$$\boxed{f(x) = \sum_{n=0}^{\infty} a_n x^n}$$

Showed

Part (c)

$x = 1$, $N = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 30, 50$.

$N = 0$

$$\boxed{f_{TS}^{0}(x) = 1}$$

and at $x=1$ same

$$\boxed{f_{TS}^{0}(x) = 1}$$

c_0

$N = 1$

$$f_{TS}^{1}(x) = \sum_{n=0}^{1} \frac{(-1)^n x^n}{n+1}$$

$$f_{TS}^{1}(x) = \frac{(-1)^0 x^0}{0+1} + \frac{(-1)^1 x^1}{1+1}$$

$$\boxed{f_{TS}^{1}(x) = \underset{c_0}{1} - \underset{c_1}{\frac{x}{2}}}$$

at $x=1$ $f_{TS}^{1}(1) = 1 - 1/2$

$$\boxed{f_{TS}^{1}(1) = 0.5}$$

$N = 2$

$$f_{TS}^{2}(x) = 1 - \frac{x}{2} + \frac{(-1)^2 x^2}{2+1}$$

$$\boxed{f_{TS}^{2}(x) = 1 - \frac{x}{2} + \frac{x^2}{3}}$$

at $x=1$, $f_{TS}^{2}(1) = 1 - \frac{1}{2} + \frac{1}{3}$

$$f_{TS}^{2}(1) = 1 - 0.5 + 0.33 \Rightarrow \boxed{f_{TS}^{2}(1) = 0.83}$$

$N=3$

$$f_{TS}^{3}(x) = 1 - \frac{x}{2} + \frac{x^2}{3} + \frac{(-1)^3 x^3}{3+1}$$

$$\boxed{f_{TS}^{3}(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4}}$$

at $x=1$

$$f_{TS}^{3}(1) = 0.83 - \frac{1}{4} \Rightarrow \boxed{f_{TS}^{3}(1) = 0.58}$$

$N=4$

$$f_{TS}^{4}(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{(-1)^4 x^4}{4+1}$$

$$\boxed{f_{TS}^{4}(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5}}$$

at $x=1$

$$f_{TS}^{4}(1) = 0.58 + \frac{1}{5} \Rightarrow \boxed{f_{TS}^{4}(1) = 0.78}$$

$N=5$

$$\boxed{f_{TS}^{5}(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \frac{x^5}{6}}$$

$$f_{TS}^{5}(1) = 0.78 - \frac{1}{6} \Rightarrow \boxed{f_{TS}^{5}(1) = 0.613}$$

$N=6$

$$\boxed{f_{TS}^{6} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \frac{x^5}{6} + \frac{x^6}{7}}$$

at $x=1$

$$f_{TS}^{6}(1) = 0.613 + \frac{1}{7} \Rightarrow \boxed{f_{TS}^{6}(1) = 0.7558}$$

$N=7$

$$\boxed{f_{TS}^{7} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \frac{x^5}{6} + \frac{x^6}{7} - \frac{x^7}{8}}$$

at $x=1$

$$f_{TS}^{7}(1) = 0.7558 - \frac{1}{8} \Rightarrow \boxed{f_{TS}^{7}(1) = 0.6308}$$

$N=8$

$$\boxed{f_{TS}^{8}(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \frac{x^5}{6} + \frac{x^6}{7} - \frac{x^7}{8} + x^8/9}$$

at $x=1$

$$f_{TS}^{8}(1) = 0.6308 + \frac{1}{9} \Rightarrow \boxed{f_{TS}^{8}(1) = 0.7419}$$

$N=9$

$$\boxed{f_{TS}^{9}(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \frac{x^5}{6} + \frac{x^6}{7} - \frac{x^7}{8} + \frac{x^8}{9} - x^9/10}$$

at $x=1$

$$f_{TS}^{9}(1) = 0.7419 - \frac{1}{10} \Rightarrow \boxed{f_{TS}^{9}(1) = 0.6419}$$

$N=10$

$$\boxed{f_{TS}^{10}(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \frac{x^5}{6} + \frac{x^6}{7} - \frac{x^7}{8} + \frac{x^8}{9} - \frac{x^9}{10} + \frac{x^{10}}{11}}$$

at $x=1$

$$f_{TS}^{10}(1) = 0.6419 + \frac{1}{11} \Rightarrow \boxed{f_{TS}^{10}(1) = 0.7328}$$

$\underline{N = 20}$

$$\boxed{f_{TS}^{20}(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \frac{x^5}{6} + \frac{x^6}{7} - \frac{x^7}{8} + \frac{x^8}{9} - \frac{x^9}{10} + \frac{x^{10}}{11} - \frac{x^{11}}{12} + \frac{x^{12}}{13} - \frac{x^{13}}{14} + \frac{x^{14}}{15} - \frac{x^{15}}{16} + \frac{x^{16}}{17} - \frac{x^{17}}{18} + \frac{x^{18}}{19} - \frac{x^{19}}{20} + \frac{x^{20}}{21}}$$

at $x = 1$

$$f_{TS}^{20}(1) = 0.7328 - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \frac{1}{15} - \frac{1}{16} + \frac{1}{17} - \frac{1}{18} + \frac{1}{19} - \frac{1}{20} + \frac{1}{21}$$

$$f_{TS}^{20}(1) = 0.71502739{\small 9}16 - \frac{1}{20} + \frac{1}{21}$$

$$\boxed{f_{TS}^{20}(1) = 0.7126}$$

(Stopping at order 20 in calculations)

Now Pade approximants

$P_N^N(x=1)$ for $N = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20$

$$\Rightarrow P_0^0 = \frac{a_0}{1} = f_{TS}^{0+0} = 1 = c_0$$

so

$$a_0 = 1 \quad \text{and} \quad \boxed{P_0^0 = 1}$$

$$\Rightarrow P_1^1(x) \overset{M=1,\,N=1}{=} \frac{a_0 + a_1 x^1}{1 + b_1 x^1} = f_{TS}^{2 \to M+N}(x) = 1 - \frac{1}{2}x + \frac{x^2}{3}$$

$$\frac{a_0 + a_1 x}{1 + b_1 x} = 1 - \frac{1}{2}x + \frac{x^2}{3}$$

$$a_0 + a_1 x = (1 + b_1 x)\left(1 - \frac{1}{2}x + \frac{1}{3}x^2\right)$$

$$(a_0 + a_1 x) = 1 - \frac{1}{2}x + \frac{1}{3}x^2 + b_1 x - \frac{b_1}{2}x^2 + \frac{b_1}{3}x^3$$

$$\boxed{a_0 = 1}$$

$$a_1 x = -\frac{1}{2}x + b_1 x \Rightarrow a_1 = \frac{-1}{2} + b_1$$

$$0 = \frac{1}{3}x^2 - \frac{b_1}{2}x^2 \Rightarrow 0 = \frac{1}{3} - \frac{b_1}{2}$$

$$\Rightarrow \boxed{b_1 = \frac{2}{3}}$$

use here

$$\Rightarrow a_1 = \frac{-1}{2} + \frac{2}{3} \Rightarrow \boxed{a_1 = \frac{1}{6}}$$

So

$$P_1'(x) = \frac{1 + 1/6\, x}{1 + 2/3\, x}$$

at $x = 1$

$$P_1'(1) = \frac{1 + 1/6}{1 + 2/3}$$

$$P_1'(1) = \frac{7}{10} \Rightarrow \boxed{P_1'(1) = 0.7}$$

$$\Rightarrow P_2^2(x) = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2}$$

and

$$f_{TS}^{4 \to 2+2}(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5}$$

we already have $\boxed{a_0 = 1}$, $\boxed{a_1 = 1/6}$,
and $\boxed{b_1 = 2/3}$

Now,

$$\Rightarrow (a_0 + a_1 x + a_2 x^2) = (1 + b_1 x + b_2 x^2)\left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5}\right)$$

$$\Rightarrow a_0 + a_1 x + a_2 x^2 = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} + b_1 x - \frac{b_1}{2} x^2 + \frac{b_1}{3} x^3 - \frac{b_1}{4} x^4 + \frac{b_1}{5} x^5 + b_2 x^2 - \frac{b_2}{2} x^3 + \frac{b_2}{3} x^4 - \frac{b_2}{4} x^5 + \frac{b_2}{5} x^6$$

now,

$$a_2 x^2 = \frac{x^2}{3} - \frac{b_1}{2} x^2 + b_2 x^2 \rightarrow eq\ 1$$

$$0 = -\frac{x^3}{4} + \frac{b_1}{3}x^3 - \frac{b_2}{2}x^3$$

$$0 = -\frac{1}{4} + \frac{b_1}{3} - \frac{b_2}{2}$$, took x^3 common here.

$$0 = -\frac{1}{4} + \frac{2}{9} - \frac{b_2}{2}$$, used $b_1 = \frac{2}{3}$ here

$$\Rightarrow \frac{b_2}{2} = -\frac{1}{4} + \frac{2}{9}$$

$$\Rightarrow b_2 = -\frac{1}{2} + \frac{4}{9} \Rightarrow \boxed{b_2 = -\frac{1}{18}}$$

use it in eq (1)

$$\Rightarrow a_2 = \frac{1}{3} - \frac{b_1}{2} + b_2$$

$$\Rightarrow a_2 = \cancel{\frac{1}{3}} - \cancel{\frac{1}{3}} - \frac{1}{18}$$, used $b_1 = 2/3$ and $b_2 = -1/18$

$$\Rightarrow \boxed{a_2 = -\frac{1}{18}}$$

Now,

$$P_2^2(1) = \frac{a_0 + a_1 + a_2}{1 + b_1 + b_2}$$

$$\Rightarrow P_2^2(1) = \frac{1 + \frac{1}{6} - \frac{1}{18}}{1 + \frac{2}{3} - \frac{1}{18}} = \frac{20}{29}$$

$$\Rightarrow \boxed{P_2^2(1) = 0.6896}$$

$$\Rightarrow P_3^3(x) = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x + b_2 x^2 + b_3 x^3}$$

and

$$f_{1.6}^{6 \to 3+3} = \overset{c_0}{1} - \overset{c_1}{\frac{x}{2}} + \overset{c_2}{\frac{x^2}{3}} - \overset{c_3}{\frac{x^3}{4}} + \overset{c_4}{\frac{x^4}{5}} - \overset{c_5}{\frac{x^5}{6}} + \overset{c_6}{\frac{x^6}{7}}$$

Now,

$$a_3 = c_3 + c_2 b_1 + c_1 b_2 + c_0 b_3 \rightarrow (*)$$

$$0 = c_4 + c_3 b_1 + c_2 b_2 + c_1 b_3 \rightarrow (**)$$

use values of $b_1 = \frac{2}{3}$, $b_2 = \frac{-1}{18}$ in eq $(**)$ where,

$c_4 = \frac{1}{5}$, $c_3 = \frac{-1}{4}$, $c_2 = \frac{1}{3}$, $c_1 = -1/2$

$$0 = \frac{1}{5} - \frac{1}{\cancel{4}_{2}}\left(\frac{\cancel{2}}{3}\right) + \frac{1}{3}\left(-\frac{1}{18}\right) - \frac{1}{2}b_3$$

$$\frac{1}{2}b_3 = \frac{1}{5} - \frac{1}{6} - \frac{1}{54}$$

$$b_3 = \frac{2}{5} - \frac{1}{3} - \frac{1}{27} \Rightarrow \boxed{b_3 = \frac{4}{135}}$$

use this in eq (*)

$$a_3 = -\frac{1}{4} + \frac{1}{3}\left(\frac{2}{3}\right) - \frac{1}{2}\left(-\frac{1}{18}\right) + (1)\left(\frac{4}{135}\right)$$

$$a_3 = -\frac{1}{4} + \frac{2}{9} + \frac{1}{36} + \frac{4}{135}$$

$$\boxed{a_3 = \frac{4}{135}}$$

so

$$P_3^3(1) = \frac{a_0 + a_1 + a_2 + a_3}{1 + b_1 + b_2 + b_3}$$

$$P_3^3(1) = \frac{1 + \frac{1}{6} - \frac{1}{18} + \frac{4}{135}}{1 + \frac{2}{3} - \frac{1}{18} + \frac{4}{135}}$$

$$P_3^3(1) = \frac{308}{443} \Rightarrow \boxed{P_3^3(1) = 0.695259}$$