Superlinear extensions of linear algebra

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1. Purpose

This poster shows the initial steps of an ongoing research on the extension of certain results on linear algebra to the category of superlinear spaces to study Grassmann-valued differential equations. The interest of the topic is mathematical and physical as illustrated by the vast literature on superspaces and their applications in supersymmetric models and quantum physics [1]-[9].

In particular, we are interested in examining conditions for the existence and uniqueness of different types of Grassmann-valued differential equations. It is also of interest for us to determine its fixed points, i.e. points that do not evolve in terms of time, and its limit points, i.e. points to which the system tends when $t \to +\infty$. These topics will be related to the kernels, eigenvalues of supermorphisms and other generalizations of linear algebra results.

2. Fundamentals on superlinear algebra

A *superspace* is a pair $(\mathbb{V}, \alpha)$ where $\mathbb{V}$ is a vector space and α is a linear endomorphism on $\mathbb{V}$ satisfying $\alpha^2 = \mathrm{Id}$. If α is known from context, the superspace will simply be denoted by $\mathbb{V}$. The elements of $\mathbb{V}_0 := \ker(\alpha - \mathrm{Id})$ and $\mathbb{V}_1 := \ker(\alpha + \mathrm{Id})$ are called *even* and *odd*, respectively. If $v \in \check{\mathbb{V}} := (\mathbb{V}_0 \cup \mathbb{V}_1)\backslash\{0\}$, then v is said to be *homogeneous*. The parity of the superspace $\mathbb{V}$ is the function $p : \check{\mathbb{V}} \to \{0,1\}$ mapping each $v \in \check{\mathbb{V}}$ to the index k of the subspace $\mathbb{V}_k$ containing it.

A *superspace morphism* is a linear morphism $f : \mathbb{V} \to \mathbb{W}$ between superspaces $\mathbb{V}, \mathbb{W}$. If $f(\mathbb{V}_0) \subset \mathbb{V}_0$ and $f(\mathbb{V}_1) \subset \mathbb{V}_1$, then f is said to be *even* and we define $p(f) := 0$. If $f(\mathbb{V}_0) \subset \mathbb{V}_1$ and $f(\mathbb{V}_1) \subset \mathbb{V}_0$, then f is called *odd* and $p(f) := 1$. In both cases we say that f is *homogenous*.

If a superspace $\mathbb{A}$ is also an algebra and $\mathbb{A}_i\mathbb{A}_j \subset \mathbb{A}_{(i+j) \bmod 2}$, then $\mathbb{A}$ is called a *superalgebra* which is *supercommutative* if $ab = (-1)^{p(a)p(b)}ba$ for all $a, b \in \check{\mathbb{A}}$. We hereafter assume $\mathbb{A}$ to be supercommutative superalgebra.

An example of superalgebra is the *Grassmann algebra* $\mathbb{R}_{S[L]}$, which is the real algebra generated by $1, \beta_1, \ldots, \beta_L$ satisfying

$$1\beta_i = \beta_i = \beta_i 1, \quad \beta_i\beta_j = -\beta_j\beta_i, \quad i, j \in \{1, \ldots, L\}.$$

Products of odd generators of $\mathbb{R}_{S[L]}$ are described by the set $M_L := M_{L0} \cup M_{L1}$ of multi-indices, where M_{L0} and M_{L1} are strings of an even and odd number of indices, respectively. The extra index $\emptyset$ in M_{L0} stands for the generator 1 so that $\beta_{[\emptyset]} = 1$. The superalgebra structure of $\mathbb{R}_{S[L]}$ is given by

$$\left(\mathbb{R}_{S[L]}\right)_k := \left\{ \sum_{\underline{\lambda} \in M_{Lk}} x_{\underline{\lambda}} \beta_{[\underline{\lambda}]} \,\middle|\, x_{\underline{\lambda}} \in \mathbb{R} \right\}.$$

A Grassmann algebra with an infinite number of generators is denoted by $\mathbb{R}_S$.

A homogeneous morphism $f : \mathbb{A} \to \mathbb{B}$ between superalgebras such that $f(a_1a_2) = (-1)^{p(f)p(a_1)} f(a_1)f(a_2)$ is a *superalgebra morphism*.

Let $\mathbb{V}$ be a right $\mathbb{A}$-module such that $va \in \mathbb{V}_{(p(v)+p(a)) \bmod 2}$ for all $a \in \check{\mathbb{A}}, v \in \check{\mathbb{V}}$. Then, $\mathbb{V}$ is said to be a *right super* $\mathbb{A}$*-module*. If $\mathbb{V}$ admits a basis of m even vectors and n odd vectors, then it is said that $\mathbb{V}$ is *free* and (m,n) is its *super dimension*.

A superspace morphism $f : \mathbb{V} \to \mathbb{W}$ between two right super $\mathbb{A}$-modules $\mathbb{V}, \mathbb{W}$ is a *supermodule morphism* if $f(va) = f(v)a$ for all $v \in \check{\mathbb{V}}$ and $a \in \check{\mathbb{A}}$. The space $\mathbf{Hom}(\mathbb{V}, \mathbb{W})$ of all such supermodule morphisms turns out to be a left super $\mathbb{A}$-module relative to the decomposition $\mathbf{Hom}(\mathbb{V}, \mathbb{W}) = \mathrm{Hom}_0(\mathbb{V}, \mathbb{W}) \oplus \mathrm{Hom}_1(\mathbb{V}, \mathbb{W})$, where $\mathrm{Hom}_0(\mathbb{V}, \mathbb{W})$ and $\mathrm{Hom}_1(\mathbb{V}, \mathbb{W})$ stand for the even and odd morphisms on $\mathbb{V}$. Hence, $f \in \mathbf{Hom}(\mathbb{V}, \mathbb{W})$ can be written uniquely as $f = f_0 + f_1$, where f_0 is even and f_1 is odd.

3. Supermatrices

An (m,n)-*supermatrix* is an $(m+n)\times(m+n)$-matrix with entries in a superalgebra $\mathbb{A}$. If $\mathbb{V}$ and $\mathbb{W}$ are super $\mathbb{A}$-modules with super dimension (m,n), then the left supermodule of (m,n)-supermatrices $\mathbf{SMat}_{(m,n)}(\mathbb{A})$ is isomorphic to the supermodule $\mathbf{Hom}(\mathbb{V}, \mathbb{W})$. Even supermatrices make up $\mathrm{SMat}_{(m,n)}(\mathbb{A}) := \left(\mathbf{SMat}_{(m,n)}(\mathbb{A})\right)_0$.

Proposition 1. *A supermatrix* $\Lambda \in \mathrm{SMat}_{(m,n)}(\mathbb{A})$ *is of the form*

$$\Lambda = \begin{bmatrix} G & P \\ Q & H \end{bmatrix},$$

where G, H *are* $\mathbb{A}_0$*-matrices of dimensions* $m \times m$ *and* $n \times n$*, respectively, and* P, Q *are* $\mathbb{A}_1$*-matrices of dimensions* $m \times n$ *and* $n \times m$*. Invertibility of* G *and* H *is equivalent to the invertibility of* Λ.

Let $\mathrm{GL}_{(m,n)}(\mathbb{A})$ stand for the invertible elements of $\mathrm{SMat}_{(m,n)}(\mathbb{A})$. The *Berezinian* [8, 7] is the unique $\mathrm{Ber} : \mathrm{GL}_{(m,n)}(\mathbb{A}) \to \mathbb{A}_0$ such that :

$$\mathrm{Ber}\begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix} = \det(GH^{-1}), \quad \forall G, H \in \mathrm{GL}(\mathbb{A}_0), \tag{1}$$

$$\mathrm{Ber}(\Lambda\Omega) = \mathrm{Ber}(\Lambda)\mathrm{Ber}(\Omega), \quad \forall \Lambda, \Omega \in \mathrm{GL}_{(m,n)}(\mathbb{A}). \tag{2}$$

4. Supersubspaces

A *supersubspace* of $\mathbb{V}$ is defined to be a vector subspace $\mathbb{S} \subset \mathbb{V}$ satisfying $\mathbb{S} = \mathbb{S} \cap \mathbb{V}_0 \oplus \mathbb{S} \cap \mathbb{V}_1$.

Proposition 2. *If* f *is homogenous, then* $\ker f$ *and* $\mathrm{im} f$ *are supersubspaces.*

Proof. Let $v = v_0 + v_1 \in \ker f$ with $v_k \in \mathbb{V}_k$. Hence $f(v_0) = -f(v_1)$ and, from homogeneity of f, it follows that $v_0, v_1 \in \ker f$. Thus $\ker f \subset (\ker f \cap \mathbb{V}_0) \oplus (\ker f \cap \mathbb{V}_1) \subset \ker f$.
Similarly, $\mathrm{im} f = f(\mathbb{V}_0) \oplus f(\mathbb{V}_1)$ is a supersubspace. □

Notwithstanding, $\ker f$ and $\mathrm{im} f$ may be supersubspaces when f is not homogenous. A counterexample can be constructed by using a superspace $\mathbb{R}^{p|q}$ where the even part has dimension p and the odd one has dimension q. Assume for simplicity $p = 2$, $q = 1$ and define $f \in \mathrm{End}(\mathbb{R}^{2|1})$ of the form

$$[f] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

in a basis $\{e_1, e_2, \theta_1\}$, where e_1, e_2 are even and θ_1 is odd.

Proposition 3. *If* $\ker f$ *is a supersubspace, then* $\ker f = \ker f_0 \cap \ker f_1$.

Proof. We note that $\ker f_0 \cap \ker f_1 \subset \ker f$. Let us prove the inverse inclusion. If $v \in \ker f$, then $v = v_0 + v_1$, where $v_k \in \ker f \cap \mathbb{V}_k$. Therefore

$$\mathbb{V}_0 \ni f_0(v_0) + f_1(v_1) = -f_0(v_1) - f_1(v_0) \in \mathbb{V}_1.$$

It follows that $f_0(v_0) = -f_1(v_1)$ and $f_1(v_0) = -f_0(v_1)$. But this is true for all $v_1 \in \ker f \cap \mathbb{V}_1$, specifically for $v_1 = 0$. Hence $f_0(v_0) = f_1(v_0) = 0$ and for the same reason $f_0(v_1) = f_1(v_1) = 0$, which means that $v \in \ker f_0 \cap \ker f_1$. □

Proposition 4. *If* f *is an endomorphism of a superspace* $(\mathbb{V}, \alpha)$*, then* $\alpha(\ker f) = \ker f$ *if and only if* $\ker f$ *is a supersubspace.*

A *supersubmodule* of a right super $\mathbb{A}$-module $\mathbb{V}$ is a supersubspace $\mathbb{W} \subset \mathbb{V}$ satisfying $\mathbb{W}\mathbb{A} \subset \mathbb{W}$.

Proposition 5. *If* $f : \mathbb{V} \to \mathbb{W}$ *is a homogeneous supermodule morphism between right* $\mathbb{A}$*-modules* $\mathbb{V}$ *and* $\mathbb{W}$*, then* $\ker f$ *and* $\mathrm{im} f$ *are supersubmodules of* $\mathbb{V}$ *and* $\mathbb{W}$.

An *eigenvector* of a supermodule endomorphism on a right super $\mathbb{A}$-module $\mathbb{V}$ is an element of $v \in \mathbb{V}\backslash\{0\}$ such that there exists an element $\lambda \in \mathbb{A}$ satisfying $f(v) = v\lambda$. The λ is called an *eigenvalue* of v. It is easy to infer properties of λ in terms of the homogeneity of f.

5. One-even-parameter semigroups

Even-reduced and *odd-reduced* supermatrices are, respectively, elements of the following sets

$$\mathrm{GL}^{\mathrm{even}}_{(p,q)}(\mathbb{A}) = \left\{ \begin{bmatrix} G & P \\ 0 & H \end{bmatrix} \right\}, \quad \mathrm{GL}^{\mathrm{odd}}_{(p,q)}(\mathbb{A}) = \left\{ \begin{bmatrix} 0 & P \\ Q & H \end{bmatrix} \right\},$$

where G, H are invertible $\mathbb{A}_0$-matrices and P, Q are $\mathbb{A}_1$-matrices. Let $\mathbb{G}_P, \mathbb{G}_Q, \mathbb{G}_H$ be subsets of matrices such that $\mathbb{S}$ is a subset of $\mathrm{GL}^{\mathrm{odd}}_{(p,q)}(\mathbb{A})$, of the form

$$\mathbb{S} = \left\{ \begin{bmatrix} 0 & P \\ Q & H \end{bmatrix} \,\middle|\, P \in \mathbb{G}_P, Q \in \mathbb{G}_Q, H \in \mathbb{G}_H \right\}.$$

On $\mathbb{S}$ we can define an *odd-reduced berezinian* [3], which is a map

$$\mathrm{Ber}^{\mathrm{odd}} : \mathbb{S} \ni \begin{bmatrix} 0 & P \\ Q & H \end{bmatrix} \mapsto \det(-PH^{-1}Q)\det H^{-1} \in \mathbb{A}_0.$$

Two examples of one-even-parameter semigroups in $\mathrm{GL}^{odd}_{(1,q)}(\mathbb{A})$ are

$$\mathbb{P}_{v_1,v_2} = \left\{ \begin{bmatrix} 0 & v_1^T \cdot t \\ v_2 & \mathbb{I}_q \end{bmatrix} \right\}, \quad \mathbb{Q}_{v_1,v_2} = \left\{ \begin{bmatrix} 0 & v_1^T \\ v_2 \cdot t & \mathbb{I}_q \end{bmatrix} \right\},$$

with $t \in \mathbb{A}_0$ and $v_1, v_2 \in \mathbb{A}_1^q$ such that $v_2 v_1^T = 0$.
Let us fix v_1 and v_2, and call elements of $\mathbb{P}_{v_1,v_2}$ and $\mathbb{Q}_{v_1,v_2}$ simply $P(t)$ and $Q(t)$. Matrices of that form are idempotent and satisfy relations

$$\begin{aligned} P(t)P(u) &= P(t), \\ Q(t)Q(u) &= P(u), \\ Q(t)P(u) &= E, \end{aligned}$$

for all $t, u \in A_0$ and $E := P(1) = Q(1)$.
Since $P(t)$ and $Q(t)$ are not identities at $t = 0$, they cannot be presented as exponents $e^{t\Lambda}$ for some supermatrix Λ. An example of one-even-parameter semigroup in $\mathrm{GL}_{(1,q)}(\mathbb{A})$ is

$$\mathbb{T}_{v_1} = \left\{ \begin{bmatrix} 1 & v_1^T \cdot t \\ 0 & \mathbb{I}_q \end{bmatrix} \right\}, \quad t \in \mathbb{A}_0.$$

6. Norms on supermodules

Let our supercommutative superalgebra $\mathbb{A}$ be $\mathbb{R}_S$. The results will also hold for any finitely generated Grassmann algebra. We define a norm on $\mathbb{R}_S$ as

$$\|a\| := \sum_{\underline{\lambda} \in M} |a_{\underline{\lambda}}|, \quad a \in \mathbb{R}_S. \tag{3}$$

From now on, the subset of elements of $\mathbb{R}_S$ finite in proposed norm is denoted by the same symbol as the whole algebra. Let us consider $\mathbf{SMat}_{(m,n)}(\mathbb{R}_S)$ and $(\mathbb{R}_S)^{p|q}$ with norms

$$\|A\|_S := \sum_{i,j=1}^{m+n} \|A_{ij}\|, \quad A \in \mathbf{SMat}_{(m,n)}(\mathbb{R}_S), \tag{4}$$

$$\|X\|_{p|q} := \sum_{i=1}^{p+q} \|X_i\|, \quad X \in (\mathbb{R}_S)^{p|q}. \tag{5}$$

The space $\mathbb{R}_S$ with the norm (3) is a Banach algebra. As a consequence, spaces $\mathbf{SMat}_{(m,n)}(\mathbb{R}_S)$ and $(\mathbb{R}_S)^{p|q}$ with norms (4) and (5) are Banach algebras too. The latter implies that exponents of supermatrices with entries in $\mathbb{R}_S$ exist.

7. Semigroups of operators

We will now focus on $\mathbb{P}_{v_1,v_2}$ and $\mathbb{T}_{v_1}$. Its elements represent a one-even-parameter semigroup of operators on $\mathbb{A}^{1|q}$. An *odd-reduced dynamical system* on $\mathbb{A}^{1|q}$ is a function $X_P(\cdot) : \mathbb{R}_+ \cup \{0\} \to \mathbb{A}^{1|q}$ such that

$$X_P(t) = \begin{cases} X(0) & \text{if } t = 0, \\ P(t)X(0) & \text{if } t > 0, \end{cases}$$

where $X(0) \in \mathbb{A}^{1|q}$ is an initial condition.
Function $X_P(t)$ is not continuous at 0 unless the first coordinate of $X(0)$ vanishes. The supermatrix

$$A := \lim_{t \to 0} \frac{P(t) - P(0)}{t} = \begin{bmatrix} 0 & v_1^T \\ 0 & 0 \end{bmatrix}, \tag{6}$$

is a *generator* of $(P(t))_{t \geq 0}$ and furthermore

$$\begin{aligned} P(t) &= P(0) + t \cdot A = \exp(tA), \\ P(t)A &= 0, \\ AP(t) &= A. \end{aligned}$$

Both $(P(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ have the same generator A and are solutions to the following linear differential equation

$$M'(t) = A \cdot M(t). \tag{7}$$

In [3, Corollary 5] it is suggested that $P(t)$ and $T(t)$ are different classes of solutions of the Cauchy problem (7). In Proposition 7 we will show that in fact they illustrate the same solution.

8. Matrix Cauchy problem

The norms given in Section 6 allow us to prove the following lemma:

Lemma 6. *The operator* $\exp(tA)$ *exists for every* $t \in \mathbb{R}$ *and* $A \in \mathbf{SMat}_{(m,n)}(\mathbb{R}_S)$.

In turn, previous lemma gives rise to the next proposition:

Proposition 7. *Let* $N : \mathbb{R} \ni t \mapsto N(t) \in \mathbf{SMat}_{(m,n)}(\mathbb{R}_S)$ *be continuous and bounded. The Cauchy problem*

$$\frac{dM}{dt} = N(t)M, \qquad M \in \mathbf{SMat}_{(m,n)}(\mathbb{R}_S), \tag{8}$$

with initial condition $M_0 \in \mathbf{SMat}_{(m,n)}(\mathbb{R}_S)$ *has a unique solution. If* $N(t) = A$*, then the general solution to* (8) *becomes*

$$M(t) = \exp(tA)M_0 \tag{9}$$

and if also $\mathrm{Ber}(M_0)$ *exists, then* $\mathrm{Ber}(M(t)) = \mathrm{Ber}\exp(tA)\mathrm{Ber}M_0$.

Proof. The function

$$\begin{aligned} F : \mathbb{R} \times \mathbf{SMat}_{(m,n)}(\mathbb{R}_S) &\longrightarrow \mathbf{SMat}_{(m,n)}(\mathbb{R}_S), \\ F(t, M) &:= N(t) \cdot M, \end{aligned}$$

is continuous and satisfies Lipschitz condition regarding the second variable (monotonously regarding the first variable) with respect to the norm (4) if $N(t)$ is continuous and bounded. The final result follows from the theorem of existence and uniqueness for differential equations. □

Expression (9) for $m = 1$ and $n = q$ provides the general solution to (7) and in particular

$$P(t) = T(t)\begin{bmatrix} 0 & 0 \\ v_2 & \mathbb{I}_q \end{bmatrix}.$$

If A is an even matrix, then $\exp(tA)$ is an even operator for every $t \in \mathbb{R}$. Nevertheless, if A is any supermatrix, then $\exp(tA)$ is a general element of $\mathbf{SMat}_{(m,n)}(\mathbb{R}_S)$.
Proposition 7 ensures that if A is even and $\mathrm{Ber}(M_0) \neq 0$, then the particular solution $M(t)$ with $M(0) = M_0$ satisfies $\mathrm{Ber}(M(t)) \neq 0$.

9. Vector Cauchy problem

Proposition 8. *Let* $N : \mathbb{R} \ni t \mapsto N(t) \in \mathrm{SMat}_{(p,q)}(\mathbb{R}_S)$ *be continuous and bounded. The Cauchy problem*

$$\frac{dX}{dt} = N(t)X, \qquad X \in (\mathbb{R}_S)^{p|q}, \tag{10}$$

with initial condition $X_0 \in (\mathbb{R}_S)^{p|q}$ *has a unique solution.*

Proof. It is analogous to the proof of the Proposition 7. □

Grassmann valued differential equation (10) makes sense only for even supermatrices $N(t)$ because otherwise $N(t)X$ need not belong to $(\mathbb{R}_S)^{p|q}$.
The general solution to (10) for a constant $N(t) = N$ is given by

$$X(t) = \exp(tN)X(0), \qquad X(0) \in (\mathbb{R}_S)^{p|q}.$$

The general method to obtain $\exp(tA)$ by making the quotient by the characteristic polynomial of N is not available as it generally holds for commutative rings satisfying that the product of two elements is zero if and only if one of them is equal to zero. In any case, it is possible to work out $X(t)$ easily for simple cases. For instance

$$N \in \mathrm{SMat}_{1,1}(\mathbb{R}_{S[1]}), \qquad \mathbb{R}_{S[1]} := \langle e_1, \theta_1 \rangle.$$

In this situation

$$X = \begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}, \; N = \begin{bmatrix} y_1 & \zeta_1 \\ \zeta_2 & y_2 \end{bmatrix}, \; N_0 := \begin{bmatrix} y_1 & 0 \\ \zeta_2 & y_2 \end{bmatrix}.$$

The nilpotency of odd elements implies $NX = N_0X$. Thus

$$X(t) = \exp(tN)X(0) = \exp(tN_0)X(0).$$

Assuming $y_1 = y_2$ and using several properties of N_0, e.g. its eigenvalues, its decomposition into commuting supermatrices or the use of superalgebra morphisms, we obtain

$$\exp(tN_0) = \begin{bmatrix} \exp(ty_1) & 0 \\ t \cdot \exp(ty_1)\zeta_2 & \exp(ty_1) \end{bmatrix}.$$

We see that $\mathrm{Ber}(\exp(tN_0)) \neq 0$ for every $t \in \mathbb{R}$. As one might expect, the mapping $\exp(tN_0)$ satisfies the equation (7) for $A = N_0$.
Proposition 9 shows that the space of solutions to a differential equation (10) can be identified with the space $(\mathbb{R}_S)^{p|q}$. The set of stable points of the equation (10) is $\bigcap_{t\in\mathbb{R}} \ker N(t)$.

11. Conclusions and outlook

The evolution of a Grassmann-valued differential equation can be determined by a certain t-parametric set of morphisms that may not be even. It is interesting to try to understand this as a certain curve in the sense of the functor of points as done in [2] for the case of differential equations on supermanifolds.

Moreover, we plan to study methods to determine $\exp(tA)$ by using the eigenvalues of A. Unfortunately, the celebrated method of the characteristic polynomial to obtain $\exp(tA)$ is no longer available as it only works well with commutative integral domains. Grassmann algebras, the most important example in physical applications, do not satisfy these properties.
Finally, we aim to study solutions to Grassmann differential equations (8) and (10) when $N(t)$ is a general t-dependent function and to the relations between their solutions. This should lead to study representations of super Lie groups and super Lie algebras.

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