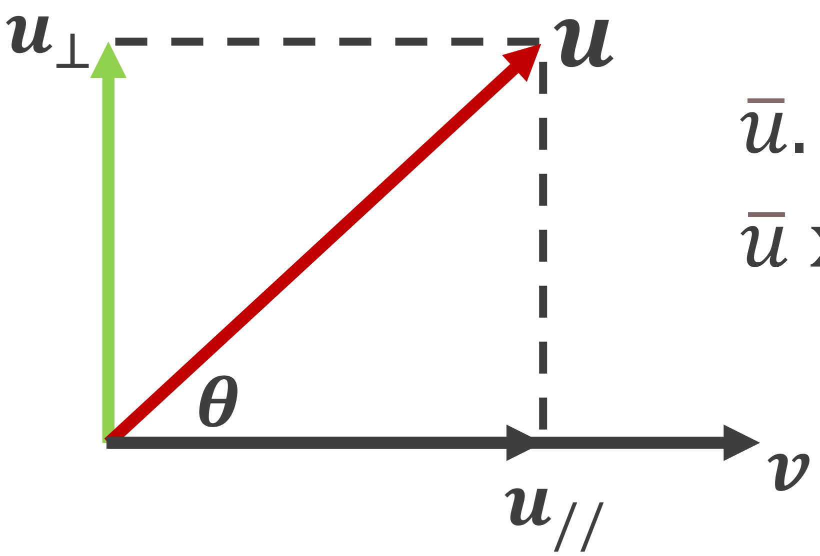
The Dot and Cross Products

Two common operations involving vectors are ***the dot product*** and ***the cross product***.

Let two vectors $\boldsymbol{u} = (u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k})$ and $\mathbf{v} = (v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k})$



$$\bar{u}.\bar{v} = u\ v\ cos\theta = u_{//}v$$

$$\bar{u}\ \mathrm{x}\ \bar{v} = u\ v\ sin\theta = u_\perp v$$

$$arah\ \perp \bar{u}, \perp \bar{v}$$

The Dot Product

The dot product of **u** and **v** is written **u** · **v** and is defined two ways:

1) $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$
2) $\mathbf{u} \cdot \mathbf{v} = |u||v| cos\theta, where\ \theta\ is\ the\ angle\ formed\ by\ \boldsymbol{u}\ and\ \boldsymbol{v}$

The two definitions are the same. They are related to one another by the Law of Cosines.

The first method of calculation is easier because it is the sum of the products of corresponding components.

The second method of calculation can be used if we know the angle θ formed by **u** and **v**.

Example

Find **u** · **v**, where **u** = 3i, −4j,1k and **v** = 5i,2j, −6k, then find the angle θ formed by **u** and **v**.

Solution:

Using the first method of calculation, we have

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (3)(5)+(-4)(2)+(1)(-6)\\ &= 15+(-8)+(-6)\\ &= 1.\end{aligned}$$

To find θ, we use the second method of calculation and solve for θ, using a calculator in degree mode for the last step.

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{26}\sqrt{65}}\right) \approx 88.61^\circ.$$

The Cross Product

The cross product of **u** and **v** is defined and best memorized as the expansion of a 3 by 3 determinant:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}.$$

The cross product of **u** and **v** is a vector, with the property that it is orthogonal to the two vectors **u** and **v**.

Example

Find $\mathbf{u} \times \mathbf{v}$, where $\mathbf{u} = \langle 3, -4, 1 \rangle$ and $\mathbf{v} = \langle 5, 2, -6 \rangle$.

Solution:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 1 \\ 5 & 2 & -6 \end{vmatrix} = \begin{vmatrix} -4 & 1 \\ 2 & -6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 1 \\ 5 & -6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -4 \\ 5 & 2 \end{vmatrix} \mathbf{k}$$

$$= 22\mathbf{i} - (-23)\mathbf{j} + 26\mathbf{k}, \text{ or } \langle 22, 23, 26 \rangle.$$

THANK YOU