V4A1 – ALGEBRAIC GEOMETRY I
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Preliminaries

These notes roughly correspond to the course **V4A1 – Algebraic Geometry I** taught by Prof. Daniel Huybrechts at the Universität Bonn in the Winter 2024/25 semester. These notes are LaTeX-ed after the fact with significant alteration and are subject to misinterpretation and mistranscription. Use with caution. Any errors are undoubtedly my own and any virtues that could be ascribed to these notes ought be attributed to the instructor and not the typist. Knowledge of commutative algebra, topology, and category theory will be assumed.

Contents

1. Lecture 1 – 7th October 2024

All rings are to be taken as commutative.

This will be a lecture course on algebraic geometry based on the abstract perspective of sheaves and schemes. The basic construction to be introduced is sheaves, which are defined as presheaves that satisfy additional properties. Sheaves keep track of both local and global information on topological spaces. We first consider the case of presheaves.

Sheaves can in fact be defined more abstractly using sites and Grothendieck topologies.

Definition 1.1 (Presheaf)**.** Let C be a category and X a topological space. A C-valued presheaf $\mathcal{F}$ on X consists of the data of

(i) An object $\mathcal{F}(U)$ of C for each $U \subseteq X$ open.
(ii) A morphism $\mathrm{res}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ in C for each $V \subseteq U \subseteq X$ open.

such that $\mathrm{res}_{U,U} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity on $\mathcal{F}(U)$ and the triangle

$$
\begin{tikzcd}
\mathcal{F}(U) \arrow[rr, "\mathrm{res}_{U,V}"] \arrow[dr, "\mathrm{res}_{U,W}"'] & & \mathcal{F}(V) \arrow[dl, "\mathrm{res}_{V,W}"] \\
& \mathcal{F}(W) &
\end{tikzcd}
\tag{1.1}
$$

is commutative for all $W \subseteq V \subseteq U \subseteq X$ open.

Remark 1.2. Commutativity of the triangle is equivalent to $\mathrm{res}_{U,W} = \mathrm{res}_{V,W} \circ \mathrm{res}_{U,V}$ as morphisms in C.

We will typically consider examples where C is the category of Abelian groups AbGrp, of rings Ring, of modules over a fixed ring A Mod_A and their appropriate generalizations in the setting of schemes. While it is typically bad categorical practice to look "within" objects of a category, we refer to elements of $\mathcal{F}(U)$ for $U \subseteq X$ open as (local) sections, with global sections being elements of $\mathcal{F}(X)$.

We will also use the notation $\Gamma(U, \mathcal{F})$ and $H^0(U, \mathcal{F})$ for $\mathcal{F}(U)$.

Example 1.3. Let X be a topological space. C_X which associates to $U \subseteq X$ open the set $C_X(U)$ of continuous functions from U to $\mathbb{R}$ and restriction maps given by restriction of continuous functions defines a presheaf on X.

Example 1.4. Let X, Y be topological spaces. $\mathcal{F}$ which associates $U \subseteq X$ to the set $\mathcal{F}(U)$ of continuous functions from U to Y and restriction maps given by restriction of continuous functions defines a presheaf on X.

Example 1.5. Let X be a topological space and G an Abelian group. We can consider the association taking each open set $U \subseteq X$ to G and restriction maps the identities. This construction defines a presheaf on X.

We can alternatively define presheaves as a type of functor from the category of open sets of a topological space to C. Recall here the definition of the category of open sets of a topological space Opens_X.

Definition 1.6 (Category of Open Sets)**.** Let X be a topological space. The category Opens_X of open sets of X has objects open sets of X and morphisms inclusions.

Remark 1.7. Spelling things out, for U, V open in X there is one morphism from V to U if $V \subseteq U$ and no morphism otherwise.

We can show that C-valued presheaves are merely contravariant functors from Opens_X to C, which is essentially a formal result.

Proposition 1.8. Let X be a topological space. The data of a presheaf on X corresponds uniquely to a functor $\mathsf{Opens}_X^{\mathsf{Opp}} \to \mathsf{C}$.

Proof. This statement follows by unwinding the definitions of functoriality. A functor associates $\mathsf{Opens}_X^{\mathsf{Opp}} \to \mathsf{C}$ associates to each open set of X an object of C and by construction of the morphisms in Opens_X there is a morphism $V \to U$ if and only if $V \subseteq U$ corresponding to a morphism $U \to V$ in $\mathsf{Opens}_X^{\mathsf{Opp}}$ inducing the corresponding restriction maps in C by functoriality. Furthermore, commutativity of the triangle is preserved since the action of functors on morphisms distributes over composition and composition is unqiue. ■

One example of special interest in the study of algebraic geometry is the construction of a canonical presheaf on the spectrum of a commutative ring A.

Definition 1.9 (Spectrum of a Ring)**.** Let A be a ring. The spectrum $\mathrm{Spec}(A)$ of A is the topological space with points given by prime ideals of A and closed sets of the form $V(a)$ for $a \in A$ consisting of the prime ideals containing a.

Note that no prime ideals contain the multiplicative unit of A so for $a_1, \ldots, a_n$ generating A as an A-module, the sets $D(a_i) = \mathrm{Spec}(A) \setminus V(a_i)$ consisting of the prime ideals not containing A form a basis for the toplogy of $\mathrm{Spec}(A)$.

We can now consider the following construction.

Proposition 1.10. Let A be a ring. Consider the association

$$U \mapsto \left\{ s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} : {}^{\forall \mathfrak{p} \in U, s(\mathfrak{p}) \in A_{\mathfrak{p}} \text{ and}}_{\exists U' \subseteq X, p \in U' \subseteq U, \exists a,b \in A \text{ s.t. } b \notin \mathfrak{q} \forall \mathfrak{q} \in U', s(\mathfrak{q}) = \frac{a}{b} \in A_{\mathfrak{q}}} \right\} \tag{1.2}$$

and for $V \subseteq U \subseteq X$ the forgetful maps. This association defines a presheaf of rings on $\mathrm{Spec}(A)$.

This defines a presheaf in terms of compatible stalks following [Har83, §2.2], which to me seems more artificial than in terms of localizations as in [Vak24, §4.1]

Proof. The category of rings is bicomplete, in particular admitting colimits and thus coproducts giving the sets of (1.2) the structure of a ring by pointwise operations. We get forgetful maps

$$\left\{ s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} : {}^{\forall \mathfrak{p} \in U, s(\mathfrak{p}) \in A_{\mathfrak{p}} \text{ and}}_{\exists V \subseteq X, p \in U' \subseteq U, \exists a,b \in A \text{ s.t. } b \notin \mathfrak{q} \forall \mathfrak{q} \in U', s(\mathfrak{q}) = \frac{a}{b} \in A_{\mathfrak{q}}} \right\}$$
$$\to \left\{ s|_V : V \to \coprod_{\mathfrak{p} \in V} A_{\mathfrak{p}} : {}^{\forall \mathfrak{p} \in U', s(\mathfrak{p}) \in A_{\mathfrak{p}} \text{ and}}_{\exists U' \subseteq X, p \in U' \subseteq U, \exists a,b \in A \text{ s.t. } b \notin \mathfrak{q} \forall \mathfrak{q} \in U', s(\mathfrak{q}) = \frac{a}{b} \in A_{\mathfrak{q}}} \right\}$$

for $U' \subseteq U$ that satisfy the commutativity of the diagram (1.1) and hence defines a presheaf. ■

We will set this as the structure presheaf $\mathcal{O}_{\mathrm{Spec}(A)}$ of $\mathrm{Spec}(A)$ since it captures important geometric phenomena that we seek to capture as motivated by the following example.

Example 1.11. Let k be an algebraically closed field and A a finite type k-algebra. Consider an open $U \subseteq \mathrm{Spec}(A)$ and $s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ as in (1.2). We can construct $\overline{s} : U \cap \mathrm{mSpec}(A) \to k$ whose value on each maximal ideal $\mathfrak{m}$ is the image of $s(\mathfrak{m}) \in A_{\mathfrak{m}}$ in the quotient $A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}$ which is isomorphic to k since k is algebraically closed. s is regular on U and thus the association of (1.2) can be thought of as a presheaf of regular functions.

Having discussed presheaves, we can define morphisms of presheaves as follows.

Definition 1.12 (Morphism of Presheaves)**.** Let X be a topological space and $\mathcal{F}, \mathcal{F}'$ C-valued presheaves on X. A morphism of presheaves $\phi : \mathcal{F} \to \mathcal{F}'$ consists of the data of morphisms $\phi_U : \mathcal{F}(U) \to \mathcal{F}'(U)$ in C such that the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi_U"] \arrow[d, "\mathrm{res}_{U,V}"'] & \mathcal{F}'(U) \arrow[d, "\mathrm{res}'_{U,V}"] \\ \mathcal{F}(V) \arrow[r, "\phi_V"'] & \mathcal{F}'(V) \end{tikzcd}$$

is commutative for $V \subseteq U \subseteq X$ open.

Remark 1.13. A morphism of presheaves is equivalent to the data of a natural transformation between the corresponding functors $\mathsf{Opens}_X^{\mathsf{Opp}} \to \mathsf{C}$.

This data allows us to define the category of C-valued presheaves $\mathsf{PSh}(X, \mathsf{C})$ for a topological space X, where we omit the "value-category" C when it is clear from context.

Example 1.14. In analogy to Example 1.3, let X be a topological space, C_X the presheaf of continuous functions on X, and C_X^{diff} the presheaf of differentiable functions on X. There is a natural morphism of presheaves $C_X^{\mathrm{diff}} \to C_X$ induced by the inclusions of differentiable into continuous functions on each open set.

Example 1.15. Let X be a topological space and S^1 the circle. We can define a presheaf of continuous functions valued in S^1 by exponentiating a locally continuous function $e^{2\pi i f}$ for f locally continuous. This defines a morphism of presheaves C_X to S^1-valued presheaves in analogue to Example 1.4 with $Y = S^1$.

Having discussed presheaves in some depth, we can now specialize to the case of sheaves, which are presheaves satisfying additional structure.

Definition 1.16 (Sheaves)**.** Let X be a topological space and C be a category admitting arbitrary products. A presheaf $\mathcal{F}$ on X is a sheaf if for all $U \subseteq X$ and all open covers $\{U_i\}_{i \in I}$ the sequence

See also the discussion in [Stacks, Tag 00VL].

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

is an equalizer where the parallel maps are given by

$$(s_i)_{i\in I} \mapsto (\mathrm{res}_{U_i,U_i\cap U_j}(s_i))_{i,j\in I} \text{ and } (s_i)_{i\in I} \mapsto (\mathrm{res}_{U_j,U_i\cap U_j}(s_j))_{i,j\in I},$$

respectively.

Remark 1.17. If C is an Abelian category, the equalizer condition can be replaced by $\mathcal{F}(U)$ being the kernel of $\prod_{i\in I}\mathcal{F}(U_i) \to \prod_{i,j\in I}\mathcal{F}(U_i\cap U_j)$ by $(s_i)_{i\in I} \mapsto (s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{i,j\in I}$.

Explicitly, the local behavior states that if there is an open set $U \subseteq X$ with an open cover $\{U_i\}_{i\in I}$, a collection of sections $(s_i)_{i\in I}$ glues to a section on U if and only if $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$. This proves to be the quality of sheaves that makes them more geometric than presheaves: while functions restrict naturally, they also glue naturally which is behavior captured by sheaves but not by presheaves.

Example 1.18. Let U be a topological space and $\{U_1, U_2\}$ an open cover of U. Consider continous functions from U to $\mathbb{R}$. The sheaf axiom allows for the construction of a continuous function $f : U \to \mathbb{R}$ from continuous functions $f_1 : U_1 \to R$ and $f_2 : U_2 \to \mathbb{R}$ that agree on the intersection $U_1 \cap U_2$.

In particular, we can see that Example 1.3 and Proposition 1.10 in fact define sheaves on X and $\mathrm{Spec}(A)$, respectively.

2. Lecture 2 – 11th October 2024

Previously, morphisms of presheaves and sheaves were defined in Definitions 1.12 and 1.16. In this way, being a sheaf is a structure on presheaves. This data determines the category of sheaves as we now define.

See [nLab-I] and the discussion therein.

Definition 2.1 (Category of Sheaves)**.** Let X be a topological space. The category $\mathsf{Sh}(X)$ of sheaves on X has objects sheaves and morphisms presheaf morphisms between them.

Remark 2.2. Since morphisms of sheaves are morphisms of the underlying presheaves, the forgetful functor $\mathsf{Sh}(X) \to \mathsf{PSh}(X)$ exhibits the category of sheaves as a full subcategory of the category of presheaves.

A natural question that arises from the consideration of a full subcategory is the existence of a left adjoint, which in this context amounts to the construction of a sheaf given a presheaf that is canonical in an appropriate sense. We will show such a construction is possible by considering stalks, the definition of which we now consider.

Definition 2.3 (Stalk)**.** Let X be a topological space, $x \in X$, and $\mathcal{F}$ a presheaf on X. The stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x is the colimit

$$\mathcal{F}_x = \operatorname{colim}_{x \in \subseteq U} \mathcal{F}(U).$$

Remark 2.4. Such a colimit is naturally filtered since in the category $\mathsf{Opens}_X^{\mathsf{Opp}}$ since it admits a final object – the empty set $\emptyset$. The subcategory of $\mathsf{Opens}_X^{\mathsf{Opp}}$ too is filtered by taking intersections. In these settings of sheaves over topological spaces, taking stalks preserves exactness [Stacks, Tag 04B0].

By the universal property of colimits, there is a unique map from $\mathcal{F}(U)$ to $\mathcal{F}_x$ for $x \in X$ which on application of the universal property of the product induces a map $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$ which in general is neither injective nor surjective. This map, however, is injective when $\mathcal{F}$ is a sheaf.

Alternatively, the stalk of a presheaf $\mathcal{F}$ on a topological space X at a point x can be described more explicitly as equivalence classes of sections that agree on sufficently small neighborhoods given by pairs (s, U) where $s \in \mathcal{F}(U)$ and U containing x modulo the equivalence relation $(s, U) \sim (s', U')$ if there exists $V \subseteq U \cap U'$ such that $s|_V = s'|_V$. This explicit construction of the stalk allows us to see that the stalks of a presheaf retains knowledge of the sheaf in an infinitesmally small neighborhood of the point.

Lemma 2.5. Let X be a topological space and $U \subseteq X$ open. If $\mathcal{F}$ is a sheaf, the canonical map $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$ is injective.

Proof. Suppose $s, s' \in \mathcal{F}(U)$ such that they map to the same element in the product. In which case we can pick open sets V_x for each x on which $s|_{V_x} = s'|_{V_x}$. These V_x form an open cover of x so $s = s'$ since $\mathcal{F}$ is a sheaf. ∎

We now describe the process of sheafification, first proving the desired construction is a sheaf before stating the definition.

Proposition 2.6. Let X be a topological space and $\mathcal{F}$ a presheaf on X. The presheaf

$$\mathcal{F}^{\#}(U) = \left\{ (s_x)_{x \in U} : \substack{\forall x \in X \text{ there is an open neighborhood } V \subseteq U \text{ of } x \\ \text{and } \sigma \in \mathcal{F}(V) \text{ such that } \forall x' \in V, s_{x'} = (\sigma, V) \in \mathcal{F}_v} \right\}$$

is a sheaf.

Proof. Suppose $\{U_i\}_{i \in I}$ is an open cover of U and $(s_{i,x})_{x \in U_i} \in \mathcal{F}^{\#}(U_i)$ where $(s_{i,x})_{x \in U_i}$ and $(s_{j,x})_{x \in U_j}$ are such that $(s_{i,x})_{x \in U_i \cap U_j} = (s_{j,x})_{x \in U_i \cap U_j}$. This glues in $\prod_{x \in U} \mathcal{F}_x$ which is a sheaf under the natural restriction maps so consider $s = (s_x)_{x \in U}$ that restricts to $(s_{i,x})_{x \in U_i}$ and $(s_{j,x})_{x \in U_j}$. For any $x \in U$ with $x \in U_i$ we can find some $V \subseteq U_i$ containing x and $\sigma \in \mathcal{F}(V)$ such that $s_{i,x'} = (\sigma, V)$ for all $x' \in V$ where since $s_{i,x'} = s_{x'}$ we have that $s = (s_x)_{x \in U}$ satisfies the condition as well. ■

In light of Proposition 2.6, we have the following.

Definition 2.7 (Sheafification)**.** Let X be a topological space and $\mathcal{F}$ a presheaf on X. The sheafification $\mathcal{F}^{\#}$ is the sheaf

$$\mathcal{F}^{\#}(U) = \left\{ (s_x)_{x \in U} : \substack{\forall x \in X \text{ there is an open neighborhood } V \subseteq U \text{ of } x \\ \text{and } \sigma \in \mathcal{F}(V) \text{ such that } \forall x' \in V, s_{x'} = (\sigma, V) \in \mathcal{F}_v} \right\}.$$

As the construction of Proposition 2.6 and the above definition suggests, sheafification does not change stalks. This expectation in fact holds.

Proposition 2.8. Let X be a topological space and $\mathcal{F}$ a presheaf on X with sheafification $\mathcal{F}^{\#}$. For all $x \in X$, $\mathcal{F}_x = \mathcal{F}_x^{\#}$.

Proof. The map $\mathcal{F}_x \to \mathcal{F}_x^{\#}$ is injective by Lemma 2.5. It remains to show surjectivity. Let $\overline{s} \in \mathcal{F}_x^{\#}$ be a section and consider $U \subseteq X$ containing x such that $\overline{s}$ is in the equivalence class of some (s, U) with $s \in \mathcal{F}^{\#}(U)$. As such, there is an open neighborhood V of U and $\sigma \in \mathcal{F}(V)$ with $s|_V$ is in the image of σ in $\prod_{x \in V} \mathcal{F}_x$ which defines a map of an element of $\mathcal{F}_x$ to $\overline{s}$. ■

This allows us to show that sheafification exhibits the expected universal property – that of the initial sheaf admitting a map from $\mathcal{F}$.

Proposition 2.9. Let X be a topological space, $\mathcal{F}$ a presheaf on X, and $\mathcal{G}$ a sheaf on X. For any morphism of presheaves $\mathcal{F} \to \mathcal{G}$ there exists a unique morphism making the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \\ \mathcal{F}^{\#} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

commute.

Proof. We have a commutative diagram of the form

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}^{\#} \arrow[r] \arrow[d] & \prod_{x \in X} \mathcal{F}_x \arrow[d] \\ \mathcal{G} \arrow[r] & \mathcal{G}^{\#} \arrow[r] & \prod_{x \in X} \mathcal{G}_x \end{tikzcd}$$

which induces the factorization as $\mathcal{G}^{\#} = \mathcal{G}$ since sheaves satisfy the compatibility condition imposed by the sheafification process. ■

The proposition easily allows us to conclude the following.

Proposition 2.10. Let X be a topological space. The forgetful functor $\mathsf{Sh}(X) \to \mathsf{PSh}(X)$ is right adjoint to the sheafification functor $\mathsf{PSh}(X) \to \mathsf{Sh}(X)$.

Proof. We want to show for a presheaf $\mathcal{F}$ and a sheaf $\mathcal{G}$ on X that there is an equivalence $\mathrm{Hom}_{\mathsf{Sh}(X)}(\mathcal{F}^{\#}, \mathcal{G}) = \mathrm{Hom}_{\mathsf{PSh}(X)}(\mathcal{F}, \mathcal{G})$ treating $\mathcal{G}$ as a presheaf in the latter. But the universal property of Proposition 2.9 implies that any morphism $\mathcal{F} \to \mathcal{G}$ uniquely extends through $\mathcal{F}^{\#}$ giving the claim. ■

3. Lecture 3 – 14th October 2024

Let us return to the example of the spectrum of a commutative ring as discussed in Proposition 1.10.

Proposition 3.1. Let A be a ring. Consider the association

$$U \mapsto \left\{ s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} : {}^{\forall \mathfrak{p} \in U, s(\mathfrak{p}) \in A_{\mathfrak{p}} \text{ and}}_{\exists U' \subseteq X, p \in U' \subseteq U, \exists a, b \in A \text{ s.t. } b \notin \mathfrak{q} \forall \mathfrak{q} \in U', s(\mathfrak{q}) = \frac{a}{b} \in A_{\mathfrak{q}}} \right\}$$

and for $V \subseteq U \subseteq X$ the forgetful maps. This association defines a sheaf of rings on $\mathrm{Spec}(A)$.

Proof. This was already shown to be a presheaf in Proposition 1.10, which is a sheaf since it satisfies the local compatibility condition of Definition 2.7. ∎

We shall denote this ring $\mathcal{O}_{\mathrm{Spec}(A)}$, and its sections admit a more explicit description as follows.

Proposition 3.2. Let A be a ring with spectrum $\mathrm{Spec}(A)$ and structure sheaf $\mathcal{O}_{\mathrm{Spec}(A)}$. Then:

(i) $\mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}} \cong A_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \subseteq A$.
(ii) $\mathcal{O}_{\mathrm{Spec}(A)}(D(f)) \cong A_f$ for $D(f) = \{\mathfrak{p} \subseteq A : f \notin \mathfrak{p}\} \subseteq \mathrm{Spec}(A)$ and all $f \in A$.

Proof of (i). Note that there is a natural homomorphism $\mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}} \to A_{\mathfrak{p}}$ by $s \mapsto s(\mathfrak{p})$ which is in $A_{\mathfrak{p}}$ by hypothesis.

This map is surjective since each element of $A_{\mathfrak{p}}$ is of the form $\frac{a}{b}$ for $b \in A \setminus \mathfrak{p}$. As such $D(b)$ gives an open neighborhood of $\mathfrak{p}$ and $\frac{a}{b}$ is an element defining a section of $\mathcal{O}_{\mathrm{Spec}(A)}(D(b))$ whose value at $\mathfrak{p}$ is exactly $\frac{a}{b}$ giving surjectivity.

For injectivity, let $U \subseteq \mathrm{Spec}(A)$ be an open set containing $\mathfrak{p}$ and $s, s' \in \mathcal{O}_{\mathrm{Spec}(A)}(U)$ such that $s(\mathfrak{p}) = s'(\mathfrak{p})$. Taking U to be sufficiently small, we have that $s = \frac{a}{b}, s' = \frac{a'}{b'}$ for $a, a' \in A$ and $b, b' \in A \setminus \mathfrak{p}$. Since these elements are equivalent in the localization, there exists $c \in A \setminus \mathfrak{p}$ such that $c(ab' - a'b) = 0$ in A. So $s = s'$ in all $A_{\mathfrak{q}}$ for $b, b', c \notin \mathfrak{q}$. But this is precisely $D(b) \cap D(b') \cap D(c)$ containing $\mathfrak{p}$ so $s = s'$ in a neighborhood of $\mathfrak{p}$ and thus give the same stalk showing injectivity, and that the map is an isomorphism. ∎

Proof of (ii). We now define a homomorphism $A_f \to \mathcal{O}_{\mathrm{Spec}(A)}(D(f))$ by $\frac{a}{f^n} \mapsto (\frac{a}{f^n} \mapsto (\frac{a}{f^n})_{\mathfrak{p} \in D(f)})$.

We first show the map is injective. Suppose there is some $\frac{a}{f^n}, \frac{a'}{f^{n'}}$ mapping to the same element in $A_{\mathfrak{p}}$ for all $\mathfrak{p} \in D(f)$. So for each such $\mathfrak{p}$ there is $c_{\mathfrak{p}} \in A \setminus \mathfrak{p}$ such that $c_{\mathfrak{p}}(af^{n'} - a'f^n) = 0$ in A. Now note that $\mathrm{Ann}(af^{n'} - a'f^n) \not\subseteq \mathfrak{p}$ for any $\mathfrak{p} \in D(f)$ since $\mathrm{Ann}(af^{n'} - a'f^n)$ contains $c_{\mathfrak{p}} \in A \setminus \mathfrak{p}$. As such, $V(\mathrm{Ann}(af^{n'} - a'f^n)) \subseteq V(f) = \mathrm{Spec}(A) \setminus D(f)$ from which we conclude that $f \in \sqrt{\mathrm{Ann}(af^{n'} - a'f^n)}$ and there is some N large such that $f^N(af^{n'} - a'f^n) = 0$ showing injectivity.

For surjectivity, take $s \in \mathcal{O}_{\mathrm{Spec}(A)}(D(f))$ with $s : D(f) \to \coprod_{\mathfrak{p} \in D(f)} A_{\mathfrak{p}}$ such that for all $\mathfrak{p} \in D(f)$ we have that $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ and there exists $U \subseteq D(f)$ containing $\mathfrak{p}$ and

$a, b \in A$ such that $b \neq \mathfrak{q}$ for all $\mathfrak{q} \in U$ and $s(\mathfrak{q}) = \frac{a}{b} \in A_{\mathfrak{q}}$. Let $\{U_i\}_{i\in I}$ be an open cover of U on which s has image $\frac{a_i}{b_i}$ with $b_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in U_i$. Since distinguished opens form a basis for the open sets of the Zariski topology on $\mathrm{Spec}(A)$, we can take $U_i = D(r_i)$ with $D(r_i) \subseteq D(b_i)$. We thus have $V((b_i)) \subseteq V((r_i))$ and thus $\sqrt{(r_i)} \subseteq \sqrt{(b_i)}$. In particular, $r_i^n = cb_i$ for some c so we can write $a_ib_i = ca_ir_i^n$ and since $D(r_i) = D(r_i^n)$ we can assume that $D(f)$ is covered by $D(r_1), \dots, D(r_m)$ given quasicompactness of the spectrum of a ring on which s is given by $\frac{a_i}{r_i}$. Now on $D(r_i) \cap D(r_j) = D(r_ir_j)$ we have the image of s given by both $\frac{a_i}{r_i}$ and $\frac{a_j}{r_j}$ giving $(h_ih_j)^N(h_ja_i - h_ia_j) = 0$. Rewriting this equation and picking N large, we have that $\frac{a}{f^n} = \frac{a_i}{r_i}$ on $D(r_i)$ giving surjectivity and the claim. ∎

We return to some generalities on sheaf theory, and discuss the kernel, cokernel, and image sheaves. This is easiest to do in the case of the kernel as justified by the following lemma.

Lemma 3.3. Let X be a topological space and $\phi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves of Abelian groups on X. The association

$$U \mapsto \ker(\mathcal{F}(U) \to \mathcal{G}(U))$$

is a sheaf on X.

Proof. For $\{U_i\}_{i\in I}$ an open cover of U, we have the following diagram

$$\begin{array}{ccccc}
\ker(\phi_U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_{i\in I} \ker(\phi_{U_i}) & \longrightarrow & \prod_{i\in I} \mathcal{F}(U_i) & \longrightarrow & \prod_{i\in I} \mathcal{G}(U_i) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_{i,j\in I} \ker(\phi_{U_i\cap U_j}) & \longrightarrow & \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j) & \longrightarrow & \prod_{i,j\in I} \mathcal{G}(U_i \cap U_j)
\end{array}$$

realizing $\mathcal{F}(U), \mathcal{G}(U)$ as the kernels of the maps between the products in the lower-right corner. As such, we have the descent condition. ∎

This defines the sheaf kernel.

Definition 3.4 (Sheaf Kernel)**.** Let X be a topological space and $\phi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves on X. The sheaf kernel $\ker(\phi)$ is the sheaf

$$U \mapsto \ker(\mathcal{F}(U) \to \mathcal{G}(U)).$$

However, in the case of the cokernel and the image, the naïvely defined presheaf is often not a sheaf as illustrated by the following example.

Example 3.5. Let $X = \{0, 1\}$ with the discrete topology, G an Abelian group, and $\mathcal{F} = \mathcal{G} = \underline{G}$. Define a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ which is the identity on G over X but the trivial map over any proper open subset of X. The cokernel is then 0 over X but G over any proper open subset of X so the sheaf condition does not hold.

As such, the definition of the cokernel and image necessitates sheafification.

Definition 3.6 (Sheaf Cokernel)**.** Let X be a topological space and $\phi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves of Abelian groups on X. The sheaf cokernel is the sheafification of the presheaf cokernel

$$U \mapsto \operatorname{coker}(\mathcal{F}(U) \to \mathcal{G}(U)).$$

Definition 3.7 (Sheaf Image)**.** Let X be a topological space and $\phi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves of Abelian groups on X. The sheaf image is the sheafification of the presheaf image

$$U \mapsto \operatorname{im}(\mathcal{F}(U) \to \mathcal{G}(U)).$$

Evidently these are sheaves. We show they satisfy the expected universal properties.

Proposition 3.8. Let X be a topological space and $\phi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves of Abelian groups on X. Then:

(i) For any sheaf $\mathcal{G}'$ admitting a morphism from $\mathcal{G}$ such that the composite $\mathcal{F} \to \mathcal{G} \to \mathcal{G}'$ is the zero morphism, there is a unique morphism making the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\phi"'] \arrow[rr, bend left, "0"] \arrow[drr, bend right, "0"'] & \mathcal{G} \arrow[r] \arrow[dr] & \operatorname{coker}(\phi) \arrow[d, dashed, "\exists!"] \\
& & \mathcal{G}'
\end{tikzcd}$$

commute.

(ii) For any sheaf $\mathcal{G}'$ admitting a morphism from $\mathcal{F}$ and a monomorphism to $\mathcal{G}$, there exists a unique morphism making the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[rr, bend left, "\phi"] \arrow[dr] & \operatorname{im}(\phi) \arrow[r] \arrow[d, dashed, "\exists!"] & \mathcal{G} \\
& \mathcal{G}' \arrow[ur] &
\end{tikzcd}$$

commute.

Proof of (i). By definition the map from $\mathcal{F}$ to the presheaf cokernel is the zero map inducing the solid diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\phi"] \arrow[rr, bend left, "0"] \arrow[rrr, bend left, "0"] \arrow[drrr] & \mathcal{G} \arrow[r] \arrow[drr] & \operatorname{coker}_{\mathsf{PSh}}(\phi) \arrow[r] \arrow[dr, dashed, "\exists!"] & \operatorname{coker}(\phi) = \operatorname{coker}_{\mathsf{PSh}}(\phi)^{\#} \arrow[d, dashed, "\exists!"] \\
& & & \mathcal{G}'
\end{tikzcd}$$

which extends to the one above by the universal property of sheafification since $\mathcal{G}'$ is a sheaf. ■

Proof of (ii). Arguing similarly, $\mathrm{im}_{\mathsf{PSh}}(\phi)$ fits into $\mathcal{F} \to \mathrm{im}_{\mathsf{PSh}}(\phi) \to \mathcal{G}$ where the morphism to $\mathcal{G}'$ is induced by the universal property of sheafification. ∎

With this language in mind, we want to be able to discuss isomorphisms of sheaves. We define these via monomorphisms and epimorphisms of sheaves.

Proposition 3.9. Let X be a topological space and $\phi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves of Abelian groups on X. The following are equivalent:

(a) ϕ is a monomorphism.
(b) $\ker(\phi)$ is the zero sheaf.
(c) For all $U \subseteq X$, ϕ_U is injective.
(d) For all $x \in X$, ϕ_x is injective.

Proof. (a)⇔(b) Suppose ϕ is a monomorphism. The map $\ker(\phi) \to \mathcal{F}$ necessarily factors through the zero object but for any $U \subseteq X$ we have $\ker(\phi_U) \hookrightarrow \mathcal{F}(U) \to \ker(\phi)(U) = 0$ showing that $\ker(\phi) = 0$. Conversely, for $\ker(\phi) = 0$, the universal property for the zero object implies that ϕ is a monomorphism.

(b)⇔(c) Since the sheaf kernel agrees with the presheaf kernel as a presheaf by Lemma 3.3 we have that ϕ_U injective implies $\ker(\phi)(U) = \ker(\phi_U) = 0$ which glues to $\ker(\phi) = 0$.

(c)⇒(d) taking colimits is left exact so the kernel of ϕ_x is $\ker(\phi)_x$ which is zero.

(d)⇒(c) Supposing that ϕ_x is injective for all x, we can take germs and find sufficiently small neighborhoods gluing to U to show ϕ_U is injective. ∎

4. Lecture 4 – 18th October 2024

Continuing the discussion of properties of morphisms of sheaves, we characterize epimorphisms and isomorphisms.

Proposition 4.1. Let X be a topological space and $\phi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves of Abelian groups on X. The following are equivalent:

(a) ϕ is an epimorphism.
(b) $\operatorname{coker}(\phi)$ is the zero sheaf.
(c) For all $x \in X$, ϕ_x is surjective.

To do.

Proof. ■

These conditions are implied by surjectivity of $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$.

Proposition 4.2. Let X be a topological space and $\phi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves of Abelian groups on X. If $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all $U \subseteq X$ open then:

(i) ϕ is an epimorphism.
(ii) $\operatorname{coker}(\phi)$ is the zero sheaf.
(iii) For all $x \in X$, ϕ_x is surjective.

To do.

Proof. ■

This allows us to characterize isomorphisms in the category of sheaves via their stalks.

Corollary 4.3. Let X be a topological space and $\phi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves of Abelian groups on X. The following are equivalent:

(a) ϕ is an isomorphism.
(b) For all $x \in X$, ϕ_x is an isomorphism.

Proof. By the definition of sheaves as compatible germs, we have (a)⇒(b).

Conversely, for all open neighborhoods $U \subseteq X$ we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}_x \arrow[r] & \mathcal{G}_x. \end{tikzcd}$$

We first show injectivity. Suppose that there is $s \in \mathcal{F}(U)$ such that $\phi_U(s) = 0$ in $\mathcal{G}(U)$. Thus for all $x \in U$ the germ $\phi_U(s)_x$ of $\phi_U(s)$ at x is zero thus ϕ_x is injective. But since $\mathcal{F}$ is a sheaf, $s = 0$ in $\mathcal{F}(U)$. For surjectivity, suppose there is some $t \in \mathcal{G}(U)$ which on shrinking to sufficiently small neighborhoods has germ t_x with preimage s_x since ϕ_x is an isomorphism and in particular surjective. The open set on which s_x is a representative section glues to a section in the preimage of t defined over U since injectivity implies that these preimages agree on overlaps. ■

These properties of morphisms of sheaves have a natural extension in the consideration of complexes.

Definition 4.4 (Exact)**.** Let X be a topological space and $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be sheaves of Abelian groups on X fitting into a diagram

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}.$$

The diagram is exact at $\mathcal{G}$ if $\operatorname{im}(\phi) = \ker(\psi)$.

Definition 4.5 (Short Exact Sequence)**.** Let X be a topological space and $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be sheaves of Abelian groups on X fitting into a diagram

$$0 \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0.$$

The diagram is a short exact sequence if $\operatorname{im}(\phi) = \ker(\psi)$.

One fundamental operation on sheaves, however, does not preserve short exact sequences.

Proposition 4.6. Let X be a topological space and

$$0 \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0$$

a short exact sequence of sheaves on X. Then

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}) \longrightarrow \Gamma(X, \mathcal{H})$$

is exact.

Proof. ■ To do.

Remark 4.7. In particular, the final morphism fails to be surjective.

However, as we have seen before, an exactness condition can be deduced on stalks.

Proposition 4.8. Let X be a topological space and

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

a diagram of sheaves on X. The following are equivalent:

(a)
$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$
is a short exact sequence of sheaves on X.

(b) For all $x \in X$ the diagram
$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x \longrightarrow 0$$
is a short exact sequence of Abelian groups.

Proof. ■ To do.

In summary, we can deduce the following theorem.

Theorem 4.9. Let X be a topolgoical space. The category $\mathsf{Sh}(X)$ of sheaves of Abelian groups on X is an Abelian category.

Proof. ■ To do.

Let us now consider how these sheaf categories behave with respect to continuous maps of topological spaces.

Definition 4.10 (Direct Image Sheaf)**.** Let $f : X \to Y$ be a continuous map of topological spaces and $\mathcal{F}$ a sheaf of Abelian groups on X. The direct image sheaf $f_*\mathcal{F}$ is the sheaf on Y given by $V \mapsto f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$.

Definition 4.11 (Inverse Image Sheaf)**.** Let $f : X \to Y$ be a continuous map of topological spaces and $\mathcal{G}$ a sheaf of Abelian groups on Y. The inverse image sheaf $f^{-1}\mathcal{G}$ is the sheaf on X given by $U \mapsto (\operatorname{colim}_{f(U)\subseteq V}\mathcal{G}(V))^\#$.

This data amalgamates into a functor in the evident way. Moreover, we can show the following exactness properties.

Proposition 4.12. Let $f : X \to Y$ be a continuous map of topological spaces. Then $f^{-1} : \mathsf{Sh}(Y) \to \mathsf{Sh}(X)$ is an exact functor.

Proposition 4.13. Let $f : X \to Y$ be a continuous map of topological spaces. Then $f^* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$ is a left exact functor.

5. Lecture 5 – 21st October 2024

We begin with a discussion of cohomology. Recall from Proposition 4.6 that the functor $\Gamma(X,-) : \mathsf{Sh}(X) \to \mathsf{AbGrp}$ from Abelian sheaves on X to the category of Abelian groups is not exact. Cohomology seeks to measure the extent to which exactness fails. We will do this by extending the functor $\Gamma(X,-)$ to a derived functor $H^i(X,-)$ that takes short exact sequences of sheaves to long exact sequences of Abelian groups in such a way that is compatible with morphisms of chain complexes (ie. short exact sequences). More precisely for a commuting diagram

Also $R^i\Gamma(X,-)$.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{H}' \arrow[r] & 0 \end{tikzcd}$$

we would expect the derived functor to induce a commuting diagram of long exact sequences

$$\begin{tikzcd} \dots \arrow[r] & H^i(X,\mathcal{F}) \arrow[r] \arrow[d] & H^i(X,\mathcal{G}) \arrow[r] \arrow[d] & H^i(X,\mathcal{H}) \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & H^i(X,\mathcal{F}') \arrow[r] & H^i(X,\mathcal{G}') \arrow[r] & H^i(X,\mathcal{H}') \arrow[r] & \dots . \end{tikzcd}$$

Example 5.1. Let $X = \mathbb{C} \setminus \{0\}$ and consider the exact sequence of sheaves

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \xrightarrow{f \mapsto \exp(2\pi i f)} \mathcal{O}_X^\times \longrightarrow 0.$$

This induces a long exact sequence in cohomology

$$0 \to H^0(X,\underline{\mathbb{Z}}) \to H^0(X,\mathcal{O}_X) \to H^0(X,\mathcal{O}_X^\times) \to H^1(X,\underline{\mathbb{Z}}) \to H^1(X,\mathcal{O}_X) = 0$$

where the vanishing of $H^1(X,\mathcal{O}_X)$ follows from X being Steinian. We can explicitly compute $H^0(X,\underline{\mathbb{Z}}) = \mathbb{Z}$ since X is connected and $H^1(X,\underline{\mathbb{Z}}) = \mathbb{Z}$ can be computed directly with some work. Alternatively, one can appeal to the agreement of sheaf and singular cohomology and the latter as the Abelianlization of $\pi_1(X) = \mathbb{Z}$ which is $\mathbb{Z}$.

We develop the theory in the generality of additive δ-functors between Abelian categories.

Definition 5.2 (δ-Functor)**.** Let A, B be Abelian categories. A δ-functor from $\mathsf{A} \to \mathsf{B}$ consists of the data of

- additive functors $H^i : \mathsf{A} \to \mathsf{B}$ for $i \geq 0$, and
- for any short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in A a morphism $\delta^i : H^i(A_3) \to H^{i+1}(A_1)$ in B.

satisfying the following properties:

- For all short exact sequences $0 \to A_1 \to A_2 \to A_3 \to 0$ in A, the induced sequence
$$\begin{tikzcd} 0 \arrow[r] & H^0(A_1) \arrow[r] & H^0(A_2) \arrow[r] & H^0(A_3) \arrow[dll, "\delta^0"'] \\ & H^1(A_1) \arrow[r] & H^1(A_2) \arrow[r] & \dots \end{tikzcd}$$
is a long exact sequence in B.
- Any morphism of short exact sequences
$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_1' \arrow[r] & A_2' \arrow[r] & A_3' \arrow[r] & 0 \end{tikzcd}$$
induces a morphism of long exact sequences
$$\begin{tikzcd} 0 \arrow[r] & H^0(A_1) \arrow[r] \arrow[d] & H^0(A_2) \arrow[r] \arrow[d] & H^0(A_3) \arrow[r] \arrow[d] & H^1(A_1) \arrow[r] \arrow[d] & \dots \\ 0 \arrow[r] & H^0(A_1') \arrow[r] & H^0(A_2') \arrow[r] & H^0(A_3') \arrow[r] & H^1(A_1') \arrow[r] & \dots. \end{tikzcd}$$

The functors defining cohomology will arise as a special class of δ-functors, known as universal δ-functors.

Definition 5.3 (Universal δ-Functor)**.** A δ-functor $(H^i)_{i\geq 0} : \mathsf{A} \to \mathsf{B}$ is universal if for any other δ-functor $(\widetilde{H}^i)_{i\geq 0}$ and any natural transformation $H^0 \Rightarrow \widetilde{H}^0$, there exists a unique natural transformation $H^i \Rightarrow \widetilde{H}^i$ for $i \geq 1$ such that for all short exact sequences $0 \to A_1 \to A_2 \to A_3 \to 0$ in A the induced diagram
$$\begin{tikzcd} 0 \arrow[r] & H^0(A_1) \arrow[r] \arrow[d] & H^0(A_2) \arrow[r] \arrow[d] & H^0(A_3) \arrow[r] \arrow[d] & H^1(A_1) \arrow[r] \arrow[d] & \dots \\ 0 \arrow[r] & \widetilde{H}^0(A_1) \arrow[r] & \widetilde{H}^0(A_2) \arrow[r] & \widetilde{H}^0(A_3) \arrow[r] & \widetilde{H}^1(A_1) \arrow[r] & \dots \end{tikzcd}$$
is commutative.

Remark 5.4. Note that if we fix the functor H^0, then any universal δ-functor will be unique.

We now introduce effacable functors which will provide a way to produce universal δ-functors.

Definition 5.5 (Effacable Functor)**.** Let $F : \mathsf{A} \to \mathsf{B}$ be a functor between Abelian categories. F is effacable if for each object A of A, there exists an object A' of A and a monomorphism $\varphi : A \to A'$ such that $F(\varphi)$ is the zero morphism.

Remark 5.6. In practice, we can often take $F(A')$ to be the zero object.

The following result is due to Grothendieck in his famed "Tôhoku" paper [Gro57].

Theorem 5.7 (Grothendieck)**.** If $(H^i)_{i\geq 0} : \mathsf{A} \to \mathsf{B}$ is a δ-functor between Abelian categories such that H^i is effacable for $i \geq 1$ then $(H^i)_{i\geq 0}$ is a universal δ-functor.

Remark 5.8. We will typically be considering the extension of a left exact additive functor between Abelian categories $F : \mathsf{A} \to \mathsf{B}$, that is, where $H^0 = F$, in which case we will denote H^i as R^iF, the ith right derived functor of F.

Cohomology is generally defined in terms of injective resolutions.

Definition 5.9 (Injective Object)**.** Let A be an Abelian category. An object I of A is injective if for all monomorphisms $A \hookrightarrow A'$ there exists a morphism $A \to I$ making the diagram

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & A' \arrow[dl, dashed, "\exists"] \\ I & \end{tikzcd}$$

commute.

Remark 5.10. Note that the map $A' \to I$ need not be unique.

Definition 5.11 (Injective Resolution)**.** Let A be an Abelian category and A an object of A. An injective resolution of A is a long exact sequence

$$0 \to A \to I_0 \to I_1 \to \dots$$

with I_j injective for all $j \geq 0$.

Definition 5.12 (Enough Injectives)**.** Let A be an Abelian category. A has enough injectives if each object A admits an injective resolution.

As a step towards defining cohomology, we make the following construction.

Proposition 5.13. Let $F : \mathsf{A} \to \mathsf{B}$ be an additive left exact functor between Abelian categories with A having enough injectives. For A an object of A with injective resolution $I_\bullet$, set

$$R^iF(A) = \frac{\ker\left(F(I_i) \to F(I_{i+1})\right)}{\operatorname{im}\left(F(I_{i-1}) \to F(I_i)\right)}.$$

Then:

(i) $R^0F(A) = F(A)$, and
(ii) the formation of $R^iF(A)$ is independent of choice of the injective resolution.

Proof of (i). $R^0F(A) = \ker(F(I_0) \to F(I_1))$ but since F is left exact, we have that $0 \to F(A) \to F(I_0) \to F(I_1)$ is exact, where the kernel of $F(I_0) \to F(I_1)$ is $F(A)$. ■

Proof of (ii). Let A be an object of A and $I_\bullet, I'_\bullet$ injective resolutions of A. The universal property of injective objects induces a diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & I_0 \arrow[r] \arrow[d] & I_1 \arrow[r] \arrow[d] & I_2 \arrow[r] \arrow[d] & \dots \\ 0 \arrow[r] & A \arrow[r] & I'_0 \arrow[r] & I'_1 \arrow[r] & I'_2 \arrow[r] & \dots \end{tikzcd}$$

and dually from $I'_\bullet \to I_\bullet$. This induces morphisms in B

$$\begin{tikzcd}
0 \arrow[r] & F(A) \arrow[r] \arrow[d] & F(I_0) \arrow[r] \arrow[d] & F(I_1) \arrow[r] \arrow[d] & F(I_2) \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & F(A) \arrow[r] & F(I'_0) \arrow[r] & F(I'_1) \arrow[r] & F(I'_2) \arrow[r] & \dots
\end{tikzcd}$$

and moreover for each i there are unique morphisms $\ker(F(I_i) \to F(I_{i+1})) \to \ker(F(I'_i) \to F(I'_{i+1}))$ and $\operatorname{im}(F(I_{i-1}) \to F(I_i)) \to \operatorname{im}\left(F(I'_{i-1}) \to F(I'_i)\right)$ induced by the universal property of kernels and images, respectively. This in turn induces a unique map between the quotients $\frac{\ker(F(I_i)\to F(I_{i+1}))}{\operatorname{im}(F(I_{i-1})\to F(I_i))} \to \frac{\ker(F(I'_i)\to F(I'_{i+1}))}{\operatorname{im}\left(F(I'_{i-1})\to F(I'_i)\right)}$. Applying the map to the map bewteen resolutions $I'_\bullet \to I_\bullet$ we get a unique map in the opposite direction and verifying that these are two-sided inverses yields the claim. ■

Definition 5.14 (Right Derived Functor of Left Exact Functor)**.** Let $F : \mathsf{A} \to \mathsf{B}$ be an additive left exact functor between Abelian categories with A having enough injectives. The ith right derived functor R^iF is given by

$$A \mapsto \frac{\ker\left(F(I_i) \to F(I_{i+1})\right)}{\operatorname{im}\left(F(I_{i-1}) \to F(I_i)\right)}$$

for some injective resolution $I_\bullet$ of A.

Remark 5.15. This is well-defined by Proposition 5.13 (ii).

The construction of the right derived functor of a left exact functor returns the zero object of B when evaluated on an injective object of A.

Lemma 5.16. Let $F : \mathsf{A} \to \mathsf{B}$ be an additive left exact functor between Abelian categories with A having enough injectives. If A is an injective object of A then $R^iF(A) = 0$ for $i \geq 1$.

Proof. An injective resolution is given by $0 \to A \to A \to 0$, in particular $I_0 = A$ and $I_i = 0$ for $i \geq 1$ giving the claim. ■

We will be most interested in the following cases, where we note here that the category of sheaves of Abelian groups on a topological space has enough injectives [Stacks, Tag 01DL].

Definition 5.17 (Sheaf Cohomology)**.** Let X be a topological space and $\mathcal{F}$ a sheaf of Abelian groups on X. The ith sheaf cohomology group of $\mathcal{F}$ is given by $R^i\Gamma(X, \mathcal{F})$.

Definition 5.18 (Derived Pushforward)**.** Let $f : X \to Y$ be a continuous map between topological spaces and $\mathcal{F}$ a sheaf on X. The ith derived pushforward of $\mathcal{F}$ is given by the sheaf $R^if_*\mathcal{F}$.

Remark 5.19. Both the global sections functor and direct image functor are left exact as shown in Propositions 4.6 and 4.13.

6. Lecture 6 – 25th October 2024

Recall that for a topological space X and a continuous map $f : X \to Y$, the induced functors $\Gamma(X, -) : \mathsf{Sh}(X) \to \mathsf{AbGrp}$ and $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$ are not in general exact, but only left exact Definitions 5.17 and 5.18. We can measure the failure of exactness by using δ-functors which is computed using injective resolutions. However, injective resolutions are quite difficult to work with. We will consider some resolutions to the difficulty of computation today.

Definition 6.1 (F-Acyclic Objects)**.** Let $F : \mathsf{A} \to \mathsf{B}$ be a left exact additive functor between Abelian categories and R^iF exists. An object A of A is F-acyclic if $R^iF(A) = 0$ for all $i \geq 1$.

Remark 6.2. Injective objects are F-acyclic per Lemma 5.16.

Cohomology can in most cases be computed on acyclic resolutions in place of injective resolutions.

Proposition 6.3. Let $F : \mathsf{A} \to \mathsf{B}$ be a left exact additive functor between Abelian categories and A having enough injectives. Then the cohomology of any F-acyclic resolution is equal to the cohomology of any injective resolution.

Proof. These are both effacable δ-functors and hence universal by Theorem 5.7. ■

Let us now turn to the case of derived pushforward.

Proposition 6.4. Let $f : X \to Y$ be a continuous map and $\mathcal{F}$ a sheaf on X. Then $R^if_*\mathcal{F}$ is the sheafification of the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{F})$ for all sheaves $\mathcal{F}$ on X.

Proof. Taking an injective resolution of $\mathcal{F}$, we know its direct image is injective, extending the left exact functor f_*. So computing the cohomology of the complex $0 \to f_*\mathcal{F} \to f_*\mathcal{I}_0 \to \dots$, we have

$$R^if_*\mathcal{F} = \frac{\ker\left(f_*\mathcal{I}_i \to f_*\mathcal{I}_{i+1}\right)}{\operatorname{im}\left(f_*\mathcal{I}_{i-1} \to f_*\mathcal{I}_i\right)}$$

with sections over $V \subseteq Y$ open are given by

$$\begin{aligned} R^if_*\mathcal{F}(V) &= \frac{\ker\left(f_*\mathcal{I}_i(V) \to f_*\mathcal{I}_{i+1}(V)\right)}{\operatorname{im}\left(f_*\mathcal{I}_{i-1}(V) \to f_*\mathcal{I}_i(V)\right)} \\ &= \frac{\ker\left(\mathcal{I}_i(f^{-1}(V)) \to \mathcal{I}_{i+1}(f^{-1}(V))\right)}{\operatorname{im}\left(\mathcal{I}_{i-1}(f^{-1}(V)) \to \mathcal{I}_i(f^{-1}(V))\right)} \\ &= \frac{\ker(\mathcal{I}_i \to \mathcal{I}_{i+1})(f^{-1}(V))}{\operatorname{im}(\mathcal{I}_{i-1} \to \mathcal{I}_i)(f^{-1}(V))} \end{aligned}$$

where the claim follows, noting that the image sheaf in the quotient is the sheafification of the presheaf image. ■

In the case of computing sheaf cohomology, a large example of Γ-acyclic sheaves is provided by flasque sheaves.

Definition 6.5 (Flasque Sheaf)**.** Let X be a topological space. A sheaf $\mathcal{F}$ on X is flasque if for all $U \subseteq X$ open and $V \subseteq U$ open $\mathrm{res}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

We consider some elementary properties of flasque sheaves.

Proposition 6.6. Let X be a topological space. Then:

(i) If $\mathcal{F}$ is flasque, then $\mathcal{F}|_U$ is flasque for all $U \subseteq X$ open.
(ii) If $\mathcal{I}$ is injective, then $\mathcal{I}$ is flasque.
(iii) If $f : X \to Y$ is a continuous map and $\mathcal{F}$ a flasque sheaf on X then $f_*\mathcal{F}$ is a flasque sheaf on Y.

Proof of (i). This is immediate from the definition as for $W \subseteq V \subseteq U$ we have $\mathcal{F}(V) = \mathcal{F}|_U(V) \to \mathcal{F}|_U(W) = \mathcal{F}(W)$ surjective. ■

Proof of (ii). See [Stacks, Tag 01EA]. ■

Proof of (iii). For $W \subseteq V \subseteq Y$ we have $f^{-1}(W) \subseteq f^{-1}(V)$ giving so $\mathcal{F}(f^{-1}(V)) = f_*\mathcal{F}(V) \to f_*\mathcal{F}(W) = \mathcal{F}(f^{-1}(W))$ is surjective. ■

As expected, flasque sheaves are Γ-acyclic. We can say more:

Proposition 6.7. Let X be a topological space. If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of sheaves and $\mathcal{F}$ is flasque then $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$ is exact for all $U \subseteq X$ open.

Proof. Taking sections is generally left exact by Proposition 4.6 giving an exact sequence

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U).$$

To show exactness on the right, then, it suffices to show that the morphism $\mathcal{G}(U) \to \mathcal{H}(U)$ is surjective. Consider the induced sequence on stalks for some $p \in U$ which is exact. So, passing to germs, for any section $s \in \mathcal{F}(U)$ there exists an open covering $\{U_i\}_{i\in I}$ for an ordered indexing set I on which s_i is the image of $t_i \in \mathcal{F}(U_i)$, though the t_i need not glue as sections of $\mathcal{G}(U)$.

We proceed by induction. For some fixed $j \in I$ suppose that t_i are such that they glue to t in $U_{(j)} = \bigcup_{i<j} U_i$. Consider $t|_{U_{(j)}\cap U_j} - t_j|_{U_{(j)}\cap U_j}$ which lies in the subsheaf $\mathcal{F}'(U_{(j)} \cap U_j)$ as its image vanishes in $\mathcal{H}(U_{(j)} \cap U_j)$. But $\mathcal{F}$ is flasque so there is $r_j \in \mathcal{F}(U_j)$ with image $t|_{U_{(j)}\cap U_j} - t_j|_{U_{(j)}\cap U_j}$ in $\mathcal{F}(U_{(j)} \cap U_j)$. Now note that $r_j + t_j$ is compatible with t: on $U_i \cap U_j$ for $i < j$ is given by

$$t|_{U_i\cap U_j} = (t|_{U_i\cap U_j} - t_j|_{U_i\cap U_j}) + t_j|_{U_i\cap U_j}.$$

The section $r_j + t_j \in \mathcal{F}(U)$ hence extends to a section on $\bigcup_{i\leq j} U_i$ by the gluability axiom of the sheaf $\mathcal{G}$. Repeating this process inductively yields a section of $\mathcal{G}(U)$ with image s, showing surjectivity, and thus exactness on the right. ■

We conclude with a final property of flasque sheaves.

Proposition 6.8. Let X be a topological space. If

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a short exact sequence of sheaves with $\mathcal{F}, \mathcal{G}$ flasque then $\mathcal{H}$ is flasque.

Proof. For suppose $V \subseteq U$ and the exactness result from Proposition 6.7 gives

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) & \longrightarrow & \mathcal{H}(U) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) & \longrightarrow & \mathcal{H}(V) & \longrightarrow & 0 \end{array}$$

a diagram with vertical maps given by restrictions. Furthermore, the diagram is commutative by the definition of morphisms of schemes which commute with restrictions. $\mathcal{G}(U) \to \mathcal{G}(V)$ is surjective by flasqueness of $\mathcal{G}$, $\mathcal{G}(U) \to \mathcal{H}(U)$ and $\mathcal{G}(V) \to \mathcal{H}(V)$ surjective by exactness of the sequence. In particular, the composite $\mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(V)$ is surjective, and hence $\mathcal{H}(U) \to \mathcal{H}(V)$ is surjective – recalling here that if $f : A \to B$, $g : B \to C$ are such that $g \circ f : A \to B \to C$ is surjective then g is surjective. ■

We introduce Čech cohomology which will be a key tool for computing sheaf cohomology, and will agree with sheaf cohomology in many cases. This will be done by computing the cohomology of the Čech complex associated to a cover.

Proposition 6.9. Let X be a topological space, $\{U_i\}_{i \in I}$ a cover of X with I a totally ordered set, and $\mathcal{F}$ a sheaf on X. Consider the data of

- An Abelian group for each $p \geq 0$

$$C^p(\{U_i\}_{i \in I}, \mathcal{F}) = \prod_{i_0 < i_1 < \cdots < i_p} \mathcal{F}(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p})$$

- Morphisms $C^p(\{U_i\}_{i \in I}, \mathcal{F}) \to C^{p+1}(\{U_i\}_{i \in I}, \mathcal{F})$ by

$$(s_{i_0,\ldots,i_p})_{i_0 < i_1 < \cdots < i_p} \mapsto \left(\sum_{j=0}^{p+1} (-1)^j s_{i_0,\ldots,\widehat{i_j},\ldots,i_{p+1}}|_{U_{i_0} \cap \cdots \cap U_{i_{p+1}}} \right)_{i_0 < i_1 < \cdots < i_p < i_{p+1}}$$

giving a diagram of Abelian groups

$$0 \to C^0(\{U_i\}_{i \in I}, \mathcal{F}) \to C^1(\{U_i\}_{i \in I}, \mathcal{F}) \to C^2(\{U_i\}_{i \in I}, \mathcal{F}) \to \ldots. \tag{6.1}$$

The diagram (6.1) is a chain complex of Abelian groups.

Proof. We verify the map $C^{p-1}(\{U_i\}_{i \in I}, \mathcal{F}) \to C^{p+1}(\{U_i\}_{i \in I}, \mathcal{F})$ is the zero map via direct computation. For a section $(s_{i_0,\ldots,i_{p-1}})_{i_0 < \cdots < i_{p-1}}$, its image in $C^{p+1}(\{U_i\}_{i \in I}, \mathcal{F})$

is given by

$$\sum_{j=0}^{p+1}(-1)^j\left(\sum_{k=0}^{p}(-1)^k s_{i_0,\dots,\widehat{i_k},\dots,i_p}|_{U_{i_0,\dots,i_p}}\right)_{i_0,\dots,\widehat{i_k},\dots,i_{p_1}}$$

$$=\sum_{j=0}^{p+1}(-1)^j\left(\sum_{k=0}^{j-1}(-1)^k s_{i_0,\dots,\widehat{i_k},\dots,\widehat{i_j},\dots,i_{p+1}}+\sum_{k=j+1}^{p+1}(-1)^{k-1}s_{i_0,\dots,\widehat{i_j},\dots,\widehat{i_k},\dots,i_{p+1}}\right)|_{U_{i_0,\dots,i_{p+1}}}$$

but the sum telescopes, giving the claim. ■

Remark 6.10. The construction of Proposition 6.9, in words, states that given a family of sections $s_{i_0,\dots,i_p}$ defined on $U_{i_0}\cap\cdots\cap U_{i_p}$ for each ordered subset of I of size $p+1$ to a section on $U_{i_0}\cap\cdots\cap U_{i_p}\cap U_{i_{p+1}}$ by taking the alternating sum of the restriction of sections $s_{i_0,\dots,i_p}|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}}$ over all $p+1$ element subsets of $i_0,\dots,i_{p+1}$.

Remark 6.11. To the end of getting better intuition for the construction, let's consider some cases of the Čech complex for $|I|$ small. Let X admit a cover by U_0,U_1,U_2 and $\mathcal{F}$ a sheaf on X. For a section $s\in\mathcal{F}(X)$ the construction of the Čech complex takes s to the tuple of sections $(s|_{U_0},s|_{U_1},s|_{U_2})$, the tuple of sections $(s|_{U_0},s|_{U_1},s|_{U_2})$ to $((s|_{U_1}-s|_{U_0})|_{U_0\cap U_1},(s|_{U_2}-s|_{U_0})|_{U_0\cap U_2},(s|_{U_2}-s|_{U_1})|_{U_1\cap U_2})$.

As such we are justified in making the following definition.

Definition 6.12 (Čech Complex)**.** Let X be a topological space, $\{U_i\}_{i\in I}$ a cover of X with I a totally ordered set, and $\mathcal{F}$ a sheaf on X. The Čech complex of $\mathcal{F}$ with respect to the cover U is the chain complex

$$0\to C^0(\{U_i\}_{i\in I},\mathcal{F})\to C^1(\{U_i\}_{i\in I},\mathcal{F})\to C^2(\{U_i\}_{i\in I},\mathcal{F})\to\dots$$

where $C^p(\{U_i\}_{i\in I},\mathcal{F})$ and the differentials given by

$$(s_{i_0,\dots,i_p})_{i_0<i_1<\cdots<i_p}\mapsto\left(\sum_{j=0}^{p+1}(-1)^j s_{i_0,\dots,\widehat{i_j},\dots,i_{p+1}}|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}}\right)_{i_0<i_1<\cdots<i_p<i_{p+1}}.$$

The Čech cohomology of a sheaf is merely the cohomology of the corresponding Čech complex.

Definition 6.13. Let X be a topological space, $\{U_i\}_{i\in I}$ a cover of X with I a totally ordered set, and $\mathcal{F}$ a sheaf on X. The Čech cohomology of the sheaf

$$\check{H}^p(X,\mathcal{F})=\frac{\ker\left(C^p(\{U_i\}_{i\in I},\mathcal{F})\to C^{p+1}(\{U_i\}_{i\in I},\mathcal{F})\right)}{\operatorname{im}\left(C^{p-1}(\{U_i\}_{i\in I},\mathcal{F})\to C^p(\{U_i\}_{i\in I},\mathcal{F})\right)}$$

is the cohomology of the corresponding Čech complex.

We can use this to compute the sheaf cohomology of $\underline{\mathbb{Z}}$ on the circle S^1.

Example 6.14. Let S^1 be the unit circle and U_0, U_1 a cover of the upper and lower semicircles intersecting around $(-1,0)$ and $(1,0)$. Then the Čech complex is given by $H^0(S^1, \underline{\mathbb{Z}}) \to C^0(\{U_0, U_1\}, \mathcal{F}) \to C^1(\{U_0, U_1\}, \mathcal{F})$ by $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ where the first map is by $s \mapsto (s|_{U_0}, s|_{U_1})$ and the second by $(s|_{U_0}, s|_{U_1}) \mapsto (s|_{U_1} - s|_{U_0}, s|_{U_1} - s|_{U_0})$ yielding $\check{H}^0(S^1, \underline{\mathbb{Z}}) = \check{H}^1(S^1, \underline{\mathbb{Z}}) = \mathbb{Z}$.

7. Lecture 7 – 28th October 2024

We continue our discussion of Čech cohomology and its relation to derived functor cohomology. We first define a sheaf-variant of Čech cohomology.

Definition 7.1 (Čech Complex of Sheaves)**.** Let X be a topological space, $\{U_i\}_{i\in I}$ a cover of X with I a totally ordered set, and $\mathcal{F}$ a sheaf on X. The Čech complex of sheaves

$$\mathcal{C}^0(\{U_i\}_{i\in I},\mathcal{F}) \to \mathcal{C}^1(\{U_i\}_{i\in I},\mathcal{F}) \to \mathcal{C}^2(\{U_i\}_{i\in I},\mathcal{F}) \to \dots$$

is the chain complex of sheaves on X with

$$\mathcal{C}^p(\{U_i\}_{i\in I},\mathcal{F}) = j_*\left(\prod_{i_0<i_1<\dots<i_p} \mathcal{F}|_{U_{i_0}\cap\dots\cap U_{i_p}}\right)$$

for $j : U_{i_0} \cap \dots \cap U_{i_p} \hookrightarrow X$ the inclusion map, and differentials those induced by taking sections on open sets.

Remark 7.2. As such, $\mathcal{C}^\bullet(\{U_i\}_{i\in I},\mathcal{F})$ is a complex of sheaves on X with the property that $\Gamma(X,\mathcal{C}^p(\{U_i\}_{i\in I},\mathcal{F})) = C^p(\{U_i\}_{i\in I},\mathcal{F})$.

This complex is in fact a long exact sequence of sheaves.

Proposition 7.3. Let X be a topological space, $\{U_i\}_{i\in I}$ a cover of X with I a totally ordered set, and $\mathcal{F}$ a sheaf on X. The Čech complex of sheaves

$$\mathcal{C}^0(\{U_i\}_{i\in I},\mathcal{F}) \to \mathcal{C}^1(\{U_i\}_{i\in I},\mathcal{F}) \to \mathcal{C}^2(\{U_i\}_{i\in I},\mathcal{F}) \to \dots$$

is a long exact sequence of sheaves on X.

Proof. See [Har83, Lem. 4.2]. ■

The formation of this complex behaves as expected on flasque sheaves.

Proposition 7.4. Let X be a topological space, $\{U_i\}_{i\in I}$ a cover of X with I a totally ordered set, and $\mathcal{F}$ a flasque sheaf on X. Then $\check{H}^p(\{U_i\}_{i\in I},\mathcal{F}) = 0$ for $p > 0$.

Proof. For $\mathcal{F}$ flasque, $\mathcal{C}^\bullet(\{U_i\}_{i\in I},\mathcal{F})$ is a complex of flasque sheaves by construction so $\check{H}^p(\{U_i\}_{i\in I},\mathcal{F}) = R^i\Gamma(X,\mathcal{C}^\bullet(\{U_i\}_{i\in I},\mathcal{F})) = 0$. ■

We anre now prepared to show the comparison theorem with derived functor cohomology.

Theorem 7.5. Let X be a topological space, $\{U_i\}_{i\in I}$ a cover of X with I a totally ordered set, and $\mathcal{F}$ a sheaf on X. There is a functorial comparison morphism $\check{H}^i(\{U_i\}_{i\in I},\mathcal{F}) \to H^i(X,\mathcal{F})$ for all i.

Proof. Using the long exact sequence with the Čech complex of sheaves $0 \to \mathcal{F} \to \mathcal{C}^0(\{U_i\}_{i\in I},\mathcal{F}) \to \mathcal{C}^1(\{U_i\}_{i\in I},\mathcal{F}) \to \dots$ and an injective resolution $0 \to \mathcal{F} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \dots$, the universal property of injective objects induces canonical maps $\mathcal{C}^i(\{U_i\}_{i\in I},\mathcal{F}) \to \mathcal{I}_i$ descending to a canonical map on cohomology. ■

In the case where a certain condition on higher cohomology is satisfied – and as we will show holds in the case of schemes – this functorial comparison morphism is an isomorphism.

Proposition 7.6. Let X be a topological space, $\{U_i\}_{i\in I}$ a cover of X with I a totally ordered set. If $\mathcal{F}$ a sheaf on X such that $H^p(U_i \cap U_j, \mathcal{F}) = 0$ for all $i, j \in I$ and $p \geq 1$ then there is an isomorphism $\check{H}^i(\{U_i\}_{i\in I}, \mathcal{F}) \cong H^i(X, \mathcal{F})$.

Proof. Let $\mathcal{F}$ be such a sheaf, in which case $R^p\Gamma(X, \mathcal{C}^k(\{U_i\}_{i\in I}, \mathcal{F})) = 0$ for $p > 0, k \geq 0$ and the result follows from Theorem 7.5. ■

We can also define a cover-independent variant of Čech cohomology as follows.

Definition 7.7 (Refinement of Cover)**.** Let X be a topological space and $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ be two covers of X. The cover $\{V_j\}_{j\in J}$ refines $\{U_i\}_{i\in I}$ if there is an order-preserving function $\rho : J \to I$ such that $V_j \subseteq U_{\rho(j)}$.

Note that for a refinement $\{V_j\}_{j\in J}$ of $\{U_i\}_{i\in I}$ the termwise map on Čech complexes

$$\prod_{\rho^{-1}(j_0)<\cdots<\rho^{-1}(j_p)} \mathcal{F}(U_{\rho^{-1}(j_0)} \cap \cdots \cap U_{\rho^{-1}(j_p)}) \to \prod_{j_0<\cdots<j_p} \mathcal{F}(U_{j_0} \cap \cdots \cap U_{j_p})$$

inducing a map on cohomology $\check{H}^p(\{U_i\}_{i\in I}, \mathcal{F}) \to \check{H}^p(\{V_j\}_{j\in J}, \mathcal{F})$. Absolute Čech cohomology is defined by passage to colimits on refinements of covers.

Definition 7.8 (Absolute Čech Cohomology)**.** Let X be a topological space and $\mathcal{F}$ a sheaf on X. The absolute Čech cohomology $\check{H}^p(X, \mathcal{F})$ is given by

$$\operatorname{colim}_{\substack{\{U_i\}_{i\in I}\to\{V_j\}_{j\in J} \\ \{V_j\}_{j\in J} \text{ refines } \{U_i\}_{i\in I}}} \check{H}^p(\{U_i\}_{i\in I}, \mathcal{F}).$$

Remark 7.9. It is often the case that Čech cohomology with respect to a cover is not equivalent to derived functor cohomology, but absolute Čech cohomology is.

Remark 7.10. It can be shown that derived functor cohomology agrees with absolute Čech cohomology in degree 1, that is, $\check{H}^1(X, \mathcal{F}) \cong H^1(X, \mathcal{F})$, though not in general.

We are now prepared to define schemes, which we will exhibit as a special class of ringed spaces.

Definition 7.11 (Ringed Space)**.** A ringed space $(X, \mathcal{O}_X)$ consists of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X known as the structure sheaf on X.

Definition 7.12 (Morphism of Ringed Spaces)**.** A morphism of ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the data of a continuous map $f : X \to Y$ and a morphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y.

Remark 7.13. Note that $f_*\mathcal{O}_X$ is automatically a sheaf of rings on Y with no need for sheafification since $\mathcal{O}_X$ is a sheaf on X (cf. Definition 4.10).

Remark 7.14. In many cases the induced map on sheaves $f^\sharp$ is obvious from the morphism f and will be left implicit.

Let us consider some examples.

Example 7.15. Let V be a $\mathbb{C}$-vector space and $U \subseteq V$ open. The sheaf of holomorphic functions on U endows U with the structure of a ringed space with the natural restriction maps.

Example 7.16. Let $f : X \to Y$ be a continuous map of topological spaces. There is a natural map $C_Y \to f_* C_X$ of sheaves of continuous functions on Y by composition, taking a function g continuous on $V \subseteq Y$ to the function $g \circ f \in C_X(f^{-1}(V))$.

8. Lecture 8 – 4th November 2024

We continue our discussion of ringed spaces. We continue with some examples, following Examples 7.15 and 7.16.

Example 8.1. Let X, Y be differentiable manifolds and $f : X \to Y$ a differentiable map. There is a natural map $f^\sharp : C_Y^{\mathrm{diff}} \to f_* C_X^{\mathrm{diff}}$ taking a differentiable function g on $V \subseteq Y$ to the differentiable function $g \circ f \in C_X(f^{-1}(V))$ making $(f, f^\sharp)$ a morphism of ringed spaces.

Example 8.2. Let X, Y be complex manifolds and $f : X \to Y$ a holomorphic map. There is a natural map $f^\sharp : C_Y^{\mathrm{diff}} \to f_* C_X^{\mathrm{diff}}$ taking a holomorphic function g on $V \subseteq Y$ to the holomorphic function $g \circ f \in C_X(f^{-1}(V))$ making $(f, f^\sharp)$ a morphism of ringed spaces.

We discuss the following example after introducing some basic notions in algebraic geometry.

Definition 8.3 (Affine Algebraic Set)**.** Let k be an alegbraically closed field. $X \subseteq k^n$ is an affine algebraic set if there exists $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ such that for all $f(x) = 0$ for all $f(x) \in \mathfrak{a}$ and all $x \in X$.

Example 8.4. Let k be an algebraically closed field and $X \subseteq k^n$ an affine algebraic set. X can be endowed with the Zariski topology. X has the structure of a ringed space where $\mathcal{O}_X(U)$ for $U \subseteq X$ open consists of the set of functions $g : U \to k$ such that for all $x \in U$ there exists an open neighborhood $V \subseteq U$ containing x and polynomials $g_1(x), g_2(x) \in k[x_1, \ldots, x_n]$ where for all $y \in V$, $g(y) = \frac{g_1(y)}{g_2(y)}$. This produces a ringed space $(X, \mathcal{O}_X)$ for each affine algebraic set, and functions of the type described above are continuous in the Zarkiski topology (here taking k in the Zariski topology as well).

Example 8.5. Let X, Y be affine algebraic sets considered as ringed spaces following Example 8.4. For $f : X \to Y$ a continuous map between affine algebraic sets, there is an induced map $f^\sharp : \mathcal{O}_Y \to f_* \mathcal{O}_X$ when f is regular, that is, for all opens $U \subseteq X$ and $g : V \to k$ in $\mathcal{O}_Y(V)$ the composite

$$f^{-1}(V) \xrightarrow{\quad f \quad} V \xrightarrow{\quad g \quad} k$$

is regular on $f^{-1}(V) \subseteq X$.

Schemes are in fact examples of locally ringed spaces, and we first turn to a discussion of this more abstract setting.

Definition 8.6 (Locally Ringed Space)**.** A ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if for all points $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring.

To discuss examples, we require the following lemma from commutative algebra.

Lemma 8.7. Let A be a commutative ring and $\mathfrak{a} \subseteq A$ an ideal. If a is invertible for all $a \in A \setminus \mathfrak{a}$ then $\mathfrak{a}$ is the unique maximal ideal of A.

Proof. Each element of $A \setminus \mathfrak{m}$ generates A as an ideal, in particular, is not contained in any proper maximal ideal. ∎

Example 8.8. Let X be a topological space and C_X its sheaf of continuous functions. For each $x \in X$, the stalk $C_{X,x}$ contains a prime ideal $\mathfrak{p}_x$ consisting of functions vanishing at x. This ideal is in fact the unique maximal ideal of $C_{X,x}$ since if $f \in C_{X,x} \setminus \mathfrak{p}_x$ then there exists a neighborhood V around x on which f is continuous. We can consider $V \setminus f^{-1}(0) \subseteq V$ open, since $f^{-1}(0)$ is closed on which f is nonzero. f is invertible on this open so the germ f is invertible, and maximality and uniqueness follow from Lemma 8.7.

Remark 8.9. Differentiable manifolds, complex manifolds, and affine algebraic sets can be argued to be locally ringed spaces in the same way.

Morphisms of locally ringed spaces will be local ring homomorphisms on stalks.

Definition 8.10 (Local Ring Homomorphism)**.** Let A, B be local rings and $\varphi : A \to B$ be a homomorphism of rings. φ is a local ring homomorphism of $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

Remark 8.11. The containment $\varphi^{-1}(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$ always holds.

Example 8.12. Let A be a local integral domain. The map $A \to \mathrm{Frac}(A)$ is a ring homomorphism between local rings but not a local ring homomorphism, for example $\mathbb{Z}_{(p)} \to \mathbb{Q}$.

We can now define morphisms of locally ringed spaces.

Definition 8.13 (Morphism of Locally Ringed Spaces)**.** Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be locally ringed spaces. A morphism of ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of locally ringed spaces if for all $x \in X$ the induced map $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to (f_*\mathcal{O}_X)_{f(x)} = \mathcal{O}_{X,x}$ is a local ring homomorphism.

This phenomena is captured in morphisms of spectra of rings.

Proposition 8.14. Let $\varphi : A \to B$ be a ring homomorphism and $\mathfrak{q} \subseteq B$ with $\mathfrak{p} = \varphi^{-1}(\mathfrak{q}) \subseteq A$. Then $A_\mathfrak{p} \to B_\mathfrak{q}$ is a local ring homomorphism.

Proof. An element of $A_\mathfrak{p} \setminus \mathfrak{p}A_\mathfrak{p}$ is of the form a/a' where $a, a' \in A \setminus \mathfrak{p}$ so its image $b/b' = \varphi(a)/\varphi(a')$ is such that $b, b' \in B \setminus \mathfrak{q}$ and hence invertible. ∎

Now for a long awaited definition, we can define affine schemes.

Definition 8.15 (Affine Scheme)**.** An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ isomorphic to $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ as a locally ringed spaec for some ring A.

We consider some examples.

Example 8.16. $\mathrm{Spec}(\mathbb{Z})$ has stalks given by $\mathbb{Z}_{(p)}$ for positive primes p and $\mathbb{Q}$ over (0).

Example 8.17. Let k be a field. $\mathrm{Spec}(k) = \{*\}$ with global sections k.

Example 8.18. $\mathrm{Spec}(A[x_1, \ldots, x_n])$ for a ring A.

Example 8.19. $\mathrm{Spec}(A)$ for a discrete valuation ring A. Affine schemes of this type can be used to test certain properties of morphisms of schemes.

Example 8.20. $\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$ consists of two points as a topological space, one closed point and one generic point. This scheme can be used to test local properties of schemes.

Remark 8.21. Note that Examples 8.19 and 8.20 have the same topological space, but have quite different properties as schemes. Grothendieck's insight was that keeping track of the sheaves of rings preserves very important information that goes missing when only considering the underlying topological spaces.

We can now define schemes more generally.

Definition 8.22 (Scheme)**.** A scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that for each $x \in X$ there exists an open set $U \subseteq X$ with $x \in U$ with $(U, \mathcal{O}_X|_U)$ an affine scheme.

Remark 8.23. By the condition of admitting neighborhoods around every point that are affine schemes, schemes are naturally locally ringed spaces.

Scheme and affine schemes form natural subcategories of locally ringed spaces.

Definition 8.24 (Category of Affine Schemes)**.** The category of affine schemes Aff has objects affine schemes and morphisms those of locally ringed spaces.

Definition 8.25 (Category of Schemes)**.** The category of schemes Sch has objects schemes and morphisms those of locally ringed spaces.

By definiton, we have $\mathsf{Aff}, \mathsf{Sch}$ as full subcategories of the category of locally ringed spaces LRS with objects locally ringed spaces and morphisms those of locally ringed spaces in the sense of Definition 8.13. Summing up, we have

$$\mathsf{Aff} \hookrightarrow \mathsf{Sch} \hookrightarrow \mathsf{LRS} \hookrightarrow \mathsf{RS},$$

here denoting RS the category of ringed spaces with objects ringed spaces and morphisms those of ringed spaces in the sense of Definition 7.12. Note, however, that LRS is not a full subcategory of RS, and in this way being a locally ringed space is a structure of a ringed space. See [nLab-I].

Example 8.26. Let A be a local integral domain and $A \to \mathrm{Frac}(A)$ the inclusion inducing the map of affine schemes $\mathrm{Spec}(\mathrm{Frac}(A)) \to \mathrm{Spec}(A)$. This is a morphism of locally ringed spaces as $\mathcal{O}_{\mathrm{Spec}(A),(0)} \to (f_*\mathcal{O}_{\mathrm{Spec}(\mathrm{Frac}(A))})_{(0)}$ is just the morphism $\mathrm{Frac}(A) \to \mathrm{Frac}(A)$.

Example 8.27. Let $X = Y = \mathbb{R}^n$ and C_X, C_Y^{diff} the sheaves of continuous and differentiable functions on X, Y, respectively. There is a natural map $(f, f^\sharp) : (X, C_X) \to (Y, C_Y^{\mathrm{diff}})$ with f the identity and $f^\sharp$ the inclusion of $C_Y^{\mathrm{diff}} \to C_X$ since the direct image f_*C_X under the identity is just C_X. Note that this is a homeomorphism of topological spaces but not an isomorphism of locally ringed spaces.

Example 8.28. Let $X = Y = \mathbb{C}^n$ and $\mathcal{O}_X, \mathcal{O}_Y$ the sheaves of holomorphic and regular functions on X, Y, respectively. There is a natural map $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ with f the identity and $f^\sharp$ the inclusion of regular functions into holomorphic functions. Note that here f is not a homeomorphism of the underlying topological spaces as X and Y have the analytic and Zariski topologies, respectively, which are not equivalent. The map remains continuous as the Zariski topology is coarser than the analytic one.

Example 8.29. Let $X = \mathbb{C}^n$, $\mathcal{O}_X$ the sheaf of holomorphic functions on X, $Y = \mathbb{A}^n_{\mathbb{C}}$ an affine scheme with structure sheaf $\mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}}$. Hilbert's Nullstellensatz gives an equivalence between $\mathrm{mSpec}(\mathbb{C}[x_1, \ldots, x_n])$ and $\mathbb{C}^n$ which induces a natural map $\mathbb{C}^n \to \mathbb{A}^n_{\mathbb{C}}$ which is once again continuous as the Zariski topology is coarser than the analytic topology. There is a natural morphism of sheaves taking a function $s : U \to \coprod_{\mathfrak{p} \in U} \mathbb{C}[x_1, \ldots, x_n]_{\mathfrak{p}}$ to

$$(U \cap \mathrm{mSpec}(\mathbb{C}[x_1, \ldots, x_n])) \xrightarrow{\quad s \quad} \coprod_{\mathfrak{p} \in U} \mathbb{C}[x_1, \ldots, x_n]_{\mathfrak{p}} \xrightarrow{\qquad} \mathbb{C}$$

which induces a morphism of locally ringed spaces since $s(\mathfrak{m}) \in A_{\mathfrak{m}}$ with $A_{\mathfrak{m}}$ local and implied by the diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\sim"] \arrow[d] & A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}} \\
A_{\mathfrak{m}} \arrow[ur] &
\end{tikzcd}$$

commuting.

9. Lecture 9 – 8th November 2024

Several of the examples presented in Section 8 suggest a close connection between scheme theory and the study of classical algebraic geometry. We begin with a brief interlude connecting these areas of study, before returning to a discussion of scheme theory.

Classical algebraic geometry begins with a an algebraically closed field $k = \overline{k}$ or $\mathbb{C}$ and an affine algebraic set $X \subseteq k^n$ which is naturally a ringed space as described in Example 8.4. For such an affine algebraic set we can consider the ideal $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ consisting of polynomials vanishing on X which by Hilbert's Nullstellensatz can be identified with $\mathrm{mSpec}(A[X])$ where $A[X] = k[x_1, \ldots, x_n]/\sqrt{\mathfrak{a}}$ and $\sqrt{\mathfrak{a}}$ is the radical ideal of $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$. We can define an affine scheme $\mathrm{Spec}(A[X])$ with its associated structure sheaf which admits a map from $X = \mathrm{mSpec}(A[X])$ by the inclusion of the maximal spectrum into the prime spectrum which is continuous in the Zariski topology by definition. See, for example, [Stacks, Tag 00FS].

The map $f : X \to \mathrm{Spec}(A[X])$ extends to a map of ringed spaces with $f^\sharp : \mathcal{O}_{\mathrm{Spec}(A[X])} \to f_*\mathcal{O}_X$ associating a section $s : U \to \coprod_{\mathfrak{p} \in U} A[X]_\mathfrak{p}$ to the section $\overline{s} : U \cap \mathrm{mSpec}(A[X]) \to \coprod_{\mathfrak{m} \in U \cap \mathrm{mSpec}(A[X])} A[X]_\mathfrak{m}$ which naturally extends to k under the isomorphism $A[X]_\mathfrak{m}/\mathfrak{m}A[X]_\mathfrak{m} \cong k$, and in fact define an isomorphism on stalks, making $f^\sharp$ a morphism of locally ringed spaces.

In fact, passing from algebra to (affine) schemes loses no information. To show this, we will require the following lemma.

Lemma 9.1. The prime spectrum of a ring $\mathrm{Spec}(A)$ is a quasicompact topological space – each open cover admits a finite subcover.

Proof. Let $\{U_i\}_{i \in I}$ be an open cover of X. Since distinguished open sets $D(f_i)$ for $f_i \in A$ form a basis for the Zariski topology we can, without loss of generality, take the cover to be $\{D(f_i)\}_{i \in I}$. As such, the ideal generated by $\{f_i\}_{i \in I}$ is A and there thus exists a finite subset $J \subseteq I$ such that $\sum_{i \in J} a_i f_i = 1$ producing $\{D(f_i)\}_{i \in J}$ a finite subcover. ■

We are now prepared to show the proposition.

Proposition 9.2. There is an equivalence of categories between $\mathsf{Ring}^{\mathsf{Opp}}$ and AffSch.

Proof. We first show that the functor $A \to \mathrm{Spec}(A)$ is fully faithful: that is, for rings A, B, there is a bijection

$$\mathrm{Mor}_{\mathsf{Ring}}(A, B) \to \mathrm{Mor}_{\mathsf{AffSch}}(\mathrm{Spec}(B), \mathrm{Spec}(A)).$$

Suppose $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a morphism of affine schemes. We define a ring morphism $A \to B$ by taking global sections on $f^\sharp$ giving

$$\mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)) = A \to B = \mathcal{O}_{\mathrm{Spec}(B)}(\mathrm{Spec}(B)) = \mathcal{O}_{\mathrm{Spec}(B)}(f^{-1}(\mathrm{Spec}(A))).$$

Conversely given a ring homomorphism $\varphi : A \to B$, we can construct a morphism of topological spaces $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ by taking $\mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q})$. We can define a map on structure sheaves $f^\sharp : \mathcal{O}_{\mathrm{Spec}(A)} \to f_*\mathcal{O}_{\mathrm{Spec}(B)}$ by considering open distinguished open subsets on which the map is given by $\mathcal{O}_{\mathrm{Spec}(A)}(D(f_i)) \to$

$f_*\mathcal{O}_{\mathrm{Spec}(B)}(D(f_i)) = \mathcal{O}_{\mathrm{Spec}(B)}(D(\varphi(f_i)))$ which is unique by the universal property of localization. This commutes with restriction as on $D(f_if_j)$ the induced map $\mathcal{O}_{\mathrm{Spec}(A)}(D(f_if_j)) \to f_*\mathcal{O}_{\mathrm{Spec}(B)}(D(\varphi(f_if_j))) = \mathcal{O}_{\mathrm{Spec}(B)}(D(\varphi(f_i)\varphi(f_j)))$. Moreover, this morphism is a local ring homomorphism in the sense of Definition 8.10 since for $\mathfrak{q} \subseteq B$ prime with $f(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q}) = \mathfrak{p} \subseteq A$ we get the commutative diagram on taking global sections and localizing

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ \mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}} \arrow[r, "f^\sharp_{\mathfrak{q}}"'] & \mathcal{O}_{\mathrm{Spec}(B),\mathfrak{q}} \end{tikzcd}$$

giving $\mathfrak{q}B_{\mathfrak{q}} = \varphi^{-1}(\mathfrak{q})A_{\varphi^{-1}(\mathfrak{q})} = \mathfrak{p}A_{\mathfrak{p}}$ showing locality.

It remains to verify these constructions are mutually inverse. For $\varphi : A \to B$, sections on $D(1)$ over $f^\sharp : \mathcal{O}_{\mathrm{Spec}(A)} \to f_*\mathcal{O}_{\mathrm{Spec}(B)}$ recovers φ. Conversely for $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ and φ the induced morphism of rings by taking global sections on the target, we can repeat the construction and produce a morphism of ringed spaces $(g, g^\sharp)$ which we want to show is equal to $(f, f^\sharp)$ where equality follows from commutativity with localization in the construction of $(f, f^\sharp)$ from $\varphi : A \to B$ above.

Finally, we note that the functor is essentially surjective by definition since each affine scheme is of the form $\mathrm{Spec}(A)$ for some ring A. This gives the claim. ■

We have just seen that affine schemes are indeed an enhancement of phenomena studied in classical algebraic geometry. To consider all schemes, however, we have to consider how affine schemes may be glued to a a general scheme. This is done by the general gluing lemma for ringed spaces.

Proposition 9.3. Let X be a topological space, $\mathcal{U} = \{U_i\}_{i\in I}$ an open cover of X, and $\mathcal{F}_i$ a sheaf on U_i with isomorphisms of sheaves $\phi_{ij} : \mathcal{F}_i|_{U_i\cap U_j} \to \mathcal{F}_j|_{U_i\cap U_j}$ such that the following hold:

(1) For each i, $\phi_{ii} : \mathcal{F}_i|_{U_i} \to \mathcal{F}_i|_{U_i}$ is $\mathrm{id}_{\mathcal{F}_i}$.
(2) For each i, j, k, $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_i \cap U_j \cap U_k$.

There is a unique sheaf $\mathcal{F}$ on X with isomorphisms $\psi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ such that for each i, j, $\psi_j = \phi_{ij} \circ \psi_i$.

Proof. Let $V \subseteq X$ open and consider

$$\mathcal{F}(V) = \left\{(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}_i(U_i \cap V) : \phi_{ij}(s_i|_{U_i\cap U_j}) = s_j|_{U_i\cap U_j} \forall i, j \in I\right\}.$$

This is evidently a presheaf since if $W \subseteq V$ we have

$$\mathcal{F}(W) = \left\{(s_i|_W)_{i\in I} : (s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}_i(U_i \cap V), \phi_{ij}(s_i|_{U_i\cap U_j\cap W}) = s_j|_{U_i\cap U_j\cap W} \forall i, j \in I\right\}$$

by restricting sectoins on each $\mathcal{F}_i(U_I \cap V)$. We show this is a sheaf. Suppose $\{V_\alpha\}_{\alpha\in A}$ is an open cover of V. (Identity) Further suppose $t, t' \in \mathcal{F}(V)$ given by $(s_i)_{i\in I}, (s'_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}_i(U_i \cap V)$ such that $t|_{V_\alpha} = t'|_{V_\alpha}$ for all $\alpha \in A$. Note further that the cover $\{U_i \cap V_\alpha\}_{\alpha\in A, i\in I}$ of V refines $\{V_\alpha\}_{\alpha\in A}$. We know that $s_i|_{U_i\cap V_\alpha} = s'_i|_{U_i\cap V_\alpha}$ for all $\alpha \in A$ and all $i \in I$ so $s_i = s'_i$ by identifying in $\mathcal{F}_i(U_i \cap V)$ on the cover $\{U_i \cap V_\alpha\}_{\alpha\in A, i\in I}$ thus showing $t = t'$ since their entries agree pointwise. (Gluability) Let $t_\alpha \in \mathcal{F}(V_\alpha)$ be sections given by $(s_{\alpha,i})_{i\in I} \in \prod_{i\in I} \mathcal{F}_i(U_i \cap V_\alpha)$ such that $t_\alpha|_{V_\alpha\cap V_\beta} = t_\beta|_{V_\alpha\cap V_\beta}$ for all $\alpha, \beta \in A$. Thus we have $s_{\alpha,i}|_{U_i\cap V_\alpha\cap V_\beta} = s_{\beta,i}|_{U_i\cap V_\alpha\cap V_\beta}$ for all $i \in I$ and $\alpha, \beta \in A$. Note that $\{U_i \cap V_\alpha\}_{\alpha\in A}$ forms an open cover of U_i and $s_{\alpha,i}$ agrees with s_{β_i} on double intersections and hence glues uniquely to a section of $s \in \mathcal{F}_i(U_i \cap V)$ so t glues by the gluing of each entry in $\mathcal{F}_i(U_i \cap V)$. The fact that $\mathcal{F}$ is unique follows from checking on stalks. In particular $\mathcal{F}_p \cong (\mathcal{F}_i)_p$ for all $p \in U_i$ and $\phi_{ij}((\mathcal{F}_i)_p) = (\mathcal{F}_j)_p = \mathcal{F}_p$ on $U_i \cap U_j$.

Take $\psi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ by $(s_i)_{i\in I} \mapsto s_i$ and on $U_i \cap U_j$ for $(s_i)_{i\in I} \in \mathcal{F}$ we have $s_j = \phi_{ij}(s_i)$ giving the desired equalities. ■

We now specialize to schemes.

Proposition 9.4. Let $\{X_i\}_{i\in I}$ be a family of schemes. For each $i \neq j$, suppose given an open subset $U_{ij} \subseteq X_i$ with the induced subscheme structure, isomorphisms $\phi_{ij} : U_{ij} \to U_{ji}$ such that:

(1) For each i, j, $\phi_{ji} = \phi_{ij}^{-1}$.
(2) For each i, j, k $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$.
(3) For each i, j, k $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$.

Show that there is a scheme X together with morphisms $\psi_i : X_i \to X$ for each i such that

(1) ψ_i is an isomorphism of X_i onto an open subscheme of X.
(2) The $\psi_i(X_i)$ cover X.
(3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$.
(4) $\psi_i = \psi_j \circ \phi_{ij}$ on U_{ij}.

Proof. We first construct X as a topological space, taking

$$X = \coprod_{i\in I} X_i / \sim$$

where $p \sim q$ if and only if $\phi_{ij}(p) = q$ for some pair of indices $i, j \in I$. Note that in X, $\psi_i(X_i)$ is open since it is obtained by gluing the open X_i along open subsets U_{ij} and thus an inclusion and isomorphism onto its image by construction. The sheaves $\mathcal{O}_{X_i}$ can be glued analogously to Proposition 9.3 yielding a locally ringed space. Furthermore, X is a scheme since it is the gluing of X_i's, each of which is a gluing of affine schemes. ■

Example 9.5. Let $X_1 = X_2 = \mathbb{A}^1_k$, $U_1 = U_2 = \mathbb{A}^1_k \setminus \{0\}$ glued along the map $x \mapsto x$. The resulting space is the line with doubled origin.

Example 9.6. Let $X_1 = X_2 = \mathbb{A}^1_k$, $U_1 = U_2 = \mathbb{A}^1_k \setminus \{0\}$ glued along the map $x \mapsto 1/x$. The resulting space is the projective line $\mathbb{P}^1_k$.

Remark 9.7. We should in some sense consider Examples 9.5 and 9.6 quite surprising since the line with doubled origin will serve as an example of a non-separated scheme which are poorly behaved, while the projective line will serve as an example of a proper scheme which are extremely well-behaved, even though they are obtained by gluing the same space $\mathbb{A}_k^1$.

We can now begin a discussion of properties of schemes, by listing definitions of properties of schemes. We first consider topological properties, which are detected on the underlying topological space.

Definition 9.8 (Connected Scheme)**.** Let X be a scheme. X is connected if there does not exist X_1, X_2 nonempty proper closed subsets of X with $X = X_1 \sqcup X_2$.

Definition 9.9 (Irreducible Scheme)**.** Let X be a scheme. X is irreducible if there does not exist U_1, U_2 nonempty proper closed subsets of X with $X_1 \cup X_2 = X$.

Definition 9.10 (Quasicompact Scheme)**.** Let X be a scheme. X is quasicompact if for every open cover of X admits a finite subcover.

Remark 9.11. We will not use the notion of "compact" in algebraic geometry which was first defined by the Bourbaki school to mean a space being both quasicompact and Hausdorff. The Hausdorff property is almost never fulfiled in the algebro-geometric setting.

Example 9.12. Any affine scheme is quasicompact by Lemma 9.1.

We conclude with the definition of a scheme-theoretic property – a property detected on the level of structure sheaves.

Definition 9.13 (Locally Noetherian Scheme)**.** Let X be a scheme. X is locally Noetherian if it admits an affine open covering by spectra of Noetherian rings.

Remark 9.14. This condition can be shown to be equivalent to every affine open subset of X being the spectrum of a Noetherian ring.

A locally Noetherian scheme that is quasicompact is said to be Noetherian.

Definition 9.15 (Noetherian Scheme)**.** Let X be a scheme. X is Noetherian if it is locally Noetherian and quasicompact.

10. Lecture 10 – 11th November 2024

We prove the statement alluded to in Remark 9.14, and more generally consider properties of schemes where the existence of an affine cover with that property implies each affine subset of the scheme has the property.

In lecture, [Har83, Prop. II.3.2] was proven. We take a more general approach via the "Affine Communication Lemma" presented in [Vak24, §5.3] and [Stacks, Tag 01OH].

The key technical step in the proof of the affine communication lemma is to produce open subsets of intersections that are basis elements of affine schemes.

Lemma 10.1. Let X be a scheme and $\mathrm{Spec}(A), \mathrm{Spec}(B) \subseteq X$. Then $\mathrm{Spec}(A) \cap \mathrm{Spec}(B)$ is the union of open sets that are simultaneously basis elements of the Zariski topologies on $\mathrm{Spec}(A), \mathrm{Spec}(B)$.

Proof. Let $\mathfrak{p} \in \mathrm{Spec}(A) \cap \mathrm{Spec}(B)$. Noting that the intersection $\mathrm{Spec}(A) \cap \mathrm{Spec}(B)$ is open in both $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$, there exists a basis open $\mathrm{Spec}(A_f)$ of $\mathrm{Spec}(A)$ in $\mathrm{Spec}(A) \cap \mathrm{Spec}(B)$ containing $\mathfrak{p}$. Arguing similarly, we can take $\mathrm{Spec}(B_g)$ a basis open of $\mathrm{Spec}(B)$ containing $\mathfrak{p}$ in $\mathrm{Spec}(A_f)$, the latter an open of $\mathrm{Spec}(A) \cap \mathrm{Spec}(B)$ and hence one of $\mathrm{Spec}(B)$. As such, the section $g \in \Gamma(\mathrm{Spec}(B), \mathcal{O}_X)$ restricts to $g' \in \Gamma(\mathrm{Spec}(A_f), \mathcal{O}_X) = A_f$ and the primes on which g and g' vanish are the same so

$$\mathrm{Spec}(B_g) = \mathrm{Spec}(A_f) \setminus \{[\mathfrak{p}] : g' \in \mathfrak{p}\} = \mathrm{Spec}((A_f)_{g'})$$

where we note that if $g' \in A_f$ of the form h/f^n for some $h \in A$ then $\mathrm{Spec}((A_f)_{g'}) = \mathrm{Spec}(A_{ah})$, giving the claim. ∎

The affine communication lemma can then be inferred as follows.

Lemma 10.2 (Affine Communication Lemma)**.** Let X be a scheme and P a property of affine open subschemes of X such that both the following conditions hold:

(i) For any $\mathrm{Spec}(A) \subseteq X$ with property P, $D(f)$ has property P for all $f \in A$.
(ii) If $f_1, \dots, f_n$ generate A and each $D(f_i)$ has property P then so does $\mathrm{Spec}(A)$.

If there is an affine open cover $\{U_i\}_{i \in I}$ of X where each U_i has P for all $i \in I$ then every affine subset of X has the property P.

Proof. Let $\{U_i\}_{i \in I}$ be an affine open cover of X such that each U_i has the property P. Let $V \subseteq X$ be another affine open subset. The $\{U_i \cap V\}_{i \in I}$ form an open cover of V. Lemma 10.1 implies each $U_i \cap V$ can be covered by affines $\{T_j\}_{j \in J}$ that are simultaneously distinguished in $\Gamma(U_i, \mathcal{O}_X|_{U_i})$ and $\Gamma(V, \mathcal{O}_X|_V)$. By (i), each of the T_j have P and for $T_j = D(g_j) \subseteq \mathrm{Spec}(\Gamma(V, \mathcal{O}_Y|_V))$ we have that the $\{D(g_j)\}_{j \in J}$ covering V so V has P by (ii). ∎

It thus suffices to show that the property of being locally Noetherian satisfies the hypotheses of Lemma 10.2.

Proposition 10.3. Let X be a locally Noetherian scheme. Then every $U \subseteq X$ affine open is the spectrum of a Noetherian ring.

Proof. The hypothesis (i) of Lemma 10.2 is clear since the property of being Noetherian is preserved under localization. For (ii), let $(f_1, \dots, f_n) = A$ and $\mathfrak{a} \subseteq A$ an ideal and $\varphi_i : A \to A_{f_i}$ the localization maps for each i. We show $\mathfrak{a} = \bigcap_{i=1}^n \varphi_i^{-1}(\varphi_i(\mathfrak{a}) A_{f_i})$.

The containment $\mathfrak{a} \subseteq \bigcap_{i=1}^n \varphi_i^{-1}(\varphi_i(\mathfrak{a})A_{f_i})$ holds on the level of sets. Conversely given some $b \in A$ contained in $\bigcap_{i=1}^n \varphi_i^{-1}(\varphi_i(\mathfrak{a})A_{f_i})$ so $\varphi_i(b) = \frac{a_i}{f_i^{n_i}}$ in A_{f_i} with $a_i \in A$ and $n_i > 0$. Let $N = \max_{1 \le i \le n}\{n_i\}$ and up to multiplication by $\frac{f_i^{N-n_i}}{f_i^{N-n_i}}$ we have $\varphi_i(b) = \frac{a_i}{f_i^N}$ with $a_i \in A$. By definition of localization, there is m_i such that $f_i^{m_i}(f_i^N - a_i) = 0$ and arguing as before, taking $M = \max_{1 \le i \le n}\{m_i\}$ and multiplying these expressions by $f_i^{M-m_i}$ for ecah i we get $f_i^{M+N}b \in \mathfrak{a}$for each i. Now since $(f_1, \dots, f_N) = 1$ we have $1 = \sum_{i=1}^n c_i f_i^{N+M}$ for $c_i \in A$ where we now wrtie $b = \sum_{i=1}^n c_i f_i^{N+M} b \in \mathfrak{a}$ showing the desired equality $\mathfrak{a} \subseteq \bigcap_{i=1}^n \varphi_i^{-1}(\varphi_i(\mathfrak{a})A_{f_i})$.

Now for any ascending chain of ideals in A, we can consider the image of that chain under φ_i in A_{f_i} but these eventually stabilize at some T_i and the preceding discussion shows that the chain in A stabilizes at $\max_{1 \le i \le n}\{T_i\}$. ∎

This implies the following corollary.

Corollary 10.4. If Spec(A) is locally Noetherian then A is a Noetherian ring.

Proof. This is Proposition 10.3 for $X = U = \operatorname{Spec}(A)$. ∎

Remark 10.5. There is a notion of a Noetherian topological space. If a scheme is Noetherian, so is its underlying space, but not conversely.

We can also state more properties of schemes that are defined in terms of the structure sheaf.

Definition 10.6 (Reduced Scheme)**.** Let X be a scheme. X is reduced if $\mathcal{O}_X(U)$ is a reduced ring for all $U \subseteq X$ open.

Definition 10.7 (Integral Scheme)**.** Let X be a scheme. X is integral if $\mathcal{O}_X(U)$ is an integral domain for all $U \subseteq X$ open.

Remark 10.8. An integral scheme has all stalks integral domains, but not conversely. $\operatorname{Spec}(k) \coprod \operatorname{Spec}(k)$ is integral on each stalk with sections k, but has global sections $k \times k$ which is not integral.

Example 10.9. The ring of dual nubmers Example 8.20 is neither reduced nor integral.

Though, in fact, integrality implies reducedness.

Proposition 10.10. Let X be a scheme. If X is integral, then X is reduced.

Proof. Every integral domain is reduced since every nilpotent is a zerodivisor. ∎

Example 10.11. The union of coordinate axes in $\mathbb{A}_k^2$ given by $\operatorname{Spec}(k[x,y]/(xy))$ is reduced but not integral.

We can say more after showing the followign lemma.

Lemma 10.12. Let X be a scheme and $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the subset of points $p \in X$ such that the stalk f_p of f at p is not contained in the maximal ideal of the ring $\mathcal{O}_{X,p}$. If $U = \operatorname{Spec}(B)$ is an affine open subscheme of X and if $\overline{f} \in B = \Gamma(U, \mathcal{O}_X|_U)$ is the restriction of f, then $U \cap X_f = D(\overline{f})$ and X_f is open in X.

Proof. We have

$$p \in X_f \cap U \iff f_p \notin \mathfrak{m}_{X,p} \iff \overline{f}_p \notin \mathfrak{m}_{U,p} \iff p \in D(\overline{f}).$$

Furthermore, X_f is open as it is the union of affine opens $\operatorname{Spec}(B_{\overline{f}})$ open in $\operatorname{Spec}(B)$ which cover X – ie. X_f is locally open in the subspace topology, hence open in the subspace topology. ∎

Proposition 10.13. Let X be a scheme. X is integral if and only if X is irreducible and reduced.

Proof. ($\Rightarrow$) Integrality implies reducedness by Proposition 10.10. Suppose to the contrary that X is reducible, in which case there are nonempty open subsets U_1, U_2 such that $X = U_1 \coprod U_2$ and $U_1 \cap U_2 = \emptyset$. Then $\mathcal{O}_X(U_1 \coprod U_2) = \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$ which is not integral, a contradiction.

($\Leftarrow$) Suppose X is irredicuble and reduced so there exists $U \subseteq X$ nonempty open and $s_1, s_2 \in \mathcal{O}_X(U)$ with $s_1 s_2 = 0$. We want to show that one of s_1, s_2 is 0. By Lemma 10.12, we have that $X_i = \{x \in U : s_{i,x} \in \mathfrak{m}_x\} \subseteq U$ is closed. For all $x \in U$, $(s_1 s_2)_x = 0$ implies $(s_1)_x (s_2)_x = 0$ so one of $(s_1)_x, (s_2)_x \in \mathfrak{m}_x$ with $X_1 \cup X_2 = U$. Since X is irreducible so too is U so take $X_1 = U$ and for $\operatorname{Spec}(A) \subseteq U$ open, we can define $t = s_1|_{\operatorname{Spec}(A)}$. Thus for all $x \in U$, and in particular all $x \in \operatorname{Spec}(A)$, $t_x \in \mathfrak{m}_x$ and all prime ideals of A contain t so t is in the nilradical and zero because X is reduced. This shows that $s_1|_{\operatorname{Spec}(A)} = 0$, but then $s_1 = 0$ by the sheaf condition. ∎

Corollary 10.14. If X is an integral scheme then there is a unique point $x \in X$ such that $\overline{\{x\}} = X$.

Proof. By hypothesis A is an integral domain for $\operatorname{Spec}(A) \subseteq X$. Taking $(0) \in \operatorname{Spec}(A)$ we have that its closure is all of X by irreducibility. For uniqueness we can consider some affine open contained in $\operatorname{Spec}(A)$ and repeat the argument to see that (0) is the generic point. ∎

We now discuss closed subschemes. Note that open subschemes of affine schemes are not necessarily affine.

Example 10.15. $\mathbb{A}^n_k \setminus \{0\}$ is an open subscheme of an affine scheme but not itself affine for $n \geq 2$.

We now state the definiton of a closed subscheme.

Definition 10.16 (Closed Subscheme)**.** Let X be a scheme. A closed subscheme is a closed locally ringed subspace $i : Z \hookrightarrow X$ such that there exists a sheaf of ideals $\mathcal{I}_Z \subset \mathcal{O}_X$ with $\mathcal{O}_X / \mathcal{I}_Z \cong i_* \mathcal{O}_Z$.

11. Lecture 11 – 15th November 2024

Extending the correspondence between radical ideals and closed algebraic subsets, we have an equivalence between closed subschemes and ideals of a ring. In particular, the generality schemes provides allows us to treat all ideals instead of just radical ideals.

Proposition 11.1. Let A be a ring and $\operatorname{Spec}(A)$ be a ring. There is a bijection between ideals of A and closed subschemes of $\operatorname{Spec}(A)$.

Proof. The maps are given by $\mathfrak{a} \mapsto \operatorname{Spec}(A/\mathfrak{a})$ and $Z \mapsto \ker(\mathcal{O}_{\operatorname{Spec}(A)}(\operatorname{Spec}(A)) \to i_*\mathcal{O}_Z(\operatorname{Spec}(A)))$ where $\mathfrak{a} = \ker(\mathcal{O}_{\operatorname{Spec}(A)}(\operatorname{Spec}(A)) \to i_*\mathcal{O}_{\operatorname{Spec}(A/\mathfrak{a})}(\operatorname{Spec}(A)))$. Conversely for $Z \subseteq X$ closed, let $\mathfrak{a}_Z = \ker(\mathcal{O}_{\operatorname{Spec}(A)}(\operatorname{Spec}(A)) \to i_*\mathcal{O}_Z(\operatorname{Spec}(A)))$ so we have an exact sequence $0 \to \mathfrak{a}_Z \to A \to \mathcal{O}_Z(Z)$ for any $f \in A$ mapping to $g \in Y$, exactness of localization we have $0 \to \mathfrak{a}_{Z,f} \to A_f \to \mathcal{O}_Z(Z_g) = \mathcal{O}_Z(Z)_g$ so $\mathfrak{a}_Z$ is an ideal sheaf where sections on $D(f)$ are $\mathfrak{a}_{Z,f} = \mathfrak{a}_Z \otimes_A A_f$ which satisfies the property of the ideal sheaf of Z so Z is $\operatorname{Spec}(A/\mathfrak{a}_Z)$. ■

We now discuss fibered products of schemes. This will lead to the construction of the category of S-schemes and k-schemes when $S = \operatorname{Spec}(k)$.

Definition 11.2 (Fibered Product)**.** Let $f : X \to S, g : Y \to S$ be morphisms of schemes. The fibered product, if it exists, is a scheme $X \times_S Y$ with maps $\mathrm{pr}_X : X \times_S Y \to X, \mathrm{pr}_Y : X \times_S Y \to Y$ such that for all schemes Z with maps to X, Y that agree on S

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_S Y \arrow[r, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_Y"'] & X \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}$$

there is a unique morphism $Z \to X \times_S Y$ making the diagram commute.

We will eventually show that the fibered product of any two schemes over any base exists. We begin in the affine case with the following preparatory statement which extends Proposition 9.2.

We learned of this argument from Ben Steffan's notes.

Proposition 11.3. There is a bijection

$$\operatorname{Mor}_{\mathsf{Ring}}(A, \Gamma(X, \mathcal{O}_X)) \to \operatorname{Mor}_{\mathsf{Sch}}(X, \operatorname{Spec}(A))$$

for all rings A and schemes X.

Proof. Let $\{\operatorname{Spec}(B_i)\}_{i \in I}$ be an affine open cover of X and $\{\operatorname{Spec}(B_{ijk})\}_{k \in K_{ij}}$ be an affine open cover of $\operatorname{Spec}(B_i) \cap \operatorname{Spec}(B_j)$. Now note the sequence

$$\operatorname{Mor}_{\mathsf{Sch}}(X, \operatorname{Spec}(A)) \longrightarrow \prod_{i,j \in I} \operatorname{Mor}_{\mathsf{Sch}}(\operatorname{Spec}(B_i), \operatorname{Spec}(A))$$

$$\rightrightarrows \prod_{i,j \in I, k \in K_{ij}} \operatorname{Mor}_{\mathsf{Sch}}(\operatorname{Spec}(B_{ijk}), \operatorname{Spec}(A))$$

is an equalizer since a morphism of schemes $X \to \mathrm{Spec}(A)$ is determined by morphisms on the open cover $\{\mathrm{Spec}(B_i)\}_{i\in I}$ which agree on the intersections $\mathrm{Spec}(B_i)\cap \mathrm{Spec}(B_j)$ and hence in particular on the affine subschemes of intersections $\mathrm{Spec}(B_{ijk})$. Applying Proposition 9.2, we have by passing to global sections

$$\begin{aligned} \mathrm{Mor}_{\mathsf{Ring}}(A, \Gamma(X, \mathcal{O}_X)) \longrightarrow {} & \textstyle\prod_{i,j\in I} \mathrm{Mor}_{\mathsf{Ring}}(A, B_i) \\ \rightrightarrows {} & \textstyle\prod_{i,j\in I, k\in K_{ij}} \mathrm{Mor}_{\mathsf{Ring}}(A, B_{ijk}) \end{aligned}$$

by the sheaf property on X. ■

We deduce that taking the spectrum of global sections is the left ajdoint of the inclusion of affine schemes to schemes.

Corollary 11.4. The functor $X \mapsto \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ is left adjoint to the inclusion $\mathsf{AffSch} \to \mathsf{Sch}$.

Proof. From Propositions 9.2 and 11.3, we have a bijection

$$\mathrm{Mor}_{\mathsf{AffSch}}(\mathrm{Spec}(\Gamma(X, \mathcal{O}_X)), \mathrm{Spec}(A)) \to \mathrm{Mor}_{\mathsf{Sch}}(X, \mathrm{Spec}(A))$$

for all rings A and schemes X. ■

We can show that fibered products over diagrams of affine schemes are just spectra of tensor products of rings.

Proposition 11.5. Let $\mathrm{Spec}(A) \to \mathrm{Spec}(R), \mathrm{Spec}(B) \to \mathrm{Spec}(R)$. Then the fibered product $\mathrm{Spec}(A) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(B)$ is canonically isomorphic to $\mathrm{Spec}(A \otimes_R B)$.

Proof. Corollary 11.4 shows that the inclusion $\mathsf{AffSch} \to \mathsf{Sch}$ is a right adjoint and thus the functor preserves limits and in particular fibered products. The fibered product as an affine scheme, if it exists, will agree with the fibered product in Sch. Under the antiequivalence of AffSch and Ring in Proposition 9.2, the fibered product in AffSch is computed as a pushout in Ring which is exactly the tensor product. ■

12. Lecture 12 – 18th November 2024

We continue with some formal properties of fibered products.

Proposition 12.1. Let $X \to S, Y \to S, W \to T, Y \to T$ be morphisms in a category admitting fibered products. Then there is a unique isomorphism $(X \times_S Y) \times_T W \cong X \times_S (Y \times_T W)$.

Proof. The ? of the diagram

$$\begin{tikzcd}
? \arrow[r] \arrow[d] & Y \times_T W \arrow[r] \arrow[d] & W \arrow[d] \\
X \times_S Y \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & T \\
X \arrow[r] & S &
\end{tikzcd}$$

with all squares cartesian can be filled by $X \times_S (Y \times_T W)$ and $(X \times_S Y) \times_T W$ by considering the vertical and horizontal rectangles, respectively. But in any such diagram, the rectangles are Cartesian as well so both of the objects above satisfy the same universal property and are thus isomorphic. ∎

We now show fibered products exist in general. We do this in a sequence of lemmas.

We follow the presentation of [Stacks, Tag 01JO] in place of [Har83, Thm. II.3.3] presented in class.

Lemma 12.2. Let $f : X \to S, g : Y \to S$ be morphisms of schemes and suppose $X \times_S Y$ exists. If $U \subseteq S, V \subseteq X, W \subseteq Y$ are open such that $f(V) \subseteq U$ and $g(W) \subseteq U$ then there is a unique morphism $V \times_U W \to X \times_S Y$ and $V \times_U W \subseteq X \times_S Y$ is open.

Proof. For any other scheme Z admitting maps to V, W whose compatible with $f|_V, g|_W$, there is a unique morphism $Z \to X \times_S Y$ which is contained in the open subscheme $\mathrm{pr}_X^{-1}(V) \cap \mathrm{pr}_Y^{-1}(W)$ of $X \times_S Y$, giving an identification $V \times_U W \cong \mathrm{pr}_X^{-1}(V) \cap \mathrm{pr}_Y^{-1}(W)$ by the uniqueness of fibered products.

Uniqueness of the map follows from the diagram

$$\begin{tikzcd}
V \times_U W \arrow[rrr] \arrow[dr, dashed, "\exists!"] \arrow[ddd] & & & V \arrow[dl] \arrow[ddd] \\
& X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] & \\
& Y \arrow[r] & S & \\
W \arrow[ur] \arrow[rrr] & & & U. \arrow[ul]
\end{tikzcd}$$

∎

Proposition 12.3. Let $X \to S, Y \to S$ be morphisms of schemes. Then the fibered product $X \times_S Y$ exists in the category of schemes.

Proof. Let $\{U_i\}_{i\in I}$ be an affine open cover of S, $\{V_j\}_{j\in J_i}$ an affine open cover of $f^{-1}(U_i)$ for each i, and $\{W_k\}_{k\in K_i}$ an affine open cover of $g^{-1}(U_i)$ for each i. By Proposition 11.5, each $V_j \times_{U_i} W_k$ is affine and satisfies the universal property of the fibered product for morphisms factoring through V_j, W_k that agree on U_i and any scheme Z admitting maps to X, Y that agree on S is given by the data of maps to each such $V_j \times_{U_i} W_k$. Moreover these schemes satisfy the hypothesis of Proposition 9.4 so these schemes glue to give the fibered product $X \times_S Y$ which satisfies the expected universal property. ∎

Note that fibered products of schemes can behave unexpectedly.

Example 12.4. Let $X = \operatorname{Spec}(k[x_1]), Y = \operatorname{Spec}(k[x_2]), S = \operatorname{Spec}(k)$ for k algebraically closed. $X \times_S Y = \operatorname{Spec}(k[x_1] \otimes_k k[x_2]) = \operatorname{Spec}(k[x_1, x_2]) = \mathbb{A}^2_k$ but the underlying topological space $|\mathbb{A}^2_k|$ is not $|\mathbb{A}^1_k| \times |\mathbb{A}^1_k|$ – the latter has points given by pairs of prime ideals in $k[x_1] \oplus k[x_2]$ but $(x_1 - x_2)$ is a point of $\mathbb{A}^2_k$ not of this form.

Example 12.5. Let $X = Y = \operatorname{Spec}(\mathbb{C})$ and $S = \operatorname{Spec}(\mathbb{R})$. $X \times_S Y = \operatorname{Spec}(\mathbb{R}[x]/(x^2 - 1) \otimes_{\mathbb{R}} \mathbb{C}) = \operatorname{Spec}(\mathbb{C}[x]/(x^2-1))$ which consists of two points given by the maximal ideals $(x-i), (x+i)$. But the product $|X| \times |Y|$ is just one point, so there is not even a natural map $|X \times Y| \to |X \times_S Y|$.

One place fibered products are ubiquitous is in the computation of fibers of a morphism.

Definition 12.6 (Fiber)**.** Let $f : X \to Y$ be a morphism of schemes and $y \in Y$ with residue field $\kappa(y)$. The fiber X_y of f over Y is the fibered product $X \times_Y \operatorname{Spec}(\kappa(y))$.

More explicitly, the fiber is induced by the following diagram.

$$\begin{tikzcd} X_y = X \times_Y \operatorname{Spec}(\kappa(y)) \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}(\kappa(y)) \arrow[r] & Y \end{tikzcd}$$

In the case of Y having a generic point, we can construct generic and closed fibers.

Definition 12.7 (Generic Fiber)**.** Let $f : X \to Y$ be a morphism of schemes and $\eta \in Y$ the unique generic point of Y with residue field $\kappa(\eta)$. The generic fiber X_η of f over Y is the fibered product $X \times_Y \operatorname{Spec}(\kappa(\eta))$.

Definition 12.8 (Closed Fiber)**.** Let $f : X \to Y$ be a morphism of schemes and $y \in Y$ a closed point with residue field $\kappa(y)$. The closed fiber X_y of f over Y is the fibered product $X \times_Y \operatorname{Spec}(\kappa(y))$.

Example 12.9. Let A be a discrete valuation ring. Then $\operatorname{Spec}(A) = \{\eta, \pi\}$ where η is the generic point and π the prime ideal corresponding to the uniformizer. A scheme X over $\operatorname{Spec}(A)$ has two fibers: the generic fiber X_η and the closed fiber X_π.

Fibered products are also a key tool in working in Grothendieck's "relative point of view."

Definition 12.10 (Category of S-Schemes)**.** The category of S-schemes Sch_S has objects morphisms of schemes $X \to S$ and morphisms commutative diagrams

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S. & \end{tikzcd}$$

Definition 12.11 (Category of k-Schemes)**.** Let k be field. The category of k-schemes Sch_k has objects morphisms of schemes $X \to \operatorname{Spec}(k)$ and morphisms commutative diagrams

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & \operatorname{Spec}(k). & \end{tikzcd}$$

Remark 12.12. When working in the setting of k-schemes and considering the fibered product of $X \to \operatorname{Spec}(k), Y \to \operatorname{Spec}(k)$ we will write $X \times_k Y$ in place of $X \times_{\operatorname{Spec}(k)} Y$.

The fibered product gives us a way to "extend scalars" on schemes defined over a field.

Definition 12.13 (Base Change)**.** Let k be a field, Sch_k the category of k-schemes, and L/k a field extension. The base change functor $(-)_L : \mathsf{Sch}_k \to \mathsf{Sch}_L$ is given by $X \mapsto X_L = X \times_k \operatorname{Spec}(L)$ and morphisms those induced morphisms of L-schemes.

More precisely, for a morphism $X \to Y$ of k-schemes, we have a diagram

$$\begin{tikzcd} X_L \arrow[dr] \arrow[ddrrr] & & Y_L \arrow[dl] \arrow[ddrrr] & & & \\ & \operatorname{Spec}(L) \arrow[ddrrr] & & & & \\ & & & X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & & & & \operatorname{Spec}(k) & \end{tikzcd}$$

with both rectangles Cartesian so there is a unique map $X_L \to Y_L$ making the diagram

$$\begin{tikzcd} X_L \arrow[r] \arrow[dr, dashed, "\exists!"] \arrow[ddr] & X \arrow[dr] \arrow[ddr] & \\ & Y_L \arrow[r] \arrow[d] & Y \arrow[d] \\ & \operatorname{Spec}(L) \arrow[r] & \operatorname{Spec}(k) \end{tikzcd}$$

commute.

These constructions are especially important in arithmetic applications.

Definition 12.14 (Rational Points)**.** Let X be a k-scheme and L/k a field extension. The set of L-rational points of X is the set $X(L) = \mathrm{Mor}_{\mathsf{Sch}_k}(\mathrm{Spec}(L), X)$.

One can easily show that this is invariant under base change of the scheme to L.

Lemma 12.15. Let X be a k-scheme and L/k a field extension. Then $X(L) = X_L(L)$ as sets.

Proof. Any morphism $\mathrm{Spec}(L) \to X$ factors over a morphism to X_L

$$\begin{tikzcd}
\mathrm{Spec}(L) \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right, "\mathrm{id}_{\mathrm{Spec}(L)}"'] & & \\
& X_L \arrow[r] \arrow[d] & X \arrow[d] \\
& \mathrm{Spec}(L) \arrow[r] & \mathrm{Spec}(k)
\end{tikzcd}$$

giving the claim. ■

The most "absolute" form of base change is the geometric fiber.

Definition 12.16 (Geometric Fiber)**.** Let $f : X \to Y$ be a morphism of schemes,$y \in Y$ with residue field $\kappa(y)$, and $\overline{\kappa(y)}$ a choice of algebraic closure of $\kappa(y)$. The geometric fiber $X_{\overline{y}}$ is defined to be $X \times_Y \mathrm{Spec}(\overline{\kappa(y)})$.

Remark 12.17. The topology may change under passage to the geometric fiber. This often has better topological behavior as the underling topological spaces of schemes over algebraically closed fields k are often identical to the set of k-rational points.

Example 12.18. Let X be a scheme over $\mathrm{Spec}(\mathbb{Z}_{(p)})$, the localization of $\mathbb{Z}$ at the prime ideal (p). $\mathbb{Z}_{(p)}$ consists of two points $\{\eta, \mathfrak{m}\}$ corresponding to the generic and maximal ideal. The generic fiber X_η is a scheme over $\mathrm{Spec}(\mathbb{Q})$ and the closed fiber $X_\mathfrak{m}$ is a scheme over $\mathrm{Spec}(\mathbb{F}_p)$. On the other hand, the generic and closed fibers $X_{\overline{\eta}}, X_{\overline{\mathfrak{m}}}$ are schemes over $\mathrm{Spec}(\overline{\mathbb{Q}}), \mathrm{Spec}(\overline{\mathbb{F}_p})$, respectively.

Returning to a discussion of arithmetic, we consider conjugate k-schemes.

Definition 12.19 (Conjugate k-Schemes)**.** Let k be a field, X a k-scheme, and $\sigma \in \mathrm{Aut}(k)$. The conjugate k-scheme is defined as the fibered product

$$\begin{tikzcd}
X_\sigma \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(k) \arrow[r, "\sigma"] & \mathrm{Spec}(k).
\end{tikzcd}$$

Note that $X_\sigma \to X$ is an isomorphism of abstract schemes, but not necessarily as k-schemes, since X_σ has a different structure map that commutes with the structure map of X up to σ.

Example 12.20. Note that $\mathrm{Aut}(\mathbb{R})$ is the trivial group. Consider the $\mathrm{Spec}(\mathbb{Q}(\sqrt{2}))$ scheme $X = \mathrm{Spec}(\mathbb{R})$ induced by the inclusion $\mathbb{Q}(\sqrt{2}) \hookrightarrow \mathbb{R}$. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ be the non-trivial element of the Galois group of the quadratic extension. We can consider X_σ obtained as the fibered product

$$\begin{tikzcd}
X_\sigma \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(\mathbb{Q}(\sqrt{2})) \arrow[r, "\sigma"'] & \mathrm{Spec}(\mathbb{Q}(\sqrt{2})).
\end{tikzcd}$$

If X and X_σ are isomorphic as $\mathbb{Q}(\sqrt{2})$-schemes, there would be an automorphism of $\mathbb{R}$ by $\sqrt{2} \mapsto -\sqrt{2}$, a contradiction.

13. Lecture 13 – 22nd November 2024

Following Example 12.20, we consider the following.

Example 13.1. Note that π, e^π are algebraically independent – there does not exist a polynomial $F \in \mathbb{Q}[x_1, x_2]$ where $F(\pi, e^\pi) = 0$. Let $X = \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty, \pi\}$ considered as a $\mathbb{C}$-scheme. Let $\sigma \in \mathrm{Aut}(\mathbb{Q}(\pi, e^\pi))$ be the element of the automorphism group $\pi \mapsto e^\pi, e^\pi \mapsto \pi$ considered as an automorphism of $\mathbb{C}$. We can compute $X_\sigma = \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty, e^\pi\}$. Any isomorphism of X_σ and X as a $\mathbb{C}$-scheme would be induced by a linear fractional transformation defining an automorphism of $\mathbb{P}^1_{\mathbb{C}}$, but this would produce an algebraic dependence between π, e^π, a contradiction.

For those more arithmetically minded, we can use the language of rational points of Definition 12.14 to understand Galois actions on schemes.

Proposition 13.2. Let X be a k-scheme and K/k a Galois extension with Galois group G. There exists a group homomorphism $G \to \mathrm{Aut}_k(X_K)$.

Proof. Consider the diagram

$$\begin{tikzcd}
X_K \arrow[dr, dashed, "\exists!"] \arrow[drr] \arrow[dd] & & \\
& X_K \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(K) \arrow[r, "\sigma"'] & \mathrm{Spec}(K) \arrow[r] & \mathrm{Spec}(k)
\end{tikzcd}$$

where the X_K of the upper-left corner is obtained by the fibered product over the $\mathrm{Spec}(K)$ of the lower-left corner. The diagram commutes since $\sigma|_k$ is the identity and the unique morphism induced by the universal property of the fibered product is an isomorphism of X_K to itself as k-schemes as its structure as a k-scheme is unchanged under σ. ■

Such methods were used by Weil to understand $\mathbb{F}_q$-points of algebraic varieties by considering fixed points of the action of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ on $X(\overline{\mathbb{F}_q})$.

The language of fibered products also allow us to describe graphs and diagonals, the latter of which will play a key role in definitions of properties of schemes.

Definition 13.3 (Graph Morphism)**.** Let $f : X \to Y$ be a morphism of S-schemes. The graph of f is the unique morphism $X \to X \times_S Y$ induced by the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "f"'] & & \\
& X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & S.
\end{tikzcd}$$

Definition 13.4 (Diagonal Morphism)**.** Let X be an S-scheme. The diagonal morphism $\Delta_{X/S}$ is the unique morphism $X \to X \times_S X$ induced by the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\
& X \arrow[r] & S.
\end{tikzcd}$$

Remark 13.5. In particular, the diagonal morphism Definition 13.4 is the graph Definition 13.3 of id_X.

Remark 13.6. The absolute variants of Definitions 13.3 and 13.4 can be recovered by taking $S = \mathrm{Spec}(\mathbb{Z})$.

The name "diagonal" is justified by the followng example.

Example 13.7. Let k be a field $X = \mathbb{A}^1_k, S = \mathrm{Spec}(k)$. The diagonal map $\mathbb{A}^1_k \to \mathbb{A}^1_k \times_k \mathbb{A}^1_k \cong \mathbb{A}^2_k$ is induced by the ring map $k[x_1, x_2] = k[x_1] \otimes_k k[x_2] \to k[x]$ with kernel $x_1 - x_2$ whose vanishing locus is the diagonal line in $\mathbb{A}^2_k$.

While in the above example the diagonal had a closed image, this is not always the case. The collection of schemes where this is satisfied are precisely the separated schemes.

Definition 13.8 (Separated Scheme)**.** Let X be an S-scheme. X is separated if the image of $\Delta_{X/S}$ is closed.

Remark 13.9. This will be the analogue of Hausdorffness in the setting of schemes.

Example 13.10. Example 13.7 shows that $\mathbb{A}^1_k$ is a separated k-scheme. On the other hand, the affine line with doubled origin Example 9.5 is not separated.

Before considering properties of morphisms of schemes, we consider stability properties of morphisms under certain operations. Thus far, we have only seen fibered products, and the appropriate notion is defined as follows.

Definition 13.11 (Stable Under Base Change)**.** Let P be a property of morphisms of schemes. The property P is stable under base change if for all $f : X \to Y$ with the property P, and all diagrams Cartesian diagrams

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

f' has the property P.

We now turn to properties of schemes. We first consider topological properties – properties determined by the map of underlying topological spaces.

Definition 13.12 (Open Morphism)**.** Let $f : X \to Y$ be a morphism of schemes. f is an open morphism if for all $U \subseteq X$ open, $f(U) \subseteq Y$ is open.

Definition 13.13 (Closed Morphism)**.** Let $f : X \to Y$ be a morphism of schemes. f is an closed morphism if for all $Z \subseteq X$ closed, $f(Z) \subseteq Y$ is closed.

Definition 13.14 (Dominant)**.** Let $f : X \to Y$ be a morphism of schemes. f is a dominant morphism if $f(X) \subseteq Y$ is dense.

Remark 13.15. If Y is integral, and thus with a unique geometric point η, a morphism $f : X \to Y$ is dense if $\eta \in f(X)$.

Definition 13.16 (Quasicompact Morphism)**.** Let $f : X \to Y$ be a morphism of schemes. f is a quasicompact morphism if for all $V \subseteq Y$ quasicompact, $f^{-1}(V) \subseteq X$ is quasicompact.

Remark 13.17. It can be shown that quasicompactness of Definition 13.16 can be verified on affine schemes.

Definition 13.18 (Quasifinite Morphism)**.** Let $f : X \to Y$ be a morphism of schemes. f is a quasifinite morphism if for all $y \in Y$, $f^{-1}(y)$ is a finite set.

We can now consider some properties of locally ringed spaces.

Definition 13.19 (Open Immersion)**.** Let $f : X \to Y$ be a morphism of schemes. f is an open immersion if there exists some $V \subseteq Y$ open such that f factors over the isomorphism of schemes $X \cong V$.

Definition 13.20 (Closed Immersion)**.** Let $f : X \to Y$ be a morphism of schemes. f is a closed immersion if there exists some $W \subseteq Y$ closed such that f factors over the isomorphism of schemes $X \cong W$.

Definition 13.21 (Immersion)**.** Let $f : X \to Y$ be a morphism of schemes. f is an immersion if there exists a closed subscheme $W \subseteq Y$ and an open subscheme $V \subseteq W$ such that f factors over the isomorphism $X \cong V$.

Example 13.22. Let $X = \mathbb{A}^1_k \setminus \{0\} = D(x) = \mathrm{Spec}(k[x^{\pm}])$. THe inclusion to $\mathbb{A}^2_k$ by $k[x_1, x_2] \to k[x_1^{\pm}]$ taking $x_1 \mapsto x_1, x_2 \mapsto 0$ is an immersion taking $W = V(x_2)$ and $V = W \setminus \{0\}$.

Remark 13.23. If f is an immersion, it factors as a closed followed by an open immersion.

Finally, we turn to scheme-theoretic properties.

Definition 13.24 (Locally Finite Type Morphism)**.** Let $f : X \to Y$ be a morphism of schemes. f is locally of finite type if there exists an affine open cover $\{V_j\}_{j \in J}$ and each affine open covering $\{U_{ij}\}_{i \in I_j}$ of $f^{-1}(V_j)$, $\Gamma(U_{ij}, \mathcal{O}_X)$ is a finite type $\Gamma(V_j, \mathcal{O}_Y)$-algebra.

Definition 13.25 (Finite Type Morphism)**.** Let $f : X \to Y$ be a morphism of schemes. f is of finite type if there exists an affine open cover $\{V_j\}_{j \in J}$ such that each $f^{-1}(V_j)$ has a finite cover $\{U_{ij}\}_{i=1}^{n_j}$ and $\Gamma(U_{ij}, \mathcal{O}_X)$ is a finite type $\Gamma(V_j, \mathcal{O}_Y)$-algebra.

Definition 13.26 (Affine Morphism)**.** Let $f : X \to Y$ be a morphism of schemes. f is an affine morphism if there exists an open cover $\{V_j\}_{j \in J}$ of Y such that $f^{-1}(V_j)$ is affine for all j.

Definition 13.27 (Finite Morphism)**.** Let $f : X \to Y$ be a morphism of schemes. f is a finite morphism if it is an affine morphism and there exists an open cover $\{V_j\}_{j \in J}$ of Y such that $\Gamma(f^{-1}(V_j), \mathcal{O}_X)$ is a finite module over $\Gamma(V_j, \mathcal{O}_Y)$.

Remark 13.28. A number of these conditions on morphisms imply others. The condition for "existence of an affine open cover" can often be replaced by "for all open covers" using Lemma 10.2.

14. Lecture 14 – 25th November 2024

Proposition 14.1. Let $f : X \to Y$ be a morphism of schemes. f is locally of finite type if and only if for all affine opens $V \subseteq Y$ and $U \subseteq f^{-1}(V)$, $\Gamma(U, \mathcal{O}_X)$ is finite type over $\Gamma(V, \mathcal{O}_Y)$

Proof. Let $f : X \to Y$ be a morphism locally of finite type. We apply the affine communication lemma Lemma 10.2 on Y. Let P be the property that "the preimage $f^{-1}(V)$ of an affine open V has a cover by affine opens $\{U_i\}_{i\in I}$ such that each $\Gamma(U_i, \mathcal{O}_X|_{U_i})$ is a finitely generated $\Gamma(V, \mathcal{O}_Y|_V)$-algebra".

We first show condition (i) of the lemma. Suppose $V, \{U_i\}_{i\in I}$ as described above with $f_i : U_i \to V$ the induced morphisms of affine schemes and take $g \in \Gamma(V, \mathcal{O}_Y|_V)$. We have $f_i^\sharp : \mathcal{O}_V \to f_{i*}\mathcal{O}_{U_i}$ where taking sections over $D(g) \subseteq V \subseteq Y$ we have $\Gamma(U_i \cap f^{-1}(D(g)), \mathcal{O}_X|_{U_i})$ the localization of $\Gamma(U_i, \mathcal{O}_X|_{U_i})$ at the image of $f_i^\sharp(g)$ when taking sections over V, and finite generation as an algebra is a property preserved by localization.

To show (ii), suppose $g_1, \dots, g_n$ generate $\Gamma(V, \mathcal{O}_Y|_V)$ so the $D(g_i)$ cover V and the $\{f^{-1}(D(g_i))\}_{i=1}^n$ cover $f^{-1}(V)$ and each of the $f^{-1}(D(g_i))$ have a cover by spectra of finitely generated $\Gamma(D(g_i), \mathcal{O}_Y|_{D(g_i)})$-algebras say by $\{T_{ij}\}_{j\in J_i}$ so the $\{T_{ij}\}_{i\in I, j\in J_i}$ form an affine open cover of $f^{-1}(V)$ by spectra of finitely generated $\Gamma(V, \mathcal{O}_Y|_V)$-algebras. In particular, any finitely generated $\Gamma(D(g_i), \mathcal{O}_Y|_{D(g_i)})$ is a finitely generated algebra over $\Gamma(V, \mathcal{O}_Y|_V)$ since $\Gamma(D(g_i), \mathcal{O}_Y|_{D(g_i)})$ is already a finitely generated algebra over $\Gamma(V, \mathcal{O}_Y|_V)$.

($\Longleftarrow$) This is essentially by definition. Y as a scheme can always be covered by affine opens. And since every such affine open has a preimage with a cover by affine opens satisfying the desired condition, f satisfies the definition for locally finite typeness. ∎

15. Lecture 15 – 29th November 2024

Following Proposition 14.1, we can prove the same generalization for the property of an affine morphism.

Proposition 15.1. Let $f : X \to Y$ be a morphism of schemes. f is an affine morphism if and only if for all $V \subseteq Y$ affine, $f^{-1}(V)$ is affine.

Proof. We apply Lemma 10.2 to the property "has affine preimage." For $\mathrm{Spec}(A) \subseteq Y$ with affine preimage $f^{-1}(\mathrm{Spec}(A)) = \mathrm{Spec}(B)$, $f^{-1}(\mathrm{Spec}(A_f)) = \mathrm{Spec}(B_{\varphi(f)})$ for $\varphi : A \to B$ the induced morphism of rings. Similarly for $(f_1, \dots, f_n) = A$, $(\varphi(f_1), \dots, \varphi(f_n)) = B$ so $\mathrm{Spec}(B)$ is covered by $\mathrm{Spec}(B_{\varphi(f_i)})$. ∎

As a corollary, affine morphisms to $\mathrm{Spec}(A)$ imply that the source is affine.

Corollary 15.2. If $f : X \to \mathrm{Spec}(A)$ is an affine morphism then X is an affine scheme.

Proof. This is immediate from the definitions: $\mathrm{Spec}(A)$ is itself affine so $f^{-1}(\mathrm{Spec}(A)) = X$ is affine. ∎

Corollary 15.3. Let $f : X \to Y$ be a morphism of schemes. f is finite if and only if for all $V \subseteq Y$ affine, $f^{-1}(V)$ is affine and $\Gamma(f^{-1}(V), \mathcal{O}_X)$ is finite type over $\Gamma(V, \mathcal{O}_Y)$.

Proof. This is immediate from Propositions 14.1 and 15.1. ∎

We now discuss separated and proper morphisms, the former first being introduced as Definition 13.8. In fact several types of morphisms we have already encountered can be seen to be separated. To do so, we utilize the following lemma.

Proposition 15.4. If $f : X \to Y$ is an affine morhpism then f is a separated morphism.

Proof. Let $\{V_j\}_{j \in J}$ be an affine open cover of Y and $\{U_j\}_{j \in J}$ an open affine cover of Y with $U_j = f^{-1}(V_j)$. The $U_j \times_{V_j} U_j$ form an affine cover of $X \times_Y X$ and each $U_j \to U_j \times_{V_j} U_j$ is closed and thus $X \to X \times_Y X$ is closed as well. ∎

Proposition 15.5. If $f : X \to Y$ is an open immersion or closed immersion, then f is a separated morphism.

Proof. Both open and closed immersions are monomorphisms, in which case $X \to X \times_Y X$ is an isomorphism and hence closed. ∎

Definition 15.6 (Proper Morphism)**.** Let $f : X \to Y$ be a morphism of schemes. f is proper if it is separated, of finite type, and universally closed.

Example 15.7. $\mathbb{A}^1_k \to \mathrm{Spec}(k)$ is separated and closed but not universally closed. It is thus not proper. To see that it is not universally closed, consider the base change along the map $\mathbb{A}^1_k \to \mathbb{A}^1_k$.

Example 15.8. $\mathbb{P}^1_k \to \mathrm{Spec}(k)$ is proper.

Separatedness and properness are preserved under composition.

Proposition 15.9. Separatedness and properness are preserved under composition.

Proof. More generally for a diagram of schemes

$$\begin{tikzcd}
X_1 \arrow[dr] & & \\
& Y \arrow[r] & Z \\
X_2 \arrow[ur] & &
\end{tikzcd}$$

there is a Cartesian diagram

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r] \arrow[d] & Y \arrow[d] \\
X_1 \times_Z X_2 \arrow[r] & Y \times_Z Y.
\end{tikzcd}$$

Taking $X_1 = X_2 = X$ and a composite $X \to Y \to Z$ we have

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Y X \arrow[r] \arrow[d] & Y \arrow[d] \\
& X \times_Z X \arrow[r] & Y \times_Z Y
\end{tikzcd}$$

where we have that $X \times_Y X \to X \times_Z X$ is closed by $Y \to Y \times_Z Y$ closed by separatedness and preservation of closed immersions under base change. So the map $X \to X \times_Z X$ is also closed showing it is separated. Additionally, being of finite type and universally closed is preserved under base change so properness is preserved under base change as well. ■

16. Lecture 16 – 2nd December 2024

For this, we follow [Vak24, §11.1].

We discuss some "cancellation" properties of morphisms of schemes.

Proposition 16.1 (Cancellation for Morphisms of Schemes)**.** Let P be a property of schemes preserved by base change and composition. Consider a diagram of schemes

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

where h is in P and the diagonal morphism induced by g is in P. Then f is in P.

Proof. We consider the diagram

$$\begin{tikzcd} X \cong X \times_Y Y \arrow[rr] \arrow[d] \arrow[rrrr, bend left, "f"] & & X \times_Z Y \arrow[rr] \arrow[dl] \arrow[dr] & & Y \times_Z Z \cong Y \arrow[d] \\ Y \arrow[r] & Y \times_Z Y & & X \arrow[r] & Z \end{tikzcd}$$

with Cartesian squares induced by the diagonal base change theorem. Both $Y \to Y \times_Z Y$ and $X \to Z$ lie in P and thus so does their composite. ∎

As a corollary, we can deduce the following results about morphisms.

Corollary 16.2. If $g \circ f$ is a separated morphism then f is a separated morphism.

Proof. It suffices to show that the diagonals of separated morphisms are separated. But closed immersions are separated giving the claim. ∎

Corollary 16.3. If $g \circ f$ is a proper morphism, g a separated morphism, and f a quasicompact morphism then f is a proper morphism.

Proof. Since g is separated, the diagonal morphism is a closed immersion and hence finite. In particular, g is proper. Applying the proposition yields the claim. ∎

We consider the valuative criteria for separatedness and properness. Recall the following.

Definition 16.4 (Valuation)**.** Let K be a field and Γ a totally ordered Abelian group. A valuation on a field K is a map $\nu : K^\times \to \Gamma$ such that

(i) $\nu(xy) = \nu(x) + \nu(y)$,
(ii) $\nu(0) = \infty$, and
(iii) $\nu(x + y) = \min\{\nu(x), \nu(y)\}$.

A valued field allows us to produce a ring of ν-integers.

Definition 16.5 (Valuation Ring)**.** Let K be a field with valuation ν. The valuation ring of K is the ring

$$\mathcal{O}_\nu = \{x \in K : \nu(x) \geq 0\}.$$

Remark 16.6. $\mathrm{Spec}(\mathcal{O}_\nu) = \{\eta, \mathfrak{m}_\nu\}$ where $\mathfrak{m}_\nu = \{x \in K : \nu(x) > 0\}$.

The results are as follows.

Theorem 16.7 (Valuative Criterion for Separatedness)**.** Let $f : X \to Y$ be a finite type morphism of schemes and Y locally Noetherian. Then f is separated if and only if for all diagrams

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed, "\leq 1"] & Y \end{tikzcd}$$

with A a discrete valuation ring with fraction field K there exists at most one map $\mathrm{Spec}(A) \to X$ making the diagram commute.

This is reflected in the well-examined example of the affine line with doubled origin.

Example 16.8. Let X be the affine line with doubled origin and consider the diagram

$$\begin{tikzcd} \mathrm{Spec}(k(t)) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(k[t]_{(t)}) \arrow[r] & \mathbb{A}^1_k. \end{tikzcd}$$

Writing $\mathbb{A}^1_k$ as $\mathrm{Spec}(k[x])$, there are two maps along the bottom $(x) \mapsto (0)$ and $(x) \mapsto (t)$, each of which induce a lift to X. In particular, there is more than one map $\mathrm{Spec}(k[t]_{(t)})$ making the diagram commute, which agrees with X not being separated.

In the case of properness, we have the following.

Theorem 16.9 (Valuative Criterion for Properness)**.** Let $f : X \to Y$ be a finite type morphism of schemes and Y locally Noetherian. Then f is proper if and only if for all diagrams

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

with A a discrete valuation ring with fraction field K there exists a unqiue map $\mathrm{Spec}(A) \to X$ making the diagram commute.

Example 16.10. Consider the square

$$\begin{tikzcd} \mathrm{Spec}(k(t)) \arrow[r] \arrow[d] & \mathbb{A}^1_k \arrow[d] \\ \mathrm{Spec}(k[t]_{(t)}) \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

where taking $\mathbb{A}^1_k = \operatorname{Spec}(k[x])$, the map $(x) \mapsto (t^{-1})$ along the top does not extend to the discrete valuation ring, verifying that $\mathbb{A}^1_k$ is not proper.

We now turn to a discussion of sheaves of modules, which we discuss in the generality of ringed spaces.

Definition 16.11 (Modules over the Structure Sheaf)**.** Let X be a ringed space. A sheaf of Abelian groups $\mathcal{F}$ on X is a sheaf of $\mathcal{O}_X$-modules if for all $U \subseteq V$ the diagram with vertical maps restrictions and horizontal maps the $\mathcal{O}_X$ action

$$\begin{tikzcd}
\mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\
\mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r] & \mathcal{F}(V)
\end{tikzcd}$$

commutes.

Naturally, one defines the morphisms of sheaves of $\mathcal{O}_X$-modules to be defined as morphisms of Abelian groups between $\mathcal{O}_X$-modules compatible with the action of $\mathcal{O}_X$. These form an Abelian category $\mathsf{Mod}_{\mathcal{O}_X}$ as shown in [Stacks, Tag 01AF].

Remark 16.12. The forgetful functor $\mathsf{Mod}_{\mathcal{O}_X} \to \mathsf{Sh}(X, \mathsf{AbGrp})$ is faithful but not full – being a morphism of $\mathcal{O}_X$-modules requires compatiblility with the $\mathcal{O}_X$ action beyond being a morphism of Abelian groups.

17. Lecture 17 – 6th December 2024

For a ringed space, we consider different types of $\mathcal{O}_X$-modules.

Definition 17.1 (Free $\mathcal{O}_X$-Module)**.** Let X be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F} \cong \mathcal{O}_X^{\oplus I}$ for some indexing set I.

Definition 17.2 (Locally Free $\mathcal{O}_X$-Module)**.** Let X be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ such that there exists an open cover $\{U_i\}_{i\in I}$ of X such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus J}$ for some J and all $i \in I$.

Definition 17.3 (Invertible $\mathcal{O}_X$-Module)**.** Let X be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ that is locally free of rank 1.

Recall that for a ring morphism $A \to B$ there is a functor taking an A-module M to its base-extension $M \otimes_A B$ which is a B-module. Moreover given a B-module, there is a forgetful functor $N \mapsto N|_A$ which satisfies the tensor-hom adjunction for extension and restriction of scalars giving an isomorphism of Abelian groups

$$\mathrm{Hom}_{\mathsf{Mod}_B}(M \otimes_A B, N) \cong \mathrm{Hom}_{\mathsf{Mod}_A}(M, N|_A)$$

for A-modules M and B-modules N. This holds for sheaves as well: for $\mathcal{O}_X, \mathcal{O}'_X$ two sheaves on a fixed topological space X and a morphism $\mathcal{O}_X \to \mathcal{O}'_X$ gives an adjunction

$$\mathrm{Hom}_{\mathsf{Mod}_{\mathcal{O}'_X}}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}'_X, \mathcal{G}) \cong \mathrm{Hom}_{\mathsf{Mod}_{\mathcal{O}_X}}(\mathcal{F}, \mathcal{G}|_{\mathcal{O}_X})$$

for $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{O}'_X$-modules $\mathcal{G}$. In particular, this extends the adjunction for a morphism $f : X \to Y$ of ringed spaces allowing us to define a functor $f^* : \mathsf{Mod}_{\mathcal{O}_Y} \to \mathsf{Mod}_{\mathcal{O}_X}$ as follows.

Definition 17.4 (Pullback Functor)**.** Let $f : X \to Y$ be a morphism of ringed spaces. The pullback functor $f^* : \mathsf{Mod}_{\mathcal{O}_Y} \to \mathsf{Mod}_{\mathcal{O}_X}$ is defined by $f^{-1}(-)\otimes_{f_*\mathcal{O}_X} \mathcal{O}_X$.

We can now define quasicoherent and coherent sheaves.

Definition 17.5 (Quasicoherent Sheaf)**.** Let X be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is a quasicoherent sheaf if for all $U \subseteq X$ open there exists an exact sequence

$$\mathcal{O}_U^{\oplus I} \to \mathcal{O}_U^{\oplus J} \to \mathcal{F}|_U \to 0.$$

Definition 17.6 (Coherent Sheaf)**.** Let X be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is a coherent sheaf if for all $U \subseteq X$ there exists a surjection

$$\mathcal{O}_U^{\oplus J} \to \mathcal{F}|_U$$

with J finite with kernel finitely generated.

Remark 17.7. Coherent sheaves are in particular quasicoherent giving full subcategories

$$\mathsf{Coh}(X) \hookrightarrow \mathsf{QCoh}(X) \hookrightarrow \mathsf{Mod}_{\mathcal{O}_X}.$$

Let us consider what happens on an affine scheme $\mathrm{Spec}(A)$ for a ring A.

Proposition 17.8. Let A be a ring. The functor $\mathsf{Mod}_A \to \mathsf{QCoh}(\mathrm{Spec}(A))$ by $M \mapsto \widetilde{M}$ is fully faithful and exact.

Proof. The construction is functorial and localization is exact, and since exactness can be verified on stalks, the claim follows. ■

Proposition 17.9. There is an equivalence of categories between $\mathsf{QCoh}(\mathrm{Spec}(A))$ and Mod_A.

Proof. The global sections functor is the inverse. ■

18. Lecture 18 – 9th December 2024

We have the following property for the cohomology of quasicoherent sheaves on affine schemes, which will allow us to compute the cohomology of quasicoherent sheaves on separated and in particular projective or proper schemes using Čech cohomology by Proposition 7.6. For this, we will need to show that any injective module gives rise to a flasque sheaf.

Lemma 18.1. Let I be an injective A-module for some ring A. Then $\widetilde{I}$ is a flasque $\mathcal{O}_{\mathrm{Spec}(A)}$-module on $\mathrm{Spec}(A)$.

Proof. We want to show that for V, U opens of $\mathrm{Spec}(A)$ and $V \subseteq U$ that $\widetilde{I}(U) \to \widetilde{I}(V)$ is a surjection. Now recall for $j_!$ the extension by zero functor, we have for V, U as above $0 \to j_!\mathcal{O}_V \to j_!\mathcal{O}_U$ exact, abusing notation and letting j denoting both inclusions. Now noting

$$\mathrm{Hom}_{\mathsf{Mod}_{\mathcal{O}_{\mathrm{Spec}(A)}}}(j_!\mathcal{O}_V, \widetilde{I}) = \widetilde{I}(V) \text{ and } \mathrm{Hom}_{\mathsf{Mod}_{\mathcal{O}_{\mathrm{Spec}(A)}}}(j_!\mathcal{O}_U, \widetilde{I}) = \widetilde{I}(U)$$

we have by injectivity of I and the exactness property of the functor $\widetilde{(-)}$ Proposition 17.8 that

$$\mathrm{Hom}_{\mathsf{Mod}_{\mathcal{O}_{\mathrm{Spec}(A)}}}(j_!\mathcal{O}_U, \widetilde{I}) = \widetilde{I}(U) \to \mathrm{Hom}_{\mathsf{Mod}_{\mathcal{O}_{\mathrm{Spec}(A)}}}(j_!\mathcal{O}_V, \widetilde{I}) = \widetilde{I}(V) \to 0$$

is exact. In particular $\widetilde{I}(U) \to \widetilde{I}(V)$ is surjective, as desired. ■

Proposition 18.2. If $\mathcal{F} \in \mathsf{QCoh}(\mathrm{Spec}(A))$ then $H^i(\mathrm{Spec}(A), \mathcal{F}) = 0$ for $i \geq 1$.

Proof. For $\mathcal{F}$ a quasicoherent sheaf, we can produce an injective resolution

$$0 \to \Gamma(\mathrm{Spec}(A), \mathcal{F}) \to I \to I/\Gamma(\mathrm{Spec}(A), \mathcal{F}) \to 0.$$

By exactness of $\widetilde{(-)}$, the first cohomology vanishes. ■

Remark 18.3. The property of Proposition 18.2 in fact characterizes affineness of schemes.

We now turn to the proj construction whcih will be a global procedure for producing proper schemes. We make some recollections from graded commutative algebra.

Definition 18.4 (Graded Ring)**.** A graded ring is a ring $A_\bullet = \bigoplus_{d \geq 0} A_d$ such that each A_d is an additive subgroup of A and $a_d \cdot a_e \in A_{d+e}$ for $a_d \in A_d, a_e \in A_e$.

Remark 18.5. A_0 contains 0_A and 1_A. Most cases of interest will have that A_0 a field.

One special ideal of $A_\bullet$ is the irrelevant ideal.

Definition 18.6 (Irrelevant Ideal)**.** Let $A_\bullet$ be a graded ring. The irrelevant ideal of $A_\bullet$ is the ideal $A_+ = \bigoplus_{d \geq 1} A_d$.

We introduce some further notation that relates to graded rings.

Definition 18.7 (Homogeneous Element)**.** Let $A_\bullet$ be a graded ring. A homogeneous element is an element $a \in A_\bullet$ such that $a \in A_d$ for some d.

Definition 18.8 (Homogeneous Ideal)**.** Let $A_\bullet$ be a graded ring. A homogeneous ideal $I_\bullet \subseteq A_\bullet$ is an ideal of $A_\bullet$ with homogeneous elements.

19. Lecture 19 – 13th December 2024

Let $A_\bullet$ be a graded ring. We can consider homogeneous localizations which we define as follows.

Definition 19.1 (Homogeneous Localization)**.** Let $A_\bullet$ be a graded ring and $S \subseteq A_\bullet$ a multiplicative subset. The homogeneous localization $(S^{-1}A)_\bullet$ is given by $\bigoplus_{d\geq 0}(S^{-1}A)_d$ where

$$(S^{-1}A)_d = \left\{ \frac{a}{b} : a \in A_e, b \in S \cap A_{e'}, e - e' = d \right\}.$$

Two special cases of interest will be the localization at a prime ideal and the locaization at a homogeneous element.

- (Homogeneous Element) Let $A_\bullet$ be a graded ring and $a \in A_d$ a homogeneous element for some d. The localization $(A_\bullet)_{(a)}$ is the degree 0 piece of the localization of Definition 19.1 for S the multiplicative set generated by a.
- (Prime Ideal) Let $A_\bullet$ be a graded ring and $\mathfrak{p} \subseteq A_\bullet$ a prime ideal. The localization $(A_\bullet)_\mathfrak{p}$ is the degree 0 piece of the localization of Definition 19.1 for S the complement of the prime ideal $\mathfrak{p}$.

This will allow us to construct $\mathrm{Proj}(A_\bullet)$ and in turn projective space and its variants. We first define $\mathrm{Proj}(A_\bullet)$ as a topological space.

Definition 19.2 (Proj Space)**.** Let $A_\bullet$ be a graded ring. The topological space $\mathrm{Proj}(A_\bullet)$ consists of the set of homogeneous prime ideals of $A_\bullet$ not containing the irrelevant ideal A_+ and closed sets of the form

$$V_+(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Proj}(A_\bullet) : \mathfrak{a} \subseteq \mathfrak{p}\}$$

where $\mathfrak{a}$ is a homogeneous ideal of $A_\bullet$.

Remark 19.3. There is an obvious inclusion $\mathrm{Proj}(A_\bullet) \to \mathrm{Spec}(A_\bullet)$, but this is often not a morphism of schemes.

Having defined the underlying topological space, we seek to define a sheaf of rings.

Proposition 19.4. Let $A_\bullet$ be a graded ring such that $(a_1, \ldots, a_n) = A_\bullet$. Then $D_+(a_i) = \mathrm{Proj}(A_\bullet) \backslash V_+(a_i)$ form an open cover of $\mathrm{Proj}(A_\bullet)$ and $D_+(a_i) \cong \mathrm{Spec}((A_\bullet)_{a_i})$.

Proof. The $D_+(a_i)$ are open by definition and the first statement follows by the argument in the affine case. The second statement follows by definition of homogeneous localization. ■

It is now clear that $\mathrm{Proj}(A_\bullet)$ is a locally ringed space, and in particular a scheme.

Corollary 19.5. Let $A_\bullet$ be a graded ring. Then $\mathrm{Proj}(A_\bullet)$ is a scheme.

Proof. The construction of Proposition 19.4 gives $\mathrm{Proj}(A_\bullet)$ a cover by affine schemes as a locally ringed space, and hence as a scheme. ■

Moreover, we can determine the stalks of the structure sheaf at stalks $\mathfrak{p}$ of $\mathrm{Proj}(A_\bullet)$.

Corollary 19.6. Let $A_\bullet$ be a graded ring and $\mathfrak{p} \in \mathrm{Proj}(A_\bullet)$. $\mathscr{O}_{\mathrm{Proj}(A_\bullet),\mathfrak{p}} = (A_\bullet)_\mathfrak{p}$.

Proof. This once again follows fromthe definition of homogeneous localization. ■

We have the following condition for Proj being empty.

Proposition 19.7. Let $A_\bullet$ be a graded ring. $\mathrm{Proj}(A_\bullet) = \emptyset$ if and only if every element of A_+ is nilpotent.

Proof. ($\Longrightarrow$) Suppose each element of A_+ is nilpotent. We show each $f \in A_+$ lies in $\mathfrak{p} \subseteq A_\bullet$ prime. Since f is nilpotent, $f^n = 0$ for some n and thus $f \in \mathfrak{p}$ since $\mathfrak{p}$ is prime. So $\mathfrak{p}$ contains A_+ and thus $\mathrm{Proj}(A_\bullet) = \emptyset$.

($\Longleftarrow$) Suppose $\mathrm{Proj}(A_\bullet) = \emptyset$ so each prime ideal $\mathfrak{p} \subseteq A_\bullet$ contains A_+. For such $\mathfrak{p}$ consider the graded prime ideal $\mathfrak{p}_\bullet = \bigoplus_{d \geq 0} \mathfrak{p} \cap A_d$ giving $A_+ \subseteq \mathfrak{p}_\bullet \subseteq \mathfrak{p}$. In particular, A_+ is contained in every prime ideal, and thus every element of A_+ is nilpotent. ■

We now make the definition of projective space.

Definition 19.8 (Projective Space)**.** Let A be a ring. Projective space $\mathbb{P}^n_A$ is $\mathrm{Proj}(A[x_0, \ldots, x_n])$.

Remark 19.9. The construction of $\mathrm{Proj}(A_\bullet)$ induces a canonical morphism of schemes $\mathrm{Proj}(A_\bullet) \to \mathrm{Spec}(A_0)$.

More generally we can consider projective space over a scheme.

Definition 19.10 (Projective Space Over Scheme)**.** Let X be a scheme. The projective space $\mathbb{P}^n_X$ is the fibered product $\mathbb{P}^n_\mathbb{Z} \times_{\mathrm{Spec}(\mathbb{Z})} X$.

Generalizing quasicoherent sheaves on modules, we can also consider the Abelian category of graded modules over a graded ring.

Definition 19.11 (Graded Modules)**.** Let $A_\bullet$ be a graded ring. The category $\mathsf{Mod}_{A_\bullet}$ is the category of graded $A_\bullet$-modules with degree-preserving homomorphisms.

The desideratum is to construct a functor from graded modules to quasicoherent sheaves analogous to Proposition 17.9, however the functor $\mathsf{Mod}_{A_\bullet} \to \mathsf{QCoh}(\mathrm{Proj}(A_\bullet))$ by $M_\bullet \mapsto \widetilde{M_\bullet}$ will not be an equivalence in general.

We consider the localization of graded modules.

Definition 19.12 (Homogeneous Localization of Modules)**.** Let $M_\bullet$ be a graded $A_\bullet$-module and $S \subseteq A_\bullet$ a multiplicative subset. The homogeneous localization $(S^{-1}M)_\bullet$ is given by $\bigoplus_{d \geq 0} (S^{-1}M)_d$ where

$$(S^{-1}M)_d = \left\{ \frac{m}{a} : m \in M_e, a \in A_{e'}, e - e' = d \right\}.$$

As in the case of Definition 19.1, we are most interested in the case of localization at a prime ideal and the localization at a homogeneous element.

- Let $M_\bullet$ be a graded $A_\bullet$-module and $a \in A_d$ a homogeneous element for some d. The localization $(M_\bullet)_{(a)}$ is the degree 0 piece of the localization of Definition 19.12 for S the multiplicative set generated by a.
- Let $M_\bullet$ be a graded $A_\bullet$-module and $\mathfrak{p} \subseteq A_\bullet$ a homogeneous prime ideal. The localization $(M_\bullet)_\mathfrak{p}$ is the degree 0 piece of the localization of Definition 19.12 for S the complement of the prime ideal $\mathfrak{p}$.

We conclude with a very important construction: that of the twist of a module $M(d)$ for a graded module $M_\bullet$.

Definition 19.13 (Twist of Graded Module)**.** Let $M_\bullet$ be a graded $A_\bullet$-module and $d \in \mathbb{Z}$. The dth twist $M(d)$ is the shift of the grading by d given by $M_{d\bullet}$.

20. Lecture 20 – 16th December 2024

Recall the construction of the twist of a graded module as in Definition 19.13. We consider the case of $M_\bullet = A_\bullet$ considered as a module over itself, which is defined to be Serre's twisting sheaf.

Definition 20.1 (Serre's Twisting Sheaf)**.** Let $A_\bullet$ be a graded ring that is finitely generated by A_1 as an A_0-algebra. Serre's twisting sheaf $\mathcal{O}_{\mathrm{Proj}(A_\bullet)}(d)$ is defined to be $\widetilde{A(d)}$ for $A_\bullet$ considered as a module over itself.

We deduce some properties of Serre's twisting sheaf.

Proposition 20.2. Let $A_\bullet$ be a graded ring that is finitely generated by A_1 as an A_0-algebra. Then:

(i) Serre's twisting sheaf $\mathcal{O}_{\mathrm{Proj}(A_\bullet)}(d)$ is an invertible sheaf on $\mathrm{Proj}(A_\bullet)$.
(ii) Let $M_\bullet$ be a graded $A_\bullet$-module. Then $\widetilde{M}(d) \cong \widetilde{M(d)}$.
(iii) There is an isomorphism

$$\mathcal{O}_{\mathrm{Proj}(A_\bullet)}(d_1) \otimes_{\mathcal{O}_{\mathrm{Proj}(A_\bullet)}} \mathcal{O}_{\mathrm{Proj}(A_\bullet)}(d_2) \cong \mathcal{O}_{\mathrm{Proj}(A_\bullet)}(d_1 + d_2).$$

Recall here that an invertible sheaf is a sheaf such that the sections over any affine open subscheme is an invertible ideal.

Proof of (i). Let $a \in A_1$ and consider the restriction $\mathcal{O}_{\mathrm{Proj}(A_\bullet)}(d)|_{D_+(a)}$. By Definition 19.12, $\mathcal{O}_{\mathrm{Proj}(A_\bullet)}(d)|_{D_+(a)} \cong A(d)_{(a)}$ is a free $(A_\bullet)_{(a)}$-module of rank 1. In particular, the sections of $\mathcal{O}_{\mathrm{Proj}(A_\bullet)}(d)|_{D_+(a)} \cong A(d)_{(a)}$ consist of elements of degree d in $(A_\bullet)_a$. There is a morphism $A(d)_{(a)} \to (A_\bullet)_{(a)}$ by $\frac{b}{a^n} \mapsto \frac{b}{a^{n+d}}$ where $b \in A_{d+n}$ with inverse given by multiplication by a^d showing it is an isomorphism of modules.

As such, $\mathcal{O}_{\mathrm{Proj}(A_\bullet)}(d)|_{D_+(a)} \cong A(d)_{(a)}$ is an invertible ideal for all $a \in A_1$ and by hypothesis the open sets $D_+(a)$ form a basis for the topology on $\mathrm{Proj}(A_\bullet)$ hence the claim. ■

Proof of (ii). Once again, we note that $D_+(a)$ form a basis for the topology on $\mathrm{Proj}(A_\bullet)$. Denoting i_a the inclusion $D_+(a) \to \mathrm{Proj}(A_\bullet)$, i_a^* is symmetric monoidal so computing affine-locally

$$\widetilde{M}(d)|_{D_+(a)} \cong M_{(a)} \otimes \mathcal{O}_{\mathrm{Proj}(A_\bullet)}(d)|_{D_+(a)} \cong \widetilde{M(d)}|_{D_+(a)}$$

as claimed. ■

Proof of (iii). This is immediate from (ii) above for $M = \mathcal{O}_{\mathrm{Proj}(A_\bullet)}(d_1)$. ■

We return to a discussion of the correspondence between graded modules and quasicoherent sheaves via the following construction.

Definition 20.3 (Graded Module of a Quasicoherent Sheaf)**.** Let $A_\bullet$ be a graded ring and $\mathcal{F}$ a quasicoherent sheaf on $\mathrm{Proj}(A_\bullet)$. The graded module $\Gamma_\bullet(\mathrm{Proj}(A_\bullet), \mathcal{F})$ associated to $\mathcal{F}$ is given by $\bigoplus_{d\in\mathbb{Z}} \Gamma(\mathrm{Proj}(A_\bullet), \mathcal{F}(d))$.

In general, this does not define an equivalence of categories with inverse $\widetilde{(-)}$ on the Proj$(-)$ of an arbitrary graded ring: any graded module that is zero in sufficiently large degrees has trivial $\widetilde{(-)}$-ification. We can show that this construction is an equivalence in our setting of interest where $A_\bullet$ is finitely generated by A_1 as an A_0-algebra. We set up the proof with the following preparatory lemmata.

Lemma 20.4. Let A be a ring and $a \in A$.

(a) If $s \in \Gamma(X, \mathcal{F})$ such that $s|_{D(a)} = 0$ then there exists some $n \in \mathbb{N}$ such that $a^n \cdot s = 0$.
(b) If $t \in \Gamma(D(f), \mathcal{F})$ then there exists $n \in \mathbb{N}$ such that $f^n t$ is a global section of $\mathcal{F}$.

Proof of (i). Let $\{\mathrm{Spec}(A_{f_i})\}_{i=1}^n$ be an affine open cover of $\mathrm{Spec}(A)$ on which $\mathcal{F}|_{\mathrm{Spec}(A_{f_i})} \cong \widetilde{M_{f_i}}$. Observe $\mathrm{Spec}(A_{f_i}) \cap D(a) = \mathrm{Spec}(A_{af_i})$ on which $(a|_{D(f_i)})^{n_i} \cdot (s|_{D(f_i)}) = 0$ for some $n_i \in \mathbb{N}$ large depending on f_i. But taking N to be larger than n_i for all $1 \le i \le n$, $a^N \cdot s = 0$. ■

Proof of (ii). Denote t_i the restriction $t|_{D(f_i)}$ that further restrict to $a^{n_i} t$ on $D(af_i)$. On intersections $D(f_i f_j)$, the sections t_i and t_j agree and hence so too they agree on the smaller $D(af_i f_j) = D(a) \cap D(f_i f_j)$ where they are both $a^{n_i} t = a^{n_j} t$. As such we can take N sufficiently large such that $a^N t = a^N t$ on all pairs $D(af_i f_j)$ and all pairs $D(f_i f_j)$ allowing us to glue to a global section. ■

This construction globalizes after a twist with a line bundle.

Lemma 20.5. Let X be a quasicompact quasiseparated scheme, $\mathcal{F}$ a quasicoherent sheaf on X, and $\mathcal{L}$ an invertible sheaf on X. Let $a \in \Gamma(X, \mathcal{L})$ and

$$X_a = \{x \in X : a_x \notin \mathfrak{m}_x \mathcal{L}_x\}.$$

(i) Let $s \in \Gamma(X, \mathcal{F})$ such that $s|_{X_a} = 0$. Then there exists $n \in \mathbb{N}$ large such that $a^n s = 0$ in $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$.
(ii) If $t \in \Gamma(X_a, \mathcal{F})$, there exists some $n \in \mathbb{N}$ large such that $a^n t \in \Gamma(X_a, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ extends to a global section of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$.

Proof of (i). Let $\{\mathrm{Spec}(A_i)\}_{i=1}^n$ be an affine open cover of X on which $\mathcal{L}$ is trivialized. For such $\mathrm{Spec}(A_i)$, we have $\Gamma(\mathrm{Spec}(A_i), \mathcal{L}) \cong A_i$ with $X_a \cap \mathrm{Spec}(A_i) = D(a|_{\mathrm{Spec}(A_i)})$ and $\Gamma(\mathrm{Spec}(A_i), \mathcal{F}) \cong M$ with s restricting to a section $s|_{\mathrm{Spec}(A_i)} \in M$. By hypothesis, $s|_{\mathrm{Spec}(A_i)} = 0$ so $(a|_{\mathrm{Spec}(A_i)})^{n_i} \cdot (s|_{\mathrm{Spec}(A_i)}) = 0$ in $\Gamma(\mathrm{Spec}(A_i), \mathcal{F} \otimes \mathcal{L}^{\otimes n_i}) = \Gamma(\mathrm{Spec}(A_i), \mathcal{F})$ for n_i sufificiently large depending on $\mathrm{Spec}(A_i)$. So taking N sufficiently large such that $(a|_{\mathrm{Spec}(A_i)})^N \cdot (s|_{\mathrm{Spec}(A_i)}) = 0$ for all $1 \le i \le n$ simultaneously, $a^N \cdot s = 0$ in $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes N})$ as desired. ■

Proof of (ii). We can extend affine locally by Lemma 20.4 and globalize the construction using (a). ■

We can now show the desired result.

Proposition 20.6. Let $A_\bullet$ be a graded ring that is finitely generated by A_1 as an A_0-algebra. Then there is an isomorphism of quasicoherent sheaves on $\operatorname{Proj}(A_\bullet)$ $\widetilde{\Gamma_\bullet(\operatorname{Proj}(A_\bullet), \mathcal{F})} \to \mathcal{F}$.

Proof. We define the morphism which we denote β on basis sets $D_+(a)$ to send a section $\frac{m}{a^d} \in \Gamma(X, \mathcal{F}(d))$ to $m \otimes a^{-d}$ as a section of $\mathcal{F}$ over $D_+(a)$ but by Lemma 20.5 (ii) the construction glues to give $\mathcal{F}$. ■

We can use the preceding discussion to understand closed subschemes of projective schemes.

Definition 20.7 (Projective Morphism)**.** Let $f : X \to Y$ be a morphsim of schemes. f is projective if it factors as $X \to \mathbb{P}^n_Y \to Y$ for some $n \geq 0$ where $X \to \mathbb{P}^n_Y$ is a closed immersion.

Remark 20.8. This does not agree with the definition of projective morphisms as stated in EGA – which requires a closed embedding into the relative proj $\underline{\operatorname{Proj}}(-)$ of some sheaf of graded algebras.

We can define quasiprojective morphisms similarly.

Definition 20.9 (Quasiprojective Morphism)**.** Let $f : X \to Y$ be a morphsim of schemes. f is quasiprojective if it factors as $X \to \overline{X} \to \mathbb{P}^n_Y \to Y$ for some $n \geq 0$ where $X \to \overline{X}$ is an immersion and $\overline{X} \to \mathbb{P}^n_Y$ is a closed immersion.

These constructions are closely related to very ample invertible sheaves.

Definition 20.10 (Very Ample Invertible Sheaf)**.** Let $f : X \to Y$ be a morphism of schemes. An invertible sheaf $\mathcal{L}$ on X is f-relatively very ample if there exists a diagram

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & \mathbb{P}^n_Y \arrow[dl] \\ & Y & \end{tikzcd}$$

such that $\mathcal{L} \cong i^* \mathcal{O}_{\mathbb{P}^n_Y}(1)$ for some $n \geq 0$.

Projective morphisms are in fact proper and separated as defined in Definitions 13.8 and 15.6.

Proposition 20.11. Let $f : X \to Y$ be a morphism of schemes. If f is projective, then f is proper and separated.

Proof. The morphism is universally closed as it is a composition of a closed immersion and a closed map which is stable under base change and separated as closed immersions are separated and f is the composition of separated maps. Finally, f is finite type since on affine opens, the morphism factors over the quotient of a finite type algebra – namely the graded polynomial ring in $n + 1$ variables. ■

On projective schemes, twisting by a very ample line bundle can make coherent sheaves finitely globally generated as we now define using the following proposition.

Definition 20.12 (Finitely Globally Generated)**.** Let X be a scheme and $\mathcal{F}$ a quasicoherent sheaf on X. $\mathcal{F}$ is finitely globally generated if there exist sections $s_0, \ldots, s_n$ such that the stalks $s_{0,x}, \ldots, s_{n,x}$ generate $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module for all $x \in X$.

21. Lecture 21 – 20th December 2024

We state several important theorems of Serre.

Theorem 21.1 (Serre – Global Generation)**.** Let A be a finite type k-algebra that is Noetherian as a ring and X a projective A-scheme. If $\mathcal{F}$ is a coherent sheaf and $\mathcal{L}$ is a very ample invertible sheaf on X, then there exists N such that for all $n > N$, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is finitely globally generated.

Theorem 21.2 (Serre – Cohomology)**.** Let A be a finite type k-algebra that is Noetherian as a ring and X a projective A-scheme. If $\mathcal{F}$ is a coherent sheaf on X then $\Gamma(X, \mathcal{F})$ is a finitely generated A-module.

In particular, if $A = k$, then $\Gamma(X, \mathcal{F})$ is a finite dimensional k-vector space.

Theorem 21.3 (Serre-Grothendieck Vanishing)**.** Let X be an A-scheme for A a Noetherian ring. If $\mathcal{O}_X(1)$ is very ample and $\mathcal{F}$ a coherent sheaf on X then there exists $N \in \mathbb{N}$ such that for all $n > N$, $H^i(X, \mathcal{F}(n)) = 0$ for all $i \geq 1$.

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