Project 3: A semilinear elliptic equation

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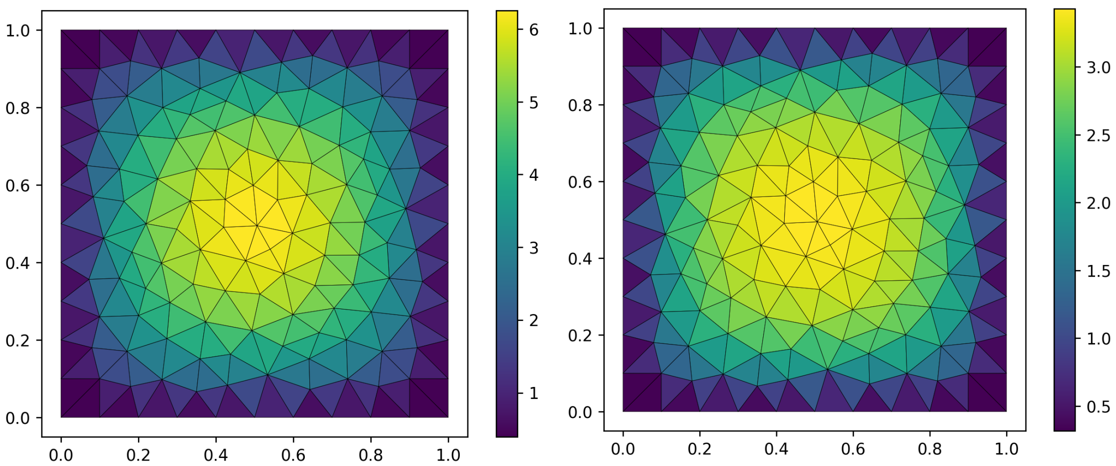
1. We multiply
$$-\Delta u_{n+1} + \alpha u_n^2 u_{n+1} = f$$
by a test function $v \in H_0^1(\Omega)$ and integrate over Ω to get
$$\int_\Omega -\Delta u_{n+1} \cdot v + \alpha u_n^2 u_{n+1} \cdot v = \int_\Omega fv.$$
Then we integrate the first term by parts and have
$$\int_\Omega -\Delta u_{n+1} \cdot v = \int_{\partial\Omega} -\mathrm{div}(\Delta u_{n+1} \cdot v) + \int_\Omega \Delta u_{n+1} \cdot \Delta v.$$
Using the divergence theorem and the Dirichlet boundary condition, we get that
$$\int_{\partial\Omega} -\mathrm{div}(\Delta u_{n+1} \cdot v) = -\oint_{\partial\Omega} \Delta u_{n+1} \cdot \mathbf{n} v = 0.$$
Finally, the weak form is
$$\int_\Omega \nabla u_{n+1} \cdot \nabla v + \alpha u_n^2 u_{n+1} \cdot v = \int_\Omega fv.$$

2. We used a mesh with size 0.1 (149 vertices). For α=0.1, the fixed-point scheme requires 12 iterations. For α=2, it requires 353 iterations. This is too many. Below is a plot of the solution, depending on the value of α.



(a) $\alpha = 0.1$ (b) $\alpha = 2$

3. With the same mesh as before, the Newton scheme requires only 5 iterations for α=0.1, 7 iterations for α=2 and 8 iterations for α=5. The Newton scheme does fare considerably better than the fixed-point scheme for larger values of α, and it also shows improvement for small values of α.