Assignment #1

Idea: use of Taylor series in regions of x-space where the series if divergent. One can devise a rational polynomial expression that is valid for a much wider range of x than the original Taylor series – yet the rational approximation (known as a Pade approximant) uses ONLY the Taylor series coefficients that lead to a divergent Taylor series!!

(A) Consider the function *f(x)* that is defined by

$$f(x) = \frac{\ln(1+x)}{x}.$$

Show that the Taylor series expansion of *f(x)* about x = 0 is

$$f_{TS}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1} \equiv \sum_{n=0}^{\infty} a_n x^n$$

(B) Evaluate, using appropriate software, the first N+1 terms of this series

$$f_{TS}^N(x) = \sum_{n=0}^{N} \frac{(-1)^n x^n}{n+1} \equiv \sum_{n=0}^{N} a_n x^n$$

at x = 1 and x = 10, for N = 1, 3, 5, 7, 9, 10, 20, 30, 50.

(C) Now, using software, determine the Pade approximants to $f_{TS}(x)$: where

$$P_N^M(x) = \frac{\text{polynomial degree M}}{\text{polynomial degree N}} = \frac{\sum_{j=0}^{M} A_j x^j}{1 + \sum_{k=1}^{N} B_k x^k}$$

The coefficients {A's, B's} are determined from

$$f_{TS}^{M+N}(x) = \frac{\sum_{j=0}^{M} A_j x^j}{\left(1 + \sum_{k=1}^{N} B_k x^k\right)} + O\left(x^{N+M+1}\right)$$

$$i.e., \quad \left(1 + \sum_{k=1}^{N} B_k x^k\right) . f_{TS}^{M+N}(x) = \sum_{j=0}^{M} A_j x^j + O\left(x^{N+M+1}\right)$$

and one equates the same powers of x on the l.h.s and r.h.s up to errors of

$$O\left(x^{N+M+1}\right).$$

[If one first determines B_N, the evaluation of all the other {B's and A's} is straightforward]

In particular, evaluate the "diagonal" Pade and the next Pade approximant approximants

$$P_{N+1}^N(x=1) \; for \; N = 0,1,2,3,4,5,6,7,8,9,10,20,30,50$$

$$P_N^N(x=1) \; for \; N = 0,1,2,3,4,5,6,7,8,9,10,20,30,50$$

Also evaluate these Pade approximants at x = 10 for the same series of N values.

Assignment #2

Given the equations

$$\begin{aligned} 10x + 2y - z &= 27 \\ -3x - 5y + 2z &= -61.5 \\ x + y + 6z &= -21.5 \end{aligned}$$

a) Solve the system by naive Gauss elimination. Show all steps of the computation;
b) Write a numerical code to solve a system of *n* equations by naive Gauss elimination. Test the code with the equation system solved in part a)
.

Assignment #3

Given the equations

$$\begin{aligned} 2x - 6y - z &= -38 \\ -3x - y + 7z &= -34 \\ -8x + y - 2z &= -20 \end{aligned}$$

a) Solve the system by Gauss elimination with partial pivoting. As part of the computation, use the diagonal elements to calculate the determinant. Show all steps of the computation.
b) Write a numerical code to solve a system of *n* equations by Gauss elimination with partial pivoting. Test the code with the equation system solved in part a).

Assignment #4

You are given the following system of equations:

$$\begin{aligned} 2x + 3y &= 3 \\ x + 2y &= 1 \end{aligned}$$

a) Determine whether the system is ill-conditioned;
b) Solve this system using a graphical method with a numerical code;
c) Solve this system using LU decomposition with a numerical code;

Assignment #5

The following second-order ODE is considered to be stiff:

$$\frac{d^2y}{dx^2} = -1001\frac{dy}{dx} - 1000y$$

with initial conditions:
$y(0) = 1$ and $y'(0) = 0$

a) Solve this differential equation (i) analytically for $x = 0$ to 5. Also, solve the ODE numerically using Euler's method with implicit function (ii), explicit function (iii) and an ode solver e.g., ode23s (iv). For case (ii) and (iii) use a step size of h = 0.5.
b) Compare the results obtained with (i), (ii), (iii) and (iv). Display the results graphically;
c) Find a step size for the fixed-step size methods that allow to have a solution like that obtained with the analytical equation.

Assignment #6

Let us consider the following integral:

$$\int_{-3}^{5} (4x-3)^3 dx$$

a) Integrate the function both analytically and numerically using two alternative methods: trapezoidal and Simpson's 1/3. Develop a computer program to solve numerically this function for both trapezoidal and Simpson's 1/3 methods with n intervals;

b) Compute the percent relative error for the numerical results obtained with different n intervals (2-100) and explain the difference between the result obtained with trapezoidal and Simpson's 1/3 methods;

Assignment #7

Use order of h^8 Romberg integration to calculate the following integral:

$$\int_{1}^{2} \left(x+\frac{1}{x}\right)^2 dx$$

with an accuracy of $\varepsilon_s = 0.5\%$ based on Eq. 22.9.

a) Develop a computer program to solve numerically this function with Romberg integration and trapezoidal method with n intervals;

b) Use the analytical solution of the integral to determine the relative percentage error obtained with Romberg integration;

c) Compare ε_s and ε_t;