V5B1 – ADVANCED TOPICS IN COMPLEX ANALYSIS
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Preliminaries

These notes roughly correspond to the course **V5B1 - Advanced Topics in Complex Analysis** taught by Prof. Ingo Lieb at the Universität Bonn in the Winter 2024/25 semester. These notes are LaTeX-ed after the fact with significant alteration and are subject to misinterpretation and mistranscription. Use with caution. Any errors are undoubtedly my own and any virtues that could be ascribed to these notes ought be attributed to the instructor and not the typist.

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1. Lecture 1 – 11th October 2024

This course will cover complex analysis in one or more variables with a view towards both number theory and algebraic geometry.

Recall the following definition.

Definition 1.1 (Affine Regular Quadric)**.** An affine regular quadric is a set

$$Q = \{(u, v) \in \mathbb{C}^2 : v^2 = f(u)\} \subseteq \mathbb{C}^2$$

where $f(u)$ is a univariate polynomial of degree 2 with distinct zeroes.

Definition 1.1

Marginal notes will follow the numbering from lecture.

We can apply a linear coordinate transformation to express the quadric Q in an especially nice form.

Proposition 1.2. There exists a linear change of coordinates such that Q is given by the set $\{(u, v) \in \mathbb{C}^2 : u^2 + v^2 = 1\} \subseteq \mathbb{C}^2$.

Proposition 1.1

A quadric given in the above form is said to be in normal form.

Definition 1.3 (Normal Form of Quadric)**.** An affine regular quadric $Q \subseteq \mathbb{C}^2$ is said to be in normal form if it is given by $\{(u, v) \in \mathbb{C}^2 : u^2 + v^2 = 1\} \subseteq \mathbb{C}^2$.

From a first course in complex analysis, we have that $\cos^2(z) + \sin^2(z) = 1$ for all $z \in \mathbb{C}$ and hence for an affine regular quadric in normal form Q, there is a map $h : \mathbb{C} \to Q$ by $z \mapsto (\cos(z), \sin(z))$.

Proposition 1.4. Let Q be an affine regular quadric in normal form. The map $h : \mathbb{C} \to Q$ by $z \mapsto (\cos(z), \sin(z))$ is holomorphic.

Proposition 1.2

Proof. It suffices to see that the coordinate functions $z \mapsto \sin(z), z \mapsto \cos(z)$ are themselves holomorphic. ■

We can also find a rational parametrization. Choose a point $(0, 1) \in Q$. For $z \in \mathbb{C}$, we can consider the complex line $v = zu + 1$ which intersects Q at one other point where $v^2 = (zu + 1)^2$. Since the quadric is in normal form, we have $v^2 = 1 - u^2$ so expanding we have $1 - u^2 = z^2u^2 + 2zu + 1$ which is a univariate quadratic equation in u with z fixed as above. Rearranging the equation, we get $(1 + z^2)u^2 + 2zu = 0$ and since we already know the intersection point $(0, 1) \in Q$ we consider the case when $u \neq 0$ so $(1 + z^2)u + 2z = 0$ and thus $u = \frac{-2z}{1+z^2}$. Substituting for v, and making similar computations, we can see $v = \frac{1-z^2}{1+z^2}$. In fact, we have the following result.

Proposition 1.5. Let Q be an affine regular quadric in normal form. The rational map $\mathbb{C} \to Q$ by $z \mapsto \left(\frac{-2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$ is injective and extends to a holomorphic map $\widehat{\mathbb{C}} \setminus \{\pm i\} \to Q$.

Proposition 1.3

Proof. The map is injective since a univariate equation of the form $(1+z^2)u^2+2zu = 0$ identified in the preceding discussion has a unique nonzero root depending on z. Furthermore, this function can be seen to be meromorphic as it is a rational function and is hence holomorphic away from its poles which are $\pm i$. ■

We can compactify the quadric by embedding it in projective space instead of $\mathbb{C}^2$.

Definition 1.6 (Complex Projective Plane)**.** The complex projective plane $\mathbb{CP}^2$ is given by

$$\left\{(z_0, z_1, z_2) \in \mathbb{C}^3 \setminus \{(0,0,0)\}\right\} / \sim$$

where $(z_0, z_1, z_2) \sim (z_0', z_1', z_2')$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $z_0 = \lambda z_0', z_1 = \lambda z_1', z_2 = \lambda z_2'$.

We will write these equivalence classes as $[z_0 : z_1 : z_2]$.

There is a map $\iota_0 : \mathbb{C}^2 \to \mathbb{CP}^2$ by $(u, v) \mapsto [1 : u : v]$ realizing $\mathbb{CP}^2$ as $\iota_0(\mathbb{C}^2) \cup \{z_0 = 0\}$, that is, by adjoining a $\mathbb{CP}^1$ at infinity. The closure of image $\widehat{Q}$ of Q under ι consists of $z_0^2 = z_1^2 + z_2^2$ where we have $\widehat{Q} = \iota_0(Q) \cup \{[0 : 1 : \pm i]\}$ which also extends to a rational parametrization by $[t_0 : t_1] \mapsto [t_0^2 + t_1^2 : -2t_0t_1 : t_0^2 - t_1^2]$ rendering the diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & \mathbb{C}^2 \arrow[d] \\
\mathbb{CP}^1 \arrow[r] & \widehat{Q} \arrow[r] & \mathbb{CP}^2
\end{tikzcd}$$

commutative. We have the following.

Proposition 1.7. The map $[t_0 : t_1] \mapsto [t_0^2 + t_1^2 : -2t_0t_1 : t_0^2 - t_1^2]$ defines a homeo-
Proposition 1.4 morphism $\mathbb{CP}^1 \to \widehat{Q}$.

Proof. On $\widehat{Q}$ the map admits an inverse $[z_0 : z_1 : z_2] \mapsto [-z_0 - z_2 : z_1]$. ■

This shows that $\widehat{Q}$ is topologically a sphere.

We now consider the case of cubics.

Definition 1.2 **Definition 1.8** (Affine Regular Cubic)**.** An affine regular cubic is a set

$$K = \{(u, v) \in \mathbb{C}^2 : v^2 = f(u)\} \subseteq \mathbb{C}^2$$

where $f(u)$ is a univariate polynomial of degree 3 with disctinct zeroes.

This too admits a normal form.

Proposition 1.9. There exists a linear change of coordinates such that K is given by the set $\{(u, v) \in \mathbb{C}^2 : v^2 = 4u^3 - g_2u - g_3\}$ with $4u^3 - g_2u - g_3$ having three distinct roots.

Proof. Suppose $f(u) = a_3u^3 + a_2u^2 + a_1u + a_0$. The desired transformation is given by $u \mapsto \sqrt[3]{\frac{1}{4}} \cdot \left(\frac{1}{\sqrt[3]{a_3}} + \frac{a_2}{3\sqrt[3]{a_3^2}} \right)$. ■

For u_1, u_2, u_3 the roots of $4u^3 - g_2u - g_3$ as above, we can solve this depressed cubic equation and observe that g_2 and g_3 are given by symmetric polynomials in the u_1, u_2, u_3. In fact, $u_1 + u_2 + u_3 = 0, -4(u_1u_2 + u_1u_3 + u_2u_3) = g_2, 4u_1u_2u_3 = g_3$. We can then write the discriminant of this polynomial as

$$16(u_1 - u_2)^2(u_2 - u_3)^2(u_3 - u_1)^2 = g_2^3 - 27g_3^2.$$

This defines an elliptic curve which is a well-studied object in number theory and arithmetic geometry.

Analogously to the case of quadrics, we can pass to the projective closure $\widehat{K}$ given by the equation $z_0 z_2^2 = 4z_1^3 - g_2 z_0^2 z_1 - g_3 z_0^3$ which is a homogeneous polynomial obtained as the one-point compactification of the regular affine cubic K at the point $[0 : 1 : 0]$.

This hints at a more general problem of finding rational parametrizations of interesting subsets of $\mathbb{C}^n$ which as Galois theory suggests is not possible in general, nor is it so in the case of cubics. Though the study of cubics will give rise to the theory of elliptic functions.

As before, we can define a quartic.

Definition 1.10 (Affine Regular Quartic)**.** An affine regular quartic is a set Definition 1.3

$$H = \{(u, v) \in \mathbb{C}^2 : v^2 = f(u)\} \subseteq \mathbb{C}^2$$

where $f(u)$ is a univariate polynomial of degree 4 with disctinct zeroes.

Without loss of generality, we can take $f(u)$ to be monic and of the form $u \prod_{i=2}^4 (u - a_i)$ by translation. Letting $x = \frac{1}{u}$ we have $v^2 = \frac{1}{x} \prod_{i=2}^4 (\frac{1}{x} - a_i)$ and multiplying with x^4 we can set $x^4 v^2 = (1 - a_2 x)(1 - a_3 x)(1 - a_4 x)$ and again taking $x^4 v^2 = y^2$ that $y^2 : (1 - a_2 x)(1 - a_3 x)(1 - a_4 x)$ which is a cubic in x with distinct zeroes that we can put in the Weierstrass normal form for cubics as in Proposition 1.9 which reduces the study of quartics to the study of cubics. The first part of the course will focus on elliptic functions and their connections to number theory.

Returning to the discussion of the cubic above, we can note that the construction of Proposition 1.5 on restriction to $\mathbb{Q}$ recovers the rational points on the circle and Proposition 2.1
by normalizing recover Pythagorean triples, those $(a, b, c) \in \mathbb{N}$ pairwise coprime such that $a^2 + b^2 = c^2$. Definition 2.1

2. Lecture 2 – 15th October 2024

We recall some results on integration theory, first treating the case of rational functions.

Proposition 2.4

Proposition 2.1. The integral of a rational function can be expresssed as a the sum of a rational function and a linear combination of logarithms of linear forms.

Proof. Without loss of generality, by taking the partial fraction decomposition of a rational function, it suffices to consider integrals of the form $\frac{f(z)}{(z-a)^m}$ where $f(z)$ is a polynomial in z. If $m > 1$ then note that $\frac{1}{1-m}\frac{d}{dz}\frac{1}{(z-a)^{m-1}} = \frac{1}{(z-a)^m}$ so we can write $\frac{d}{dz}\frac{1}{1-m}\frac{f(z)}{(z-a)^{m-1}} = \frac{f(z)}{(z-a)^m} + \frac{f'(z)}{(1-m)(z-a)^{m-1}}$ reducing the integral to the integration of $\frac{f'(z)}{(1-m)(z-a)^{m-1}} + \frac{d}{dz}\frac{1}{1-m}\frac{f(z)}{(z-a)^{m-1}}$. Iterating this procedure, we get to the case of $\frac{f(z)}{(z-a)}$ which can be treated via integration by parts and that $\int \frac{1}{z-a} dz = \log(z-a)$. ∎

We now turn to the case of integrals related to cubics. Consider the integral

$$\int \frac{1}{\sqrt{4z^3 - g_2 z - g_3}} dz$$

which arises from the affine regular cubic in normal form as discussed in Proposition 1.9. We will soon be able to find a parameterization of the above integral to solve it via the study of elliptic functions.

We begin with a more general treatment of periods of meromorphic functions.

Definition 1.1

Definition 2.2 (Period)**.** Let $f(z)$ be a meromorphic function. A complex number $\omega \in \mathbb{C}$ is a period of f if $f(z+\omega) = f(z)$ for all z.

Remark 2.3. It can be seen that $0 \in \mathbb{C}$ is a period for all meromorphic functions.

To the end of showing that these periods form a discrete additive subgroup of $\mathbb{C}$, let us recall the following result concerning the classification of subgroups of discrete subgroups of $\mathbb{C}$.

Proposition 1.1

Lemma 2.4. If G is a discrete additive subgroup of $\mathbb{C}$ then G is one of the following:

- The trivial group 0.
- $\omega \cdot \mathbb{Z}$ for $\omega \in \mathbb{C}$.
- $\Omega = \{a_1\omega_1 + a_2\omega_2 : a_1, a_2 \in \mathbb{Z}\}$ with $\frac{\omega_2}{\omega_1} \in \mathbb{C} \setminus \mathbb{R}$.

Proof. If G is trivial, we are done. Otherwise, suppose G is non-trivial and consider $r > 0$ such that the closed unit ball $\overline{B_r(0)}$ of radius r contains a non-identity element. Since $\overline{B_r(0)}$ is compact and G is discrete the intersection $\overline{B_r(0)} \cap G$ consists of only finitely many elements of G. Take ω_1 nonzero in this intersection such that $|\omega_1|$ is minimal. This shows $\mathbb{Z} \cdot \omega_1 \leq G$ and we are done if this is an equality.

If $\mathbb{Z} \cdot \omega_1 < G$ then consider some $\omega_2 \in G \setminus \mathbb{Z} \cdot \omega_1$ with $|\omega_2|$ minimal. We first show that $\frac{\omega_2}{\omega_1} \notin \mathbb{R}$. Suppose to the contrary that $\frac{\omega_2}{\omega_1} \in \mathbb{R}$. Then there is an integer n such that $n < \frac{\omega_2}{\omega_1} < n+1$ where the inequalities are strict since $\omega_2 \in G \setminus \mathbb{Z} \cdot \omega_1$. As such $|n\omega_1 - \omega_2| < |\omega_1|$ contradicting minimality of $|\omega_1|$ showing $\mathbb{Z} \cdot \omega_1 \oplus \mathbb{Z} \cdot \omega_2 \leq G$.

Note that since ω_1, ω_2 are $\mathbb{R}$-linearly independent, any complex number can be written as an $\mathbb{R}$-linear combination of ω_1, ω_2. For $z \in \mathbb{C}$ expressed as $z = \lambda_1\omega_1 + \lambda_2\omega_2$ with $\lambda_1, \lambda_2 \in \mathbb{R}$, take $m_1 m_2 \in \mathbb{Z}$ such that $|\lambda_1 - m_1|, |\lambda_2 - m_2| \leq \frac{1}{2}$. In particular, for $z \in G$ possibly not in $\mathbb{Z} \cdot \omega_1 \oplus \mathbb{Z} \cdot \omega_2$ we have that $z' = z - m_1\omega_1 - m_2\omega_2 \in G$ as well, but

$$|z'| = |(\lambda_1 - m_1)\omega_1 + (\lambda_2 - m_2)\omega_2| < \frac{1}{2}|\omega_1| + \frac{1}{2}|\omega_2| \leq |\omega_2|$$

where the strictness of the middle inequality follows from $\mathbb{R}$-linear independence of ω_1, ω_2 and the second inequality from $\frac{1}{2}|\omega_1| \leq \frac{1}{2}|\omega_2|$ by minimality of $|\omega_1|$ in $\overline{B_r(0)} \cap G$. Now noting that $|\omega_2|$ was minimal among $G \setminus \mathbb{Z} \cdot \omega_1$, we have that $z' \in \mathbb{Z} \cdot \omega_1$ so for $z' = n\omega_1$ we can rewrite $z = (n + m_1)\omega_1 + m_2\omega_2$ showing $z \in \mathbb{Z} \cdot \omega_1 \oplus \mathbb{Z} \cdot \omega_2$, giving the claim. ∎

We can now show that periods form an additive subgroup as follows.

Proposition 2.5. Let $f(z)$ be a meromorphic function on $\mathbb{C}$. If f is not constant, then the periods of f form a discrete additive subgroup Ω of $\mathbb{C}$.

Proof. Let ω, ω' be periods and $-\omega$ the additive inverse of ω in $\mathbb{C}$. We can compute

$$\begin{aligned} f(z + (\omega + \omega')) = f((z + \omega) + \omega') = f(z + \omega') &= f(z) \\ f(z - \omega) = f((z + \omega) - \omega) &= f(z) \end{aligned}$$

showing that periods are closed under addition and inversion, and contain zero per Remark 2.3. Associativity follows from associativity on the group additive group of complex numbers, showing that the periods form a subgroup.

It remains to show discreteness. Let ω be a period and consider $B_r(\omega)$ the open ball of radius r with r chosen small enough that f is analytic on $B_r(\omega)$. Suppose to the contrary for each $n \in \mathbb{N}$ there exists a period $\omega_n \in B_{r/n}(\omega)$. Then $|\omega - \omega_n| < \frac{r}{n}$ showing that the sequence $\omega_n \to \omega$ as $n \to \infty$. By the identity theorem Theorem A.1, f is constant on $B_r(\omega)$, a contradiction, as f is non-constant. ∎

Elliptic functions are defined in terms of their period group.

Definition 2.6 (Elliptic Function)**.** A meromorphic function $f(z)$ is elliptic if its period group contains a lattice.

Remark 2.7. Elliptic functions are often also known as doubly periodic functions.

Remark 2.8. Defining elliptic functions in terms of their period group containing a lattice allows constant functions to be elliptic – since constant functions have period group $\mathbb{C}$.

Elliptic functions are determined by their values on their open period parallelogram since for Ω a lattice, we can define an equivalence relation $z \sim z'$ if $z - z' \in \Omega$.

Definition 2.9 (Period Parallelogram)**.** Let $\Omega = \{a_1\omega_1 + a_2\omega_2 : a_1, a_2 \in \mathbb{Z}\} \subseteq \mathbb{C}$ be a lattice. The period parallelogram is given by

$$P_\Omega = \{t_1\omega_1 + t_2\omega_2 : 0 \leq t_1, t_2 < 1\} \subseteq \mathbb{C}.$$

Under the equivalence relation described above, the quotient space $\mathbb{C}/\Omega$ is an additive group where each point of the period parallelogram is a representative of the quotient. Moreover, this can be seen to be a complex torus using the standard cut-and-paste diagram for a torus in topology which is a compact topological space, and in fact a compact Riemann surface. As such, any Ω-periodic meromorphic function is determined by its values on the torus.

We now consider a general property of lattices before returning to a discussion of periodic functions.

Proposition 2.10. Let $\Omega = \{a_1\omega_1 + a_2\omega_2 : a_1, a_2 \in \mathbb{Z}\} \subseteq \mathbb{C}$ be a lattice. Then
Proposition 1.2 $\sum_{\omega \in \Omega \setminus \{0\}} \frac{1}{\omega^k}$ is absolutely convergent for $k > 2$.

Proof. Denote P_ℓ the parallelogram given by the lattice points of the convex hull of $\pm\ell\omega_1, \pm\ell\omega_2$ and ∂P_ℓ its boundary. We have that $|\partial P_\ell| = 8\ell$ and for $C = \max_{\omega \in \partial P_1} |\omega|$ and that $(\ell C)^k \leq |\omega|^k$ for $\omega \in \partial P_\ell$. We can compute

$$\begin{aligned} \sum_{\omega \setminus \{0\}} \frac{1}{|\omega|^k} &= \sum_{\ell=1}^{\infty} \sum_{\omega \in \partial P_\ell} \frac{1}{|\omega|^k} \\ &\leq \sum_{\ell=1}^{\infty} \frac{8\ell}{(\ell C)^k} = \frac{8}{C^k} \sum_{\ell=1}^{\infty} \frac{1}{\ell^{k-1}} \end{aligned}$$

which is only convergent if $k > 2$. ■

Let us return to a discussion of ellitpic functions.

Proposition 2.11. Let $\Omega = \{a_1\omega_1 + a_2\omega_2 : a_1, a_2 \in \mathbb{Z}\} \subseteq \mathbb{C}$ be a lattice. Then the elliptic functions with respect to Ω form a field that is closed under differentiation.

Proof. Double periodicity is preserved under sums, products, differences, and quotients, and the operations distribute in the expected way. Preservation under differentiation follows from the chain rule. ■

However, holomorphic elliptic functions are uninteresting.

Proposition 2.12. Let f be an elliptic function with respect to a lattice Ω. If f is
Proposition 2.2 holomorphic, then f is constant.

Proof. The closure of the open period parallelogram P_Ω is compact on which $|f(z)|$ admits a maximum. By periodicity, $f(z)$ is a bounded holomorphic function on $\mathbb{C}$, from which the claim follows by Liouville's theorem A.2. ■

Generalizing our discussion to elliptic functions with poles, we can show the following.

Proposition 2.13. Let f be a nonconstant elliptic function and $z_1, \ldots, z_m$ the poles
Proposition 2.3 of f in P_Ω. Then $\sum_{i=1}^{m} \operatorname{res}_{z_i} f = 0$.

Proof. Without loss of generality, we can take the poles to lie in the interior of the open period parallelogram. By Cauchy's residue theorem, we have $\frac{1}{2\pi i}\int_{\partial P_\Omega} f(z) = \sum_{i=1}^m \mathrm{res}_{z_i} f$. Denoting $[a,b]$ the oriented straight line path from a to b we compute

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial P_\Omega} f(z)dz &= \int_{[0,\omega_1]} f(z)dz + \int_{[\omega_1,\omega_1+\omega_2]} f(z)dz \\ &\quad + \int_{[\omega_1+\omega_2,\omega_2]} f(z)dz + \int_{[\omega_2,0]} f(z)dz \\ &= \left(\int_{[0,\omega_1]} f(z)dz - \int_{[\omega_2,\omega_1+\omega_2]} f(z)dz\right) + \\ &\quad \left(\int_{[\omega_1,\omega_1+\omega_2]} f(z)dz - \int_{[0,\omega_2]} f(z)dz\right)\end{aligned}$$

where both of the summands vanish by periodicity of f yielding the claim. ∎

Remark 2.14. Proposition 2.13 implies that an elliptic function has at least two poles when counting with multiplicity.

Given a non-constant – and hence meromorphic – elliptic function, it can be shown that its image is all of $\widehat{\mathbb{C}}$.

Proposition 2.15. Let f be a non-constant elliptic function. Then f assumes every value in $\widehat{\mathbb{C}}$ equally often, counting with multiplicity.

Proof. Let $\lambda \in \widehat{\mathbb{C}}$. Note that the integral $\frac{1}{2\pi i}\int_{\partial P_\Omega} \frac{f'(z)}{f(z)-\lambda}dz$ computes the difference between the number of times f assumes the value λ and f assumes the value ∞ which is zero since the integrand is an elliptic function by Proposition 2.11 and zero by Proposition 2.13. ∎

3. Lecture 3 – 17th October 2024

We consider some additional properties of elliptic functions.

Proposition 2.5

Proposition 3.1. Let Ω be a lattice with period parallelogram P_Ω and f an elliptic function with respect to Ω. Let $a_1, \ldots, a_n$ and $a_{n+1}, \ldots, a_\ell$ be the zeroes and poles of f, respectively, of orders $m_1, \ldots, m_n$ and $m_{n+1}, \ldots, m_\ell$, respectively. Then

$$\left(\sum_{i=1}^{n} m_i a_i\right) - \left(\sum_{i=n+1}^{\ell} m_i a_i\right) \in \Omega. \tag{3.1}$$

Proof. Without loss of generality, we can take these zeroes and poles to lie in the interior of the period parallelogram. Consider the function $g(z) = z \cdot \frac{f'(z)}{f(z)}$ which has simple poles at both the poles and zeroes of f with residues $m_i a_i$ and $-m_i a_i$, respectively. In particular, the sum of (3.1) is given by $\frac{1}{2\pi i}\int_{\partial P_\Omega} g(z)dz$. Decomposing this as an integral over segments as in Proposition 2.13, we have

$$\int_{\partial P_\Omega} g(z)dz = \int_{[0,\omega_1]} g(z)dz + \int_{[\omega_1,\omega_1+\omega_2]} g(z)dz + \int_{[\omega_1+\omega_2,\omega_2]} g(z)dz + \int_{[\omega_2,0]} g(z)dz.$$

Considering the integral over the oriented segments $[0,\omega_1]$ and $[\omega_1+\omega_2,\omega_2]$ we can write this integral

$$\begin{aligned}
\int_{[0,\omega_1]} g(z)dz + \int_{[\omega_1+\omega_2,\omega_2]} g(z)dz &= \int_{[0,\omega_1]} g(z)dz - \int_{[\omega_2,\omega_1+\omega_2]} g(z)dz \\
&= \int_{[0,\omega_1]} \frac{zf'(z)}{f(z)}dz - \int_{[0,\omega_1]} \frac{(z+\omega_2)f'(z+\omega_2)}{f(z+\omega_2)}dz \\
&= \omega_2 \int_{[0,\omega_1]} \frac{f'(z)}{f(z)}dz
\end{aligned}$$

where we know that $f(z) = f(z+\omega_2), f'(z) = f'(z+\omega_2)$ with $f'(z)$ elliptic by Proposition 2.11. In particular, this an integer multiple of ω_2. Arguing similarly, we can see that

$$\int_{[\omega_1,\omega_1+\omega_2]} g(z)dz + \int_{[\omega_2,0]} g(z)dz = \omega_1 \int_{[0,\omega_2]} \frac{f'(z)}{f(z)}dz$$

which is also an integer multiple of ω_1, giving the claim. ■

We return to a consideration of the field of elliptic functions more generally.

Proposition 3.1

Proposition 3.2. Let Ω be a lattice. The function

$$f_k(z) = \sum_{\omega\in\Omega} \frac{1}{(z-\omega)^k}$$

is a nonconstant elliptic function for $k \geq 3$.

Proof. The function is elliptic by inspection so it remains to show that the function is locally uniformly convergent. For this, fix $r > 0$ and consider the disc $B_{2r} = \{z \in$

$\mathbb{C} : |z| < 2r\}$ and note that $|\Omega \cap B_{2r}| < \infty$ by discreteness of Ω. For any z with $|z| < r$ and ω with $|\omega| > 2r$ we have $\frac{|\omega|}{2} \leq |\omega| - |z| \leq |\omega - z|$ so we have

$$\sum_{|\omega| \geq 2r} \frac{1}{|z-\omega|^k} \leq 2^k \sum_{\omega \in \Omega \setminus \{0\}} \frac{1}{|\omega|^k}$$

where the latter is convergent by Proposition 2.10, giving the claim. ■

This allows us to produce elliptic functions of orders at least 3. A natural question arises if there are elliptic functions of lower orders. By Proposition 2.13 it is clear that elliptic functions of order 1 are not possible. Though, as it turns out, elliptic functions of order 2 will play an important role in the theory.

We can rearrange the equation $f_3(z)$ as

$$f_3(z) - \frac{1}{z^3} = \sum_{\omega \in \Omega \setminus \{0\}} \frac{1}{(z-\omega)^3} \tag{3.2}$$

and we note that this function has poles at all $\Omega \setminus \{0\}$ with residue zero. As such, we can form the integral $\int_0^z \sum_{\omega \in \Omega \setminus \{0\}} \frac{1}{(w-\omega)^3} dw$ which by convergence of the sum is given by

$$\sum_{\omega \in \Omega \setminus \{0\}} \int_0^z \frac{1}{(w-\omega)^3} dw = -\frac{1}{2} \sum_{\omega \in \Omega \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

and similarly $\int_0^z \frac{1}{w^3} dw = -\frac{1}{2z^2}$ so we have by (3.2) that

$$\int f_3(z) dz = -\frac{1}{2z^2} - \frac{1}{2} \sum_{\omega \in \Omega \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

This yields the Weierstrass $\wp$-function.

Definition 3.3 (Weierstrass $\wp$-Function)**.** Let Ω be a lattice. The Weierstrass $\wp$-function of Ω is given by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

Moreover, this function has the expected properties.

Proposition 3.4. Let Ω be a lattice. Then:

(i) $\wp(z)$ is elliptic of order 2 and
(ii) $\wp'(z)$ is elliptic of order 3.

Proof of (i). $\wp(z)$ is of order 2 by construction, with double poles at the lattice points. We now show it is elliptic. Noting that $\wp'(z+\omega) - \wp(z) = 0$ we have that $\wp(z+\omega) - \wp(z) = C_\omega$ where C_ω is a constant depending on $\omega \in \Omega$. For the basis ω_1, ω_2 of Ω we can consider for $z = -\frac{\omega_i}{2}$ that $\wp(\frac{\omega_i}{2}) - \wp(-\frac{\omega_i}{2}) = C_{\omega_i}$ but $\wp$ is even so $C_{\omega_i} = 0$ for $i \in \{1, 2\}$ showing that it is elliptic. ■

Proof of (ii). This follows from the discussion above, for the Weierstrass $\wp$-function arises as an integral of $f_3(z)$ which is an elliptic function of order 3. ■

The Weierstrass $\wp$-function is extremely important to the study of elliptic functions, since every elliptic function can be written as a rational function in $\wp, \wp'$. We first prove the following preparatory lemma.

Lemma 3.5. Let Ω be a lattice and $f(z)$ elliptic with respect to Ω with poles, if any, in Ω. Then $f(z) = \sum_{i=0}^{n} a_i \wp(z)^i$ for $a_i \in \mathbb{C}$.

Proof. If $f(z)$ is a constant, we are done. Otherwise, we can take the Laurent series expansion of $f(z)$ around the origin which is of the form $\frac{a_{-2n}}{z^{2n}} + \dots$ since f is even. Now note that $f(z) - \frac{a_{-2n}}{\wp(z)^n}$ is even and elliptic with a pole of order at most $2n-2$. Thus, repeating this process finitely many times, we eventually arrive at a constant function where we can arrive at the desired claim by multiplying the expressions by $\wp(z)^{2n}$ to clear denominators. ■

Using the above lemma, we can in fact show that it suffices to use a rational function in $\wp(z)$ multiplied by $\wp'(z)$.

Proposition 3.6. Let Ω be a lattice and $f(z)$ elliptic with respect to Ω. Then $f(z)$ can be written as the product of $\wp'(z)$ and a rational function in $\wp(z)$.

Proposition 3.2

Proof. Note that if $f(z)$ is an elliptic function of odd order, then $f(z)/\wp(z)$ is an elliptic function of even order. So to prove the claim, it suffices to treat the case of $f(z)$ an elliptic function of even order.

Let $f(z)$ be even with poles $a_1, \dots, a_n$ in $P_\Omega \setminus \{0\}$. Noting that $\wp(z) - \wp(a_i)$ vanishes for all a_i, the function $(\wp(z) - \wp(a_i))^{m_i} f(z)$ will have pole nowhere in $P_\Omega \setminus \{0\}$ for n_i sufficiently large. Then observing that $f(z) \prod_{i=1}^{n} (\wp(z) - \wp(a_i))^{m_i}$ is an even elliptic function with poles only at lattice points. Thus by Lemma 3.5, it can be written as a polynomial in $\wp(z)$ allowing us to rewrite $f(z)$ as the quotient of this polynomial by the product $\prod_{i=1}^{n} (\wp(z) - \wp(a_i))^{m_i}$, that is, as a rational function in $\wp(z)$. ■

Remark 3.7. The expression of an ellitpic function in terms of $\wp(z), \wp'(z)$ is not unique.

Consider the elliptic function $\wp'(z)^2$. This is an even elliptic function of order 6 with poles on Ω. By Lemma 3.5, we can write this as a degree 3 polynomial in $\wp(z)$, say $a_0 + a_1\wp(z) + a_2\wp(z)^2 + a_3\wp(z)^3$ with $a_i \in \mathbb{C}$. The coefficients a_i, however, are in fact highly structured and can be deduced from the lattice.

4. Lecture 4 – 22nd October 2024

Following our previous dicussion, we seek to express $\wp'(z)^2$ the elliptic function of order 6 in terms of a degree 3 polynomial in $\wp(z)$ with coefficients determined by the lattice. In particular, we have the following theorem.

Theorem 4.1. Let Ω be a lattice. Then the Weierstrass $\wp$-function of Ω satisfies the nonlinear differential equation Theorem 3.4

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\Omega)\wp(z) - g_3(\Omega) \tag{4.1}$$

where $g_2(\Omega) = 60\sum_{\omega\in\Omega\setminus\{0\}}\frac{1}{\omega^4}$ and $g_3(\Omega) = 140\sum_{\omega\in\Omega\setminus\{0\}}\frac{1}{\omega^6}$.

Proof. We take Laurent expansions of $\wp(z), \wp'(z)^2, \wp(z)^3$ to observe

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + c_2z^2 + c_4z^4 + \dots\\ \wp'(z)^2 &= \frac{4}{z^6} - 8c_2z + 4c_4z^3 + \dots\\ \wp(z)^3 &= \frac{1}{z^6} + 3c_2\frac{1}{z^2} + 3c_4 + \dots\end{aligned}$$

where we compute

$$\wp'(z) - 4\wp(z)^3 = -20c_2\frac{1}{z^2} - 28c_4 + \dots$$

and iterating the process once more to use $\wp(z)$ write the pole of order 2 that

$$\wp'(z)^2 - 4\wp(z)^3 + 20c_2\wp(z) = 28c_4 + \dots$$

where we now note that the function on the right is a holomorphic elliptic function and hence $28c_4$ itself. Rearranging, we yield $\wp'(z)^2 = 4\wp(z)^3 - 20c_2\wp(z) - 28c_4$. To find c_2, c_4 in terms of the lattice, we write $h(z) = \wp(z) - \frac{1}{z^2} = \sum_{\omega\in\Omega\setminus\{0\}}\left(\frac{1}{(z-\omega^2)} - \frac{1}{\omega^2}\right)$ so taking the derivative we get $h^{(m)}(z) = (-1)^m(m-1)!\sum_{\omega\in\Omega\setminus\{0\}}\frac{1}{(z-\omega)^{m+2}}$. Now noting that $(2m)!c_{2m} = h^{(2m)}(0)$ we conclude that $c_{2m} = (2m+1)\sum_{\omega\in\Omega\setminus\{0\}}\frac{1}{\omega^{2m+2}}$ giving the claim. ■

We deduce the following as an immediate corollary.

Corollary 4.2. Let Ω be a lattice with Weierstrass function $\wp(z)$. Then
 (i) $2\wp'(z)\wp''(z) = 12\wp(z)^2\wp'(z) - g_2(\Omega)$.
 (ii) $\wp''(z) = 12\wp(z)\wp'(z)$.

Proof. This follows from a direct computation of the derivative of the function (4.1) in Theorem 4.1. ■

The field of elliptic functions with respect to a lattice Ω can be explicitly described as follows.

Theorem 4.3. Let Ω be a lattice. Then the field of elliptic functions $K(\Omega)$ with respect to Ω is isomorphic to

$$\mathbb{C}(X)[Y]/(Y^2 - 4X^3 + g_2(\Omega)X + g_3(\Omega)).$$

Proof. By Proposition 3.6, the map $\mathbb{C}(X)[Y]$ by $X \mapsto \wp(z), Y \mapsto \wp'(z)$ is surjective with kernel given by the relation imposed by the differential equation of Theorem 4.1. ∎

In particular, this algebraically realizes the field of elliptic functions as a degree 2 extension of the field of rational functions on $\mathbb{C}$.

By Proposition 2.13, we know that elliptic functions satisfy strong constraints on their zeroes and poles. However, given a set of zeroes and poles alongside their multiplicities, it is unclear if the inverse problem can be solved – constructing an elliptic function with those zeroes and poles. This, however, turns out to be possible, and will necessitate the development of the theory of elliptic modular functions.

We begin with the following lemma, which produces an example of such a function.

Lemma 4.4. Let Ω be a lattice. The function

$$f(z) = \frac{1}{z} + \sum_{\omega \in \Omega \setminus \{0\}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right)$$

is convergent with derivative $-\wp(z)$.

Proof. This function arises as the integral $\int_0^z \wp(w)dw$ and can be seen to be convergent using the termwise estimate

$$\left| \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right| \leq \frac{|z|^2}{|\omega^3| \left(1 - \frac{|z|}{|\omega|}\right)} \leq \frac{2|z|^2}{|\omega|^3}$$

with the rightmost term convergent by Proposition 2.10. ∎

This is the Weierstrass ζ function.

Definition 4.1 **Definition 4.5** (Weierstrass ζ-Function)**.** Let Ω be a lattice. The Weierstrass ζ-function of Ω is given by

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \Omega \setminus \{0\}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).$$

Continuing along this line, we consider the the primitive of $\log \zeta(z)$.

Lemma 4.6. Let Ω be a lattice. The function

$$f(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right)$$

has derivative $\log \zeta(z)$.

Proof. The logarithm of the product is given by $\sum_{\omega \in \Omega \setminus \{0\}} \left(\log\left(1 - \frac{z}{\omega}\right) + \frac{z}{\omega} + \frac{z^2}{\omega^2} \right)$. Note that this is equal to $\int_0^z (\zeta(w) - \frac{1}{w})dw = \log\left(\frac{f(z)}{z}\right)$ and thus $\zeta(z) - \frac{1}{z} = \frac{\sigma'(z)}{\sigma(z)} - \frac{1}{z}$ as desired. ∎

This is the Weierstrass σ-function.

Definition 4.1

Definition 4.7 (Weierstrass σ-Function)**.** Let Ω be a lattice. The Weierstrass σ-function of Ω is given by

$$\sigma(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right).$$

The following property of the ζ and σ functions will be used to construct an elliptic function given a compatible set of zeroes and poles with multiplicities.

Proposition 4.8. Let Ω be a lattice and $\zeta(z), \sigma(z)$ the Weierstrass ζ and σ functions with respect to Ω. Then:

(i) $\zeta(z + \omega_i) = \zeta(z) + \eta_i$ for ω_i a generating element of the lattice and η_i some constant depending on ω_i.

(ii) $\sigma(z + \omega_i) = \sigma(z) \exp(\eta_i z + c_i)$ for ω_i a generating element of the lattice, η_i the constant depending on ω_i identified above, and c_i some other constant depending on ω_i.

Proof of (i). We know the derivative of $\zeta(z+\omega_i) - \zeta(z)$ is given by $-\wp(z+\omega_i) + \wp(z) = 0$ so integrating, we have $\zeta(z + \omega_i) = \zeta(z) + \eta_i$ for some constant ω_i. ■

Proof of (ii). From $\frac{\sigma'(z)}{\sigma(z)} = \zeta(z)$ we have

$$\frac{\sigma'(z + \omega_i)}{\sigma(z + \omega)} = \zeta(z + \omega_i) = \zeta(z) + \eta_i = \frac{\sigma'(z)}{\sigma(z)} + \eta_i$$

so integrating, we get $\sigma(z + \omega_i) = \sigma(z) \exp(\eta_i z + c_i)$, as desired. ■

We can now prove the desired result.

Theorem 4.9. Let Ω be a lattice, $a_1, \dots, a_n, a_{n+1}, \dots, a_\ell$ points in the interior of the fundamental period parallelogram P_Ω labeled by natural numbers $m_1, \dots, m_n, m_{n+1}, \dots, m_\ell$ such that

$$\left(\sum_{i=1}^{n} m_i a_i\right) - \left(\sum_{i=n+1}^{\ell} m_i a_i\right) = 0.$$

Then there exists an elliptic function $f(z)$ with respect to Ω with zeroes at $a_1, \dots, a_n$ and poles at $a_{n+1}, \dots, a_\ell$ of multiplicities $m_1, \dots, m_n$ and $m_{n+1}, \dots, m_\ell$, respectively.

Proof. The function

$$f(z) = \frac{\prod_{i=1}^{n} \sigma(z - a_i)^{m_i}}{\prod_{i=n+1}^{\ell} \sigma(z - a_i)^{m_i}}$$

suffifces. By construction of $\sigma(z)$, this function is meromorphic with the desired zeroes and poles. To see that it is periodic, we note

$$f(z + \omega_j) = f(z) \exp(A_j)$$

where

$$A_i = \sum_{i=1}^{n} m_i(\eta_j(z - a_i) + c_j) - \sum_{i=n+1}^{\ell} m_i(\eta_j(z - a_i) + c_j)$$

which is zero modulo Ω by Proposition 4.8 (ii). ■

We conclude with the following remark.

Remark 4.10. A lattice can be normalized to have $\Omega = \langle 1, \tau \rangle$ with τ having positive imaginary part. Can define another function $\Theta(z, \tau) = \sum_{n \in \mathbb{Z}} \exp(\pi i \tau n^2 + 2\pi i n z + \pi i n)$ which is known as the Θ-function of the lattice Ω. Further exposition of this function can be found in a standard graduate text on complex analysis, but we will not discuss these functions in the course.

5. Lecture 5 – 24th October 2024

The differential equation of Theorem 4.1 suggests a close link between the theory of elliptic functions and cubic curves. In fact, we will show that the Weierstrass $\wp$-function and its derivative determine a map from $\mathbb{C}/\Omega \to \mathbb{C}^2$ with image a regular affine cubic Definition 1.8. We now set up some requisite results.

Proposition 5.1. Let $z_1, z_2 \in \mathbb{C}/\Omega$ and $\wp(z_1) \neq \wp(z_2)$. Then Theorem 5.1

$$\wp(z_1 + z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2.$$

Proof. Consider the function $f(z) = \wp'(z) - a\wp(z) - b$ for $a, b \in \mathbb{C}$ such that $f(z_1) = 0 = f(z_2)$. In this case, we have $\wp'(z_1) - a\wp(z_1) - b = \wp'(z_2) - a\wp(z_2) - b$ which yields $\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}$ on solving for a.

Note also that $f(z)$ is an elliptic function of order 3 since it is a $\mathbb{C}$-linear combination of elliptic functions with $\wp'(z)$ of order 3 so by Proposition 2.13 there is some $z_3 \in P_\Omega$ which is a zero of f satisfying $z_1 + z_2 + z_3 = 0$ up to the quotient of Ω. As such, $f(-z_1 - z_2) = 0$ which by substitution yields $\wp'(-z_1 - z_2) = a\wp(-z_1 - z_2) + b$ and thus $-\wp'(z_1+z_2) = a\wp(z_1+z_2)+b$ by $\wp'(z)$ an odd function as in Proposition 3.4.

Now observe that the points $(\wp(z_1), \wp'(z_1)), (\wp(z_2), \wp'(z_2)), (\wp(z_3), -\wp'(z_3))$ lie on the complex line $v = au + b$. But by the differential equation in Theorem 4.1, $\wp(z_1), \wp(z_2), \wp(z_3)$ are the zeroes of the cubic polynomial and hence satisfy the symmetric polynomial identity $a^2 = 4(\wp(z_1) + \wp(z_2) + \wp(z_3))$. So

$$\wp(z_3) = \wp(z_1 + z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)$$

as required. ∎

We deduce the following as a corollary.

Corollary 5.2. If $z \in \mathbb{C}/\Omega$ is such that $2z \notin \Omega$ then Theorem 5.1

$$\wp(2z) = -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 = -2\wp(z) + \frac{1}{4}\left(\frac{12\wp^2(z) - g_2}{2\wp'(z)}\right)^2.$$

Proof. The first follows from passing to the limit as $z_2 \to z_1$ and the second from the first by writing $\wp''(z)$ in terms of the derivative of (4.1). ∎

The construction of the embedding proceeds as follows.

Proposition 5.3. Let Ω be a lattice. Then $\mathbb{C}/\Omega \to \mathbb{C}^2$ by $z \mapsto (\wp(z), \wp'(z))$ is a holomorphic parametrization of the regular affine plane cubic

$$E = \{(u, v) \in \mathbb{C}^2 : v^2 = 4u^3 - g_2(\Omega)u - g_3(\Omega)\}.$$

Proof. The function is holomorphic on the quotient since the Weierstrass function and its derivative have poles at the lattice points. The map is surjective since for any $(u, v) \in E$ there is $z \in \mathbb{C}/\Omega$ with $\wp(z) = u$ by Proposition 2.15 and $\wp'(z) = \pm v$ satiesfies the equation for E by (4.1). The map is also injective since if $(\wp(z_1), \wp'(z_1)) = (\wp(z_2), \wp'(z_2))$ we have $z_2 - z_1 \in \Omega$ since $\wp(z_1) = \wp'(z_2)$ excludes

$z_1 + z_2 \in \Omega$ from Proposition 5.1 so $z_2 - z_1 \in \Omega$, that is, they are identified in the quotient, showing the map is injective. ■

This map to an affine regular plane cubic can be extended to a regular projective cubic.

Proposition 5.4. Let Ω be a lattice and

$$E = \{(u, v) \in \mathbb{C}^2 : v^2 = 4u^3 - g_2(\Omega)u - g_3(\Omega)\} \subseteq \mathbb{C}^2$$

the image of $z \mapsto (\wp(z), \wp'(z))$. Then the closure $\overline{E} \subseteq \mathbb{CP}^2$ has the structure of an Abelian group with identity element the image of a lattice point.

Proof. $\overline{E}$ is given by the projective closure of E which takes elements of Ω to the point $[0 : 0 : 1]$, extending the group structure on the quotient $\mathbb{C}/\Omega$. ■

Remark 5.5. Explicitly, the sum of points p, p' on $\overline{E}$ for an embedding $\overline{\varphi} : \mathbb{C} \to \mathbb{CP}^2$ by $z \mapsto [1 : \wp(z) : \wp'(z)]$ is given by $\varphi(\varphi^{-1}(p) + \varphi^{-1}(p'))$.

As an aside, we show that cubics do not admit a rational parametrization.

Proposition 5.6. A regular projective cubic $\overline{E}$ does not admit a rational parametrization.

Proof. A rational parametrization would necessarily factor as a holomorphic map $\widehat{\mathbb{C}} \to \mathbb{C} \to \overline{E}$, but any map $\widehat{\mathbb{C}} \to \mathbb{C}$ cannot be holomorphic. ■

6. Lecture 6 – 29th October 2024

Recall the construction of the regular affine cubic using the Weierstrass $\wp$-functions Proposition 5.3. A natural question of the solution to the inverse problem arises. In particular: given an affine regular cubic $E \subseteq \mathbb{C}^2$, is there a lattice Ω such that the Weierstrass functions of Ω parametrize E?

This can be shown using elliptic modular functions as hinted at in Section 4 preceding Lemma 4.4 as further developed in [Ahl79]. We address the easier question of conditions of lattices that result in $K(\Omega), K(\Omega')$ being isomorphic. Theorem 4.3 connects this question to the former. We have the following preparatory lemmata.

Lemma 6.1. Let Ω be a lattice. If $f(z) \in K(\Omega)$ an elliptic function then $f(z)$ satisifes an algebraic differential equation.

Proof. For a lattice Ω, $K(\Omega)$ is a function field of dimension 1, that is, of transcendence degree over $\mathbb{C}$ is 1. This implies that for any $f, g \in K(\Omega)$ there is an algebraic equation $F(x, y)$ such that $F(f, g) = 0$ so applying this to an elliptic function $f(z)$ and its elliptic derivative $f'(z)$, we have the claim. ■

This extends to an algebraic relation between the elliptic functions $f(z), f(z+\omega)$ and the constant $f(\omega)$ to yield for $\omega \in \Omega$ an algebraic relation betewen $f(z)$ and $f(z+\omega)$.

Let us now consider an alternative way to define elliptic functions.

Proposition 6.2. Let $q \in \mathbb{C}^\times$ with $0 < |q| < 1$. The set of meromorphic functions $f(z)$ on $\mathbb{C}^\times$ satisfying $f(z) = f(qz)$ for all $z \in \mathbb{C}^\times$ forms a field K_q.

Proof. By inspection, the set of such functions is closed under addition and multiplication, and multiplicative inversion. ■

Let $A_{q,r}$ be the annulus $\{z \in \mathbb{C}^\times : r < |z| < \frac{r}{|q|}\}$.

Lemma 6.3. Let $q \in \mathbb{C}^\times$ with $0 < |q| < 1$ and $f(z)$ meromorphic on $\mathbb{C}^\times$ such that $f(z) = f(qz)$ for all $z \in \mathbb{C}^\times$. Then f assumes all values in $\widehat{\mathbb{C}}$ already in the annulus $A_{q,r}$. Furthermore, if f is holomorphic on $A_{q,r}$ then f is constant.

Proof. If it does not assume all values of $\widehat{\mathbb{C}}$ then it is bounded on the closure of the annulus and hence constant by Liouville's theorem Theorem A.2. ■

We can thus form the torus $T = \mathbb{C}^\times/\langle q\rangle$ where $\langle q\rangle$ is the subgroup of $\mathbb{C}^\times$ generated by multiples of q. This is a torus with fundamental domain represented by the annulus, identifying the inner boundary circle with the outer.

Lemma 6.4. Let f be holomorphic on $\mathbb{C}^\times$ such that there is a constant $c \neq 0$ satisfying $f(qz) = c \cdot f(z)$. Then $c = q^k$ for some $k \in \mathbb{Z}$ and $f(z) = f(1) \cdot z^k$.

Proof. Consider the Laurent series expansion of $f(z)$ given by $\sum_{n=-N}^{\infty} a_n z^n$ for $N \in \mathbb{N}$. We can compare coefficients

$$\sum_{n=-N}^{\infty} c a_n z^n = \sum_{n=-N}^{\infty} a_n q^n z^n$$

which are necessarily equal for all n, and which holds if and only if $a_n = 0$ for all but one $n = k$ at which we have $ca_k z^k = a_k q^k z^k$ with the coefficent a_k recovered as $f(1)$, giving the claim. ■

The subsequent lemma allows us to produce a class of meromorphic functions on $\mathbb{C}^\times$ that are invariant under q-multiplication.

Lemma 6.5. Let $q \in \mathbb{C}^\times$ with $0 < |q| < 1$. Then the function

$$p(z) = \left(\prod_{n=1}^{\infty}(1 - q^n z)\right)\left(\prod_{n=0}^{\infty}\left(1 - \frac{q^n}{z}\right)\right)$$

is holomorphic on $\mathbb{C}^\times$, and satisfies $p(qz) = -\frac{1}{qz}p(z)$.

Proof. By inspection, the only pole of the function is at 0, and the second statement follows from a direct computation. ■

We can now show the main result.

Theorem 6.6. Let $a_1, \ldots, a_n$ and $a_{n+1}, \ldots, a_\ell$ be lists of distinct elements in the annulus

$$A_{q,r} = \{z \in \mathbb{C}^\times : r < |z| < \frac{r}{|q|}\}$$

and $m_1, \ldots, m_n, m_{n+1}, \ldots, m_\ell$ a list of positive integers. The following are equivalent:

Theorem 6.2

(a) There exists a function $f \in K_q$ with zeroes at $a_1, \ldots, a_n$ with multiplicities $m_1, \ldots, m_n$ and poles at $a_{n+1}, \ldots, a_\ell$ with multiplicities $m_{n+1}, \ldots, m_\ell$, respectively.
(b) $\ell = 2n$ and $a_1 \ldots a_n q^k = a_{n+1} \ldots a_\ell$ for some integer k.

Proof. We make some preliminary observations. Note p has simple zeroes at each positive q-power by inspection. And by passage to the inverse of a function, can assume that counted with multiplicity there are more zeroes than poles. Setting $h(z) = \frac{p(z/a_1)\ldots p(a/a_m)}{p(z/a_{n+1})\ldots p(z/a_\ell)}$, we have that $h(z)$ is meromorphic on $\mathbb{C}^\times$ as it is obtained as a quotient of holomorphic functions, and we can set $\lambda = \frac{a_1 \ldots a_n}{a_{n+1} \ldots a_\ell}$.

(a)⇒(b) Let $f \in K_q$ be given with prescribed zeroes and poles and set $F(z) = f(z)/h(z)$. The zeroes of h are some q-power multiplied by a so they cancel in the annulus gainst zeroes of f, that is, F is holomorphic in the annulus. Now $F(qz) = \frac{1}{\lambda}F(z)$ implying F is holomorphic on $\mathbb{C}^\times$. Applying Lemma 6.4, we get $\frac{1}{\lambda}q^k$ which gives the claim.

(b)⇒(a) The function $f(z) = c \cdot z^k \cdot \left(\frac{p(z/a_1)\ldots p(a/a_m)}{p(z/a_{n+1})\ldots p(z/a_\ell)}\right)$ has the desired properties. ■

Returning to the discussion of elliptic functions, the construction above gives a map $K_q \to K(\Omega)$ by $f(z) \mapsto f(\exp(2\pi i z))$ where the image of $p(z)$ is the Weierstrass σ function up to a correction factor. We additionally remark that elliptic functions are used to parametrize parabolas and ellipses, and are at times used in physical mathematics.

7. Lecture 7 – 31st October 2024

We turn to the study of analytic number theory, and in particular the prime number theorem. Recall Riemann's zeta function.

Definition 7.1 (Riemann Zeta Function)**.** The Riemann zeta function is defined as the infinite sum

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Remark 7.2. $\zeta(s)$ converges for $\mathrm{Re}(s) > 1$: $n^{-s} = e^{-s\log(n)}$ here taking the real logarithm, so $|n^{-s}| = n^{-\mathrm{Re}(s)}$ and the series converges – in fact, uniformly – on $\mathrm{Re}(s) > 1$.

We now define the Mellin transform which will be a crucial construction, which will be justified by the following lemma.

Lemma 7.3. Let $f : [1, \infty) \to \mathbb{R}$ be a locally integrable function such that $|f(x)| \leq C \cdot x^k$ for some constant C and $k \in \mathbb{N}$. Then the integral $\int_1^\infty f(x)x^{-s-1}dx$ exists for $\mathrm{Re}(s) > k$.

Proof. For s such that $\mathrm{Re}(s) > k$ we have that $|f(x)|x^{-s-1} \leq C \cdot x^{k-s-1}$ where $\mathrm{Re}(k - s - 1) < 0$ so the function is bounded above by C, giving the claim. ■

Remark 7.4. Recall that a locally integrable function is a function where the integral over any finite interval exists.

Definition 7.5 (Mellin Transform)**.** Let $f : [1, \infty) \to \mathbb{R}$ be a locally integrable function such that $|f(x)| \leq C \cdot x^k$ for some constant C and $k \in \mathbb{N}$. The Mellin transform of f is given by

$$\mathcal{M}_f(s) = s \int_1^\infty f(x)x^{-s-1}dx.$$

The Mellin transform for the function x and the Gauss bracket will play a key role in estimating the distribution of primes.

Lemma 7.6. The Mellin transform of $f(x) = x$ is given by $1 + \frac{1}{s-1}$.

Proof. We compute

$$\begin{aligned} s \int_1^\infty x \cdot x^{-s-1}dx &= s \int_1^\infty x^{-s}dx \\ &= \frac{s}{s-1}. \end{aligned}$$

■

We now define the Gauss bracket and compute its Mellin transform.

Definition 7.7 (Gauss Bracket)**.** The Gauss bracket $[x]$ is given by the largest integer at most x.

Lemma 7.8. The Mellin transform of the Gauss bracket $f(x) = [x]$ is given by the Riemann zeta function $\zeta(s)$.

Proof. We compute

$$\begin{aligned} s\int_1^\infty [x]x^{-s-1}dx &= \sum_{n=1}^\infty s\int_n^{n+1} [x]x^{-s-1}dx \\ &= \sum_{n=1}^\infty ns\int_n^{n+1} x^{-s-1}dx \\ &= \sum_{n=1}^\infty ns\left(\frac{x^{-s}}{-s}\,\Big|_{x=n}^{n+1}\right) \\ &= \sum_{n=1}^\infty n(n^{-s} - (n+1)^{-s}) \end{aligned}$$

note the series covnerges for $\mathrm{Re}(s) > 2$ where we decompose the sum

$$\begin{aligned} \sum_{n=1}^\infty n(n^{-s} - (n+1)^{-s}) &= \sum_{n=1}^\infty n^{1-s} - \sum_{n=1}^\infty (n+1-1)(n+1)^{-s} \\ &= \sum_{n=1}^\infty n^{1-s} - \sum_{n=2}^\infty n^{1-s} + \sum_{n=2}^\infty n^{-s} \\ &= 1 + (\zeta(s) - 1) \\ &= \zeta(s) \end{aligned}$$

as desired. ■

From this, we can deduce the Mellin transform $\mathcal{M}_{[x]-x}(s)$.

Proposition 7.9. The Mellin transform of the difference $[x] - x$ is given by

$$\mathcal{M}_{[x]-x}(s) = \zeta(s) - \frac{1}{s-1} - 1$$

on $\mathrm{Re}(s) > 1$ and is meromorphic in that halfspace with pole at $s = 1$ with residue
Theorem 1 1.

Proof. For $\mathrm{Re}(s) > 2$ the statement follows from Lemmas 7.6 and 7.8 and noting that $\frac{s}{s-1} = \frac{1}{s-1} + 1$, and extends to $\mathrm{Re}(s) > 1$ by the identity theorem Theorem A.1 as the functions agree in the right halfspace $\mathrm{Re}(s) > 2$ but are holomorphic for $\mathrm{Re}(s) > 1$ away from the pole at 1. The function $\frac{1}{s-1}$ is the sole contributor to the residue at 1 and has residue 1 by inspection. ■

The Riemann zeta function admits an alternative description via infinite products.

Theorem 7.10. The Riemann zeta function $\zeta(s)$ is equal to the infinite product
Theorem 2 $\prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$ for $\mathrm{Re}(s) > 1$.

Proof. The convergence of the sum $\sum_{n=1}^{\infty} \frac{1}{n^s}$ implies the convergence of the product since it contains $\sum_{p \text{ prime}} \frac{1}{p^s}$ is a subsequence. Now let $\mathcal{P}_k \subseteq \mathcal{P}$ be the finite set of the first k primes and consider the restricted product $\prod_{p \in \mathcal{P}_k} \frac{1}{1-p^{-s}}$ which is equal to $\prod_{\mathcal{P}_k} \sum_{m=0}^{\infty} \frac{1}{(p^m)^s}$. Moreover since the product and sum are absoltuely convergent, we can rewrite this as $\sum_{n \in S_k} \frac{1}{n^s}$ where S is the set of natural numbers with all factors in $\mathcal{P}_k$. Passing to the limit as $k \to \infty$, we get the claim. ■

Remark 7.11. This already implies that there are infinitely many primes, since if not we would have a finite product equalling a divergent harmonic series, a contradiction.

We also show the following structure result for zeroes of the Riemann zeta function of real part 1.

Theorem 7.12. The Riemann zeta function $\zeta(s)$ has no zeroes with $\mathrm{Re}(s) = 1$. Theorem 3

Proof. Suppose to the contrary there is a zero s with $\mathrm{Re}(s) = 1$, say of the form $1 + ti$. Consider the function

$$F(s) = \zeta(s)^3 \zeta(s+it)^4 \zeta(s+2it).$$

Observe that if $\zeta(1+it) = 0$ then at the pole of $\zeta(s)$ at $s = 1$ is canceled by the zero of order four in the second factor. The function $F(s)$ thus vanishes at $s = 1$ and is holomorphic in a neighborhood around that point. Passing to logarithms, we have that $\lim_{s\to 1} \log |F(s)| = -\infty$.

Now consider the product expression of Theorem 7.10 where we compute

$$\begin{aligned} \log |\zeta(s)| &= \mathrm{Re}\left(\sum_{p \text{ prime}} \log(1-p^{-s})^{-1} \right) \\ &= \mathrm{Re}\left(\sum_{p \text{ prime}} \sum_{n=1}^{\infty} \frac{1}{n \cdot p^{ns}} \right) \\ &= \mathrm{Re}\left(\sum_{n=1}^{\infty} \frac{a_n}{n^s} \right) \end{aligned}$$

where the coefficient a_n is given by

$$a_n = \begin{cases} \frac{1}{r} & n = p^r \\ 0 & n \neq p^r \end{cases}$$

where on substituting to our sum we have for $s > 1$ real

$$\begin{aligned}
\log|F(s)| &= 3\log|\zeta(s)| + 4\log|\zeta(s+ti)| + \log|\zeta(s+2ti)| \\
&= \sum_{n=1}^{\infty} a_n \mathrm{Re}\left(3n^{-s} + 4n^{-s-ti} + n^{-s-2ti}\right) \\
&= \sum_{n=1}^{\infty} \frac{a_n}{n^s}\left(3 + 4cos(t\cdot\log(n)) + cos(2t\cdot log(n))\right)
\end{aligned}$$

where we note that $\frac{a_n}{n^s} \geq 0$ and

$$\begin{aligned}
3 + 4\cos(t\cdot\log(n)) + \cos(2t\cdot\log(n)) &= 3 + 4\cos(t\cdot\log(n)) + 2\cos^2(t\cdot\log(n)) - 1 \\
&= 2(1+\cos(t\cdot\log(n)))^2
\end{aligned}$$

which is also positive. In particular $\lim_{s\to 1}\log|F(s)| \neq -\infty$, a contradiction. ■

We now introduce some language relating to the prime number theorem.

Definition 7.13 (Prime Counting Function)**.** The prime counting function $\pi(x)$ is the number of primes at most x.

Definition 7.14 (Theta Function)**.** The theta function is given by

$$\vartheta(x) = \sum_{p \text{ prime}, p\leq x} \log(p).$$

The prime number theorem concerns the asymptotics of $\pi(x)$, though it is often easier to consider $\vartheta(x)$. These are linked by the following result.

Proposition 7.15. The existence of the following limits are equivalent

(a) $\lim_{x\to\infty}\frac{\pi(x)\cdot\log(x)}{x}$
(b) $\lim_{x\to\infty}\frac{\vartheta(x)}{x}$

and they are equal if they exist.

Proof. $\left(\frac{\vartheta(x)}{x} \leq \frac{\pi(x)\log(x)}{x}\right)$ Observe

$$\begin{aligned}
\vartheta(x) &= \sum_{p \text{ prime},\ p\leq x} \log(p) \\
&\leq \sum_{p \text{ prime},\ p\leq x} \frac{\log(x)}{\log(p)}\cdot\log(p) \\
&= \log(x)\sum_{p \text{ prime},\ p\leq x} 1 = \pi(x)\log(x).
\end{aligned}$$

$\left(\frac{\pi(x)\log(x)}{x} \leq \frac{\vartheta(x)}{x}\right)$ Conversely, we can estimate for $1 < y < x$

$$\begin{aligned}\pi(x) = \pi(y) + \sum_{p \text{ prime, } y<p\leq x} 1 \\ \leq \pi(y) + \frac{1}{\log(y)} \sum_{p \text{ prime, } y<p\leq x} \log(p) \\ y + \frac{1}{\log(y)}\vartheta(x).\end{aligned}$$

We now have

$$\frac{\pi(x)\log(x)}{x} \leq \frac{y\log(x)}{x} + \frac{\log(x)}{\log(y)}\frac{\vartheta(x)}{x}.$$

Taking $y = \frac{x}{(\log(x))^2}$ we yield

$$\frac{\pi(x)\log(x)}{x} \leq \frac{1}{\log(x)} + \frac{\log(x)}{\log(x) - 2\log(\log(x))}\frac{\vartheta(x)}{x}$$

where we have

$$\begin{aligned}\lim_{x\to\infty}\frac{1}{\log(x)} = 0 \\ \lim_{x\to\infty}\frac{\log(x)}{\log(x) - 2\log(\log(x))} = 1\end{aligned}$$

giving the first claim.

The second claim follows immediately from the inequalities. ∎

8. Lecture 8 – 5th November 2024

We continue towards our proof of the prime number theorem by estimating the ratio $\frac{\vartheta(x)}{x}$.

Lemma 8.1. The function $\frac{\vartheta(x)}{x}$ is bounded on $[1, \infty)$.

Proof. Consider the binomial coefficient $\binom{2k}{k}$. Let k be a postive integer. If $k < p < 2k$, then $p | \binom{2k}{k}$ since writing the factorial as

$$\binom{2k}{k} = \frac{(2k) \dots (k+1)}{k!}$$

we have that p is in the numerator, but not canceled out by the denominator as p is prime. In particular, this holds for each p for $k < p < 2k$ so we have

$$\prod_{k<p<2k} p \leq \binom{2k}{k} \leq 2^{2k}$$

where the second inequality follows from $\sum_{r=0}^{2k} \binom{2k}{r} = 2^{2k}$. Passing to the logarithm, we have

$$\sum_{k<p\leq 2k} \log(p) \leq 2k \cdot \log(2).$$

Taking $k = 2^{\ell-1}$ for some ℓ, we obtain

$$\sum_{p\leq 2^{\ell}} \log(p) \leq 2^{\ell+1} \log(2)$$

by summing inequalities of the above form. Noting for any $x \in \mathbb{R}_{\geq 0}$ an ℓ such that $2^{\ell-1} < x \leq 2^{\ell}$ we have

$$\begin{aligned} \vartheta(x) &\leq \sum_{p\leq 2^{\ell}} \log(p) \\ &\leq 2^{\ell+1} \log(2) \\ &\leq 4x \log(2) \end{aligned}$$

giving $\frac{\vartheta(x)}{x} \leq 4 \log(2)$ hence the claim. ■

Recalling the construction of the Mellin transform from Definition 7.5, we show the following.

Lemma 8.2. The Mellin transform $\mathcal{M}_{\vartheta(x)}(s)$ of the Theta function $\vartheta(x)$ is given by

$$\mathcal{M}_{\vartheta(x)}(s) = \sum_{p \text{ prime}} \frac{\log(p)}{p^s} = -\frac{\zeta'(s)}{\zeta(s)} - \sum_{p \text{ prime}} \frac{\log(p)}{p^s(p^s-1)}$$

for $\mathrm{Re}(s) > 1$.

Proof. We compute

$$\begin{aligned}
\mathcal{M}_{\vartheta(x)}(s) &= s\int_1^\infty \vartheta(x)x^{-s-1}dx \\
&= \sum_{n=1}^\infty s\int_n^{n+1} \vartheta(x)x^{-s-1}dx && \vartheta \text{ constant on interval} \\
&= \sum_{n=1}^\infty s\vartheta(n)\int_n^{n+1} x^{-s-1}dx && \vartheta(x) = \vartheta(n) \text{ on interval } [n, n+1] \\
&= \sum_{n=1}^\infty s\vartheta(n)\left(\frac{x^{-s}}{-s}|_{s=n}^{n+1}\right) \\
&= \sum_{n=1}^\infty \vartheta(n)\left(n^{-s} - (n+1)^{-s}\right)
\end{aligned}$$

where we note this is convergent for $\mathrm{Re}(s) > 1$. Taking $\mathrm{Re}(s) > 2$ we can decompose the sum

$$\begin{aligned}
&= \sum_{n=1}^\infty \vartheta(n)n^{-s} - \sum_{n=1}^\infty \vartheta(n)(n+1)^{-s} \\
&= \sum_{n=1}^\infty \vartheta(n)n^{-s} - \sum_{n=2}^\infty \vartheta(n-1)n^{-s} && \text{changing index of summation in 2nd sum} \\
&= \sum_{n=1}^\infty (\vartheta(n) - \vartheta(n-1))n^{-s}
\end{aligned}$$

but noting

$$\vartheta(n) - \vartheta(n-1) = \begin{cases} \log(n) & n \text{ prime} \\ 0 & \text{otherwise} \end{cases}$$

the above expression simplifies to $\sum_{p \text{ prime}} \frac{\log(p)}{p^s}$. Since this also converges for $\mathrm{Re}(s) > 1$, the identity theorem Theorem A.1 allows us to extend this function to agree with $\mathcal{M}_{\vartheta(x)}(s)$ on this halfspace.

For the second equality, we consider the logarithmic derivative of $\zeta(s)$ as an Euler product in Theorem 7.10 to observe

$$\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= \sum_{p \text{ prime}} \frac{p^{-s}\log(p)}{1-p^{-s}} \\
&= \sum_{p \text{ prime}} \frac{\log(p)}{p^s - 1}
\end{aligned}$$

with

$$\sum_{p \text{ prime}} \frac{\log(p)}{p^s - 1} - \sum_{p \text{ prime}} \frac{\log(p)}{p^s(p^s-1)} = \sum_{p \text{ prime}} \frac{\log(p)}{p^s}$$

giving the claim. ∎

In summary, we have the following.

Proposition 8.3. The Mellin transform $\mathcal{M}_{\vartheta(x)-x}(s)$ of the function $\vartheta(x) - x$ is given by

$$\mathcal{M}_{\vartheta(x)-x}(s) = -\frac{\zeta'(s)}{\zeta(s)} - 1 - \frac{1}{s-1} - \sum_{p \text{ prime}} \frac{\log(p)}{p^s(p^s - 1)}$$

for $\mathrm{Re}(s) \geq 1$.

Proof. The equality follows from Lemma 7.6 and the second equality of Lemma 8.2. The domain of defnition can be extended to the closed halfspace $\mathrm{Re}(s) \geq 1$ since the summation term of (8.3) converges for $\mathrm{Re}(s) > \frac{1}{2}$ and the pole of $\zeta(s)$ at $s = 1$ of order 1 is negated by the sum with the Mellin transform of x, so the function admits a holomorphic extension to an open neighborhood of the closed halfspace $\mathrm{Re}(s) \geq 1$. ■

To conclude the proof of the prime number theorem, we will provide an estimate of this function by extending the domain of definition for the Mellin transform to the closed halfspace $\mathrm{Re}(s) \geq 1$.

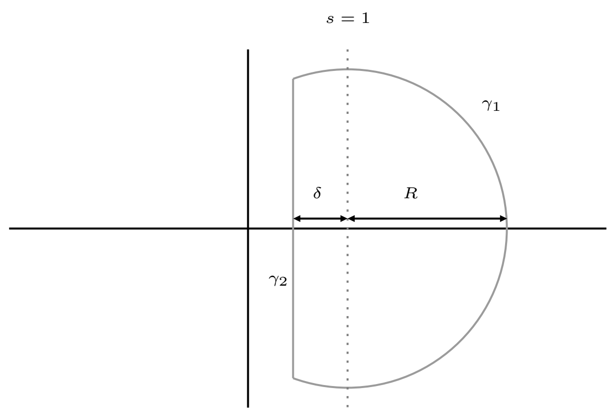
9. Lecture 9 – 7th November 2024

We conclude the proof of the prime number theorem by considering a sequence of estimation results.

Proposition 9.1. Let $f : [1, \infty) \to \mathbb{R}$ be a bounded locally integrable function and $F(s) = \int_1^\infty f(x)x^{-s}dx$. If F admits a holomorphic extension to a neighborhood of the closed halfspace $\mathrm{Re}(s) \geq 1$ then Theorem 8

$$F(1) = \lim_{\mu\to\infty} \int_1^\mu \frac{f(x)}{x} dx.$$

Proof. By Cauchy's integral formula, we have $F(a) - F_\mu(a) = \frac{1}{2\pi i}\int_\gamma \frac{F(s)-F_\mu(s)}{s-a}ds$ where $F_\mu(s) = \int_1^\mu \frac{f(x)}{x}dx$. We consider the integral over the contour γ



traversed counterclockwise where $\delta > 0$ sufficiently small that the complement γ_2 of the semicircular arc γ_1 lies within the domain of holomorphy of F. So we have

$$F(1) - F_\mu(1) = \frac{1}{2\pi i} \int_\gamma \frac{F(s) - F_\mu(1)}{s-1} ds$$

where we decompose the integral

$$\int_\gamma \frac{F(s) - F_\mu(1)}{s-1} ds = \int_{\gamma_1} \frac{F(s) - F_\mu(s)}{s-1} ds - \int_{\gamma_2} \frac{F_\mu(s)}{s-1} ds + \int_{\gamma_2} \frac{F(s)}{s-1} ds. \tag{9.1}$$

For the first two summands of (9.1), we consider the function

$$h_{\mu,R}(s) = \frac{(s-1)^2 + R^2}{R^2} \mu^{s-1}$$

which satisfies $h_{\mu,R}(1) = 1$ and consider instead

$$\int_{\gamma_1} \frac{(F(s) - F_\mu(s))h_{\mu,R}(s)}{s-1} ds, \int_{\gamma_2} \frac{F_\mu(s)h_{\mu,R}(s)}{s-1} ds.$$

Now for the first summand of (9.1), we have

$$\begin{aligned}|F(s)-F_\mu(s)| &= \left|\int_\mu^\infty f(x)x^{-s}dx\right| \\ &\leq C\int_\mu^\infty x^{-\mathrm{Re}(s)}dx && f(x)\leq C \\ &= \frac{C}{\mathrm{Re}(s)-1}\mu^{1-\mathrm{Re}(s)} \\ &= \left|\frac{2C}{(s-1)+(\overline{s}-1)}\mu^{1-s}\right|\end{aligned}$$

so for $\mathrm{Re}(s)>1$ and $|s-1|=R$ we get

$$\left|\frac{F(s)-F_\mu(s)}{s-1}\right| \leq \left|\frac{2C}{(s-1)^2+R^2}\mu^{1-s}\right|$$

and thus

$$\left|\frac{(F(s)-F_\mu(s))h_{\mu,R}(s)}{s-1}\right| \leq \left|\frac{2C}{(s-1)^2+R^2}\mu^{1-s}\right|\cdot\left|\frac{(s-1)^2+R^2}{R^2}\mu^{s-1}\right|$$

showing

$$\left|\frac{(F(s)-F_\mu(s))h_{\mu,R}(s)}{s-1}\right| \leq \frac{2C}{R^2}.$$

For the second summand of (9.1) we have on γ_2

$$\begin{aligned}|F_\mu(s)| &\leq C\int_1^\mu x^{-\mathrm{Re}(s)}dx && f(x)\leq C \\ &= \frac{C}{1-\mathrm{Re}(s)}(\mu^{1-\mathrm{Re}(s)}-1) \\ &\leq \frac{C}{1-\mathrm{Re}(s)}\mu^{1-\mathrm{Re}(s)}\end{aligned}$$

and applying the same argument as before for $\mathrm{Re}(s)\leq 1$ and $|s-1|\leq R$ we get

$$\left|\frac{F_\mu(s)}{s-1}\right| \leq \frac{2C}{R^2}.$$

Finally, for the third summand of (9.1), we further decompose the contour γ_2 into γ_2' consisting of the segment in $\mathrm{Re}(s)\leq 1-\delta'$ for $0<\delta'<\delta$ and γ_2^+,γ_2^- the arc segments $[1+Ri,\gamma_2'(0)]$ and $[\gamma_2'(1),1-Ri]$ so that the concatenation of $\gamma_2^+,\gamma_2',\gamma_2^{-1}$ is all of γ_2 as previously defined. $F(s)$ is bounded on γ_2 so

$$\left|F(s)\left(\frac{s-1}{R^2}+\frac{1}{s-1}\right)\right| \leq A(R,\delta)$$

for some constant $A(R,\delta)$ depending only on R and δ and on γ_2' we get the estimate

$$\left|\frac{1}{2\pi i}\int_{\gamma_2'}F(s)\left(\frac{s-1}{R^2}+\frac{1}{s-1}\right)\mu^{s-1}ds\right| \leq \frac{R\cdot A(R,\delta)}{2}\cdot\sup_{s\in\gamma_2'}\mu^{\mathrm{Re}(s)-1} = \frac{RA}{2}\mu^{-\delta'}.$$

The segments γ_2^+, γ_2^- have length at most $\frac{\pi\delta'}{2}$ so since $|\mu^{\mathrm{Re}(s)-1}| \leq 1$ we have

$$\left|\frac{1}{2\pi i}\int_{\gamma_2'} F(s)\left(\frac{s-1}{R^2}+\frac{1}{s-1}\right)\mu^{s-1}ds\right| \leq \frac{1}{2\pi}\cdot \pi\delta' A(R,\delta) = \frac{\delta_1 A}{2}.$$

Now for $\varepsilon > 0$ we can take $R = \frac{1}{\varepsilon}$ and δ sufficiently small such that each of the first two summands of (9.1) is less than ε by taking μ large and for the third summand taking δ' small such that the integral over γ_2' and the sum of integrals on γ_2^+, γ_2^- are both at most ε by taking μ large and δ' small. In sum we have that $|F(1) - F_\mu(1)| \leq 4\varepsilon$ bounded by a function decreasing in μ, giving the claim. ■

The prime number theorem is a consequence of Proposition 9.1 and the following lemma.

Lemma 9.2. Let $f : [1, \infty) \to \mathbb{R}$ be a nondecreasing function such that

$$\lim_{\mu\to\infty}\int_1^\mu \left(\frac{f(x)}{x^2} - \frac{1}{x}\right)dx$$

exists. Then $\lim_{x\to\infty}\frac{f(x)}{x} = 1$.

Proof. Suppose to the contrary that the limit is not 1, then for $\eta > 0$ there is an increasing sequence $\{x_n\}_{n\geq 0}$ such that $|\frac{f(x_n)}{x_n} - 1| \geq \eta$ in which case we can assume, by rearranging the equation, that $f(x_n) \geq (1+\eta)x_n$ for all n by passing to a subsequence. Taking $\rho = \frac{1+\eta}{1+\eta/2}$ we have for $x_n \leq x \leq \rho x_n$ that

$$\left(1+\frac{\eta}{2}\right)x \leq (1+\eta)x_n \leq f(x_n) \leq f(x)$$

with the last inequality by f increasing. So integrating on the interval $[x_n, \rho x_n]$ we get

$$0 < \frac{\eta\log(\rho)}{2} \leq \int_{x_n}^{\rho x_n}\left(\frac{f(x)}{x^2} - \frac{1}{x}\right)dx$$

a contradiction to existence fo the improper integra. ■

The above suffices to prove the prime number theorem.

Theorem 9.3 (Prime Number Theorem)**.** Let $\pi(x)$ be the prime counting function. Then

$$\lim_{x\to\infty}\frac{\pi(x)\cdot\log(x)}{x} = 1.$$

Proof. By Proposition 7.15 it suffices to show $\lim_{x\to\infty}\frac{\vartheta(x)}{x} = 1$. Let $f(x) = \frac{\vartheta(x)}{x} - 1$ and

$$F(s) = s\int_1^\infty\left(\frac{\vartheta(x)}{x} - 1\right)x^{-s}dx = s\int_1^\infty(\vartheta(x) - x)x^{-s-1}dx = \mathcal{M}_{\vartheta(x)-x}(s)$$

which admits a holomorphic extension to a neighborhood of the closed halfspace $\mathrm{Re}(s) \geq 1$ by Proposition 8.3. Further noting $\frac{\vartheta(x)}{x}$ is bounded by Lemma 8.1 and is locally integrable, Proposition 9.1 applies and we have existence of the improper integral $F_\mu(1)$ and the conclusion follows by Lemma 9.2. ■

We now begin a discussion of complex analysis in several variables.

Consider $\mathbb{C}^n$ which is an n-dimensional complex vector space and thus a $2n$-dimensional real vector space.

Definition 9.4 (Orientation)**.** An orientation on $\mathbb{C}^n$ is a choice of coordinates on $\mathbb{C}^n$ as a $2n$-dimensional real vector space.

Remark 9.5. We will use the orientation

$$(a_1, b_1, a_2, b_2, \dots, a_n, b_n) \in \mathbb{R}^{2n}$$

for $(z_1, \dots, z_n) \in \mathbb{C}^n$ with $z_j = a_j + b_j i$. This behaves well with respect to taking products of complex vector spaces.

$\mathbb{C}^n$ is a normed vector space.

Definition 9.6 (Maximum Norm)**.** The maximum norm on $\mathbb{C}^n$ is given by $\|z\|_{\max} = \max_{1 \leq j \leq n}\{|z_j|_{\mathbb{C}}\}$ where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $|\cdot|_{\mathbb{C}}$ is the standard norm on $\mathbb{C}$.

Definition 9.7 (Euclidean Norm)**.** The Euclidean norm on $\mathbb{C}^n$ is given by $\|z\| = \sqrt{\sum_{j=1}^n |z_j|_{\mathbb{C}}}$ where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $|\cdot|_{\mathbb{C}}$ is the standard norm on $\mathbb{C}$.

Remark 9.8. We will often just denote the standard norm $|\cdot|_{\mathbb{C}}$ on $\mathbb{C}$ as $|\cdot|$.

Open discs plays a central role in univariate complex analysis. Its analogue in complex analysis of several variables is the polydisc.

Definition 9.9 (Polydisc)**.** Let $\tau \in \mathbb{C}^n$ and $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$. The polydisc $D_r(\tau)$ of polyradius r is given by the set

$$\{z \in \mathbb{C}^n : |\tau_j - z_j| < r_j, \forall 1 \leq j \leq n\}.$$

Remark 9.10. We can also define the polydisc $D_r(\tau)$ of radius $r \in \mathbb{R}_{>0}$ as

$$\{z \in \mathbb{C}^n : |\tau_j - z_j| < r, \forall 1 \leq j \leq n\}.$$

We also have an analogue of the open ball.

Definition 9.11 (Multivariate Ball)**.** Let $\tau \in \mathbb{C}^n$ and $r \in \mathbb{R}_{>0}$ the ball $B_r(\tau)$ is given by the set

$$\{z \in \mathbb{C}^n : \|\tau - z\| < r\}.$$

Remark 9.12. The open balls form a basis for the analytic topology on $\mathbb{C}^n$.

Let us define the multivariate analogue of holomorphic functions via real and complex differentiable functions.

Definition 9.13 (Real Differentiable)**.** Let $U \subseteq \mathbb{C}^n$ be an open set and $f : U \to \mathbb{C}$ a function. f is real differentiable at $\tau \in U$ if there exist continuous functions $\Delta_1, \dots, \Delta_n, E_1, \dots, E_n$ continuous at τ such that

$$f(z) - f(\tau) = \sum_{j=1}^n \Delta_j(z)(z_j - \tau_j) + \sum_{j=1}^n E_j(z)(\overline{z}_j - \overline{\tau}_j)$$

for all $z \in U$.

Definition 9.14 (Complex Differentiable)**.** Let $U \subseteq \mathbb{C}^n$ be an open set and $f : U \to \mathbb{C}$ a function. f is complex differentiable at $\tau \in U$ if there exist functions $\Delta_1, \ldots, \Delta_n : U \to \mathbb{C}$ continuous at z such that

$$f(z) - f(\tau) = \sum_{j=1}^{n} \Delta_j(z)(z_j - \tau_j)$$

for all $z \in U$.

Definition 9.15 (Holomorphic Function)**.** Let $U \subseteq \mathbb{C}^n$ be an open set and $f : U \to \mathbb{C}$ a function. f is holomorphic on U if f is complex differentiable at all $z \in U$.

The functions Δ_j, η_j play a special role.

Definition 9.16 (Wirtinger Derivatives)**.** Let $U \subseteq \mathbb{C}^n$ be an open set and $f : U \to \mathbb{C}$ a function. The Wirtinger derivatives of f at τ are given by $\Delta_j(z), \eta_j(z)$ where

$$f(z) - f(a) = \sum_{j=1}^{n} \Delta_j(z)(z_j - \tau_j) + \sum_{j=1}^{n} E_j(z)(\overline{z}_j - \overline{\tau}_j)$$

with $\Delta_1, \ldots, \Delta_n, E_1, \ldots, E_n : U \to \mathbb{C}$ continuous for all $z \in U$.

10. Lecture 10 – 12th November 2024

Continuing our discussion of real and complex differentiation, we note that Wirtinger derivatives are closely related to the multivariable Cauchy-Riemann equations.

Proposition 10.1. Let $U \subseteq \mathbb{C}^n$ be an open set and $f : U \to \mathbb{C}$ a real differentiable function of the form

$$f(z) - f(\tau) = \sum_{j=1}^{n} \Delta_j(z)(z_j - \tau_j) + \sum_{j=1}^{n} E_j(z)(\overline{z}_j - \overline{\tau}_j).$$

Then

$$\partial_{z_j} f(z) = \Delta_j(z) = \frac{1}{2}\left(\partial_{x_j} f(z) - i \cdot \partial_{y_j} f(z)\right)$$

and

$$\partial_{\overline{z_j}} f(z) = E_j(z) = \frac{1}{2}\left(\partial_{x_j} f(z) + i \cdot \partial_{y_j} f(z)\right).$$

Proof. Suppsose $z_1 = \tau_1, \ldots, z_{j-1} = \tau_{j-1}, z_{j+1} = \tau_{j+1}, \ldots, z_n = \tau_n$ so $f'(z) = \lim_{z_j \to \tau_j} \frac{f(z)-f(\tau)}{z_j - \tau_j} = \Delta_j(\tau)$, the second equality in the first line follows from the first by applying the chain rule, and the second set of equalities follows by an analogous computation. ■

As such, we can show the following theorem.

Theorem 10.2. Let $U \subseteq \mathbb{C}^n$ be an open set and $f : U \to \mathbb{C}$ a function. f is complex holomorphic if and only if it is real differentiable and satisfies the system of partial differential equations

$$\partial_{\overline{z_1}} f(z) = \cdots = \partial_{\overline{z_n}} f(z) = 0.$$

Proof. Proposition 10.1 expresses $\partial_{\overline{z_j}} f(z)$ in terms of $E_j(z)$ which vanish identically for a holomorphic function by Definition 9.14. ■

Remark 10.3. Evidently a holomoprhic function is holomorphic in each variable. That is, for $f : U \to \mathbb{C}$ holomorphic, the function $f(\tau_1, \ldots, \tau_{j-1}, z_j, \tau_{j+1}, \ldots, \tau_n)$ is a univariate holomorphic function in z_j. Furthermore, while in real analysis there exist functions that are differentiable in each variable but are not even continuous, differentiability in each variable implies global differentiability in complex analysis by Hartogs' theorem. This is highly subtle and is beyond the scope of the course, and an account can be found in the text of Hörmander [H90].

We can show that holomorphic functions on a domain behave well algebraically and form a ring.

Proposition 10.4. Let $U \subseteq \mathbb{C}^n$ be open. The set of holomorphic functions $\mathcal{O}_U$ is a $\mathbb{C}$-algebra that contains $\mathbb{C}[z_1, \ldots, z_n]$.

Proof. Elements of $\mathbb{C}[z_1, \ldots, z_n]$ are holomorphic on $\mathbb{C}^n$ and hence on U, and the constants are holomorphic on U. Complex differentiable functions are preserved sums and products, and thus so too are holomorphic functions. ■

We now state and prove the Cauchy integral formula for functions of several complex variables. To do so, we will define integrals over the distinguished boundary of a polydisc.

Definition 10.5 (Distinguished Boundary of Polydisc)**.** Let $D_r(\tau)$ be a polydisc of polyradius r around τ. The distinguished boundary $T_r(\tau)$ of $D_r(\tau)$ is given by

$$\{z \in \mathbb{C}^n : |z_j - \tau_j| = r_j\}.$$

Remark 10.6. The distinguished boundary is the product of n copies of the topological circle S^1, that is, is of real dimension n.

We can now turn to a discussion of the multivariate Cauchy integral formula.

Theorem 10.7 (Multivariate Cauchy Integral)**.** Let $U \subseteq \mathbb{C}^n$ be an open set and $f : U \to \mathbb{C}$ a holomorphic function. If $D_r(\tau) \subseteq U$ is a polydisc with distinguished boundary $T_r(\tau)$ then for all $\alpha \in U$

$$f(\alpha) = \frac{1}{(2\pi i)^n} \int_{T_r(\tau)} \frac{f(z)}{(z_1 - \alpha_1) \dots (z_n - \alpha_n)} dz.$$

Proof. The $n = 1$ case is the univariate Cauchy integral formula. We proceed by induction on n, supposing it holds for the case k. Consider the case $k + 1$ with a function $f(z_1, \dots, z_{k+1})$. Let $g(z_1) = f(z_1, \alpha_2, \dots, \alpha_{k+1})$ be a univariate function. By Cauchy's integral formula in one dimension, we have

$$f(\alpha) = f(\alpha_1, \dots, \alpha_{k+1}) = \int_{T_{r_1}(\tau_1)} \frac{g(z_1)}{z_1 - \alpha_1} dw.$$

But by the induction hypothesis, for fixed w, we have

$$f(z_1, \alpha_2, \dots, \alpha_{k+1}) = \int_{T_{(r_2,\dots,r_{k+1})}((\tau_2,\dots,\tau_{k+1}))} \frac{f(z_1, z_2, \dots, z_n)}{(z_2 - \alpha_2) \dots (z_n - \alpha_n)}$$

and combining the two integrals yields the claim. ■

We can alternatively phrase Theorem 10.7 in terms of Cauchy kernels.

Definition 10.8 (Cauchy Kernel)**.** Let $U \subseteq \mathbb{C}^n$ be an open set, $f : U \to \mathbb{C}$ a continuous function, and $D_r(\tau) \subseteq U$ is a polydisc with distinguished boundary $T_r(\tau)$. Then the Cauchy kernel is given by

$$C_f(z) = \frac{1}{(2\pi i)^n} \int_{T_r(\tau)} \frac{f(w)}{(w_1 - z_1) \dots (w_n - z_n)} dw_1 \dots dw_n.$$

Remark 10.9. By Theorem 10.7, the Cauchy kernel agrees with f if f is holomorphic.

The Cauchy kernel allows us to deduce that holomorphic functions of several variables are infinitely differentiable.

Proposition 10.10. Let $U \subseteq \mathbb{C}^n$ be an open set containing a polydisc $D_r(\tau)$ with distinguished boundary $T_r(\tau)$ and $f : U \to \mathbb{C}$ a holomorphic function. Then f admits holomorphic partial derivatives of all orders. Theorem 1.2

Proof. Differentiating the Cauchy kernel under the integral sign, we have

$$\partial_z^\nu C_f(z) = \frac{\nu_1! \dots \nu_n!}{(2\pi i)^n} \int_{T_r(\tau)} \frac{f(w)}{(w_1 - z_1)^{\nu_1+1} \dots (w_n - z_n)^{\nu_n+1}} d_{w_1} \dots dw_n$$

giving the claim. ∎

We will now consider analogues of key results in complex analysis in the multivariate setting. This discussion will require the following lemma which often allows us to reduce to the single variable case.

Lemma 10.11. Let $\alpha, \beta \in \mathbb{C}^n$ and $\lambda : \mathbb{C} \mapsto \mathbb{C}^n$ by $t \mapsto \alpha + t\beta$. If f is holomorphic on $U \subseteq \mathbb{C}^n$ open then $f \circ \lambda$ is holomorphic on $\lambda^{-1}(U)$.

Proof. We compute

$$\frac{d}{dt}(f \circ \lambda)(t) = \sum_{j=1}^n \partial_{z_j}(\lambda(t))\beta_j$$

which is a β_j-weighted sum of holomorphic functions and hence holomorphic on $\lambda^{-1}(U) = f|_L$. ∎

We can now prove the multivariate analogues of the identity theorem and maximum principle.

Theorem 1.5

Theorem 10.12 (Multivariate Identity)**.** Let f be holomorphic on a domain G and identically zero on a nonempty open subset U of G. Then $f \equiv 0$ on G.

Proof. Let L be a line the image of λ through G and consider $f \circ \lambda$ the induced holomorphic function of one variable. By the identity theorem in one variable Theorem A.1, f is identically zero on all of L. Writing G as the union of all lines passing through it gives the claim. ∎

Theorem 1.6

Theorem 10.13 (Multivariate Maximum Modulus)**.** Let f be holomophric on a domain G and $|f|$ has local maximum at $\tau \in G$. Then $f(z) = f(\tau)$ is constant.

Proof. Arguing as before, let L be a line the image of λ through G and consider $f \circ \lambda$ the induced holomorphic function of one variable. By the maximum maximum modulus principle in one variable, f is constant on all of L. Writing G as the union of all lines passing through it gives the claim. ∎

We now turn to a consequence of Theorem 10.7 and some surprising consequences.

Proposition 10.14. Let $D_r(\tau) \subseteq \mathbb{C}^n$ for $n \geq 2$. If f is holomorphic in a punctured neighborhood of the origin in $D_r(\tau)$, then f is holomorphic on $D_r(\tau)$.

Proof. By translation, it suffices to consider the case of a function f holomorphic on $D_1(0) \setminus \overline{D_{1/2}(0)}$ in $\mathbb{C}^n$, here considering polydiscs of fixed radius. By the univariate Cauchy integral formula we have

$$f(\alpha) = \frac{1}{(2\pi i)^n} \int_{T_r(\tau)} \frac{f(z)}{(z_1 - \alpha_1) \dots (z_n - \alpha_n)} dz$$

but fixing any α_j the Cauchy integral formula in one variable holds but α_j is arbitrary so the function extends in each variable to all of $D_1(0)$ and so it does overall. ∎

This implies that holomorphic functions have zeroes and thus poles along a subset of $\mathbb{C}^n$ of codimension at least 1.

Corollary 10.15. Let f be a holomorphic function on $U \subseteq \mathbb{C}^n$ open. Then $\{z \in \mathbb{C}^n : f(z) = 0\}$ and $\{z \in \mathbb{C}^n : f(z) = \infty\}$ are non-isolated.

Proof. The poles of f are non-isolated by Proposition 10.14 and thus so too zeroes are non-isolated under passage to $1/f(z)$. ∎

With the language of holomorphic functions in hand, we can discuss holomorphic maps.

Definition 10.16 (Holomorphic Map)**.** Let $U \subseteq \mathbb{C}^n, V \subseteq \mathbb{C}^m$ be open sets and $f = (f_1, \dots, f_m) : U \to V$ be a continuous function. f is a holomorphic map if each component function $f_k : U \to \mathbb{C}$ is holomorphic.

The chain rule generalizes to the multivariate setting.

Proposition 10.17 (Multivariate Chain Rule)**.** Let $U \subseteq \mathbb{C}^n, V \subseteq \mathbb{C}^m$ be open sets and $f = (f_1, \dots, f_m) : U \to V$ and $g : V \to \mathbb{C}$ holomorphic functions. Then

$$\partial z_j(g \circ f) = \sum_{k=1}^{m} \frac{\partial g}{\partial w_k} \frac{\partial f}{\partial z_j}.$$

Proof. This is immediate from the chain rule and the Cauchy-Riemann equations. ∎

11. Lecture 11 – 14th November 2024

Recall the analytic continuation result of Proposition 10.14. We will build up to showing a more general analogue that allows us to extend a holomorphic function on the complement of a compact set $U \setminus K \subseteq \mathbb{C}^n$ to $U \subseteq \mathbb{C}^n$ known as Hartogs' Kugelsatz. Hartogs' original proof relied on Cauchy's integral formula paired with intricate geometric arguments. We provide a simplified proof that uses more technology. In particular, we continue our discussion of holomorphic maps.

Definition 11.1 (Holomorphic Jacobian)**.** Let $U \subseteq \mathbb{C}^n, V \subseteq \mathbb{C}^m$ be open sets and $f : U \to V$ a holomorphic map. The holomorphic Jacobian is defined to be the matrix

$$J_f^{\mathsf{Hol}}(z) = \begin{bmatrix} \frac{\partial f_1}{\partial z_1}(\tau) & \dots & \frac{\partial f_1}{\partial z_n}(\tau) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial z_1}(\tau) & \dots & \frac{\partial f_m}{\partial z_n}(\tau) \end{bmatrix} \in \mathbb{C}^{m\times n}$$

and $\tau \in U$.

We consider some examples.

Example 11.2. Let $U \subseteq \mathbb{C}^n$ be open and $\mathrm{id}_U : U \to U$ be the holomorphic identity map. The holomorphic Jacobian $J_{\mathrm{id}_U}^{\mathsf{Hol}}$ is the identity matrix.

Formation of the Jacobian is a linear operation.

Proposition 11.3. Let $U \subseteq \mathbb{C}^n, V \subseteq \mathbb{C}^m, W \subseteq \mathbb{C}^k$ be open sets and $f : U \to V, g : V \to W$ holomorphic maps. Then $J_{g\circ f}^{\mathsf{Hol}} = J_g^{\mathsf{Hol}} \times J_f^{\mathsf{Hol}}$.

Proof. We compute, denoting the coordinates on V by $w_1, \dots, w_m$, we have

$$\begin{aligned} J_g^{\mathsf{Hol}}(f(\tau)) \times J_f^{\mathsf{Hol}}(\tau) &= \begin{bmatrix} \frac{\partial g_1}{\partial w_1}(f(\tau)) & \dots & \frac{\partial g_1}{\partial w_m}(f(\tau)) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial w_1}(f(\tau)) & \dots & \frac{\partial g_k}{\partial w_m}(f(\tau)) \end{bmatrix} \begin{bmatrix} \frac{\partial f_1}{\partial z_1}(\tau) & \dots & \frac{\partial f_1}{\partial z_n}(\tau) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial z_1}(\tau) & \dots & \frac{\partial f_m}{\partial z_n}(\tau) \end{bmatrix} \\ &= \begin{bmatrix} \sum_{r=1}^m \frac{\partial g_1}{\partial w_r}(f(\tau)) \cdot \frac{\partial f_r}{\partial z_1}(\tau) & \dots & \sum_{r=1}^m \frac{\partial g_1}{\partial w_r}(f(\tau)) \cdot \frac{\partial f_r}{\partial z_n}(\tau) \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^m \frac{\partial g_k}{\partial w_r}(f(\tau)) \cdot \frac{\partial f_r}{\partial z_1}(\tau) & \dots & \sum_{r=1}^m \frac{\partial g_k}{\partial w_r}(f(\tau)) \cdot \frac{\partial f_r}{\partial z_n}(\tau) \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial (g_1\circ f)}{\partial z_1}(\tau) & \dots & \frac{\partial (g_1\circ f)}{\partial z_n}(\tau) \\ \vdots & \ddots & \vdots \\ \frac{\partial (g_k\circ f)}{\partial z_1}(\tau) & \dots & \frac{\partial (g_k\circ f)}{\partial z_n}(\tau) \end{bmatrix} \\ &= J_{f\circ g}^{\mathsf{Hol}}(\tau) \end{aligned}$$

as desired. ∎

We can define biholomorphisms which will be the appropriate notion of invertible maps in complex analysis of several variables.

Definition 11.4 (Biholomorphism)**.** Let $U \subseteq \mathbb{C}^n, V \subseteq \mathbb{C}^m$ and $f : U \to V$ a
Definition 2.1 holomorphic map. f is a biholomorphism if f^{-1} exists and is holomorphic.

A complex function $f : U \to V$ can be written as $u + iv$ where $u, v : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$. We can use this to produce the real Jacobian of the complex-valued function f.

Definition 11.5 (Real Jacobian)**.** Let $U \subseteq \mathbb{C}^n, V \subseteq \mathbb{C}^m$ and $f : U \to V$ a differentiable map. For $f = u + iv$ with $u, v : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$, the real Jacobian of f is defined as

$$J_f^{\mathbb{R}}(\tau) = \begin{bmatrix} J_{u,x}(\tau) & J_{u,y}(\tau) \\ J_{v,x}(\tau) & J_{v,y}(\tau) \end{bmatrix} \in \mathbb{C}^{2m \times 2n}.$$

We can also define the complex Jacobian using Wirtinger derivatives and their conjugates.

Definition 11.6 (Complex Jacobian)**.** Let $U \subseteq \mathbb{C}^n, V \subseteq \mathbb{C}^m$ and $f : U \to V$ a differentiable map. The complex Jacobian of f is defined as

$$J_f^{\mathbb{C}}(\tau) = \begin{bmatrix} J_{f,z}(\tau) & J_{\overline{f},z}(\tau) \\ J_{f,\overline{z}}(\tau) & J_{\overline{f},\overline{z}}(\tau) \end{bmatrix} \in \mathbb{C}^{2m \times 2n}.$$

The holomorphic, real, and complex Jacobians are related in the following way. We omit the linear-algebraic proofs.

Proposition 11.7. Let $U \subseteq \mathbb{C}^n, V \subseteq \mathbb{C}^m$ and $f : U \to V$ a differentiable map. Then $\operatorname{rank}(J_f^{\mathbb{R}}(\tau)) = \operatorname{rank}(J_f^{\mathbb{C}}(\tau))$ and $\det(J_f^{\mathbb{R}}) = \det(J_f^{\mathbb{C}})$.

If f is holomorphic and in particular a biholomorphism, we can say more.

Proposition 11.8. Let $U \subseteq \mathbb{C}^n, V \subseteq \mathbb{C}^m$ and $f : U \to V$ a biholomorphism. Then
$\det(J_f^{\mathbb{C}}) = |\det(J_f^{\mathsf{Hol}})|$. Proposition 2.2

We will show that a holomorphic map is a biholomorphism when it is bijective on sets. This is not true for differentiable maps. We set out some preparations for the theorem.

Proposition 11.9. Let $U \subseteq \mathbb{C}^n, V \subseteq \mathbb{C}^m$ and $f : U \to V$ a biholomorphism such
that $J_f^{\mathsf{Hol}}(z)$ is invertible for all z. Then f is biholomorphic. Proposition 2.4

Proof. Note that $J_f^{\mathbb{R}}(z)$ is everywhere regular showing the existence of a real differentiable inverse f^{-1} by the implicit function theorem. It remains to show f^{-1} is holomorphic. We can compute the complex Jacobian of f^{-1} which we write as $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and since f is holomorphic we have that its complex Jacobian is block diagonal $\begin{bmatrix} M & 0 \\ 0 & \overline{M} \end{bmatrix}$, we have the product

$$\begin{bmatrix} M & 0 \\ 0 & \overline{M} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \mathrm{id}$$

showing $B, C = 0$ and thus f^{-1} is holomorphic. ∎

We make some further recollections from analysis and differential topology.

Theorem 11.10 (Implicit Function)**.** Let $U \subseteq \mathbb{C}^n, V \subseteq \mathbb{C}^m$ and $f : U \times V \to \mathbb{C}^m$ a holomorphic map. If $f(0,0) = 0$ and the holomorphic Jacobian $J^{\mathsf{Hol}}_{f,w}(0)$ is regular in the variables w on V then there exist open subsets $U_1 \subseteq U, V_1 \subseteq V$ and a holomorphic map $\phi : U_1 \to V_1$ such that $f(z,w) = 0$ if and only if $w = \phi(z)$.

Proposition 2.5

Theorem 11.11 (Submanifold)**.** Let $A \subseteq \mathbb{C}^n$, $a \in A$, $1 \leq k \leq n$. The following are equivalent:

Proposition 2.6

(a) There is a neighborhood U of a, a neighborhood V of $0 \in \mathbb{C}^k$, and a holomorphic homeomorphism $\phi : V \to \mathbb{C}^n$ such that $\phi(V) = A \cap U$ and J^{Hol}_{ϕ} is of rank k.
(b) There exists a neighborhood U of a and a holomorphic function $f : U \to \mathbb{C}^{n-k}$ such that $U \cap A = \{z \in U : f(z) = 0\}$ and J^{Hol}_{f} is of rank $n-k$.
(c) There exists a neighborhood U of a and $W \subseteq \mathbb{C}^m$ of 0 and a holomorphic map $\phi : U \to W$ such that $\phi(A \cap U) = \{w \in W : w_{k+1} = \cdots = w_m = 0\}$.

In analogy to differential topology, we have the following definition.

Definition 11.12 (Locally Analytic Submanifold)**.** Let $A \subseteq \mathbb{C}^n$, $a \in A$, $1 \leq k \leq n$. A is a locally analytic submanifold of dimension k if one and thus all of the following conditions hold:

Definition 2.2

(a) There is a neighborhood U of a, a neighborhood V of $0 \in \mathbb{C}^k$, and a holomorphic homeomorphism $\phi : V \to \mathbb{C}^n$ such that $\phi(V) = A \cap U$ and J^{Hol}_{ϕ} is of rank k.
(b) There exists a neighborhood U of a and a holomorphic function $f : U \to \mathbb{C}^{n-k}$ such that $U \cap A = \{z \in U : f(z) = 0\}$ and J^{Hol}_{f} is of rank $n-k$.
(c) There exists a neighborhood U of a and $W \subseteq \mathbb{C}^m$ of 0 and a holomorphic map $\phi : U \to W$ such that $\phi(A \cap U) = \{w \in W : w_{k+1} = \cdots = w_m = 0\}$.

This definition globalizes.

Definition 11.13 (Analytic Submanifold)**.** Let $A \subseteq \mathbb{C}^n$, $a \in A$, $1 \leq k \leq n$. A is an analytic manifold of dimension k if it is a locally analytic submanifold of dimension k for all $a \in A$.

Definition 2.3

We will require the subsequent lemma to prove the desired result.

Lemma 11.14. Let $U \subseteq \mathbb{C}^n$ open, $f : U \to \mathbb{C}$ a holomorphic function, and $M = \{z \in U : f(z) = 0\} \subseteq U$. If M is nonempty, there is a point $a \in M$ such that M is a locally analytic manifold of dimension $n-1$.

Lemma 2.7

Proof. We proceed by cases. Let $a \in M$ and suppose $\nabla f(a) \neq 0$. Then Theorem 11.11 (a) applies showing it is a locally analytic submanifold. Otherwise, we can take sufficiently many derivatives such that the derivative is nonzero – all derivatives being zero implies the function is identically zero. Now let $\Lambda = \{r \in \mathbb{N} : D^\alpha f \equiv 0, |\alpha| \leq r\}$. This set is finite since the function is zero if all derivatives vanish. Now choose $\lambda \in \Lambda$ maximal so that there is a point $a \in M$ and $D^\beta f$ with $|\beta| = \lambda$ and

$\nabla(D^\beta(f))(a) \neq 0$. So $M_0 = \{a \in M : D^\beta f(a) = 0\}$ is a locally analytic submanifold of dimension $n-1$. We show that $M = M_0$ close to a. By Theorem 11.11, we can take new coordinates on M_0 such that $M_0 = \{z_n = 0\}$ and $a = 0$ and consider a holomorphic homeomorphism $\phi : V \to \mathbb{C}^n$ for V an open neighborhood of a by $\phi(z', z_n)$ where $\phi(0, z_n) \neq 0$ for $z_n \neq 0$ which has a zero of order k in $z_n = 0$. By continuity of zeroes for ramilies of holomorphic functions, $\phi(z', z_n)$ must have k zeroes for z' close to 0. But $M \subseteq M_0$ implies that all these zeroes are in M_0 so M satisfies $z_n = 0$ and is thus a locally analytic manifold of dimension $n-1$. ■

12. Lecture 12 – 19th November 2024

We complete the proof of Osgood's theorem alluded to in Section 11.

Theorem 12.1 (Osgood)**.** Let $U \subseteq \mathbb{C}^n, V \subseteq \mathbb{C}^m$ and $f : U \to V$ a bijective holomorphic map. Then f is a biholomorphism.

Proof. We proceed by induction on dimension of the source. If $n = 1$ then the open mapping theorem implies that f is an open map so f^{-1} is defined, continuous, and bijective. $V(f')$ is discrete if it is nonempty and the Riemann extension theorem implies that the inverse can be extended to a holomorphic function.

Assume the statement holds for maps from open subsets of $\mathbb{C}^{n-1}$ to $V \subseteq \mathbb{C}^m$. We show that $J_f^{\mathsf{Hol}}(z)$ is either regular or nonzero. Let τ be a point where $J_f^{\mathsf{Hol}}(\tau)$ is not the zero matrix. Up to permuting the coordinates in both source and target, we can assume that the lower right corner of the matrix is nonzero. Using the implicit function theorem Theorem 11.10, we can introduce new coordinates $w_1 = z_1, \ldots, w_{n-1} = z_{n-1}$ and $w_n = f_m(z_1, \ldots, z_n)$ so that $f = (g_1, \ldots, g_{m-1}, w_n)$ where the g_j are functions of $w_1, \ldots, w_{n-1}$. We can now consider

$$\widetilde{f} = (g_1(w_1, \ldots, w_{n-1}, 0), \ldots, g_{m-1}(w_1, \ldots, w_{m-1}, 0))$$

which is bijective and by the inductive hypothesis has $J_{\widetilde{f}}^{\mathsf{Hol}}(0)$ regular with nonzero determinant. This gives us an expression of the Jacobian determinant of f as a block diagonal with $J_{\widetilde{f}}^{\mathsf{Hol}}(0)$ in the upper left corner and 1 in the lower right corner. But these determinants agree, so $J_f^{\mathsf{Hol}}(0)$ is nonzero.

Now consider $h(z) = \det(J_f^{\mathsf{Hol}}(z))$ which is a holomorphic function in U. Assume $V(f) \subseteq M$ is nonempty where $M = V(J_f^{\mathsf{Hol}}(z))$. By Lemma 11.14, there is τ suc that M is an $n-1$ dimensional local analytic submanifold so $J_f^{\mathsf{Hol}}(z) = 0$ for all $z \in M \cap U_\tau$ where U_τ is a sufficiently small neighborhood of τ. So f is constant on $M \cap U_\tau$ and thus not injective, a contradiction. ∎

We now recall the definition and some elementary properties of differential forms. Proofs can be found in [Lee13] for the real theory and [Lee24] for the complex theory. In

Definition 12.2 (Tangent Vector)**.** Let $x_0 \in \mathbb{R}^n$, $S(x_0)$ the set of functions in a neighborhood of x_0, and $D(x_0)$ the subset of differentiable functions. A tangent vector is a map $D : D(x_0) \to \mathbb{R}$ is a tangent vector if $D(1) = 0$ and $D(gf) = 0$ for all $g \in S(x_0), f \in D(x_0)$ such that $f(x_0) = g(x_0) = 0$.

These maps naturally form a vector space.

Proposition 12.3. Let $x_0 \in \mathbb{R}^n$. If D is a tangent vector and $f, g \in D(x_0)$ then $D(fg) = g(x_0)D(f) + f(x_0)D(g)$. Furthermore, the tangent vectors form a vector space with basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$.

This leads to the following definitions.

Definition 12.4 (Tangent Space)**.** Let $x_0 \in \mathbb{R}^n$. The tangent space $T_{x_0}\mathbb{R}^n$ is the $\mathbb{R}$-vector space of tangent vectors at x_0.

Definition 12.5 (Cotangent Space)**.** Let $x_0 \in \mathbb{R}^n$. The cotangent space $(T_{x_0}\mathbb{R}^n)^\vee$ is the dual of the tangent space.

Since the tangent space is itself a vector space, we can define linear forms on it.

Definition 12.6 (Total Differential)**.** Let $x_0 \in \mathbb{R}^n$ and $f \in D(x_0)$ a differentiable function at x_0. The total differential d is the map $T_{x_0}\mathbb{R}^n \to \mathbb{R}$ by $D \mapsto D(f)$.

Remark 12.7. Total differentials of coordinate functions dx_j form a basis of $(T_{x_0}\mathbb{R}^n)^\vee$ dual to the basis $\frac{\partial}{\partial x_j}$ of $T_{x_0}\mathbb{R}^n$. As such, each cotangent vector v can be written as $\sum_{i=1}^n a_i(x)dx_i$ where $a_i(x)$ are functions on $\mathbb{R}^n$. A cotangent vector is continuous/differentiable/integrable if all the constituent a_is are so.

Definition 12.8 (Pfaffian Form)**.** Let $M \subseteq \mathbb{R}^n$ be an open subset. A Pfaffian form is the data of a cotangent vector $D_{x_0} \in (T_{x_0}M)^\vee$ for all $x_0 \in M$.

The Pfaffian forms naturally form a module $E^1(M)$ over the ring of functions $E^0(M)$. Taking the exterior algebra, we can define $E^p(M) = \bigwedge^p E^1(M)$ and write $E(M) = \bigoplus_{p=0}^n E^p(M)$.

Definition 12.9 (Differential p-Form)**.** Let $M \subseteq \mathbb{R}^n$ be open. A differential p-form on M is an element of $E^p(M) = \bigwedge^p E^1(M)$.

Note a p-form can be written as

$$\sum_{1\le i_1<i_2<\cdots<i_p\le n} f_{i_1,\ldots,i_p} \cdot dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}$$

with $f_{i_1,\ldots,i_p} \in E^0(M)$ so we can define the differential of a p-form as the $p+1$-form

$$\sum_{1\le i_1<i_2<\cdots<i_p\le n} df_{i_1,\ldots,i_p} \cdot dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}.$$

These transform with respect to maps $\mathbb{R}^n \to \mathbb{R}^m$ as follows.

Proposition 12.10. Let $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ be open and $f : U \to V$ be a continuous map. Let $g \in E^p(V)$

$$g = \sum_{0\le i_1<i_2<\cdots<i_p\le m} g_{i_1,\ldots,i_p} \cdot dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}$$

and

$$g \circ f = \sum_{0\le i_1<i_2<\cdots<i_p\le m} (g_{i_1,\ldots,i_p} \circ f) \cdot dg_{i_1} \wedge dg_{i_2} \wedge \cdots \wedge dg_{i_p}.$$

If f and the $g_{i_1,\ldots,i_p}$ are differentiable then $d(g) \circ f = d(g \circ f)$.

Of special concern to us will be differential n-forms in n-dimensional objects.

Definition 12.11 (Integration over Differential Form)**.** Let f be an n-form on a measurable subset M of $\mathbb{R}^n$. Then

$$\int_M f = \int_M f_{1,\ldots,n}(x)dx$$

where $f = f_{1,\ldots,n}(x) \cdot dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$.

In the complex setting the constructions and results carry over verbatim. However, instead of merely treating differentiable functions, we also consider holomorphic functions and thus holomorphic differential forms. These are defined as follows.

Definition 12.12 (Holomorphic Differential Form)**.** Let $U \subseteq \mathbb{C}^n$ be open. A differential 1-form f on U is holomorphic if $\partial_{\overline{z_j}} f \cdot d\overline{z_j} = 0$ for all j and

$$f = \sum_{j=1}^{n} \partial_{z_j} f \cdot dz_j + \sum_{j=1}^{n} \partial_{\overline{z_j}} d\overline{z_j}.$$

13. Lecture 13 – 21st November 2024

We consider power series in several complex variables.

Definition 13.1 (Multivariate Complex Power Series)**.** The multivariate complex power series ring is the ring $\mathbb{C}[z_1, \ldots, z_n]$ with $f \in \mathbb{C}[[z_1, \ldots, z_n]]$ of the form

$$f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha z^\alpha.$$

Remark 13.2. Here we use multi-index notation, where c_α is a constant depending on $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$.

Remark 13.3. We can also write f as an infinite sum of homogeneous polynomials of fixed degree.

Convergence is defined as follows.

Definition 13.4 (Series Convergence)**.** Let $\sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha$ is an infinite sum and $\tau \in \mathbb{C}^n$. The sum converges to τ if for all $\varepsilon > 0$, there exists a finite subset $J \subseteq \mathbb{Z}^n_{\geq 0}$ such that

$$\left| \tau - \sum_{\alpha \in J_0} c_\alpha \right| < \varepsilon$$

for all $J_0 \subseteq J$. Definition 3.1

Definition 13.5 (Formal Power Series Convergence)**.** Let $f \in \mathbb{C}[[z_1, \ldots, z_n]]$. f is a convergent power series if there exists $\tau \in \mathbb{C}^n \setminus \{0\}$ such that the series $\sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha \tau^\alpha$ is convergent in the sense of Definition 13.4.

Definition 13.6 (Ring of Convergent Power Series)**.** The ring of convergent power series is the subring $\mathbb{C}\{z_1, \ldots, z_n\} \subseteq \mathbb{C}[[z_1, \ldots, z_n]]$ consisting of convergent power series. Definition 3.2

We can provide a sufficient criterion for convergence.

Proposition 13.7. Let $f(z) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha z^\alpha$ be a formal complex power series and $\tau \in \mathbb{C}^n \setminus \{0\}$ such that $|c_\alpha \tau^\alpha|$ is bounded. Then $f(z)$ is uniformly convergent for all $z \in B_{|\tau|}(0)$. Proposition 3.1

Proof. For any $z \in B_{|\tau|}(0)$, we have $|z| = \lambda|\tau|$ for some $0 < \lambda < 1$. Thus by boundedness of $|c_\alpha z^\alpha| \leq |c_\alpha \tau^\alpha| \cdot \lambda^{|\alpha|}$ and the claim follows from $\sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} \lambda^{|\alpha|}$ being convergent. ■

More generally, we can write holomorphic functions as a convergent power series in the following way.

Theorem 13.8. Let f be a holomorphic function on a polydisc $D_r(\tau)$. Then

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} \frac{\partial^\alpha f}{\alpha!} (z - \tau)^\alpha.$$

for $z \in D_r(\tau)$. Theorem 3.3

Proof. We have that $z \mapsto \frac{1}{w-z}$ can be written as a geometric series $\sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} \frac{(z-\tau)^\alpha}{(w-\tau)^{\alpha+1}}$ which is uniformly convergent for w on the boundary of a proper sub-polydisc by Proposition 13.7. Now applying Cauchy's integral formula Theorem 10.7, we can interchange integration and summation to yield

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} \left(\frac{1}{(2\pi i)^n} \int_{\partial T_{r'}(\tau)} \frac{f(w)}{(w-\tau)^{\alpha+1}} dw \right) (z-\tau)^\alpha$$

but the integral is precisely the partial derivative, which was the claim. ■

We now build up to the definition of the structure sheaf on $\mathbb{C}^n$.

Definition 3.3 **Definition 13.9** (Germ of Functions)**.** Let $z \in \mathbb{C}^n$. The ring of germs $\mathcal{O}_z$ at z is given by equivalence classes of functions f holomorphic in a neighborhood of z under the pointwise operations where $f \sim g$ if and only if there exists an open set $V \subseteq U_f \cap U_g$ of the neighborhoods U_f, U_g on which f, g are holomorphic, respectively, on which $f|_V = g|_V$.

We can define the structure sheaf as the disjoint union of these rings.

Definition 3.4 & 3.6 **Definition 13.10** (Structure Sheaf)**.** The structure sheaf $\mathcal{O}$ on $\mathbb{C}^n$ is given by $\coprod_{z \in \mathbb{C}^n} \mathcal{O}_z$.

There is an evident map $p : \mathcal{O} \to \mathbb{C}^n$ by $(f, z) \mapsto z$.

Definiton 3.5 **Definition 13.11** (Continuous Section of Structure Sheaf)**.** Let $U \subseteq \mathbb{C}$. A continuous section of $\mathcal{O}$ over U is a continuous function $\sigma : U \to \mathcal{O}$ such that $\sigma(z) \in \mathcal{O}_z$ and there exists a holomorphic function f on U such that $\sigma(z) = f_z$ for all $z \in U$.

Remark 13.12. The definition above allows us to endow $\mathcal{O}$ with the structure of a topological space with basis elements given by $\sigma(U)$ for $U \subseteq \mathbb{C}^n$

Let us return to more general considerations of holomorphic functions.

Proposition 4.1 **Proposition 13.13.** Let $U \subseteq \mathbb{C}^n$ be open, $K \subseteq \mathbb{C}$ an annulus around the origin of internal and external radii r, R, respectively, and $f : U \times K \to \mathbb{C}$ a holomorphic function. There exist unique holomorphic functions f_0 on $U \times D_R(0)$ and f_∞ on $U \times (\mathbb{C} \setminus \overline{D_r(0)})$ such that $f = f_0 + f_\infty$ and $\lim_{w \to \infty} f_\infty(z, w) = 0$.

Proof. We can write

$$f(z, w) = \frac{1}{2\pi i} \int_{|u|=R'} \frac{f(z,u)}{u-w} du - \frac{1}{2\pi i} \int_{|u|=r'} \frac{f(z,u)}{u-w} du$$

for $r' < |w| < R'$ which are holomorphic in z and w in $|w| < R'$ and $|w| > r'$, respectively. We can take

$$\widetilde{f}(z) = \begin{cases} f_0(z, w) & |w| < R \\ -f_\infty(z, w) & |w| > r \end{cases}$$

which is an entire function that goes to 0 as $w \to \infty$. Thus $\widetilde{f}$ is an entire function limiting to 0 as $w \to \infty$ showing uniqueness. ■

We can also count the number of zeroes with respect to w.

Proposition 13.14. Let $U \subseteq \mathbb{C}^n$ be open, $K \subseteq \mathbb{C}$ an annulus around the origin of internal and external radii r, R, respectively, and $f : U \times K \to \mathbb{C}$ a holomorphic function. If f has no zeroes in K then the number of zeroes of $f(z, w)$ as a function of w is independent of z.

Proof. The number of zeroes of $f(z, w)$ as a function of w is given by the integral

$$\frac{1}{2\pi i} \int_{\partial D_r(0)} \frac{f_w(z, w)}{f(z, w)} dw$$

which is continuous in z and integer-valued, and thus constant. ∎

We conclude with a statement highly reminiscent of the Weierstrass preparation lemma, which we will soon prove.

Proposition 13.15. Let $U \subseteq \mathbb{C}^n$ be open, $K \subseteq \mathbb{C}$ an annulus around the origin of internal and external radii r, R, respectively, and $f : U \times K \to \mathbb{C}$ a holomorphic function with k zeroes in w. If f has no zeroes in K then there exists a nonzero function c on U and a holomorphic function h on $U \times K$ such that

$$w^k e^{h(z,w)} = c(z) f(z, w).$$

Proof. For each $z \in U$, the function $w \mapsto \frac{f(z,w)}{w^k} = g(z, w)$ admits a holomorphic logarithm with respect to w. Thus $\frac{g_w(z,w)}{g(z,w)}$ has a holomorphic logarithm with respect and we have $h(z, w) = \int_u^w \frac{g_u(z,u)}{g(z,u)} du$ on $U \times K$ which is holomorphic on K and depends holomorphically on z, w.

We can take the derivative with respect to w given by

$$\frac{g(z, w)e^{h(z,w)} \frac{g_w(z,w)}{g(z,w)} - e^{h(z,w)} g_w(z, w)}{g(z, w)^2} = \frac{e^{h(z,w)}}{g(z, w)}$$

and thus for any $z \in U$ there is a constant $c(z)$ such that $e^{h(z,w)} = c(z)g(z, w)$ which on substituting $g(z, w) = \frac{f(z,w)}{w^k}$ yields the claim. ∎

14. Lecture 14 – 26th November 2024

We now show the Weierstrass preparation theorem.

Theorem 14.1 (Weierstrass Preparation)**.** Let $U \subseteq \mathbb{C}^n$ be open and $D_R(0) = D \subseteq \mathbb{C}$ a disc. If $f : U \times D \to \mathbb{C}$ is holomorphic of and of order k with respect to the coordinate w on D then there are functions $e(z,w)$ holomorphic nonvanishing on $U \times D$ and $\omega(z,w)$ holomorphic on $U \times D$ of the form $w^k + a_{k-1}(z)w^{k-1} + \cdots + a_1(z)w + a_0(z)$ such that $f(z,w) = e(z,w) \cdot \omega(z,w)$.

Proof. Take r such that $f|_{U\times K}$ on the annulus $K = K(r,R)$ satisfies the hypotheses for Proposition 13.15. As such, taking $h(z,w), c(z)$ such that $w^k e^{h(z,w)} = c(z) \cdot f(z,w)$. Now applying the Laurent decomposition Proposition 13.13 we get $h(z,w) = h_0(z,w) + h_\infty(z,w)$ with h_0, h_∞ holomorphic on $|w| < R, |w| > r$, respectively, and $\lim_{w\to\infty} h_\infty(z,w) = 0$.

We have

$$f(z,w) = \frac{w^k e^{h_0(z,w)+h_\infty(z,w)}}{c(z)} = \frac{w^k e^{h_0(z,w)}}{c(z)} e^{h_\infty(z,w)}.$$

Taking $e(z,w) = \frac{e^{h_0(z,w)}}{c(z)}$, we have

$$f(z,w) = e(z,w)e^{h_\infty(z,w)}.$$

Moreover, since $\lim_{w\to\infty} h_\infty(z,w) = 0$, $\lim_{w\to\infty} e^{h_\infty(z,w)} = 1$ for fixed z. We can thus write

$$e^{h_\infty(z,w)} = 1 + \sum_{m=1}^{\infty} a_m(z)w^{-m}$$

and decomposing the sum we have

$$\begin{aligned} w^k e^{h_\infty(z,w)} &= w^k \left(\sum_{m=1}^{k} a_m(z)w^{-m} \right) + w^k \cdot \sum_{m=k+1}^{\infty} a_m(z)w^{-m} \\ &= \omega(z,w) + \mathcal{R}_\infty(z,w) \end{aligned}$$

where the summand $\omega(z,w) = w^k \left(\sum_{m=1}^k a_m(z)w^{-m}\right)$ of the desired form. To prove the result, it suffices to show that $\mathcal{R}_\infty(z,w)$ is of the desired form.

Now we have

$$0 = \frac{f(z,w)}{e(z,w)} - \omega(z,w) - \mathcal{R}_\infty(z,w) = \mathcal{R}_0(z,w) - \mathcal{R}_\infty(z,w)$$

where by uniqueness of Laurent decompositions on the annulus as $\omega(z,w)$ has the same zeroes as f. ■

We now consider some special types of functions in the convergent power series ring $\mathbb{C}\{z_1, \ldots, z_n\}$.

Definition 14.2 (Regular of Fixed Order)**.** Let $f \in \mathbb{C}\{z_1, \ldots, z_n\}$. f is z_n-regular of order k if $f(0, z_n)$ is not identically zero and has a zero of order k at $z_n = 0$.

Definition 4.1

The Weierstrass theorem shows that zeroes of holomorphic functions are highly structured.

Lemma 14.3. If f a holomorphic function on an open neighborhood U around the origin and vanishing at the origin in $\mathbb{C}^{n+1}$, then there exists a linear change of coordinates T such that $f \circ T$ is z_n regular of some order. Lemma 4.5

Proof. Consider a complex line L on which f is not identically zero. Setting this line L as the w-coordinate, we get that f vanishes at $w = 0$. ■

Example 14.4. $f(z_1, z_2)$ is neither z_1 nor z_2-regular. But under the transformation $z_1 = u_1, z_2 = u_1 + u_2$, we have that $f(u_1, u_2) = u_1^2 + u_1 u_2$ is u_1-regular of order 2.

Lemma 14.5. Let f be a z_n-regular function of order k. There are constants $0 < r', 0 < r < R$ such that f converges on $D_{r'}(0) \times D_R(0) \subseteq \mathbb{C}^{n-1} \times \mathbb{C}$, has no zeroes outside the disc $D_{r'}(0) \times D_r(0) \subseteq \mathbb{C}^{n-1} \times \mathbb{C}$, and $f(z', z_n)$ is of order k with respect to z_n. Lemma 4.6

Proof. This is a direct application of Theorem 14.1 as the function $f(0, z_n)$ has an order k zero at the origin. ■

More generally, we can define Weierstrass polynomials as follows.

Definition 14.6 (Weierstrass Polynomial)**.** A Weierstrass polynomial of order k is a function in $(z_1, \dots, z_{n-1}, z_n) = (z', z_n)$ given by

$$\omega(z', z_n) = z_n^k + a_{k-1}(z') z_n^{k-1} + \cdots + a_1(z') z_n + a_0(z')$$

where $a_k(0) = 0$.

We will require the following result about z_n-regular functions of order k in the following exposition.

Theorem 14.7 (Weierstrass Formal)**.** Let f be a z_n-regular function of order k. For each $g \in \mathbb{C}\{z_1, \dots, z_n\}$, there exists a unique decomposition $g = qf + r$ where $q \in \mathbb{C}\{z_1, \dots, z_n\}$ and $r \in \mathbb{C}\{z_1, \dots, z_{n-1}\}[z_n]$ of degree smaller than k in z_n. Theorem 4.8

Proof. By the Weierstrass preparation theorem, we can write $f = e\omega$ for ω a Weierstrass polynomial of order k. Up to setting $g = g/e$, we can take f to be a Weierstrass polynomial. Now consider g/ω and let r, R, r' be such that g/ω is holomorphic on $D_{r'}(0) \times K(r, R)$, the product of a disc and annulus. By the Laurent decomposition proposition 13.13, we can write $\frac{g}{\omega} = q + h$ where q, h are holomorphic in $D_{r'}(0) \times D_R(0)$ and $D_{r'}(0) \times K(r, R)$, respectively. Now recalling ω is given as $z_n^k + a_{k-1}(z') z_n^{k-1} + \cdots + a_1(z') z_n + a_0(z')$, we have that for $h(z', z_n) = \sum_{m=1}^{\infty} c_m(z') z_n^{-m}$, the product ωh is given by

Once again denoting z' for the coordinates $z_1, \dots, z_{n-1}$.

$$\omega h = b_1(z') z_n^{k-1} + \cdots + b_0(z') + \sum_{m=1}^{\infty} b_m(z') z_n^{-m}.$$

Writing $r(z', z_n) = b_1(z') z_n^{k-1} + \cdots + b_0(z'), \mathcal{R}(z', z_n) = \sum_{m=1}^{\infty} b_m(z') z_n^{-m}$, we have $g = q\omega + r(z', z_n) + \mathcal{R}(z', z_n)$ and it remains to show the latter summand vanishes.

Rewriting this $0 = -g + q\omega + r(z', z_n) + \mathcal{R}(z', z_n)$, we have that $-g + q\omega + r$ is holomorphic on $D_{r'}(0) \times D_R(0)$ and $\mathcal{R}(z', z_n)$ is holomorphic outside $D_r(0)$ and hence the above is the Laurent decomposition of the zero function showing $\mathcal{R}(z', z_n)$ vanishes. Uniqueness of q, r follows from uniqueness of the Laurent decomposition. ■

15. Lecture 15 – 28th November 2024

We have the following strengthening of the Weierstrass preparation theorem in the cases where the function f lies in $\mathbb{C}\{z_1, \ldots, z_{n-1}\}[z_n]$.

Proposition 15.1. Let $U \subseteq \mathbb{C}^n$ be open and $D_R(0) = D \subseteq \mathbb{C}$ a disc and $f \in \mathbb{C}\{z_1, \ldots, z_{n-1}\}[z_n]$. If $f : U \times D \to \mathbb{C}$ is holomorphic of and of order k with respect to the coordinate w on D then there are functions $e(z, w) \in \mathbb{C}\{z_1, \ldots, z_{n-1}\}[z_n]$ holomorphic nonvanishing on $U \times D$ and $\omega(z, w)$ holomorphic on $U \times D$ of the form $w^k + a_{k-1}(z)w^{k-1} + \cdots + a_1(z)w + a_0(z)$ such that $f(z, w) = e(z, w) \cdot \omega(z, w)$.

Proof. By Theorems 14.1 and 14.7 we have $f = e\omega = q\omega + r$ so $r = 0, e = q$ with uniqueness and f is of the desired form. ■

We now consider some algebraic properties of the ring of convergent power series.

Theorem 15.2. The ring of convergent power series $\mathbb{C}\{z_1, \ldots, z_n\}$ is Noetherian.

Proof. We proceed by induction on the number of variables n. This holds for $n = 0$ as $\mathbb{C}$ is a field and hence Noetherian.

Now suppose $\mathbb{C}\{z_1, \ldots, z_{n-1}\}$ is Noetherian and thus $\mathbb{C}\{z_1, \ldots, z_{n-1}\}[z_n]$ is Noetherian. Consider $\mathfrak{a} \subseteq \mathbb{C}\{z_1, \ldots, z_{n-1}\}[z_n]$ a nonzero ideal and $f \in \mathfrak{a}$. By Lemma 14.5, we can change coordinates such that f is z_n-regular. Now for $g \in \mathbb{C}\{z_1, \ldots, z_n\}$, Theorem 14.7 implies that we can write $g = qf + r$ with $r \in \mathbb{C}\{z_1, \ldots, z_{n-1}\}[z_n]$ of degree less than k. This yields a homomorphism of $\mathbb{C}\{z_1, \ldots, z_{n-1}\}$-modules with kernel f given by $\mathbb{C}\{z_1, \ldots, z_n\} \mapsto \mathbb{C}\{z_1, \ldots, z_{n-1}\}^{\oplus k}$ by $g \mapsto (a_0(z'), \ldots, a_{k-1}(z'))$ where $r = a_{k-1}(z')z_n^{k-1} + \cdots + a_0(z')$. Let $\overline{\mathfrak{a}}$ be the image of $\mathfrak{a}$ under this map. Now noting that $\mathbb{C}\{z_1, \ldots, z_{n-1}\}^{\oplus k}$ is Noetherian as a module $\overline{\mathfrak{a}}$ is generated by finitely many elements $\overline{g_1}, \ldots, \overline{g_s}$, the images of $g_1, \ldots, g_s \in \mathfrak{a}$. But this shows that g is a linear combination of $g_1, \ldots, g_s, f$ for $g \in \mathfrak{a}$ so this forms a finite generating set of the ideal showing that the ring is Noetherian. ■

We can also show that the ring of convergent power series $\mathbb{C}\{z_1, \ldots, z_n\}$ is factorial, that is, a unique factorization domain. Doing so will require the following proposition.

Proposition 15.3. Let $U \subseteq \mathbb{C}^n$ open containing the origin and $f, g \in \mathcal{O}(U)$. If f_0, g_0 are relatively prime in $\mathcal{O}_0$ then there is a neighborhood V of the origin such that f_z, g_z are relatively prime for all $z \in V$.

In turn:

Theorem 15.4. The ring of convergent power series $\mathbb{C}\{z_1, \ldots, z_n\}$ is a unique factorization domain.

We now discuss meromorphic functions.

Definition 15.5 (Sheaf of Meromorphic Functions)**.** The sheaf of meromorphic functions $\mathcal{K}$ on $\mathbb{C}^n$ is given by $\coprod_{z \in \mathbb{C}^n} \mathcal{K}_z$ where $\mathcal{K}_z$ is the ring of germs of meromorphic functions at z.

Sections of $\mathcal{K}$ give rise to meromorphic functions in the same way as Definition 13.11.

16. Lecture 16 – 3rd December 2024

Recall that meromorphic functions are constructed locally, so a meromorphic function on an open set $U \subseteq \mathbb{C}^n$ as a section of the sheaf of Definition 15.5 is given by the data of triples $\{(U_i, f_i, g_i)\}_{i \in I}$ where $\{U_i\}_{i \in I}$ form an open cover of U, $f_i, g_i \in \mathcal{O}(U_i)$ with g_i not identically zero and satisfying the conditions that $(\frac{f_i}{g_i})_z = \frac{f_{i,z}}{g_{i,z}} \in \mathcal{K}_z$ for $z \in U_i$ and $f_i g_j = f_j g_i$ on $U_{ij} = U_i \cap U_j$. In particular, the value of a meromorphic function h is defined at z if its representative $\frac{f_z}{g_z}$ has nonvanishing denominator. This in turn allows us to define the polar set of a meromorphic function.

Definition 16.1 (Polar Set)**.** Let $U \subseteq \mathbb{C}^n$ be open and $h \in \mathcal{K}(U)$. The polar set M_h of h in U is the set of points of U on which h is undefined.

Remark 16.2. If h is already holomorphic, then its polar set M_h is empty.

We now define the notion of an analytic hypersurface and show the polar set of a single meromorphic function is an analytic hypersurface.

Definition 16.3 (Analytic Hypersurface)**.** Let $U \subseteq \mathbb{C}^n$ be open. An analytic hypersurface is the vanishing locus of a holomorphic function $f \in \mathcal{O}(U)$ in U.

We can now show the desired result.

Proposition 16.4. Let $U \subseteq \mathbb{C}^n$ be open and $h \in \mathcal{K}(U)$ be a meromorphic function. The polar M_h is an analytic hypersurface in U.

Proof. Without loss of generality, let $V \subseteq U$ be sufficiently small that $h = \frac{f}{g}$ with $f, g \in \mathcal{O}(V)$. We have that $M_h \cap V = \{g = 0\}$ since if f_z, g_z are relatively prime for all $z \in U$ and $a \in \{g = 0\}$ with h holomorphic at a implies $h_a g_a = f_a$, a contradiction to relatively primiality. ∎

We now discuss a number of extension and convexity properties which is well-known in the univariate setting using the tools of Hadamard factorization and Blaschke products. In particular, we seek to characterize the domains of existence of holomorphic functions on these domains, and to produce holomorphic and meromorphic functions with prescribed properties.

Definition 16.5 (Analytic Set)**.** Let $U \subseteq \mathbb{C}^n$ be open. $M \subseteq U$ is an analytic set if there exists a presentation of M as the vanishing locus of finitely many holomorphic functions.

Remark 16.6. In particular, M will be closed and for all $z \in M$ there is a neighborhood V and $f_1, \ldots, f_k \in \mathcal{O}(V)$ such that $V \cap M = \{f_1 = \cdots = f_k = 0\}$.

One of the most elementary examples of an extension proerty is the extension over analytic sets.

Theorem 16.7 (First Riemann Extension)**.** Let $U \subseteq \mathbb{C}^n$ be open, $M \subsetneq U$ an analytic set, and h holomorphic on $U \setminus M$ and locally bounded on U. Then h extends to a holomorphic function on all of U.

Proposition 1.1

Proof. Without loss of generality suppose $0 \in M$ and $h = \frac{f}{g}$ with $\{g = 0\} \subseteq M$ so g is a Weierstrass polynomial $\omega(z', z_n)$. Now consider the univariate function $z_n \mapsto h(z', z_n)$ for fixed z'. $\omega(z', z_n)$ has finitely many zeroes so $z_n \mapsto h(z', z_n)$ is bounded at these zeros and thus extends holomorphically over these zeroes given by Cauchy integrals that vary holomorphically in z'. ∎

We now introduce the Bochner-Martinelli kernel, which will play a key role in what follows.

Definition 16.8 (Bochner-Martinelli Kernel)**.** The Bochner-Martinelli kernel is given by Definition 1.2

$$\beta(w, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} \frac{1}{\|w_j - z_j\|^{2n}} \mathrm{d}\overline{w_1} \wedge \mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_{j-1} \wedge \mathrm{d}w_j \wedge \mathrm{d}\overline{w_{j+1}} \wedge \cdots \wedge \mathrm{d}\overline{w_n} \wedge \mathrm{d}w_n.$$

Remark 16.9. That is, in the expression of Definition 16.8, the wedge-factor $\mathrm{d}\overline{w_j}$ is omitted, making the Bochner-Martinelli kernel a form of type $(n, n-1, 0, 0)$ in the $\mathrm{d}w, \mathrm{d}\overline{w}, \mathrm{d}z, \mathrm{d}\overline{z}$'s, respectively.

We prove an elementary property of the Bochner-Martinelli kernel.

Lemma 16.10. Let $U \subseteq \mathbb{C}^n$ be a bounded open set with smooth boundary and $\tau \in U$ and f continuously differentiable on ∂U. Then $\mathrm{d}(f(w)\beta(w, z)) = \overline{\partial} f(w) \wedge \beta(w, z)$.

Proof. Note that $f(w)\beta(w, \tau)$ contain the holomorphic forms $\mathrm{d}w_j$ so we can compute

$$\begin{aligned}
\mathrm{d}(f(w)\beta(w,\tau)) &= \overline{\partial}(f(w)\beta(w,\tau)) \\
&= \overline{\partial} f(w) \wedge \beta(w,\tau) \\
&\qquad + f(w)\frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} \partial_{\overline{w_j}} \left(\frac{\overline{w_j} - \overline{\tau_j}}{\|w - \tau\|^{2n}} \right) \mathrm{d}\overline{w_1} \wedge \mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}\overline{w_n} \wedge \mathrm{d}w_n
\end{aligned}$$

but

$$\begin{aligned}
\sum_{j=1}^{n} \partial_{\overline{w_j}} \left(\frac{\overline{w_j} - \overline{\tau_j}}{\|w - \tau\|^{2n}} \right) &= \sum_{j=1}^{n} \left(\frac{1}{\|w - z\|^{2n}} - n \frac{|w_j - \tau_j|^2}{\|w - \tau\|^{2n+2}} \right) \\
&= 0
\end{aligned}$$

giving the claim. ∎

17. Lecture 17 – 5th December 2024

We prove the Bochner-Martinelli integral formula.

Theorem 17.1 (Bochner-Martinelli Integral Formula)**.** Let $G \subseteq \mathbb{C}^n$ be a bounded domain with smooth boundary. If f is a continuously differentiable function on the boundary ∂G then

$$f(\tau) = \int_{\partial G} f(w)\beta(w,\tau) - \int_G \overline{\partial} f(w) \wedge \beta(w,\tau)$$

Theorem 1.2 for all $\tau \in G$.

Proof. For fixed $\tau \in G$ and $r > 0$ sufficiently small that $\overline{B_r(\tau)} \subsetneq G$. We have by Stokes' theorem and Lemma 16.10

$$\begin{aligned}\int_{\partial G} f(w)\beta(w,\tau) - \int_{\partial B_r(\tau)} f(w)B(w,z) &= \int_{G\setminus\overline{B_r(\tau)}} \mathrm{d}(f(w)\beta(w,\tau)) \\ &= \int_{G\setminus\overline{B_r(\tau)}} \overline{\partial} f(w) \wedge \beta(w,\tau).\end{aligned}$$

Note further that

$$\lim_{r\to 0} \int_{G\setminus\overline{B_r(\tau)}} \overline{\partial} f(w) \wedge \beta(w,\tau) = \int_G \overline{\partial} f(w) \wedge \beta(w,\tau)$$

so we can rearrange the equations above to observe

$$\int_{\partial B_r(\tau)} f(w)\beta(w,\tau) = f(\tau)\int_{\partial B_r(\tau)} \beta(w,\tau) - \int_{\partial B_r(\tau)} (f(w)-f(\tau))\beta(w,\tau).$$

To prove the result, we show $\int_{\partial B_r(\tau)} \beta(w,\tau) = 1$ and that the second summand of the expression above vanishes.

For the first claim, we apply Stokes' theorem once more, and recall that the volume of $B_r(\tau)$ is $\frac{\pi^n}{n!} r^{2n}$ and compute

$$\begin{aligned}\int_{\partial B_r(\tau)} \beta(w,\tau) &= \int_{\partial B_r(\tau)} \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^n \frac{\overline{w_j}-\overline{z_j}}{\|w-\tau\|^{2n}} \mathrm{d}\overline{w_1} \wedge \mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_{j-1} \wedge \mathrm{d}w_j \wedge \mathrm{d}\overline{w_{j+1}} \wedge \cdots \wedge \mathrm{d}\overline{w_n} \wedge \mathrm{d}w_n \\ &= \frac{(n-1)!}{r^{2n}(2\pi i)^n} \int_{\partial B_r(\tau)} \sum_{j=1}^n (\overline{w_j}-\overline{z_j}) \mathrm{d}\overline{w_1} \wedge \mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_{j-1} \wedge \mathrm{d}w_j \wedge \mathrm{d}\overline{w_{j+1}} \wedge \cdots \wedge \mathrm{d}\overline{w_n} \wedge \mathrm{d}w_n \\ &= \frac{(n-1)!}{r^{2n}(2\pi i)^n} \int_{B_r(\tau)} \mathrm{d}\left(\sum_{j=1}^n (\overline{w_j}-\overline{z_j}) \mathrm{d}\overline{w_1} \wedge \mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_{j-1} \wedge \mathrm{d}w_j \wedge \mathrm{d}\overline{w_{j+1}} \wedge \cdots \wedge \mathrm{d}\overline{w_n} \wedge \mathrm{d}w_n\right) \\ &= \frac{(n-1)!}{r^{2n}(2\pi i)^n} \int_{B_r(\tau)} n \cdot \mathrm{d}\overline{w_1} \wedge \mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_j \wedge \mathrm{d}\overline{w_{j+1}} \\ &= \frac{(n-1)!}{r^{2n}(2\pi i)^n} \int_{B_r(\tau)} n(2i)^n \mathrm{d}V \\ &= 1\end{aligned}$$

For the second claim, the computation above yields

$$\frac{(n-1)!}{r^{2n}(2\pi i)^n}\left(\int_{B_r(\tau)} n\cdot \mathrm{d}\overline{w_1}\wedge \mathrm{d}w_1\wedge\cdots\wedge \mathrm{d}w_n\wedge \mathrm{d}\overline{w_n}\right) + \frac{(n-1)!}{r^{2n}(2\pi i)^n}\int_{B_r(\tau)}\sum_{j=1}\partial_{\overline{w_j}}f(w)(\overline{w_j}-\overline{\tau_j})\cdot \mathrm{d}\overline{w_1}\wedge \mathrm{d}w_1\wedge\cdots\wedge \mathrm{d}w_n\wedge \mathrm{d}\overline{w_n}.$$

By boundedness of U, $|f(w)-f(\tau)|\leq C\|w-\tau\|$ and $|\partial_{\overline{w_j}}f(w)(\overline{w_j}-\overline{z_j})|\leq C\|w-z\|$ for some large constant C. So for all $w\in\partial B_r(\tau)$, the quantities $|f(w)-f(\tau)|, |\partial_{\overline{w_j}}f(w)(\overline{w_j}-\overline{z_j})|$ are bounded above by Cr so

$$\left|\int_{\partial B_r(\tau)}(f(w)-f(\tau))\beta(w,\tau)\right|\leq \frac{(n-1)!}{r^{2n}(2\pi)^n}\left(2\int_{B_r(\tau)} n\cdot 2^n Cr\mathrm{d}V\right) = 2Cr$$

which vanishes as $r\to 0$ proving the claim. ■

Remark 17.2. Note that the Bochner-Martinelli kernel is not a holomorphic form.

The Bochner-Martinelli integral formula allows us to prove Hartogs' so-called "Kugelsatz," allowing us to extend holomorphic functions over compact domains. For this, we will require the following lemmata, only one of which we prove.

Lemma 17.3. Let $G\subseteq\mathbb{C}^n$ be a domain and $\beta(w,z)$ the Bochner-Martinelli kernel on G and f continuously differentiable on ∂G. Then $\overline{\partial}_z\beta(w,z) = -\overline{\partial}_w\beta_1(w,z)$ where Lemma 1.5

$$\beta_1(w,z) = (n-1)\beta(w,z)\wedge(\overline{\partial}_w\beta)^{n-2}\wedge\overline{\partial}_z\beta\cdot\left(\frac{1}{2\pi i}\right)^n\frac{1}{\|w-z\|^{2n}}.$$

Proof. See [Ran86, §4, Prop. 4.9]. ■

Lemma 17.4. Let $G\subseteq\mathbb{C}^n$ and f holomorphic on ∂G. Then

$$F(z) = \int_{\partial G} f(w)B(w,z)$$

is holomorphic in z for $z\notin\partial G$. Lemma 1.6

Proof. We compute

$$\begin{aligned}\overline{\partial}_z F(z) &= \int_{\partial G} f(w)\overline{\partial}_z B(w,z)\\ &= -\int_{\partial G} f(w)\overline{\partial}_w\beta_1(w,z)\\ &= -\int_{\partial G}\overline{\partial}_w(f(w)\beta_1(w,z))\\ &= -\int_{\partial G}\mathrm{d}_w(f(w)\beta_1(w,z))\\ &= 0\end{aligned}$$

as desired. ■

In turn:

Theorem 17.5 (Hartogs' Kugelsatz)**.** Let $U \subseteq \mathbb{C}^n$ be an open, $K \subseteq U$ compact, and $U \setminus K$ connected. If $n \geq 2$ and f is holomorphic on $U \setminus K$ then f extends holomorphically to U.

Proof. Let $G \subsetneq U$ be a domain and G_0 a subdomain of G fully containing K. By hypothesis f is holomorphic in $G \setminus \overline{G_0}$ and by the Bochner-Martinelli integral formula Theorem 17.1, we have

$$f(z) = \int_{\partial G} f(w)\beta(w,z) - \int_{\partial G_0} f(w)\beta(w,z).$$

Denoting the integrals $F_1(z), F_2(z)$, respectively, F_1 is holomorphic for all $z \notin \partial G$ by Lemma 17.4 and F_2 holomorphic for all $z \notin G_0$. In particular, F_1 is also holomorphic on all of G and thus holomorphic in K and $F_2(z) \to 0$ as $|z| \to \infty$.

Since $n \geq 2$, there exist hyperplanes that do not meet G_0 on which F_2 is zero, showing F_2 is zero by the identity theorem Theorem 10.12 and F_1 is a holomorphic extension of f since it coincides with f on $G \setminus \overline{G_0}$ once again using the identity theorem. ∎

We now return to a consideration of the multivariate Cauchy-Riemann equations, in particular considering their role in producing holomorphic functions corresponding to fixed differential form data.

The motivating problem is the existence of holomorphic functions satisfying Cousin-I distributions.

Definition 17.6 (Cousin-I Distribution)**.** Let $U \subseteq \mathbb{C}^n$ be open. A Cousin-I distribution on U is given by $\{(U_i, f_i)\}_{i \in I}$ where $\{U_i\}_{i \in I}$ form an open cover of U, $f_i \in \mathcal{K}(U_i)$ holomorphic such that $f_i - f_j$ is holomorphic on $U_{ij} = U_i \cap U_j$.

A solution to the Cousin-I problem is a meromorphic function compatible with the data of a Cousin-I distribution in the following sense.

Definition 17.7 (Solution to Cousin-I Problem)**.** Let $U \subseteq \mathbb{C}^n$ be open and $\{(U_i, f_i)\}_{i \in I}$ be a Cousin-I distribution on U. A solution to the Cousin-I problem is a meromorphic function $f \in \mathcal{K}(U)$ such that $f|_{U_i} - f_i$ is holomorphic for all $i \in I$.

Solutions to problems of this type are constructed by finidng a smooth solution and "altering" it to a holomorphic solution by solving a $\overline{\partial}$-equation.

We discuss a sequence of reductions for this problem, first recalling the following notions from the theory of smooth manifolds.

Definition 17.8 (Partition of Unity)**.** Let $U \subseteq \mathbb{R}^n$ be open and $\{U_i\}_{i \in I}$ a locally finite open cover of U such that each U_i is relatively compact in U. A partition of unity $\{\psi_i\}_{i \in I}$ subordinate to the cover $\{U_i\}_{i \in I}$ is the data of smooth functions $\psi_i : U_i \to \mathbb{R}$ such that $\mathrm{supp}(\psi_i) \subseteq U_i$ and $\sum_{i \in I} \psi_i(x) = 1$ for all $x \in U$.

Partitions of unity exist for open sets of $\mathbb{C}^n$.

Proposition 17.9. Let $U \subseteq \mathbb{R}^n$ be open and $\{U_i\}_{i \in I}$ a locally finite open cover of U such that each U_i is relatively compact in U. There exists a partition of unity $\{\psi_i\}_{i \in I}$ subordinate to the cover $\{U_i\}_{i \in I}$.

A proof of Proposition 17.9 can be found in [Lee13, §2]. Note in particular that holomorphic phenomena are in particular smooth which justifies their use in what follows.

Proposition 17.10. Let $U \subseteq \mathbb{C}^n$ be open and $\{(U_i, f_i)\}_{i\in I}, \{(V_j, g_j)\}_{j\in J}$ be Cousin-I distributions such that their union is a Cousin-I distribution. If $f \in \mathcal{K}(U)$ is a solution to the Cousin-I distribution $\{(U_i, f_i)\}_{i\in I}$ then f is a solution to the Cousin-I distribution $\{(V_j, g_j)\}_{j\in J}$ as well.

Proof. Let $f \in \mathcal{K}(U)$ be a solution to the Cousin-I problem for the distribution $\{(U_i, f_i)\}_{i\in I}$ and $\{\psi_j\}_{j\in J}$ be a partition of unity subordinate to the cover $\{V_j\}_{j\in J}$. We want to show that $f|_{V_j} - g_j \in \mathcal{O}(V_j)$ for all $j \in J$. For this, we note that V_j admits a cover by $\{U_i \cap V_j\}_{i\in I}$ on which $f_i|_{U_i\cap V_j} - g_j|_{U_i\cap V_j}$ is holomorphic so $f_i\psi_j - g_j$ is holomorphic on all V_j and similarly $f - f_i|_{U_i\cap V_j}$ is holomorphic for all $U_i \cap V_j$ and all $i \in I$ with fixed $j \in J$ which extends holomorphically to $f|_{V_j} - f_i\psi_j$ on V_j But their sum is $f|_{V_j} - g_j$ and holomorphic as it is the finite sum of holomorphic functions by local finiteness of the cover. ■

This motivates the following definition.

Definition 17.11 (Equivalent Cousin-I Distributions)**.** Let $U \subseteq \mathbb{C}^n$ be open and $\{(U_i, f_i)\}_{i\in I}, \{(V_j, g_j)\}_{j\in J}$ be Cousin-I distributions. $\{(U_i, f_i)\}_{i\in I}, \{(V_j, g_j)\}_{j\in J}$ are equivalent Cousin-I distributions if their union is a Cousin-I distribution.

This reduces solving the Cousin-I problem to the following.

Proposition 17.12. Let $U \subseteq \mathbb{C}^n$ be open and $f \in C^\infty(U)$, $f = \sum_{j=1}^n f_j \mathrm{d}\overline{z_j}$, and there exists u a smooth function such that $\overline{\partial} u = f$ and $\overline{\partial} f = 0$. Then the Cousin-I problem admits a solution. Proposition 1

Proof. Let $\{(U_i, f_i)\}_{i\in I}$ be a Cousin-I distribution as above and set $g_i = \sum_{j\in I} \phi_j f_{ji}$ which is defined on U_i. Note that on U_{ik} the difference $g_i - g_k$ is given by f_{ik} as expected and that $\overline{\partial} g_i - \overline{\partial} g_j = 0$. We can define function f as in the hypothesis of the theorem by taking $f|_{U_i} = \overline{\partial} g_i$ which statisfies the hypothesis by inspection and admits a solution u to the Cousin-I problem. ■

18. Lecture 18 – 10th December 2024

We first show that problems of the type in Proposition 17.12 can be solved. These are known as distributions of type $(0,1)$ since it is a linear combination of degree 1 antiholomorphic forms.

Theorem 18.1. Let $G \subseteq \mathbb{C}^n$ be a bounded domain with smooth boundary for $n \geq 2$ and $f = \sum_{j=1}^n f_j \mathrm{d}\overline{z_j}$ such that $\overline{\partial} f = 0$. Then

$$u(z) = -\int_G f(w) \wedge B(w,z)$$

is smooth and satisfies $\overline{\partial} u = f$.

Proof. By boundedness of G, the integral defining u exists and is given by a $\mathbb{C}$-linear combination of

$$u_j(z) = \int_G \frac{f_j(w)(\overline{w_j} - \overline{z_j})}{\|w-z\|^{2n}} \mathrm{d}V(w) = \int_G \frac{f_j(w)(z+y)\overline{y_j}}{\|y\|^{2n}} \mathrm{d}V(y)$$

with the second equality following by translation and denoting $y = w - z$. Now letting L_z be a differential operator with constant coefficients, we have

$$L_z u = \int_G \frac{L_z f_j(z+y)\overline{y_j}}{\|y\|^{2n}} \mathrm{d}V(w) = \int_G \frac{L_w f_j(w)(\overline{w_j} - \overline{z_j})}{\|w-z\|^{2n}} \mathrm{d}V(w).$$

Taking $u(z) = -\int_G f(w) \wedge B(w,z)$, we have $u(z)$ as a $\mathbb{C}$-linear combination of the u_j's and thus u is smooth. It remains to show that $\overline{\partial} u = f$ for which it suffices that $\partial_{\overline{z_j}} u = f_j$. But we get

$$\begin{aligned}
\partial_{\overline{z_1}} u &= -\int_G \partial_{\overline{w_1}} f \wedge B(w,z) \\
&= -\int_G (\partial_{\overline{w_1}} f_1 \cdot \mathrm{d}\overline{w_1} + \cdots + \partial_{\overline{w_1}} f_1 \cdot \mathrm{d}\overline{w_n}) \wedge B(w,z) \\
&= -\int \overline{\partial} f_1 \wedge B(w,z) \\
&= f_1(z)
\end{aligned}$$

with the final equality following from Theorem 17.1. Analogous computations hold for $\partial_{\overline{z_j}} u$. Moreover, u is holomorphic outside a large ball so $u \to 0$ as $\|z\| \to \infty$ so u vanishes identically away from the origin, showing compact support of u. ■

This result in fact implies Theorem 17.5 and shows that Theorem 18.1 which does not hold for $n = 1$.

19. Lecture 19 – 12th December 2024

We want to consider the $\overline{\partial}$-equation more generally.

Theorem 19.1. Let $G \subseteq \mathbb{C}^n$ be a bounded domain with smooth boundary, $f \in C^1(\overline{G})$, and

$$u(z) = \frac{1}{2\pi i}\int_G \frac{f(w)}{w-z}\mathrm{d}w \wedge \mathrm{d}z.$$

Then $\overline{\partial} u = f$ on G. Theorem 3.2

Proof. Let $\tau \in G$ and $D_r(\tau) \subseteq G$. Consider an indicator function $\chi : G \to \mathbb{R}$ such that

$$\chi(z) = \begin{cases} 1 & z \in D_r(\tau) \\ 0 & z \notin U \end{cases}$$

for U an open neighborhood of $\overline{D_r(\tau)}$. We then compute

$$u(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\chi(w)f(w)}{w-z}\mathrm{d}w \wedge \mathrm{d}\overline{w} + \frac{1}{2\pi i}\int_G \frac{(1-\chi(w))f(w)}{w-z}\mathrm{d}w \wedge \mathrm{d}\overline{w}.$$

by Theorem 17.1. In particular the second summand is holomorphic in z. Letting teh first summand be $u_1(z)$ and $z \in D_r(\tau)$ we have $\overline{\partial} u = \overline{\partial} u_1 = \chi(z)f(z) = f(z)$ on G. ■

We now show that each form of type $(0,q)$ arises as the antiholomorphic differential of a form of type $(0, q-1)$. This is the statement of the Dolbeault lemma. We proceed using the following lemma.

Lemma 19.2. Let $D_r(\tau)$ be a polydisc and f a smooth $(0,q)$-form on $D_r(\tau)$ such that $\overline{\partial} f = 0$. Then there exists a smooth $(0, q-1)$-form u on $D_{r'}(\tau)$ with $r' < r$ having relatively compact closure such that $\overline{\partial} u = f$.

Proof. Let $D_{r''}(\tau) \subseteq D_{r'}(\tau)$ and $D_{r'}(\tau) \subseteq D_r(\tau)$ be polydiscs with relatively compact closure. Define E_k to be the set of smooth $(0,s)$-forms

$$E_k = \left\{ f = \sum_{|J|=s} f_J \mathrm{d}\overline{z}^J : f_J = 0 \text{ if } l \in J, l > k \right\}.$$

We induct over k.

Note if $f \in E_0$ then f is identically 0 and each form lies in E_n. For the base case, observe that if f is a $(0,1)$-form on $D_{r'}(\tau)$ with $\overline{\partial} f = 0$ wtih $f \in E_0$ we are done. Now suppose the assumption holds for E_{k-1} where we can write

$$f = \sum_{|J|=s} f_J \mathrm{d}\overline{z}^J = \mathrm{d}\overline{z_k} \wedge g + h$$

with $g, h \in E_{k-1}$. We can compute to observe $0 = \overline{\partial} f = -\mathrm{d}\overline{z_k} = -\mathrm{d}\overline{z}^k \wedge \overline{\partial} g + \overline{\partial} h$ where in particular $\mathrm{d}\overline{z_k} \wedge \overline{\partial} g = \overline{\partial} h$. Now let

$$g = \sum_{|T|=q-1} g_T \mathrm{d}\overline{z}^T.$$

We have that g_T are holomorphic in $z_{k+1}, \dots, z_n$ since if $\partial_{\overline{z_l}} g_T \neq 0$ or some $l > k$ then $\mathrm{d}\overline{z_l}$ is one of the summands of $\overline{\partial} g$ and thus $\mathrm{d}\overline{z}^K \wedge \overline{\partial} g$ has $\mathrm{d}\overline{z}^K \wedge \mathrm{d}\overline{z_l}$ as a summand but $\overline{\partial} h$ does not, giving the claim. Now using Theorem 19.1, we can set $\widetilde{g}_T$ to be the $(0,1)$-forms for the problem $\overline{\partial}\widetilde{g}_T = g_T$ which are holomorphic in $z_{k+1}, \dots, z_n$ by the preceding discussion. We then set

$$\widetilde{g} = \sum_{|T|=s-1} \widetilde{g}_T \mathrm{d}\overline{z}^T$$

and compute

$$\begin{aligned}\overline{\partial}\widetilde{g} &= \sum_{|T|=s-1} \sum_{l=1}^{n} \partial_{\overline{z_l}} \widetilde{g}_T \cdot \mathrm{d}\overline{z_l} \wedge \mathrm{d}\overline{z}^T \\ &= h_1 + \overline{\partial}\overline{z_k} \wedge g\end{aligned}$$

with $g, h \in E_{k-1}$ so taking u such that $\overline{\partial} u = h - h_1$ we have $f = \overline{\partial} g + \overline{\partial} u$ as desired. ■

We can now show the Dolbeault lemma in general.

Theorem 3.3

Theorem 19.3 (Dolbeault Lemma)**.** Let $D_r(\tau) \subseteq \mathbb{C}^n$ be a polydisc, f a smooth $(0,q)$-form on $D_r(\tau)$ such that $\overline{\partial} f = 0$. Then there exists a smooth $(0, q-1)$-form u on $D_r(\tau)$ such that $\overline{\partial} u = f$.

Proof. Let $D_{r_0}(\tau) \subseteq D_{r_1}(\tau) \subseteq \dots$ be a sequence of polydiscs each $D_{r_t}(\tau)$ having compact closure in $D_{r_{t+1}}(\tau)$ and $D_r(\tau) = \bigcup_{t=0}^{\infty} D_{r_t}(\tau)$. We proceed by induction on the index t.

By Lemma 19.2, there exist functions u_0 smooth on $D_{r_0}(\tau)$ satisfying $\overline{\partial} u_0 = f$ on $D_{r_0}(\tau)$. Now suppose that there are functions $u_0, \dots, u_l$ which are $(0, q-1)$-forms on $D_{r_\lambda}(\tau)$ for $0 \leq \lambda \leq l$ such that $\overline{\partial} u_\lambda = f$ on $D_{r_\lambda}(\tau)$ and $u_\lambda = u_{\lambda-1}$ on $D_{r_{\lambda-2}}(\tau)$. We can then construct u'_{l+1} on $D_{r_{l+1}}(\tau)$ such that $\overline{\partial} u'_{l+1} = f, \overline{\partial}(u'_{l+1} - u_l) = 0$ on $D_{r_l}(\tau)$.

We can thus find h on $D_{r_{l-1}}(\tau)$ such that $\overline{\partial} h = u'_{l+1} + u_l$ with h smooth on $D_{r_{l+1}}(\tau)$ and taking $u_{l+1} = u'_{l+1} - \overline{\partial} h$ we have $\overline{\partial} u_{l+1} = f$. We can then take $u = \lim_{l \to \infty} u_l$ which is a solution on all of $D_r(\tau)$. The case $q = 1$ follows by estimating using the Taylor development and bounding the growth differences on polydiscs. ■

Appendix A. Basic Results in Complex Analysis

In this appendix, we collect some basic results of complex analysis, largely following the text of Stein and Shakarchi [SS03].

Theorem A.1 (Identity)**.** Let $D \subseteq \mathbb{C}$ be a domain and f, g holomorphic functions on D. If the set $\{z \in D : f(z) = g(z)\}$ contains an accumulation point, then $f(z) = g(z)$ for all $z \in D$.

Theorem A.2 (Liouville)**.** If $f(z)$ is a holomorphic function such that $|f(z)| \leq M$ for some $M \in \mathbb{R}_{\geq 0}$ then f is constant.

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