V5A2 – RIGID ANALYTIC GEOMETRY
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Preliminaries

These notes roughly correspond to the course **V5A2 - Rigid Analytic Geometry** taught by Prof. Jens Franke at the Universität Bonn in the Summer 2025 semester. These notes are LaTeX-ed after the fact with significant alteration and are subject to misinterpretation and mistranscription. Use with caution. Any errors are undoubtedly my own and any virtues that could be ascribed to these notes ought be attributed to the instructor and not the typist. These notes assume knowledge of the course on the same topic held in the Winter 2024-25 semester.

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Contents

1. Lecture 1 – 17th April 2025

We fix the following notation.

Notation 1.1. (i) K is a field complete with respect to a non-Archimedean norm.
(ii) We denote the Tate algebra

$$\mathbb{T}_n = \left\{ f \in \sum_{\alpha \in \mathbb{N}^n} f_\alpha X^\alpha : \forall \varepsilon > 0, |\{\alpha : |f_\alpha|_K \geq \varepsilon\}| < \infty \right\} \subseteq K[[X_1, \ldots, X_n]]$$

the subring of convergent power series, with norm $\|f|\mathbb{T}_n\| = \max_{\alpha \in \mathbb{N}^n} |f_\alpha|$.

Remark 1.2. (i) The norm $|\cdot|_K$ extends uniquely to any algebraic extension of K.
(ii) The Tate algebra $\mathbb{T}_n$ is Noetherian, hence all ideals are closed.

Affinoid algebras are quotients of Tate algebras.

Definition 1.3 (Affiniod Algebra)**.** A K-algebra A is an affinoid K-algebra if it is of the form $\mathbb{T}_n/I$.

There is an induced norm on the Tate algebra known as the residual norm.

Definition 1.4 (Residual Norm)**.** Let A be an affinoid K-algebra. The residue norm of $a \in A$ is

$$\|a\| = \inf\{\|f|\mathbb{T}_n\| : \overline{f} = a\}.$$

Remark 1.5. Definition 1.4 is independent of the choice of representative.

As in algebraic geometry, affinoid algebras give rise to ringed spaces via the Tate spectrum. We discuss the construction by first defining the space, and the sheaf of rings on it.

Definition 1.6 (Tate Spectrum – Set)**.** Let A be an affinoid K-algebra. The set underlying the Tate spectrum $\mathrm{Sp}(A)$ is $\mathrm{mSpec}(A)$.

Remark 1.7. The Tate spectrum is endowed with the property that $[\kappa(x) : K] < \infty$, where $\kappa(x) = A/\mathfrak{m}_x$ is a field as the ideal $\mathfrak{m}_x$ corresponding to x is maximal.

The topology on the set is defined by rational sieves.

Definition 1.8 (Rational Open Set)**.** Let $\langle f_0, \ldots, f_n \rangle_A = A$. The rational open associated to the generators $R_A(f_0|f_1, \ldots, f_n)$ is given by

$$R_A(f_0|f_1, \ldots, f_n) = \{x \in \mathrm{Sp}(A) : |f_i(x)| < |f_0(x)|, 1 \leq i \leq n\}.$$

Remark 1.9. Rational open subsets are preserved under finite intersection. For $\langle f_0, \ldots, f_n \rangle_A, \langle g_0, \ldots, g_m \rangle_A$ generators of A, the intersection

$$R_A(f_0|f_1, \ldots, f_n) \cap R_A(g_0|g_1, \ldots, g_m) = R_A(f_0 g_0|f_i g_j, 1 \leq i \leq n, 1 \leq j \leq m).$$

These rational open sets form the basis for the topology on the Tate spectrum $\mathrm{Sp}(A)$.

Definition 1.10 (Tate Spectrum – Topology)**.** Let A be an affinoid K-algebra. The set underlying the Tate spectrum $\mathrm{Sp}(A)$ has a topology with basis consisting of the rational open sets $R_A(f_0|f_1, \ldots, f_n)$ and with Grothendieck topology obtained by enforcing quasicompactness of the rational open sets.

In some simple cases, the underlying space of the Tate spectrum admits a description.

Example 1.11. Let $K = \overline{K}$. $\mathrm{Sp}(\mathbb{T}_n) = (K^\circ)^n$, where K° is the subring of powerbounded elements of K. Each point $x \in \mathrm{Sp}(A)$ is taken to $(\xi_i)_{i=1}^n$ where ξ_i is the image of X_i in $K \cong \kappa(x)$ and an n-tuple of powerbounded elements of K is taken to the ideal of $\mathbb{T}_n$ consisting of functions vanishing at that tuple. In this case, the basis for the ordinary topology on the Tate spectrum is identified with non-Archimedean balls $d(\xi, \nu) = \max_{1 \leq i \leq n} |\xi_i - \nu_i|$.

We now want to define the structure sheaf on $\mathrm{Sp}(A)$ which will be valued in the category affinoid K-algebras Aff_K. This is a full subcategory of the category of K-algebras as all maps between affinoid K-algebras are automatically continuous.

The structure sheaf is defined as follows.

Definition 1.12 (Tate Spectrum – Structure Sheaf)**.** Let A be an affinoid K-algebra. The functor

$$R_A(f_0|f_1, \ldots, f_n) \mapsto A\left\langle \frac{\varepsilon}{f_0} \right\rangle \left\langle \frac{f_1}{f_0}, \ldots, \frac{f_n}{f_0} \right\rangle$$

where $\varepsilon \in K^\times$ such that $\max_{0 \leq i \leq n} |f_i(x)| \geq |\varepsilon|$ for all $x \in \mathrm{Sp}(A)$ represents the functor $\mathsf{Rat}_A^{\mathsf{Opp}} \to \mathsf{Aff}_K$

$$F_\Omega(B) = \{\varphi \in \mathrm{Hom}_{\mathsf{Aff}_K}(A, B) : \mathrm{Sp}(\varphi)(\mathrm{Sp}(B)) \subseteq \Omega\},$$

where $\Omega = \mathcal{O}_{\mathrm{Sp}(A)}(A\left\langle \frac{\varepsilon}{f_0} \right\rangle \left\langle \frac{f_1}{f_0}, \ldots, \frac{f_n}{f_0} \right\rangle$.

Summing up the preceding constructions, we have:

Definition 1.13 (Tate Spectrum – Ringed Space)**.** Let A be an affinoid K-algebra. The Tate spectrum is given by:

- Topological space $\mathrm{mSpec}(A)$ with basis for the topology given by rational open subsets $R_A(f_0|f_1, \ldots, f_n)$ with $\langle f_0, \ldots, f_n \rangle_A = A$.
- Sheaf of rings given by $R_A(f_0|f_1, \ldots, f_n) \mapsto A\left\langle \frac{\varepsilon}{f_0} \right\rangle \left\langle \frac{f_1}{f_0}, \ldots, \frac{f_n}{f_0} \right\rangle$.

Here we used the fact that any sheaf on the base extends to a sheaf on the space.

Remark 1.14. (i) There are identifications $\mathcal{O}_{\mathrm{Sp}(\mathcal{O}_{\mathrm{Sp}(A)}(\Omega))} \cong \mathcal{O}_{\mathrm{Sp}(A)}|_\Omega$.
(ii) By Tate acyclicity, the higher cohomology of $\mathcal{O}_{\mathrm{Sp}(A)}$ vanishes.

We state some additional results surrounding Tate acyclicity.

Definition 1.15 (Laurent Order)**.** Let $\mathcal{S}$ be a sieve on $\mathrm{Sp}(A)$. We define the Laurent order $\mathfrak{o}_L(\mathcal{S})$ inductively as follows:

- $\mathfrak{o}_L(\mathcal{S}) = 0$ if and only if it is the all sieve.

- $\mathfrak{o}_L(\mathcal{S}) \leq k$ if there is $g \in \mathcal{O}_X(\Omega)$ such that the restriction sieves $\mathcal{S}|_{R_\Omega(g|1)}$ and $\mathcal{S}|_{R_\Omega(1|g)}$ have Laurent order at most k.
- $\mathcal{S}$ is of Laurent order k if k is the smallest number such that $\mathcal{S}$ is of Laurent order at most k

Finiteness of the Laurent order characterizes covering sieves.

Proposition 1.16. Let $\mathcal{S}$ be a sieve on $\mathrm{Sp}(A)$ for A an affinoid K-algebra. $\mathcal{S}$ is a covering sieve if and only if $\mathfrak{o}_L(\mathcal{S}) < \infty$.

This immediately gives a simple sufficient condition for Tate acyclicity.

Corollary 1.17. Let $\mathcal{F}$ be a sheaf of Abelian groups on $\mathrm{Sp}(A)$. If

$$0 \to \mathcal{F}(\Omega) \to \mathcal{F}(R_\Omega(g|1)) \oplus \mathcal{F}(R_\Omega(1|g)) \to \mathcal{F}(R_\Omega(g|1) \cap R_\Omega(1|g)) \to 0$$

is exact for all $\Omega \subseteq \mathrm{Sp}(A)$ rational and $g \in \mathcal{O}_{\mathrm{Sp}(A)}(\Omega)$ then $\mathcal{F}$ is acyclic.

We state two additional results concerning the unviersality of certain affinoid K-algebras. We first recall the following definitions.

Definition 1.18 (nat Ring)**.** Let A be a topological ring. A is a nat ring if it has a basis of neighborhoods of zero consisting of open subgroups.

Definition 1.19 (Tate Ring)**.** A nat ring A is Tate if if it has a powerbounded neighborhood of zero and has a topologically nilpotent unit known as a quasi-uniformizer.

In turn:

Proposition 1.20. Let A be a Tate ring and

$$A\langle f_1, \dots, f_n\rangle = A\langle X_1, \dots, X_n\rangle / \langle X_1 - f_1, \dots, X_n - f_n\rangle.$$

$A\langle f_1, \dots, f_n\rangle$ is initial among nat A-algebras B where $f_1, \dots, f_n$ are powerbounded. Furthermore, $A\langle f_1, \dots, f_n\rangle$ contains A as a dense subring.

Proposition 1.21. Let A be a Tate ring and

$$A\left\langle \frac{1}{f_1}, \dots, \frac{1}{f_n}\right\rangle = A\langle X_1, \dots, X_n\rangle / \left\langle X_1 - \frac{1}{f_1}, \dots, X_n - \frac{1}{f_n}\right\rangle.$$

$A\langle \frac{1}{f_1}, \dots, \frac{1}{f_n}\rangle$ is initial among nat A-algebras B where $f_1, \dots, f_n$ are units with $\frac{1}{f_1}, \dots, \frac{1}{f_n}$ powerbounded. Furthermore, $A\langle \frac{1}{f_1}, \dots, \frac{1}{f_n}\rangle$ contains $A[\frac{1}{f_1}, \dots, \frac{1}{f_n}]$ as a dense subring.

We are now ready to define coherent sheaves.

We now begin marginal labeling, which follows the lecture.

We begin with the following preparatory result.

Proposition 1.22. Let A be an affinoid algebra and $\Omega \subseteq \mathrm{Sp}(A)$ a rational subset. Then:

Proposition 2.1

(i) For $B = \mathcal{O}_{\mathrm{Sp}(A)}(\Omega)$ and $\mathfrak{m} \in \mathrm{Sp}(B)$, there is an isomorphism of K-algebras $B^\wedge_{\mathfrak{m}} \cong A^\wedge_{\widetilde{\mathfrak{m}}}$ where $\widetilde{\mathfrak{m}}$ is the preimage of $\mathfrak{m}$ under the map $A \to B$ and $(-)^\wedge_I$ is the completion of a ring with respect to the ideal I.
(ii) B is flat as an A-algebra.

Proof of (i). We first show a claim:

(†) For all $n \in \mathbb{N}$, $A/\widetilde{\mathfrak{m}}^n \to B/\mathfrak{m}^n$ is an isomorphism.

Note that $B/\mathfrak{m}^n$ is initial amongst B-algebras C such that $\mathfrak{m}^n C = 0$, while $A/\mathfrak{m}^n$ is initial amongst A-algebras $\tilde{C}$ such that $\mathfrak{m}^n \tilde{C} = 0$. For C as above, the image of $\mathrm{Sp}(C)$ in $\mathrm{Sp}(B)$ is $\mathfrak{m}$, while the image of $\mathrm{Sp}(\tilde{C})$ in $\mathrm{Sp}(A)$ is $\widetilde{\mathfrak{m}} \in \Omega$. Applying the universal property twice, $\tilde{C}$ can be endowed uniquely with the structure of a B-algebra, and by $\kappa(\mathfrak{m}) \cong \kappa(\widetilde{\mathfrak{m}})$ it follows that $\tilde{C}$ is annihilated by $\widetilde{\mathfrak{m}}$. Thus both $A/\widetilde{\mathfrak{m}}^n, B/\mathfrak{m}^n$ satisfy the same universal property, hence are isomorphic.

The desired claim follows from (†) by passage to the limit. ■

Proof of (ii). By a standard result in commutative algebra, it suffices to show $B_{\mathfrak{m}}$ is A-flat for all $\mathfrak{m} \in \mathrm{mSpec}(B)$. B being Noetherian, $B_{\mathfrak{m}}^{\wedge}$ is a faithfully flat $B_{\mathfrak{m}}$-algebra, whereby it is sufficient to show that $B_{\mathfrak{m}}^{\wedge}$ is flat over A. But $B_{\mathfrak{m}}^{\wedge} \cong A_{\widetilde{\mathfrak{m}}}^{\wedge}$ by (i), which is a flat A-module as A is Noetherian, giving the claim. ■

As in the case of algebraic geometry, coherent sheaves are defined as $\widetilde{(-)}$-ifications of finitely generated modules.

Definition 2.1

Definition 1.23 ($\widetilde{(-)}$)**.** Let A be an affinoid K-algebra and M a finitely generated A-module. The sheaf $\widetilde{M}$ is the sheafification of the presheaf $\Omega \mapsto M \otimes_A \mathcal{O}_{\mathrm{Sp}(A)}(\Omega)$ on rational open subsets.

Since each module in the exact sequence

$$0 \to \mathcal{O}_{\mathrm{Sp}(A)}(\Omega) \to \mathcal{O}_{\mathrm{Sp}(A)}(R_\Omega(1|g)) \oplus \mathcal{O}_{\mathrm{Sp}(A)}(R_\Omega(g|1)) \to \mathcal{O}_{\mathrm{Sp}(A)}(R_\Omega(g|1, g^2)) \to 0$$

is flat, exactness is preserved under $- \otimes_A M$. In particular, by Corollary 1.17 we have:

Proposition 2.2

Proposition 1.24. Let A be an affinoid K-algebra and M a finitely generated A-module with associated sheaf $\widetilde{M}$. Then $H^p(\Omega, \widetilde{M}) = 0$ for all $p > 0$ and for all rational Ω.

Proof. By flatness of $\mathcal{O}_{\mathrm{Sp}(A)}(\Omega)$ as an A-algebra, exactness of the sequence for Ω rational, we have a short exact sequence

$$0 \to \mathcal{O}_{\mathrm{Sp}(A)}(\Omega) \to \mathcal{O}_{\mathrm{Sp}(A)}(\Omega_1) \oplus \mathcal{O}_{\mathrm{Sp}(A)}(\Omega_2) \to \mathcal{O}_{\mathrm{Sp}(A)}(\Omega_1 \cap \Omega_2) \to 0$$

and by flatness of $\mathcal{O}_{\mathrm{Sp}(A)}(\Omega)$ (vis. [Stacks, Tag 00M5]), we get that

$$0 \to \mathcal{O}_{\mathrm{Sp}(A)}(\Omega) \otimes_A M \to (\mathcal{O}_{\mathrm{Sp}(A)}(\Omega_1) \oplus \mathcal{O}_{\mathrm{Sp}(A)}(\Omega_2)) \otimes_A M \to \mathcal{O}_{\mathrm{Sp}(A)}(\Omega_1 \cap \Omega_2) \otimes_A M \to 0$$

is exact, so by noting that $M \otimes_A \mathcal{O}_{\mathrm{Sp}(A)}(\Omega) \cong \widetilde{M}(\Omega)$, we have that $\widetilde{M}$ is acyclic. ■

2. Lecture 2 – 24th April 2025

We define coherent sheaves.

Definition 2.2

Definition 2.1 (Coherent Sheaves)**.** Let A be an affinoid K-algebra and $\mathcal{F}$ an $\mathcal{O}_{\mathrm{Sp}(A)}$-module. $\mathcal{F}$ is coherent if it is of the form $\widetilde{M}$ for some finitely generated A-module M.

The coherence condition can be shown to be local in the sense that any sheaf of modules for which there exists a covering sieve consisting of a trivialization by a cover by rational opens on which the modules are finitely generated is coherent. We show this as a consequence of a sequence of results.

Proposition 2.3

Proposition 2.2. Let A be an affinoid K-algebra and Ω_1, Ω_2 a rational cover of $\mathrm{Sp}(A)$ with intersection Ω_{12}. If $f_{12} \in \mathcal{O}_{\mathrm{Sp}(A)}(\Omega_{12})$ then $f_{12} = f_1|_{\Omega_{12}} + f_2|_{\Omega_{12}}$ for $f_i \in \mathcal{O}_{\mathrm{Sp}(A)}(\Omega_i)$ with $\|f_i\| = O(\|f_{12}\|)$.

Proof. Omitted. ∎

Lemma 2.1

Lemma 2.3. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_{\mathrm{Sp}(A)}$-modules on $\mathrm{Sp}(A)$ and Ω_1, Ω_2 a rational cover of $\mathrm{Sp}(A)$ on which $\mathcal{M}|_{\Omega_i} = \widetilde{M_i}$ with M_i finitely generated. If $m_{12} \in \mathcal{M}(\Omega_{12})$ then there are $m_i \in \mathcal{M}(\Omega_i)$ with $\|m_i|\mathcal{M}(\Omega_i)\| = O(\|m_{12}\|)$ (the constant independent of m_{12}) and such that

$$\|m_{12} - m_1|_{\Omega_{12}} - m_2|_{\Omega_{12}}\| \leq \frac{1}{2}\|m_{12}\|.$$

Finish proof.

Lemma 2.2

Lemma 2.4. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_{\mathrm{Sp}(A)}$-modules on $\mathrm{Sp}(A)$ and Ω_1, Ω_2 a rational cover of $\mathrm{Sp}(A)$ on which $\mathcal{M}|_{\Omega_i} = M_i$ with M_i finitely generated. If $m_{12} \in \mathcal{M}(\Omega_{12})$ then

$$m_{12} = m_1|_{\Omega_{12}} + m_2|_{\Omega_{12}}$$

where $m_i \in \mathcal{M}(\Omega_i)$ and $\|m_i|\mathcal{M}(\Omega_i)\| = O(\|m_{12}|\mathcal{M}(\Omega_{12})\|)$ with the implied constant independent of m_{12}.

Proof. Let C be the implied constant of Lemma 2.3. We can define recursively

$$\begin{aligned} m_{12} = m_{12}^{(0)} &= m_1^{(0)}|_{\Omega_{12}} + m_2^{(0)}|_{\Omega_{12}} + m_{12}^{(1)} \\ m_{12}^{(1)} &= m_1^{(1)}|_{\Omega_{12}} + m_2^{(0)}|_{\Omega_{12}} + m_{12}^{(2)} \\ \vdots \quad & \qquad\quad \vdots \end{aligned}$$

where $\|m_{12}^{(i+2)}|\mathcal{M}(\Omega_{12})\| \leq \frac{1}{2}\|m_{12}^{(i)}|\mathcal{M}(\Omega_{12})\|$ and $\|m_j^{(i)}|\mathcal{M}(\Omega_j)\| \leq C\|m_{12}^{(i)}|\mathcal{M}(\Omega_{12})\|$, hence $\|m_{12}^{(i)}|\mathcal{M}(\Omega_{12})\| \leq \frac{1}{2^i}\|m_{12}|\mathcal{M}(\Omega_{12})\|$ and $\|m_j^{(i)}|\mathcal{M}(\Omega_j)\| \leq \frac{C}{2^i}\|m_{12}|\mathcal{M}(\Omega_{12})\|$ and the assertion follows with $m_j = \sum_{i=0}^{\infty} m_j^{(i)} \in \mathcal{M}(\Omega_j)$ with the implied constant C. ∎

From this we deduce:

Corollary 2.1

Corollary 2.5. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_{\mathrm{Sp}(A)}$-modules on $\mathrm{Sp}(A)$ and Ω_1, Ω_2 a rational cover of $\mathrm{Sp}(A)$ on which $\mathcal{M}|_{\Omega_i} = M_i$ with M_i finitely generated. If $m_{12} \in \mathcal{M}(\Omega_{12})$ and $\varepsilon > 0$ then there are $m_i \in \mathcal{M}(\Omega_i)$ such that $m_{12} = m_1|_{\Omega_{12}} + m_2|_{\Omega_{12}}$ and $\|m_2|\mathcal{M}(\Omega_2)\| < \varepsilon$.

Proof. Choose $m_2' \in \mathcal{M}(\Omega_2)$ such that $\|m_{12} - m_2|_{\Omega_{12}}|\mathcal{M}(\Omega_{12})\| < \delta$ then $m_{12} - m_2' = m_1 + m_2''$ with $\|m_1|\mathcal{M}(\Omega_1)\| + \|m_2''|\mathcal{M}(\Omega_2)\| \leq C \cdot \delta$ with C as in Lemma 2.4. Then choose δ such that $C \cdot \delta < \varepsilon$. ∎

Corollary 2.2

Corollary 2.6. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_{\mathrm{Sp}(A)}$-modules on $\mathrm{Sp}(A)$ and Ω_1, Ω_2 a rational cover of $\mathrm{Sp}(A)$ on which $\mathcal{M}|_{\Omega_i} = M_i$ with M_i finitely generated. There are $(\mu_j)_{j=1}^{N_1} \in \mathcal{M}(\Omega)$ such that the $\mu_j|_{\Omega_1}$ generate $\mathcal{M}(\Omega_1)$.

Proof. By Corollary 2.5, $m_j^{(1)}|_{\Omega_{12}} = \mu_j^1|_{\Omega_{12}} + \mu_j^{(2)}|_{\Omega_{12}}$ where $\mu_j^{(k)} \in \mathcal{M}(\Omega_k)$ and $\|\mu_j^{(1)}|\mathcal{M}(\Omega) \leq \varepsilon$ for any ε. Let

$$\mu_j|_{\Omega_1} = m_j^{(1)} - \mu_j^{(1)}$$
$$\mu_j|_{\Omega_2} = \mu_j^{(2)}$$

then $\|\mu_j|_{\Omega_1} - m_j^{(1)}|\mathcal{M}(\Omega_j)\| \leq \varepsilon$ and when $\varepsilon = \frac{1}{2}$ the assertion follows. ∎

It follows that the set

$$\{g^{-k} \cdot m_1|_{\Omega_{12}} | m_1 \in \mathcal{M}(\Omega_1)\}$$

is dense in $\mathcal{M}(\Omega_{12})$, recalling here that $\Omega_1 = R_A(1|g)$.

Corollary 2.3

Corollary 2.7. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_{\mathrm{Sp}(A)}$-modules on $\mathrm{Sp}(A)$ and $\Omega_1 = R_A(1|g), \Omega_2 = R_A(g|1)$ for $g \in A$ a rational cover of $\mathrm{Sp}(A)$ on which $\mathcal{M}|_{\Omega_i} = M_i$ with M_i finitely generated. If $m_{12} \in \mathcal{M}(\Omega_{12})$ and $\varepsilon > 0$ then

$$m_{12} = g^{-k} \cdot m_1|_{\Omega_{12}} + m_2|_{\Omega_{12}}$$

where $m_i \in \mathcal{M}(\Omega_i)$ and $\|m_2|\mathcal{M}(\Omega_2)\| \leq \varepsilon$.

Proof. One need only repeat the arguments of Corollary 2.5 ∎

Corollary 2.4

Corollary 2.8. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_{\mathrm{Sp}(A)}$-modules on $\mathrm{Sp}(A)$ and $\Omega_1 = R_A(1|g), \Omega_2 = R_A(g|1)$ for $g \in A$ a rational cover of $\mathrm{Sp}(A)$ on which $\mathcal{M}|_{\Omega_i} = M_i$ with M_i finitely generated. There are $(\mu_j^{(2)})_{j=1}^{N_2} \in \mathcal{M}(\Omega)$ such that $\mu_j|_{\Omega_2}$ generate $\mathcal{M}(\Omega_2)$.

Proof. Write

$$m_j^{(2)}|_{\Omega_{12}} = g^{-k}\mu_j^{(1)}|_{\Omega_{12}} + \mu_j^{(2)}|_{\Omega_{12}}$$
$$\mu_j|_{\Omega_2} = \mu_j^{(1)}$$
$$\mu_j|_{\Omega_2} = g^k(m_j^{(2)} - \mu_j^{(2)})$$

and the assertion follows when $\varepsilon \leq \frac{1}{2}$ in which case $\|m_j^{(2)} - g^k \cdot \mu_j|\mathcal{M}(\Omega_2)\| \leq \frac{1}{2}$. ∎

3. Lecture 3 – 8th May 2025

We prove the locality statement earlier alluded to. As before $X = \mathrm{Sp}(A)$ for an affinoid K-algebra A.

Proposition 2.4

Proposition 3.1. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$-modules and $\mathcal{S}$ the sieve on X generated by these $\Omega \in \mathsf{Rat}_X$ for which $\mathcal{M}|_\Omega \cong \widetilde{M_\Omega}$ where M_Ω is a finitely generated $\mathcal{O}_X(\Omega)$-module. If $\mathcal{S}$ is a covering sieve, then $\mathcal{M}$ is coherent.

Proof. By induction on Laurent order, it suffices to show that for $g \in A$ and $\Omega_1 = R_A(1|g), \Omega_2 = R_A(g|1)$ rational opens of X on which $\mathcal{M}|_{\Omega_1} = \widetilde{M_1}, \mathcal{M}|_{\Omega_2} = \widetilde{M_2}$ where M_1, M_2 are finitely generated $\mathcal{O}_X(\Omega_1), \mathcal{O}_X(\Omega_2)$-modules respectively, that $\mathcal{M}$ is finitely generated too.

By Corollaries 2.6 and 2.8 there are sections $(m_i)_{i=1}^n$ generating $\mathcal{M}$ as an $\mathcal{O}_X$-module such that their restrictions to Ω_1, Ω_2 generate $\mathcal{M}|_{\Omega_1}, \mathcal{M}|_{\Omega_2}$. Consider

$$\mathcal{K} = \ker\left(\mathcal{O}_X^n \xrightarrow{(m_i)_{i=1}^N} \mathcal{M}\right).$$

We have $\mathcal{K}|_{\Omega_j} = \widetilde{K}_j$ where

$$K_j = \ker\left(\mathcal{O}_X(\Omega_j)^n \xrightarrow{(m_i|_{\Omega_j})_{i=1}^n} M_j\right).$$

Applying the same reasoning, we have that there are $(k_i)_{i=1}^m$ that generate $\mathcal{K}$ as an $\mathcal{O}_X$-module. It follows that $\mathcal{M}$ is the cokernel of

$$\mathcal{O}_X^m \xrightarrow{(k_i)_{i=1}^m} \mathcal{O}_X^n.$$

The universal property shows that $\mathcal{M}$ is isomorphic to the cokernel as it is an isomorphism of each Ω_j. Since $\widetilde{(-)}$ is exact, we obtain $\mathcal{M} = \widetilde{M}$ where M is the cokernel of $A^{\oplus m} \to A^{\oplus n}$ by the k_i's. ■

Remark 3.2. In general if $\mathcal{R}$ is a sheaf of rings on a site, we say an $\mathcal{R}$-module is finitely generated if there are fintely many global sections such that $\mathcal{R}^n \to \mathcal{M}$ is an epimorphism of sheaves. We say $\mathcal{M}$ is locally finitely generated if the objects on which $\mathcal{M}$ is finitely generated form a covering sieve, and $\mathcal{M}$ is coherent if it is locally finitely generated and the kernels of the local maps $\mathcal{R}|_X^n \to \mathcal{M}|_X$ are finitely generated.

Remark 3.3. On a one point space, this is the condition of the kernel being finitely generated. That is, that the ring is a coherent ring.

Example 3.4. Let us consider Remark 3.2 in the setting of $X = \mathrm{Sp}(A), \mathcal{R} = \mathcal{O}_X, \mathcal{M} = \widetilde{M}$ for M a finitely generated A-module. In this case, the kernel of the map $A^{\oplus n} \to M$ generate the kernel sheaf $\mathcal{O}_{\mathrm{Sp}(A)}^n \to \mathcal{M}$ so sheaves coherent in the sense of Definition 2.1 are coherent in the sense of Remark 3.2.

Dually, if $\mathcal{M}$ is coherent in the sense of Remark 3.2, we can use locality of coherence Proposition 3.1 to observe that the global sections generating $\mathcal{M}$ and the kernel sheaf $\mathcal{K}$ give rise to A-modules M, K such that M is the cokernel of $K \to A^n$.

This concludes our discussion of coherent sheaves.

Recall that $\mathrm{Sp}(\mathbb{T}_1)$ for K algebraically closed has van der Put points ξ given by the balls of radius $\leq R$ for $R \in |K| \subseteq \mathbb{R}_{\geq 0}$. Denote $\mathfrak{K}_{\leq R}$ of all balls $K_{\leq R}(X)$ for $x \in \mathrm{Sp}(A)$. For a van der Put point ξ of $\mathrm{Sp}(A)$, we can define M_ξ to be the set of all $r \in [0,1) \cap |K^\times|$ for which there exists an $x \in \mathfrak{K}_{\leq r} \cap \xi$ – that is, ξ contains a ball of radius r. Denote $K_{\leq R}(\xi)$ be set of rational open sets of radius at most R in the van der Put point ξ.

Example 3.5. If R is arbitrarily small, then $K_{\leq R}(\xi) = \{x\}$. In this case, $x \in \Omega$ if and only if $R(f_0|f_1, \ldots, f_n) = \Omega \in \xi$ if and only if $\nu(f_0) \geq \nu(f_i)$ where $\nu(f) = |f(x)|$ for all $1 \leq i \leq n$.

4. Lecture 4 – 15th May 2025

We consider the van der Put points of a Tate spectrum in terms of the adic spectrum.

The construction of the adic spectrum begins with Huber pairs.

Definition 4.1 (Huber Pair)**.** A Huber pair is a pair (A, A^+) where:

- A is a nat ring with an open bounded subring whose topology is I-adic for some finitely generated ideal I.
- $A^+ \subseteq A^\circ$ is an open integrally closed subring.

Definition 4.2 (Morphism of Huber Pairs)**.** A morphism of Huber pairs $\varphi : (A, A^+) \to (B, B^+)$ is a continuous homomorphism of rings $\varphi : A \to B$ such that $\varphi(A^+) \subseteq B^+$.

Remark 4.3. The subring of powerbounded elements A° is always an integrally closed subring of A. In particular, if (A, A^+) is a Huber pair, then also (A, A°) is a Huber pair. If further A is Tate, then for any morphism of Huber pairs $\varphi : (A, A^\circ) \to (B, B^\circ)$ the condition that $\varphi(A^\circ) \subseteq B^\circ$ is automatic and B is also Tate.

The adic spectrum is defined as a certain subspace of the space of valuations.

Definition 4.4 (Valuation)**.** A valuation on a ring A is a map $\nu : A \to \Gamma \cup \{0\}$, where Γ is written multiplicatively, such that:

(i) $0 < \gamma$ for all $\gamma \in \Gamma$.
(ii) $\nu(ab) = \nu(a) \cdot \nu(b)$.
(iii) $\nu(a + b) \leq \max\{\nu(a), \nu(b)\}$.
(iv) $\nu(1) = 1$.

Definition 4.5 (Support of Valuation)**.** Let ν be a valuation on a ring A. The support $\mathrm{supp}(\nu)$ of ν is the prime ideal $\{a \in A : \nu(a) = 0\} \subseteq A$.

Remark 4.6. We assume that Γ is generated by the image of $R \setminus \mathrm{supp}(\nu)$. In this case, $\nu \simeq \widetilde{\nu}$ if and only if there is a unique isomorphism of groups $\tau : \Gamma \to \widetilde{\Gamma}$ of groups $\widetilde{\nu} = \tau \circ \nu$.

We consider some additional constructions related to valuations.

Definition 4.7 (Convex Subset)**.** A subset X of Γ is convex if and only if for all $\gamma, \gamma' \in X$, $[\gamma, \gamma']_\Gamma \subseteq X$.

Definition 4.8 (Rank of Valuation)**.** Let ν be a valuation on a ring A. The rank of ν is the number of convex subgroups.

Remark 4.9. The set of convex subgroups is linearly ordered by inclusion.

We can define continuous valuations by topologizing $\Gamma \cup \{0\}$ with the "convergest sequences" topology as described below.

Definition 4.10 (Continuous Valuation)**.** We endow $\Gamma \cup \{0\}$ with the topology where all elements of Γ are open points and the collection of half-open intervals $\{[0, \gamma)_\Gamma : \gamma \in \Gamma\}$ form a neighborhood basis of 0.

Let ν be a valuation on a ring A. We call ν a continuous valuation if it is continuous as a map where $\Gamma \cup \{0\}$ is equipped with above described topology.

Remark 4.11. The following conditions are equivalent to a valuation being continuous:

- If $\nu(a) \neq 0$ then $\{a' \in A : \nu(a') < \nu(a)\}$ is open.
- The set $\{a \in A : \nu(a) < \gamma\}$ is open for $\gamma \in \Gamma$.

Definition 4.12 (Space of Continuous Valuations)**.** Let A be a Huber ring and let $\mathrm{Cont}(A)$ be the set of continuous valuations equipped with the topology that the rational open subsets

$$R_{\mathrm{Cont}(A)}(a_0|a_1, \dots, a_n) = \{\nu/\sim, \nu \text{ cts.} : \nu(a_0) \neq 0, \nu(a_i) \leq \nu(a_0)\ \forall 1 \leq i \leq n\}$$

such that $(a_0, \dots, a_n)$ is an open ideal in A form a basis of open sets.

Remark 4.13. For general topological rings, these rational open subsets may fail to be closed under finite intersection, hence we would require the rational open subsets to be merely a sub-basis of the topology. As for Huber rings rational open subsets are closed under finite intersections, this destinction is not necessary for our purposes.

We recall the following result.

Proposition 4.14. Let A be a Huber ring and let $\mathrm{Cont}(A)$ be the space of continuous valuations of A. $\mathrm{Cont}(A)$ is a spectral space with quasicompact topology base given by the rational open subsets.

This allows us to construct the adic spectrum.

Definition 4.15 (Affinoid Adic Space)**.** Let (A, A^+) be a Huber pair. The adic spectrum $\mathrm{Spa}(A, A^+)$ of (A, A^+) is the subspace

$$\bigcap_{a \in A^+} R_{\mathrm{Cont}(A)}(1|a) = \{\nu/\sim, \nu \text{ cts.} : \nu(a) \leq 1\ \forall a \in A^+\}$$

of $\mathrm{Cont}(A)$.

Corollary 4.16. Let A^+ be an integrally closed subring of a Huber ring A. The subspace

$$\bigcap_{a \in A^+} R_{\mathrm{Cont}(A)}(1|a) = \{\nu/\sim, \nu \text{ cts.} : \nu(a) \leq 1\ \forall a \in A^+\}$$

is spectral.

Proof. This is a proconstructible subset of a spectral space, hence spectral. ∎

Remark 4.17. In particular if A is Tate then the only open ideal is the ring itself, and $(a_0, \dots, a_n) = A$. Hence,

$$R_{\mathrm{Cont}(A)}(f_0|f_1, \dots, f_n) = \{\nu/\sim, \nu \text{ cts.} : \nu(f_i) \leq \nu(f_0)\ \forall 1 \leq i \leq n\}.$$

Henceforth we take A to be an affinoid K-algebra.

Example 4.18. If $\kappa \in K$ with $|\kappa| = 1$ and since $K^\circ \subseteq A^\circ$ also $1/\kappa \in K^\circ \subseteq A^\circ$. In particular, since $\nu(\kappa) \leq 1, \nu(1/\kappa) \leq 1$ we get $\nu(\kappa) = 1$ yielding that $|K^\times|$ is a subgroup of Γ. Up to replacing ν by an equivalent valuation as in Remark 4.6, we can assume that $|K^\times| \subseteq \Gamma$ which is typically not convex.

Definition 4.19. If $t \in \mathbb{R}$ such that $t^n \in |K^\times|$ for some $n \in \mathbb{N}$, say $\gamma \leq t$ (resp. $\gamma < t$) if and only if $\gamma^n \leq t^n$ (resp. $\gamma^n < t^n$). If $t \in \mathbb{R}_{>0}$ and there is no positive integer n with $t^n \in |K^\times|$ exists, say $\gamma \leq t$ (resp. $\gamma < t$) if and only if there exists $n \in \mathbb{N}$ and $\kappa \in K^\times$ such that $\gamma^n \leq |\kappa|^n < t^n$ (resp. $\gamma^n < |\kappa|^n < t^n$).

In what follows, we will use the following fact.

Lemma 4.20. Let A be an affinoid K-algebra. If $\| - |A\|$ is a residual norm, then a valuation ν on A such that $\nu((K^\circ)^\times) \subseteq \{1\}$ satisfies: ν is continuous if and only if there exists $c \in \mathbb{R}$ such that $\nu(a) \leq c\|a|A\|$ and $|K^\times|$ is cofinal in Γ (i.e. for all $\gamma \in \Gamma$, there is $\varepsilon \in K^\times$ such that $|\varepsilon| \leq \gamma$).

We will soon show that the space of van der Put points of the Tate spectrum $\mathrm{Sp}(A)$ is homeomorphic as a G_+-space to $\mathrm{Spa}(A, A^\circ)$ by constructing a specific bijection, and show that $\mathrm{Spa}(A, A^\circ)$ has Krull dimension that of A.

Example 4.21. Let $A = \mathbb{T}_1$ and $A^+ = K^\circ + A^{\circ\circ}$. (A, A^+) is a Huber pair and $\mathrm{Spa}(A, A^\circ) \setminus \mathrm{Spa}(A, A^+)$ has exactly one element ν where $\nu(f) = (\|f|\mathbb{T}_1\| - \kappa_f)$ where for $f = \sum_{j \geq 0} f_j T^j \in \mathbb{T}_1 \setminus \{0\}$, $\kappa_f = \max\{j : |f_j| = \|f|\mathbb{T}_1\|\}$.

Remark 4.22. The pair (A, A^+) is not necessarily topologically of finite type over (K, K°).

The proofs of the abovementioned result will require a careful treatment of power-bounded elements and the proof of the result on Krull dimension is also fairly subtle.

5. Lecture 5 – 22nd May 2025

We make preparations towards showing the result of van der Put-Schneider and Huber [vdPS95, Thm. 4] showing that the space of van der Put points on $\mathrm{Sp}(A)$ for A affinoid is in bijection with the points of the adic spectrum $\mathrm{Spa}(A, A^\circ)$ of the correct Krull dimension.

Proposition 5.1. Let $A \to B$ be a morphism of affinoid K-algebras.

(i) If B is finite over A, then B° is integral – but not necessarily finite – over A°.

(ii) $\widetilde{A} = A^\circ/A^{\circ\circ}$ is finite type over $K^\circ/K^{\circ\circ}$ with Krull dimension equal to that of A. Moreover, $\widetilde{B}$ is finite over $\widetilde{A}$.

Example 5.2. If $A = \mathbb{T}_n$, then $\widetilde{A} = A^\circ/A^{\circ\circ} \cong K[X_1, \ldots, X_n]$ the ordinary polynomial ring in n variables.

Proposition 5.3. Let A be a nat ring. Then $A\langle X_1, \ldots, X_n\rangle^\circ \cong A^\circ\langle X_1, \ldots, X_n\rangle$.

Lemma 5.4. Let A be an affinoid K-algebra, $\Omega = R_{\mathrm{Sp}(A)}(f_0|f_1, \ldots, f_n)$, and $B = \mathcal{O}_{\mathrm{Sp}(A)}(\Omega)$. Consider the following conditions: Lemma 3.1

(a) $\nu \in R_{\mathrm{Spa}(A,A^\circ)}(f_0|f_1, \ldots, f_n)$.

(b) ν can be extended to an element of $\mathrm{Spa}(B, B^\circ)$.

(c) ν admits a unique continuous extension to B.

then the following implications hold:

(i) (a) and (b) are equivalent.

(ii) (a) and (b) imply (c).

The statement of Lemma 5.4 reduces to showing this for Weierstrass domains (where $f_0 = 1$) and Laurent domains (where $f_1 = 1$).

Lemma 5.5. The statement of Lemma 5.4 holds for $f_0 = 1$. Proof ok?

Proof. Note that in this case $f_0 = 1$ so $\mathcal{O}_{\mathrm{Sp}(A)}(R_{\mathrm{Sp}(A)}(1|f_1, \ldots, f_n))$ is of the form $A\langle f_1, \ldots, f_n\rangle$.

(a)⇒(b) Suppose $\nu \in R_{\mathrm{Spa}(A,A^\circ)}(1|f_1, \ldots, f_n)$ so $\nu(f_i) \leq 1$ for all i. For $g \in A\langle X_1, \ldots, X_n\rangle$, we set $\nu_1(g) = \lim_{n\to\infty} v_n$ where $v_n = \nu(a_n)$ and

$$a_n = \sum_{|\alpha|\leq n} g_\alpha f^\alpha. \tag{5.1}$$

We observe that the limit $\nu_1(g)$ converges as by definition of the Tate algebra $A\langle X_1, \ldots, X_n\rangle$ there is an $N \in \mathbb{N}$ such that for each $\varepsilon \in \mathbb{R}_{>0}$ we have $|g_\alpha| < \varepsilon$ for $|\alpha| \geq N$. This implies that

$$\nu\left(\sum_{k<|\alpha|\leq l} g_\alpha f^\alpha\right) \leq \varepsilon \tag{5.2}$$

when $\min\{k, l\} \geq N$. If $\liminf_{n\to\infty} v_n = 0$ in (5.1) then $\lim_{n\to\infty} v_n = 0$. For any $\varepsilon > 0$, we can find N such that (5.2) holds, and the assertion follows. However, if

$\varepsilon = \liminf_{n\to\infty} |v_n|$ is positive, then we can find N such that (5.2) holds and $k > N$ with $|v_k| \geq \varepsilon$. For $l \geq k$ we have $\nu(a_l - a_k) < v_k = \nu(a_k)$ so $v_l = v_k$ and the sequence converges. This gives a continuous extension of ν to a continuous valuation which is bounded by Proposition 5.3.

(b)⇒(a),(c) Denote $B = \mathcal{O}_{\mathrm{Sp}(A)}(R_{\mathrm{Sp}(A)}(1|f_1, \ldots, f_n))$ and suppose that we have $\mu : A \to \Gamma_\mu$ a continuous extension of ν to an element of $\mathrm{Spa}(B, B^\circ)$. By Proposition 1.20, A is dense in B so for any $b \in B$ there is a convergent $(a_n)_{n=1}^\infty$ converging to b. By continuity of μ, we have

$$\mu(b) = \lim_{n\to\infty} \mu(a_n) = \lim_{n\to\infty} \nu(a_n)$$

as sequences in $\Gamma_\mu \cup \{0\}$. The topology of $\Gamma_\nu \subseteq \Gamma_\mu$ is discrete we have in the case when $\mu(b) \neq 0$ that the sequence $\nu(a_n)$ stabilize for n large. So the values of ν and μ coincide giving $\Gamma_\mu = \Gamma_\nu$. Alternatively, viewing the convergence in $\Gamma_\nu \cup \{0\}$, we have that μ is uniquely determined by ν. This shows that (b) implies (c). Now observe that since the images of f_i lie in B°, $\nu(f_i) = \mu(f_i) \leq 1$ so ν is indeed in $R_{\mathrm{Spa}(A,A^\circ)}(1|f_1, \ldots, f_n)$. ■

Lemma 5.6. The statement of Lemma 5.4 holds for $f_1 = 1$.

Proof. This is essentially the same as Lemma 5.5, using the universal property of the localization Proposition 1.21 instead. ■

Proof of Lemma 5.4. By induction on the Laurent order on the sieve, it suffices to prove the statement in the case of Weierstrass and Laurent domains, which are precisely the cases treated in Lemmas 5.5 and 5.6. ■

6. Lecture 6 – 27th May 2025

We continue with the proof of the result of van der Put-Schneider and Huber stating that the space of van der Put points on $\mathrm{Sp}(A)$ is homeomorphic to $\mathrm{Spa}(A, A^\circ)$.

We can define a morphism $\mathrm{Spa}(A, A^\circ) \to \mathrm{Sp}(A)^*$ by

$$(\nu/\sim) \mapsto \xi_\nu = \left\{R_{\mathrm{Sp}(A)}(f_0|f_1, \dots, f_n) : \nu(f_i) \leq \nu(f_0), \forall 1 \leq i \leq n\right\}. \tag{6.1}$$

This can be shown to be well-defined.

Proposition 6.1. Let A be an affinoid K-algebra, $\Omega = R_{\mathrm{Sp}(A)}(f_0|f_1, \dots, f_n)$, and $B = \mathcal{O}_{\mathrm{Sp}(A)}(\Omega)$. Then the map (6.1) exists and is well-defined.

Proof. We first show that ξ_ν is a van der Put point. If $\Omega \subseteq \Theta \subseteq X$ and $\Omega \in \xi_\nu$ then $\nu \in R_{\mathrm{Spa}(A,A^\circ)}(f_0|f_1, \dots, f_n)$ extends to $\nu_\Omega \in \mathrm{Spa}(\mathcal{O}_{\mathrm{Sp}(A)}(\Omega), \mathcal{O}_{\mathrm{Sp}(A)}(\Omega)^\circ)$. Consider the restriction map $\mathcal{O}_{\mathrm{Sp}(A)}(\Theta) \to \mathcal{O}_{\mathrm{Sp}(A)}(\Omega)$. The map is continuous and the image of $\mathcal{O}_{\mathrm{Sp}(A)}(\Theta)^\circ$ is in $\mathcal{O}_{\mathrm{Sp}(A)}(\Omega)^\circ$. By Lemma 5.4, there is a unique extension to $\nu_\Theta \in \mathrm{Spa}(\mathcal{O}_{\mathrm{Sp}(A)}(\Theta), \mathcal{O}_{\mathrm{Sp}(A)}(\Theta)^\circ)$ so $\Theta \in \xi_\nu$.

Having shown the upwards closure property, it remains to show the sieve properties. Let $\Omega = R_{\mathrm{Sp}(A)}(f_0|f_1, \dots, f_n), \Theta = R_{\mathrm{Sp}(A)}(g_0|g_1, \dots, g_m) \in \xi_\nu$. We have $\nu(f_0 g_0) \geq \nu(f_i g_j)$ for all $1 \leq i \leq n, 1 \leq j \leq m$ so $\Omega \cap \Theta = R_{\mathrm{Sp}(A)}(f_0 g_0|f_i g_j) \in \xi_\nu$. This shows verifies the second van der Put point condition. Now for $\Omega \in \xi_\nu$, let $\mathcal{S}$ be a covering sieve of Ω. We need to show that $\mathcal{S} \cap \xi_\nu$ is nonempty. We proceed by induction on the (necessarily finite) Laurent order of the sieve. If $\mathfrak{o}_L(\mathcal{S}) = 0$ then $\Omega \in \mathcal{S}$ and the assertion follows. Observe that there is $g \in \mathcal{O}_{\mathrm{Sp}(A)}(\Omega)$ such that $\mathfrak{o}_L(\mathcal{S}|_{\Omega_1}) < \mathfrak{o}_L(\mathcal{S})$ or $\mathfrak{o}_L(\mathcal{S}|_{\Omega_2}) < \mathfrak{o}_L(\mathcal{S})$ where $\Omega_1 = R_\Omega(g|1), \Omega_2 = R_\Omega(1|g)$. If $\nu_\Omega(g) \leq 1$ then by Lemma 5.4 replacing $\mathrm{Sp}(A)$ by Ω and ν by ν_Ω we get an extension to an element of $\mathrm{Spa}(\mathcal{O}_{\mathrm{Sp}(A)}(\Omega_2), \mathcal{O}_{\mathrm{Sp}(A)}(\Omega_2)^\circ)$. Arguing similarly in the case of $\nu_\Omega(g) > 1$ we get an extension to an element of $\mathrm{Spa}(\mathcal{O}_{\mathrm{Sp}(A)}(\Omega_1), \mathcal{O}_{\mathrm{Sp}(A)}(\Omega_1)^\circ)$. The Ω_i are rational open subsets and by the induction assumption $\emptyset \neq (\mathcal{S}|_{\Omega_i}) \cap \xi_\nu \subseteq \mathcal{S} \cap \xi_\nu$. The former is nonemtpy, and thus so is the latter. This too shows the well-definedness of the map. ∎

Proposition 6.1 does the bulk of the work of proving the theorem of Huber and van der Put-Schneider. We first show that this map is a bijection.

Definition 6.2 (ξ-Infinitesmal)**.** Let A be an affinoid K-algebra and $\xi \in \mathrm{Sp}(A)^*$ be a van der Put point. An element $a \in A$ is ξ-infinitesmal if and only if $R_{\mathrm{Sp}(A)}(\varepsilon|a) \in \xi$ for all $\varepsilon \in K^\times$.

Definition 6.3 (ξ-Ordering)**.** Let A be an affinoid K-algebra and $\xi \in \mathrm{Sp}(A)^*$ be a van der Put point. If a is not ξ-infinitesmal, then we say $b \preceq_\xi a$ if and only if the following equivalent conditions hold:

(i) $R_{\mathrm{Sp}(A)}(a|b, \varepsilon) \in \xi$ for some $\varepsilon \in K^\times$.
(ii) $R_{\mathrm{Sp}(A)}(a|b, \varepsilon) \in \xi$ for all $\varepsilon \in K^\times$ with $|\varepsilon|$ small enough.

Remark 6.4. We have for a ξ-infinitesmal, $b \preceq_\xi a$ if and only if b is ξ-infinitesmal and we say $a \simeq_\xi b$ if and only if $b \preceq_\xi a$ and $a \preceq_\xi b$.

We can show that the ordering of Definition 6.3 defines a linear partial order on the non-ξ-infintesmal elments of A.

Proposition 6.5. Let ξ be a van der Put point on $\mathrm{Sp}(A)$. The ξ-ordering is a linear partial order on the monoid $\{a \in A : a \text{ not } \xi\text{-infinitesmal}\}$.

Proof. If a, b are not ξ-infinitesmal then $R_{\mathrm{Sp}(A)}(a|\varepsilon), R_{\mathrm{Sp}(A)}(b|\varepsilon) \in \xi$ for some $\varepsilon \in K^\times$. Their intersection $R_{\mathrm{Sp}(A)}(a|\varepsilon) \cap R_{\mathrm{Sp}(A)}(b|\varepsilon) = R_{\mathrm{Sp}(A)}(ab|\varepsilon^2)$ is in ξ and the covering $R_{\mathrm{Sp}(A)}(a|b,\varepsilon) \cup R_{\mathrm{Sp}(A)}(b|a,\varepsilon)$ is admissable (i.e. in the Grothendieck topology) as it is finite, hence one of $R_{\mathrm{Sp}(A)}(a|b,\varepsilon), R_{\mathrm{Sp}(A)}(b|a,\varepsilon)$ lie in ξ giving one of $b \preceq_\xi a$ or $a \preceq_\xi b$.

To show this partial order is linear, suppose we have a, b, c non-infinitesmal and $ac \preceq_\xi bc$. Then $R_{\mathrm{Sp}(A)}(bc|ac,\varepsilon) \in \xi$ and $R_{\mathrm{Sp}(A)}(c|\delta) \in \xi$ for $\varepsilon, \delta \in K^\times$. When $\widetilde{\varepsilon} \in K^\times$ such that $|\widetilde{\varepsilon}| \cdot \|c|A\|_{\max} \leq |\varepsilon|$ we have $R_{\mathrm{Sp}(A)}(c|\delta) \cap R_{\mathrm{Sp}(A)}(bc|ac,\epsilon) \subseteq R_{\mathrm{Sp}(A)}(b|a,\widetilde{\varepsilon})$ so $a \preceq_\xi b$, that is, the monoid is cancellative. ∎

The data of the linear order allows us to define a map to a linearly ordered group, and in fact prescribes a point of the adic spectrum.

Proposition 6.6. Let ξ be a van der Put point on $\mathrm{Sp}(A)$ and Γ denote the quotient of the linearly ordered monoid $\{a \in A : a \text{ not } \xi\text{-infinitesmal}\}$ by the equivalence $\simeq_\xi$. There is a continuous map

$$\nu_\xi : A \to \Gamma \cup \{0\} \text{ by } \nu_\xi(a) = \begin{cases} 0 & a \text{ is } \xi\text{-infinitesmal} \\ [a]_\xi & \text{otherwise} \end{cases} \tag{6.2}$$

with $\nu_\xi \in \mathrm{Spa}(A, A^\circ)$.

Proof. (TODO: the proof makes no sense) It is clear that ν_ξ is a valuation. If $\gamma \in \Gamma$ with $\gamma = \frac{[a]_\xi}{[b]_\xi}$ then for $R_{\mathrm{Sp}(A)}(a|\varepsilon) \cap R_{\mathrm{Sp}(A)}(b|\varepsilon) \in \xi$ and δ such that $\delta \cdot \|b|A\|_{\max} < \varepsilon$ then $\|f|A\|_{\max} < \varepsilon\delta$ implies that $\nu_\xi(f) < \gamma$ showing continuity. ν_ξ satisfies the correct compatibility condition with respect to A° as powerbounded elements are ξ-infinitesmal. ∎

This is inverse to the construction (6.1).

Proposition 6.7. Let A be an affinoid K-algebra. The constructions (6.1) and (6.2) are mutually inverse.

Proof. Suppose ξ is given. Let $\Omega = R_{\mathrm{Sp}(A)}(f_0|f_1, \ldots, f_n) \in \xi$ then

$$R_{\mathrm{Sp}(A)}(f_0|f_1, \ldots, f_n, \varepsilon) = R_{\mathrm{Sp}(A)}(f_0|f_1, \ldots, f_n) \in \xi$$

when ε is small enough, showing $\nu_\xi(f_i) \leq \nu_\xi(f_0)$ and $\Omega \in \xi_{\nu_\xi}$. Suppose that $\nu \in \xi_{\nu_\xi}$, that is, $\nu_\xi(f_i) \leq \nu_\xi(f_0)$. Since not all $f_i \in \mathrm{supp}(\nu_\xi) = \{f \in A : f \text{ is } \xi\text{-infinitesmal}\}$ we have $f_0 \in \mathrm{supp}(\nu_\xi)$ hence f_0 is not ξ-infinitesmal. Taking $\varepsilon \in K^\times$ small, all $R_{\mathrm{Sp}(A)}(f_0|f_i,\varepsilon) \in \xi$ hence

$$\bigcap_{i=1}^{n} R_{\mathrm{Sp}(A)}(f_0|f_1, \ldots, f_n) \subseteq R_{\mathrm{Sp}(A)}(f_0|f_1, \ldots, f_n)$$

and $\bigcap_{i=1}^n R_{\mathrm{Sp}(A)}(f_0|f_1,\dots,f_n) \in \xi$. This shows $\xi = \xi_{\nu_\xi}$.

Dually suppose ν is given. It follows from the definitions and continuity that $a \in A$ is ξ-infinitesmal if and only if $\nu(a) = 0$ and $a \geq b$ if and only if $\nu(a) \geq \nu(b)$ so indeed $\nu_{\xi_\nu} = \nu$. ■

This proves the bijection in full. It remains to show the assertion regarding the Krull dimension.

For the statement on Krull dimension, we consider some general properties of specializations in $\mathrm{Spa}(A, A^+)$ for A Tate.

Lemma 6.8. Let (A, A^+) be a Huber pair with A Tate. There is a continuous morphism $S : \mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A)$ by $\nu \mapsto \mathrm{supp}(\nu)$

Proof. We have $S^{-1}(\mathrm{Spec}(A_f)) = \bigcup_{m\geq 0} R_{\mathrm{Spa}(A,A^+)}(f|s^m)$ for a topologically nilpotent unit $s \in A^\times \cap A^{\circ\circ}$. ■

The map S detects when points have disjoint neighborhoods in $\mathrm{Spa}(A, A^+)$.

Lemma 6.9. Let (A, A^+) be a Huber pair with A Tate. Then $S(x) \neq S(y)$ if and only if $x, y \in \mathrm{Spa}(A, A^+)$ have disjoint open neighborhoods.

Proof. Without loss of generality, let $a \in A$ be such that $|a|_x > 0$ but $|a|_y = 0$. By continuity of $|\cdot|_x$ and $\lim_{m\to\infty} s^m = 0$, $|a|_x \geq |s^m|_x$ when m is sufficiently large. Thus $x \in R_{\mathrm{Spa}(A,A^+)}(a|s^m), y \in R_{\mathrm{Spa}(A,A^+)}(s^{m+1}|a)$ and these rational open subsets are disjoint. ■

Recall that for X a space, a point $y \in X$ is a generification of $x \in X$ if $x \in \overline{\{y\}}$.

Remark 6.10. Since $\mathrm{Spa}(A, A^+)$ is spectral, each irreducible component has a unique generic point so each point of $\mathrm{Spa}(A, A^+)$ has a maximal generification.

Definition 6.11 (Vertical generification)**.** Let $(\nu/\sim) \in \mathrm{Spa}(A, A^\circ)$, Γ the group of values of ν, and $\Lambda \subseteq \Gamma$ a convex subgroup. We define a valuation

$$(\nu/\Lambda)(a) = \begin{cases} 0 & \nu(a) = 0 \\ \nu(a)\Lambda & \text{otherwise.} \end{cases}$$

These generifications are vertical in the sense that they occur within a single fiber of the map $S : \mathrm{Spa}(A, A^\circ) \to \mathrm{Spec}(A)$ of Lemma 6.8

7. Lecture 7 – 5th June 2025

We continue with the proofs of results relating to vertical generifications in adic spaces.

Lemma 7.1. Let A be a Tate ring with topologically nilpotent unit s and $(\nu/\sim) \in \mathrm{Spa}(A, A^\circ)$.

(i) $\Lambda_{\max} = \{\gamma \in \Gamma : \nu(s) \leq \gamma^n \leq \nu(s)^{-1}, \forall n \in \mathbb{N}\}$ is the largest proper convex subgroup of Γ.
(ii) The set of convex subgroups of Γ is linearly ordered.
(iii) The most vertical generification of $(\nu/\sim)$ is $((\nu/\Lambda_{\max})/\sim)$.
(iv) All generifications of ν are vertical.

Proof of (i). By continuity $\Gamma = \bigcup_{m\in\mathbb{N}}[\nu(s)^m, \nu(s)^{-m}]_\Gamma$ so $\Gamma = \bigcup_{m\in\mathbb{N}}[\gamma^m, \gamma^{-m}]_\Gamma$ for any $\gamma \in \Gamma \setminus \Lambda_{\max}$ with $|\gamma| < 1$, hence $\Lambda_{\max}$ is indeed maximal. ■

Proof of (ii). Let $\widetilde{\nu}$ be a generification of ν. By Lemma 6.9 their supports coincide $\mathrm{supp}(\nu) = \mathrm{supp}(\tilde{\nu}) = \mathfrak{p}$, and $\tilde{\nu} = \nu/\Lambda$ where $\Lambda = \bigcup_{a,b\in A\setminus\mathfrak{p}, \tilde{\nu}\in R_{\mathrm{Spa}(A,A^\circ)}(a|b,s^m)} \left[\frac{\nu(a)}{\nu(b)}, \frac{\nu(b)}{\nu(a)}\right]_\Gamma$ so $\widetilde{\nu} \in R_{\mathrm{Spa}(A,A^\circ)}(a|b, s^m)$. ■

Proof of (iii). This is immediate from maximality of $\Lambda_{\max}$. ■

Proof of (iv). Let $\mu/\sim$ be a generification of $\nu/\sim$. By Lemma 6.9, the two valuations have identical supports. We claim that if $\nu(a) \leq \nu(b)$ then $\mu(a) \leq \mu(b)$. If $\nu(a) = 0$ then the statement is trivial as it follows from equality of supports. Otherwise if $\nu(a) \neq 0$, then also $\nu(b) \neq 0$ and by continuity there is some natural number n such that $\nu(s^n) \leq \nu(b)$. Then $(\nu/\sim) \in R_{\mathrm{Spa}(A,A^\circ)}(b|a, s^n)$. Since $\mu/\sim$ is a generification, it lies in $R_{\mathrm{Spa}(A,A^\circ)}(b|a, s^n)$ as well. So $\mu(a) \leq \mu(b)$ as desired. It then follows that $\mu \simeq \nu/\Lambda$ where $\Lambda \subseteq \Gamma$ is the convex subgroup generated by $\{\nu(a) : a \in A, \nu(a) = 1\}$. ■

Remark 7.2. Note that Lemma 6.9 (which implies that generifications have identical supports) only holds in the Tate case.

Proposition 7.3. Let (A, A^+) be a Huber pair with A Tate and $x, y \in \mathrm{Spa}(A, A^+)$. Then x, y have disjoint open neighborhoods if and only if their most generic vertical generifications differ.

Proof. Suppose that x, y have disjoint open neighborhoods. Then they obviously cannot have a common vertical generification.

Conversely suppose ξ, ν are the most generic vertical generifications of x, y, repsectively. By Lemma 6.8 their supports agree which we denote $\mathfrak{p} \subseteq A$. Without loss of generality, there are $a, b \in A \setminus \mathfrak{p}$ such that $|a|_\xi \leq |b|_\xi$ but $|b|_\nu \leq |a|_\nu$. Then $\frac{|a|_\nu}{|b|_\nu}$ is not an element of the largest proper convex subgroup of Γ_y. Hence by Lemma 7.1 there is $m \in \mathbb{N}$ such that $|b|_y^m \leq |s|_y \cdot |a|_y^m$. Since $|a|_\xi \geq |b|_\xi$ but this cannot happen for ξ: since $|a|_\xi \geq |b|_\xi$ we have $|b|_x^{2m} \geq |s|_x |a|_x^{2m}$. Since $a, b \notin \mathfrak{p}$ there is $\ell \in \mathbb{N}$ such that $|c|_z \geq |s^\ell|_z$ when $z \in \{x, y\}$ and $c \in \{a, b\}$. Then $x \in R_{\mathrm{Spa}(A,A^\circ)}(b^{2m}|sa^{2m}, s^{2\ell m})$ and $y \in R_{\mathrm{Spa}(A,A^\circ)}(sa^m|b^m, s^{\ell m+1})$ and these open subsets are disjoint as $|b^{2m}|_z \geq |sa^{2m}|_z$ and $|sa^m|_z \geq |b^m|_z$ imply $|s^2a^{2m}|_z \geq |b^{2m}|_z$ hence $|a| = |b| = 0$ and $0 < s < 1$. ■

We now recollect some additional results from the theory of affinoid algebras.

Proposition 7.4. If $A \to B$ is a morphism of affinoid K-algebras such that B is finite over A then there is $D_{B/A} \in \mathbb{N}$ such that for all $b \in B$ there is a polynomial $P(T) = T^d + \sum_{i=0}^{d-1} p_i T^i$ with $p_i \in A$ such that $P(b) = 0$ and $\|p_i|A\|_{\max} \leq \|b|B\|_{\max}^{d-i}$ and $d < D_{B/A}$.

Corollary 7.5. Let A be an affinoid K-algebra. Then

$$A^\circ = \{a \in A : \|a|A\|_{\max} \leq 1\}$$
$$A^\circ = \{a \in A : \|a|A\|_{\max} \leq 1\}.$$

In what follows, we denote $\mathfrak{K} = K^\circ/K^{\circ\circ}$ and for any affinoid algebra A, $\widetilde{A} = A^\circ/A^{\circ\circ}$ considered as an algebra over $\mathfrak{K}$.

Proposition 7.6. Let $A \to B$ be a morphism of affinoid K-algebras such that B is fintiely generated as an A-module. Then B° is integral over A°.

Proposition 7.7. Let $A \to B$ be a morphism of affinoid K-algebras such that B is finitely generated as an A-module. Then $\widetilde{B}$ is integral over $\widetilde{A}$.

Proof. This is merely the observation that the property of being integral over persists under quotients. ■

Remark 7.8. B° may fail to be finite over A°, even when $A = K$ and B is a finite field extension of A.

Corollary 7.9. Let A be an affinoid K-algebra of dimension d. There is a homomorphism $\mathfrak{K}[X_1, \ldots, X_d] \to \widetilde{A}$ of $\mathfrak{K}$-algebras such that $\widetilde{A}$ is integral over $\mathfrak{K}[X_1, \ldots, X_d]$. In particular, if $\mathfrak{p}$ is a prime ideal of $\widetilde{A}$ then the transcendence degree of $\kappa(\mathfrak{p})/\mathfrak{K}$ is d.

Proof. Apply Proposition 7.7 in the case of the map $\mathbb{T}_d \to A$ which exists by Noether normalization. ■

For a valuation ν, let $\mathrm{supp}(\nu) = \mathfrak{p} \subseteq A$ be its support. Denote $\mathfrak{p}_0 = \{a \in A^\circ : \exists \varepsilon \in K^{\circ\circ} \text{ s.t. } \nu(a) < \nu(\varepsilon)\}$. The quotient $\widetilde{\mathfrak{p}} = \mathfrak{p}_0/A^{\circ\circ}$ is prime in $\widetilde{A}$.

Proposition 7.10. Let ν be a valuation and $0 = \Gamma_0 \subsetneq \Gamma_1 \subsetneq \Gamma_2 \subsetneq \cdots \subsetneq \Gamma_d \subsetneq \Gamma$ a tower of convex subgroups. Then $d \leq \dim(A)$.

Proof. Without loss of generality, we can assume there is $b_i \in A^\times$ and $\alpha_i = \frac{a_i}{b_i} \in A^\circ$. Using that $\nu(\alpha_i) \in \Gamma_i \backslash \Gamma_{i-1}$, it follows that for $P \in K^\circ[X_1, \ldots, X_d]/K^{\circ\circ}[X_1, \ldots, X_d]$ we have $\nu(P(\alpha_1, \ldots, \alpha_d)) \in \Gamma_d$ a proper convex subgroup. It follows that $P(\alpha_1, \ldots, \alpha_d) \notin \mathfrak{p}_0 \subseteq A^\circ$. Thus the images in $\widetilde{A}$ are algebraically independent over $\mathfrak{K}$. When $d > \dim(A)$ this contradicts a corollary from the Winter semester course. ■

In sum, we have shown the following.

Theorem 7.11 (Huber-Schneider-van der Put)**.** Let A be an affinoid K-algebra. There is a bijection between the van der Put points of $\mathrm{Sp}(A)$ and $\mathrm{Spa}(A, A^\circ)$ via (6.1):

$$(\nu/\sim) \mapsto \xi_\nu = \left\{R_{\mathrm{Sp}(A)}(f_0|f_1, \ldots, f_n) : \nu(f_i) \leq \nu(f_0), \forall 1 \leq i \leq n\right\}.$$

In particular, $\mathrm{Spa}(A, A^\circ)$ is a spectral space with Krull dimension at most that of A.

Proof. The map was shown to be well-defined in Proposition 6.1 and was shown to be a bijection in Proposition 6.7. Spectrality of $\mathrm{Spa}(A, A^\circ)$ is immediate from Corollary 4.16. Finally the statement of the Krull dimension is Proposition 7.10. ■

8. Lecture 8 – 17th June 2025

We start with a

Remark 8.1. The degree of $\tilde{A}$ over $\underline{K}$ is less-equal the Krull-dimension of A.

We want to pursue the problem of how to write down a rank $d+1$ valuations on A, where $d = \dim(A)$. Recall from end of last term the rank 2 examples for $\mathbb{T}_1$ reduced to inspecting $\|\cdot|\mathbb{T}_1\|$ or the maximum norm. In our case of interest the maximum norm will coincide with the spectral norm, so we want to understand how to deal with the maximum norm.

Proposition 8.2. When A is reduced, then $\|\cdot|A\|_{max}$ is equivalent to the residual norm.

Proof. Last term. ∎

As $\|\cdot|\mathbb{T}_1\|$ is multiplicative (i.e. $\|fg|\mathbb{T}_1\| = \|f|\mathbb{T}_1\|\|g|\mathbb{T}_1\|$), it defines an absolute value on the field of fractions $\mathbb{K}_1$. But $\mathbb{K}_1$ is not complete w.r.t. the unique extension $\|\cdot|\mathbb{K}_1\|$.

Herein, we let K be a field with non-Archimedean absolute value (not necessarily complete). Let $f, g\colon V \to W$ be linear maps between normed K-vector spaces. We say $f >> g$ if there is $C > 0$ such that $|g| \leq C \cdot |f|$.

Proposition 8.3. Let V be a finite dimensional K-vector space and $\|\cdot\|_V$ satisfying $\|\lambda v\|_V = |\lambda| \cdot \|v\|_V, \|v+w\|_V \leq \max\{\|v\|, \|w\|\}$. The following conditions are equivalent:

(i) If $(b_i)_{i=1}^n$ is any basis then $\|\sum_{i=1}^n \lambda_i b_i\|_V >> \max_{1\leq i\leq n} |\lambda_i|$ (viewed as linear functions in the λ_i).

(ii) For every $v \in V \setminus \{0\}$ there is a bounded linear functional $\ell : V \to K$ such that $\ell(v) \neq 0$.

Proof. (a)⇒(b) Suppose there is some basis where (i) holds. Let $\ell_i \in V^\vee$ such that $\ell_i\left(\sum_{j=1}^n \lambda_i b_i\right) = \lambda_i$. These are bounded.

(b)⇒(a) We proceed by induction on the dimension of V. The cases where $\dim(V) \leq 1$ are clear. Let $n > 1$ and suppose the assertion is shown for $\dim(V') < n$ and $v \in V \setminus \{0\}$ such that $V = V' \oplus Kv$. Let $\ell \in V^\vee$ be such that $\ell(v) \neq 0$ with $\|(v', \lambda v)\|_V = \max\{|\lambda|, \|v'\|_V\}$ and the induction assumption applies. ∎

Remark 8.4. This does not hold in general when K is not complete. In this case, a K-linear functional is bounded if and only if it is continuous.

We now consider the general case without the condition on multiplicativity of the norm.

Proposition 8.5. Let V be a normed K-vector space. The following are equivalent:

(a) Every finite-dimensional subspace of V satisfies the equivalent conditions of Proposition 8.3.

(b) Every finite dimensional subspace of V is closed in V.

Proof. (a)⇒(b) Suppose $W \subseteq V$ is a finite dimensional subspace and $v \in \overline{W} \setminus W$. Let $\widetilde{W} = W \oplus Kv$ equipped with a norm $\|\cdot\|_V|_{\widetilde{W}}$ the restriction of the one on V. For v in the closure of W in $\widetilde{W}$ the norms $\|\cdot\|_V$ and $\sum_{i=1}^n \lambda_i b_i \mapsto \lambda_i$ for a basis $(b_i)_{i=1}^n$ fails to be eqiuvalent to

(b)⇒(a) Let $W \subseteq V$ be finite dimensional, $w \in W \setminus \{0\}$, and W' a subspace of W such that $W = W' \oplus Kw$ algebraically. If $\ell : W \to K$ is a linear functional, set $\ell(\lambda w - w') = \lambda$ when $w \in W'$ and all $\lambda \in K$. This produces a linear functional nonvanishing at w'. ■

This leads to the notion of weakly Cartesian norms.

Definition 8.6 (Weakly Cartesian Norm)**.** Let V be a normed K-vector space. V is weakly Cartesian if either (and thus both) of the conditions of Proposition 8.5 hold.

Remark 8.7. By Proposition 8.3, two weakly Cartesian norms on any finite-dimensional vector space are equivalent.

Lemma 8.8. Let $|\cdot|$ be an ultrametric absolute value on L and K a subfield such that $[L : K] < \infty$ and L is weakly Cartesian as a K-vector space. Then every weakly Cartesian L-norm on an L-vector space V is weakly Cartesian as a K-norm.

This shows that there are many bounded K-linear functionals on L.

Proposition 8.9. Let $|\cdot|$ be an ultrametric absolute value on K and L/K a finite field extension. Then:

(i) $\|\ell|L\|_s = \lim_{n\to\infty} \sqrt[n]{\|\ell\|^n}$ does not depend on the choice of a weakly Cartesian K-norm $\|\cdot\|$ on L and is a power multiplicative ring norm and a K-vector space norm on L.

(ii) Every power multiplicative norm on L is bounded by the spectral norm $\|\cdot\|_s$.

(iii) If the minimal polynomial of s is of the form $T^d + \sum_{i=0}^{d-1} p_i T^i$ then $\|\ell\| = \max_{0\le i<d} \sqrt[d-i]{|p_i|}$.

Proof of (i). If $\|\cdot\|$ is given by a basis $(b_i)_{i=1}^n$ as in Proposition 8.3 and $b_i b_j = m_{ij}^k b_k$, then $\|xy\| \le C\|x\| \cdot \|y\|$ with $C = \max\{m_{ij}^k\}$. Such a C exists by being weakly Cartesian. We have that $\|x^n\| \le C^k \|x\|^n$ and applying this iteratively by induction we have k depends on n sublinearly so $n \le 2^k$ and we have $\lim_{n\to\infty} \sqrt[n]{\|x\|^n} \le \sqrt[n]{\|x^m\|}$ when $m > 0$ showing convergence. Checking that the axioms of the norm hold shows that $\|x\|_s \neq 0$ when $x \neq 0$. ■

Proof of (ii). If N is any norm and $\|\cdot\|$ is a weakly Cartesian norm, then $N(\ell) \le \|\ell\|$ – every norm is bounded by a weakly Cartesian one, but not conversely. ■

Proof of (iii). Let $M = \max_{0\le i<d} \sqrt[d-i]{|p_i|}$ then $\|x\|_s^n = \|x^n\| \le M^{n+1-d}$ where $n \ge d$ by induction on n. Hence $\|\cdot\|_s \le M$. For the reverse inequality, the arguments of (i) give that without loss of generality we can take L/K to be normal. If $(x_i)_{i=1}^m$ are the images of x under the elements of $\mathrm{Aut}(L/K)$ then the minimal polynomial of x

over K is of the form $(\prod_{i=1}^{a}(T - x_i))^q$ for some positive integer q as $\|x_i\|_s = \|x\|_s$ by (i) so $M \leq \|x\|_s$ giving the equality. ■

We will consider $\|\cdot\|_s$ for algebraic extensions of L over K.

Proposition 8.10. Let L/K be a finite field extension and $|\cdot|$ an ultrametric absolute value on K.

(i) There is a bijection between extensions of $|\cdot|$ to some absolute value of L and the isomorphism classes of embeddings $L \to \widehat{L}$ with dense image and $\widehat{L}$ finite over $\widehat{K}$ by $|\cdot|_L \mapsto \widehat{L}$ the completion of L with respect to that absolute value and conversely $L \hookrightarrow \widehat{L}$ to the unique norm $|\cdot|_L$ where $|\ell|_L = |\ell|_{\widehat{L}}$.

(ii) For a list of representatives of isomorphism classes in (i) and $d_i = [\widehat{L}_i : \widehat{K}]$ then $\sum_{i=1}^{n} d_i < [L : K]$. In particular, n is finite and the spectral norm admits an explicit description $\|x\|_s = \max_{1 \leq i \leq n} |x|_{\widehat{L}_i}$.

Proof of (i). This is immediate from the construction. ■

Proof of (ii). Let $\mathbb{L}$ be the completion of L with respect to any weakly Cartesian norm. This is a vector space over $\widehat{K}$ of dimension equal to the degree of the field extension $[L : K]$ and $\mathbb{L} \twoheadrightarrow \bigoplus_{i=1}^{n} \widehat{L}_i$ is reduced. That $\|x\|_s \geq \|x\|_{\widehat{L}_i}$ follows from Proposition 8.9 (ii). For the opposite inequality, we can take L/K to be a normal field extension, wherein $\mathrm{Aut}(L/K)$ permutes L transitively. For $n > 0$ let $\widehat{L}$ be a finite extension of $\widehat{K}$ in which L can be embedded. Then this follows from Proposition 8.9 (iii). ■

Check

9. Lecture 9 – 26th June 2025

Proposition 9.1. Let L/K be a finite field extension and $\mathbb{L}$ be the completion of L with respect to some weakly Cartesian norm. Let $(\mathbb{L}_i)_{i=1}^n$ be the set of isomorphism classes of dense K-embeddings $L \to \mathbb{L}_i$ where $\mathbb{L}_i$ is a finite field extension of the completion $\mathbb{K}$ of K. The following are equivalent.

(a) $\sum_{i=1}^n d_i = d$ where $d_i = [\mathbb{L}_i : \mathbb{K}]$.
(b) $\| \cdot |\mathbb{L}\|_s$ is weakly Cartesian.
(c) $\mathbb{L}$ is reduced.

Proof. (a)⇒(b) Let $\sum_{i=1}^n d_i = d$ and denote $B^{(i)} = (b_j^{(i)})_{j=1}^{d_i}$ and $B = (b_i)_{i=1}^d$ be bases of $\mathbb{L}_i/K$ and L/K. By Artin-Whaples weak approximation, $\mathbb{L} \to \bigoplus_{i=1}^n \mathbb{L}_i$ is a surjective linear map, hence an isomorphism as $\mathbb{K}$-vector spaces. The image of B and the direct sum $\bigoplus_{i=1}^n B^{(i)}$ is a basis for the $\mathbb{L}$-vector space $\bigoplus_{i=1}^n \mathbb{L}_I$. Since two bases define the same norm, the spectral norm on L is equivalent to the norm defined by B and hence weakly Cartesian.

(b)⇒(c) $\mathbb{L}$ is the completion of L with respect to the spectral norm which is power multiplicative. Assume $\ell \in \mathbb{L}$ is nilpotent, ie. $\ell^n = 0$. We have $\ell = \lim_{j\to\infty} \ell_j$ with $\ell_j \in L$. Hence
$$0 = \|\ell^n\|_s = \lim_{j\to\infty} \|\ell_j^n\|_s = \lim_{j\to\infty} \|\ell_j\|_s^n$$
so $\lim_{j\to\infty} \|\ell_j\|_s = 0$ and thus $\ell = 0$.

(c)⇒(a) By the basic theory of finite-dimensional K-algebras, $\mathbb{L} = \bigoplus_{j=1}^m \mathbb{L}^{(j)}$ where the $\mathbb{L}^{(j)}$ are fields as $\mathbb{K}$ is reduced. By the choice of the $\mathbb{L}_i$, every $\mathbb{L}^{(j)}$ is amongst the $\mathbb{L}_i$ as if $\mathbb{L}_i$ occured twice, say as $\mathbb{L}^{(j)}$ and $\mathbb{L}^{(k)}$ then $L \to \mathbb{L} \to \mathbb{L}^{(j)} \oplus \mathbb{L}^{(k)}$ would factor over $L \to \mathbb{L}_i \xrightarrow{\Delta} \mathbb{L}^{(j)} \oplus \mathbb{L}^{(k)}$ which would fail to have dense image as L is dense in its completion. It follows that
$$d = \dim_K L = \dim_{\mathbb{K}} \mathbb{L} = \sum_{j=1}^m \mathbb{L}^{(j)} = \sum_{j=1}^n d_j.$$
as desired. ■

Example 9.2. Let $K = \mathbb{Q}$ and $L = \mathbb{Q}(i)$. Then there are two $L \to \mathbb{Q}_5$ with dense image, in particular $n = 2, d_1 = d_2 = 1$.

Definition 9.3 (Weakly Stable)**.** Let K be a field with non-Archimedean absolute value. K is weakly stable if the conditions of Proposition 9.1 hold for all finite extensions L/K.

It is useful to understand explicitly when these conditions hold.

Proposition 9.4. Let K be a field with non-Archimedean absolute value.

(i) If L/K is separable, the equivalent conditions of Proposition 9.1 hold.
(ii) Every perfect, and thus every characteristic zero field, is weakly stable.
(iii) If K has positive characteristic p, the following are equivalent:
  (a) K is weakly stable.

(b) $K^{1/p}/K$ is weakly Cartesian.
(c) For every $q = p^N$ $K^{1/q}/K$ is weakly Cartesian.

Proof of (i). As L/K is separable, $\mathrm{Tr}_{L/K}$ does not vanish identically. Let $\varepsilon \in L$ be such that $\mathrm{Tr}_{L/K}(\varepsilon) = 1$. If $\ell \in L^\times$ then we can define a functional λ by $\lambda(t) = \mathrm{Tr}_{L/K}(\frac{t\varepsilon}{\ell})$, which is a $\|\cdot\|_s$-bounded K-linear functional on L such that $\lambda(\ell) = 1$. By Proposition 8.3, the spectral norm is weakly Cartesian. ■

Proof of (ii). This is immediate from (i). ■

Proof of (iii). This can be shown using (ii) and Lemma 8.8. ■

Proposition 9.5. Let K be a complete non-Archimedean field and V a K-Banach space containing a dense subspace of at most countable dimension over K. Let $W \subseteq V$ be a subspace and $\ell : W \to K$ a linear functional. Then for every $B > \|\ell|W^\vee\|$ there is $\lambda \in V^\vee$ such that $\|\lambda|V^\vee\| \leq B$ and $\lambda|_W = \ell$.

Proof. Let $(b_i)_{i=1}^\infty$ be a set of K-linear generators of a dense K subspace of V and W_i the closure in V of $W + \sum_{j=1}^i Kb_j$. Then W_0 is the closure of W and hence ℓ extends to $\lambda_0 : W_0 \to K$ such that $\|\ell|W^\vee\| = \|\lambda_0|W_0^\vee\|$ and $\lambda_i : W_i \to K$ will be constructed by induction such that $\|\lambda_i|W_i^\vee\| \leq B_i$ where $\|\ell|W^\vee\| = B_0 < B_1 < \dots$. Note $\dim(W_{i+1}/W_i) \leq 1$ and if $W_{i+1} = W_i$ then the extension is trivial. Otherwise, choose some $x \in W_{i+1}$ such that $\|x\| \leq (1+\varepsilon)\cdot \mathrm{dist}(x, W_i)$. Then $W_{i+1} = W_i \oplus Kx$ defines a projection $\pi : W_{i+1} \to W_i$ of norm at most $(1+\varepsilon)$. Choosing ε small enough, we can define $\lambda_{i+1} = \lambda_i \circ \pi$. Finally, λ_∞ on $\bigcup_{i=1}^\infty W_i$ extends to V in which W_∞ is dense by continuity in a unique norm-preserving way. ■

Remark 9.6. Recall that the following conditions on a complete non-Archimedean field are equivalent:

(i) K is spherically complete: the intersection of any decreasing sequence of balls is nonempty.
(ii) The Hahn-Banach theorem holds for all normed K-vector spaces: for any K-vector space V with norm $\|\cdot|V\|$ and $W \subseteq V$ a vector subspace, any linear functional $\ell : W \to K$ such that $|\ell(w)| \leq \|w|V\|$ extends to a linear functional $\widetilde{\ell} : V \to K$ bounded by $\|\cdot|V\|$.
(iii) Let $\ell^\infty = \{(x_i)_{i=0}^\infty : \text{all } x_i \in K, \sup|x_i| < \infty\}$ there is $\lambda : \ell^\infty \to K$ such that $\|\lambda|(\ell^\infty)^\vee\| = 1$ and such that $\lambda(x) = \lim_{i\to\infty} x_i$.

Example 9.7. $\mathbb{Q}_p$ is spherically complete, but $\mathbb{C}_p$ is not spherically complete.

Let A be a nat ring Definition 1.18 equipped with a multiplicative ring norm. We consider norms on M.

Definition 9.8 (Module Norm)**.** Let A be a nat ring and M an A-module. A map $\|\cdot|M\|$ to a totally ordered Abelian group is a norm if

(i) $\|m|M\| = 0$ if and only if $m = 0$,
(ii) $\|m+n|M\| \leq \max\{\|m|M\|, \|n|M\|\}$,
(iii) $\|am|M\| \leq \|a|A\| \cdot \|m|M\|$.

Definition 9.9 (Faithful Module Norm)**.** Let A be a nat ring and M an A-module. A module norm on M is faithful if the inequality in (iii) of definition 9.8 is always an equality.

We consider the bounded homomorphisms between normed modules over a nat ring A.

Definition 9.10 (Bounded Module Homomorphisms)**.** Let A be a nat ring and M, N A-modules with norms. We define the set of bounded homomorphisms

$$\mathrm{BHom}_A(M,N) = \left\{ \varphi : M \to N \in \mathrm{Hom}_A(M,N) : \substack{\exists C \in \mathbb{R}_{>0} \text{ s.t.} \\ \|\varphi(m)|N\| \leq C \cdot \|m|M\|} \right\}. \tag{9.1}$$

Definition 9.11 (Norm of Homomorphism)**.** Let A be a nat ring and M, N A-modules with norms. Let $\varphi \in \mathrm{BHom}_A(M,N)$. We define $\|\varphi|\mathrm{BHom}_A(M,N)\|$ to be the smallest $C \in \mathbb{R}_{>0}$ such that $\|f(m)|N\| \leq C \cdot \|m|M\|$ holds for all $m \in M$.

Lemma 9.12. Let K be a field of characteristic p complete with respect to a non-Archimedean absolute value, $q = p^N$, and $M = K^{1/q}$. For $\mathfrak{X} \subseteq M$ countable, the closure of the subfield $L \subseteq M$ generated by K and $\mathfrak{X}$. Let $A = K\langle X_1, \dots, X_n\rangle, B = L\langle Y_1, \dots, Y_n\rangle$ where B has the A-algebra structure by $X_i \mapsto Y_i^q$. For every $b \in B$ nonzero and $\varepsilon \in \mathbb{R}_{>0}$ there is a bounded homomorphism $\varphi \in \mathrm{BHom}_A(B,A)$such that $\varphi(b) \neq 0$ and

$$\|b|B\| \cdot \|\varphi|\mathrm{BHom}_A(B,A)\| \leq (1-\varepsilon) \cdot \|\varphi(b)|A\|.$$

Proof. Let $b = \sum_{\alpha \in \mathbb{N}^n} b_\alpha Y^\alpha$ with $b_\alpha \in L$. Choose α such that $|b_\alpha|_L = \|b|B\|$. By Proposition 9.5, there is a K-linear functional $\lambda : L \to K$ such that $\lambda(b_\alpha) \neq 0$ and such that $|\lambda(\ell)| \cdot |b_\alpha| \leq (1-\varepsilon)|\ell| \cdot |\lambda(b_\alpha)|$. We have $\alpha = q\alpha' + \beta$ where all $0 \leq \beta_i \leq q-1$. Then $\ell(c) = \sum_{\gamma \in \mathbb{N}^N} \lambda(c_{\beta + q\gamma}) X^\gamma$ where $c = \sum_{\delta \in \mathbb{N}^n} c_\delta Y^\delta \in B$. ∎

Remark 9.13. In the statement of Lemma 9.12, M may be a field extesion of infinite degree, but is complete with respect to the absolute value of $|\cdot|_K$.

Lemma 9.14. Let K be a field of characteristic p with respect to a non-Archimedean absolute value, $q = p^N$, $E = \mathrm{Frac}(K\langle X_1, \dots, X_n\rangle)$, and F/E a finite field extension such that $F^q \subseteq E$. There is $\mathfrak{X} \subseteq E$ such that

$$F \cap K^{1/q}\langle X_1, \dots, X_n\rangle \subseteq B = K(\mathfrak{X})\langle Y_1, \dots, Y_n\rangle.$$

Proof. Let $(f^{(i)})_{i=1}^m$ generate F/E as a field extension. Then $(f^{(i)})^q = \sum_{\alpha \in \mathbb{N}^n} \varphi_\alpha^{(i)} X^\alpha$ where $\varphi_\alpha^{(i)} \in K$. Hence $\xi_\alpha^{(i)} = \sqrt[q]{\varphi_\alpha^{(i)}} \in M$ and

$$\mathfrak{X} = \{\xi_\alpha^{(i)} : \alpha \in \mathbb{N}^N, 1 \leq i \leq m\}$$

suffices. ∎

Proposition 9.15. Let K be a field complete with respect to an ultrametric norm and $E = \mathrm{Frac}(\mathbb{T}_n)$ equipped with $|\frac{f}{g}|_E = \frac{\|f|\mathbb{T}_n\|}{\|g|\mathbb{T}_n\|}$ for $f \in \mathbb{T}_n, g \in \mathbb{T}_n \setminus \{0\}$. Then E is weakly stable.

Proof. For K of characteristic zero, this is merely the statement of Lemma 9.14. In the charcteristic p case, we can use the same lemma to observe that in the hypotheses, the spectral norm of F is weakly Cartesian. But using the construction of Lemma 9.12 the spectral norm of F is weakly Cartesian. The construction therein in conjunction with Proposition 9.4 can be used to give for each $f \in F \setminus \{0\}$ an E-linear functional $\ell : F \to E$ such that $\ell(t) \neq 0$. Since F is in the fraction field of B, the claim follows from Proposition 9.1. ■

Remark 9.16. Every finite field extension of E is also weakly stable with respect to every extension of the absolute value on E.

10. Lecture 10 – 3rd July 2025

Recall the following definition.

Definition 10.1 (Universally Japanese)**.** Let A be an integral domain. A is universally Japanese if for all $\mathfrak{p} \in \mathrm{Spec}(A)$ and all finite extensions $L/\kappa(\mathfrak{p})$ the integral closure of A in L is a finitely generated A-module.

This is a property that holds for all affinoid algebras.

Proposition 10.2. If A is an affinoid K-algebra then A is universally Japanese.

Proof. Omitted. ∎

Proposition 10.3. Let A be an affinoid K-algebra. The norm on $A/\mathrm{Nil}(A)$ by $\| \cdot | A\|_{\max}$ is equivalent to the residual norm. In particular, it is complete.

Proof. Suppose that A is an integral domain. By Noether normalization, we can take A to be a finite $\mathbb{T}_d$-algebra. Let $(a_i)_{i=1}^n$ be a basis of L/E where L and E are the fields of fractions of A and $\mathbb{T}_d$, respectively. Since A is a finitely generated $\mathbb{T}_d$-module, there is $f \in \mathbb{T}_d \setminus \{0\}$ such that $f \cdot A \subseteq M$ where $M = \oplus_{i=1}^n \mathbb{T}_d a_i \subseteq L$. We put $\| \sum_{i=1}^n b_i a_i | M\| = \max\{\|b_i|\mathbb{T}_d\| : 1 \leq i \leq n\}$. By a result from the Winter, $\|a|A\|$ and $\|fa|M\|$ are equivalent norms on A. Since $\|fa|M\|$ is a weakly Cartesian K-norm on L, it follows that $\| \cdot |A\|$ is the restriction to A of the spectral norm on L. By Proposition 9.1, this is equivalent to any weakly Cartesian norm on L – recall here that the maximum norm is bounded by the spectral norm. In particular, the norm restricting to $\|fa|M\|$ on A. Hence $\| \cdot |A\|_s = \| \cdot |A\|_{\max}$ as stated. The general case can be reduced to this one by induction on the number of minimal prime ideals of A. ∎

Example 10.4. $\| \cdot |A\|_{\max}$ may fail to be multiplicative even when A is an integral domain. For example, let $A = \mathbb{T}_2/(X_1X_2 - \pi)$ for $\pi \in K^\times \cap K^{\circ\circ}$ a topoligically nilpotent unit. Then

$$\begin{aligned} \|X_1|A\|_{\max} &= 1 \\ \|X_2|A\|_{\max} &= 1 \\ \|X_1X_2|A\|_{\max} &= |\pi| < 1. \end{aligned}$$

Proposition 10.5. Let A be an affinoid K-algebra. The spectral norm on A is multiplicative if and only if A is reduced and $\widetilde{A} = A^\circ/A^{\circ\circ}$ is an integral domain.

Proof. ($\Rightarrow$) Since $\| \cdot |A\|_s$ is power-multiplicative, the nilradical of A is trivial when $\| \cdot |A\|_s$ is a norm.

($\Leftarrow$) Suppose A is reduced and $\widetilde{A}$ is an integral domain. Let $a_1, a_2 \in A \setminus \{0\}$. We need to show $\|a_1a_2|A\|_s = \|a_1|A\|_s \cdot \|a_2|A\|_s$. Since $\| \cdot |A\|_s$ is power multiplicative, it suffices to prove the statement for a_1^n, a_2^n where $n \in \mathbb{N}$. By all residual norms on an affinoid K-algebra being equivalent, there are $k_i \in K^\times$ such that $|k_i| = \|a_i|A\|_s$. Replacing a_i by a_i/k_i we have $\|a_i|A\|_s = 1$ and the inequality $\|a_1a_2|A\|_s < 1$ would imply that the images of a_i in $\widetilde{A}$ are nonzero nilpotents. ∎

Remark 10.6. Let A be an affinoid K-algebra. We can use these results to construct continuous valuations of A hence showing that the dimension of $\mathrm{Spa}(A, A^\circ)$ is equal to $\dim(A)$. For a domain A, $\mathbb{T}_d \subseteq A$ for some d by Noether normalization such that $A/\mathbb{T}_d$ is finite. Then $\dim(A) = d$ by going up.

Let E and F be the fields of fractions of $\mathbb{T}_d$ and A, respectively. Equip E with the absolute value defined by the completely multiplicative $\mathbb{T}_d$-norm. $|\cdot|_E$ admits an extension to $|\cdot|_F$.

Example 10.7. Let $Y = V(X_1X_2 - \pi) \subseteq \mathrm{Sp}(\mathbb{T}_2)$. Define

$$|f|^{(i)} = \max\{|f(x)| : |X_i| = 1\}$$

which is an absolute value on $A = \mathbb{T}_2/(X_1X_2 - \pi)$. $F^\circ/F^{\circ\circ}$ is finite over $E^\circ/E^{\circ\circ}$ so the transcendence degree of the residue field over the ground field is d and thus defines a rank d valuation ν such that $\nu(\widetilde{A}) \leq 1$ recalling here that $\widetilde{A} = A^\circ/A^{\circ\circ}$. The construction above can be used to augument $|\cdot|_F$ to a rank $d+1$ valuation on F, for example, taking

$$\mu(f) = \begin{cases} 0 & f = 0 \\ (|f|_F, \nu(f \pmod{F^{\circ\circ}})) & \text{otherwise.} \end{cases}$$

Its restriction to A gives a rank $d+1$ valuation, that is, a point of $\mathrm{Spa}(A, A^\circ)$.

We recall the following propositions.

Proposition 10.8 (Proposition 7.7)**.** Let $A \to B$ be a morphism of affinoid K-algebras such that B is finitely generated as an A-module. Then $\widetilde{B}$ is finitely generated over $\widetilde{A}$.

Corollary 10.9. If A is an affinoid K-algebra then $\widetilde{A}$ is of finite type over $\mathfrak{K} = K^\circ/K^{\circ\circ}$.

Recall that

$$\mathcal{F}_\Omega(B) = \{f \in \mathrm{Hom}_{\mathsf{Aff}_K}(A, B) : \mathrm{Sp}(f)(\mathrm{Sp}(B)) \subseteq \Omega \subseteq \mathrm{Sp}(A)\}\,. \tag{10.1}$$

This functor was used to define sections over a rational open subset, but can be applied to arbitrary open subsets of the Tate spectrum $\mathrm{Sp}(A)$.

Definition 10.10 (Affinoid Open Subset)**.** An open subset $\Omega \subseteq \mathrm{Sp}(A)$ is affinoid if the functor (10.1) is representable.

Definition 10.11 (Admissable Open Subset)**.** An open subset of $\Omega \subseteq \mathrm{Sp}(A)$ is admissable if and only if for every morphism $A \to B$ in Aff_K the map $\mathrm{Sp}(B) \to \mathrm{Sp}(A)$ is continuous over Ω.

Remark 10.12. The continuity condition is that

$$\{V \in \mathsf{Opens}_{f^{-1}(\Omega)} : \exists U \subseteq \mathrm{Sp}(A) \text{ s.t. } f(V) \subseteq U\}$$

is a covering sieve of $f^{-1}(\Omega)$.

We can now state without proof the Gerritzen-Grauert theorem.

Theorem 10.13 (Gerritzen-Grauert)**.** Let $\Omega \subseteq \mathrm{Sp}(A)$ be affinoid then Ω can be written as a finite union of rational open subsets of $\mathrm{Sp}(A)$. In particular, any affinoid open subset is a G_+-quasicompact open subset of $\mathrm{Sp}(A)$.

Proof. Omitted. ■

We consider the simple example of a finite union of rational open subsets.

Proposition 10.14. Let A be an affinoid K-algebra. Any finite union of rational open subsets is admissable.

Proof. Let $\Omega = \bigcup_{i=1}^n \Omega_i$ be such that each $\Omega_i \in \mathsf{Rat}_{\mathrm{Sp}(A)}$ and $\mathcal{S}$ a covering sieve for Ω. There are finitely many $\Omega_{ij} \in \mathcal{S} \cap \mathcal{B}_{\Omega_i}$ such that $\Omega_i = \bigcup_{j=1}^{m_i} \Omega_{ij}$. If $\varphi : A \to B$ inducing $f : \mathrm{Sp}(B) \to \mathrm{Sp}(A)$ and $\Omega_{ij} = R_{\mathrm{Sp}(A)}(a_{ij,0}|a_{ij,1}, \dots, a_{ij,n})$ we have $f^{-1}(\Omega_{ij}) = R_{\mathrm{Sp}(B)}(\varphi(a_{ij,0})|\varphi(a_{ij,1}), \dots, \varphi(a_{ij,n})) \in \mathsf{Rat}_{\mathrm{Sp}(B)} \cap f^{-1}\mathcal{S}$. Since $f^{-1}(\Omega) = \bigcup_{i=1}^n \bigcup_{j=1}^{m_i} \Omega_{ij}$, this shows $f^{-1}\mathcal{S}$ is a covering sieve of $f^{-1}(\Omega)$. ■

11. Lecture 11 – 10th July 2025

Recall from the Gerritzen-Grauert representability theorem Theorem 10.13, every affinoid subset is admissable in the sense of Definition 10.11. We continue our investigations of these notions.

Lemma 11.1. Let $U \subseteq \mathrm{Sp}(A)$ be open. If for every morphism $\alpha : A \to B$ of affinoid K-algebras such that $\mathrm{Sp}(\alpha)(\mathrm{Sp}(B)) \subseteq U \subseteq \mathrm{Sp}(A)$ there exists a rational open subset Ω in U containing $\mathrm{Sp}(B)$ then U is admissable (i.e. in the Grothendieck topology).

Proof. Let $\alpha : A \to B$ be any morphism of affiniod K-algebras and $f = \mathrm{Sp}(\alpha) : \mathrm{Sp}(B) \to \mathrm{Sp}(A)$ the induced map. We want to show that $f^{-1}\mathcal{S}$ is a covering sieve for $f^{-1}(U)$ for any covering sieve $\mathcal{S}$ of U. The rational open subsets $\Theta \subseteq f^{-1}(U)$ admissably cover $f^{-1}(U)$ so by locality of covering sieves it suffices to show $f^{-1}\mathcal{S}|_\Theta$ covers Θ. Since $\Theta = \mathrm{Sp}(\mathcal{O}_{\mathrm{Sp}(B)}(\Theta))$, we may replace B by $\mathcal{O}_{\mathrm{Sp}(B)}(\Theta)$. This gives $f(\mathrm{Sp}(B)) \subseteq U$ and we have to show that $f^{-1}\mathcal{S}$ covers $\mathrm{Sp}(B)$. By assumption, there is $\Omega \subseteq \mathrm{Sp}(A)$ such that $f(\mathrm{Sp}(B)) \subseteq \Omega$. Then $f^{-1}\mathcal{S} = f^{-1}(\mathcal{S}|_\Omega)$ covers $\mathrm{Sp}(B)$ as $\mathcal{S}|_\Omega$ covers Ω and such Ω are admissable. ∎

Lemma 11.2. Let A be an affinoid K-algebra. Then $\mathrm{Sp}(A) \setminus V(a)$ is admissable.

Proof. Keeping the notations of Lemma 11.1 and its proof, if $f(\mathrm{Sp}(B)) \subseteq U$ then $a \in B^\times$ and there is $r \in K^\times$ such that $\frac{1}{ar} \in B^\times$ so $f(\mathrm{Sp}(B)) \subseteq \mathrm{Sp}(A)$ is contained in the rational open subset $R_{\mathrm{Sp}(A)}(ar|1)$. ∎

Example 11.3. Every Zariski open subset is admissable. This is immediate from Lemma 11.2 and that every Zariski open is a union of such opens.

It is, however, not the case that all open subsests are admissable.

Example 11.4. Let $A = \mathbb{T}_2$, $c \in K$ such that $0 < |c| < 1$, $\Theta_i = R_{\mathrm{Sp}(A)}(c^i|X_2) \cap R_{\mathrm{Sp}(A)}(c|X_1^i)$, $\Xi = R_{\mathrm{Sp}(A)}(X_1|1)$. The open set $U = \Xi \cup \bigcup_{i=1}^\infty \Theta_i$ is not admissable. Indeed, let $Y = \mathrm{Sp}(\mathbb{T}_1)$ identified with $V(X_2) \subseteq \mathrm{Sp}(A)$ with $\pi : X \to Y$ the projection induced by the inclusion. For every rational open $\Omega \subseteq X$ there is $\varepsilon \in K^\times$ such that $\Omega \cap R_{\mathrm{Sp}(A)}(\varepsilon|X_2) = \pi^{-1}(\Omega \cap Y) \cap R_{\mathrm{Sp}(A)}(\varepsilon|X_2)$. Let n be chosen smallest such that $|c|^{n+1} < |\varepsilon|$. Then $U \cap R_{\mathrm{Sp}(A)}(X_2|\varepsilon) \cap \Theta_i = \emptyset$ for $i > n$ as $R_{\mathrm{Sp}(A)}(X_2|\varepsilon) \cap \Theta_i = \emptyset$. And if $\Omega \subseteq U$ then $\Omega \cap R_{\mathrm{Sp}(A)}(X_2|\varepsilon) \subseteq \Xi \cup \bigcup_{i=1}^n \Theta_i$. So $\Omega \subseteq U$. Since these Ω cover U admissably, $\mathcal{S} = [\Xi, \Theta_1, \dots]$ covers U. If $i : Y \to X$ is the inclusion, $i^*\mathcal{S}$ would imply that Y decomposes $R_Y(X_1|1) \cup (Y \setminus R_Y(X_1|1))$ producing $e \in \mathbb{T}_1$ which is 1 on one component and 0 on the other, contradicting $\mathbb{T}_1$ being a domain.

In this proof, we have relied on the following lemma.

Lemma 11.5. Let $X = \mathrm{Sp}(\mathbb{T}_n), Y = \mathrm{Sp}(\mathbb{T}_{n-1}), \pi : X \to Y$ the inclusion and $i : Y \to X$ the quotient of $\mathbb{T}_n$ by X_n. If $\Omega \subseteq X$ is rational there is $\varepsilon \in K^\times$ such that $\Omega \cap R_X(\varepsilon|X_n) = (\pi^{-1}i^{-1}(\Omega)) \cap R_X(\varepsilon|X_n)$.

We are finally able to define rigid spaces.

Definition 11.6 (Para-Rigid Space)**.** Let X be a G_+-space with a sheaf of K-algebras. X is para-rigid if the set of affinoid opens admissably cover X.

Definition 11.7 (Continuous Morphism of Para-Rigid Spaces)**.** A morphism $f : Y \to X$ of para-rigid spaces is the data of a continuous topological map with a morphism $f^* : \mathcal{O}_X \to f_\flat \mathcal{O}_Y$ of presheaves such that the composite

$$\mathcal{O}_{X,f(y)} \to (f_\flat \mathcal{O}_Y)_{f(y)} \to \mathcal{O}_{Y,y}$$

is a local homomorphism of local rings.

The opposite category of affinoid K-algebras embeds fully faithfully into the category of para-rigid spaces.

Proposition 11.8. There is a bijection

$$\mathrm{Hom}_{\mathsf{ParRigid}}(\mathrm{Sp}(B), \mathrm{Sp}(A)) \leftrightarrow \mathrm{Hom}_{\mathsf{Aff}_K}(A, B)$$

by the induced map on global sections and the $\mathrm{Sp}(-)$ of a morphism of algebras natural in A and B.

We consider some continuity properties of para-rigid spaces.

Proposition 11.9. Let X be a para-rigid space. The following are equivalent.

(a) Every morphism of para-rigid spaces $T \to X$ is continuous over X.
(b) Every $\mathrm{Sp}(A) \to X$ is continuous over X.

In particular, continuity is local on the source.

Definition 11.10 (Intrinsically Admissable Para-Rigid Space)**.** Let X be a para-rigid space. X is intrinsically admissable if it satisfies either (and thus both) of the conditions of Proposition 11.9.

This already holds for G_+ quasicompact spaces.

Proposition 11.11. Let X be a para-rigid space. If X is G_+-quasicompact then X is intrinsically admissable.

Proposition 11.12. Let X be a para-rigid space. The following are equivalent:

(a) The intrinsically admissable open subsets of a para-rigid space form a G_{++} basis of X.
(b) The intersection of intrinsically admissable opens of X are intrinsically admissable.
(c) The affinoid open subspaces of X are intrinsically admissable.

We now make a long-awaited definition.

Definition 11.13 (Rigid Analytic Space)**.** Let X be a para-rigid space. X is a rigid analytic space if it satisfies the equivalent conditions of Proposition 11.12.

Definition 11.14 (Admissable Open)**.** Let $U \subseteq X$ be an open subset of a rigid analytic space. U is admissable if every morphism of rigid analytic spaces is continuous over U.

References

[vdPS95] Marius van der Put and P. Schneider. “Points and topologies in rigid geometry”. English. In: *Math. Ann.* 302.1 (1995), pp. 81–103. ISSN: 0025-5831. DOI: 10.1007/BF01444488. URL: https://eudml.org/doc/165324.

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