proposition 1.4.8.** Let  $a = \sum a_k X^k$  be a power series with positive radius of convergence  $\rho_a$ . Then the function  $\underline{a}$  represented by  $a$  is continuous on  $\rho_a \mathbb{B}$ .

**Definition 1.4.9.**

$$\exp : \mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

converges absolutely for all  $z \in \mathbb{C}$ .

**Corollary 1.4.10.**

$$\exp(x + y) = \exp(x) \exp(y)$$

for  $x, y \in \mathbb{C}$

**Definition 1.4.11.**

$$\cos : \mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

and

$$\sin : \mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

are called the cosine and sine functions.

**Definition 1.4.12.** The sequence  $((1 + 1/n)^n)$  converges and its limit

$$e := \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

the Euler number, satisfies  $2 < e \leq 3$ . Moreover, we can show that

$$e = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!}$$

**Proposition 1.4.13.** As an application of this property of the exponential function, we determine the values of the exponential function for rational arguments. Namely,

$$\exp(r) = e^r, \quad r \in \mathbb{Q}$$

that is, for a rational number  $r$ ,  $\exp(r)$  is the  $r^{\text{th}}$  power of  $e$ .

**Proposition 1.4.14** (Euler's formula).

$$e^{iz} = \cos z + i \sin z, \quad z \in \mathbb{C}$$

**Theorem 1.4.15.** For all  $z, w \in \mathbb{C}$  we have <sup>3</sup>

$$\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w$$

$$\sin(z \pm w) = \sin z \cos w \pm \cos z \sin w$$

$$\sin z - \sin w = 2 \cos \frac{z+w}{2} \sin \frac{z-w}{2}$$

$$\cos z - \cos w = -2 \sin \frac{z+w}{2} \sin \frac{z-w}{2}$$

**Proposition 1.4.16.** Define  $\exp_{\mathbb{R}} := e^z|_{\mathbb{R}}$ .

(1) If  $x < 0$ , then  $0 < e^x < 1$ . If  $x > 0$ , then  $1 < e^x < \infty$ .

(2)  $\exp_{\mathbb{R}} : \mathbb{R} \rightarrow \mathbb{R}^+$  is strictly increasing.

(3) For each  $\alpha \in \mathbb{Q}$ ,

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^\alpha} = \infty$$

that is, the exponential function increases faster than any power function.

(4)  $\lim_{x \rightarrow -\infty} e^x = 0$ .

**Proposition 1.4.17.** For all  $a > 0$  and  $r \in \mathbb{Q}$

$$a^r = e^{r \log a}.$$

**Proposition 1.4.18.** For all  $\alpha > 0$ ,

$$\lim_{x \rightarrow \infty} \frac{\log x}{x^\alpha} = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} x^\alpha \log x = 0.$$

In particular, the logarithm function increases more slowly than any (arbitrarily small) positive power function.

**Definition 1.4.19** (Definition of  $\pi$ ). Firstly, notice that

$$|e^{it}|^2 = e^{it}(\overline{e^{it}}) = e^{it}e^{-it} = e^0 = 1, \quad t \in \mathbb{R},$$

Hence, define

$$\text{cis} : \mathbb{R} \rightarrow \mathbb{S}^1, \quad t \mapsto e^{it}$$

Now we show that  $\text{cis}$  is surjective:

Next step, we claim the set  $M := \{t > 0; e^{it} = 1\}$  has a minimum element. First we show that  $M$  is nonempty. Since, there is some  $t \in \mathbb{R}^\times$  such that  $e^{it} = -1$ . Because

$$e^{-it} = \frac{1}{e^{it}} = \frac{1}{-1} = -1$$

we can suppose that  $t > 0$ . Then  $e^{2it} = (e^{it})^2 = (-1)^2 = 1$  and  $M$  is nonempty. Next we show that  $M$  is closed in  $\mathbb{R}$ . To prove this, choose a sequence  $(t_n)$  in  $M$  which converges to  $t^* \in \mathbb{R}$ . Since  $t_n$  is positive for all  $n$ , we have  $t^* \geq 0$ . In addition, the continuity of  $\text{cis}$  implies

$$e^{it^*} = \text{cis}(t^*) = \text{cis}(\lim t_n) = \lim \text{cis}(t_n) = 1$$

To prove that  $M$  is closed, it remains to show that  $t^*$  is positive. Suppose, to the contrary, that  $t^* = 0$ . Then there is some  $m \in \mathbb{N}$  such that  $t_m \in (0, 1)$ . From Euler's formula we have  $1 = e^{it_m} = \cos t_m + i \sin t_m$  and so  $\sin t_m = 0$ .

$$\sin t = t - \frac{t^3}{6} + \left(\frac{t^5}{5!} - \frac{t^7}{7!}\right) + \cdots \geq t - \frac{t^3}{6}$$

we get

$$\sin t \geq t(1 - t^2/6), \quad 0 < t < 1.$$

For  $t_m$ , this yields  $0 = \sin t_m \geq t_m(1 - t_m^2/6) > 5t_m/6$ , a contradiction. Thus  $M$  is closed. Since  $M$  is a nonempty closed set which is bounded below, it has minimum element.

The preceding lemma makes it possible to define a number  $\pi$  by

$$\pi := \frac{1}{2} \min \{t > 0; e^{it} = 1\}$$

**Definition 1.4.20.** The tangent and cotangent functions are defined by

$$\tan z := \frac{\sin z}{\cos z}, \quad z \in \mathbb{C} \setminus \left(\frac{\pi}{2} + \pi\mathbb{Z}\right), \quad \cot z := \frac{\cos z}{\sin z}, \quad z \in \mathbb{C} \setminus \pi\mathbb{Z}.$$

### 1.4.3 Differentiation in One Variable

Setting:  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ,  $X$  be a subset of  $\mathbb{K}$  and  $E$  be a normed vector space over  $\mathbb{K}$ .  $a$  is a limit point of  $X$ .

**Definition 1.4.21.** A function  $f : X \rightarrow E$  is called differentiable at  $a$  if the limit

$$f'(a) := \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

exists in  $E$ . When this occurs,  $f'(a) \in E$  is called the derivative of  $f$  at  $a$ . Besides the symbol  $f'(a)$ , many other notations for the derivative are used:

$$\dot{f}(a), \quad \partial f(a), \quad Df(a), \quad \frac{df}{dx}(a).$$



**Proposition 1.4.22.** (1) (linearity) Let  $f, g : X \rightarrow E$  be differentiable at  $a$  and  $\alpha, \beta \in \mathbb{K}$ . Then the function  $\alpha f + \beta g$  is also differentiable at  $a$  and

$$(\alpha f + \beta g)'(a) = \alpha f'(a) + \beta g'(a).$$

In other words, the set of functions which are differentiable at  $a$  forms a subspace  $V$  of  $E^X$ , and the function  $V \rightarrow E$ ,  $f \mapsto f'(a)$  is linear.

(2) (product rule) Let  $f, g : X \rightarrow \mathbb{K}$  be differentiable at  $a$ . Then the function  $f \cdot g$  is also differentiable at  $a$  and

$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a).$$

The set of functions which are differentiable at  $a$  forms a subalgebra of  $\mathbb{K}^X$ .

(3) (quotient rule) Let  $f, g : X \rightarrow \mathbb{K}$  be differentiable at  $a$  with  $g(a) \neq 0$ . Then the function  $f/g$  is also differentiable at  $a$  and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}$$

**Definition 1.4.23.**  $X$  be a topological space,  $A$  is perfect if for all  $a \in A$ ,  $a$  is a limit point.

**Proposition 1.4.24.** Suppose that  $f : X \rightarrow \mathbb{K}$  is differentiable at  $a$ , and  $f(a)$  is a limit point of  $Y$  with  $f(X) \subseteq Y \subseteq \mathbb{K}$ . If  $g : Y \rightarrow E$  is differentiable at  $f(a)$ , then  $g \circ f$  is differentiable at  $a$  and

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

**Proposition 1.4.25** (differentiability of inverse functions). Let  $f : X \rightarrow \mathbb{K}$  be injective and differentiable at  $a$ . Then,  $f(a)$  is a limit point of  $Y = f(X)$ . Suppose that  $f^{-1} : f(X) \rightarrow X$  is continuous at  $b := f(a)$ . Then  $f^{-1}$  is differentiable at  $b$  if and only if  $f'(a)$  is nonzero. In this case,

$$(f^{-1})'(b) = \frac{1}{f'(a)}, \quad b = f(a)$$

*Proof:*

Setting: Let  $X \subseteq \mathbb{K}$  be perfect.

**Definition 1.4.26.**  $f : X \rightarrow E$  is called differentiable on  $X$  if  $f$  is differentiable at each point of  $X$ . The function

$$f' : X \rightarrow E, \quad x \mapsto f'(x)$$

is called the derivative of  $f$ . It is also denoted by  $\dot{f}, \partial f, Df$  and  $df/dx$ .

If  $f : X \rightarrow E$  is differentiable, then it is natural to ask whether the derivative  $f'$  is itself differentiable. When this occurs  $f$  is said to be twice differentiable and we call  $\partial^2 f := f'' := \partial(\partial f)$  the second derivative of  $f$ . Repeating this process we can define further higher derivatives of  $f$ . Specifically, we set

$$\begin{aligned} \partial^0 f &:= f^{(0)} := f, & \partial^1 f(a) &:= f^{(1)}(a) := f'(a) \\ \partial^{n+1} f(a) &:= f^{(n+1)}(a) := \partial(\partial^n f)(a) \end{aligned}$$

for all  $n \in \mathbb{N}$ . The element  $\partial^n f(a) \in E$  is called the  $n^{\text{th}}$  derivative of  $f$  at  $a$ . The function  $f$  is called  $n$ -times differentiable on  $X$  if the  $n^{\text{th}}$  derivative exists at each  $a \in X$ . If  $f$  is  $n$ -times differentiable and the  $n^{\text{th}}$  derivative

$$\partial^n f : X \rightarrow E, \quad x \mapsto \partial^n f(x)$$

is continuous, then  $f$  is  $n$ -times continuously differentiable. The space of  $n$ -times continuously differentiable functions from  $X$  to  $E$  is denoted by  $C^n(X, E)$ . In particular,  $C^0(X, E) = C(X, E)$  is the space of continuous  $E$ -valued functions on  $X$ . Finally

$$C^\infty(X, E) := \bigcap_{n \in \mathbb{N}} C^n(X, E)$$

If  $E = \mathbb{K}$ , we simply write  $C^n(X, E)$  by  $C^n(X)$ .

**Proposition 1.4.27.** Let  $X \subseteq \mathbb{K}$  be perfect,  $k \in \mathbb{N}$  and  $n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ .

(1) (linearity) For all  $f, g \in C^k(X, E)$  and  $\alpha, \beta \in \mathbb{K}$ ,

$$\alpha f + \beta g \in C^k(X, E) \quad \text{and} \quad \partial^k(\alpha f + \beta g) = \alpha \partial^k f + \beta \partial^k g.$$

Hence  $C^n(X, E)$  is a subspace of  $C(X, E)$  and the differentiation operator

$$\partial : C^{n+1}(X, E) \rightarrow C^n(X, E), \quad f \mapsto \partial f$$

is linear.

(2) (Leibniz' rule) Let  $f, g \in C^k(X)$ . Then  $f \cdot g$  is in  $C^k(X)$  and

$$\partial^k(fg) = \sum_{j=0}^k \binom{k}{j} (\partial^j f) \partial^{k-j} g$$

Hence  $C^n(X)$  is a subalgebra of  $\mathbb{K}^X$ .

**Theorem 1.4.28** (mean value theorem). If  $f \in C([a, b], \mathbb{R})$  is differentiable on  $(a, b)$ , then there is some  $\xi \in (a, b)$  such that

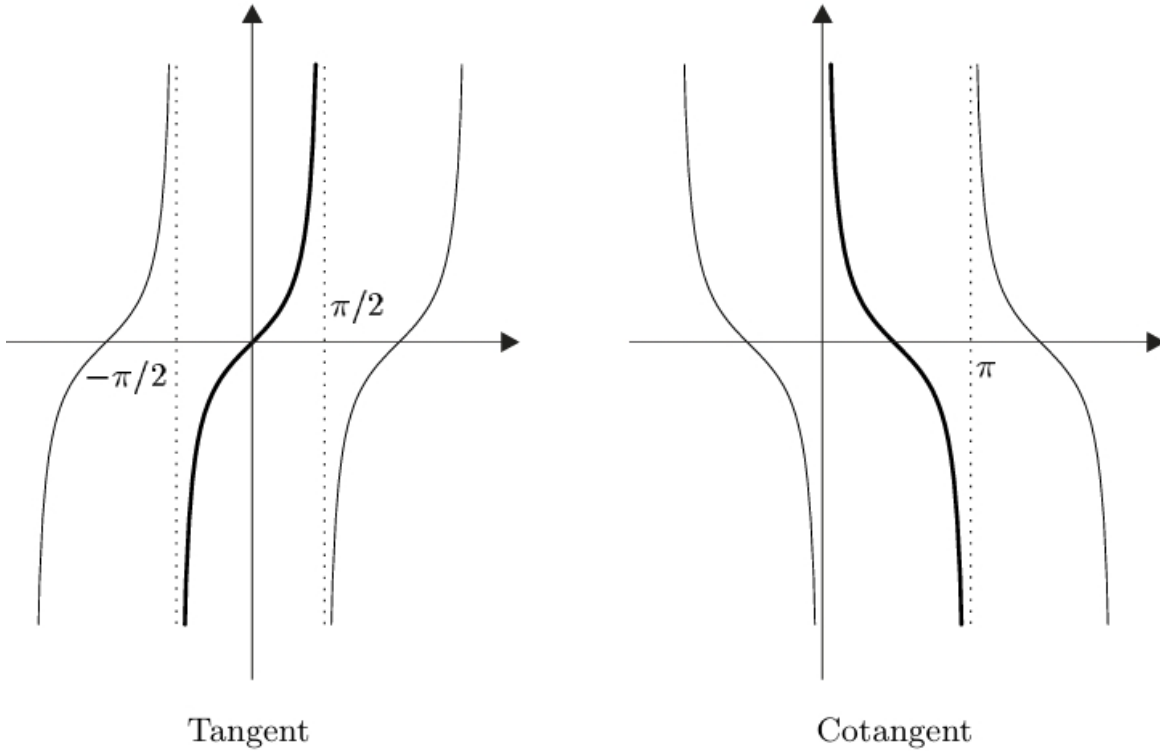
$$f(b) = f(a) + f'(\xi)(b - a).$$

**Theorem 1.4.29** (mean value theorem, second version). Suppose that  $E$  is a normed vector space and  $f \in C([a, b], E)$  is differentiable on  $(a, b)$ . Then

$$\|f(b) - f(a)\| \leq \sup_{t \in (a, b)} \|f'(t)\| (b - a).$$

**Proposition 1.4.30.** Suppose that  $I$  is a perfect interval and  $f \in C(I, \mathbb{R})$  is differentiable on  $I^\circ$ .

(1)  $f$  is increasing (or decreasing) if and only if  $f'(x) \geq 0$  (or  $f'(x) \leq 0$ ) for all  $x \in I^\circ$ .



(2) If  $f'(x) > 0$  (or  $f'(x) < 0$ ) for all  $x \in I^\circ$ , then  $f$  is strictly increasing (or strictly decreasing).

**Proposition 1.4.31.** Suppose that  $I$  is a perfect interval and  $f : I \rightarrow \mathbb{R}$  is differentiable with  $f'(x) \neq 0$  for all  $x \in I$ .

- (1)  $f$  is strictly monotone.
- (2)  $J := f(I)$  is a perfect interval.
- (3)  $f^{-1} : J \rightarrow \mathbb{R}$  is differentiable and  $(f^{-1})'(f(x)) = 1/f'(x)$  for all  $x \in I$ .

**Definition 1.4.32.** Define

$$\arcsin := (\sin \mid (-\pi/2, \pi/2))^{-1} : (-1, 1) \rightarrow (-\pi/2, \pi/2),$$

$$\arccos := (\cos \mid (0, \pi))^{-1} : (-1, 1) \rightarrow (0, \pi),$$

$$\arctan := (\tan \mid (-\pi/2, \pi/2))^{-1} : \mathbb{R} \rightarrow (-\pi/2, \pi/2),$$

$$\operatorname{arccot} := (\cot \mid (0, \pi))^{-1} : \mathbb{R} \rightarrow (0, \pi).$$

To calculate the derivatives of the inverse trigonometric functions we use Theorem 2.8(iii). For the arcsine function this gives

$$\arcsin' x = \frac{1}{\sin' y} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}, \quad x \in (-1, 1),$$

where we have set  $y := \arcsin x$  and used  $x = \sin y$ . Similarly, for the arctangent function,

$$\arctan' x = \frac{1}{\tan' y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}, \quad x \in \mathbb{R},$$

where  $y \in (-\pi/2, \pi/2)$  is determined by  $x = \tan y$ .

The derivatives of the arccosine and arccotangent functions can be calculated the same way and, summarizing, we have

$$\begin{aligned}\arcsin' x &= \frac{1}{\sqrt{1-x^2}}, & \arccos' x &= \frac{-1}{\sqrt{1-x^2}}, & x &\in (-1, 1), \\ \arctan' x &= \frac{1}{1+x^2}, & \operatorname{arccot}' x &= \frac{-1}{1+x^2}, & x &\in \mathbb{R}.\end{aligned}$$

Setting:  $D$  convex perfect subset of  $\mathbb{K}$ ,  $E$  be a Banach space,  $f : D \rightarrow E$  be a function.

**Theorem 1.4.33.** Let  $D$  be convex and  $n \in \mathbb{N}^\times$ . Then, for each  $f \in C^n(D, E)$  and  $a \in D$ , there is a function  $R_n(f, a) \in C(D, E)$  such that

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + R_n(f, a)(x), \quad x \in D$$

The remainder function  $R_n(f, a)$  satisfies

$$\|R_n(f, a)(x)\| \leq \frac{1}{(n-1)!} \sup_{0 < t < 1} \|f^{(n)}(a + t(x-a)) - f^{(n)}(a)\| |x-a|^n$$

for all  $x \in D$ .

Setting:  $\mathbb{K} = \mathbb{R}$  and  $E = \mathbb{R}$ .

**Theorem 1.4.34** (Schlömlich remainder formula). Let  $I$  be a perfect interval,  $a \in I$ ,  $p > 0$  and  $n \in \mathbb{N}$ . Suppose that  $f \in C^n(I, \mathbb{R})$  and  $f^{(n+1)}$  exists on  $I^\circ$ . Then, for each  $x \in I \setminus \{a\}$ , there is some  $\xi := \xi(x) \in (\min\{x, a\}, \max\{x, a\})$  such that

$$R_n(f, a)(x) = \frac{f^{(n+1)}(\xi)}{pn!} \left( \frac{x-\xi}{x-a} \right)^{n-p+1} (x-a)^{n+1}.$$

In particular, take  $p = n+1$  and  $p = 1$  respectively, we obtain

$$R_n(f, a)(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1} \quad (\text{Lagrange})$$

and

$$R_n(f, a)(x) = \frac{f^{(n+1)}(\xi)}{n!} \left( \frac{x-\xi}{x-a} \right)^n (x-a)^{n+1} \quad (\text{Cauchy})$$

**Example 1.4.35.** Taylor series expansion for the general power function:

$$(1+x)^s = \sum_{k=0}^{\infty} \binom{s}{k} x^k, \quad x \in (-1, 1)$$

*Proof:* Step 1: Notice that we only need to check the case when  $s < 0$ .

Step 2: Show that

$$\binom{s}{n} = \prod_{k=1}^n (1 - (s+1)/n) = \mathcal{O}(n)$$

.

Step 3: Show that, for each  $x > -1$ , there are some  $\tau \in (0, 1)$  and  $\tau' \in (0, 1)$  such that

$$(1+x)^s = \sum_{k=0}^n \binom{s}{k} x^k + \binom{s}{n+1} \frac{x^{n+1}}{(1+\tau x)^{n+1-s}} \quad (\text{Lagrange})$$

and

$$(1+x)^s = \sum_{k=0}^n \binom{s}{k} x^k + \binom{s}{n+1} (n+1)(1+\tau' x)^{s-1} x \left( \frac{x-\tau' x}{1+\tau' x} \right)^n \quad (\text{Cauchy})$$

Step 4: For  $-1 < x < 0$ ,

$$\frac{x-\tau' x}{1+\tau' x} = 1 - \frac{x+1}{1+\tau' x} < -x < 1$$

### 1.4.4 Sequences of Functions

Setting:  $X$  is a set and  $E$  be a Banach space over  $\mathbb{K}$ .

**Definition 1.4.36.** An  $E$ -valued sequence of functions on  $X$  is simply a sequence  $(f_n)$  in  $E^X$ . If the choice of  $X$  and  $E$  is clear from the context (or irrelevant) we say simply that  $(f_n)$  is a sequence of functions.

The sequence of functions  $(f_n)$  converges pointwise to  $f \in E^X$  if, for each  $x \in X$ , the sequence  $(f_n(x))$  converges to  $f(x)$  in  $E$ .

**Definition 1.4.37.** A sequence of functions  $(f_n)$  converges uniformly to  $f$  if, for each  $\varepsilon > 0$ , there is some  $N = N(\varepsilon) \in \mathbb{N}$  such that

$$|f_n(x) - f(x)| < \varepsilon, \quad n \geq N, \quad x \in X.$$

**Definition 1.4.38.** Define  $B(X, E)$  be the space of bounded functions. There's a natural norm

$$f \mapsto \sup_{x \in X} |f(x)|$$

on  $B(X, E)$  and we denote it by  $\|\cdot\|_\infty$ . It's easy to check  $(B(X, E), \|\cdot\|_\infty)$  forms a Banach space.

If  $f_n$  and  $f$  are in  $B(X, E)$ , then  $(f_n)$  converges uniformly to  $f$  if and only if  $(f_n)$  converges to  $f$  in  $B(X, E)$ .

**Proposition 1.4.39.** The following are equivalent:

- (1) The sequence of functions  $(f_n)$  converges uniformly.
- (2) For each  $\varepsilon > 0$ , there is some  $N := N(\varepsilon) \in \mathbb{N}$  such that

$$\|f_n - f_m\|_\infty < \varepsilon, \quad n, m \geq N.$$

**Definition 1.4.40.** Let  $(f_k)$  be an  $E$ -valued sequence of functions on  $X$ , that is, a sequence in  $E^X$ . Then

$$s_n := \sum_{k=0}^n f_k \in E^X, \quad n \in \mathbb{N}$$

and so we have a well defined sequence  $(s_n)$  in  $E^X$ .

The series  $\sum f_k$  is called

pointwise convergent  $:\Leftrightarrow \sum f_k(x)$  converges in  $E$  for each  $x \in X$ ,

absolutely convergent  $:\Leftrightarrow \sum |f_k(x)| < \infty$  for each  $x \in X$ ,

uniformly convergent  $:\Leftrightarrow (s_n)$  converges uniformly,

norm convergent  $:\Leftrightarrow \sum \|f_k\|_\infty < \infty$ .

**Remark 1.4.41.** If  $f_k$  is norm convergent, then by Proposition 4.1.4,  $f_k \mapsto f$  for some  $f \in B(X, E)$ . Hence,  $f_k$  is uniformly convergent and absolutely convergent.

**Theorem 1.4.42** (Weierstrass majorant criterion). Suppose that  $f_k \in B(X, E)$  for all  $k \in \mathbb{N}$ . If there is a convergent series  $\sum \alpha_k$  in  $\mathbb{R}$  such that  $\|f_k\|_\infty \leq \alpha_k$  for almost all  $k \in \mathbb{N}$ , then  $\sum f_k$  is norm convergent. In particular,  $\sum f_k$  converges absolutely and uniformly.

**Theorem 1.4.43.** Let  $\sum a_k Y^k$  be a power series with positive radius of convergence  $\rho$  and  $0 < r < \rho$ . Then the series  $\sum a_k Y^k$  is norm convergent on  $r\overline{\mathbb{B}}_{\mathbb{K}}$ . In particular, it converges absolutely and uniformly.

Setting:  $X$  metric space,  $E$  be a Banach space.

**Definition 1.4.44.** A sequence of functions  $(f_n)$  is called locally uniformly convergent if each  $x \in X$  has a neighborhood  $U$  such that  $(f_n|_U)$  converges uniformly. A series of functions  $\sum f_n$  is called locally uniformly convergent if the sequence of partial sums  $(s_n)$  converges locally uniformly.

**Theorem 1.4.45.** If a sequence of continuous functions  $(f_n)$  converges locally uniformly to  $f$ , then  $f$  is also continuous. In other words, locally uniform limits of continuous functions are continuous.

**Theorem 1.4.46** (differentiability of the limits of sequences of functions). Let  $X$  be an open subset of  $\mathbb{K}$  and  $f_n \in C^1(X, E)$  for all  $n \in \mathbb{N}$ . Suppose that there are  $f, g \in E^X$  such that

(1)  $(f_n)$  converges pointwise to  $f$ , and

(2)  $(f'_n)$  converges locally uniformly to  $g$ .

Then  $f$  is in  $C^1(X, E)$ , and  $f' = g$ . In addition,  $(f_n)$  converges locally uniformly to  $f$ .

*Proof:* Step 1:  $f'_n \in C(X, E)$  converges locally uniformly to  $g$ , then  $g \in C(X, E)$

Step 2: Let  $a \in X$ . Then there is some  $r > 0$  such that  $(f'_n)$  converges uniformly to  $g$  on  $B_r := \mathbb{B}_{\mathbb{K}}(a, r) \subset X$ . Thus, for each  $x \in B_r$ , we can apply the mean value theorem to the function

$$[0, 1] \rightarrow E, \quad t \mapsto f_n(a + t(x - a)) - tf'_n(a)(x - a)$$

to get

$$|f_n(x) - f_n(a) - f'_n(a)(x - a)| \leq \sup_{0 < t < 1} |f'_n(a + t(x - a)) - f'_n(a)| |x - a|.$$

Taking the limit  $n \rightarrow \infty$  we get

$$|f(x) - f(a) - g(a)(x - a)| \leq \sup_{0 < t < 1} |g(a + t(x - a)) - g(a)| |x - a|$$

for each  $x \in B_r$ . Step 1 shows that  $g$  is in  $C(X, E)$ , so it follows from (2.1) that

$$f(x) - f(a) - g(a)(x - a) = o(|x - a|)(x \rightarrow a).$$

Hence  $f$  is differentiable at  $a$  and  $f'(a) = g(a)$ . Since this holds for all  $a \in X$ , we have shown that  $f \in C^1(X, E)$ .

Step 3: Now we show that  $f$  is locally uniquely continuous: for  $x \in B_r$ ,

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f(x) - (f_n(a) - f(a))| + |f_n(a) - f(a)| \\ &\leq r \sup_{0 < t < 1} |f'_n(a + t(x - a)) - f'(a + t(x - a))| + |f_n(a) - f(a)| \end{aligned}$$

**Corollary 1.4.47.** Suppose that  $X \subseteq \mathbb{K}$  is open, and  $(f_n)$  is a sequence in  $C^1(X, E)$  for which  $\sum_n f_n$  converges pointwise and  $\sum f'_n$  converges locally uniformly. Then the sum  $\sum_{n=0}^{\infty} f_n$  is in  $C^1(X, E)$  and

$$\left( \sum_{n=0}^{\infty} f_n \right)' = \sum_{n=0}^{\infty} f'_n$$

In addition,  $\sum f_n$  converges locally uniformly.

Setting: Let  $a = \sum_k a_k X^k \in \mathbb{K}[[X]]$  be a power series with radius of convergence  $\rho = \rho_a > 0$ , and  $\underline{a}$  the function on  $\rho\mathbb{B}_{\mathbb{K}}$  represented by  $a$ . When no misunderstanding is possible, we write  $\mathbb{B}$  for  $\mathbb{B}_{\mathbb{K}}$ .

**Theorem 1.4.48.** Let  $a = \sum_k a_k X^k$  be a power series. Then  $\underline{a}$  is continuously differentiable on  $\rho\mathbb{B}$ . The 'termwise differentiated' series  $\sum_{k \geq 1} k a_k X^{k-1}$  has radius of convergence  $\rho$  and

$$\underline{a}'(x) = \left( \sum_{k=0}^{\infty} a_k x^k \right)' = \sum_{k=1}^{\infty} k a_k x^{k-1}, \quad x \in \rho\mathbb{B}.$$

**Corollary 1.4.49.** If  $a = \sum a_k X^k$  is a power series with positive radius of convergence  $\rho$ , then  $\underline{a} \in C^\infty(\rho\mathbb{B}, \mathbb{K})$  and  $a_k = \underline{a}^{(k)}(0)/k!$ .

**Definition 1.4.50.** Let  $D$  be open in  $\mathbb{K}$ . A function  $f : D \rightarrow \mathbb{K}$  is called analytic (on  $D$ ) if, for each  $x_0 \in D$ , there is some  $r = r(x_0) > 0$  such that  $\mathbb{B}(x_0, r) \subseteq D$  and a power series  $\sum_k a_k X^k$  with radius of convergence  $\rho \geq r$ , such that

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k, \quad x \in \mathbb{B}(x_0, r).$$

In this case, we say that  $\sum_k a_k (X - x_0)^k$  is the power series expansion for  $f$  at  $x_0$ . The set of all analytic functions on  $D$  is denoted by  $C^\omega(D, \mathbb{K})$ , or by  $C^\omega(D)$  if no misunderstanding is possible. Further,  $f \in C^\omega(D)$  is called real (or complex) analytic if  $\mathbb{K} = \mathbb{R}$  (or  $\mathbb{K} = \mathbb{C}$ ).

## 1.5 Functions of Several Variables



# Chapter 2

## Measure

### 2.1 Measure Space

**Definition 2.1.1.** Let  $X$  be a nonempty set. An algebra of sets on  $X$  is a nonempty collection  $\mathcal{A}$  of subsets of  $X$  that is closed under finite unions and complements; in other words, if  $E_1, \dots, E_n \in \mathcal{A}$ , then  $\bigcup_1^n E_j \in \mathcal{A}$ ; and if  $E \in \mathcal{A}$ , then  $E^c \in \mathcal{A}$ . A  $\sigma$ -algebra is an algebra that is closed under countable unions.

We observe that since  $\bigcap_j E_j = \left(\bigcup_j E_j^c\right)^c$ , algebras (resp.  $\sigma$ -algebras) are also closed under finite (resp. countable) intersections. Moreover, if  $\mathcal{A}$  is an algebra, then  $\emptyset \in \mathcal{A}$  and  $X \in \mathcal{A}$ , for if  $E \in \mathcal{A}$  we have  $\emptyset = E \cap E^c$  and  $X = E \cup E^c$ .

**Definition 2.1.2.** If  $X$  is any topological space, the  $\sigma$ -algebra generated by the family of open sets in  $X$  is called the Borel  $\sigma$ -algebra on  $X$  and is denoted by  $\mathcal{B}_X$ . Its members are called Borel sets.  $\mathcal{B}_X$  thus includes open sets, closed sets, countable intersections of open sets, countable unions of closed sets, and so forth.

There is a standard terminology for the levels in this hierarchy. A countable intersection of open sets is called a  $G_\delta$  set; a countable union of closed sets is called an  $F_\sigma$  set.

**Definition 2.1.3.** Let  $\{X_\alpha\}_{\alpha \in A}$  be an indexed collection of nonempty sets,  $X = \prod_{\alpha \in A} X_\alpha$ , and  $\pi_\alpha : X \rightarrow X_\alpha$  the coordinate maps. If  $\mathcal{M}_\alpha$  is a  $\sigma$ -algebra on  $X_\alpha$  for each  $\alpha$ , the product  $\sigma$ -algebra on  $X$  is the  $\sigma$ -algebra generated by

$$\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}.$$

We denote this  $\sigma$ -algebra by  $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ . (If  $A = \{1, \dots, n\}$  we also write  $\bigotimes_1^n \mathcal{M}_j$  or  $\mathcal{M}_1 \otimes \dots \otimes \mathcal{M}_n$ .)

**Proposition 2.1.4.** If  $A$  is countable, then  $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$  is the  $\sigma$ -algebra generated by

$$\left\{ \prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha \right\}$$

.

**Proposition 2.1.5.** Let  $X_1, \dots, X_n$  be topological spaces and let  $X = \prod_1^n X_j$ , equipped with the product topology. Then  $\bigotimes_1^n \mathcal{B}_{X_j} \subset \mathcal{B}_X$ . If the  $X_j$ 's are second countable, then  $\bigotimes_1^n \mathcal{B}_{X_j} = \mathcal{B}_X$ .

**Proposition 2.1.6.** Define an elementary family to be a collection  $\mathcal{E}$  of subsets of  $X$  such that

- (1)  $\emptyset \in \mathcal{E}$ ,
- (2) If  $E, F \in \mathcal{E}$  then  $E \cap F \in \mathcal{E}$ ,
- (3) If  $E \in \mathcal{E}$  then  $E^c$  is a finite disjoint union of members of  $\mathcal{E}$ .

If  $\mathcal{E}$  is an elementary family, the collection  $\mathcal{A}$  of finite disjoint unions of members of  $\mathcal{E}$  is an algebra.

**Proposition 2.1.7.**  $X$  is a topological space,  $Y \in \mathcal{B}_X$  be a measurable set. Give  $Y$  the subspace topology from  $X$ , then  $\mathcal{B}_Y$  equals to the  $\sigma$ -algebra  $\{Y \cap E : E \in \mathcal{B}_X\}$ .

**Definition 2.1.8.** Let  $X$  be a set equipped with a  $\sigma$ -algebra  $\mathcal{M}$ . A measure on  $\mathcal{M}$  (or on  $(X, \mathcal{M})$ , or simply on  $X$  if  $\mathcal{M}$  is understood) is a function  $\mu : \mathcal{M} \rightarrow [0, \infty]$  such that

- (1)  $\mu(\emptyset) = 0$ ,
- (2) if  $\{E_j\}_1^\infty$  is a sequence of disjoint sets in  $\mathcal{M}$ , then  $\mu(\bigcup_1^\infty E_j) = \sum_1^\infty \mu(E_j)$ .

If  $X$  is a set and  $\mathcal{M} \subset \mathcal{P}(X)$  is a  $\sigma$ -algebra,  $(X, \mathcal{M})$  is called a measurable space and the sets in  $\mathcal{M}$  are called measurable sets. If  $\mu$  is a measure on  $(X, \mathcal{M})$ , then  $(X, \mathcal{M}, \mu)$  is called a measure space.

**Definition 2.1.9.** Let  $(X, \mathcal{M}, \mu)$  be a measure space. Here is some standard terminology concerning the "size" of  $\mu$ . If  $\mu(X) < \infty$  (which implies that  $\mu(E) < \infty$  for all  $E \in \mathcal{M}$  since  $\mu(X) = \mu(E) + \mu(E^c)$ ),  $\mu$  is called finite. If  $X = \bigcup_1^\infty E_j$  where  $E_j \in \mathcal{M}$  and  $\mu(E_j) < \infty$  for all  $j$ ,  $\mu$  is called  $\sigma$ -finite. More generally, if  $E = \bigcup_1^\infty E_j$  where  $E_j \in \mathcal{M}$  and  $\mu(E_j) < \infty$  for all  $j$ , the set  $E$  is said to be  $\sigma$ -finite for  $\mu$ .

If for each  $E \in \mathcal{M}$  with  $\mu(E) = \infty$  there exists  $F \in \mathcal{M}$  with  $F \subset E$  and  $0 < \mu(F) < \infty$ ,  $\mu$  is called semifinite. ( $\sigma$ -finite is semi-finite)

**Example 2.1.10.** Let  $X$  be any nonempty set,  $\mathcal{M} = \mathcal{P}(X)$ , and  $f$  any function from  $X$  to  $[0, \infty]$ . Then  $f$  determines a measure  $\mu$  on  $\mathcal{M}$  by the formula  $\mu(E) = \sum_{x \in E} f(x)$ . Two special cases are of particular significance: If  $f(x) = 1$  for all  $x$ ,  $\mu$  is called counting measure; and if, for some  $x_0 \in X$ ,  $f$  is defined by  $f(x_0) = 1$  and  $f(x) = 0$  for  $x \neq x_0$ ,  $\mu$  is called the point mass or Dirac measure at  $x_0$ .

**Proposition 2.1.11.** Let  $(X, \mathcal{M}, \mu)$  be a measure space.

- (1) (Monotonicity) If  $E, F \in \mathcal{M}$  and  $E \subset F$ , then  $\mu(E) \leq \mu(F)$ .
- (2) (Subadditivity) If  $\{E_j\}_1^\infty \subset \mathcal{M}$ , then  $\mu(\bigcup_1^\infty E_j) \leq \sum_1^\infty \mu(E_j)$ .

- (3) (Continuity from below) If  $\{E_j\}_1^\infty \subset \mathcal{M}$  and  $E_1 \subset E_2 \subset \cdots$ , then  $\mu(\bigcup_1^\infty E_j) = \lim_{j \rightarrow \infty} \mu(E_j)$ .
- (4) (Continuity from above) If  $\{E_j\}_1^\infty \subset \mathcal{M}$ ,  $E_1 \supset E_2 \supset \cdots$ , and  $\mu(E_1) < \infty$ , then  $\mu(\bigcap_1^\infty E_j) = \lim_{j \rightarrow \infty} \mu(E_j)$ .

**Definition 2.1.12.** If  $(X, \mathcal{M}, \mu)$  is a measure space, a set  $E \in \mathcal{M}$  such that  $\mu(E) = 0$  is called a null set. By subadditivity, any countable union of null sets is a null set, a fact which we shall use frequently. If a statement about points  $x \in X$  is true except for  $x$  in some null set, we say that it is true almost everywhere (abbreviated a.e.), or for almost every  $x$ . (If more precision is needed, we shall speak of a  $\mu$ -null set, or  $\mu$ -almost everywhere).

**Definition 2.1.13.** If  $\mu(E) = 0$  and  $F \subset E$ , then  $\mu(F) = 0$  by monotonicity provided that  $F \in \mathcal{M}$ , but in general it need not be true that  $F \in \mathcal{M}$ . A measure whose domain includes all subsets of null sets is called complete. Completeness can sometimes obviate annoying technical points, and it can always be achieved by enlarging the domain of  $\mu$ , as follows.

**Theorem 2.1.14.** Suppose that  $(X, \mathcal{M}, \mu)$  is a measure space. Let  $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$  and  $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}$ . Then  $\overline{\mathcal{M}}$  is a  $\sigma$ -algebra, and there is a unique extension  $\bar{\mu}$  of  $\mu$  to a complete measure on  $\overline{\mathcal{M}}$ .

**Definition 2.1.15** (outer measure). The abstract generalization of the notion of outer area is as follows. An outer measure on a nonempty set  $X$  is a function  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  that satisfies

- (1)  $\mu^*(\emptyset) = 0$ ,
- (2)  $\mu^*(A) \leq \mu^*(B)$  if  $A \subset B$ ,
- (3)  $\mu^*(\bigcup_1^\infty A_j) \leq \sum_1^\infty \mu^*(A_j)$ .

**Proposition 2.1.16.** Let  $\mathcal{E} \subset \mathcal{P}(X)$  and  $\rho : \mathcal{E} \rightarrow [0, \infty]$  be such that  $\emptyset \in \mathcal{E}$ ,  $X \in \mathcal{E}$ , and  $\rho(\emptyset) = 0$ . For any  $A \subset X$ , define

$$\mu^*(A) = \inf \left\{ \sum_1^\infty \mu(E_j) : E_j \in \mathcal{E} \text{ and } A \subset \bigcup_1^\infty E_j \right\}.$$

Then  $\mu^*$  is an outer measure.

**Proposition 2.1.17.** If  $\mu^*$  is an outer measure on  $X$ , a set  $A \subset X$  is called  $\mu^*$ -measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \text{ for all } E \subset X.$$

**Theorem 2.1.18** (Carathéodory's Theorem). If  $\mu^*$  is an outer measure on  $X$ , the collection  $\mathcal{M}$  of  $\mu^*$ -measurable sets is a  $\sigma$ -algebra, and the restriction of  $\mu^*$  to  $\mathcal{M}$  is a complete measure.

**Definition 2.1.19.** If  $\mathcal{A} \subset \mathcal{P}(X)$  is an algebra, a function  $\mu_0 : \mathcal{A} \rightarrow [0, \infty]$  will be called a premeasure if

- (1)  $\mu_0(\emptyset) = 0$ ,
- (2) if  $\{A_j\}_1^\infty$  is a sequence of disjoint sets in  $\mathcal{A}$  such that  $\bigcup_1^\infty A_j \in \mathcal{A}$ , then  $\mu_0(\bigcup_1^\infty A_j) = \sum_1^\infty \mu_0(A_j)$ .

In particular, a premeasure is finitely additive since one can take  $A_j = \emptyset$  for  $j$  large. The notions of finite and  $\sigma$ -finite premeasures are defined just as for measures.

**Theorem 2.1.20.** If  $\mu_0$  is a premeasure on  $\mathcal{A} \subset \mathcal{P}(X)$ , it induces an outer measure on  $X$ , namely,

$$\mu^*(E) = \inf \left\{ \sum_1^\infty \mu_0(A_j) : A_j \in \mathcal{A}, E \subset \bigcup_1^\infty A_j \right\}.$$

then every set in  $\mathcal{A}$  is  $\mu^*$  measurable and  $\mu^*|_{\mathcal{A}} = \mu_0$ .

**Theorem 2.1.21.** Let  $\mathcal{A} \subset \mathcal{P}(X)$  be an algebra,  $\mu_0$  a premeasure on  $\mathcal{A}$ , and  $\mathcal{M}$  the  $\sigma$ -algebra generated by  $\mathcal{A}$ . There exists a measure  $\mu$  on  $\mathcal{M}$  whose restriction to  $\mathcal{A}$  is  $\mu_0$  - namely,  $\mu = \mu^*|_{\mathcal{M}}$  where  $\mu^*$  is given by Proposition 2.1.15. If  $\mu_0$  is  $\sigma$ -finite, then  $\mu$  is the unique extension of  $\mu_0$  to a measure on  $\mathcal{M}$  and the completion of  $\mu$  is  $\mu^*|_{M^*}$  where  $M^*$  is the  $\mu^*$ -measurable sets.

**Example 2.1.22** (Lebesgue-Stieltjes measure). Consider sets of the form  $(a, b]$  or  $(a, \infty)$  or  $\emptyset$ , where  $-\infty \leq a < b < \infty$ . In this section we shall refer to such sets as h-intervals (h for "half-open"). Clearly the intersection of two h-intervals is an h-interval, and the complement of an h-interval is an h-interval or the disjoint union of two h-intervals. Hence the collection  $\mathcal{A}$  of finite disjoint unions of h-intervals is an algebra. Notice that the  $\sigma$ -algebra generated by  $\mathcal{A}$  is  $\mathcal{B}_{\mathbb{R}}$ .

Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing and right continuous. If  $(a_j, b_j]$  ( $j = 1, \dots, n$ ) are disjoint h-intervals, let

$$\mu_0 \left( \bigcup_1^n (a_j, b_j] \right) = \sum_1^n [F(b_j) - F(a_j)]$$

and let  $\mu_0(\emptyset) = 0$ . Then  $\mu_0$  is a premeasure on the algebra  $\mathcal{A}$ .

**Example 2.1.23.** If  $F : \mathbb{R} \rightarrow \mathbb{R}$  is any increasing, right continuous function, there is a unique Borel measure  $\mu_F$  on  $\mathbb{R}$  such that  $\mu_F((a, b]) = F(b) - F(a)$  for all  $a, b$ . If  $G$  is another such function, we have  $\mu_F = \mu_G$  iff  $F - G$  is constant. Conversely, if  $\mu$  is a Borel measure on  $\mathbb{R}$  that is finite on all bounded Borel sets and we define

$$F(x) = \begin{cases} \mu((0, x]) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu((-x, 0]) & \text{if } x < 0 \end{cases}$$

then  $F$  is increasing and right continuous, and  $\mu = \mu_F$ .

**Example 2.1.24** (Lebesgue measure). This is the complete measure  $\mu_F$  associated to the function  $F(x) = x$ , for which the measure of an interval is simply its length. We shall denote it by  $m$ . The domain of  $m$  is called the class of Lebesgue measurable sets, and we shall denote it by  $\mathcal{L}$ . We shall also refer to the restriction of  $m$  to  $\mathcal{B}_{\mathbb{R}}$  as Lebesgue measure.

**Proposition 2.1.25.** If  $E \in \mathcal{L}$ , then  $E + s \in \mathcal{L}$  and  $rE \in \mathcal{L}$  for all  $s, r \in \mathbb{R}$ . Moreover,  $m(E + s) = m(E)$  and  $m(rE) = |r|m(E)$ .

## 2.2 Intergration

**Proposition 2.2.1.**  $f : X \rightarrow Y$  between two sets induces a mapping  $f^{-1} : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ , defined by  $f^{-1}(E) = \{x \in X : f(x) \in E\}$ , which preserves unions, intersections, and complements. Thus, if  $\mathcal{N}$  is a  $\sigma$ -algebra on  $Y$ ,  $\{f^{-1}(E) : E \in \mathcal{N}\}$  is a  $\sigma$ -algebra on  $X$ . If  $(X, \mathcal{M})$  and  $(Y, \mathcal{N})$  are measurable spaces, a mapping  $f : X \rightarrow Y$  is called  $(\mathcal{M}, \mathcal{N})$ -measurable, or just measurable when  $\mathcal{M}$  and  $\mathcal{N}$  are understood, if  $f^{-1}(E) \in \mathcal{M}$  for all  $E \in \mathcal{N}$ .

If  $\mathcal{N}$  is generated by  $\mathcal{E}$ , then  $f : X \rightarrow Y$  is  $(\mathcal{M}, \mathcal{N})$ -measurable iff  $f^{-1}(E) \in \mathcal{M}$  for all  $E \in \mathcal{E}$ .

**Definition 2.2.2.** If  $(X, \mathcal{M})$  is a measurable space, a real- or complex-valued function  $f$  on  $X$  will be called  $\mathcal{M}$ -measurable, or just measurable, if it is  $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$  or  $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$  measurable.  $\mathcal{B}_{\mathbb{R}}$  or  $\mathcal{B}_{\mathbb{C}}$  is always understood as the  $\sigma$ -algebra on the range space unless otherwise specified. In particular,  $f : \mathbb{R} \rightarrow \mathbb{C}$  is Lebesgue (resp. Borel) measurable if is  $(\mathcal{L}, \mathcal{B}_{\mathbb{C}})$  ( resp.  $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{C}})$ ) measurable;

**Proposition 2.2.3.** If  $(X, \mathcal{M})$  is a measurable space and  $f : X \rightarrow \mathbb{R}$ , the following are equivalent:

- (1)  $f$  is  $\mathcal{M}$ -measurable.
- (2)  $f^{-1}((a, \infty)) \in \mathcal{M}$  for all  $a \in \mathbb{R}$ .
- (3)  $f^{-1}([a, \infty)) \in \mathcal{M}$  for all  $a \in \mathbb{R}$ .
- (4)  $f^{-1}((-\infty, a)) \in \mathcal{M}$  for all  $a \in \mathbb{R}$ .
- (5)  $f^{-1}((-\infty, a]) \in \mathcal{M}$  for all  $a \in \mathbb{R}$ .

**Corollary 2.2.4.**  $f : X \rightarrow Y$  is continuous, then  $f$  is  $(B_X, B_Y)$ -measurable.

**Proposition 2.2.5.** A function  $f : X \rightarrow \mathbb{C}$  is  $\mathcal{M}$ -measurable iff  $\operatorname{Re} f$  and  $\operatorname{Im} f$  are  $\mathcal{M}$ -measurable.

**Definition 2.2.6.** It is sometimes convenient to consider functions with values in the extended real number system  $\overline{\mathbb{R}} = [\infty, \infty]$  (with order topology). It is easily verified that  $\mathcal{B}_{\overline{\mathbb{R}}}$  is generated by the rays  $(a, \infty]$  or  $[-\infty, a)$  ( $a \in \mathbb{R}$ ), and we define  $f : X \rightarrow \overline{\mathbb{R}}$  to be  $\mathcal{M}$ -measurable if it is  $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$ -measurable. And we always define  $0 \cdot \infty$  to be 0.

**Proposition 2.2.7.** If  $f, g : X \rightarrow \mathbb{C}$  are  $\mathcal{M}$ -measurable, then so are  $f + g$  and  $fg$ .

**Proposition 2.2.8.** If  $\{f_j\}$  is a sequence of  $\overline{\mathbb{R}}$ -valued measurable functions on  $(X, \mathcal{M})$ , then the functions

$$\begin{aligned} g_1(x) &= \sup_j f_j(x), & g_3(x) &= \overline{\lim}_{j \rightarrow \infty} f_j(x), \\ g_2(x) &= \inf_j f_j(x), & g_4(x) &= \underline{\lim}_{j \rightarrow \infty} f_j(x) \end{aligned}$$

are all measurable.

**Corollary 2.2.9.** If  $f, g : X \rightarrow \overline{\mathbb{R}}$  are measurable, then so are  $\max(f, g)$  and  $\min(f, g)$ .

If  $\{f_j\}$  is a sequence of complex-valued measurable functions and  $f(x) = \lim_{j \rightarrow \infty} f_j(x)$  exists for all  $x$ , then  $f$  is measurable.

**Definition 2.2.10.** We now discuss the functions that are the building blocks for the theory of integration. Suppose that  $(X, \mathcal{M})$  is a measurable space. If  $E \subset X$ , the characteristic function  $\chi_E$  of  $E$  (sometimes called the indicator function of  $E$  and denoted by  $1_E$ ) is defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

It is easily checked that  $\chi_E$  is measurable iff  $E \in \mathcal{M}$ . A simple function on  $X$  is a finite linear combination, with complex coefficients, of characteristic functions of sets in  $\mathcal{M}$ . (We do not allow simple functions to assume the values  $\pm\infty$ .) Equivalently,  $f : X \rightarrow \mathbb{C}$  is simple iff  $f$  is measurable and the range of  $f$  is a finite subset of  $\mathbb{C}$ . Indeed, we have

$$f = \sum_1^n z_j \chi_{E_j}, \text{ where } E_j = f^{-1}(\{z_j\}) \text{ and } \text{range}(f) = \{z_1, \dots, z_n\}.$$

We call this the standard representation of  $f$ . It exhibits  $f$  as a linear combination, with distinct coefficients, of characteristic functions of disjoint sets whose union is  $X$ . Note: One of the coefficients  $z_j$  may well be 0, but the term  $z_j \chi_{E_j}$  is still to be envisioned as part of the standard representation, as the set  $E_j$  may have a role to play when  $f$  interacts with other functions.

**Theorem 2.2.11.** Let  $(X, \mathcal{M})$  be a measurable space. If  $f : X \rightarrow [0, \infty]$  is measurable, there is a sequence  $\{\phi_n\}$  of simple functions such that  $0 \leq \phi_1 \leq \phi_2 \leq \dots \leq f$ ,  $\phi_n \rightarrow f$  pointwise, and  $\phi_n \rightarrow f$  uniformly on any set on which  $f$  is bounded.

If  $f : X \rightarrow \mathbb{C}$  is measurable, there is a sequence  $\{\phi_n\}$  of simple functions such that  $0 \leq |\phi_1| \leq |\phi_2| \leq \dots \leq |f|$ ,  $\phi_n \rightarrow f$  pointwise, and  $\phi_n \rightarrow f$  uniformly on any set on which  $f$  is bounded.

**Definition 2.2.12.** The following implications are valid iff the measure  $\mu$  is complete:

- (1) If  $f$  is measurable and  $f = g$   $\mu$ -a.e., then  $g$  is measurable.
- (2) If  $f_n$  is measurable for  $n \in \mathbb{N}$  and  $f_n \rightarrow f$   $\mu$ -a.e., then  $f$  is measurable.

**Proposition 2.2.13.** Let  $(X, \mathcal{M}, \mu)$  be a measure space and let  $(X, \overline{\mathcal{M}}, \bar{\mu})$  be its completion. If  $f$  is an  $\overline{\mathcal{M}}$ -measurable function on  $X$ , there is an  $\mathcal{M}$ -measurable function  $g$  such that  $f = g\bar{\mu}$ -almost everywhere.

**Definition 2.2.14.** In this section we fix a measure space  $(X, \mathcal{M}, \mu)$ , and we define

$L^+$  = the space of all measurable functions from  $X$  to  $[0, \infty]$ .

If  $\phi$  is a simple function in  $L^+$  with standard representation  $\phi = \sum_1^n a_j \chi_{E_j}$ , we define the integral of  $\phi$  with respect to  $\mu$  by

$$\int \phi d\mu = \sum_1^n a_j \mu(E_j)$$

**Proposition 2.2.15.** Let  $\phi$  and  $\psi$  be simple functions in  $L^+$ .

- (1) If  $c \geq 0$ ,  $\int c\phi = c \int \phi$ .
- (2)  $\int(\phi + \psi) = \int \phi + \int \psi$ .
- (3) If  $\phi \leq \psi$ , then  $\int \phi \leq \int \psi$ .

**Definition 2.2.16.** We now extend the integral to all functions  $f \in L^+$  by defining

$$\int f d\mu = \sup \left\{ \int \phi d\mu : 0 \leq \phi \leq f, \phi \text{ simple} \right\}.$$

**Theorem 2.2.17.** If  $\{f_n\}$  is a sequence in  $L^+$  such that  $f_j \leq f_{j+1}$  for all  $j$ , and  $f = \lim_{n \rightarrow \infty} f_n (= \sup_n f_n)$ , then  $\int f = \lim_{n \rightarrow \infty} \int f_n$ .

**Corollary 2.2.18.** If  $\{f_n\}$  is a finite or infinite sequence in  $L^+$  and  $f = \sum_n f_n$ , then  $\int f = \sum_n \int f_n$ .

**Proposition 2.2.19.** If  $f \in L^+$ , then  $\int f = 0$  iff  $f = 0$  a.e.

**Lemma 2.2.20** (Fatou's lemma,). If  $\{f_n\}$  is any sequence in  $L^+$ , then

$$\int (\liminf f_n) \leq \liminf \int f_n.$$

**Proposition 2.2.21.** The two definitions of  $\int f$  agree when  $f$  is simple, as the family of simple functions over which the supremum is taken includes  $f$  itself and

$$\int f \leq \int g \text{ whenever } f \leq g, \text{ and } \int cf = c \int f \text{ for all } c \in [0, \infty).$$

**Definition 2.2.22.** If  $f^+$  and  $f^-$  are the positive and negative parts of  $f$  and at least one of  $\int f^+$  and  $\int f^-$  is finite, we define

$$\int f = \int f^+ - \int f^-.$$

We shall be mainly concerned with the case where  $\int f^+$  and  $\int f^-$  are both finite; we then say that  $f$  is integrable. Since  $|f| = f^+ + f^-$ , it is clear that  $f$  is integrable iff  $\int |f| < \infty$ .

Next, if  $f$  is a complex-valued measurable function, we say that  $f$  is integrable if  $\int |f| < \infty$ . More generally, if  $E \in \mathcal{M}$ ,  $f$  is integrable on  $E$  if  $\int_E |f| < \infty$ . Since  $|f| \leq |\operatorname{Re} f| + |\operatorname{Im} f| \leq 2|f|$ ,  $f$  is integrable iff  $\operatorname{Re} f$  and  $\operatorname{Im} f$  are both integrable, and in this case we define

$$\int f = \int \operatorname{Re} f + i \int \operatorname{Im} f.$$

It follows easily that the space of complex-valued integrable functions is a complex vector space and that the integral is a complex-linear functional on it. We denote this space - provisionally - by  $L^1(\mu)$  (or  $L^1(X, \mu)$ , or  $L^1(X)$ , or simply  $L^1$ , depending on the context).

**Proposition 2.2.23.** If  $f \in L^1$ , then  $|\int f| \leq \int |f|$ .

**Proposition 2.2.24.** (1) If  $f \in L^1$ , then  $\{x : f(x) \neq 0\}$  is  $\sigma$ -finite.

(2) If  $f, g \in L^1$ , then  $\int_E f = \int_E g$  for all  $E \in \mathcal{M}$  iff  $\int |f - g| = 0$  iff  $f = g$  a.e.

**Remark 2.2.25.**  $(X, M, \mu)$  is a measurable space. Take  $E \in M$  and  $(E, M \cap E, \mu|_E)$  is also measurable space. If  $f \in L^1(E)$ , then

$$f' = \begin{cases} f & x \in E \\ 0 & x \in E^c \end{cases}$$

is a function in  $L^1(X)$  and  $\int_X f' = \int_E f$ .

**Remark 2.2.26.**  $(X, M, \mu)$  is a measurable space.  $(X, M', \tau)$  is another measurable space such that  $M' \supset M$  and  $\tau|_M = \mu$ . Then if  $f \in L^1(X, M)$ ,  $f \in L^1(X, M')$  and values of integration of  $f$  on both measurable spaces are the same. This follows from Theorem 2.2.11 and Monotone Convergence Theorem.

**Theorem 2.2.27** (Dominated Convergence Theorem). Let  $\{f_n\}$  be a sequence in  $L^1$  such that (a)  $f_n \rightarrow f$ , and (b) there exists a nonnegative  $g \in L^1$  such that  $|f_n| \leq g$  for all  $n$ . Then  $f \in L^1$  and  $\int f = \lim_{n \rightarrow \infty} \int f_n$ .

*Proof:* By Fatou's lemma.

**Theorem 2.2.28.** Suppose that  $\{f_j\}$  is a sequence in  $L^1$  such that  $\sum_1^\infty \int |f_j| < \infty$ . Then  $\sum_1^\infty f_j$  converges a.e. to a function in  $L^1$ , and  $\int \sum_1^\infty f_j = \sum_1^\infty \int f_j$ .

**Theorem 2.2.29.** If  $f \in L^1(\mu)$  and  $\epsilon > 0$ , there is an integrable simple function  $\phi = \sum a_j \chi_{E_j}$  such that  $\int |f - \phi| d\mu < \epsilon$ . (That is, the integrable simple functions are dense in  $L^1$  in the  $L^1$  metric.) If  $\mu$  is a Lebesgue measure on  $\mathbb{R}$ , the sets  $E_j$  in the definition of  $\phi$  can be taken to be finite unions of open intervals; moreover, there is a continuous function  $g$  that vanishes outside a bounded interval such that  $\int |f - g| d\mu < \epsilon$ .



**Definition 2.2.30.** Let  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$  be measure spaces. We have already discussed the product  $\sigma$ -algebra  $\mathcal{M} \otimes \mathcal{N}$  on  $X \times Y$ ; we now construct a measure on  $\mathcal{M} \otimes \mathcal{N}$  that is, in an obvious sense, the product of  $\mu$  and  $\nu$ .

To begin with, we define a (measurable) rectangle to be a set of the form  $A \times B$  where  $A \in \mathcal{M}$  and  $B \in \mathcal{N}$ . Clearly

$$(A \times B) \cap (E \times F) = (A \cap E) \times (B \cap F), \quad (A \times B)^c = (X \times B^c) \cup (A^c \times B).$$

Therefore, by Proposition 2.1.6, the collection  $\mathcal{A}$  of finite disjoint unions of rectangles is an algebra, and of course the  $\sigma$ -algebra it generates is  $\mathcal{M} \otimes \mathcal{N}$ .

If we integrate with respect to  $x$

$$\begin{aligned} \mu(A) \chi_B(y) &= \int \chi_A(x) \chi_B(y) d\mu(x) = \sum \int \chi_{A_j}(x) \chi_{B_j}(y) d\mu(x) \\ &= \sum \mu(A_j) \chi_{B_j}(y). \end{aligned}$$

In the same way, integration in  $y$  then yields

$$\mu(A) \nu(B) = \sum \mu(A_j) \nu(B_j).$$

It follows that if  $E \in \mathcal{A}$  is the disjoint union of rectangles  $A_1 \times B_1, \dots, A_n \times B_n$ , and we set

$$\pi(E) = \sum_{j=1}^n \mu(A_j) \nu(B_j)$$

then  $\pi$  is well defined on  $\mathcal{A}$  (since any two representations of  $E$  as a finite disjoint union of rectangles have a common refinement), and  $\pi$  is a premeasure on  $\mathcal{A}$ . Therefore,  $\pi$  generates an outer measure on  $X \times Y$  whose restriction to  $\mathcal{M} \times \mathcal{N}$  is a measure that extends  $\pi$ . We call this measure the product of  $\mu$  and  $\nu$  and denote it by  $\mu \times \nu$ . Moreover, if  $\mu$  and  $\nu$  are  $\sigma$ -finite - say,  $X = \bigcup_1^\infty A_j$  and  $Y = \bigcup_1^\infty B_k$  with  $\mu(A_j) < \infty$  and  $\nu(B_k) < \infty$  - then  $X \times Y = \bigcup_{j,k} A_j \times B_k$ , and  $\mu \times \nu(A_j \times B_k) < \infty$ , so  $\mu \times \nu$  is also  $\sigma$ -finite. Then  $\mu \times \nu$  is the unique measure on  $\mathcal{M} \otimes \mathcal{N}$  such that  $\mu \times \nu(A \times B) = \mu(A) \nu(B)$  for all rectangles  $A \times B$ .

The same construction works for any finite number of factors. That is, suppose  $(X_j, \mathcal{M}_j, \mu_j)$  are measure spaces for  $j = 1, \dots, n$ . If we define a rectangle to be a set of the form  $A_1 \times \dots \times A_n$  with  $A_j \in \mathcal{M}_j$ , then the collection  $\mathcal{A}$  of finite disjoint unions of rectangles is an algebra, and the same procedure as above produces a measure  $\mu_1 \times \dots \times \mu_n$  on  $\mathcal{M}_1 \otimes \dots \otimes \mathcal{M}_n$  such that

$$\mu_1 \times \dots \times \mu_n(A_1 \times \dots \times A_n) = \prod_{j=1}^n \mu_j(A_j).$$

Moreover, if the  $\mu_j$ 's are  $\sigma$ -finite so that the extension from  $\mathcal{A}$  to  $\bigotimes_1' \mathcal{M}_j$  is uniquely determined.

**Proposition 2.2.31.** If  $(X_j, \mathcal{M}_j)$  is a measurable space for  $j = 1, 2, 3$ , then  $\bigotimes_1^3 \mathcal{M}_j = (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$ . Moreover, if  $\mu_j$  is a  $\sigma$ -finite measure on  $(X_j, \mathcal{M}_j)$ , then  $\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$ .

*Proof:* Consider  $\{A \subset M_1 \otimes M_2 : A \times E_3 \in M_1 \otimes M_2 \otimes M_3\}$  for some  $E_3 \in M_3$  is a  $\sigma$ -algebra.

**Definition 2.2.32.** We return to the case of two measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ . If  $E \subset X \times Y$ , for  $x \in X$  and  $y \in Y$  we define the  $x$ -section  $E_x$  and the  $y$ -section  $E^y$  of  $E$  by

$$E_x = \{y \in Y : (x, y) \in E\}, \quad E^y = \{x \in X : (x, y) \in E\}.$$

Also, if  $f$  is a function on  $X \times Y$  we define the  $x$ -section  $f_x$  and the  $y$ -section  $f^y$  of  $f$  by

$$f_x(y) = f^y(x) = f(x, y).$$

**Proposition 2.2.33.** If  $E \in \mathcal{M} \otimes \mathcal{N}$ , then  $E_x \in \mathcal{N}$  for all  $x \in X$  and  $E^y \in \mathcal{M}$  for all  $y \in Y$ .

If  $f$  is  $\mathcal{M} \otimes \mathcal{N}$ -measurable, then  $f_x$  is  $\mathcal{N}$ -measurable for all  $x \in X$  and  $f^y$  is  $\mathcal{M}$ -measurable for all  $y \in Y$ .

**Theorem 2.2.34.** Suppose  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$  are  $\sigma$ -finite measure spaces. If  $E \in \mathcal{M} \otimes \mathcal{N}$ , then the functions  $x \mapsto \nu(E_x)$  and  $y \mapsto \mu(E^y)$  are measurable on  $X$  and  $Y$ , respectively, and

$$\mu \times \nu(E) = \int \nu(E_x) d\mu(x) = \int \mu(E^y) d\nu(y).$$

**Theorem 2.2.35** (The Fubini-Tonelli Theorem). Suppose that  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$  are  $\sigma$  finite measure spaces.

- (1) (Tonelli) If  $f \in L^+(X \times Y)$ , then the functions  $g(x) = \int f_x d\nu$  and  $h(y) = \int f^y d\mu$  are in  $L^+(X)$  and  $L^+(Y)$ , respectively, and

$$\begin{aligned} \int f d(\mu \times \nu) &= \int \left[ \int f(x, y) d\nu(y) \right] d\mu(x) \\ &= \int \left[ \int f(x, y) d\mu(x) \right] d\nu(y). \end{aligned}$$

- (2) (Fubini) If  $f \in L^1(\mu \times \nu)$ , then  $f_x \in L^1(\nu)$  for a.e.  $x \in X$ ,  $f^y \in L^1(\mu)$  for a.e.  $y \in Y$ , Define

$$g(x) = \begin{cases} \int f_x & \text{if } f_x \in L^1(\nu) \\ 0 & \text{otherwise} \end{cases}$$

$$h(y) = \begin{cases} \int f^y & \text{if } f^y \in L^1(\mu) \\ 0 & \text{otherwise} \end{cases}$$

, we have  $g(x) \in L^1(\mu)$ ,  $h(y) \in L^1(\nu)$  and  $\int g(x) d\mu = \int h(y) d\nu = \int f d(\mu \times \nu)$ .

**Definition 2.2.36.** Lebesgue measure  $m^n$  on  $\mathbb{R}^n$  is the completion of the  $n$ -fold product of Lebesgue measure on  $\mathbb{R}$  with itself, that is, the completion of  $m \times \cdots \times m$  on  $\mathcal{B}_{\mathbb{R}} \otimes \cdots \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^n}$ .

**Proposition 2.2.37.** Lebesgue measure is translation-invariant. More precisely, for  $a \in \mathbb{R}^n$  define  $\tau_a : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $\tau_a(x) = x + a$ .

- (1) If  $E \in \mathcal{L}^n$ , then  $\tau_a(E) \in \mathcal{L}^n$  and  $m(\tau_a(E)) = m(E)$ .
- (2) If  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  is Lebesgue measurable, then so is  $f \circ \tau_a$ . Moreover, if either  $f \geq 0$  or  $f \in L^1(m)$ , then  $\int (f \circ \tau_a) dm = \int f dm$ .

**Theorem 2.2.38.** Suppose  $T \in GL(n, \mathbb{R})$ .

- (1) If  $f$  is a Lebesgue measurable function on  $\mathbb{R}^n$ , so is  $f \circ T$ . If  $f \geq 0$  or  $f \in L^1(m)$ , then

$$\int f(x) dx = |\det T| \int f \circ T(x) dx.$$

- (2) If  $E \in \mathcal{L}^n$ , then  $T(E) \in \mathcal{L}^n$  and  $m(T(E)) = |\det T| m(E)$ .

**Theorem 2.2.39** (Change of Variables). Let  $G = (g_1, \dots, g_n)$  be a map from an open set  $\Omega \subset \mathbb{R}^n$  into  $\mathbb{R}^n$  whose components  $g_j$  are of class  $C^1$ .  $G$  is called a  $C^1$  diffeomorphism if  $G$  is injective and  $D_x G$  is invertible for all  $x \in \Omega$ . In this case, the inverse function theorem guarantees that  $G(\Omega)$  is open and  $G^{-1} : G(\Omega) \rightarrow \Omega$  is also a  $C^1$  diffeomorphism and that  $D_x(G^{-1}) = [D_{G^{-1}(x)} G]^{-1}$  for all  $x \in G(\Omega)$ .

Suppose that  $\Omega$  is an open set in  $\mathbb{R}^n$  and  $G : \Omega \rightarrow \mathbb{R}^n$  is a  $C^1$  diffeomorphism.

- (1) If  $f$  is a Lebesgue measurable function on  $G(\Omega)$ , then  $f \circ G$  is Lebesgue measurable on  $\Omega$ . If  $f \geq 0$  or  $f \in L^1(G(\Omega), m)$ , then

$$\int_{G(\Omega)} f(x) dx = \int_{\Omega} f \circ G(x) |\det D_x G| dx.$$

- (2) If  $E \subset \Omega$  and  $E \in \mathcal{L}^n$ , then  $G(E) \in \mathcal{L}^n$  and  $m(G(E)) = \int_E |\det D_x G| dx$ .

## 2.3 Signed Measure and Complex Measure

**Definition 2.3.1.** Let  $(X, \mathcal{M})$  be a measurable space. A signed measure on  $(X, \mathcal{M})$  is a function  $\nu : \mathcal{M} \rightarrow [-\infty, \infty]$  such that

- (1)  $\nu(\emptyset) = 0$
- (2)  $\nu$  assumes at most one of the values  $\pm\infty$ ;
- (3) if  $\{E_j\}$  is a sequence of disjoint sets in  $\mathcal{M}$ , then  $\nu(\bigcup_1^\infty E_j) = \sum_1^\infty \nu(E_j)$ , where the latter sum converges absolutely if  $\nu(\bigcup_1^\infty E_j)$  is finite.

**Definition 2.3.2.** If  $\nu$  is a signed measure on  $(X, \mathcal{M})$ , a set  $E \in \mathcal{M}$  is called positive (resp. negative, null) for  $\nu$  if  $\nu(F) \geq 0$  (resp.  $\nu(F) \leq 0, \nu(F) = 0$ ) for all  $F \in \mathcal{M}$  such that  $F \subset E$ .

**Definition 2.3.3** (mutually singular). Two signed measures  $\mu$  and  $\nu$  on  $(X, \mathcal{M})$  are mutually singular, or that  $\nu$  is singular with respect to  $\mu$ , if there exist  $E, F \in \mathcal{M}$  such that  $E \cap F = \emptyset, E \cup F = X, E$  is null for  $\mu$ , and  $F$  is null for  $\nu$ . We express this relationship symbolically with the perpendicularity sign:

$$\mu \perp \nu.$$

**Theorem 2.3.4** (Jordan Decomposition Theorem). If  $\nu$  is a signed measure, there exist unique positive measures  $\nu^+$  and  $\nu^-$  such that  $\nu = \nu^+ - \nu^-$  and  $\nu^+ \perp \nu^-$ .

Moreover, if  $\nu$  omits  $-\infty$ ,  $\mu^-$  is finite and if  $\nu$  omits  $\infty$ ,  $\nu^+$  is finite.

**Remark 2.3.5.** The measures  $\nu^+$  and  $\nu^-$  are called the positive and negative variations of  $\nu$ , and  $\nu = \nu^+ - \nu^-$  is called the Jordan decomposition of  $\nu$ . Furthermore, we define the total variation of  $\nu$  to be the measure  $|\nu|$  defined by

$$|\nu| = \nu^+ + \nu^-.$$

**Definition 2.3.6.** Integration with respect to a signed measure  $\nu$  is defined in the obvious way: We set

$$\begin{aligned} L^1(\nu) &= L^1(\nu^+) \cap L^1(\nu^-) \\ \int f d\nu &= \int f d\nu^+ - \int f d\nu^- \quad (f \in L^1(\nu)). \end{aligned}$$

One more piece of terminology: a signed measure  $\nu$  is called finite (resp.  $\sigma$ -finite) if  $|\nu|$  is finite (resp.  $\sigma$ -finite).

**Proposition 2.3.7.**  $E \in \mathcal{M}$  is  $\nu$ -null iff  $|\nu|(E) = 0$ , and  $\nu \perp \mu$  iff  $|\nu| \perp \mu$  iff  $\nu^+ \perp \mu$  and  $\nu^- \perp \mu$ .

**Definition 2.3.8.** Suppose that  $\nu$  is a signed measure and  $\mu$  is a positive measure on  $(X, \mathcal{M})$ . We say that  $\nu$  is absolutely continuous with respect to  $\mu$  and write

$$\nu \ll \mu$$

if  $\nu(E) = 0$  for every  $E \in \mathcal{M}$  for which  $\mu(E) = 0$ . Absolute continuity is in a sense the

**Proposition 2.3.9.**  $\nu \ll \mu$  iff  $|\nu| \ll \mu$  iff  $\nu^+ \ll \mu$  and  $\nu^- \ll \mu$ .

**Proposition 2.3.10.** If  $\nu \perp \mu$  and  $\nu \ll \mu$ , then  $\nu = 0$ .

**Proposition 2.3.11.** Let  $\nu$  be a finite signed measure and  $\mu$  a positive measure on  $(X, \mathcal{M})$ . Then  $\nu \ll \mu$  iff for every  $\epsilon > 0$  there exists  $\delta > 0$  such that  $|\nu(E)| < \epsilon$  whenever  $\mu(E) < \delta$ .

*Proof:* If there's  $\epsilon > 0$  such that for all  $n > 0$ , there's  $E_n \in \mathcal{M}$  such that  $\mu(E_n) < 2^{-n}$  and  $\nu(E_n) \geq \epsilon$ . Consider the set

$$\limsup E_n = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k$$

**Corollary 2.3.12.** If  $f \in L^1(\mu)$ , for every  $\epsilon > 0$  there exists  $\delta > 0$  such that  $|\int_E f d\mu| < \epsilon$  whenever  $\mu(E) < \delta$ .

**Definition 2.3.13.** A complex measure on a measurable space  $(X, \mathcal{M})$  is a map  $\nu : \mathcal{M} \rightarrow \mathbb{C}$  such that

(1)  $\nu(\emptyset) = 0$ ;

(2) if  $\{E_j\}$  is a sequence of disjoint sets in  $\mathcal{M}$ , then  $\nu(\bigcup_1^\infty E_j) = \sum_1^\infty \nu(E_j)$ , where the series converges absolutely.

**Example 2.3.14.** If  $\mu$  is a positive measure and  $f \in L^1(\mu)$ , then  $f d\mu$  is a complex measure.

If  $\nu$  is a complex measure, we shall write  $\nu_r$  and  $\nu_i$  for the real and imaginary parts of  $\nu$ . Thus  $\nu_r$  and  $\nu_i$  are signed measures that do not assume the values  $\pm\infty$ ; hence they are finite, and so the range of  $\nu$  is a bounded subset of  $\mathbb{C}$ .

The notions we have developed for signed measures generalize easily to complex measures. For example, we define  $L^1(\nu)$  to be  $L^1(\nu_r) \cap L^1(\nu_i)$ , and for  $f \in L^1(\nu)$ , we set  $\int f d\nu = \int f d\nu_r + i \int f d\nu_i$ . If  $\nu$  and  $\mu$  are complex measures, we say that  $\nu \perp \mu$  if  $\nu_a \perp \mu_b$  for  $a, b = r, i$ , and if  $\lambda$  is a positive measure, we say that  $\nu \ll \lambda$  if  $\nu_r \ll \lambda$  and  $\nu_i \ll \lambda$ .

**Theorem 2.3.15** (Lebesgue-Radon-Nikodym Theorem). If  $\nu$  is a complex measure and  $\mu$  is a  $\sigma$ -finite positive measure on  $(X, \mathcal{M})$ , there exist a complex measure  $\lambda$  and an  $f \in L^1(\mu)$  such that  $\lambda \perp \mu$  and  $d\nu = d\lambda + f d\mu$ . If also  $\lambda' \perp \mu$  and  $d\nu = d\lambda' + f' d\mu$ , then  $\lambda = \lambda'$  and  $f = f' \mu$ -a.e.

**Definition 2.3.16** (total variation of complex measure). If  $\nu$  is a complex measure and  $\nu_r$  and  $\nu_i$  be the real part and imaginary part of  $\nu$ . Take a  $\sigma$ -finite positive measure  $\mu$  on  $X$ , for example  $|\nu_r| + |\nu_i|$ , such that  $\nu \ll \mu$ . By Lebesgue-Radon-Nikodym Theorem,  $\nu = f d\mu$  for some  $f \in L^1(\mu)$ . Define total variation of  $\nu$  by  $|f| d\mu$ . This definition is independent of the choice of  $f$  and  $\mu$ .

**Proposition 2.3.17.** Let  $\nu$  be a complex measure on  $(X, \mathcal{M})$ .

- (1)  $|\nu(E)| \leq |\nu|(|E|)$  for all  $E \in \mathcal{M}$ .
- (2)  $\nu \ll |\nu|$
- (3)  $L^1(\nu) = L^1(|\nu|)$ , and if  $f \in L^1(\nu)$ , then  $|\int f d\nu| \leq \int |f| d|\nu|$ .

**Definition 2.3.18.** A measurable function  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  is called locally integrable (with respect to Lebesgue measure) if  $\int_K |f(x)| dx < \infty$  for every bounded measurable set  $K \subset \mathbb{R}^n$ .

We denote the space of locally integrable functions by  $L^1_{\text{loc}}$ . If  $f \in L^1_{\text{loc}}$ ,  $x \in \mathbb{R}^n$ , and  $r > 0$ , we define  $A_r f(x)$  to be the average value of  $f$  on  $B(r, x)$ :

$$A_r f(x) = \frac{1}{m(B(r, x))} \int_{B(r, x)} f(y) dy$$

**Theorem 2.3.19** (The Lebesgue Differentiation Theorem). Let us define the Lebesgue set  $L_f$  of  $f$  to be

$$L_f = \left\{ x : \lim_{r \rightarrow 0} \frac{1}{m(B(r, x))} \int_{B(r, x)} |f(y) - f(x)| dy = 0 \right\}.$$

If  $f \in L^1_{\text{loc}}$ , then  $m((L_f)^c) = 0$ .

**Definition 2.3.20.** A family  $\{E_r\}_{r>0}$  of Borel subsets of  $\mathbb{R}^n$  is said to shrink nicely to  $x \in \mathbb{R}^n$  if  $E_r \subset B(r, x)$  for each  $r$ ; and there is a constant  $\alpha > 0$ , independent of  $r$ , such that  $m(E_r) > \alpha m(B(r, x))$ .

**Theorem 2.3.21.** Let  $\nu$  be a regular complex Borel measure on  $\mathbb{R}^n$ , and let  $d\nu = d\lambda + f dm$  be its Lebesgue-Radon-Nikodym representation. Then for  $m$ -almost every  $x \in \mathbb{R}^n$ ,

$$\lim_{r \rightarrow 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

where  $E_r$  shrinks nicely to 0.

## 2.4 Function of bounded variation

**Definition 2.4.1.** If  $F : \mathbb{R} \rightarrow \mathbb{C}$  and  $x \in \mathbb{R}$ , we define

$$T_F(x) = \sup \left\{ \sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x \right\}.$$

$T_F$  is called the total variation function of  $F$ .

$T_F$  is an increasing function with values in  $[0, \infty]$ . If  $T_F(\infty) = \lim_{x \rightarrow \infty} T_F(x)$  is finite, we say that  $F$  is of bounded variation on  $\mathbb{R}$ , and we denote the space of all such  $F$  by  $BV$ .

**Proposition 2.4.2.** We observe that the sums in the definition of  $T_F$  are made bigger if the additional subdivision points  $x_j$  are added. Hence, if  $a < b$ , the definition of  $T_F(b)$  is unaffected if we assume that  $a$  is always one of the subdivision points. It follows that

$$T_F(b) = T_F(a) + \sup \left\{ \sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, a = x_0 < \cdots < x_n = b \right\}$$

**Definition 2.4.3.** Define  $BV([a, b])$  to be the set of all functions on  $[a, b]$  whose total variation

$$\sup \left\{ \sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, a = x_0 < \cdots < x_n = b \right\}$$

is finite.

If  $F \in BV$ , the restriction of  $F$  to  $[a, b]$  is in  $BV([a, b])$  for all  $a, b$ ; indeed, its total variation on  $[a, b]$  is nothing but  $T_F(b) - T_F(a)$ . Conversely, if  $F \in BV([a, b])$  and we set  $F(x) = F(a)$  for  $x < a$  and  $F(x) = F(b)$  for  $x > b$ , then  $F \in BV$ . By this device the results that we shall prove for  $BV$  can also be applied to  $BV([a, b])$ .

**Proposition 2.4.4.** (1) If  $F : \mathbb{R} \rightarrow \mathbb{R}$  is bounded and increasing, then  $F \in BV$ .

(2) If  $F, G \in BV$  and  $a, b \in \mathbb{C}$ , then  $aF + bG \in BV$ .

(3) If  $F$  is differentiable on  $\mathbb{R}$  and  $F'$  is bounded, then  $F \in BV([a, b])$  for  $-\infty < a < b < \infty$  (by the mean value theorem).

**Proposition 2.4.5.** Define the normalized bounded variation function space to be

$$NBV = \{F \in BV : F \text{ is right continuous and } F(-\infty) = 0\}$$

If  $F \in BV$ , then  $T_F(-\infty) = 0$ . If  $F$  is also right continuous, then so is  $T_F$ . If  $F \in NBV$ ,  $T_F$  is also in  $NBV$ .

**Proposition 2.4.6.** If  $\mu$  is a complex Borel measure on  $\mathbb{R}$  and  $F(x) = \mu((-\infty, x])$ , then  $F \in NBV$ . Conversely, if  $F \in NBV$ , there is a unique complex Borel measure  $\mu_F$  such that  $F(x) = \mu_F((-\infty, x])$ . Moreover,  $|\mu_F| = \mu_{T_F}$ .

**Proposition 2.4.7.** A function  $F : \mathbb{R} \rightarrow \mathbb{C}$  is called absolutely continuous if for every  $\epsilon > 0$  there exists  $\delta > 0$  such that for any finite set of disjoint intervals  $(a_1, b_1), \dots, (a_N, b_N)$ ,

$$\sum_1^N (b_j - a_j) < \delta \implies \sum_1^N |F(b_j) - F(a_j)| < \epsilon.$$

More generally,  $F$  is said to be absolutely continuous on  $[a, b]$  if this condition is satisfied whenever the intervals  $(a_j, b_j)$  all lie in  $[a, b]$ . Clearly, if  $F$  is absolutely continuous, then  $F$  is uniformly continuous. On the other hand, if  $F$  is everywhere differentiable and  $F'$  is bounded, then  $F$  is absolutely continuous, for  $|F(b_j) - F(a_j)| \leq (\max |F'|)(b_j - a_j)$  by the mean value theorem.

If  $F \in NBV$ , then  $F$  is absolutely continuous iff  $\mu_F \ll m$ .

**Theorem 2.4.8.** If  $F \in NBV$ , then  $F$  is differentiable almost  $m$ -everywhere (In this section we only consider Borel measure). Take a set  $M \subset \mathbb{R}$  such that  $F$  is differentiable on  $M$  and its complement is Borel null-set. We have

$$F'(x) = \lim_{n \rightarrow \infty} \frac{F(x + \frac{1}{n}) - F(x)}{\frac{1}{n}} \cdot \chi_M \in L^1(m)$$

Moreover,  $\mu_F \perp m$  iff  $F' = 0$  a.e., and  $\mu_F \ll m$  iff  $F(x) = \int_{-\infty}^x F'(t)dt$ .

*Proof:* Since total variation of  $\mu_F$  is regular, the theorem follows from Theorem 2.3.21.

**Theorem 2.4.9.** If  $f \in L^1(m)$ , then the function  $F(x) = \int_{-\infty}^x f(t)dt$  is in  $NBV$  and is absolutely continuous, and  $f = F'$  a.e. Conversely, if  $F \in NBV$  is absolutely continuous, then  $F' \in L^1(m)$  and  $F(x) = \int_{-\infty}^x F'(t)dt$ .

**Theorem 2.4.10.** If  $-\infty < a < b < \infty$  and  $F : [a, b] \rightarrow \mathbb{C}$ , the following are equivalent:

- (1)  $F$  is absolutely continuous on  $[a, b]$ .
- (2)  $F(x) - F(a) = \int_a^x f(t)dt$  for some  $f \in L^1([a, b], m)$ .
- (3)  $F$  is differentiable a.e. on  $[a, b]$ ,  $F' \in L^1([a, b], m)$ , and  $F(x) - F(a) = \int_a^x F'(t)dt$ .

**Theorem 2.4.11** (integrate by part). If  $F$  and  $G$  are in  $NBV$  and at least one of them is continuous, then for  $-\infty < a < b < \infty$ ,

$$\int_{(a,b]} FdG + \int_{(a,b]} GdF = F(b)G(b) - F(a)G(a).$$

## 2.5 $L^p$ Space

## 2.6 Radon measure

**Definition 2.6.1.** Let  $\mu$  be a Borel measure on  $X$  and  $E$  a Borel subset of  $X$ . The measure  $\mu$  is called outer regular on  $E$  if

$$\mu(E) = \inf \{ \mu(U) : U \supset E, U \text{ open} \}$$

and inner regular on  $E$  if

$$\mu(E) = \sup \{ \mu(K) : K \subset E, K \text{ compact} \}.$$

If  $\mu$  is outer and inner regular on all Borel sets,  $\mu$  is called **regular**.

A **Radon measure** on  $X$  is a Borel measure that is finite on all compact sets, outer regular on all Borel sets, and inner regular on all open sets.

**Definition 2.6.2.** A complex measure is regular if its total variation is regular.

**Proposition 2.6.3.** Every  $\sigma$ -finite Radon measure is regular.

**Lemma 2.6.4.** In  $C_2$  LCH space  $X$ , every open subset is  $\sigma$ -compact.

*Proof:* Since in  $C_2$  space, every open subspace is still  $C_2$  hence Lindelöf.

By Proposition 1.2.37, for all  $x \in U$ , there's  $V_x$  open and precompact such that  $x \in V_x \subset \bar{V}_x \subset U$ . Take a countable subcovering of  $\{V_x\}$  indexed by  $J$ , we have

$$\bigcup_{x \in J} \bar{V}_x = U$$

.

**Proposition 2.6.5.** Let  $X$  be a  $C_2$  LCH space. Then every Borel measure on  $X$  that is finite on compact sets is regular and hence Radon.

**Proposition 2.6.6.** If  $\mu$  is a Radon measure on  $X$ ,  $C_c(X)$  is dense in  $L^p(\mu)$  for  $1 \leq p < \infty$

**Theorem 2.6.7** (The Riesz Representation Theorem). If  $U$  is open in  $X$  and  $f \in C_c(X)$ , we shall write

$$f \prec U$$

to mean that  $0 \leq f \leq 1$  and  $\text{supp}(f) \subset U$ .

If  $I$  is a positive linear functional on  $C_c(X)$ , there is a unique Radon measure  $\mu$  on  $X$  such that  $I(f) = \int f d\mu$  for all  $f \in C_c(X)$ . Moreover,  $\mu$  satisfies

$$\mu(U) = \sup \{ I(f) : f \in C_c(X), f \prec U \} \text{ for all open } U \subset X$$

and  $\mu(K) = \inf \{ I(f) : f \in C_c(X), f \geq \chi_K \}$  for all compact  $K \subset X$ .



**Corollary 2.6.8.** There's one-to-one correspondence between bounded positive linear functional  $C_c(X)$  and finite Radon measure on  $X$ . Moreover, since  $C_c(X)$  is dense subset of Banach space  $C_0(X)$ , by Theorem 1.2.66, every bounded positive linear functional on  $C_c(X)$  can be extended to  $C_0(X)$  continuously.

*Proof:* If  $I$  is a bounded positive linear functional, by Riesz Representation Theorem,

$$\mu(X) = \sup \left\{ \int f d\mu : f \in C_c(X), 0 \leq f \leq 1 \right\} < \infty$$

**Proposition 2.6.9.** If  $\mu$  is a  $\sigma$ -finite Radon measure on  $X$  and  $A \in \mathcal{B}_X$ , the Borel measure  $\mu_A$  defined by  $\mu_A(E) = \mu(E \cap A)$  is a Radon measure.

**Proposition 2.6.10.** Suppose that  $\mu$  is a Radon measure on  $X$ . If  $\phi \in L^1(\mu)$  and  $\phi \geq 0$ , then  $\nu(E) = \int_E \phi d\mu$  is a Radon measure.

**Proposition 2.6.11.** Suppose that  $\mu$  is a Radon measure on  $X$  and  $\phi \in C(X, (0, \infty))$ . Let  $\nu(E) = \int_E \phi d\mu$ , and let  $\nu'$  be the Radon measure associated to the functional  $f \mapsto \int f \phi d\mu$  on  $C_c(X)$ , then  $\nu = \nu'$ , and hence  $\nu$  is a Radon measure.

**Definition 2.6.12.** A complex measure is Radon if its real and imaginary parts are difference of finite Radon measure.

**Definition 2.6.13.**  $M(X)$  is the space of all the complex Radon measures and for  $\mu \in M(X)$ , define

$$\|\mu\| = |\mu|(X)$$

Then,  $\|\cdot\|$  is a norm on vector space  $M(X)$ .

**Theorem 2.6.14.** Let  $X$  be an LCH space, and for  $\mu \in M(X)$ ,  $I_\mu : f \in C_0(X) \mapsto \int f d\mu$  is a bounded linear functional on  $C_0(X)$ . Then  $\mu \mapsto I_\mu$  is an bijective isometry between  $M(X)$  the space of complex Radon measure and space of bounded linear functional on  $C_0(X)$ .

**Proposition 2.6.15.** Suppose  $X, Y$  are LCH spaces.

- (1)  $\mathcal{B}_X \otimes \mathcal{B}_Y \subset \mathcal{B}_{X \times Y}$ .
- (2) If  $X$  and  $Y$  are second countable, then  $\mathcal{B}_X \otimes \mathcal{B}_Y = \mathcal{B}_{X \times Y}$ .
- (3) If  $X$  and  $Y$  are second countable and  $\mu$  and  $\nu$  are Radon measures on  $X$  and  $Y$ , then  $\mu \times \nu$  is a Radon measure on  $X \times Y$ .
- (4) If  $E \in \mathcal{B}_{X \times Y}$ , then  $E_x \in \mathcal{B}_Y$  for all  $x \in X$  and  $E^y \in \mathcal{B}_X$  for all  $y \in Y$ .
- (5) If  $f : X \times Y \rightarrow \mathbb{C}$  is  $\mathcal{B}_{X \times Y}$ -measurable, then  $f_x$  is  $\mathcal{B}_Y$ -measurable for all  $x \in X$  and  $f^y$  is  $\mathcal{B}_X$ -measurable for all  $y \in Y$ .

**Definition 2.6.16** (Radon product). Every  $f \in C_c(X \times Y)$  is  $\mathcal{B}_X \otimes \mathcal{B}_Y$ -measurable. Moreover, if  $\mu$  and  $\nu$  are  $\sigma$ -finite Radon measures on  $X$  and  $Y$ , then  $C_c(X \times Y) \subset L^1(\mu \times \nu)$ , and

$$\int f d(\mu \times \nu) = \iint f d\mu d\nu = \iint f d\nu d\mu \quad (f \in C_c(X \times Y)).$$

The formula  $I(f) = \int f d(\mu \times \nu)$  defines a positive linear functional on  $C_c(X \times Y)$ , so it determines a Radon measure on  $X \times Y$  by the Riesz representation theorem. We call this measure the Radon product of  $\mu$  and  $\nu$  and denote it by  $\mu \hat{\times} \nu$ .

**Proposition 2.6.17.** Suppose that  $\mu$  and  $\nu$  are  $\sigma$ -finite Radon measures on  $X$  and  $Y$ . If  $E \in \mathcal{B}_{X \times Y}$ , then the functions  $x \mapsto \nu(E_x)$  and  $y \mapsto \mu(E^y)$  are Borel measurable on  $X$  and  $Y$ , and

$$\mu \hat{\times} \nu(E) = \int \nu(E_x) d\mu(x) = \int \mu(E^y) d\nu(y).$$

Moreover, the restriction of  $\mu \hat{\times} \nu$  to  $\mathcal{B}_X \otimes \mathcal{B}_Y$  is  $\mu \times \nu$ .

**Theorem 2.6.18.** Suppose that, for each  $\alpha \in A$ ,  $\mu_\alpha$  is a Radon measure on the compact Hausdorff space  $X_\alpha$  such that  $\mu_\alpha(X_\alpha) = 1$ . Then there is a unique Radon measure  $\mu$  on  $X = \prod_{\alpha \in A} X_\alpha$  such that for any  $\alpha_1, \dots, \alpha_n \in A$  and any Borel set  $E$  in  $\prod_1^n X_{\alpha_j}$ ,

$$\mu\left(\pi_{(\alpha_1, \dots, \alpha_n)}^{-1}(E)\right) = (\mu_{\alpha_1} \hat{\times} \dots \hat{\times} \mu_{\alpha_n})(E).$$

# Chapter 3

## Complex Analysis

### 3.1 Line Integration

**Theorem 3.1.1** (Open mapping Theorem). If  $f$  is a holomorphic function and non-constant in a connected open set  $\Omega \subset \mathbb{C}$ , then  $f$  is open.

**Proposition 3.1.2.**  $U$  is an open subset of  $\mathbb{C}$ ,  $f : U \rightarrow \mathbb{C}$  is a injective holomorphic map, then  $f'(z) \neq 0$  for all  $z \in U$ . By Open Mapping Theorem, the image of  $f$  is still open in  $\mathbb{C}$ , we denote it by  $V$ . Then  $f : U \rightarrow V$  is a holomorphic bijective function.  $f^{-1}$  is also holomorphic and  $(f^{-1})'(z) = \frac{1}{f'(z)}$ .

**Proposition 3.1.3.**  $f$  holomorphic,  $f'(a) \neq 0$ , then  $f$  is local biholomorphic at  $a$ .

*Proof:* By inverse function theorem and Proposition 3.1.3.

## 3.2 Multiple Variables

# Chapter 4

## Functional Analysis

### 4.1 Foundation

**Definition 4.1.1.** Let  $K$  denote either  $\mathbb{R}$  or  $\mathbb{C}$ , and let  $X$  be a vector space over  $K$ . A seminorm on  $X$  is a function  $x \mapsto \|x\|$  from  $X$  to  $[0, \infty)$  such that

- (1)  $\|x + y\| \leq \|x\| + \|y\|$  for all  $x, y \in X$  (the triangle inequality),
- (2)  $\|\lambda x\| = |\lambda| \|x\|$  for all  $x \in X$  and  $\lambda \in K$ .

The second property clearly implies that  $\|0\| = 0$ . A seminorm such that  $\|x\| = 0$  only when  $x = 0$  is called a norm, and a vector space equipped with a norm is called a normed vector space (or normed linear space).

**Definition 4.1.2.** Banach space is a complete normed vector space.

**Definition 4.1.3** (quotient space). A related construction is that of quotient spaces. If  $\mathcal{M}$  is a vector subspace of the vector space  $X$ , it defines an equivalence relation on  $X$  as follows:  $x \sim y$  iff  $x - y \in \mathcal{M}$ . The equivalence class of  $x \in X$  is denoted by  $x + \mathcal{M}$ , and the set of equivalence classes, or quotient space, is denoted by  $X/\mathcal{M}$ .  $X/\mathcal{M}$  is a vector space with vector operations  $(x + \mathcal{M}) + (y + \mathcal{M}) = (x + y) + \mathcal{M}$  and  $\lambda(x + \mathcal{M}) = (\lambda x) + \mathcal{M}$ . If  $X$  is a normed vector space and  $\mathcal{M}$  is closed,  $X/\mathcal{M}$  inherits a norm from  $X$  called the quotient norm, namely

$$\|x + \mathcal{M}\| = \inf_{y \in \mathcal{M}} \|x + y\|$$

**Proposition 4.1.4.** A normed vector space is complete if and only if every absolutely convergent series converges.

**Proposition 4.1.5.** A linear map  $T : X \rightarrow Y$  between two normed vector spaces is called bounded if there exists  $C \geq 0$  such that

$$\|Tx\| \leq C\|x\| \text{ for all } x \in X$$

If  $X$  and  $Y$  are normed vector spaces and  $T : X \rightarrow Y$  is a linear map, the following are equivalent:

- (1)  $T$  is continuous.
- (2)  $T$  is continuous at 0 .
- (3)  $T$  is bounded.

**Definition 4.1.6.** If  $X$  and  $Y$  are normed vector spaces, we denote the space of all bounded linear maps from  $X$  to  $Y$  by  $L(X, Y)$ . It is easily verified that  $L(X, Y)$  is a vector space and that the function  $T \mapsto \|T\|$  defined by

$$\begin{aligned}\|T\| &= \sup\{\|Tx\| : \|x\| = 1\} \\ &= \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0\right\} \\ &= \inf\{C : \|Tx\| \leq C\|x\| \text{ for all } x\}\end{aligned}$$

is a norm on  $L(X, Y)$ , called the operator norm.

**Proposition 4.1.7.** If  $Y$  is complete, so is  $L(X, Y)$ .

**Corollary 4.1.8.** Let  $X$  be a vector space over  $K$ , where  $K = \mathbb{R}$  or  $\mathbb{C}$ . A linear map from  $X$  to  $K$  is called a linear functional on  $X$ . If  $X$  is a normed vector space, the space  $L(X, K)$  of bounded linear functionals on  $X$  is called the dual space of  $X$ . Then  $X^*$  is a Banach space with the operator norm.

**Proposition 4.1.9.** Let  $X$  be a vector space over  $\mathbb{C}$ . If  $f$  is a complex linear functional on  $X$  and  $u = \operatorname{Re} f$ , then  $u$  is a real linear functional, and  $f(x) = u(x) - iu(ix)$  for all  $x \in X$ . Conversely, if  $u$  is a real linear functional on  $X$  and  $f : X \rightarrow \mathbb{C}$  is defined by  $f(x) = u(x) - iu(ix)$ , then  $f$  is complex linear. In this case, if  $X$  is normed, we have  $\|u\| = \|f\|$ .

**Definition 4.1.10.** If  $X$  is a real vector space, a sublinear functional on  $X$  is a map  $p : X \rightarrow \mathbb{R}$  such that

$$p(x + y) \leq p(x) + p(y) \text{ and } p(\lambda x) = \lambda p(x) \text{ for all } x, y \in X \text{ and } \lambda \geq 0$$

**Theorem 4.1.11** (The Hahn-Banach Theorem). Let  $X$  be a real vector space,  $p$  a sublinear functional on  $\mathcal{X}$ ,  $\mathcal{M}$  a subspace of  $\mathcal{X}$ , and  $f$  a linear functional on  $\mathcal{M}$  such that  $f(x) \leq p(x)$  for all  $x \in \mathcal{M}$ . Then there exists a linear functional  $F$  on  $\mathcal{X}$  such that  $F(x) \leq p(x)$  for all  $x \in \mathcal{X}$  and  $F|_{\mathcal{M}} = f$ .

**Definition 4.1.12** (complex Hahn-Banach Theorem). Let  $X$  be a complex vector space,  $p$  a seminorm on  $\mathcal{X}$ ,  $\mathcal{M}$  a subspace of  $\mathcal{X}$ , and  $f$  a complex linear functional on  $\mathcal{M}$  such that  $|f(x)| \leq p(x)$  for  $x \in \mathcal{M}$ . Then there exists a complex linear functional  $F$  on  $X$  such that  $|F(x)| \leq p(x)$  for all  $x \in \mathcal{X}$  and  $F|_{\mathcal{M}} = f$ .

**Corollary 4.1.13.** Let  $X$  be a normed vector space.

- (1) If  $\mathcal{M}$  is a closed subspace of  $X$  and  $x \in X \setminus \mathcal{M}$ , there exists  $f \in X^*$  such that  $f(x) \neq 0$  and  $f|_{\mathcal{M}} = 0$ . In fact, if  $\delta = \inf_{y \in \mathcal{M}} \|x - y\|$ ,  $f$  can be taken to satisfy  $\|f\| = 1$  and  $f(x) = \delta$ .

- (2) If  $x \neq 0 \in X$ , there exists  $f \in X^*$  such that  $\|f\| = 1$  and  $f(x) = \|x\|$ .
- (3) The bounded linear functionals on  $X$  separate points.
- (4) If  $x \in X$ , define  $\hat{x} : X^* \rightarrow \mathbb{C}$  by  $\hat{x}(f) = f(x)$ . Then the map  $x \mapsto \hat{x}$  is a linear isometry from  $X$  into  $X^{**}$  (the dual of  $X^*$ ).

**Theorem 4.1.14** (open mapping theorem). Let  $X$  and  $Y$  be Banach spaces. If  $T \in L(X, Y)$  is surjective, then  $T$  is open.

Now we assume all the Banach spaces are over  $\mathbb{C}$ .

**Definition 4.1.15.** If  $X$  and  $Y$  are normed vector spaces and  $T$  is a linear map from  $X$  to  $Y$ , we define the graph of  $T$  to be

$$\Gamma(T) = \{(x, y) \in X \times Y : y = Tx\}$$

which is a subspace of  $X \times Y$ . We say that  $T$  is closed if  $\Gamma(T)$  is a closed subspace of  $X \times Y$ . Clearly, if  $T$  is continuous, then  $T$  is closed.

**Theorem 4.1.16** (The Closed Graph Theorem). If  $X$  and  $Y$  are Banach spaces and  $T : X \rightarrow Y$  is a closed linear map, then  $T$  is bounded.

**Theorem 4.1.17** (The Uniform Boundedness Principle). Suppose that  $X$  and  $Y$  are normed vector spaces and  $\mathcal{A}$  is a subset of  $L(X, Y)$ . If  $X$  is a Banach space and  $\sup_{T \in \mathcal{A}} \|Tx\| < \infty$  for all  $x \in X$ , then  $\sup_{T \in \mathcal{A}} \|T\| < \infty$ .

## 4.2 Topological vector space

**Definition 4.2.1.** A topological vector space is a vector space  $X$  over the field  $K (= \mathbb{R} \text{ or } \mathbb{C})$  which is endowed with a topology such that the maps  $(x, y) \rightarrow x + y$  and  $(\lambda, x) \rightarrow \lambda x$  are continuous from  $X \times X$  and  $K \times X$  to  $X$ . A topological vector space is called locally convex if there is a base for the topology consisting of convex sets (that is, sets  $A$  such that if  $x, y \in A$  then  $tx + (1 - t)y \in A$  for  $0 < t < 1$ ). Most topological vector spaces that arise in practice are locally convex and Hausdorff.

**Proposition 4.2.2.** Let  $\{p_\alpha\}_{\alpha \in A}$  be a family of seminorms on the vector space  $X$ . If  $x \in X$ ,  $\alpha \in A$ , and  $\epsilon > 0$ , let

$$U_{x\alpha\epsilon} = \{y \in X : p_\alpha(y - x) < \epsilon\}$$

and let  $\mathcal{T}$  be the topology generated by the sets  $U_{x\alpha\epsilon}$ .

- (1) For each  $x \in X$ , the finite intersections of the sets  $U_{x\alpha\epsilon} (\alpha \in A, \epsilon > 0)$  form a neighborhood base at  $x$ .
- (2)  $x_i \rightarrow x$  iff  $p_\alpha(x_i - x) \rightarrow 0$  for all  $\alpha \in A$ .
- (3)  $(X, \mathcal{T})$  is a locally convex topological vector space.

**Proposition 4.2.3.** Suppose  $X$  and  $Y$  are vector spaces with topologies defined, respectively, by the families  $\{p_\alpha\}_{\alpha \in A}$  and  $\{q_\beta\}_{\beta \in B}$  of seminorms, and  $T : X \rightarrow Y$  is a linear map. Then  $T$  is continuous iff for each  $\beta \in B$  there exist  $\alpha_1, \dots, \alpha_k \in A$  and  $C > 0$  such that  $q_\beta(Tx) \leq C \sum_{j=1}^k p_{\alpha_j}(x)$

**Proposition 4.2.4.** Let  $X$  be a vector space equipped with the topology defined by a family  $\{p_\alpha\}_{\alpha \in A}$  of seminorms.  $X$  is Hausdorff iff for each  $x \neq 0$  there exists  $\alpha \in A$  such that  $p_\alpha(x) \neq 0$ .

**Definition 4.2.5.** A topological vector space whose topology is defined by a countable family of seminorms is called a Fréchet space if it is Hausdorff and complete. (every Cauchy sequence converges.)

### 4.3 Hilbert Space

**Definition 4.3.1.** Let  $\mathcal{H}$  be a complex vector space. An inner product (or scalar product) on  $\mathcal{H}$  is a map  $(x, y) \mapsto \langle x, y \rangle$  from  $X \times X \rightarrow \mathbb{C}$  such that:

- (1)  $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$  for all  $x, y, z \in \mathcal{H}$  and  $a, b \in \mathbb{C}$ .
- (2)  $\langle y, x \rangle = \overline{\langle x, y \rangle}$  for all  $x, y \in \mathcal{H}$ .
- (3)  $\langle x, x \rangle \in (0, \infty)$  for all nonzero  $x \in X$ .

A Hilbert space is a vector space over  $\mathbb{C}$  with a inner product such that the norm induced by this inner product is complete.

**Proposition 4.3.2.** If  $\mathcal{M}$  is a closed subspace of  $\mathcal{H}$ , then  $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ ; that is, each  $x \in \mathcal{H}$  can be expressed uniquely as  $x = y + z$  where  $y \in \mathcal{M}$  and  $z \in \mathcal{M}^\perp$ . Moreover,  $y$  and  $z$  are the unique elements of  $\mathcal{M}$  and  $\mathcal{M}^\perp$  whose distance to  $x$  is minimal.

**Theorem 4.3.3.** If  $f \in \mathcal{H}^*$ , there is a unique  $y \in \mathcal{H}$  such that  $f(x) = \langle x, y \rangle$  for all  $x \in X$ .

**Definition 4.3.4.** If  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are Hilbert spaces with inner products  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$ , a unitary map from  $\mathcal{H}_1$  to  $\mathcal{H}_2$  is an invertible linear map  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  that preserves inner products:

$$\langle Ux, Uy \rangle_2 = \langle x, y \rangle_1 \text{ for all } x, y \in \mathcal{H}_1$$

### 4.4 Spectrum of Operator



# Chapter 5

## Harmonic Analysis

### 5.1 Fourier Transform

**Definition 5.1.1** (Schwartz space). The Schwartz space  $\mathcal{S}(\mathbb{R}^n)$  consisting of those  $C^\infty$  functions which, together with all their derivatives, vanish at infinity faster than any power of  $|x|$ . More precisely, for any nonnegative integer  $N$  and any multi-index  $\alpha$  we define

$$\|f\|_{(N,\alpha)} = \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha f(x)|$$

then

$$\mathcal{S} = \mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty : \|f\|_{(N,\alpha)} < \infty \text{ for all } N, \alpha\}$$

**Proposition 5.1.2.**  $\mathcal{S}(\mathbb{R}^n)$  is a Fréchet Space and Fourier Transform is a linear bi-continuous bijection between Schwartz space.