F4D1 – ANALYSIS AND GEOMETRY ON MANIFOLDS WINTER SEMESTER 2024/25

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Preliminaries

These notes roughly correspond to the course **F4D1 – Analysis and Geometry on Manifolds** taught by Prof. Laurent Côté at the Universität Bonn in the Winter 2024/25 semester. These notes are LaTeX-ed after the fact with significant alteration and are subject to misinterpretation and mistranscription. Use with caution. Any errors are undoubtedly my own and any virtues that could be ascribed to these notes ought be attributed to the instructor and not the typist.

Contents

1. Lecture 1 – 8th October 2024

We first set the following conventions to be used throughout these notes:

- Let X be a topological space. A neighborhood of a point $p \in X$ is an open set $U \subseteq X$ containing p.
- For $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ and $r \in \mathbb{R}_{\geq 0}$, $B_r(p) = \{x \in \mathbb{R}^n : |x - p|^2 < r\}$ is the open ball of radius r centered at p.

We begin with a review of point set topology.

Recall the definition of locally Euclidean spaces.

Definition 1.1 (Locally Euclidean Space)**.** Let X be a topological space. X is locally euclidean if each point $x \in X$ has a neighborhood homeomorphic to $\mathbb{R}^n$ for some fixed n.

Remark 1.2. Note that the definition above does not permit topological spaces with points $x, y \in X$ such that x admits a neighborhood homeomorphic to $\mathbb{R}^n$ and y admits a neighborhood homeomorphic to $\mathbb{R}^m$ for $m \neq n$.

On a locally Euclidean topological space, we can take the neighborhoods homeomorphic to $\mathbb{R}^n$ and consider open subsets of such neighborhoods which also possess a map to $\mathbb{R}^n$.

Definition 1.3 (Chart)**.** Let X be a locally Euclidean topological space. A chart (U, ϕ) consists of an open set $U \subseteq X$ and a continuous map $\phi : U \to \mathbb{R}^n$ that is a homeomorphism onto its image.

Given a point $x \in X$ and a neighborhood, we can consider charts with a prescribed image $\phi(x) \in \mathbb{R}^n$. An especially nice case is when $\phi(x) = 0 \in \mathbb{R}^n$.

Definition 1.4 (Centered Chart)**.** Let X be a locally Euclidean topological space. A chart (U, ϕ) is centered at $x \in U$ if $\phi(x) = 0 \in \mathbb{R}^n$.

In fact, one can show that locally Euclidean topological spaces have charts centered at x for all points $x \in X$.

Proposition 1.5. Let X be a topological space. The following are equivalent:

(a) X is locally Euclidean.
(b) For any point $x \in X$, there is a chart centered at x with image the unit ball of $\mathbb{R}^n$.
(c) For any point $x \in X$, there is a chart centered at x with image $\mathbb{R}^n$.

Proof. (b)$\Longleftrightarrow$(c) by composing appropriately with the homeomorphism $B_1(0) \to \mathbb{R}^n$ by fixing the origin and the map on the complement defined by $x \mapsto \frac{1}{1-\|x\|}$. Furthermore (c)$\Rightarrow$(a) since (c) is a homeomorphisms of a neighborhood to $\mathbb{R}^n$ are in particular continuous maps to $\mathbb{R}^n$ homeomorphic onto its image.

It remains to show (a)$\Rightarrow$(b). Consider a chart (U, ϕ). For $x \in U$, we can consider the map $U \to \mathbb{R}^n$ by $y \mapsto y - \phi(x)$ yielding a chart centered at x. By scaling this map by some $\lambda \in \mathbb{R}_{>0}$ we can consider a map $\widetilde{\phi}$ by $y \mapsto \lambda y - \lambda\phi(x)$ with image containing $B_1(0)$. Restriction to the preimage of $B_1(0)$ under $\widetilde{\phi}$ yields a chart centered at x with image the unit ball $(U|_{\widetilde{\phi}^{-1}(B_1(0))}, \widetilde{\phi})$. ∎

We now introduce the notion of Hausdorff spaces, which include the spaces of concern in this course, as well as a large proportion of spaces one will encounter over the course of one's mathematical life.

Definition 1.6 (Hausdorff)**.** Let X be a topological space. X is Hausdorff if for any two distinct points $x, x' \in X$ there exist open neighborhoods U, U' of x, x', respectively, such that $U \cap U' = \emptyset$.

Example 1.7. Euclidean space $\mathbb{R}^n$ is Hausdorff.

Example 1.8. CW complexes are Hausdorff.

Example 1.9. Let X be the topological space given by the set $\{0, 1\}$ and open sets $\emptyset, \{0\}, \{0, 1\}$. This space is not Hausdorff since the points 0 and 1 cannot be separated by open sets. This space is in fact the quotient space $\mathbb{R}/\mathbb{R}^\times$ with $\mathbb{R}^\times$ acting on $\mathbb{R}$ by multiplication.

Remark 1.10. As suggested by Example 1.9, quotient spaces are the prototypical example of non-Hausdorff spaces.

We can show the following properties of Hausdorff spaces.

Proposition 1.11. Let X be a Hausdorff topological space. Then:

(i) Compact sequences have unique limits.
(ii) Compact subsets are closed.
(iii) One-point subsets are closed.

Proof of (a). Suppose to the contrary that there is a sequence $\{x_i\}_{i=1}^\infty$ with limit points x, x' distinct. Since X is Hausdorff, we can take open neighborhoods U, U' of x, x', respectively, such that $U \cap U' = \emptyset$. However we can take N large we have both $x_i \in U$ and $x_i \in V$, a contradiction as U, U' are disjoint. ■

Proof of (b). Let $K \subseteq X$ be compact. We want to show that its complement $X \setminus K$ is open. Let $x \in X \setminus K$. Since X is Hausdorff, we can consider a neighborhood V_y for each $y \in K$ disjoint from (possibly varying) neighborhoods U_y of x. Since K is compact, K is covered by finitely many V_y's say $V_{y_1}, \ldots, V_{y_n}$ and set $U = \bigcap_{i=1}^n U_{y_i}$. Note that each U_{y_i} is an open set of X containing x in the complement of V_{y_i} in X and as such their intersection contains x and is in the complement of K. As such, any $x \in X \setminus K$ admits an open neighborhood disjoint from K showing K is closed. ■

Proof of (c). This is immediate from (b), for one-point sets are compact. ■

We now discuss bases and covers of topological spaces.

Definition 1.12 (Basis for a Topological Space)**.** Let X be a topological space. A collection $\mathcal{B}$ of arbitrary subsets of X is a basis of X if for any $p \in X$ and any neighborhood U of p there exists an element of B containing p and contained in U.

It can be shown that any open set of a topological space can be written as a union of basis sets.

Proposition 1.13. Let X be a topological space and $\mathcal{B}$ an arbitrary collection of subsets of X. $\mathcal{B}$ is a basis of X if and only if every open set of X can be written as a union of sets of $\mathcal{B}$.

Proof. ($\Rightarrow$) Suppose $\mathcal{B}$ is a basis of X and let $U \subseteq X$ be open. For $x \in U$ consider $V_x \in \mathcal{B}$ containing x but contained in U where we have $U = \bigcup_{x\in U} V_x$, writing U as a union of basis sets.

($\Leftarrow$) Suppose for each open $U \subseteq X$ we can write $U = \bigcup_{i\in I} V_i$. As such, for each point $x \in U$ there is some V_i contained in U containing X thus forming a basis. ∎

We want to focus our attention on topological spaces that are appropriately "small" by imposing size conditions on the basis.

Definition 1.14 (Second Countable Space)**.** Let X be a topological space. X is a second countable space if X admits a countable basis $\mathcal{B}$.

The countability property is preserved under the following conditions.

Proposition 1.15. Let X be a topological space. Then:

(i) If X is second countable, then any subspace of X with the subspace topology is second countable.
(ii) If $\{U_i\}_{i\in I}$ is a countable open cover of X with each each U_i second countable then X is countable.
(iii) If X is locally Euclidean and $\{K_i\}_{i=1}^{\infty}$ is a sequence of compact subsets such that $X = \bigcup_{i=1}^{\infty} K_i$ then X is second countable.

Proof of (i). For $Y \subseteq X$ a subspace with the induced topology, we seek to produce a countable basis for Y. Let $\mathcal{B}$ be a countable basis for X and $\mathcal{B}' = \{\overline{V} = V \cap Y | V \in \mathcal{B}\}$ which is countable since it is cardinality is at most that of $\mathcal{B}$ which is countable by hypothesis. We show $\mathcal{B}'$ is a basis for Y. Let $\overline{U} \subseteq Y$ be open so $\overline{U} = U \cap Y$ for some $U \subseteq X$ open by definition of the subspace topology. Let $y \in \overline{V}$ and note that since $\mathcal{B}$ is a basis for X, there is a set V of $\mathcal{B}$ contained in U with $y \in V$ giving a set $V \cap Y = \overline{V} \in \mathcal{B}'$ that is contained in $\overline{U}$ hence producing a basis for Y. ∎

Proof of (ii). Let $\{U_i\}_{i\in I}$ be a countable open cover of X and for some fixed $i \in I$ consider $\{U_{ij}\}_{j\in J_i}$ a countable basis for U_i. Note that $\{U_{ij}\}_{i\in I, j\in J_i}$ is countable, a countable union of countable sets which we seek to show is a basis for X. We show that any open $V \subseteq X$ can be written as a union of some U_{ij}'s. Since the U_i's are open in X, we can set $V_i = U_i \cap V$ which are open in U_i. As we have shown in class, V_i can be written as a union $V_i = \bigcup_{j\in J_i'} U_{ij}$ for fixed i and $J_i' \subseteq J_i$ and we have $V = \bigcup_{i\in I, j\in J_i'} U_{ij}$ expressing an open of X as a union of elements of $\{U_{ij}\}_{i\in I, j\in J_i}$ making $\{U_{ij}\}_{i\in I, j\in J_i}$ basis of X which we have shown to be countable above. ∎

Proof of (iii). For each K_i consider an open cover of K_i by open sets $\{U_{ij}\}_{j\in J_i}$ hoemeomorphic to $\mathbb{R}^n$ which exist by X being locally Euclidean. Without loss of generality, we can consider this cover to be finite taking $\{U_{ij}\}_{j=1}^{m}$ since K_i is compact. Each U_{ij} has basis given by preimages of the countable basis of open balls in $\mathbb{R}^n$ by taking balls of rational radius centered at rational points endowing each U_{ij} with a countable basis. Thus each K_i has a countable basis by taking finite unions over the countable bases of U_{ij} for $j \in J_i$ and X has countable basis by taking countable union over the countable bases of each K_i as desired. ■

Remark 1.16. The property of being second countable is not preserved under arbitrary quotients, though this holds when the quotient map is open.

We can describe the second countability property in terms of covers.

Proposition 1.17. Let X be a topological space. If X is second countable then any open cover of X admits a countable subcover.

Proof. Let $\mathcal{B}$ be a countable basis for X and $\{U_i\}_{i \in I}$ an open cover of X. Consider $\widetilde{\mathcal{B}}$ consisting of those basis elements of X contained in some U_i. Note that $\widetilde{\mathcal{B}}$ is a cover of X since for any point $x \in U_i$ there is an element of $\mathcal{B}$ containing x contained in U_i. For each $V \in \widetilde{\mathcal{B}}$ of which there are countably many, consider $U_V \in \{U_i\}_{i \in I}$ such that $V \subseteq U_V$. These form a cover of X indexed by a countable set $\widetilde{\mathcal{B}}$ giving the claim. ■

We also introduce the following notion of compact exhaustability.

Definition 1.18 (Compact Exhaustability)**.** Let X be a topological space. X is compactly exhaustible if there exists a sequence of compact subsets $\{K_i\}_{i=1}^{\infty}$ of X such that $K_i \subseteq K_{i+1}^{\circ}$ and $X = \bigcup_{i=1}^{\infty} K_i$.

The condition of compact exhaustability is satisfied under relatively mild hypotheses.

Proposition 1.19. Let X be a topological space. If X is locally Euclidean, Hausdorff, and second countable, X admits an exhaustion by compact subsets.

Proof. We first note that since X is locally Euclidean, it admits a basis $\mathcal{B}$ of open subsets having compact closure: for each chart (U, ϕ) we can take some $x \in U$ and set the image of the chart to be centered at x homeomorphic onto the open unit ball by Proposition 1.5 and produce a countable basis of the ball by smaller balls wich have compact closure. By taking preimages, we can consider the countable union of countable balls with compact closures inducing the respective property for each open of X.

Furthermore, since X is second countable, it is covered – up to a choice of bijection of the countable indexing set with the natural numbers – by countably many sets $\{U_i\}_{i=1}^{\infty}$ with compact closure. Suppose $K_1 = \overline{U_1}$. We proceed by induction and suppose that there are compact sets $K_1, \ldots, K_m$ such that $U_i \subseteq K_i$ for each i and $K_i \subseteq K_{i+1}^{\circ}$ for $2 \leq i \leq m-1$. Since K_m is compact, there is some $N_m \geq m+1$ large such that $K_m \subseteq U_1 \cup \cdots \cup U_{N_m}$. If $K_{m+1} = \overline{U_1} \cup \cdots \cup \overline{U_{N_m}}$ then K_{m+1} is closed and thus compact with interior containing K_m giving the claim. ■

2. Lecture 2 – 11th October 2024

We continue our discussion of topological manifolds in general and bases and covers in particular.

Definition 2.1 (Locally Finite)**.** Let X be a topological space and $\mathcal{C}$ a collection of subsets of X. $\mathcal{C}$ is locally finite if for every $x \in X$ there exists a neighborhood U of X such that U intersects only finitely many elements of $\mathcal{C}$.

Example 2.2. Let $X = \mathbb{R}$ in the usual topology and $\mathcal{C} = \{(a-1, a+1) : a \in \mathbb{Z}\}$. This is locally finite since every sufficiently small ball will intersect at most two elements in $\mathcal{C}$.

Example 2.3. Let $X = \mathbb{R}$ in the usual topology and $\mathcal{C} = \{(a-1, a+1) : a \in \mathbb{Q}\}$. This is not locally finite since $\mathbb{Q}$ is dense in $\mathbb{R}$.

We can now define paracompactness in terms of a refinement condition.

Definition 2.4 (Refinement)**.** Let X be a topological space and $\{U_i\}_{i\in I}$ an open cover of X. A cover $\{V_j\}_{j\in J}$ is a refinement of $\{U_i\}_{i\in I}$ if for all elements U_i there is some $V_j \subseteq U_i$.

Definition 2.5 (Paracompact)**.** Let X be a topological space. X is paracompact if each cover of X has a refinement by a locally finite cover.

This is a weaker condition than compactness but still captures a number of desirable properties.

Lemma 2.6. Let X be a Hausdorff topological space admitting a compact exhaustion. Then for any basis $\mathcal{B}$ of X, any open cover admits a locally finite subcover by basis elements. In particular, X is paracompact.

Proof. By assumption, there is a sequence $\{K_i\}_{i=1}^{\infty}$ of compact sets with $K_i \subseteq K_{i+1}^{\circ}$ and $\bigcup_{i=1}^{\infty} K_i = X$. Let $\{U_j\}_{j\in J}$ be an open cover. For $m \in \mathbb{Z}$, set $V_m = K_{m+1} \setminus K_m^{\circ}$ for $m \geq 0$ and $\emptyset$ otherwise. First note that the V_m are compact as it is a closed set of a compact set and that $\bigcup_{m\in\mathbb{Z}} V_m = X$, and that $V_m \cap V_{m-1} = \partial K_m$ is compact it being a closed subset of a compact space. Further noting that $\{U_j \cap K_{m+1}^{\circ} \cap K_{m-1}^{c}\}_{j\in J}$ forms an open cover of V_j. Moreover, since $\mathcal{B}$ is a basis, we can find a refinement of this cover by basis elements $W_1, \ldots, W_n$. This cover suffices as it is a refinement of $\{U_j\}_{j\in J}$ and is locally finite since for any $x \in X$ we have that $x \in V_m$ for some m and thus $x \in K_{m+2}^{\circ} \cap K_{m-1}^{c}$ hence intersecting only finitely many of the W's.

The latter claim follows immediately from the former. ∎

From this, we conclude the following corollary.

Corollary 2.7. If X is a locally Euclidean, Hausdorff, and second countable topological, then X is paracompact.

Proof. This follows from previous results. Being locally Euclidean and second countable implies compact exhaustion by Proposition 1.19, which in turn implies paracompactness by Lemma 2.6. ∎

We can now begin a discussion of topological manifolds.

Definition 2.8 (Topological Manifold)**.** A topological space M is a topological manifold if it is locally Euclidean, Hausdorff, and second countable.

Remark 2.9. The Hausdorffness condition is required here to ensure the collection of objects we are considering is not too large.

These objects naturally assemble into a category, in fact a full subcategory of the category of topological spaces.

Definition 2.10 (Category of Topological Manifolds)**.** The category of topological manifolds Mfld consists of objects topological manifolds and morphisms continuous maps.

Remark 2.11. Fullness as a subcategory follows from the definition, and as such equivalences in Mfld are homeomorphisms.

We have already encountered a number of examples.

Example 2.12. $\mathbb{R}^n$ is a topological manifold.

Example 2.13. A finite-dimensional $\mathbb{R}$-vector space is a topological manifold under the metric topology.

Example 2.14. Any open subset of $\mathbb{R}^n$ is a topological manifold.

Example 2.15. Let $U \subseteq \mathbb{R}^n$ open and $f : U \to \mathbb{R}^m$ be a continuous function. Set $\Gamma(f) = \{(x, y) \in U \times \mathbb{R}^m : f(x) = y\}$. Then $\Gamma(f)$ is a manifold.

Example 2.16. The n-sphere $S^n \subseteq \mathbb{R}^{n+1}$ is a smooth manifold.

Example 2.17. Let C^n be the boundary of the n-cube. Then C^n is homeomorphic to the sphere S^n and hence a manifold.

Example 2.18. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ with the quotient topology be the n-torus. Then $\mathbb{T}^n$ is a manifold.

Example 2.19. Real projective space $\mathbb{RP}^n$ is a manifold.

Example 2.20. The Klein bottle is a manifold.

Remark 2.21. The examples of Examples 2.19 and 2.20 are examples of non-orientable manifolds.

We can also define manifolds with boundary, where charts are taken to be homeomorphic to the upper-half space.

Definition 2.22 (Upper-Half Space)**.** The upper-half space $\mathbb{H}^n$ is given by

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}.$$

Manifolds with boundary are then defined as follows.

Definition 2.23 (Manifold with Boundary)**.** A topological space M is a manifold with boundary if it is Hausdorff, second countable, and each point has a neighborhood homeomorphic to an open subset of $\mathbb{H}^n$.

Remark 2.24. As such, every manifold is a manifold with boundary, translating the image of charts such that it does not intersect $x_1 = \cdots = x_n = 0$ in $\mathbb{H}^n$.

Example 2.25. $\mathbb{H}^n$ is a manifold with boundary.

Example 2.26. $S^n \cap \mathbb{H}^{n+1}$ is a manifold with boundary, and is in fact homeomorphic to the closed unit disc.

Interior and boundary points of manifolds with boundary are defined as follows.

Definition 2.27 (Interior Point)**.** Let M be a manifold with boundary. A point $x \in M$ is an interior point if it has a neighborhood homeomorphic to $\mathbb{R}^n$.

Definition 2.28 (Boundary Point)**.** Let M be a manifold with boundary. A point $x \in M$ is an interior point if it does not have a neighborhood homeomorphic to $\mathbb{R}^n$.

3. Lecture 3 – 15th October 2024

To the end of motivating the forthcoming discussion of smooth manifolds, we turn to an exposition about properties of general topological manifolds.

The classification of one-dimensional manifolds is extremely simple, and is given as follows.

Theorem 3.1. If M is a topological manifold of dimension 1, then M is homeomorphic to $\mathbb{R}$ or to S^1.

Permitting manifolds with boundary, we have the following.

Theorem 3.2. If M is a topological manifold with boundary of dimension 1, then M is homeomorphic to one of $\mathbb{R}$, S^1, $[0, 1]$, or $[0, 1)$.

The case of compact connected 2-manifolds is also well-known.

Theorem 3.3. If M is a compact connected topological manifold of dimension 2, then M is homeomorphic to S^2, a connected sum of tori $\mathbb{T}^2$, or a connected sum of real projective planes $\mathbb{RP}^2$.

In the cases of dimension at least three, the classification theory is extremely poorly understood, and in fact provably so.

The theory of smooth manifolds refines the theory of topological manifolds, and as such allows us to say more about them. We begin by introducing some basic notions that illustrate the local theory of smooth manifolds via charts and atlases.

Definition 3.4 (Smooth Function)**.** Let $U \subseteq \mathbb{R}^n$ be an open subset. A function $f = (f_1, \ldots, f_m) : U \to \mathbb{R}^m$ is smooth if each f_i admits all continuous partial derivatives of all orders.

Remark 3.5. Smooth functions are at times denoted C^∞ functions, in contrast to continuous functions which are denoted by C^0.

In other words, for all multiindices $\alpha \in \mathbb{N}^n$, the partial derivative $\partial_{x^\alpha} f_i = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} f_i$ exists and is continuous.

Smooth manifolds are defined in terms of smooth atlases that are built out of smoothly compatible charts.

Definition 3.6 (Smoothly Compatible Charts)**.** Let M be a topological manifold. A pair of charts (U_1, ϕ_1) and (U_2, ϕ_2) are smoothly compatible if the transition function $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is a smooth function $\mathbb{R}^n \to \mathbb{R}^n$.

We can now define smooth atlases.

Definition 3.7 (Smooth Atlas)**.** Let M be a topological manifold. A smooth atlas $\mathcal{A}$ on M consists of the data of pairwise smoothly compatible charts (U_α, ϕ_α) such that $\bigcup_{\alpha \in \mathcal{A}} U_\alpha = M$.

On a fixed topological manifold, however, there may be multiple choices of smooth atlases. This is somewhat remedied by the following.

Definition 3.8 (Equivalent Atlases)**.** Let M be a topological manifold and $\mathcal{A}, \mathcal{A}'$ two smooth atlases on M. $\mathcal{A}$ and $\mathcal{A}'$ are equivalent smooth atlases if $\mathcal{A} \cup \mathcal{A}'$ is a smooth atlas.

We finally arrive at the definition of a smooth manifold.

Definition 3.9 (Smooth Manifold)**.** A smooth manifold M consists of the data of a topological manifold M and an equivalence class of smooth atlases $[\mathcal{A}]$ on M.

Remark 3.10. Smooth atlases naturally have an ordering by containment, and one can show using Zorn's lemma that an atlas is contained in a maximal atlas. In practice, however, it is rarely useful to work with the maximal atlas.

4. Lecture 4 – 18th October 2024

We consider some examples of smooth manifolds in parallel to Examples 2.12 to 2.20.

Example 4.1. $\mathbb{R}^n$ is canonically a smooth manifold. The canonical atlas is the tautological chart given by the identity morphism.

Example 4.2. A finite dimensional $\mathbb{R}$-vector space is a smooth manifold by choosing a vector space basis B which induces a homeomorphism with $\mathbb{R}^n$. Between two choices of bases, the transition maps are given by an element of $\mathrm{GL}_{\dim(V)}(\mathbb{R})$ which is linear and hence smooth.

Example 4.3. The n-sphere $S^n \subseteq \mathbb{R}^{n+1}$ is a smooth manifold.

Example 4.4. Let $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function and for $\lambda \in \mathbb{R}$ the level set $\Phi^{-1}(\lambda) = \{x \in \mathbb{R}^{n+1} : \Phi(x) = \lambda\}$ such that for all $x \in \Phi^{-1}(\lambda)$ the derivative $D\Phi(x) \neq 0$ is a smooth manifold.

Example 4.5. Let M, N be smooth manifolds. Then their product $M \times N$ is a smooth manifold.

Example 4.6. Consider again $\mathbb{R}$ with the smooth structure given by a global chart $x \mapsto x^3$. This is a smooth manifold, albeit not one compatible with the canonical chart.

Having defined and considered some examples of smooth manifolds, we discuss smooth maps, to the end of defining the category of smooth manifolds.

Definition 4.7 (Smooth Maps - $\mathbb{R}^m$)**.** Let M be a smooth manifold. A map $f : M \to \mathbb{R}^m$ is smooth if for all $p \in M$ there exists a chart (U, ϕ_p) containing p such that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is a smooth function.

This allows us to define, more generally, smooth maps between smooth manifolds.

Definition 4.8 (Smooth Maps - Manifolds)**.** Let M, N be smooth manifolds. A continuous map $f : M \to N$ is smooth if for all $p \in M$ there exists a chart (U, ϕ) containing p and (V, ψ) containing $\phi(U)$ such that $\psi \circ f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is smooth.

We can consider some properties of smooth maps.

Proposition 4.9. A continuous map between smooth manifolds $f : M \to N$ is smooth if and only if each $p \in M$ has a neighborhood U such that $f|_U$ is smooth.

Moreover, we have the following.

Proposition 4.10. Let $f : M \to N$ be a continuous map. Then:

(i) If f is constant, f is smooth.
(ii) If $U \subseteq M$ is open, then $U \hookrightarrow M$ is smooth.
(iii) Smoothness is preserved by composition.

In particular, Proposition 4.10 (iii) is necessary to make the category of smooth manifolds well-defined as we soon see.

Definition 4.11 (Diffeomorphism)**.** Let $f : M \to N$ be a continuous map between smooth manifolds. f is a diffeomorphism if f is bijective with smooth inverse.

Remark 4.12. In particular, f is a homeomorphism on the underlying topological manifolds.

Here are some examples of smooth maps.

Example 4.13. $f : \mathbb{R} \to \mathbb{R}$ by $x \mapsto x + 2$ is a diffeomorphism.

Example 4.14. Let $A \in \mathrm{GL}_n(\mathbb{R})$. The map $\mathbb{R}^n \to \mathbb{R}^n$ by $x \mapsto Ax$ is a diffeomorphism.

We can now define the category of smooth manifolds.

Definition 4.15 (Category of Smooth Manifolds)**.** The category of smooth manifolds SmMfld consists of objects smooth manifolds and morphisms smooth maps.

Remark 4.16. One can observe that diffeomorphisms are exactly the isomorphisms in the category of smooth manifolds SmMfld.

Remark 4.17. The forgetful functor from smooth manifolds to topological manifolds is neither full nor essentially surjective – there are far fewer smooth morphisms between smooth manifolds M, N than there are continuous maps between the underlying topological manifolds and not every topological manifold is homeomorphic to a smooth manifold.

Henceforth, we will focus our attention to the study of SmMfld.

Remark 4.18. Returning to the motivating discusion of the classification of manifolds initiated in Section 3, we note that classification of smooth manifolds is more manageble due to recent progress in differential geometry, though far from being trivial.

5. Lecture 5 – 22nd October 2024

Having considered smooth manifolds, we turn to a discussion of smooth manifolds with boundary. We begin with our building blocks of smooth functions in this setting.

Definition 5.1 (Smooth Functions on $\mathbb{H}^n$)**.** Let $U \subseteq \mathbb{H}^n$ be an open subset of the upper-half space. A function $f : U \to \mathbb{R}^m$ is smooth if every $p \in U$ admits an open neighborhood $U_p \subseteq \mathbb{R}^n$ such that f extends to a smooth function $\widetilde{f} : U_p \to \mathbb{R}^m$.

Considering some examples, we have the following.

Example 5.2. For $n = 1$, $\mathbb{H}^1 = [0, \infty)$ and $f(x) = x^2$ fulfils this property.

Example 5.3. $\sqrt{x}$ does not work as there is no smooth extension of the function at the origin – derivatives get arbitrarily large.

Given the above definition, one can extend the definition of smooth manifolds to smooth manifolds with boundary, taking smooth compatibility to be defined using Definition 5.1 above. In particular, every smooth manifold is a smooth manifold with boundary. However, we will largely restrict our attention to the boundaryless case.

We now turn to a discussion of partitions of unity, which will serve as a key tool in the study of smooth manifolds. Indeed, it is the fact that smooth manifolds admit partitions of unity that makes the study of them differ from the study of geometric objects such as those found in algebraic geometry. We begin with some preparatory lemmata, in particular the following.

Lemma 5.4. The function $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

is smooth.

Proof. The function is continuous as it is piecewise continuous and agrees at the origin. It thus suffices to show that f has well-defined and continuous derivatives of all order. We proceed by induction on the hypothesis that $f^{(k)}$ exists, is identically 0 on $(-\infty, 0]$, and is of the form $P_k(\frac{1}{t})e^{-\frac{1}{t}}$ where P_k is a univariate polynomial. For the first derivative, we can explicitly compute that the derivative of $e^{-frac1t}$ is given by $\frac{1}{t^2}e^{-\frac{1}{t}}$ and the function satisfies the hypotheses. Suppose this holds for the first k derivatives. We can compute the $k + 1$st derivative at the origin as $\lim_{t \to 0^+} \frac{f^{(k)}(t) - f^{(k)}(0)}{t} = \lim_{t \to 0^+} \frac{f^{(k)}(t)}{t}$ using that $f^{(k)}(t)$ is of the form $P_k(\frac{1}{t})\frac{1}{t}e^{-\frac{1}{t}}$ we can show using substitution that this limits to 0 and thus that $f^{(k+1)}(t)$ is differentiable at the origin. Verification that it is of the desired form on $(0, \infty)$ is a direct computation of the derivative of $P_k(\frac{1}{t})e^{-\frac{1}{t}}$. In particular, $f^{(k+1)}(t)$ is differentiable and continuous, yielding the claim. ■

This function allows us to construct smooth functions that take prescribed values on specific open intervals, which we will soon generalize to balls in $\mathbb{R}^n$.

Lemma 5.5. Let $r_1 < r_2$. There exists a smooth function $g : \mathbb{R} \to \mathbb{R}$ with the following properties:

- $g = 1$ on $(-\infty, r_1]$.
- $0 < g < 1$ on (r_1, r_2).
- $g = 0$ on $[r_2, \infty)$.

Proof. The function $g(t) = \frac{f(r_2-t)}{f(r_2-t)+f(t-r_1)}$ suffices since the denominator is nonvanishing as both summands are bounded below and one of them is strictly positive for all $t \in \mathbb{R}$. ■

This phenomena has a natural generalization as cutoff functions.

Definition 5.6 (Cutoff Function)**.** Let $0 < r_1 < r_2$. A cutoff function is a function $H : \mathbb{R}^n \to \mathbb{R}$ with the following properties:

- $H = 1$ on $\overline{B_{r_1}}$.
- $0 < H < 1$ on $B_{r_2} \setminus \overline{B_{r_1}}$.
- $H = 0$ on $\mathbb{R}^n \setminus B_{r_2}$.

These functions are easily seen to exist by inputting the norm of a point into the function of Lemma 5.5.

Lemma 5.7. Let $0 < r_1 < r_2$. There exists a cutoff function $H : \mathbb{R}^n \to \mathbb{R}$.

Proof. The function $H(x) = g(|x|)$ with g as in Lemma 5.5 suffices. It is smooth as it is piecewise smooth and infinitely differentiable at the boundary of each piece. ■

We define one final notion before considering partitions of unity.

Definition 5.8 (Support)**.** Let X be a topological space and $f : X \to \mathbb{R}$ a real-valued function on X. The support of f is given by

$$\operatorname{supp}(f) = \overline{\{x \in X : f(x) \neq 0\}}.$$

Example 5.9. A univariate polynomial has support $\mathbb{R}$ since the closure of the complement of its finitely many roots is $\mathbb{R}$.

Example 5.10. The support of the function g in Lemma 5.5 is $(-\infty, r_2]$.

Partitions of unity are defined as follows.

Definition 5.11 (Partition of Unity)**.** Let M be a smooth manifold and $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ an open cover of M. A partition of unity subordinate to the cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is a collection of smooth functions $\{\psi_\alpha\}_{\alpha\in\mathcal{A}}$ such that:

- $0 < \psi_\alpha < 1$,
- $\operatorname{supp}(\psi_\alpha) \subseteq U_\alpha$,
- $\{\operatorname{supp}(\psi_\alpha)\}_{\alpha\in\mathcal{A}}$ is locally finite, and
- $\sum_{\alpha\in\mathcal{A}} \psi_\alpha = 1$.

These partitions of unity exist for any smooth manifold.

Theorem 5.12. Let M be a smooth manifold and $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ an open cover of M. There exists a partition of unity subordinate to the cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$.

6. Lecture 6 – 25th October 2024

We consider some applications of partitions of unity.

Definition 6.1 (Bump Function)**.** Let X be a topological space and $A, U \subseteq X$ be a closed and open subspace, respectively. A bump function for A supported in U is a function $\psi : X \to \mathbb{R}$ such that $\psi|_A \equiv 1$ and $\operatorname{supp}(\psi) \subseteq U$.

Using partitions of unity, we can construct a bump function supported on any clsoed subset of a smooth manifold.

Proposition 6.2. *Let M be a smooth manifold, $A \subseteq M$ closed and $U \subseteq M$ an open neighborhood of A. There exists a smooth bump function for A supported in U.*

Proof. Let U be as above and $V = M \setminus A$. These form an open cover $\{U, V\}$ of M. By Theorem 5.12, there exists a partition of unity ψ_U, ψ_V subordinate to this cover. ψ_U suffices since $\psi_V \equiv 0$ on A and $\psi_U + \psi_V \equiv 1$ so it is idenitcally 1 on A. ■

We also make the following definition, allowing us to consider smooth functions on a closed subset of a smooth manifold.

Definition 6.3 (Smooth Function on Closed Subset)**.** Let M, N be smooth manifolds and $A \subseteq M$ a closed subset. A continuous map $f : M \to N$ is smooth on A if it admits a smooth extension in an open neighborhood of each $x \in A$.

Remark 6.4. Recall from Definition 5.1 that we require that for each $x \in A$ there is an open neighborhood U_x of x such that $\widetilde{f} : U_x \to N$ is smooth and $\widetilde{f}|_{U_x \cap A} = f|_{U_x \cap A}$.

In particular, we can construct smmooth functions from a closed set of a smoth manifold to $\mathbb{R}^m$ whose support is contained in an fixed open neighborhood of A.

Proposition 6.5. *Let M be a smooth manifold, $A \subseteq M$ closed, and $f : A \to \mathbb{R}^m$ a smooth function. For $U \subseteq M$ containing A, there exists a smooth extension $\widetilde{f} : M \to \mathbb{R}^m$ such that $\widetilde{f}|_A = f$ and $\operatorname{supp}(\widetilde{f}) \subseteq U$.*

Proof. For each $x \in A$ consider a neighborhood $W_x \subseteq U$ containing x. By hypothesis, there exists $\widetilde{f}_x : W_x \to \mathbb{R}^m$ such that $\widetilde{f}_x|_{W_x \cap A} = f|_{W_x \cap A}$. Now note that $\{W_p\}_{p \in A} \cup \{M \setminus A\}$ is an open cover of M. Let $\{\psi_x\}_{x \in A} \cup \{\psi_{M \setminus A}\}$ be a partition of unity subordinate to the cover. Set $\widetilde{f}(y) = \sum_{x \in A} \widetilde{f}_x(y)\psi_x(y)$. Observe that since $\{\operatorname{supp}(\psi_x)\}_{x \in A}$ is locally finite, only fintely many terms of the sum are nonzero in a neighborhood of any point of M showing $\widetilde{f}$ is smooth. Additionally, if $y \in A$ then $\psi_{M \setminus A}(y) = 0$ and thus $\widetilde{f}(y) = \psi_{M \setminus A}(y) + \sum_{x \in A} \widetilde{f}_x(y)\psi_x(y) = f(y)$ showing $\widetilde{f}$ is an extension of f. Finally, the condition on the support holds as $\operatorname{supp}(\psi_x) \subseteq U$ for each p ■

Remark 6.6. Proposition 6.5 fails in the category of topological manifolds. Take $A = S^1 \subseteq \mathbb{R}^2$ and $f : S^1 \to S^1$ the identity map. f does not admit a continuous extension to $\mathbb{R}^2$ obstructed by the homotopy groups.

Partitions of unity can also be used to construct smooth functions with prescribed properties. One example of this is in exhaustion functions.

Definition 6.7 (Exhaustion Function)**.** Let X be a topological space. An exhaustion function $f : X \to \mathbb{R}$ is a continuous function such that for all $c \in \mathbb{R}$ $f^{-1}((-\infty, c]) \subseteq X$ is open.

Remark 6.8. This recovers the notion of an exhaustion by compact sets indexing over the natural numbers.

Example 6.9. $f : \mathbb{R} \to \mathbb{R}$ by $x \mapsto x^2$ has closed and hence compact preimage.

Example 6.10. $f : \mathbb{R} \to \mathbb{R}$ by $x \mapsto x$ is a non-example since the preimage of $(-\infty, c]$ which is $(-\infty, c]$ is non-compact.

It can be shown that every smooth manifold admits a smooth exhaustion function.

Proposition 6.11. If M is a smooth manifold, M admits a smooth positive exhaustion function.

Proof. Let $\{U_i\}_{i=1}^{\infty}$ be a countable open cover having compact closure and $\{\psi_i\}_{i=1}^{\infty}$ a partition of unity subordinate to this cover. Set $f(x) = \sum_{i=1}^{\infty} i \cdot \psi_i(x)$. f is smooth since only finitely many terms of the sum are nonzero in the neighborhood of any point and positive by construcion.

To show that f is an exhaustion function, we have for any $c \in \mathbb{R}$ and a natural number $N > c$ that $f^{-1}((-\infty, c])$ is a closed subset of the compact set $f^{-1}((-\infty, N]) = \bigcup_{i=1}^{N} \overline{U_i}$ and hence compact. If $x \notin \bigcup_{i=1}^{N} \overline{U_i}$ then

$$f(x) = \sum_{i=N+1}^{\infty} i\psi_i(x) \geq \sum_{i=N+1}^{\infty} N\psi_i(x) = N \sum_{i=1}^{\infty} \psi_i(x) = N > c$$

showing that $x \notin f^{-1}((-\infty, c])$ and conversely if $f(x) \leq c$ then $x \in \bigcup_{i=1}^{N} \overline{U_i}$ showing that $f^{-1}((-\infty, c]) \subseteq \bigcup_{i=1}^{N} \overline{U_i}$ yielding the claim. ∎

The construction of exhaustion functions is closely linked to the fact that any closed subset of a smooth manifold can be obtained as the preimage of 0 of a smooth function $f : M \to \mathbb{R}$. We deduce the general case as a consequence of the following lemma.

Lemma 6.12. Let $A \subseteq \mathbb{R}^n$ be a closed subset. There exists a nonnegative smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f^{-1}(0) = A$.

Proof. Let $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$ be a countable open cover of $M \setminus A$ by balls. By Lemma 5.7, there exists a cutoff function H that is equal to 1 on $\overline{B_{1/2}(0)}$ and supported in $B_1(0)$. Now for each i, let $C_i \geq 1$ be a constant such that $C_i > \sup_{x \in \mathbb{R}^n} \{\partial^{\alpha} H : |\alpha| \leq i\}$. We show

$$f(x) = \sum_{i=1}^{\infty} \frac{r_i^i}{2^i C_i} H\left(\frac{x - x_i}{r_i}\right)$$

suffices.

First note that each term $\frac{r_i^i}{2^i C_i} H\left(\frac{x-x_i}{r_i}\right)$ is bounded by $\frac{1}{2^i}$ since $r_i < 1$ and that the sequence $\sum_{i=1}^{\infty} \frac{1}{2^i}$ converges so the function is continuous. To see that f is smooth, we proceed by induction on the hypothesis that partial derivatives of order up to k exist and are continuous. Noting that an order $k+1$ partial derivative is of the form

$$\partial^\alpha \left(\frac{r^i}{2^i C_i} H\left(\frac{x-x_i}{r_i}\right)\right) = \frac{r^i}{2^i C_i} \partial^\alpha \left(H\left(\frac{x-x_i}{r_i}\right)\right)$$

which is once again bounded above by $\frac{1}{2^i}$ by construction of H so repeating the argument above, this is continuous showing f is smooth. ■

We deduce the following general statement.

Theorem 6.13. Let M be a smooth manifold and $A \subseteq M$ closed. There exists a nonnegative smooth function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = A$.

Proof. Let M be a smooth manifold as stated and $A \subseteq M$ closed. Let $(\phi_\alpha, U_\alpha)_{\alpha \in \mathcal{A}}$ be an atlas of M. Without loss of generality, we can take $\phi_\alpha(U_\alpha) = \mathbb{R}^n$ for some fixed n. Furthermore, $A \cap U_\alpha$ is closed in the subspace topology of U_α and thus has closed image $\phi_\alpha(A \cap U_\alpha) \subseteq \phi_\alpha(U_\alpha) = \mathbb{R}^n$. By Lemma 6.12 there are functions $f_\alpha : \mathbb{R}^n \to [0, \infty)$ such that $f_\alpha^{-1}(0) = \phi_\alpha(A \cap U_\alpha)$. Moreover, the functions $f_\alpha \circ \phi_\alpha : U_\alpha \to \mathbb{R}$ are smooth since they are the composite of smooth functions. It remains to glue these $f_\alpha \circ \phi_\alpha$ into a smooth function $M \to [0, \infty)$. Let $(\psi_\alpha)_{\alpha \in \mathcal{A}}$ be a partition of unity subordinate to the cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$. We show that the function $f(x) = \sum_{\alpha \in \mathcal{A}} (\psi_\alpha(x) \cdot (f_\alpha \circ \phi_\alpha)(x))$ which is smooth as the product and sum of smooth functions. We have $A = f^{-1}(0)$ since the summand $\psi_\alpha(x) \cdot (f_\alpha \circ \phi_\alpha)(x)$ is zero if and only if $x \in A \cap U_\alpha$ and undefined on the complement $M \setminus U_\alpha$ so preimage of 0 is given by the union of the component functions. ■

Having discussed partitions of unity, a central technical tool in the study of smooth manifolds, we begin a consideration of tangent vectors.

Definition 6.14 (Tangent Space)**.** Let M be a smooth manifold and $x \in M$. The tangent space $T_x M$ of M at x is the set of equivalence classes of smooth curves $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = x$ where $\gamma \sim \gamma'$ if and only if for every smooth function f defined near p there is an equality $(f \circ \gamma)'(0) = (f \circ \gamma')'(0)$.

Remark 6.15. This more abstract definition may seem quite foreign to those encountered in single and multivariable calculus. However, providing an embedding-independent description of tangent spaces is required for a study of smooth manifolds in the appropriate generality.

The formation of these tangent spaces behave well with respect to smooth maps.

Definition 6.16 (Differential of a Smooth Map)**.** Let $f : M \to N$ be a morphism of smooth manifolds. The differential of f at x is the map $df_x : T_x M \to T_{f(x)} N$ by $[\gamma] \mapsto [f \circ \gamma]$.

Remark 6.17. The map is well-defined and in fact functorial.

7. Lecture 7 – 29th October 2024

Let us consider some elementary properties of tangent spaces.

Lemma 7.1. Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n, \sigma : (-\delta, \delta) \to \mathbb{R}^n$ such that $\gamma(0) = \sigma(0)$ in $\mathbb{R}^n$. Then $\gamma \sim \sigma$ if and only if $\gamma'(0) = \sigma'(0)$.

Proof. ($\Rightarrow$) Suppose $\gamma \sim \sigma$. The ith component function is smooth so we have that the ith components $\gamma'(0)_i$ and $\sigma'(0)_i$ agree, showing that $\gamma'(0) = \sigma'(0)$. ($\Leftarrow$) Conversely, suppose $\gamma'(0) = \sigma'(0)$ then given any smooth function f defined on $\mathbb{R}^n$ near $p = \gamma(0) = \sigma(0)$, we have by the chain rule

$$\begin{aligned}(f \circ \gamma)'(0) &= (\partial_{x_1} f(p), \dots, \partial_{x_n} f(p))(\gamma'(0)_1, \dots, \gamma'(0)_n) \\ &= (\partial_{x_1} f(p), \dots, \partial_{x_n} f(p))(\sigma'(0)_1, \dots, \sigma'(0)_n) \\ &= (f \circ \sigma)'(0)\end{aligned}$$

yielding the claim. ∎

We can specialize to $\mathbb{R}$-vector spaces to get the following.

Corollary 7.2. Let V be a finite dimensional $\mathbb{R}$-vector space. For any $p \in V$, the canonical map $V \to T_pV$ by $w \mapsto [t \mapsto p + tw]$ is a bijection.

Proof. If $V = \mathbb{R}^n$, then this is immediate from the previous lemma, for the construction produces a unique equivalence class of curves centered at p. Otherwise, choosing a basis of V as an $\mathbb{R}$-vector space, we get an isomorphism of $\mathbb{R}$-vector spaces $F : V \to \mathbb{R}^n$ whereby the diagram

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[d, mapsto] & \mathbb{R}^n \arrow[r, "F^{-1}"] \arrow[d, mapsto] & V \arrow[d, mapsto] \\ T_pV \arrow[r, "dF_p"'] & T_{F(p)}V \arrow[r, "dF_p^{-1}"'] & T_pV \end{tikzcd}$$

commutes with the horizontal maps along the top row being isomorhisms, so by functoriality, so is the bottom row. ∎

Remark 7.3. The same can be shown to be a morphism of vector spaces, not merely a map of sets.

We can alternatively define tangent spaces using derivations of smooth functions. Let us recall the definition.

Definition 7.4 (Derivation)**.** Let M be a smooth manifold. A derivation at $p \in M$ is an $\mathbb{R}$-linear map $v : C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule:

$$v(fg) = f(p)v(g) + v(f)g(p)$$

for all and $f, g \in C^\infty(M)$.

For some fixed point p on a smooth manifold M, we will construct a bijection of sets between the tangent space T_pM of Definition 6.14 and the set of derivations at that point. Furthermore we will show that the set of derivations forms a $\mathbb{R}$-vector space and use the derivations perspective going forward. With a non-insignificant amount of work, one can endow the space of equivalence classes of curves with the structure of an $\mathbb{R}$-vector space and show that the bijection described above is in fact an isomorphism of $\mathbb{R}$-vector space. We refer to the text of Warner for a complete account [War83]. We set up the proof of this proposition with the following lemmata, denoting D_p for the set of derivations up until we identify them with the tangent space defined previously.

Lemma 7.5. Let M be a smooth manifold with $p \in M$ and D_pM the set of derivations at p. Then:

(i) If $f \in C^\infty(M)$ is constant, then $v(f) = 0$.
(ii) If $f(p) = g(p) = 0$ for $f, g \in C^\infty(M)$ then $v(fg) = 0$.

Proof of (i). By linearity of v, it suffices to show this for f being the constant function 1. In which case we have $f = f^2$ so

$$\begin{aligned} v(f) &= v(f^2) \\ &= f(p)v(f) + v(f)f(p) \\ &= 2v(f) \end{aligned}$$

which holds if and only if $v(f) = 0$. ■

Proof of (ii). Once again we compute

$$\begin{aligned} v(fg) &= f(p)v(g) + v(f)g(p) \\ &= 0 \cdot v(g) + v(f) \cdot 0 \\ &= 0 \end{aligned}$$

as desired. ■

Lemma 7.6. Let V be a finite dimensional $\mathbb{R}$-vector space. A derivation $v \in D_pV$ is determined uniquely by its action on a dual basis $\xi_1, \ldots, \xi_n$.

Proof. Let $e_1, \ldots, e_n$ be a basis of V idnucing an isomorphism to $\mathbb{R}^n$. It suffices to show that $v(f) = 0$ if $\partial_{x_1} f(p), \ldots, \partial_{x_n} f(p)$. Writing f using Taylor's formula

$$\begin{aligned} f(x) = f(p) &+ \sum_{i=1}^n \partial_{x_i} f(p)(x_i - p_i) \\ &+ \sum_{i,j=1}^n (x_i - p_i)(x_j - p_j) \int_0^1 (1-t)\partial_{x_i x_j} f(p + t(x-p))dt. \end{aligned}$$

The second summand and the $(x_i - p_i)$ factor of the third summand vanish at p, so the function is constant, and hence has trivial derivation by Lemma 7.5 (i). ■

We deduce the subsequent statement as a corollary.

Corollary 7.7. Let V be a finite dimensional $\mathbb{R}$-vector space. The map $V \to D_pV$ by $w \mapsto [f \mapsto \frac{d}{dt}|_{t=0} f(p + tw)]$ is an isomorphism of $\mathbb{R}$-vector spaces.

Proof. We construct an inverse map taking a derivation v to $\sum_{i=1}^n v(\xi_i)e_i$ for a dual basis element $\xi_i : V \to \mathbb{R}$. By Lemma 7.6, this construction is injective so it suffices to show that the map $V \to D_pV$ is injective as well. Suppose to the contrary $w \in V$ is nonzero but maps to the zero derivation, in which case $0 = \frac{d}{dt}f(p+tw)$ for all $f \in C^\infty(V)$ and thus in particular holds for the dual vector $w^\vee \in C^\infty(M)$ where $\frac{d}{dt}w^\vee(p+tw) = 1$, a contradiction. ∎

Remark 7.8. The space of derivations at a point in a vector space is canonically isomorphic to the vector space itself.

We are now ready to show the main result.

Proposition 7.9. Let M be a smooth manifold and $p \in M$. Let D_pM be the set of derivations at p and T_pM the tangent space of M at p. There is a bijection $T_pM \to D_pM$ by $\gamma \mapsto [f \mapsto (f \circ \gamma)'(0)]$.

Proof. Let M be as above and let (ϕ, U) be a chart centered at p. We have a commuting diagram

$$\begin{tikzcd} T_pM \arrow[r] \arrow[d, "K_p"'] & T_{\phi(p)}\phi(U) \arrow[d, "K_{\phi(p)}"] \\ D_pM \arrow[r] & D_{\phi(p)}\phi(U) \end{tikzcd}$$

denoting the map $\gamma \mapsto [f \mapsto (f \circ \gamma)'(0)]$ by K_p. The horizontal maps are equalities since ϕ is a diffeomorhphism and so is $K_{\phi(p)}$ by Corollaries 7.2 and 7.7 implying the right vertical arrow is as well. ∎

We are thus justified in making the following definition (cf. Definition 6.14).

Definition 7.10 (Tangent Space)**.** Let M be a smooth manifold and $p \in M$. The tangent space T_pM of M at x is the set of derivations at p.

Indeed, the derivations $D_pM = T_pM$ form a $\mathbb{R}$-vector space.

Proposition 7.11. The tangent space T_pM is a vector subspace of the dual space $C^\infty(M)^\vee$.

Proof. It suffices to show that for $v_1, v_2 \in T_pM$ that for all $\lambda \in \mathbb{R}$, $\lambda v_1 + v_2 \in T_pM$. We compute for $f, g \in C^\infty(M)$

$$\begin{aligned}(\lambda v_1 + v_2)(fg) &= \lambda v_1(fg) + v_2(fg) \\ &= \lambda\left(v_1(f)g(p) + f(p)v_1(g)\right) + \left(v_2(f)g(p) + f(p)v_2(g)\right) \\ &= f(p)\left(\lambda v_1 + v_2\right)(g) + \left(\lambda v_1 + v_2\right)(f)g(p)\end{aligned}$$

as desired. ∎

We conclude with a discussion of coordinates.

Definition 7.12 (Coordinates on Tangent Space of $\mathbb{R}^n$)**.** Let $p \in \mathbb{R}^n$. The coordinate $(\partial_{x_i})_p \in T_p\mathbb{R}^n$ is represented by the curve $p + te_i$ with e_i the ith standard basis vector of $\mathbb{R}^n$.

Definition 7.13 (Coordinates on Smooth Manifold)**.** Let M be a smooth manifold and $p \in M$. Thhe coordinate $(\partial_{x_i})_p$ is given by $d\phi^{-1}_{\phi(p)}(\partial_{x_i})_p$ for some chart (ϕ, U) containing p.

These behave well under morphisms of smooth manifolds.

Proposition 7.14. Let M, N be smooth m, n manifolds, respectively, and $F : M \to N$ a smooth map. Let $(U, \phi), (V, \psi)$ be charts on M, N respectively with $F(U) \subseteq V$, and $p \in M$. Then the map $T_pM \to T_{F(p)}N$ is given by the Jacobian matrix

$$\begin{bmatrix} \partial_{x_1} F_1(p) & \dots & \partial_{x_m} F_1(p) \\ \vdots & \ddots & \vdots \\ \partial_{x_1} F_n(p) & \dots & \partial_{x_m} F_n(p) \end{bmatrix}.$$

Proof. Up to composition with ϕ, ψ we get a map from the basis $(\partial_{x_1})_p, \dots, (\partial_{x_m})_p$ of $T_pM = T_{\phi(p)}\mathbb{R}^m$ to the basis $(\partial_{y_1})_{F(p)}, \dots, (\partial_{y_n})_{F(p)}$ of $T_{F(p)}N = T_{\psi(F(p))}\mathbb{R}^n$. But using the chain rule we have

$$dF_p((\partial_{x_i})_p) = \sum_{j=1}^n \partial_{x_i} F_j(p)(\partial_{y_j})_{F(p)}$$

giving the Jacobian matrix. ■

8. Lecture 8 – 5th November 2024

We continue with the discussion of tangent spaces.

Definition 8.1 (Tangent Bundle)**.** Let M be a smooth manifold. The tangent bundle TM of M is given by $\coprod_{p\in M} T_pM$.

Remark 8.2. Elements of the tangent bundle are denoted by pairs $(p, v) \in M \times T_pM$ and there is a natural forgetful map $TM \to M$ by $(p, v) \mapsto p$.

Proposition 8.3. Let M be a smooth manifold. The tangent bundle can be endowed with a structure of a smooth manifold such that the map $TM \to M$ is smooth.

Remark 8.4. We will soon see that the data of a map of smooth manifolds with fiberes fector spaces forms a vector bundle. In fact, this map is sufficiently functorial and is a functor from the category of smooth manifolds to the category of vector bundles.

We now discuss submersions, immersions, and embeddings, which are special classes of smooth maps.

Definition 8.5 (Rank of Smooth Map)**.** Let $F : M \to N$ be a morphism of smooth manifolds. The rank of F at p is the rank of the linear map $dF_p : T_pM \to T_{F(p)}N$.

Smooth maps of full rank are particularly important.

Definition 8.6 (Submersion)**.** Let M, N be smooth m, n manifolds, respectively, and $F : M \to N$ a smooth map. F is a submersion if dF_p is surjective for all $p \in M$.

Definition 8.7 (Immersion)**.** Let M, N be smooth m, n manifolds, respectively, and $F : M \to N$ a smooth map. F is an immersion if dF_p is injective for all $p \in M$.

Remark 8.8. A necessary but insufficient condition for a submersion is that $m \geq n$, and a necessary but insufficient condition for an immersion is that $m \leq n$.

To define immersions, we require the following lemma.

Lemma 8.9. Let $m, n \in \mathbb{N}$. The set of matrices of rank $\min\{m, n\}$ is open in $\mathrm{Mat}_{m\times n}(\mathbb{R})$.

Proof. Fix some full rank matrix $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$. Up to transposition, it suffices to consider the case $m \leq n$. Now note that if A is of full rank, there is an invertible submatrix A' of A obtained by deleting $m - n$ columns from A. Consider the map $\mathrm{Mat}_{m\times n}(\mathbb{R}) \to \mathrm{Mat}_{m\times m}(\mathbb{R}) \to \mathbb{R}$ by deleting the columns and taking the determinant, respectively. Observe that this map is continuous and that the image of A is nonzero. As such, the preimage of any open neighborhood of the image of A not intersecting zero gives an open neighborhood of A in $\mathrm{Mat}_{m\times n}(\mathbb{R})$ of matrices of full rank, as desired. ∎

We can show that immersions and submersions are local conditions.

Proposition 8.10. Let M, N be smooth m, n manifolds, respectively, and $F : M \to N$ a smooth map. Then:

(i) If dF_p is injective, there exists an open neighborhood of p on which $dF_{(-)}$ is injective.
(ii) If dF_p is surjective, there exists an open neighborhood of p on which $dF_{(-)}$ is surjective.

Proof. On passage to charts, we can reduce to the case of $M \subseteq \mathbb{R}^m, N \subseteq \mathbb{R}^n$ where we note that $dF_{(-)} : M \to \mathrm{Mat}_{m\times n}(\mathbb{R})$ has image in the full rank matrices – of full column rank in the case of injectivity and of full row rank in the case of surjectivity – both of which are open conditions by Lemma 8.9 yielding the claim. ■

We can now define local diffeomorphisms.

Definition 8.11 (Local Diffeomorphism)**.** Let M, N be smooth m, n manifolds, respectively, and $F : M \to N$ a smooth map. F is a local diffeomorphism if it is both an immersion and a submersion.

The rank theorem will give a necessary and sufficient conditions for a smooth map to be a local diffeomorphism.

Example 8.12. Let $S^1 \subseteq \mathbb{C}$. The map $S^1 \to S^1$ by $z \mapsto z^2$ is a local diffeomorphism but not a (global) diffeomorphism.

We define embeddings as follows.

Definition 8.13. Let M, N be smooth m, n manifolds, respectively, and $F : M \to N$ a smooth map. F is an embedding if it is an immersion and a homeomorphism onto its image endowed with the subspace topology.

Example 8.14. The inclusion $S^1 \to \mathbb{C}$ is an embedding.

Example 8.15. More generally, the inclusion $S^n \to \mathbb{R}^{n+1}$ is an embedding.

Example 8.16. The map $\mathbb{R} \to \mathbb{T}^2$ the 2-torus by $t \mapsto (t, \alpha t)$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is an immersion since the map on tangent spaces is injective, but not an embedding as it is not a homeomorphism onto its image with the subspace topology.

9. Lecture 9 – 8th November 2024

Proposition 8.10 shows that immersions and submersions are local conditions. The rank theorem implies that there exist coordinates such that the induced map on tangent spaces is the projection to the first n coordinates or the inclusion of the first m coordinates in the case of submersions and immersions, respectively.

Theorem 9.1 (Rank)**.** Let M, N be smooth m, n manifolds, respectively, and $F : M \to N$ a smooth map of constant rank r. Then, there exist charts (U, ϕ) around p and (V, ψ) around $F(p)$ with $F(U) \subseteq V$ such that $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is of the form

$$(x_1, \dots, x_r, x_{r+1}, \dots, x_m) \mapsto (x_1, \dots, x_r, 0, \dots, 0)$$

as a map $\mathbb{R}^m \to \mathbb{R}^n$.

Proof. Without loss of generality, we can reduce to the case $M = U \subseteq \mathbb{R}^m, N = V \subseteq \mathbb{R}^n$ and by assumption we have that a rank $r = \min\{m, n\}$ submatrix of the Jacobian matrix of F is invertible. Up to rearranging coordinates, we can assume the matrix $(\partial_{x_i} F_j(p))_{1 \le i,j \le r}$ is invertible.

Now labeling coordinates on the source $(x_1, \dots, x_r, y_1, \dots, y_{m-r})$ and on the target $(v_1, \dots, v_r, w_1, \dots, w_{n-r})$ we can decompose $F = F(x, y)$ as $Q(x, y)$ and $R(x, y)$ where $Q : \mathbb{R}^m \to \mathbb{R}^r$ and $R : \mathbb{R}^m \to \mathbb{R}^{n-r}$ onto the v and w coordinates, respectively. The above shows that $(\partial_{x_i} Q_j(p))_{1 \le i,j \le r}$ is invertible.

Note that for $\phi : U \to \mathbb{R}^m$ we have $d\phi_{(0,0)}$ given by the block matrix

$$\begin{bmatrix} \partial_{x_i} Q_j & \partial_{y_i} Q_j \\ 0 & \mathrm{id}_{(n-r)\times(n-r)} \end{bmatrix}$$

and thus invertible. The inverse function then implies that there are connected open neighborhoods $U_0 \subseteq U$ and $\widetilde{U_0}$ of $(0, 0) \in \mathbb{R}^m$ on which $\phi|_{U_0} : U_0 \to \widetilde{U_0}$ is a diffeomorphism. Up to further restriction, we can take $\widetilde{U_0}$ to be an open cube $(-\varepsilon, \varepsilon)^m$. Decompose the inverse function similarly with $\phi^{-1}(x, y) = (A(x, y), B(x, y))$ where $A : \widetilde{U_0} \to \mathbb{R}^r, B : \widetilde{U_0} \to \mathbb{R}^{m-r}$ onto the x and y coordinates, respectively. Computing the composition, we have

$$\begin{aligned}(\phi \circ \phi^{-1})(x, y) &= \phi(A(x, y), B(x, y)) \\ &= (Q(A(x, y), B(x, y)), B(x, y))\end{aligned}$$

showing $B(x, y) = y$ and thus $\phi^{-1}(x, y) = (A(x, y), y), Q(A(x, y), y) = x$.

We now compute $F \circ \phi^{-1}$ given by

$$(F \circ \phi^{-1})(x, y) = (Q(A(x, y), y), R(A(x, y), y))$$

where we write $\widetilde{R} : U_0 \to \mathbb{R}^{n-r}$ the map given by $R(A(x, y), y)$. We can compute the Jacobian $d(F \circ \phi^{-1})$ at x, y to be given by the matrix

$$\begin{bmatrix} \mathrm{id}_{r\times r} & 0 \\ \partial_{x_i} \widetilde{R}_j & \partial_{y_i} \widetilde{R}_j \end{bmatrix}.$$

Note that $F \circ \phi^{-1}$ is a diffeomorphism and hence of rank r since composition with a diffeomorphism preserves rank, so the matrix above is of rank r, implying the submatrix $\partial_{y_i}\widetilde{R_j}$ is the zero $(n-r)\times(n-r)$ matrix. In particular, $\widetilde{R}(x,y)$ does not depend on y. Writing $\widetilde{R}(x,y)$ as $S(x)$, we have $(F\circ\phi^{-1})(x,y)=(x,S(x))$.

We can now consider a subset $V_0\subseteq V$ where

$$V_0=\{(v,w)\in V:(F\circ\phi^{-1})^{-1}(v,w)\in\widetilde{U_0}\}=\{(v,w)\in V:(v,0)\in\widetilde{U_0}\}$$

with the latter equality from $(F\circ\phi^{-1})(x,y)=(x,S(x))$. We thus have $F(U_0)\subseteq V_0$. Now taking $\psi:V_0\to\mathbb{R}^n$ to be $(v,w)\mapsto(v,w-S(v))$ which is a diffeomorphism as it is coordinatewise smooth and with inverse $(v,w)\mapsto(v,w+S(v))$ so (V_0,ψ) is a smooth chart and

$$(\psi\circ F\circ\phi^{-1})(x,y)=\psi(x,S(x))=(x,S(x)-S(x))=(x,0)$$

as desired. ■

We now turn to a discussion of submanifolds.

Definition 9.2 (Submanifold)**.** Let M be a topological manifold. A subset $S\subseteq M$ is a topological submanifold if S is a topological manifold in the subspace topology.

Example 9.3. S^n is a submanifold of $\mathbb{R}^{n+1}$.

Example 9.4. The union of the coordinate axes in $\mathbb{R}^2$ is not a submanifold.

This specializes to smooth manifolds in the following way.

Definition 9.5 (Smooth Submanifold)**.** Let M be a smooth manifold. A topological submanifold $S\subseteq M$ is a smooth submanifold if S admits a smooth structure such that the inclusion $S\hookrightarrow M$ is a smooth map.

Remark 9.6. Any open submanifold of a smooth manifold M is a smooth submanifold.

The subsequent lemma produces a number of examples.

Lemma 9.7. Let M,N be smooth m,n manifolds, respectively, and $F:M\to N$ a smooth map that is an embedding. Then $F(M)$ admits a unique smooth structure making it a smooth submanifold.

Proof. Since f is an embedding, it is a homeomorphism onto its image in the subspace topology. In particular, $F(M)$ is a topological submanifold of N. We can define a smooth atlas on $F(M)$ by taking $(F(U),\phi\circ F^{-1})$ over all charts (U,ϕ) of M. Since F is a diffeomorphism on its image, any two charts $(U,\phi),(U',\phi')$ on M are smoothly compatible by considering $\phi\circ F\circ F^{-1}\circ\phi'^{-1}=\phi\circ\phi'^{-1}$ which is smooth as ϕ,ϕ',ϕ'^{-1} are smooth.

To see uniqueness, if $\mathcal{A}'$ is another smooth atlas on $F(M)$ for which F is a diffeomorphism, its preimage must agree with the smooth atlas on M showing that they are equivalent atlases. ■

We now restrict attention to embedded submanifolds. Recall the definiton of properness from point set topology.

Definition 9.8 (Proper)**.** Let $f : X \to Y$ be a continuous map between topological spaces. f is proper if for all $K \subseteq Y$ compact, $f^{-1}(K) \subseteq X$ is compact.

This allows us to define embedded submanifolds.

Definition 9.9 (Embedded Submanifold)**.** Let M be a topological manifold and S a submanifold of M. S is an embedded submanifold if the inclusion $S \hookrightarrow M$ is proper.

Example 9.10. The inclusion $S^n \setminus \{N\} \hookrightarrow \mathbb{R}^{n+1}$ is not an embedded submanifold.

10. Lecture 10 – 12th November 2024

We state and prove the slice theorem.

Theorem 10.1 (Slice)**.** Let M be a smooth n-manifold and $S \subseteq M$ a subset such that for all $p \in S$ there exists a chart (V, ψ) of M satisfying

$$\psi(U \cap S) = \left\{ (x_0, \dots, x_k, x_{k+1}, \dots, x_m) \in \mathbb{R}^m : {}^{x_{k+1}=c_{k+1}, \dots, x_m = c_m}_{c_j \in \mathbb{R} \text{ constant}} \right\}.$$

Then S admits a smooth structure making it a smooth submanifold of M.

Proof. S is second countable and Hausdorff under the subspace topology. S is also locally Euclidean under the restriction of ψ to the first k coordinates. S is therefore a topological submanifold. Let $\pi : \mathbb{R}^m \to \mathbb{R}^k$ be the projection onto the first k coordinates and (U, ψ) a chart of M. Note that $\psi(V) = (\pi \circ \phi)(U \cap S)$ and in particular $\phi(U \cap S)$ is the interesesction of the $\phi(U)$ with some slice of $\mathbb{R}^n$ which is a diffeomorphism since projection maps are diffeomorphisms and has an invese given by $\phi|_{U \cap S} \circ \psi^{-1}$. This shows the inclusion map is a topological embedding.

To get a smooth structure on S, we verify that the charts are smoothly compatible. But ϕ is smooth in the first k coordinates because ϕ is and constant in the remaining hence infinitely diferentiable. This shows that the transition functions as the composition of two such maps is smooth. ■

Remark 10.2. Recall the discussion of level sets in Example 4.4. Using the slice slice theorem, $\varphi^{-1}(0) \subseteq \mathbb{R}^{n+1}$ is a smooth submanifold for $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}$ with $d\Phi$ nonzero on $\varphi^{-1}(0)$.

Lemma 10.3. Let $F : M \to N$ be a smooth map that factors through the inclusion $S \hookrightarrow N$ as a continuous map. Then $F : M \to S$ is smooth.

Proof. By Theorem 10.1 on N, there is a commutative diagram of the form

$$\begin{tikzcd}
S \arrow[d, hook] & U \arrow[l, hook'] \arrow[r, "\sim"] \arrow[d] & \phi(U) \subseteq \mathbb{R}^k \arrow[d] \\
N & V \arrow[l, hook'] \arrow[r, "\sim"'] & \psi(V) \subseteq \mathbb{R}^n
\end{tikzcd}$$

So taking charts on S to be $U \subseteq S$ such that $F^{-1}(U) \subseteq M$ is open and charts the composition of $\mu \circ F^{-1} : U \to \mathbb{R}^m$ for (W, μ) a chart of M and W containing $F^{-1}(U)$, F being a smooth map to M implies that the charts are smoothly compatible producing a smooth map $F : M \to S$. ■

This lemma shows that the smooth structure produced in Theorem 10.1 is unique.

Proposition 10.4. Let M be a smooth n-manifold and $S \subseteq M$ smooth k-submanifold such that for all $p \in S$ there exists a chart (V, ψ) of M satisfying

$$\psi(U \cap S) = \left\{ (x_0, \dots, x_k, x_{k+1}, \dots, x_m) \in \mathbb{R}^m : {}^{x_{k+1}=c_{k+1}, \dots, x_m = c_m}_{c_j \in \mathbb{R} \text{ constant}} \right\}.$$

Then the smooth structure on S is unique.

Proof. For S' the same space with a possibly different smooth structure, we have maps

$$S' \xrightarrow{\mathrm{id}_S} S \hookrightarrow N$$

$$S \xrightarrow{\mathrm{id}_S} S' \hookrightarrow N$$

and the smooth structure on S is equivalent to that on S' by Lemma 10.3. ■

We discuss a weak variant of Whitney's embedding theorem. We will return to a more general variant of this theorem with the language of Sard's theorem in hand.

Theorem 10.5 (Whitney Embedding)**.** Let M be a compact smooth m-manifold. M admits an embedding into $\mathbb{R}^N$ for $N >> 0$.

Proof. Let $B_1, \ldots, B_k$ be a finite open cover of M and without loss of generality consider charts (U_i, ϕ_i) for $1 \leq i \leq k$ such that $\overline{U_i} \subseteq B_i$ and $\phi(U_i) = B_1(0) \subseteq \mathbb{R}^m$. Let $\rho_i : M \to \mathbb{R}$ be cutoff functions for $\overline{U_i} \subseteq B_i$ in the sense of Definition 5.6 and which exist by Lemma 5.7. Define a map $F : M \to \mathbb{R}^{mk+k}$ by

$$p \mapsto (\rho_1(p)\phi_1(p), \ldots, \rho_k(p)\phi_k(p), \rho_1(p), \ldots, \rho_k(p))$$

here noting that ϕ are functions to $\mathbb{R}^m$ and hence contribute an m-tuple to the above.

This is injective as if $F(p) = F(q)$ then $\rho_i(p) = \rho_i(q)$ for all $1 \leq i \leq k$ in which case $\phi_i(p) = \phi_i(q)$ for all $1 \leq i \leq k$ showing $p = q$ since ϕ_i is a homeomorphism.

To show it is an immersion, for any $v \in T_p(M)$ we know $dF_p(v) = 0$ implies $v(\rho_i) = 0$ for all i and thus $\rho_i(p)(d\phi_i)_p(v) = 0$ for all i. So $(d\phi_i)_p(v) = 0$ since and ϕ_i is a diffeomorphism $v = 0$ too and thus giving injectivity of the tangent space. ■

11. Lecture 11 – 15th November 2024

We begin our discussion of transversality.

Definition 11.1 (Transverse Submanifolds at a Point)**.** Let M be a smooth manifold and S, S' submanifolds of M. M and M' are transverse at $p \in S \cap S'$ if the subspaces T_pS and T_pS' span T_pM.

Definition 11.2 (Transverse Submanifolds)**.** Let M be a smooth manifold and S, S' submanifolds of M. S and S' are transverse submanifolds S – $S \pitchfork S'$ – if they are transverse at each point $p \in S \cap S'$.

Example 11.3. The union of coordinate axes in $\mathbb{R}^2$ is transverse.

Example 11.4. The generic intersection of a circle and a line is transverse.

Example 11.5. A line tangent to a circle is not transverse, as the tangent spaces at the intersection point is just the line, while the tangent space of $\mathbb{R}^2$ at that point is $\mathbb{R}^2$.

Transverse intersections behave especially nicely, insofar as their intersections are smooth submanifolds. On the other hand, non-transverse intersections might not even be a topological manifold.

Example 11.6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x^2 - y^2$ and $g(x, y) = 0$. Let S be the graph of f in $\mathbb{R}^3$ given by $\{(x, y, z) \in \mathbb{R}^3 : z = f(x, y)\}$ and S' the graph of g in $\mathbb{R}^3$ given by $\{(x, y, z) \in \mathbb{R}^3 : z = g(x, y) = 0\}$. The intersection $S \cap S'$ is given by $\{(x, y, z) \in \mathbb{R}^3 : x^2 - y^2 = z = 0\}$ which is two lines intersecting transversely in $\mathbb{R}^3$ and not a topological manifold.

We now show the lemma.

Lemma 11.7. Let M be a smooth manifold and S, S' submanifolds of M. If S, S' are transverse, then $S \cap S'$ is a smooth submanifold.

Proof. By composing with a chart centered around zero in $\mathbb{R}^m$ and the slice Theorem 10.1, it suffices to show that the intersection is a smooth submanifold ina neighborhood of 0. Using the rank theorem 9.1, after possibly shrinking U, that $S = f^{-1}(0)$ for $f : U \to \mathbb{R}^{m-\dim(S)}$ and $S' = g^{-1}(0)$ for $g : U \to \mathbb{R}^{m-\dim(S')}$ with f, g of full rank.

Consider $H : U \to \mathbb{R}^{m-\dim(S)} \oplus \mathbb{R}^{m-\dim(S')}$ by $p \mapsto (f(p), g(p))$. It suffices to show that H is surjective, where injectivity follows from S, S' being submanifolds. We first observe that $H^{-1}(0) = f^{-1}(0) \cap g^{-1}(0) = S \cap S'$. To see the surjectivity of H at the origin, note that $T_0S + T_0S' \to T_0U$ is a linear isomorphism since S, S' are transverse and that there is a map $dH_0 : T_0U \to \mathbb{R}^{m-\dim(S)} \oplus \mathbb{R}^{m-\dim(S')}$ fitting into

$$\begin{tikzcd}
T_0S + T_0S' \arrow[r] \arrow[d, "\wr"'] & T_0S/(T_0S \cap T_0S') \oplus T_0S'/(T_0S \cap T_0S') \arrow[d, "{(df_0, dg_0)}"] \\
T_0U \arrow[r] & \mathbb{R}^{m-\dim(S)} \oplus \mathbb{R}^{m-\dim(S')}
\end{tikzcd}$$

The top horizontal map is well-defined as if $v + w = v' + w'$ then $v - v' = w - w'$ is in $T_0S \cap T_0S'$. It remains to show the right vertical map is surjective, we show in particular it is an isomorphism. Observe the map is injective as as the kernels of df_0, dg_0 in T_0S, T_0S' is exactly the intersection. We then consider the short exact sequence

$$0 \longrightarrow T_0S \cap T_0S' \longrightarrow T_0S \oplus T_0S' \longrightarrow T_0U \longrightarrow 0$$

where the maps are $v \mapsto (v, v)$ and $(a, b) \mapsto a - b$. Computing the dimensions, $\dim(T_0S \cap T_0S') + m = \dim(S) + \dim(S')$ and thus $\dim(T_0S/(T_0S \cap T_0S')) = \dim(S') - (\dim(S) + \dim(S') - m) = m - \dim(S)$ and similarly $\dim(T_0S'/(T_0S \cap T_0S')) = \dim(S) - (\dim(S) + \dim(S') - m) = m - \dim(S')$ showing that the final map is surjective. The claim follows. ■

We can generalize transversality of manifolds to transversality of maps. This is motivated by the goal of producing fibered products in the category of smooth manifolds.

Definition 11.8 (Fibered Product)**.** Let $f : X \to Z, g : Y \to Z$ be continuous maps of topological spaces. The fibered product $X \times_Z Y$ is given by

$$\{(x, y) \in X \times Y : f(x) = g(y) \in \mathbb{Z}\} \subseteq X \times Y$$

with the subspace topology on the product.

Remark 11.9. Definition 11.8 implies the universal property of fibered products. For all topological spaces W making the solid diagram commute

$$\begin{tikzcd}
W \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

there is a unique map $W \to X \times_Z Y$ making the entire diagram commute.

Note that neither the categories Mfld of Definition 2.10 nor SmMfld of Definition 4.15 admit fibered products.

Example 11.10. In the setup of Example 11.6, the fibered product $S \times_{\mathbb{R}^3} S'$ is the union of transversely intersecting lines in $\mathbb{R}^3$ which is not a topological manifold even though S, S' are topological and even smooth manifolds (cf. Example 2.15).

Extending Definitions 11.1 and 11.2, we can define transverse maps as follows.

Definition 11.11 (Transverse Maps at a Point)**.** Let M_1, M_2, N be smooth manifolds and $F : M_1 \to N, G : M_2 \to N$ be smooth maps. F and G are transverse at $F(p_1) = F(p_2) \in N$ if $\operatorname{im}(dF_{p_1}) + \operatorname{im}(dG_{p_2})$ spans T_qN.

Definition 11.12 (Transverse Maps)**.** Let M_1, M_2, N be smooth manifolds and $F : M_1 \to N, G : M_2 \to N$ be smooth maps. F and G are transverse – $F \pitchfork G$ – if it is transverse at all $q \in Z$.

Remark 11.13. Transversality of maps generalizes transversality of manifolds by taking $F : S \to M, G : S' \to M$ for smooth submanifolds $S, S' \subseteq M$.

12. Lecture 12 – 19th November 2024

We prove that fibered products exist over a diagram with legs given by transverse maps. We first recall the following topological statement.

Proposition 12.1. Let X, Y, Z be Hausdorff spaces and $f : X \to Z, g : Y \to Z$ continuous maps. Then $X \times_Z Y$ is a closed subset of $X \times Y$ and the inclusion map $X \times_Z Y \hookrightarrow X \times Y$ is a proper map.

We now begin the proof in earnest.

Theorem 12.2. Let M_1, M_2, N be smooth manifold and $f_1 : M_1 \to N, f_2 : M_2 \to N$ smooth maps. If f_1 is transverse to f_2 then $M_1 \times_N M_2$ admits a smooth embedding into $M_1 \times M_2$. In particular, $M_1 \times_N M_2$ is a smooth manifold.

Proof. Note that smooth manifolds are Hausdorff so Proposition 12.1 implies the inclusion into the product is proper, and it thus suffices to verify that the inclusion is an immersion as it is injective being defined as subset of the product in Definition 11.8. Let Δ be the set

$$\Delta = \{(x, y, z_1, z_2) : z_1 = z_2\} \subseteq M_1 \times M_2 \times N \times N$$

and

$$W = \{(x, y, z_1, z_2) : f(x) = z_1, g(y) = z_2\} \subseteq M_1 \times M_2 \times N \times N.$$

By definition, $W \cap \Delta$ can be identificed with the fiber product $M_1 \times_N M_2$. In particular, the constructions induce the commutative diagram

$$\begin{tikzcd}
W \cap \Delta \arrow[rr] \arrow[d, "\wr"'] \arrow[dr, "i"] & & M_1 \times M_2 \times N \times N \arrow[dl, "\pi"] \\
M_1 \times_N M_2 \arrow[r, hook] & M_1 \times M_2 &
\end{tikzcd}$$

where we wish to show that the inclusion of the top row is a smooth embedding which would imply i and thus the inclusion of the fibered product is smooth by Lemma 11.7. For this, we show that W and Δ are transverse. Let $(x, y, z_1, z_2) \in M_1 \times M_2 \times N \times N$. We have $T_pW = \{(v, w, df_x(v), df_y(w))\} \subseteq T_xM_1 \times T_yM_2 \times T_{z_1}N \times T_{z_2}N$ and $T_p\Delta = \{(v', w', u, u)\} \subseteq T_xM_1 \times T_yM_2 \times T_{z_1}N \times T_{z_2}N$. For $T_{(a,b,c,d)}(M_1 \times M_2 \times N \times N)$ we exhibit a solution for the system

$$\begin{aligned}
a &= v + v' \\
b &= w + w' \\
c &= u + df_x(v) \\
d &= u + dg_y(w)
\end{aligned}$$

for some $(v, w, df_x(v), dg_y(v)) \in T_pW$ and $(v', w', u, u) \in T_p\Delta$. But the solution to $c - d = df_x(v) - dg_y(w)$ has a solution by the transversality hypothesis, and thus so does $c + d = 2u + df_x(v) + dg_y(w)$ by taking a suitable tangent vector in $T_{z_1}N$ and thus so too do the equations $a - v = v', b - w = w'$ showing the equation can indeed be solved. The map is injective as the vanishing of the first two coordinates imply the vanishing of all coordinates in the image.

The second statement follows from the first by Lemma 9.7. ∎

We now discuss some measure theory with the goal of building up towards Sard's theorem. Recall that a rectangle is a set of the form

$$\prod_{i=1}^{n}(a_i - \varepsilon_i, a_i + \varepsilon_i)$$

for $(a_1, \dots, a_n) \in \mathbb{R}^n, (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{R}^n_{>0}$ which has volume $\prod_{i=1}^{\infty} 2\varepsilon_i$

Definition 12.3 (Set of Measure Zero)**.** A subset $S \subseteq \mathbb{R}^n$ is of measure zero if for any $\varepsilon > 0$ there exists a countable family of rectangles $\{C_i\}_{i=1}^{\infty}$ such that $S \subseteq \bigcup_{i=1}^{\infty} C_i$ and $\sum_{i=1}^{\infty} \mathrm{vol}(C_i) < \varepsilon$.

Some elementary properties of measurable sets in $\mathbb{R}^n$ are as follows.

Lemma 12.4.
(i) Let $A, B \subseteq \mathbb{R}^n$. If $A \subseteq B \subseteq \mathbb{R}^n$ and B has measure zero then A has measure zero as well.
(ii) If $A \subseteq \mathbb{R}^n$ is countable union of measure zero subsets, then A has measure zero as well.

Proof of (i). A can be covered by a subcollection of cubes, the volume of which can be made arbitrarily small since the volume can be made arbitrarily small for B. ∎

Proof of (ii). The measure of a countable union of measurable sets is at most the sum of the measures of each subset, each of which can be made arbitrarily small. ∎

Being measure zero on "slices" in fact implies being measure zero.

Lemma 12.5. Let $A \subseteq \mathbb{R}^n$ be compact. If $A \cap (\{c\} \times \mathbb{R}^{n-1}) \subseteq \mathbb{R}^n$ has measure zero in $\mathbb{R}^{n-1}$ for each $c \in \mathbb{R}$ then A has measure zero in $\mathbb{R}^n$.

Proof. Let $[a, b] \subseteq \mathbb{R}$ such that $A \subseteq [a, b] \times \mathbb{R}^{n-1}$ and for each $c \in \mathbb{R}$, let $A_c \subseteq \mathbb{R}^{n-1}$ be the compact subset $\{x \in \mathbb{R}^{n-1} : (c, x) \in A\}$. For some fixed $\delta > 0$, there is a cover of A_c by finitely many rectangles of dimension $n - 1$ with total volume less than δ. Let $U_c \subseteq \mathbb{R}^{n-1}$ be the union of the cubes for fixed c. Since A is compact, there is an open interval J_c of $\mathbb{R}$ containing c such that $A \cap (J_c \times \mathbb{R}^{n-1})$ is contained in $J_c \times U_c$. If not, there would be a sequence of points (c_i, x_i) in A and $c_i \to c$ but $x_i \notin U_c$ that on passage to a convergent subsequence produecs a sequence converging to $A_c \setminus U_c$ contradicting $A_c \subseteq U_c$.

Now the intervals J_c form an open cover of $[a, b]$ and by compactness of this interval it suffices to consider a cover of the interval by $J_{c_1}, \dots, J_{c_m}$ which on shrinking the intersections can be taken to have total length at most $2|b - a|$. Thus A is contained in $(J_{c_1} \times U_{c_1}) \cup \dots (J_{c_m} \times U_{c_m})$ which is of volume at most $2\delta|b - a|$ which can be made arbitrarily small, yielding the claim. ∎

Corollary 12.6. Let $A \subseteq \mathbb{R}^n$ be a countable union of compact subsets and $f : A \to \mathbb{R}$ a continuous function. Then the graph $\{(x, y) \in A \times \mathbb{R} : f(x) = y\}$ is a measure zero subset of $\mathbb{R}^{n+1}$.

Proof. We argue by induction on the cardinality of the union. If A is compact, then this follows immediately from Lemma 12.5. Suppose it holds for m. Then for each of $K_1, \dots, K_m$, the graph of K_i has measure zero and the graph of A is the union of these graphs which is zero by Lemma 12.4 (ii). ■

Images of measure zero sets under smooth maps are measure zero.

Lemma 12.7. Let $A \subseteq \mathbb{R}^n$ be a subset and $f : A \to \mathbb{R}^n$ be a smooth map. If A is measure zero then $f(A)$ is measure zero.

Proof. Without loss of generality, we can consider a collection of balls $\{U_p\}_{p \in A}$ where the extension $\widetilde{f}_p$ of f at p restricts to f on $U_p \cap A$. In particular, $A \subseteq \bigcup_{p \in A} U_p$ and by Proposition 1.17 we can consider a countable subcover $\{U_i\}_{i=1}^{\infty}$ of the union of the U_p's. It suffices to prove that $F(A \cap U_i)$ is of measure zero.

Note that $\overline{U}_i$ is compact so there is a constant Q such that $|f(x) - f(y)| < Q|x - y|$ for all $x, y \in \overline{U}_i$. Now fixing $\delta > 0$, we can cover $A \cup \overline{U_i}$ by a countable union of rectangles C_j with total volume at most δ but by the inequality above, the diameter of $F(\overline{U}_i \cap C_j)$ is at most a λ-multiple of the diameter of C_j where λ can be made smaller than some large natural multiple of Q. So $f(A \cap \overline{U}_i)$ is contained in a countable union of balls of diameter at most λ-times of the diameter of C_j which is bounded above by a constant multiple of δ which can then be made arbitrarily small. ■

We can generalize our discussion on Euclidean space to manifolds.

Definition 12.8 (Measure Zero Subset)**.** Let M be a smooth manifold and $A \subseteq M$ a subset. A is of measure zero if for all smooth charts (U_α, ϕ_α) containing A, $\phi_\alpha(A \cap U_\alpha)$ has measure zero in $\mathbb{R}^n$.

The converse holds.

Proposition 12.9. Let M be a smooth m-manifold and $A \subseteq M$ a subset. If there is a collection of charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ where $A \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ and $\phi_\alpha(A \cap U_\alpha) \subseteq \mathbb{R}^m$ of measure zero for each α then $A \subseteq M$ is of measure zero.

Proof. It suffices to show that for any smooth chart (ψ, V), $\psi(A \cap V)$ is of measure zero. We can write $\psi(A \cap V)$ as $\bigcup_{\alpha \in \mathcal{A}} \psi(A \cap U_\alpha \cap V)$ but we have $\psi(A \cap U_\alpha \cap V)$ is the image of $\phi_\alpha(A \cap U_\alpha \cap V)$ under the smooth map $\mathbb{R}^m \to \mathbb{R}^m$ given by $\psi \circ \phi^{-1}$ so each $\psi(A \cap U_\alpha \cap V)$ is measure zero by Lemma 12.7 and the union is measure zero by Lemma 12.4. ■

13. Lecture 13 – 22nd November 2024

We contiue our discussion of measure theory on manifolds and introduce Sard's theorem and consequences.

Lemma 13.1. Let M, N be smooth manifolds and $F : M \to N$ a smooth map. If $A \subseteq M$ is of measure zero then $F(A) \subseteq M$ has measure zero.

Proof. Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ be a countable atlas for M. We want to show that any chart (V, ψ) on N is such that $\psi(V \cap F(A))$ is measure zero which will imply the result by Proposition 12.9.

Without loss of generality, let $F(A) \subseteq V$. Note $\psi(F(A))$ is a countable union of the sets $\psi((F \circ \phi_\alpha^{-1} \circ \phi_\alpha)(A \cap U_\alpha))$. Note that each of these is measure zero by Lemma 12.7 since each of the maps are smooth and their union is measure zero by Lemma 12.4. ■

We now state Sard's theorem after introducing the requisite language.

Definition 13.2 (Critical Point)**.** Let M, N be smooth manifolds and $F : M \to N$ a smooth map. $p \in M$ is a critical point of F if the differential $dF_p : T_pM \to T_{F(p)}N$ is not surjective.

Definition 13.3 (Critical Value)**.** Let M, N be smooth manifolds and $F : M \to N$ a smooth map. $q \in N$ is a critical value of F if $F^{-1}(q)$ contains a critical point of F.

Remark 13.4. This recovers the notion of critical points from multivariate calculus.

Sard's theorem is stated as follows.

Theorem 13.5 (Sard)**.** Let M, N be smooth manifolds and $F : M \to N$ a smooth map. The set of critical values of F is a measure zero subset of N.

Deferring the proof, let us consider some examples.

Example 13.6. Let M be a smooth manifold and $F : M \to \mathbb{R}$ a smooth map by $p \mapsto 0$ for all $p \in M$. Then the set of critical points of M is full measure – consisting of all of M – but the set of critical values is measure zero, being $\{0\} \subseteq \mathbb{R}^n$.

In fact, we can show that images of smooth maps between smooth manifolds have measure zero when the dimension of the source is strictly less than the dimension of the target.

Corollary 13.7. Let M, N be smooth m, n-manifolds, respectively, and $F : M \to N$ a smooth map. If $m < n$ then $F(M) \subseteq N$ is of measure zero.

Proof. Since $m < n$, the differential $dF_{(-)}$ is never surjective so every point of M is a critical point and every point of the image is a critical value, which is of measure zero by Theorem 13.5. ■

In fact, Sard's theorem allows us to show the strong Whitney embedding theorem.

Lemma 13.8. Let M be an M-submanifold of $\mathbb{R}^n$, $\mathbb{R}^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\}$, $v \in \mathbb{R}^n \setminus \mathbb{R}^{n-1}$, and $\pi_v : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection with kernel $v \cdot \mathbb{R}$. If $n > 2m+1$ then there is a dense set of vectors $v \in \mathbb{R}^n \setminus \mathbb{R}^{n-1}$ such that $\pi_v|_M$ is an injective immersion into $\mathbb{R}^{n-1}$.

Proof. Note that $\pi_v|_M$ is injective if and only if for all $p \in M$, $T_pM \cap \ker(d\pi_v) = p$ and is an immersion if and only if for all $p \in M$, $T_pM \cap \ker(d\pi_v)$ is trivial, which is equivalent to T_pM not containing v.

Consider the diagonal $\Delta \subseteq M \times M$ and the zero section of the tangent bundle 0_M. Consider maps

$$\begin{aligned} \alpha : (M \times M) \setminus \Delta &\to \mathbb{RP}^{n-1} \\ (p,q) &\mapsto [p-q] \\ \beta : TM \setminus 0_M &\to \mathbb{RP}^{n-1} \\ (p,w) &\mapsto [w] \end{aligned}$$

which are the composition of a linear and a quotient map, and a projection map, respectively, and hence smooth. Since $n > 2m-1$, $(M \times M) \setminus \Delta$ and $TM \setminus 0_M$ are both of dimension $2m$. The union of the images has measure zero by Corollary 13.7 ■

We now show the strong Whitney embedding theorem.

Theorem 13.9 (Strong Whitney Embedding)**.** Let M be a compact smooth m-manifold. M admits an embedding into $\mathbb{R}^{2m+1}$.

Proof. By Theorem 10.5, M admits an embedding into $\mathbb{R}^N$ for sone N large. But iteratively applying Lemma 13.8, we can consider the image of M under the successive smooth embeddings to $\mathbb{R}^{2m+1}$. ■

14. Lecture 14 – 26th November 2024

We complete the proof of Theorem 13.5 by way of various lemmata.

Lemma 14.1. Let $U \subseteq \mathbb{R}^m$, $F : U \to \mathbb{R}^n$ be a smooth map with critical points $C \subseteq U$, and $C_k \subseteq C$ on which the ith partial derivatives of F for $1 \leq i \leq k$ functions of vanish. If $k > \frac{m}{n} - 1$ then $F(C_k)$ has measure zero.

Proof. For each $p \in U$, there exists a closed cube $E \subseteq U$ containing p. By second countability, we can cover C_k by countably many such cubes. We show $F(C_k \cap E)$ is of measure zero. Let $A > \sup_{q \in E} |\partial_x^\alpha(q)|$ be a constant and $|\alpha| \leq k+1$. Let $L > 0$ be the side length of E and $K >> 1$ large a natural number. We can subdivide E into cubes of sidelength L/K, of which there are K^m. Consider an enumeration of these subcubes and an index i_0 such that $p \in E_{i_0}$.

Since $p \in C_k$, we know by Taylor's theorem that

$$|F(x) - F(a)| < A'|x-p|^{k+1}$$

since the first k-terms of the Taylor expansion vanishes, which holds for all $x \in E_{i_0}$ and A' depending on A. Thus $F(E_{i_0})$ is contained in a ball centered around $F(p)$ of radius $A'(L/K)^{k+1}$.

Now

$$F(C_k \cap E) = \bigcup_{\{i_0 : E_{i_0} \cap C_k \neq \emptyset\}} F(C_k \cap E_{i_0}).$$

but each $F(C_k \cap E_{i_0})$ is covered in a union of balls of volume $\Lambda \left(A'(L/K)^{k+1}\right)^n$ and there are at most K^m of such cube images, so $F(C_k \cap E)$ is covered by a union of balls of total volume $K^m \cdot \Lambda \left(A'(L/K)^{k+1}\right)^n = \Lambda A'^n L^{kn+n} K^{m-kn-n}$ but by hypothesis $m - kn - n < 0$ so by making K arbitrarily large, we can make the volume of $F(C_k \cap E)$ arbitrarily small, showing it is of measure zero. ■

The proof of Theorem 13.5 proceeds by induction, and we show that the induction step holds by considering C_k as a telescoping sum.

Lemma 14.2. Let $U \subseteq \mathbb{R}^m$, $F : U \to \mathbb{R}^n$ be a smooth map with critical points $C \subseteq U$, and $C_k \subseteq C$ on which the ith partial derivatives of F for $1 \leq i \leq k$ functions of vanish. If Sard's theorem holds for domains of dimension strictly less than m then $F(C \setminus C_1)$ has measure zero.

Proof. Note that C_1 is closed in U and up to replacing U by $U \setminus C_1 \subseteq U$ open, we can take $C_1 = \emptyset$. We show $F(C)$ is of measure zero. Up to reordering coordinates x, y in source and target, respectively, we can assume that $\partial_{x_1} F_1(p) \neq 0$. Now taking $u(x) = F_1(x), v_i(x) = F_i(x)$ for $2 \leq i \leq m$, the inverse function theorem shows that the functions u, v form a coordinate system around V_p of p with transition matrix

$$\begin{bmatrix} \partial_{x_1} F_1 & \dots \\ 0 & \mathrm{id} \end{bmatrix}$$

which extend smoothly to coordinates on $\overline{V_p}$. With respect to the new coordinates, we write $F(u, v) = (u, F_2(u, v), \ldots, F_n(u, v))$ and thus the differential is of the form

$$\begin{bmatrix} 1 & 0 \\ \vdots & \partial_{v_j} F_i \end{bmatrix}$$

and observing that $C \cap \overline{V_p}$ is precisely the set of points such that the Jacobian submatrix $\partial_{v_j} F_i$ is not full.

We show $F(C \cap \overline{V_p})$ is measure zero by showing it is measure zero on slices. We show $F(C \cap \overline{V_p}) \cap \{y_1 = \ell\}$ is measure zero for $\ell \in \mathbb{R}$. Let $B_\ell = \{v : (\ell, v) \in \overline{V_p}\} \subseteq \mathbb{R}^{m-1}$ and set $F_\ell(v) = (F_2(\ell, v), \ldots, F_n(\ell, v))$ and since $F(\ell, v) = (\ell, F_\ell(v))$ we have that the critical values of $F|_{\overline{V_p}}$ in $\{y_1 = \ell\}$ are precisely the pairs (ℓ, v') such that v' is a critical value of F_ℓ.

Now since Sard's theorem holds on the domain of F_ℓ of dimension $m - 1$, the critical values of F_ℓ are measure zero for each ℓ. And thus $F(C)$ is of measure zero by Lemma 12.5. ■

Lemma 14.3. Let $U \subseteq \mathbb{R}^m$, $F : U \to \mathbb{R}^n$ be a smooth map with critical points $C \subseteq U$, and $C_k \subseteq C$ on which the ith partial derivatives of F for $1 \leq i \leq k$ functions of vanish. If Sard's theorem holds for domains of dimension strictly less than m then for all $k \geq 1$, $F(C_k \setminus C_{k-1})$ has measure zero.

Proof. As in the proof of Lemma 14.2, we can consider $U \setminus C_{k+1}$ and prove that $F(C_k)$ is of measure zero. For $p \in C_k$ and $\sigma : U \to \mathbb{R}$ such that σ is a partial derivative that has at least one nonvanishing partial derivative at p, that is, $\sigma = \partial^\alpha_{x_i} F_j$ with $|\alpha| = k$ and $\partial_{x_i} \sigma(p) \neq 0$ for all i.

Let V_p be a neighborhood of p consisting of regular points and $\Sigma = \{\sigma^{-1}(0)\} \cap V_p$. Then Σ is a smooth submanifold of V_p. By definition of C_k, we have that $(C_k \cap V_p) \subseteq \sigma^{-1}(0) \cap V_p$. Moreover, $F(C_k \cap V_p)$ is contained in the set of critical values of $F|_\Sigma$ as if all $\partial_{x_i} F_j = 0$ then $dF|_{T\Sigma} = 0$. But $\dim(\Sigma) = \dim(U) - 1$ so the set of $F|_\Sigma$ is measure zero by hypothesis. ■

The lemmata imply the proof as follows.

Proof of Theorem 13.5. We proceed by induction on dimension of the source. If $m = 0$, we are done. Now assume Theorem 13.5 holds for all $m < \widetilde{m}$. By covering the target with charts and considering the preimage of the fixed chart, we can assume that the source is $U \subseteq \mathbb{R}^m$ and the target is $\mathbb{R}^n$. We have

$$\cdots \subseteq C_2 \subseteq C_1 \subseteq C_0 = C$$

and thus $F(C) \subseteq \bigcup_{k \geq 0} F(C_k \setminus C_{k+1})$ which is the union of measure zero sets by Lemmas 14.1 to 14.3 and thus $F(C)$ is of measure zero. ■

15. Lecture 15 – 29th November 2024

Recall that for M a smooth manifold, the tangent bundle Definition 8.1 is a smooth manifold by Proposition 8.3.

Definition 15.1 (Vector Field)**.** Let M be a smooth manifold. A smooth vector field X on M is a smooth map $X : M \to TM$ such that $\pi \circ X = \mathrm{id}_M$.

Remark 15.2. In particular, X is a smooth section of the vector bundle. We can also consider arbitrary sections which will be known as rough vector fields.

In other words, X is the assignment of a tangent vector $v \in T_pM$ to each $p \in M$.

Example 15.3. We have that $T_p\mathbb{R}^n = \mathbb{R}^n$ so $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ so a smooth section is just a smooth map that is the identity on the first coordinate.

Example 15.4. The vector field ∂_{x_i} is the vector field of the ith coordinate for each point of a smooth manifold M.

Example 15.5. There is a canonical vector field 0_M on a smooth manifold M that is the zero tangent vector for each $p \in M$.

In fact, we can show that the collection of all possible smooth vector fields assemble into a vector space.

Lemma 15.6. Let M be a smooth manifold and $\mathfrak{X}(M)$ the collection of all smooth vector fields. Then:

(i) $\mathfrak{X}(M)$ is a $\mathbb{R}$-vector space.
(ii) $\mathfrak{X}(M)$ is a module over $C^\infty(M)$.

Proof of (i). These extend over the pointwise operations on vector spaces. ■

Proof of (ii). A smooth section can be represented by a tuple of m-smooth functions which naturally inherits a module action by $C^\infty(M)$. ■

This leads to the following definition.

Definition 15.7 (Space of Vector Fields)**.** Let M be a smooth manifold. $\mathfrak{X}(M)$ is the $\mathbb{R}$-vector space of smooth vector fields on M.

Given a smooth map $F : M \to N$, we can relate vector fields on M, N in the following way.

Definition 15.8 (F-Related)**.** Let M, N be smooth manifolds, $F : M \to N$ a smooth map, and X, Y smooth vector fields on M, N. X and Y are F-related if the diagram

$$\begin{tikzcd} TM \arrow[r, "dF"] \arrow[d, shift left, "\pi_M"] & TN \arrow[d, shift right, "\pi_N"'] \\ M \arrow[r, "F"'] \arrow[u, shift left, "X"] & N \arrow[u, shift right, "Y"'] \end{tikzcd}$$

commutes.

Remark 15.9. As suggested by the name, the relationship between vector fields is not only dependent on the manifolds but also on the smooth map F.

We can also produce a vector field on the target manifold given a vector field on the source in a construction known as pushing forward.

Definition 15.10 (Pushforward Vector Field)**.** Let $F : M \to N$ be a diffeomorphism between smooth manifolds and X is a vector field on M. The pushforward vector field is the vector field F_*X such that the diagram

$$\begin{tikzcd} TM \arrow[r, "dF"] \arrow[d, shift left, "\pi_M"] & TN \arrow[d, shift right, "\pi_N"'] \\ M \arrow[u, bend left, "X"] \arrow[r, "F"'] & N \arrow[u, bend right, "F_*X"'] \end{tikzcd}$$

commutes.

Remark 15.11. More explicitly, for each point $p \in M$, $(F_*X)_p = dF_{F^{-1}(p)}(X_{F^{-1}(p)})$.

Given that tangent spaces can be understood in terms of derivations, so too can vector fields.

Lemma 15.12. Let M be a smooth manifold.

(i) If X is a smooth tangent field on M then Xf is a smooth function.
(ii) If X is a coarse vector field and Xf is smooth for all $f \in C^\infty(M)$ then X is smooth.

Generalizing Definition 7.4, we have the following.

Definition 15.13 (Derivation)**.** Let M be a smooth manifold. A derivation is a map $X : C^\infty(M) \to C^\infty(M)$ if for all $f, g \in C^\infty(M)$,

$$X(fg) = f \cdot X(g) + g \cdot X(f).$$

We show that derivations arise from vector fields in the following sense.

Proposition 15.14. Let M be a smooth manifold and X a smooth vector field on M. Then the map $f \mapsto X(f)$ is a derivation and each derivation arises in this way.

Proof. For all $p \in M$ we have $X(fg)(p) = f(p)X(g)(p)+g(p)X(f)(p) = f(p)X(g(p))+g(p)X(f(p))$ so the map is a derivation. Now for a derivation $Y : C^\infty(M) \to C^\infty(M)$ we can produce a vector field $Y(f)$ for all $f \in C^\infty(M)$ but this is a smooth vector field by Lemma 15.12. ■

We can also consider vector fields as elements of a Lie algebra.

Definition 15.15 (Lie Bracket)**.** Let M be a smooth manifold and X, Y smooth vector fields on M. The Lie bracket $[X, Y]$ is a map $C^\infty(M) \to C^\infty(M)$ by $f \mapsto (XY)(f) - (YX)(f)$.

This in fact defines a derivation, making $[X, Y]$ a smooth vector field.

Lemma 15.16. The action of the Lie bracket defines a derivation on $C^\infty(M)$.

Proof. We compute

$$
\begin{aligned}
[X,Y](f,g) &= (XY)(fg) - (YX)(fg) \\
&= X(f \cdot Y(g) + g \cdot Y(f)) - Y(f \cdot X(g) + g \cdot X(f)) \\
&= X(f) \cdot Y(g) + f \cdot (XY)(g) + g \cdot (XY)(f) + X(f) \cdot X(g) \\
&\qquad - Y(f) \cdot X(g) - f(YX)(g) - g(YX)(f) - X(f) \cdot Y(g) \\
&= f((XY) - (YX))(g) - g((XY) - (YX))(f) \\
&= f \cdot [X,Y](g) + g \cdot [Y,X](f)
\end{aligned}
$$

as desired. ■

16. Lecture 16 – 3rd December 2024

We show that the formation of Lie brackets is natural in the following sense.

Lemma 16.1. Let M, N be smooth manifolds and $F : M \to N$ a smooth map. Let $X_1, X_2 \in \mathfrak{X}(M), Y_1, Y_2 \in \mathfrak{X}(N)$ with X_i F-related to Y_i for $i \in \{1, 2\}$. Then $[X_1, X_2]$ and $[Y_1, Y_2]$ are F-related.

Proof. Let $f \in C^\infty(N)$ and consider $X_1X_2(f \circ F)$ which by F-relatedness yields $X_1((Y_2f) \circ F) = (Y_1Y_2f) \circ F$ and similarly $X_2X_1(f \circ F)$ which by F-relatedness and an analogous calculation yields $X_2X_1(f \circ F) = (Y_2Y_1f) \circ F$. Thus in the Lie bracket we have

$$\begin{aligned}[X_1, X_2](f \circ F) &= (X_1X_2 - X_2X_1)(f \circ F) \\ &= (Y_1Y_2 - Y_2Y_1)(f) \circ F \\ &= [Y_1, Y_2](f) \circ F\end{aligned}$$

showing F-relatedness, as desired. ■

We now consider coordinates on vector fields, in analogy to coordinates on the tangent bundle Definition 7.13. Recall that on a smooth manifold M we have coordinates ∂_{x_i} on the tangent bundle TM which are the preimages of ith coordinate functions of a chart (U, ϕ) on M under the canonical identification of the Euclidean space with its tangent bundle.

Lemma 16.2. Let M be a smooth manifold. The map $M \to TM$ by $p \mapsto (\partial_{x_i})_p$ defines a smooth vector field on an open set $U \subseteq M$.

Proof. This is a coarse section by inspection, and smoothness follows from the smoothness of charts and projection maps. ■

Lemma 16.2 justifies the following definition.

Definition 16.3 (Coordinate Vector Field)**.** Let M be a smooth manifold. The section $p \mapsto (\partial_{x_i})_p$ defines the coordinate vector field ∂_{x_i}.

Using this, we can define frames.

Definition 16.4 (Local Frame)**.** Let M be a smooth m-manifold and $X^1, \ldots, X^m \in \mathfrak{X}(M)$ be vector fields. $(X^1, \ldots, X^m)$ is a local frame at p if the tangent vectors $X^1_p, \ldots, X^m_p$ span T_pM.

Definition 16.5 (Global Frame)**.** Let M be a smooth m-manifold and $X^1, \ldots, X^m \in \mathfrak{X}(M)$ be vector fields. $(X^1, \ldots, X^m)$ is a global frame if the tangent vectors $X^1_p, \ldots, X^m_p$ span T_pM for all $p \in M$.

As expected, being a local frame is a local condition.

Lemma 16.6. Let M be a smooth m-manifold and $X^1, \ldots, X^m$ a local frame at $p \in M$. Then there exists an open subset $U \subseteq M$ containing p such that $(X^1, \ldots, X^m)$ is a global frame on U.

Proof. The $(X^1, \ldots, X_p^m)$ considered as a matrix is of full rank, and by openness of the full rank condition in p and Lemma 8.9 shows that there exists such U. ∎

Example 16.7. The vector fields ∂_{x_i} is a form a global frame, though not all vector fields arise as coordinate vector fields – that is, admit a representation as a coordinate vector field for some chart.

Now define integral curves.

Definition 16.8 (Integral Curve)**.** Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. An integral curve for X is a curve $\gamma : (a, b) \to M$ with $(a, b) \subseteq \mathbb{R}$ containing zero and such that $d\gamma_t(\partial_t) = X_{\gamma(t)}$.

Definition 16.9 (Starting Point)**.** Let M be a smooth manifold and $\gamma : (a, b) \to M$ an integral curve for $X \in \mathfrak{X}(M)$. The starting point of γ is $\gamma(0) \in M$.

To describe integral curves of Definition 16.8 more explicitly, we consider the diagram

$$\begin{tikzcd} T\mathbb{R} = T(a,b) \arrow[r, "d\gamma"] \arrow[d, shift left] & TM \arrow[d] \\ (a,b) \arrow[u, shift left, "\partial_t"] \arrow[ur, "\dot{\gamma}"] \arrow[r, "\gamma"'] & M. \end{tikzcd}$$

For a fixed vector field $X \in \mathfrak{X}(M)$, an integral curve is a curve in M that "flows along" the vector field in the sense that for each time t, the derivative of the curve $\dot{\gamma}(t)$ at a time t is precisely the vector $X_{\gamma(t)}$ – the element of $T_{\gamma(t)}M$ corresponding to $X_{\gamma(t)}$.

Example 16.10. Let $M = \mathbb{R}^2$ and consider the vector field ∂_x which associates to each point $(x, y) \in \mathbb{R}^2$ the vector $(1, 0) \in \mathbb{R}^2 = T_p\mathbb{R}^2$ for all $p \in \mathbb{R}^2$. The integral curves are curves of the form $t \mapsto p + t(1, 0)$ with starting point p.

Example 16.11. Let $M = \mathbb{R}^2$ and consider the vector field $x\partial_y - y\partial_x$ which associates to each point (x, y) the vector $(-y, x)$. Suppose that $\gamma : \mathbb{R} \to \mathbb{R}^2$ is an integral curve. Given as a component function, γ necessarily satisfies $\dot{\gamma}^1(t) = -\gamma^2(t)$ and $\dot{\gamma}^2(t) = \gamma^1(t)$. For $p = (a, b)$ this system of ordinary differential equations is satisfied by the path $t \mapsto (a\cos(t) - b\sin(t), a\sin(t) + b\cos(t))$.

More generally, the existence of integral curves is given by the solution to a system of ordinary differential equations.

Theorem 16.12 (Existence-Uniqueness Theorem for Integral Curves)**.** Let M be a smooth m-manifold and $X \in \mathfrak{X}(M)$. For $p \in M$ there is an open interval $J \subseteq \mathbb{R}$ containing 0 and an integral curve $\gamma : J \to M$ with starting point p. Moreover, γ is unique.

Proof. Let $M \subseteq \mathbb{R}^n$ be open and we solve the system of differential equations

$$\begin{cases} \dot{\gamma}^1(t) = X^1_{\gamma(t)} \\ \vdots \\ \dot{\gamma}^m(t) = X^m_{\gamma(t)} \end{cases}$$

which exists and is unique by the Picard-Lindelöf existence-uniqueness theorem for solutions to a system of differential equations. ■

The construction of integral curves is preserved under F-related vector fields.

Lemma 16.13. Let $F : M \to N$ be a morphism of smooth manifolds, $X \in \mathfrak{X}(M), Y \in \mathfrak{X}(N)$ with X, Y F-related. Then F takes integral curves of X to integral curves of Y.

Proof. Let $\gamma : (a, b) \to M$ be an integral curve. We compute $(F \circ \gamma)'(t) = dF_{\gamma(t)}\dot{\gamma}(t)$ which by F-relatedness is $Y_{F\circ\gamma(t)}$ as desired. ■

We conclude with the following definition.

Definition 16.14 (Complete Vector Field)**.** Let M be a smooth manifold and $X \in \mathfrak{X}(M)$ a vector field. X is a complete vector field if for all $p \in M$ the maximal integral curve at p is defined on all of $\mathbb{R}$.

Example 16.15. The vector field ∂_x on $M = \mathbb{R} \setminus \{0\}$ is an incomplete vector field. For $p = -1$ in $\mathbb{R}$, teh integral curve starting at p is the map $t \mapsto -1 + t$ which is only defined on $(-\infty, 1)$.

17. Lecture 17 – 7th December 2024

We begin with a discussion of flows.

Definition 17.1 (Flow Domain)**.** Let M be a smooth manifold. A flow domain is an open set $\mathcal{D} \subseteq \mathbb{R} \times M$ such that for all $p \in M$

$$\mathcal{D}^{(p)} = \{t \in \mathbb{R} : (t, p) \in \mathcal{D}\} \subseteq \mathbb{R}$$

is an open interval containing 0.

More explicitly, $\mathcal{D}$ admits two projection maps

$$\begin{tikzcd} & \mathcal{D} \arrow[dl] \arrow[dr] & \\ \mathbb{R} & & M \end{tikzcd}$$

where the condition states that the fiber of the projection map $\mathcal{D} \to M$ over $p \in M$ is an open interval of $\mathbb{R}$ containing 0.

Definition 17.2 (Flow)**.** Let M be a smooth manifold and $\mathcal{D}$ a flow domain. A flow is a map $\Psi : \mathcal{D} \to M$ such that $\Psi(0, p) = p$ and $\Psi(t, \Psi(s, p)) = \Psi(s+t, p)$ when $s, t, s+t \in \mathcal{D}^{(p)}$.

Remark 17.3. Note that for fixed t, $\Psi(t, -) = \Psi_t : M \to M$ is a diffeomorphism as it is smooth with smooth inverse $\Psi(-t, -) : M \to M$ and for fixed $p \in M$, $\psi(-, p) = \Psi^{(p)} : \mathcal{D}^{(p)} \to M$ is a path in M as $\mathcal{D}^{(p)}$ is an interval in $\mathbb{R}$ containing 0.

We can recover smooth vector fields from flows.

Lemma 17.4. Let M be a smooth manifold, $\mathcal{D}$ a flow domain, and $\Psi : \mathcal{D} \to M$ a flow. The map $p \mapsto \frac{d}{dt}|_{t=0}\Psi^{(p)}(t)$ defines a smooth vector field X^Ψ on M such that $\Psi^{(p)} : \mathcal{D}^{(p)} \to M$ are integral curves of X^Ψ starting at p.

Proof. By Lemma 15.12 suffices to show that for all smooth functions $f \in C^\infty(M)$ that $X^\Psi f$ is smooth. We then compute for $p \in M$ that $X^\Psi f(p) = X^\Psi_p f = \frac{d}{dt}|_{t=0}(f \circ \Psi^{(p)}(t)) = \partial_t(f \circ \Psi)(0, p)$ which is a smooth vector field by inspection.

To show that $\Psi^{(p)}$ are the integral curves of X^Ψ we fix some $t_0 \in \mathcal{D}^{(p)}$, $f \in C^\infty(M)$ and $q = \Psi(t_0, p)$. We then compute

$$\begin{aligned} X^\Psi_q f &= \Psi^{(q)}(0) f \\ &= \frac{d}{dt}|_{t=0} f(\Psi^{(q)}(t)) \\ &= \frac{d}{dt}|_{t=0} f(\Psi(t, \Psi(t_0, p))) \\ &= \frac{d}{dt}|_{t=0} f(\Psi(t + t_0, p)) \\ &= \frac{d}{dt}|_{t=0} f(\Psi^{(p)}(t + t_0)) \\ &= \dot{\Psi}^{(p)}(t_0) f \end{aligned}$$

as desired. ■

We now show the main result on flows.

Theorem 17.5 (Flow)**.** Let M be a smooth manifold and $X \in \mathfrak{X}(M)$ a smooth vector field. There exists a unique flow domain $\mathcal{D}$ and a unqiue flow $\Psi : \mathcal{D} \to M$ such that $X^\Psi = X$ and $\Psi^{(p)} : \mathcal{D}^{(p)} \to M$ are the maximal integral curves of X.

Proof. To define Ψ, we take $(t,p) \in \mathcal{D}$ to $\gamma^{(p)}$ the unique integral curve starting at p, the existence of which is given by Theorem 16.12. This satisfies the hypotheses of Lemma 17.4 which gives the desired construction. ■

We now turn to a discussion of the Lie derivative, which seeks to generalize derivatives of functions on an arbitrary manifold.

Definition 17.6 (Lie Derivative)**.** Let M be a smooth manifold and $X, Y \in \mathfrak{X}(M)$ smooth vector fields. The Lie derivative of Y with respect to X at a point $p \in M$ is given by

$$\lim_{t\to 0}(\mathcal{L}_X Y)_p = \frac{d}{dt}|_{t=0}\left(\frac{Y_{p+tX_p} - Y_p}{t}\right).$$

This in fact defines a smooth vector field.

Lemma 17.7. Let M be a smooth manifold and $X, Y \in \mathfrak{X}(M)$ smooth vector fields. The Lie derivative $\mathcal{L}_X Y$ of Y with respect to X is a smooth vector field.

Proof. Let θ be the flow for X and for $p \in M$, let (U, ϕ) be a smooth chart of M containing p. Let $J \subseteq \mathbb{R}$ be an open interval containing 0 and $U_0 \subseteq U$ containing p such that $\theta(J_0 \times U_0) \subseteq U$. For $(t,q) \in J_0 \times U_0$, the Jacobian matrix of the differential is given by

$$(d\Psi_{-t})_{\Psi(t,x)} = (\partial_{x_i}\Psi^j(-t,x))_{1\le i,j\le n}.$$

As such, we have $d\Psi_{-t}Y_{\Psi(t,x)} = (\partial_{x_i}\Psi^j(-t,x))_{1\le i,j\le n}$ giving the claim. ■

We will in fact show that the Lie derivative can be computed in terms of the Lie bracket, for which we will require the following lemma.

Lemma 17.8. Let M be a smooth manifold and let $X \in \mathfrak{X}(M)$ a smooth vector field. If $X_p \neq 0$ for some $p \in M$ then there exists a chart (U, ϕ) around p with respect to which $X = \partial_{x_1}$.

Proof. Let (U, ϕ) be a chart around p with coordinates $(x_1, \dots, x_m)$ and let S be the hypersurface defined by $x_j = 0$ for $X_p^j \neq 0$ with X the vector field as above. Shrinking S such that X is nowhere tangent to S the flowout theorem [Lee13, Thm. 9.20] implies that there is a flow domain $\mathcal{O}_\delta \subseteq \mathbb{R} \times S$ such that the flow of X restricts to a diffeomorphism Φ from $\mathcal{O}_\delta$ to an open subset $W \subseteq M$ containing S.

There is a product neighborhood $(-\varepsilon, \varepsilon) \subseteq W_0$ of $(0,p)$ in $\mathcal{O}_\delta$. Now taking a smooth local parametrization $X : \Omega \to S$ with image contained in W_0 and Ω open in $\mathbb{R}^{m-1}$ with coordinates $x_2, \dots, x_n$. It follows that the map $\Psi : (-\varepsilon, \varepsilon) \times \Omega \to M$ by $\Psi(t, x_2, \dots, x_n) = \Phi(t, X(x_2, \dots, x_m))$ iwhich is a diffeomprhpism onto a neighborhood of p in M. Moreover this map pushes ∂_t to itself with $\Phi_*(\partial_t) = X$ so

$\Psi_*(\partial_t) = X$ with Ψ a smooth coordinate chart with the desired representation. The claim follows by relabeling. ∎

18. Lecture 18 – 10th December 2024

We show the desired result, that Lie derivatives can be computed in terms of Lie brackets.

Proposition 18.1. Let M be a smooth manifold and $X, Y \in \mathfrak{X}(M)$ smooth vector fields. Then $\mathcal{L}_X Y = [X, Y]$.

Proof. It suffices to show that $(\mathcal{L}_Y X)_p = [X, Y]_p$. We consider three cases: where p is a nonvanishing point of X, p is in the support of X, and p is outside the support of X.

In the first case, Lemma 17.8 allows us to choose ∂_{x_1} as the coordinate representation for X in which case the flow is given by $\theta_t(u) = (u^1 + t, u^2, \dots, u^n)$ and for each fixed t, the Jacobian of $d(\theta_{-t})_{\theta_t(x)}$ is the identity and an explicit computation gives the pointwise equality.

In the second case if p is in the support of X the claim holds by the above. Otherwise X is zero at p so θ_t is the identity on a neighborhood of p for all p so the Lie derivative is zero as well, which agrees with the Lie bracket. ∎

Moreover, we can show the following result for local frames.

Theorem 18.2. Let M be a smooth m-manifold and let $X^1, \dots, X^m \in \mathfrak{X}(M)$ be a local frame at p. There exists a chart (U, ϕ) around p such that $X^i = \partial_{x_i}$ if and only if $[X^i, X^j] = 0$ near p for all i, j.

Proof. $(\Rightarrow)$ is clear since coordinate vector fields commute.

$(\Leftarrow)$ is the construction of Lemma 17.8. ∎

As a prelude to a discussion of vector bundles, we make some recollections on linear algebra.

We fix a field $\mathbb{R}$ and consider the category $\mathsf{Vect}_{\mathbb{R}}^{\mathsf{fd}}$ of finite dimensional $\mathbb{R}$-vector spaces and morphisms linear maps. This category has especially nice properties as we now describe.

Theorem 18.3. The category $\mathsf{Vect}_{\mathbb{R}}^{\mathsf{fd}}$ is a closed symmetric monoidal Abelian category.

To be more explicit, all finite limits and colimits exist, implying the existence of kernels and cokernels, the category is preserved under finite direct sums which agree with finite coproducts, and admits a tensor product which satisfies the tensor-hom adjunction.

Definition 18.4 (Tensor)**.** Let $V \in \mathsf{Vect}_{\mathbb{R}}^{\mathsf{fd}}$. A tensor of type (a, b) over V is an element of

$$\underbrace{V \otimes \cdots \otimes V}_{a \text{ times}} \otimes \underbrace{V^\vee \otimes \cdots \otimes V^\vee}_{b \text{ times}}.$$

This leads us to the definition of alternating and symmetric tensors.

Definition 18.5 (Alternating Tensor)**.** Let $V \in \mathsf{Vect}_{\mathbb{R}}^{\mathsf{fd}}$ and $\alpha \in T^{0,b}V$. α is an alternating tensor if the induced map $V \times \cdots \times V \to \mathbb{R}$ is such that

$$\alpha(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_b) = (-1)^{j-i}\alpha(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_b).$$

Definition 18.6 (Symmetric Tensor)**.** Let $V \in \mathsf{Vect}_{\mathbb{R}}^{\mathsf{fd}}$ and $\alpha \in T^{0,b}V$. α is a symmetric tensor if the induced map $V \times \cdots \times V \to \mathbb{R}$ is such that

$$\alpha(x_1, \ldots, x_b) = \alpha(x_{\sigma(1)}, \ldots, x_{\sigma(b)})$$

for all permutations $\sigma \in S_b$.

Definition 18.7 (Space of Alternating Tensors)**.** The space of alternating tensors $\bigwedge^b V \subseteq T^{0,b}V$ is the linear subspace consisting of alternating tensors.

Definition 18.8 (Space of Symmetric Tensors)**.** The space of alternating tensors $\mathrm{Sym}^b V \subseteq T^{0,b}V$ is the linear subspace consisting of symmetric tensors.

19. Lecture 19 – 13th December 2024

Let us recall the following definition from category theory.

Definition 19.1 (Split Monomorphism)**.** Let C be a category and $f : A \to B$ a monomorphism. f is a split monomorphism if there exists $s : B \to A$ such that $f \circ s = \mathrm{id}_B$.

The inclusion of both $\bigwedge^b V$ and $\mathrm{Sym}^b V$ into $T^{0,b}V$ are of this type.

Lemma 19.2. The inclusions $\bigwedge^b V \hookrightarrow T^{0,b}V$ and $\mathrm{Sym}^b V \hookrightarrow T^{0,b}V$ are split via the maps

$$\alpha \mapsto \frac{1}{b!} \sum_{\sigma \in S_b} \mathrm{sign}(\sigma)\alpha_\sigma \text{ and } \alpha \mapsto \frac{1}{b!} \sum_{\sigma} \alpha_\sigma$$

with α_σ the permutation of the inputs of α by σ.

Proof. The composite maps

$$\bigwedge^b V \hookrightarrow T^{0,b}V \twoheadrightarrow \bigwedge^b V, \mathrm{Sym}^b V \hookrightarrow T^{0,b}V \twoheadrightarrow \mathrm{Sym}^b V$$

are easily seen to be inclusions, from which the claim follows. ■

The preceding discussions of linear algebra can be globalized to the setting of vector bundles, as we now define.

Definition 19.3 (Vector Bundle)**.** Let M be a smooth manifold. A vector bundle is the data of a smooth manifold E and a smooth map $\pi : E \to M$ such that for all $p \in M$, $E_p = \pi^{-1}(p)$ is a k-dimensional real vector space and an open neighborhood U of p on which $\pi^{-1}(U) \cong U \times \mathbb{R}^k$.

Remark 19.4. We will henceforth assume $\dim_{\mathbb{R}} E_p$ is constant for all p.

Example 19.5. Consider the Möbius bundle on the circle S^1. This is not a trivial bundle – of the form $S^1 \times \mathbb{R}^k$ for some k, but is locally of this form.

We define morphisms of vector bundles as follows.

Definition 19.6 (Morphism of Vector Bundles)**.** Let $\pi : E \to M, \pi' : E' \to M'$ be vector bundles over smooth manifolds M, M'. A moprhism of vector bundles is a pair of smooth maps $\widetilde{F} : E \to E', F : M \to M'$ such that

$$\begin{tikzcd} E \arrow[r, "\widetilde{F}"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "F"'] & M' \end{tikzcd}$$

commutes and $\widetilde{F}|_{E_b} : E_p \to E'_{f(p)}$ is a linear map for all $p \in M$.

Definition 19.7 (Subbundle)**.** Let $\pi : E \to M$ be a vector bundle over a smooth manifold M. A subbundle of E is the data of a vector bundle $\pi' : E' \to M$ and a morphism of vector bundles $\widetilde{F} : E' \to E$ such that

$$\begin{tikzcd} E' \arrow[r, "\widetilde{F}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ M' \arrow[r, "\mathrm{id}_M"'] & M \end{tikzcd}$$

commutes and $E'_p \to E_p$ is an inclusion of vector spaces for all $p \in M$.

Moreover, Definitions 19.3 and 19.6 allow us to define a category of vector bundles and a category of vector bundles over a fixed base.

Definition 19.8 (Category of Vector Bundles)**.** The category of vector bundles VB is a category with objects vector bundles and morphisms those of vector bundles.

Definition 19.9 (Category of Vector Bundles With Fixed Base)**.** Let M be a smooth manifold. The category of smooth vector bundles over M $\mathsf{VB}(M)$ has objects vector bundles with base M and morphisms those of vector bundles with $F = \mathrm{id}_M$.

Remark 19.10. The category of vector bundles admits a functor to the category of smooth manifolds SmMfld taking a vector bundle to its base.

Remark 19.11. There is an equivalence of categories between $\mathsf{VB}(*)$ the category of smooth vector bundles over a point and $\mathsf{Vect}_{\mathbb{R}}$ the category of real vector spaces.

The following lemma will allow us to make a functorial construction with vector bundles – pulling them back along morphisms.

Lemma 19.12. Let $F : M \to M'$ be a morphism of smooth manifolds and $\pi' : E' \to M'$ be a vector bundle. The fibered product $M \times_{M'} E'$ exists and has the structure of a smooth vector bundle.

Proof. Since $\pi' : E' \to M'$ is surjective, F, π' are a transversal pair, and the fibered product $M \times_{M'} E'$ exists by Theorem 12.2. Denote the projection from the fibered product to M by $\pi : M \times_{M'} E' \to M$. That π is a smooth vector bundle follows from functoriality of the construction of vector bundles. ∎

We now make the desired construction.

Definition 19.13 (Pullback of Vector Bundles)**.** Let $F : M \to M'$ be a morphism of smooth manifolds and $\pi' : E' \to M'$ be a vector bundle. The pullback bundle F^*E is the bundle induced by the fibered product $\pi : M \times_{M'} E' \to M$.

Moreover, we can construct vector bundles from gluing data.

Lemma 19.14. Let M be a smooth manifold, $\{U_i\}_{i \in I}$ an open cover of M, and $\{E_i\}_{i \in I}$ a collection of k-dimensional real vector spaces, and $\psi_{ij} \in \mathrm{GL}_k(\mathbb{R})$ such that the following hold:

(1) For each i, $\psi_{ii} : E_i \to E_i$ is id_{E_i}.

(2) For each i, j, k, $\psi_{ik} = \psi_{jk} \circ \psi_{ij}$ on $U_i \cap U_j \cap U_k$.

There is a unqiue vector bundle E on M with isomorphisms $\rho_i : E|_{U_i} \to E_i$ such that for each i, j, $\rho_j = \psi_{ij} \circ \rho_i$. Moreover, every vector bundle arises in this way.

Proof. Let $V \subseteq M$ the construction

$$E|_V = \left\{ (v_i)_{i \in I} \in \prod_{i \in I} E_i|_V : \psi_{ij}(v_i|_{U_i \cap U_j}) = \psi_{ji}(v_j|_{U_i \cap U_j}), \forall i, j \in I \right\}$$

produces a vector bundle. For E' any vector bundle on M, taking its associated trivializing cover $\{U_i'\}_{i \in I'}$ and $\psi_{ij} = \rho_j \circ \rho_i^{-1}$. ■

Remark 19.15. Different gluing data as in the hypothesis of Lemma 19.14 can give rise to the same vector bundle.

20. Lecture 20 – 17th December 2024

The linear algebra constructions described in Section 19 globalize to vector bundles.

Theorem 20.1 (Omnibus Linear Algebra)**.** The category of vector bundles over a fixed base is symmetric monoidal category, and the formation of $\bigwedge^k E, \mathrm{Sym}^k E, E \otimes E'$ are compatibile with pullback.

Proof. This follows from the functoriality of the linear algebra constructions and of pullback. ■

Remark 20.2. Neither the category of vector bundles nor the category of vector bundles over a fixed base are Abelian. Consider the vector bundle $\mathbb{R} \times \mathbb{R}$ over $\mathbb{R}$ and the morphism of vector bundles over $\mathbb{R}$ by $(t, v) \mapsto (t, t \cdot v)$. The kernel is the union of the two axes in $\mathbb{R} \times \mathbb{R} \cong \mathbb{R}^2$ which is not a vector bundle.

We also remark on the distinction between vector subbundles and subobjects in a category.

Definition 20.3 (Subbundle)**.** Let E, E' be vector bundles over a fixed base M. E is a subbundle of E' if there is a morphism $i : E \to E'$ over M such that $E_p \to E'_p$ is an injection of vector spaces for all points $p \in M$.

Note that monomorphisms in the category of vector bundles over a fixed base may not be subbundles.

Generalizing Lemma 15.6, we can make show the following.

Lemma 20.4. Let M be a smooth manifold and $\pi : E \to M$ be a vector bundle. Then:

(i) The space of sections $\Gamma(M, E)$ is an $\mathbb{R}$-vector space.
(ii) The space of sections $\Gamma(M, E)$ is a module over $C^\infty(M)$.

Proof of (i). These extend over the pointwise operations on vector spaces. ■

Proof of (ii). A smooth section can be represented by a tuple of smooth functions which naturally inherits a module action by $C^\infty(M)$. ■

The main constructions we will consider are as follows.

Definition 20.5 (1-Form)**.** Let M be a smooth manifold. The space of 1-forms is $\Omega^1(M) = \Gamma(T^*M)$.

Definition 20.6 (k-Form)**.** Let M be a smooth manifold. The space of 1-forms is $\Omega^k(M) = \Gamma(\bigwedge^k T^*M)$.

Sections over closed submanifolds of M can be defined in the following way.

Lemma 20.7. Let M be a smooth manfiold, $A \subseteq M$ closed, U an open neighborhood of A, and $\pi : E \to M$ be a vector bundle. If $\pi : A \to E$ is a smooth section, then there exists $\widetilde{\sigma} \in \Gamma(M, E)$ such that $\widetilde{\sigma}|_A = \sigma$ and $\mathrm{supp}(\widetilde{\sigma}) \subseteq U$.

Proof. Since sections are locally pointwise given by smooth functions on a closed subsets, the result follows by the extension lemma for smooth functions. ■

We append some results from Section 21 to this section, leaving that lecture for the introduction to Riemannial geometry.

Lemma 20.8. Let $F : M \to N$ be a morphism of smooth manifolds and E a vector bundle on N. There is an $\mathbb{R}$-linear map $\Gamma(N, E) \to \Gamma(M, F^*E)$ given by precomposition.

Proof. This can be deduced by the universal property of fibered products and the map is given by $\sigma \mapsto \sigma \circ F$. ■

Example 20.9. If $E' = T^*N$, the universal property induces the diagram

$$\begin{tikzcd} TM \arrow[d] & F^*T^*N \arrow[l, "dF^\vee"'] \arrow[r, "F^*"] \arrow[d] & T^*N \arrow[d] \\ M & M \arrow[l, "\mathrm{id}_M"] \arrow[r, "F"'] & N \end{tikzcd}$$

so we have $dF^\vee(p, v) = (p, dF_p^\vee(v))$ giving a map $F^*\sigma = dF^\vee \circ \sigma \circ F$ which produces pullbacks for tensors defined using T^*N more generally.

We conclude with the definition of local and global frames for vector bundles more generally.

Definition 20.10 (Local Frame)**.** Let $\pi : E \to M$ be a rank k vector bundle over a smooth manifold M. A collection of sections $X^1, \dots, X^k$ of E is a local frame at $p \in M$ if the span of the vectors $X_p^1, \dots, X_p^k$ span E_p.

Definition 20.11 (Global Frame)**.** Let $\pi : E \to M$ be a rank k vector bundle over a smooth manifold M. A collection of sections $X^1, \dots, X^k$ of E is a global frame at if the span of the vectors $X_p^1, \dots, X_p^k$ span E_p for all $p \in M$.

21. Lecture 21 – 20th December 2024: Some Riemannian Geometry

Some of the material covered on this day has been placed in the preceding Section 20.

We begin with the following definition.

Definition 21.1 (Inner Product)**.** Let V be a finite-dimensional real vector space. An inner product is a bilinear map $g \in \mathrm{Sym}^2(V)$ which is symmetric and positive semidefinite.

This globalizes to Riemannian metrics.

Definition 21.2 (Riemannian Metric)**.** Let M be a smooth manifold. A Riemannian metric on M is a section $g \in \Gamma(M, \mathrm{Sym}^2(T^*M))$ such that for all $p \in M$, g_p is an inner product.

In other words, a Riemannian metric is a smoothly varying metric on the tangent space of a smooth manifold.

Definition 21.3 (Riemannian Manifold)**.** A Riemannian manifold is a pair (M, g) consisting of a smooth manifold M and a Riemannian metric g.

Remark 21.4. We can replace TM by a vector bundle E and consider metrics on E as sections $g \in \mathrm{Sym}^2(E^\vee)$ such that $(E_p, g_p(-,-))$ is an inner product for each $p \in M$.

Note that on $\mathbb{R}^n$, we have a frame $dx_1, \ldots, dx_n$ on $T^*\mathbb{R}^n$, the bundle $\mathrm{Sym}^2(T^*M)$ admits a global frame by $\{dx_i \otimes dx_j\}_{1 \leq i,j \leq n}$ hence any section σ of this vector bundle can be written as $\sigma = \sum_{1 \leq i,j \leq n} \sigma_{i,j} \cdot dx_i \otimes dx_j$ where $\sigma_{i,j} : \mathbb{R}^n \to \mathbb{R}$. σ is symmetric if $\sigma_{i,j} = \sigma_{j,i}$ for all pairs $1 \leq i, j \leq n$ in which case it is represented by a symmetric matrix.

Example 21.5. On $\mathbb{R}^n$, $g_0 = \sum_{i=1}^n dx_i \otimes dx_j = \sum_{i=1}^n \delta_{ij} \cdot dx_i \otimes dx_j$ is the Euclidean metric on $\mathbb{R}^n$.

We make the following definitions for constructions on Riemannian manifolds.

Definition 21.6 (Length of Tangent Vector)**.** Let (M, g) be a Riemannian manifold. The length of a tangent vector $v \in T_pM$ is $|v|_g = g_p(v, v)^{1/2}$.

Definition 21.7 (Angle Between Tangent Vectors)**.** Let (M, g) be a Riemannian manifold. The angle between two nonzero tangent vectors $v, w \in T_pM$ is the unique $\theta \in (0, \pi)$ such that $\cos(\theta) = \frac{g_p(v,w)}{|v|_g \cdot |w|_g}$.

Definition 21.8 (Orthogonal Tangent Vectors)**.** Let (M, g) be a Riemannian manifold. Two nonzero tangent vectors $v, w \in T_pM$ are orthogonal if the angle between them is zero.

In the space of $\mathbb{R}^n$, this specializes to the following.

Example 21.9. If $M = \mathbb{R}^n$ and $g = g_0$ the Euclidean metric as in Example 21.5, then Definitions 21.6 to 21.8 coincide with the definitions that arise in linear algebra and Euclidean geometry.

In this way, we see the endowment of a smooth manifold with a Euclidean metric as a way to do geometry on an arbitrary smooth manifold.

Definition 21.10 (Length of Curve)**.** Let (M, g) be a Euclidean manifold and $\gamma : [a, b] \to M$ a curve. The length of the curve is defined to be the integral $L_g(\gamma) = \int_a^b |\dot{\gamma}(t)|dt$.

Remark 21.11. $|\dot{\gamma}(t)|$ is the pushforward of ∂_t along γ.

Definition 21.12 (Geodesic Length)**.** Let (M, g) be a Riemannian manifold and $p, q \in M$. The distance between p and q is

$$\inf_{\{\gamma:[a,b]\to M:\gamma(a)=p,\gamma(b)=q\}} L_g(\gamma).$$

It can in fact be shown that the length is independent of the parametrization, hence making the manifold (M, g) a metric space with the geodesic length Definition 21.12 as the metric.

More generally, the construction of Example 21.5 globalizes.

Theorem 21.13. Let M be a smooth manifold. There exists a Riemannian metric g on M, making (M, g) a Riemannian manifold.

This allows us to deduce the following corollary.

Corollary 21.14. Let M be a smooth manifold. The space of Riemannian metrics on M is convex and nonempty, hence contractible.

Moreover, using the fact that inner products restrict on vector subspaces, we can show that pullbacks define Riemannian metrics.

Lemma 21.15. Let $i : M \to N$ be an immersion. If g is a Riemannian metric on N, then i^*g is a Riemannian metric on M.

We conclude with the Gram-Schmidt process for Riemannian manifolds.

As commented by the instructor, this proposition is most naturally placed here.

Proposition 21.16 (Gram-Schmidt for Riemannian Manifolds)**.** Let (M, g) be a Riemannian manifold. Given $p \in M$, there is a local orthonormal frame on a neighborhood U of p such that for all $q \in U$ and $e_1, \dots, e_n \in \Gamma(U, TM)$, $g_q(e_i, e_j) = \delta_{ij}$.

22. Lecture 22 – 7th January 2025

We seek to extract information about a smooth manifold M from its space of k-forms $\Omega^k(M)$.

Definition 22.1 (Form of a Function)**.** Let M be a smooth manifold and $f \in C^\infty(M)$. $df \in \Omega^1(M)$ is defined to be $df_p(v) = v(f)$ for $v \in T_pM$ and all $p \in M$.

We can in fact show that on local coordinate charts and f a coordinate function x_i, the dx_i are dual to $\partial_{x_i} \in TM$.

Lemma 22.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. With respect to the canonical identification $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$, $df = (\partial_{x_1} f, \dots, \partial_{x_n} f)$.

Proof. It suffices to observe that $df_p((\partial x_i)_p) = (\partial_{x_i})_p(f) = (\partial_{x_i} f)(p)$ since the coordinate vector field $(\partial_{x_i})_p$ is represented by the path at p in direction e_i the ith basis vector. ■

This is also compatible with the notation for differentials.

Lemma 22.3. Let M, N be smooth manifolds and $F : M \to N, f : N \to \mathbb{R}$ be smooth maps. Then $F^*(df) = d(f \circ F)$ in $\Omega^1(M)$.

Proof. Let $v \in T_pM$ for some $p \in M$. We compute

$$F^*(df)_p(v) = df_{F(p)}(dF(v)) = (dF_p(v))(f) = v(f \circ F) = d(f \circ F)(v)$$

as desired. ■

Corollary 22.4. Let M be a smooth manifold. If $f : M \to \mathbb{R}$ is smooth, then df is smooth.

Proof. Let (ϕ, U) be a local chart on M on which $f|_U$ is locally given by $f \circ \phi^{-1} \circ \phi$ but $df = d((f \circ \phi^{-1}) \circ \phi) = \phi^{-1} d(f \circ \phi^{-1})$ with the equalities by Lemmas 22.2 and 22.3, respectively. ■

We now turn to a discussion of line and path integrals.

Definition 22.5 (Line Integral)**.** Let $\omega \in \Omega^1([a, b])$ where $\omega = f(t) \cdot dt$ be a 1-form on a closed interval $[a, b] \subseteq \mathbb{R}$. The line integral $\int_a^b \omega$ is defined to be $\int_a^b f(t)dt$.

This can be generalized to path intervals on manifolds.

Definition 22.6 (Path Integral)**.** Let M be a smooth manifold, $\omega \in \Omega^1(M)$, and $\gamma : [a, b] \to M$ a path. Then the path integral $\int_\gamma \omega$ is defined to be $\int_a^b \gamma^*\omega$.

The definition of Definition 22.6 is invariant under reparametrization of a path.

Lemma 22.7. Let $\sigma : [c, d] \to [a, b]$ be an endpoint-preserving function with positive derivative. Then $\int_c^d (\gamma \circ \sigma)^*\omega = \int_a^b \gamma^*\omega$.

Proof. We have that $(\gamma \circ \sigma)^*\omega = \sigma^*(\gamma^*\omega)$ so it suffices to show $\int_c^d \sigma^*\eta = \int_a^b \eta$ for $\eta \in \Omega^1([a,b])$. But

$$\int_c^d \sigma^*\eta = \int_c^d f(\sigma(t))\sigma'(t)dt = \int_a^b f(s)ds$$

writing $\eta = f(t)dt$ and the univaraite chain rule in the second quantity. ■

We can show the following for line integrals on manifolds.

Lemma 22.8. Let M be a smooth manifold, $f : M \to \mathbb{R}$ smooth, and $\gamma : [a,b] \to M$ a path. Then $\int_a^b \gamma^* df = \int_a^b (f \circ \gamma)dt$.

Proof. This is an immediate corollary of Lemma 22.3 for $F = \gamma$. ■

From this we deduce that integrals of forms over closed paths vanish.

Corollary 22.9. Let M be a smooth manifold, $f : M \to \mathbb{R}$ smooth, and γ a closed path on M. Then $\int_\gamma df = 0$.

Proof. Apply Lemma 22.8 to observe that the path integral is computed as a trivial definite integral. ■

The following example shows there are 1-forms not necessarily of the form df for $f \in C^\infty(M)$ on a smooth manifold M.

Example 22.10. Let

$$\omega = \frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy \in \Omega^1(M)$$

and $\gamma : [0, 2\pi] \to M$ by $t \mapsto (\cos(t), \sin(t))$ which is a closed path traversing the unit circle. We compute

$$\int_\gamma \omega = \int_0^{2\pi} \frac{\cos(t) \cdot d(\sin(t)) - \sin(t) \cdot d(\cos(t))}{\cos^2(t) + \sin^2(t)} = \int_0^{2\pi} \frac{\cos^2(t) + \sin^2(t)}{1} dt = 2\pi.$$

This is not df for $f \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ since the integrals of such forms are zero by Corollary 22.9.

We can define special types of 1-forms as follows. We will later generalize these to arbitrary differential forms.

Definition 22.11 (Exact 1-Form)**.** Let M be a smooth manifold and $\omega \in \Omega^1(M)$. ω is exact if $\omega = df$ for $f \in C^\infty(M)$.

Definition 22.12 (Closed 1-Form)**.** Let M be a smooth manifold and $\omega \in \Omega^1(M)$. ω is closed if $\partial_{x_i}\omega^j = \partial_{x_j}\omega^i$ on a local chart of M.

Evidently any exact 1-form is a closed 1-form by Clariault's theorem on commutativity of partial derivatives, but closed forms are not necessarily exact for ω as in Example 22.10 is an example of such.

On open balls of Euclidean spaces, the converse holds.

Proposition 22.13 (Poincaré Lemma)**.** Let $\omega \in \Omega^1(B_1(0))$ for $B_1(0) \subseteq \mathbb{R}^n$. ω is a closed 1-form if and only if it is an exact 1-form.

Proof. ($\Rightarrow$) Let γ_x be the straight line path from origin to $x \in B_1(0)$ by $t \mapsto tx$. Let $\omega = \omega^1 dx_1 + \cdots + \omega^n dx_n$ and

$$f(x) = \int_0^1 \left(\sum_{i=1}^n \omega^i(tx) x_i \right) dt.$$

We compute

$$\begin{aligned}
(\partial x_j f)(x) &= \partial_{x_j} \left(\int_0^1 \left(\sum_{i=1}^n \omega^i(tx) x_i \right) dt \right) \\
&= \int_0^1 \left(\sum_{i=1}^n \partial x_j \omega^i(tx) \cdot t \cdot x_i + \omega^j(tx) \right) dt \\
&= \int_0^1 \left(\sum_{i=1}^n \partial x_i \omega^j(tx) \cdot t \cdot x_i + \omega^j(tx) \right) dt && \text{closedness} \\
&= \int_0^1 \frac{d}{dt}(t\omega^j(tx)) dt \\
&= t\omega^j(x)|_{t=0}^{t=1} \\
&= \omega^j(x)
\end{aligned}$$

so $df = \omega$ showing the form is exact.

($\Leftarrow$) Any exact 1-form is closed by the preceding discussion. ∎

We conclude with the following linear algebraic construction.

Definition 22.14 (Evaluation of Element of $\bigwedge^k V^*$)**.** Let $\varepsilon^1, \ldots, \varepsilon^n$ be a basis of V^* with V an n-dimensional $\mathbb{R}$-vector space. Let $I = (i_1, \ldots, i_k)$ be an increasing sequence. For $\varepsilon^I \in \bigwedge^k V^*$, its evaluation on $v_1, \ldots, v_k \in V$ is given by

$$\varepsilon^I(v_1, \ldots, v_k) = \det \left(\begin{bmatrix} \varepsilon^{i_1}(v_1) & \ldots & \varepsilon^{i_k}(v_1) \\ \vdots & \ddots & \vdots \\ \varepsilon^{i_1}(v_k) & \ldots & \varepsilon^{i_k}(v_k) \end{bmatrix} \right).$$

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We can consider the construction of Definition 22.14 in a special case to make it more explicit.

Example 23.1. Let $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3), z = (z_1, z_2, z_3) \in \mathbb{R}^3$ and $\varepsilon^{123} \in \bigwedge^3(\mathbb{R}^3)^*$.
$$\varepsilon^{123}(v, w, z) = \det\left(\begin{bmatrix} v_1 & w_1 & z_1 \\ v_2 & w_2 & z_2 \\ v_3 & w_3 & z_3 \end{bmatrix}\right).$$

We further deduce some properties of this construction.

Lemma 23.2. Let V be a finite-dimensional $\mathbb{R}$-vector space with basis $\{e_1, \dots, e_n\}$ and dual basis $\{\varepsilon^1, \dots, \varepsilon^n\}$. Let $I = (i_1, \dots, i_k)$ be a multiindex. Then:

(i) If I has a repeated index, then $\varepsilon^I = 0$.

(ii) If $J = I_\sigma$ is a permutation of I by the symmetric group of order k then $\varepsilon^J = \operatorname{sgn}(\sigma)\varepsilon^I$.

(iii) Given $J = (j_1, \dots, j_k)$ another multiindex of length k, $\varepsilon^I(e_{j_1}, \dots, e_{j_k})$ is given by the determinant
$$\det\left(\begin{bmatrix} \delta_{i_1 j_1} & \dots & \delta_{i_1 j_k} \\ \vdots & \ddots & \vdots \\ \delta_{i_k j_1} & \dots & \delta_{i_k j_k} \end{bmatrix}\right).$$

Proof. Up to a choice of isomorphism $V \cong \mathbb{R}^n$, the properties are immediate from the definition of the construction Definition 22.14 via determinants. ∎

As a corollary, we deduce that the space $\bigwedge^k V^*$ has a basis given by wedges of dual bases along increasing indices.

Lemma 23.3. Let V be a finite-dimensional $\mathbb{R}$-vector space with basis $\{e_1, \dots, e_n\}$ and dual basis $\{\varepsilon^1, \dots, \varepsilon^n\}$. The vector space $\bigwedge^k V^*$ has a basis given by ε^I where I is an increasing multiindex of length k.

Proof. By Lemma 23.2, a permutation preserves the factors of the wedge product up to a sign, so we can always use a permutation to consider elements of the desired form. ∎

Once again considering the case of $\mathbb{R}^3$, we can give a basis of $\bigwedge^2(\mathbb{R}^3)^*$ explicitly.

Example 23.4. A basis of $\bigwedge^2(\mathbb{R}^3)^*$ is given by $\{\varepsilon^{12}, \varepsilon^{13}, \varepsilon^{23}\}$.

We seek to endow $\Omega^\bullet(M) = \bigoplus_{k\geq 0} \Omega^k(M)$ with the structure of an algebra over $C^\infty(M) = \Omega^0(M)$. This is done via the construction of the wedge product which we first construct locally on $\bigwedge^\bullet V^* = \bigoplus_{k\geq 0} \bigwedge^k V^*$.

Definition 23.5 (Exterior Product)**.** Let V be a finite-dimensional vector space. Given $\omega_1 \in \bigwedge^{k_1} V^*, \omega_2 \in \bigwedge^{k_2} V^*$, the exterior product is given by
$$\omega_1 \wedge \omega_2 = \frac{(k+\ell)!}{k!\ell!}\operatorname{Alt}(\omega_1 \otimes \omega_2) \in \bigwedge^{k_1+k_2} V^*.$$

Recall that the $\mathrm{Alt}(-)$ construction makes an arbitrary tensor an alternating one by Lemma 19.2. In the case of a finite-dimensional real vector space, the wedge product can be computed fairly explicitly.

Lemma 23.6. Let V be a finite-dimensional $\mathbb{R}$-vector space with basis $\{e_1, \dots, e_n\}$ and dual basis $\{\varepsilon^1, \dots, \varepsilon^n\}$. For $\varepsilon^I \in \bigwedge^{|I|} V^*, \varepsilon^J \in \bigwedge^{|J|} V^*$, $\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}$ where IJ is the concatenation of the multiindices I and J.

We can now deduce some properties of the exterior product.

Proposition 23.7. Let V be a finite-dimensional vector space. Let $\omega, \omega', \xi, \xi' \in \bigwedge^\bullet V^*$ be forms of fixed degree. Then:

(i) $(a \cdot \omega + a' \cdot \omega') \wedge \xi = a \cdot \omega \wedge \xi + a' \cdot \omega' \wedge \xi$.
(ii) $\omega \wedge (\omega' \wedge \xi) = (\omega \wedge \omega') \wedge \xi$.
(iii) $\omega \wedge \xi = (-1)^{|\omega| \cdot |\xi|} \xi \wedge \omega$.
(iv) For a multiindex $I = (i_1, \dots, i_k)$ and a basis $\{\varepsilon^1, \dots, \varepsilon^n\}$ of V^*,
$$\varepsilon^I = \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}.$$
(v) Given $\omega^1, \dots, \omega^k \in V^*$, $v_1, \dots, v_k \in V$, we have
$$(\omega^1 \wedge \cdots \wedge \omega^k)(v_1, \dots, v_k) = \det\left(\begin{bmatrix} \omega^1(v_1) & \dots & \omega^1(v_k) \\ \vdots & \ddots & \vdots \\ \omega^k(v_1) & \dots & \omega^k(v_k) \end{bmatrix}\right).$$

Proof of (i). The exterior product is defined using linear operations Definition 23.5 hence is bilinear. ■

Proof of (ii). Fixing a dual basis of V, this is clear from Lemma 23.6. ■

Proof of (iii). Once again after a choice of basis, this is clear from Lemma 23.6 and Lemma 23.2 (ii). ■

Proof of (iv). In the case $k = 2$, this is immediate from Lemma 23.6, and the general case follows by induction. ■

Proof of (v). If ω^i are basis elements, then this follows from (d) and the construction of ε^I, from which the general case can be obtained by extending multilinearly. ■

This construction globalizes to differential forms using the omnibus linear algebra theorem Theorem 20.1.

Definition 23.8 (Algebra of Forms)**.** Let M be a smooth manifold. The exterior algebra of forms on M is given by $\Omega^\bullet(M) = \bigoplus_{k \geq 0} \bigwedge^k \Omega^1(M)$ with operation $- \wedge -$.

Let us look at a few examples.

Example 23.9. Let $M = \mathbb{R}$, $\Omega^0(\mathbb{R}) = C^\infty(\mathbb{R})$, and $\Omega^1(M)$ consists of expressions of the form $f(x) \cdot dx$ for $f \in C^\infty(M)$.

Example 23.10. Let $M = \mathbb{R}^3$. $\Omega^2(M)$ is the vector space of elements of the form $f_{12}(x) \cdot dx_1 \wedge dx_2 + f_{13}(x) \cdot dx_1 \wedge dx_3 + f_{23}(x) \cdot dx_2 \wedge dx_3$.

This construction is in fact functorial and compatible with pullback.

Proposition 23.11. Let $F : M \to N$ be a smooth map between smooth manifolds and $F^* : \Omega^k(N) \to \Omega^k(M)$ the pullback of k-forms. F^* has the following properties:

(i) F^* is linear.

(ii) $F^*(\omega \wedge \omega') = F^*(\omega) \wedge F^*(\omega')$.

(iii) If $y_1, \dots, y_n$ are local coordinates on N,
$$F^*\left(\sum_I \omega_I \cdot dy_1 \wedge \dots \wedge dy_n\right) = \sum_I (\omega_I \circ F) \cdot dF^{i_1} \wedge \dots \wedge dF^{i_k}.$$

(iv) If $\dim(M) = \dim(N)$ and $x_1, \dots, x_m$ and $y_1, \dots, y_m$ coordinates on M, N, respectively, then
$$F^*(v \cdot dy_1 \wedge \dots \wedge dy_m) = \det(dF) \cdot (v \circ F) \cdot dx_1 \wedge \dots \wedge dx_m.$$

Proof of (i) and (ii). This is immediate from the omnibus linear algebra theorem Theorem 20.1. ■

Proof of (iii). This holds for 1-forms by (ii) and Lemma 22.3. ■

Proof of (iv). This is immediate from (iii) and noting that
$$(dF^1 \wedge \dots \wedge dF^n)(\partial_{x_1}, \dots, \partial_{x_n}) = \det(dF).$$
■

Remark 23.12. The generalised operator d produces a chain complex
$$C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots$$
whose cohomology, surprisingly, computes $\mathbb{R}$-singular cohomology of M. This is the content of the de Rham comparison theorem.

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