WORKSHOP ON DUALIZABLE CATEGORIES AND CONTINUOUS K-THEORY
MAX PLANCK INSTITUTE FOR MATHEMATICS, BONN – SEPTEMBER 2024

WERN JUIN GABRIEL ONG

Preliminaries

This document contains notes from the 2024 Workshop on Dualizable Categories and Continuous K-Theory held at the Max Planck Institute for Mathematics in Bonn in September 2024. Short courses were given by A. Efimov (HUJI) and A. Matthew (Chicago). These notes are LaTeX-ed after the fact with significant alteration and are subject to misinterpretation and mistranscription. Use with caution. Any errors are undoubtedly my own and any virtues ought to be attributed to the speakers and not the typist. Note that the presentation here differs from the order of the lectures, and these notes should probably be read in said order which can be found on the conference page alongside the recordings [DC24].

Contents

A. Efimov – Dualizable Categories and Localizing Motives

1. Lecture I (10th September)

We show that dualizability is equivalent to flatness. We work here over the absolute base but the same proof applies over any rigid category.

Recall the following definition.

Definition 1.1 (Flat)**.** Let $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$. $\mathcal{C}$ is flat if for all fully faithful continuous functors $\mathcal{B} \to \mathcal{E}$, $\mathcal{B} \otimes \mathcal{C} \to \mathcal{E} \otimes \mathcal{B}$ is fully faithful.

We introduce some language necessary to show the desired result.

Definition 1.2 ($\mathsf{Pr}^{\mathsf{acc}}_{\mathsf{St}}$)**.** $\mathsf{Pr}^{\mathsf{acc}}_{\mathsf{St}}$ is the ∞-categyr with objects presentable stable ∞-categories and exact accessible functors.

Remark 1.3. There is a non-full inclusion $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}} \hookrightarrow \mathsf{Pr}^{\mathsf{L}}_{\mathsf{acc}}$.

Definition 1.4 (Oplax 2-Functor)**.** Let $F : \mathcal{B} \to \mathcal{E}$ be an accessible exact functor in $\mathsf{Pr}^{\mathsf{acc}}_{\mathsf{St}}$ and $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$. An oplax 2-functor $\mathcal{C} \otimes \mathcal{B} \to \mathcal{C} \otimes \mathcal{E}$ is the given by the composition

$$\mathrm{Fun}^{\kappa-\mathsf{lex}}((\mathcal{C}^{\kappa})^{\mathsf{Opp}}, \mathcal{B}) \hookrightarrow \mathrm{Fun}((\mathcal{C}^{\kappa})^{\mathsf{Opp}}, \mathcal{B}) \xrightarrow{F\circ -} \mathrm{Fun}((\mathcal{C}^{\kappa})^{\mathsf{Opp}}, \mathcal{E}) \xrightarrow{i^L} \mathrm{Fun}^{\kappa-\mathsf{lex}}((\mathcal{C}^{\kappa})^{\mathsf{Opp}}, \mathcal{E})$$

with the identifications $\mathcal{C} \otimes \mathcal{B} = \mathrm{Fun}^{\kappa-\mathsf{lex}}((\mathcal{C}^{\kappa})^{\mathsf{Opp}}, \mathcal{B}), \mathcal{C} \otimes \mathcal{E} = \mathrm{Fun}^{\kappa-\mathsf{lex}}((\mathcal{C}^{\kappa})^{\mathsf{Opp}}, \mathcal{E})$ and i^L the left adjoint to the inclusion $i : \mathrm{Fun}^{\kappa-\mathsf{lex}}((\mathcal{C}^{\kappa})^{\mathsf{Opp}}, \mathcal{E}) \to \mathrm{Fun}((\mathcal{C}^{\kappa})^{\mathsf{Opp}}, \mathcal{E})$

Remark 1.5. We can think of an oplax structure on a 2-functor as the data of a comparison map $\mathcal{C} \otimes (F \circ G) \to (\mathcal{C} \otimes F) \circ (\mathcal{C} \otimes G)$.

Definition 1.6 (Category of Correspondences)**.** The category of correspondences $\mathsf{Corr}(\mathcal{B}, \mathcal{E})$ is the ∞-category of triples (T, i_1, i_2) where T is a presentable stable ∞-category $i_1 : \mathcal{B} \to T, i_2 : \mathcal{E} \to T$ fully faithful and continuous, and T admits a semiorthogonal decomposition $\langle i_2(\mathcal{E}), i_1(\mathcal{B})\rangle$.

We can define compositions of correspondences as follows. Suppose $T_{12} \in \mathsf{Corr}(\mathcal{B}_1, \mathcal{B}_2)$ and $T_{2,3} \in \mathsf{Corr}(\mathcal{B}_2, \mathcal{B}_3)$ we can define T_{123} as the pushout $T_{12} \cup_{\mathcal{B}_2} T_{23}$ in $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$ which has a semiorthogonal decomposition $\langle \mathcal{B}_3, \mathcal{B}_2, \mathcal{B}_1\rangle$ and take T_{13} as the subcategory generated by the images of $\mathcal{B}_1, \mathcal{B}_3$ and as such $T_{13} \in \mathsf{Corr}(\mathcal{B}_1, \mathcal{B}_3)$. In fact we can show the following.

Proposition 1.7. There is an equivalence of ∞-categories $\mathrm{Fun}^{\mathsf{acc}}(\mathcal{B}, \mathcal{E}) \to \mathsf{Corr}(\mathcal{B}, \mathcal{E})$ by $(T, i_1, i_2) \mapsto (i_2^R i_1 : \mathcal{D} \to \mathcal{E})$ and $(F : \mathcal{B} \to \mathcal{E}) \mapsto \mathcal{E} \uplus_F \mathcal{B}$ the semiorthogonal gluing of $\mathcal{E}, \mathcal{B}$ along F.

We will use the following proposition to prove the nontrivial direction of the equivalence of flatness and dualizability.

Proposition 1.8. Let $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$. There is a natural oplax 2-functor $\mathcal{C} \otimes - : \mathsf{Pr}^{\mathsf{acc}}_{\mathsf{St}} \to \mathsf{Pr}^{\mathsf{acc}}_{\mathsf{St}}$ inducing the Lurie tensor product on restriction to $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$. Furthermore, if $\mathcal{C}$ is flat, then $\mathcal{C} \otimes - : \mathsf{Pr}^{\mathsf{acc}}_{\mathsf{St}} \to \mathsf{Pr}^{\mathsf{acc}}_{\mathsf{St}}$ is a 2-functor.

Proof Outline. We can define a oplax 2-endofunctor on correspondences on objects by the usual tensor product and on morphisms $(T, i_1, i_2) \mapsto (\mathcal{C} \otimes T, \mathcal{C} \otimes i_1, \mathcal{C} \otimes i_2)$ and we have $\mathcal{C} \otimes T_{123} = (\mathcal{C} \otimes T_{12}) \bigcup_{\mathcal{C} \otimes \mathcal{B}_2} (\mathcal{C} \otimes T_{23})$ by commutativity of the tensor product with all colimits and thus with pushouts in $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$. This object is the target of a canonical map from $\mathcal{C} \otimes T_{13}$ which is not fully faithful. As such, there is a map $\mathcal{C} \otimes T_{13} \to (\mathcal{C} \otimes T_{23}) \circ (\mathcal{C} \otimes T_{23})$. The precise argument uses the theory of complete 2-Segal spaces. If $\mathcal{C}$ is flat, then the functor $\mathcal{C} \otimes T_{13} \to \mathcal{C} \otimes T_{123}$ is an actual 2-functor. Then using the equivalence between the 2-category of correspondences and the 2-category of accessible functors, we get the claim. ■

Theorem 1.9. Let $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$. The following are equivalent:

(a) $\mathcal{C}$ is flat.
(b) $\mathcal{C}$ is dualizable.

Proof. (b)⇒(a) If $\mathcal{C}$ is dualizable then $\mathcal{C} \otimes \mathcal{B} = \mathrm{Fun}^{\mathsf{L}}(\mathcal{C}^\vee, \mathcal{B})$ which preserves fully faithful functors.

(a)⇒(b) We will verify Grothendieck's AB6 axiom which is equivalent to dualizability per Theorem 7.14. Consider a family of posets J_i over an indexing set $i \in I$. Let $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$ and consider a commutative square

$$\begin{tikzcd}
\mathrm{Fun}\left(\prod_{i\in I} J_i, \mathcal{C}\right) \arrow[r, "\mathcal{U}^{\mathrm{colim}}"] \arrow[d, "\prod_{i\in I} F"'] & \mathcal{C} \arrow[d, "\delta"] \\
\prod_{i\in I} \mathrm{Fun}(J_i, \mathcal{C}) \arrow[r, "\prod_{i\in I} \mathrm{colim}"'] & \prod_{i\in I} \mathcal{C}
\end{tikzcd}$$

where F is a left Kan extension. The AB6 axiom is equivalent to the fact that composition with the right adjoints of the vertical maps give isomorphisms of functors $\prod_{i\in I} \mathrm{Fun}(J_i, \mathcal{C}) \to \mathcal{C}$. We can write this as the tensor product of $\mathcal{C}$ with a diagram of spectra, and Sp being a dualizable category, AB6 holds so the Beck-Chevalley condition of the composition with right adjoints of the vertical functors. Then assuming flatness of $\mathcal{C}$, we can apply $\mathcal{C} \otimes -$ to show that it holds for $\mathcal{C}$ showing $\mathcal{C}$ is dualizable. ■

Remark 1.10. The same works over a rigid $\mathbb{E}_1$-(symmetric) monoidal base $\mathcal{E}$.

We now want to consider some examples of non-dualizable categories. In particular, a natural question arises if dualizable categories are closed under extensions (of compactly generated categories). Consider, for example, a short exact sequence of categories

$$0 \longrightarrow \mathsf{Ind}(\mathcal{A}) \xrightarrow{F} \mathcal{C} \xrightarrow{G} \mathsf{Ind}(\mathcal{B}) \longrightarrow 0$$

with F, G strongly continuous. Is it true that $\mathcal{C}$ is dualizable?

Note in the situation above $\mathcal{C}$ has a semiorthogonal decomposition $\langle G^R(\mathsf{Ind}(\mathcal{B})), F(\mathsf{Ind}(\mathcal{A}))\rangle$ so this corresponds to the composition $G^{\mathsf{RR}} \circ F : \mathsf{Ind}(\mathcal{A}) \to \mathsf{Ind}(\mathcal{B})$ where G^{RR} is the twice right adjoint functor to G. Now observe that $\mathrm{Fun}^{\mathsf{acc,ex}}(\mathsf{Ind}(\mathcal{A}), \mathsf{Ind}(\mathcal{B}))$ is equivalent to $\mathrm{Fun}^{\mathsf{ex}}(\mathcal{B}, \mathsf{Pro}(\mathsf{Ind}(\mathcal{A})))$. We have the following proposition.

Proposition 1.11. Let

$$0 \longrightarrow \mathsf{Ind}(\mathcal{A}) \xrightarrow{F} \mathcal{C} \xrightarrow{G} \mathsf{Ind}(\mathcal{B}) \longrightarrow 0$$

be a short exact sequence of categories and $G^{\mathsf{RR}} \circ F : \mathsf{Ind}(\mathcal{A}) \to \mathsf{Ind}(\mathcal{B})$ the composition of the twice right adjoint of G with F. The following are equivalent:

(a) $\mathcal{C}$ is dualizable.
(b) $\mathcal{C}$ is compactly generated.
(c) The image of $\mathcal{B}$ lies in the category of Tate objects of $\mathsf{Pro}(\mathsf{Ind}(\mathcal{A}))$ – the idempotent complete subcategory generated by $\mathsf{Pro}(\mathcal{A}), \mathsf{Ind}(\mathcal{A})$ in $\mathsf{Pro}(\mathsf{Ind}(\mathcal{A}))$.

Proof. See [CD+23, Rmk. A.3.13]. ■

This description leads to a few examples of non-dualizable categories obtained by extensions as above.

Proposition 1.12. Let k be a field and $\mathcal{C}$ the category of triples (V, W, φ) are dg $\mathbb{C}$-vector spaces and $\varphi : \bigoplus_{\mathbb{N}} V \to \bigoplus_{\mathbb{N}}(W)$. Then $\mathcal{C}$ is non-dualizable.

Proof Outline. We need to show that $V \mapsto \prod_{\mathbb{N}} \bigoplus_{\mathbb{N}} V$ is not contained in the Tate objects of $\mathsf{Perf}(k)$ in the sense of Proposition 1.11 (c). Applying Grothendieck's AB6, we have

$$\prod_{\mathbb{N}} \bigoplus_{\mathbb{N}} V = \lim_{\{f:\mathbb{N}\to\mathbb{N}\}} \prod_{i\in\mathbb{N}} \bigoplus_{j=0}^{f(i)-1} V = \lim_{\{f:\mathbb{N}\to\mathbb{N}\}} \mathrm{Hom}(X_f, V)$$

where $X_f = \prod_{i\in\mathbb{N}} k^{f(i)}$. We want to show that $(\overline{X_f})_f$ is not an essentially constant system in Calk_k of k-vector spaces. For $f \leq g$ the map $X_g \to X_f$ is split surjective and admits a section. If $(\overline{X_f})_f$ is essentially constant, then it is eventually constant since the maps are split surjections. This fails since the fiber of $X_{f+1} \to X_f$ is a vector space of countable dimension which is nonzero in the Calkin category. ■

For R a commutative Noetherian ring, a result of Neeman describes all the localizing subcategories of $\mathcal{D}(R)$ as in bijection with subsets of $\mathrm{Spec}(R)$. If $S \subseteq \mathrm{Spec}(R)$ then S corresponds to $\mathcal{D}_S(R)$ which is a localizing subcategory generated by residue fields $\langle R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} : \mathfrak{p} \in S\rangle$.

This leads to the following.

Theorem 1.13. Let R be a commutative Noetherian ring and $S \subseteq \mathrm{Spec}(R)$. The following are equivalent:

(a) $\mathcal{D}_S(R)$ is dualizable.
(b) $\mathcal{D}_S(R)$ is compactly generated.
(c) S is convex – for a short exact sequence $X \to Y \to Z$ with $X, Z \in S$ then $Y \in S$.

2. Lecture II (11th September)

We turn to a discussion of localizing invariants of sheaves. In fact, the general case follows from a statement about sheaves on the real line.

Definition 2.1 (Sheaves with Non-Negative Singular Support)**.** Let $\mathcal{F} \in \mathsf{Sh}(\mathbb{R}, \mathsf{Sp})$. $\mathcal{F}$ has non-negative singular support if for any $a < b$ $\mathcal{F}((-\infty, b)) \to \mathcal{F}((a, b))$ is an isomorphism.

Remark 2.2. Equivalently, the singular support of $\mathcal{F}$ has singular support contained in the non-negative part of the cotangent bundle.

We can then define the following category of non-negative sheaves.

Definition 2.3 (Non-Negative Sheaves)**.** The category of non-negative sheaves on $\mathbb{R}$ is the subcategory $\mathsf{Sh}_{\geq 0}(\mathbb{R}, \mathsf{Sp}) \subseteq \mathsf{Sh}(\mathbb{R}, \mathsf{Sp})$ such with non-negative singular support.

To be able to compute localizing invariants, we want to form a resolution by compactly generated categories which is given by the following proposition.

Proposition 2.4. There is a short exact sequence in $\mathsf{Cat}^{\mathsf{dual}}$:

$$0 \longrightarrow \mathsf{Sh}_{\geq 0}(\mathbb{R}, \mathsf{Sp}) \longrightarrow \mathrm{Fun}(\mathbb{Q}_{\leq}^{\mathsf{Opp}}, \mathsf{Sp}) \longrightarrow \textstyle\prod_{\mathbb{Q}} \mathsf{Sp} \longrightarrow 0 \tag{2.1}$$

Proof Outline. The essential content of the proof is that any sheaf $\mathcal{F} \in \mathsf{Sh}_{\geq 0}(\mathbb{R}, \mathsf{Sp})$ is determined by its value on rays $(-\infty, a)$ for $a \in \mathbb{Q}$ and the sheaf condition reduces to $\mathcal{F}((-\infty, a)) = \lim_{b<a} \mathcal{F}((-\infty, b))$ by Definition 8.9 (iii). Denoting the functor $\varphi : \mathrm{Fun}(\mathbb{Q}_{\leq}^{\mathsf{Opp}}, \mathsf{Sp}) \to \prod_{\mathbb{Q}} \mathsf{Sp}$, we have $\varphi(\mathcal{F})_a = \mathrm{cone}(\mathrm{colim}_{b>a} \mathcal{F}(b) \to \mathcal{F}(a))$. Taking $\mathbb{Q}$ to be a dense linearly ordered set in $\mathbb{R}$, φ is a localization of categories. φ admits a right adjoint φ^R defined by $\varphi^R((X_a)_{a\in\mathbb{Q}})(b) = X_b$ with transition maps given by zero maps. As such $\varphi \circ \varphi^R = \mathrm{id}_{\mathrm{Fun}(\mathbb{Q}_{\leq}^{\mathsf{Opp}}, \mathsf{Sp})}$. We can now observe that $\mathsf{Sh}_{\geq 0}(\mathbb{R}, \mathsf{Sp}) = \varphi^R(\prod_{\mathbb{Q}} \mathsf{Sp})^{\perp}$ the right orthogonal of the essential image of the right adjoint, but this agrees with the left orthogonal since the inclusion of the category has both left and right adjoints giving an equivalence with the left orthogonal which is precisely the kernel of φ. ■

Remark 2.5. φ can be thought of as a variant of extension of scalars in the setting of almost mathematics. D. Vaintrob gives an interpretation of the middle term $\mathrm{Fun}(\mathbb{Q}_{\leq}^{\mathsf{Opp}}, \mathsf{Sp})$ as quasicoherent sheaves on the infinite root stack and the functor φ as a pullback to the infinite root stack.

Observe that the latter two terms of the exact sequence (2.1) are compactly generated. Denoting

$$\mathcal{A} = \mathrm{Fun}(\mathbb{Q}_{\leq}^{\mathsf{Opp}}, \mathsf{Sp})^{\omega} \text{ and } \mathcal{B} = \left(\prod_{\mathbb{Q}} \mathsf{Sp}\right)^{\omega} = \bigoplus_{\mathbb{Q}} \mathsf{Sp}^{\omega}, \tag{2.2}$$

we can show the functor $\varphi^{\omega} : \mathcal{A} \to \mathcal{B}$ is a K-equivalence.

Definition 2.6 (K-Equivalence)**.** Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a functor. f is a K-equivalence if there exists $\psi : \mathcal{B} \to \mathcal{A}$ such that $[\varphi \circ \psi] \cong [\mathrm{id}_{\mathcal{B}}] \in K_0(\mathrm{Fun}(\mathcal{B}, \mathcal{B}))$ and $[\psi \circ \varphi] \cong [\mathrm{id}_{\mathcal{A}}] \in K_0(\mathrm{Fun}(\mathcal{A}, \mathcal{A}))$.

To show the desired K-equivalence statement, we will further require the following lemma concerning completed semiorthogonal decompositions. This lemma will also be pertinent to forthcoming discussions of the K-theory of nuclear modules.

Lemma 2.7. Let $\mathcal{C} \in \mathsf{Cat}^{\mathsf{Perf}}$ be a small category with infinite semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{C}_0, \mathcal{C}_1, \dots \rangle$ and $\mathcal{B}_n = \langle \mathcal{C}_0, \dots, \mathcal{C}_n \rangle \subseteq \mathcal{C}$. Then there are isomorphisms

$$K_0(\lim_n \mathcal{B}_n) \cong \prod_n K_0(\mathcal{C}_n) \cong \lim_n K_0(\mathcal{B}_n).$$

In other words, the formation of K_0 commutes with this especially nice limit.

Proof. There is a natural map $\lim_n \mathcal{B}_n \to \prod_{\mathbb{N}} \mathcal{C}_n$. Denoting objects of $\mathcal{B}_n$ by tuples $(x_0, x_1, \dots, x_n)$ with $x_i \in \mathcal{C}_i$ and $\mathcal{E} = \lim_n \mathcal{B}_n$ there is a functor $F : \mathcal{E} \to \mathcal{C}_n$ by $(x_0, x_1, \dots) \mapsto x_n$ and a functor $G : \prod_{\mathbb{N}} \mathcal{C}_n \to \mathcal{E}$ by $(x_n)_{n \in \mathbb{N}} \mapsto (0, \dots, 0, x_n, 0, \dots, 0)$ where all the transition maps are zero maps. $F \circ G \cong \mathrm{id}_{\mathcal{C}_n}$ and we want to show $G \circ F \cong \mathrm{id}_{\mathcal{E}}$. We define endofunctors $\Phi_n : \mathcal{E} \to \mathcal{E}$ by $\Phi_n(x) = (0, \dots, x_n, x_{n+1}, \dots)$ with the transition maps as in $\mathcal{E}$. This yields an exact triangle in $\mathrm{Fun}(\mathcal{E}, \mathcal{E})$

$$\textstyle\bigoplus_{n \geq 1} \Phi_n \longrightarrow \bigoplus_{n \geq 0} \Phi_n \longrightarrow G \circ F$$

but then employing an Eilenberg Swindle-type argument, we can further observe that in K-groups

$$[G \circ F] \cong \left[\bigoplus_{n \geq 0} \Phi_n\right] - \left[\bigoplus_{n \geq 1} \Phi_n\right] \cong [\Phi_0] \cong [\mathrm{id}_{\mathcal{E}}].$$

■

The result is as follows.

Proposition 2.8. Let $\mathcal{A}, \mathcal{B}$ be as in (2.2). The induced functor on compact objects $\varphi^\omega : \mathcal{A} \to \mathcal{B}$ K-equivalence.

Proof. Let $\psi : \mathcal{B} \to \mathcal{A}$ given by h_a for $a \in \mathbb{Q}$ where h_a are representable presheaves

$$h_a(b) \begin{cases} \mathbb{S} & b \leq a \\ 0 & b > a \end{cases}$$

and $\mathbb{S}$ is the sphere spectrum. This is a section of φ^ω so $\varphi \circ \psi \cong \mathrm{id}_{\mathcal{A}}$. Conversely consider a bijection $\mathbb{N} \to \mathbb{Q}$ and let $\mathcal{A}_n \subseteq \mathcal{A}$ be sequences of representable presheaves h_{a_n} as above so we have that $\mathrm{Fun}(\mathcal{A}, \mathcal{A}) \cong \lim_n \mathrm{Fun}(\mathcal{A}_n, \mathcal{A})$ but the restriction functors $\mathrm{Fun}(\mathcal{A}_{n+1}, \mathcal{A}) \to \mathrm{Fun}(\mathcal{A}_n, \mathcal{A})$ have a fully faithful left and right adjoints. As such, from Lemma 2.7 we have that $K_0(\mathrm{Fun}(\mathcal{A}, \mathcal{A})) \cong \lim_n K_0(\mathrm{Fun}(\mathcal{A}_n, \mathcal{A})) = \mathrm{End}_{\mathbb{Z}}(\bigoplus_{\mathbb{Q}} \mathbb{Z})$ since $K_0(\mathbb{S}) = \mathbb{Z}$. As such $[\psi \circ \varphi] = \mathrm{id}_{\mathcal{A}}$. ■

These results are sufficient to show that the image of non-negative sheaves on the real line under a localizing invariant is always trivial.

Theorem 2.9. For all localizing invariants $F : \mathsf{Cat}^{\mathsf{Perf}} \to \mathcal{E}$, $F^{\mathsf{cont}}(\mathsf{Sh}_{\geq 0}(\mathbb{R}, \mathsf{Sp})) = 0$.

Remark 2.10. It is quite surprising that Theorem 2.9 can be proved for arbitrary localizing invariants, without the additional hypothesis that the functor commutes with filtered colimits.

Theorem 2.11. Let $F : \mathsf{Cat}^{\mathsf{Perf}} \to \mathcal{E}$ be a localizing invariant with $\mathcal{E}$ accessible and X a finite CW complex. If $\mathcal{C} \in \mathsf{Cat}^{\mathsf{dual}}_{\mathsf{St}}$ is a dualizable category, then $F^{\mathsf{cont}}(\mathcal{C}) = F^{\mathsf{cont}}(\mathsf{Sh}(X, \mathcal{C}))$.

As a corollary, we have the following as a special case of the above.

Corollary 2.12. Let R be an $\mathbb{E}_1$-ring and X a finite CW complex. Then

$$K_0^{\mathsf{cont}}(\mathsf{Sh}(X, \mathsf{Mod}_R)) = [X, \Omega^\infty K(R)].$$

Using the above, we can also show that K-theory commutes with infinite products.

Turning to the category of sheaves on a locally compact Hausdorff space, we have the following.

Theorem 2.13. Let $F, G : \mathsf{Cat}^{\mathsf{Perf}} \to \mathcal{E}$ be localizing invariants with $\mathcal{E}$ having a non-degenerate t-structure and $\varphi : F \to G$ be a map in $\mathrm{Fun}(\mathsf{Cat}^{\mathsf{Perf}}, \mathcal{E})$. If φ is an isomorphism on π_0, then φ is an isomorphism in $\mathrm{Fun}(\mathsf{Cat}^{\mathsf{Perf}}, \mathcal{E})$.

Proof Outline. Noting that $\pi_n F(\mathcal{C}) \cong \pi_{n+1} F(\mathsf{Calk}_{\omega_1}(\mathcal{C}))$ and similarly for G, it follows by induction that $\pi_n \varphi$ is an isomorphism from $\pi_n F \to \pi_n G$ for $n \geq 0$. The same proof applies in the continuous setting for $n \leq -1$.

For $\mathcal{C} \in \mathsf{Cat}^{\mathsf{dual}}_{\mathsf{St}}$ there is a short exact sequence

$$0 \longrightarrow \mathsf{Sh}_{>0}(\mathbb{R}, \mathsf{Sp}) \longrightarrow \mathsf{Sh}_{\geq 0}(\mathbb{R}, \mathsf{Sp}) \longrightarrow \mathcal{C} \xrightarrow{\Gamma_c} 0$$

inducing an isomorphism $\pi_n F^{\mathsf{cont}}(\mathcal{C}) \cong \pi_{n-1}(\mathsf{Sh}_{>0}(\mathbb{R}, \mathsf{Sp}))$ by the vanishing of the middle term by Theorem 2.9 so by induction $\pi_n \varphi$ is an isomorphism for all n. ■

Corollary 2.14. The K-theory functor $K : \mathsf{Cat}^{\mathsf{Perf}} \to \mathsf{Sp}$ commutes with products.

Proof. Let $\mathcal{C} \in \mathsf{Cat}^{\mathsf{Perf}}$ be a small category and a set I, the map $K(\prod_I \mathcal{C}) \to \prod_I K(\mathcal{C})$ can be considered as a morphism of localizing invariants which induces an isomorphism on K_0 so the result follows from Theorem 2.13. ■

The following statement gives a variant of homotopy invariance.

Theorem 2.15. Let X be a finite CW complex. If $F : \mathsf{Cat}^{\mathsf{Perf}} \to \mathcal{E}$ is a localizing invariant and $\mathcal{C} \in \mathsf{Cat}^{\mathsf{dual}}_{\mathsf{St}}$ then $F^{\mathsf{cont}}(\mathsf{Sh}(X, \mathcal{C})) \cong F^{\mathsf{cont}}(\mathcal{C})^X = \Gamma(X, F^{\mathsf{cont}}(\mathcal{C}))$.

3. Lecture III (12th September)

4. Lecture IV (13th September)

5. Lecture V (13th September)

A. Matthew – Dualizable Categories and their K-Theory

6. Lecture I (9th September)

We refer to Lurie's foundational texts [HTT] and [HA] for background on ∞-categories and higher algebra.

Recall that the ∞-categorical analogue of an Abelian category is a stable ∞-category. Grothendieck groups can be defined for stable ∞-categories analogously to how Grothendieck groups are defined for Abelian categories.

Definition 6.1 (Grothendieck Group)**.** Let $\mathcal{C}$ be a stable ∞-category. The Grothendieck group $\mathsf{k}_0(\mathcal{C})$ of $\mathcal{C}$ is defined to be the quotient

$$\mathsf{k}_0(\mathcal{C}) = \frac{\mathbb{Z}\left[\{[X] : X \in \mathcal{C}\}\right]}{\sim}$$

where $[X] \sim [X'] + [X'']$ if there exists a cofiber sequence $X' \to X \to X''$ in $\mathcal{C}$.

Remark 6.2. The construction of the Grothendieck group depends only on the underlying homotopy category $\mathsf{h}\mathcal{C}$. The homotopy category $\mathsf{h}\mathcal{C}$ is in fact triangulated by [HA, Thm. 1.1.2.4].

A more contemporary interpretation of algebraic K-theory following work of Quillen, Waldhausen, and others, is to define K-theory as a spectrum $K(\mathcal{C})$ such that $\pi_0(K(\mathcal{C})) = \mathsf{k}_0(\mathcal{C})$. In some sense, this is the "right homotopical enhancement" of $\mathsf{k}_0(-)$. Formalizing how this is the "right homotopical enhancement" will require a discussion of localizing invariants.

Recall the following definitions.

Definition 6.3 (Idempotent Complete)**.** Let $\mathcal{C}$ be an ∞-category. $\mathcal{C}$ is idempotent complete if its image under the Yoneda embedding $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathsf{Opp}}, \mathsf{Grpd}_\infty)$ is closed under retracts.

Definition 6.4 (Exact Functor)**.** Let $f : \mathcal{E} \to \mathcal{C}$ be a functor between ∞-categories. f is an exact functor if f preserves finite colimits.

As such, we make the following definitions.

Definition 6.5 ($\mathsf{Cat}^{\mathsf{Perf}}$)**.** $\mathsf{Cat}^{\mathsf{Perf}}$ is the ∞-category of small idempotent-complete ∞-categories and exact functors.

Remark 6.6. Note that exact functors preserve finite colimits.

To define localizing invariants, we need to discuss Karoubi quotients, which are the analogue of Verdier quotients in $\mathsf{Cat}^{\mathsf{Perf}}$.

Definition 6.7 (Karoubi Quotient)**.** Let $\mathcal{C} \in \mathsf{Cat}^{\mathsf{Perf}}$ and $\mathcal{B} \subseteq \mathcal{C}$ an inclusion in $\mathsf{Cat}^{\mathsf{Perf}}$. The Karoubi quotient $\mathcal{C}/\mathcal{B}$ is defined to be the pushout in $\mathsf{Cat}^{\mathsf{Perf}}$

$$\begin{tikzcd} \mathcal{B} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ 0 \arrow[r] & 0 \bigcup_{\mathcal{B}} \mathcal{C} = \mathcal{C}/\mathcal{B}. \end{tikzcd} \tag{6.1}$$

Remark 6.8. The pushout square (6.1) is also a pullback square.

Informally, we can think of the Karoubi quotient using the quotient functor $p : \mathcal{C} \to \mathcal{C}/\mathcal{B}$ where given $x, y \in \mathcal{C}$ the hom-objects in the quotient can be computed by

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{B}}(p(x), p(y)) = \mathrm{colim}\,\mathrm{Hom}_{\mathcal{C}}(x, y')$$

where $y \to y'$ is a morphism in $\mathcal{C}$ such that its cofiber $\mathrm{cofib}(y \to y') \in \mathcal{B}$.

Remark 6.9. The quotient functor $p : \mathcal{C} \to \mathcal{C}/\mathcal{B}$ is not essentially surjective but it is up to retracts/idempotent completion. As a consequence of a result of Thomason, a class $Z \in \mathcal{C}/\mathcal{B}$ lifts if and only if the corresponding class in the Grothendieck group lifts. As such, $Z \oplus Z[1]$ always lifts.

Example 6.10. Let X be a quasicompact and quasiseparated scheme and $U \subseteq X$ a quasicompact open. By arguments of Thomason-Trabough there is a Karoubi sequence

$$\mathsf{Perf}(X \setminus U) \longrightarrow \mathsf{Perf}(X) \longrightarrow \mathsf{Perf}(U).$$

Another important example of Karoubi quotients is that of the Calkin category which we define as follows.

Definition 6.11 (Calkin Category)**.** Let $\mathcal{A} \in \mathsf{Cat}^{\mathsf{Perf}}$ and κ a regular cardinal. The Calkin category $\mathsf{Calk}_\kappa(\mathcal{A})$ is defined to be the Karoubi quotient $\mathsf{Ind}(\mathcal{A})^\kappa/\mathcal{A}$ of the κ-compact objects of $\mathsf{Ind}(\mathcal{A})$ by $\mathcal{A}$.

Remark 6.12. The most relevant case will be when $\kappa = \omega_1$ in which case the Calkin category will be the quotient of the subcategory of $\mathsf{Ind}(\mathcal{A})$ generated by sequential colimits modulo the constant colimits.

Denoting $\lim^\kappa, \mathrm{colim}^\kappa$ κ-filtered (co)limits, we can compute hom-objects in the Calkin category as the quotient

$$\mathrm{Hom}_{\mathsf{Calk}(\mathcal{A})}(\mathrm{colim}^\kappa_{i\in I} x_i, \mathrm{colim}^\kappa_{j\in J} y_j) = \frac{\lim_{i\in I}\mathrm{colim}_{j\in J}\mathrm{Hom}_{\mathcal{A}}(x_i, y_j)}{\mathrm{colim}_{j\in J}\lim_{i\in I}\mathrm{Hom}_{\mathcal{A}}(x_i, y_j)}.$$

Example 6.13. Let R be a ring. For $\mathcal{A} = \mathsf{Perf}(R)$, $\mathsf{Calk}(\mathcal{A}) = \mathcal{D}(R)/\mathsf{Perf}(R)$ and

$$\mathrm{Hom}_{\mathsf{Calk}(\mathcal{A})}(M, M') = \frac{\mathrm{Hom}_{\mathsf{Mod}_R}(M, M')}{\mathrm{Hom}_{\mathsf{Mod}_R}(M, R) \otimes_R M'}.$$

If $R = k$ a field and M, M' k-vector space then the morphisms in the Calkin category would be the quotient of all linear maps $M \to M'$ by the linear maps of finite rank.

Let us turn to semiorthogonal decompositions of ∞-categories.

Definition 6.14 (Semiorthogonal Decomposition)**.** Let $\mathcal{C}$ be a stable idempotent complete ∞-category. A semiorthogonal decomposition of $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ of $\mathcal{C}$ is the data of stable subcategories $\mathcal{C}_1, \mathcal{C}_2$ such that

(i) If $x_1 \in \mathcal{C}_1, x_2 \in \mathcal{C}_2$ then $\mathrm{Hom}_{\mathcal{C}}(x_2, x_1) = 0$.
(ii) For all $y \in \mathcal{C}$ there is a fiber sequence $y_2 \to y \to y_1$ with $y_1 \in \mathcal{C}_1, y_2 \in \mathcal{C}_2$.

Remark 6.15. For (ii) of Definition 6.14, y uniquely determines contractible choices of y_1, y_2 by (i). It is enough to require that $\mathcal{C}_1, \mathcal{C}_2$ generate $\mathcal{C}$ as a stable ∞-category and this stable generation property is in fact equivalent to (ii).

Given a stable idempotent complete ∞-category $\mathcal{C}$ with semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ there is a Karoubi sequence

$$\mathcal{C}_2 \hookrightarrow \mathcal{C} \longrightarrow \mathcal{C}_1$$

where the map $\mathcal{C} \to \mathcal{C}_1$ is given by $y \mapsto y_1$ as in Definition 6.14 (ii).

More generally, given a Karoubi sequence

$$\mathcal{B} \xrightarrow{i} \mathcal{C} \xrightarrow{p} \mathcal{C}/\mathcal{B} \tag{6.2}$$

and either i admits a right adjoint i^R or p admits a right adjoint p^R then $\mathcal{C} = \langle \ker(i^R), i(\mathcal{B}) \rangle$ or $\mathcal{C} = \langle p^R(\mathcal{C}/\mathcal{B}), i(\mathcal{B}) \rangle$, respectively.

Example 6.16. $\mathsf{Perf}(\mathbb{P}^1_k) = \langle \mathcal{O}_{\mathbb{P}^1_k}, \mathcal{O}_{\mathbb{P}^1_k}(1) \rangle$.

Revisiting Karoubi quotients, we would expect Karoubi quotients to remain so after ind-completion. Under ind-completion functors have right adjoints by the adjoint functor theorem and as such we would expect that for a Karoubi sequence as in (6.2) above, the category $\mathcal{C}$ will admit a semiorthogonal decomposition described using these adjoint functors. In particular, ind-completion induces a Karoubi sequence

$$\mathsf{Ind}(\mathcal{B}) \xrightarrow{i} \mathsf{Ind}(\mathcal{C}) \xrightarrow{p} \mathsf{Ind}(\mathcal{C}/\mathcal{B})$$

from which we conclude that $\mathcal{C}/\mathcal{B} = \ker(i^R : \mathsf{Ind}(\mathcal{C}) \to \mathcal{D})^\omega$, the compact objects of the kernel of the right adjoint of i.

Remark 6.17. There is an equivalence of ∞-categories $\mathsf{Ind}(\mathcal{C}) \to \mathrm{Fun}^{\mathsf{Ex}}(\mathcal{C}^{\mathsf{Opp}}, \mathsf{Sp})$ by $\mathrm{colim}^{\kappa}_{i\in I}\, x_i \mapsto \mathrm{colim}_{i\in I}\, \mathsf{y}_{x_i}$ with y_i the spectra-valued Yoneda functor $\mathrm{Hom}_{\mathcal{C}}(-, x_i)$ and the right adjoint given by restriction. This construction is analogous to the computation of colimits in Pr^{L} as limits on passage to right adjoints.

We can now define localizing invariants.

Definition 6.18 (Localizing Invariant)**.** Let $\mathcal{E}$ be a stable ∞-category. A localizing invariant is a functor $F : \mathsf{Cat}^{\mathsf{Perf}} \to \mathcal{E}$ such that:

(i) $F(0) = 0$.
(ii) If $\mathcal{B} \subseteq \mathcal{C}$ is an inclusion in $\mathsf{Cat}^{\mathsf{Perf}}$ then

$$\begin{tikzcd} F(\mathcal{B}) \arrow[r] \arrow[d] & F(\mathcal{C}) \arrow[d] \\ 0 \arrow[r] & F(\mathcal{C}/\mathcal{B}) \end{tikzcd}$$

is a pushout.

Remark 6.19. One usually requires $\mathcal{E}$ to be accessible and F to preserve κ-filtered colimits for some regular cardinal κ.

Localizing invariants give rise to additive invariants via semiorthogonal decompositions.

Example 6.20. Let $\mathcal{C} \in \mathsf{Cat}^{\mathsf{Perf}}$ with semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$. If F is a localizing invariant then there is an isomorphism $F(\mathcal{C}_1) \oplus F(\mathcal{C}_2) \to F(\mathcal{C})$. Functors satisfying this condition are known as additive invariants.

Example 6.21. Let $\mathcal{C} \in \mathsf{Cat}^{\mathsf{Perf}}$ and F a localizing invariant. Consider $\mathsf{Ind}(\mathcal{C})^{\omega_1}$ and $\mathsf{Calk}_{\omega_1}(\mathcal{C}) = \mathsf{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C}$. There is a fiber sequence

$$F(\mathcal{C}) \longrightarrow F(\mathsf{Ind}(\mathcal{C})^{\omega_1}) \longrightarrow F(\mathsf{Calk}_{\omega_1}(\mathcal{C})).$$

However, by the Eilenberg Swindle, the middle term vanishes so $F(\mathcal{C})$ is $\Omega F(\mathsf{Calk}_{\omega_1}(\mathcal{C}))$.

Returning to K-theory, K-theory can in fact be defined as a universal additive invariant.

Theorem 6.22 (Blumberg-Gepner-Tabuada, Barwick)**.** The K-theory functor $K : \mathsf{Cat}^{\mathsf{Perf}} \to \mathsf{Sp}$ is the initial localizing invariant with a natural map from the underlying groupoid $\mathcal{C}^{\simeq} \to \Omega^\infty K(\mathcal{C})$ for all $\mathcal{C} \in \mathsf{Cat}^{\mathsf{Perf}}$.

Other well-known examples of localizing invariants are topological Hochschild homology and topological cyclic homology.

Definition 6.23 (Localizing Motives)**.** MotLoc is the initial stable presentable ∞-category equipped with a localizing invariant $\mathcal{U}_{\mathsf{Loc}} : \mathsf{Cat}^{\mathsf{Perf}} \to \mathsf{MotLoc}$ that preserves filtered colimits.

In other words, MotLoc can be obtained by a quotient of the "free stable presentable ∞-category" $\mathrm{Fun}((\mathsf{Cat}^{\mathsf{Perf},\omega})^{\mathsf{Opp}}, \mathsf{Sp})$.

Theorem 6.24 (Blumberg-Gepner-Tabuada)**.** For all $\mathcal{C} \in \mathsf{Cat}^{\mathsf{Perf}}$,

$$\mathrm{Hom}_{\mathsf{MotLoc}}(\mathcal{U}_{\mathsf{Loc}}(\mathsf{Sp}^\omega), \mathcal{U}_{\mathsf{Loc}}(\mathcal{C})) = K(\mathcal{C}).$$

7. Lecture II (9th September)

We begin with some recollections on Pr^{L}. See [HTT, §5] for further exposition. Recall the following definition.

Definition 7.1 (Presentable Category)**.** An ∞-category $\mathcal{C}$ is presentable if $\mathcal{C}$ has all colimits and for some regular cardinal κ, the subcategory of κ-compact objects $\mathcal{C}^\kappa$ is small and $\mathcal{C} = \mathsf{Ind}_\kappa(\mathcal{C}^\kappa)$.

Remark 7.2. Recall that the κ-compact objects are those $x \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathsf{Grpd}_\infty$ that preserve κ-filtered colimits and $\mathcal{C} = \mathsf{Ind}_\kappa(\mathcal{C}^\kappa)$ states that $\mathcal{C}$ is κ-compactly generated.

Example 7.3. Let $\kappa = \omega_0$. If $\mathcal{C} = \mathsf{Ind}(\mathcal{C}^\omega)$ then $\mathcal{C}$ is a compactly generated category, the ind-completion of the subcategory of compact objects. For this to be presentable, the compact objects will have all finite colimits and thus $\mathcal{C}$ will have all colimits. We will be most concerned with $\kappa = \omega_1$.

We can now define Pr^{L} as follows.

Definition 7.4 (Pr^{L})**.** Pr^{L} is the ∞-category with objects presentable ∞-categories and morphisms colimit preserving functors.

Remark 7.5. By the adjoint functor theorem, colimit preserving functors are left adjoints, justifying the L in the notation.

We can in fact show that Pr^{L} has all limits and these limits are computed on the underlying ∞-categories.

Proposition 7.6. Let Pr^{L} be the ∞-category of presentable ∞-categories with left adjoint functors. Then the forgetful functor to (big) ∞-categories preserves limits and colimits in Pr^{L} are computed as limits in Pr^{L} on passage to right adjoint functors.

More explicitly, a diagram in Pr^{L} is a diagram with transition maps left adjoint functors. Passage to right adjoints gives a diagram of presentable ∞-categories with transition maps right adjoint functors. Under the equivalence $\mathsf{Pr}^{\mathsf{L}} \cong (\mathsf{Pr}^{\mathsf{R}})^{\mathsf{Opp}}$ it then suffices to compute the limit on the underlying (big) ∞-categories.

Example 7.7. Let $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$ be the subcategory of Pr^{L} in Definition 7.4 spanned by those stable categories. To compute the pushout

$$\begin{tikzcd} \mathcal{B} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ 0 \arrow[r] & \mathcal{C}/\mathcal{B} \end{tikzcd}$$

in $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$ the limit of the passage to right adjoints then gives an isomorphism $\mathcal{C}/\mathcal{B} = \ker(\mathcal{C} \to \mathcal{B})$ where the functor $\mathcal{C} \to \mathcal{B}$ is the right adjoint of the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$.

Recall that Pr^{L} has a symmetric monoidal structure given by the Lurie tensor product where for $\mathcal{B}, \mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}$ the tensor product $\mathcal{B} \otimes \mathcal{C}$ is determined by the universal property that for any ∞-category $\mathcal{E}$, a map $\mathcal{B} \times \mathcal{C} \to \mathcal{E}$ preserving colimits in each factor of the source is uniquely extended from a colimit-preserving map $\mathcal{B} \otimes \mathcal{C} \to \mathcal{E}$. This is in fact a closed symmetric monoidal structure with the internal-hom $\mathrm{Hom}_{\mathsf{Pr}^{\mathsf{L}}}(\mathcal{B}, \mathcal{C}) = \mathrm{Fun}^{\mathsf{L}}(\mathcal{B}, \mathcal{C})$ where $\mathrm{Fun}^{\mathsf{L}}(\mathcal{B}, \mathcal{C})$ are the colimit preserving functors between the categories. By unwinding the definitions, one finds $\mathcal{B} \otimes \mathcal{C} = \mathrm{Fun}^{\mathsf{L}}(\mathcal{B}, \mathcal{C}^{\mathsf{Opp}})^{\mathsf{Opp}}$.

One can observe that Pr^{L} has many idempotent algebra objects, those $\mathcal{A} \in \mathsf{Pr}^{\mathsf{L}}$ such that $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is an isomorphism.

Example 7.8. The ∞-category of spectra Sp is an idempotent object of Pr^{L}. The modules over Sp are exactly the presentable stable ∞-categories $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}} \subseteq \mathsf{Pr}^{\mathsf{L}}$.

As such, on restriction, $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$ is symmetric monoidal with unit Sp.

We can now define dualizable categories as follows.

Definition 7.9. Let $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$. $\mathcal{C}$ is dualizable if there exists $\mathcal{C}^\vee = \mathrm{Hom}^{\mathsf{L}}(\mathcal{C}, \mathsf{Sp})$ and maps $\mathrm{ev} : \mathcal{C} \otimes \mathcal{C}^\vee \to \mathsf{Sp}$ and $\mathrm{coev} : \mathsf{Sp} \to \mathcal{C} \otimes \mathcal{C}^\vee$ such that

$$\mathcal{C} \xrightarrow{\mathrm{id}_{\mathcal{C}} \otimes \mathrm{coev}} \mathcal{C} \otimes \mathcal{C}^\vee \otimes \mathcal{C} \xrightarrow{\mathrm{ev} \otimes \mathrm{id}_{\mathcal{C}}} \mathcal{C}$$

$$\mathcal{C}^\vee \xrightarrow{\mathrm{coev} \otimes \mathrm{id}_{\mathcal{C}^\vee}} \mathcal{C}^\vee \otimes \mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{\mathrm{id}_{\mathcal{C}^\vee} \otimes \mathrm{ev}} \mathcal{C}^\vee$$

are homotopic to the identities on $\mathcal{C}, \mathcal{C}^\vee$, respectively.

Example 7.10. For $\mathcal{C} = \mathsf{Ind}(\mathcal{C}_0)$ for $\mathcal{C}_0 \in \mathsf{Cat}^{\mathsf{Perf}}$, then $\mathcal{C}$ is dualizable. In fact, $\mathcal{C}^\vee = \mathsf{Ind}((\mathcal{C}_0)^{\mathsf{Opp}})$. For $\mathcal{C} = \mathcal{D}(A)$ the derived ∞-category of a discrete ring A, $\mathcal{C}^\vee = \mathcal{D}(A^{\mathsf{Opp}})$ where the evaluation map is the tensor product of modules and the coevaluation map is the diagonal bimodule.

The dualizable categories amalgamate into an ∞-category $\mathsf{Cat}^{\mathsf{dual}}_{\mathsf{St}}$.

Definition 7.11 ($\mathsf{Cat}^{\mathsf{dual}}_{\mathsf{St}}$)**.** $\mathsf{Cat}^{\mathsf{dual}}_{\mathsf{St}}$ is the ∞-category with objects dualizable categorise and morphisms strongly continuous functors – colimit-preserving functors with colimit-preserving right adjoints.

Remark 7.12. A continuous functor of compactly generated categories is strongly continuous if and only if it preserves compact objects.

Example 7.13. There is a functor $\mathsf{Ind} : \mathsf{Cat}^{\mathsf{Perf}} \to \mathsf{Cat}^{\mathsf{dual}}_{\mathsf{St}}$ by $\mathcal{C} \mapsto \mathsf{Ind}(\mathcal{C})$

In fact, we have the following characterization of dualizable categories.

Theorem 7.14 (Lurie, Clausen-Scholze, Efimov)**.** Let $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$. The following are equivalent.

(1) $\mathcal{C}$ is dualizable.
(2) $\mathcal{C}$ is a retract of a compactly generated category in $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$.
(3) The colimit functor $\mathsf{Ind}(\mathcal{C}) \to \mathcal{C}$ sending an ind-object to its colimit admits a left adjoint.
(4) $\mathcal{C}$ is an ω_1-compactly generated category and $\mathrm{colim} : \mathsf{Ind}(\mathcal{C}^{\omega_1}) \to \mathcal{C}$ has a left adjoint.

(5) $\mathcal{C}$ is generated by compactly exhaustible objects.
(6) There exists compactly generated $\mathcal{A}, \mathcal{B} \in \mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$ and a strongly continuous localization functor $L : \mathcal{A} \to \mathcal{B}$ – admitting a colimit-preserving fully faithful right adjoint – such that $\mathcal{C} = \ker(L)$.
(7) If $\mathcal{B} \subseteq \mathcal{B}'$ is a fully faithful inclusion in $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$ then $\mathcal{C} \otimes \mathcal{B} \to \mathcal{C} \to \mathcal{B}'$ is fully faithful.
(8) $\mathcal{C}$ satisfies Grothendieck's (AB6) axiom: for an indexing set I and a filtered category J_i and functors $f_i : J_i \to \mathcal{C}$ then

$$\prod_{i \in I} \operatorname{colim}_{j \in J_i} f_i(j_i) = \operatorname{colim}_{(j_i) \in \prod_{J_i}, i \in I} \prod_{i \in I} f_i(j_i).$$

(9) There exists a fully faithful strongly continuous map $\mathcal{C} \to \mathcal{B}$ where $\mathcal{B} \in \mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$ is compactly generated.
(10) $\operatorname{colim} : \mathsf{Ind}(\mathcal{C}) \to \mathcal{C}$ preserves all products.

Proof as a sequence of propositions.

8. Lecture III (10th September)

We turn to a discussion of compact morphisms.

Definition 8.1 (Compact Morphism)**.** Let $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$. A morphism $f : X \to Y$ in $\mathcal{C}$ is compact if for every ind-object $\{Z_i\}_{i\in I}$ in $\mathcal{C}$ such that $\operatorname{colim}_I Z_i = 0$ then

$$\pi_* \operatorname{colim}_I \operatorname{Hom}(Y, Z_i) \to \pi_* \operatorname{colim}_I \operatorname{Hom}(X, Z_i)$$

is the zero map.

Remark 8.2. X is compact if and only if id_X is a compact morphism.

Remark 8.3. Compact morphisms form a 2-sided ideal. In particular, a morphism that factors through a compact object is a compcat morphism. In fact, the converse is true if $\mathcal{C}$ is compactly generated.

Definition 8.4 (Compactly Exhaustible)**.** Let $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}_{\mathsf{St}}$. $X \in \mathcal{C}$ is compactly exhaustible if and only if X is a sequential colimit along compact maps – $\operatorname{colim}_{\mathbb{N}} X_n$ where each transition map $X_n \to X_{n+1}$ compact.

Before considering some examples of dualizable categories, we recall the following definition.

Definition 8.5 (Homological Epimorphism)**.** Let $f : A \to B$ be a map of $\mathbb{E}_1$-rings. $f : A \to B$ is a homological epimorphism if $\mathcal{D}(B) \to \mathcal{D}(A)$ is fully faithful.

Remark 8.6. Equivalently, B is idempotent or that the kernel of the extension of scalars functor $- \otimes_A B : \mathcal{D}(A) \to \mathcal{D}(B)$ is dualizable.

Note that there are few classical examples of this in classical ring theory, but a number of examples arise from Faltings and Gabber-Ramero's almost ring theory.

Example 8.7. Let R be a commutative ring and $I \subseteq R$ an ideal such that I is flat as an R-module and $I^2 = I$ then the map $R \to R/I$ is a homological epimorphism.

Another arithmetic example is as follows.

Example 8.8. Let A, B be perfect $\mathbb{F}_p$-algebras. If $A \to B$ is a surjection then this is a homological epimorphism. In this case are no higher tor of B with itself over A with $B \otimes^L_A B \to B$ is an isomorphism.

Yet another important example is the case of sheaves on a locally compact Hausdorff space.

Definition 8.9 (Sheaves on a LCH Space)**.** Let X be a locally compact Hausdorff space. $\mathsf{Sh}(X, \mathsf{Sp})$, the ∞-category of sheaves of spectra on X is a subcategory of $\operatorname{Fun}(N(\mathsf{Opens}(X))^{\mathsf{Opp}}, \mathsf{Sp})$ consisting of functors $\mathcal{F}$ such that:

(i) $\mathcal{F}(\emptyset) = 0$.
(ii) For $U, V \subseteq X$ open the square

$$\begin{tikzcd} \mathcal{F}(U \cup V) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r] & \mathcal{F}(U \cap V) \end{tikzcd}$$

is a pullback.

(iii) For a filtered union of open subsets $\bigcup_{i\in I} U_i$ then

$$\mathcal{F}\left(\bigcup_{i\in I} U_i\right) \longrightarrow \lim_I \mathcal{F}(U_i)$$

is an isomorphism.

Example 8.10. Let X be a locally compact Hausdorff space. The ∞-category of sheaves of spectra $\mathsf{Sh}(X, \mathsf{Sp})$ is dualizable.

9. Lecture IV (11th September)

10. Lecture V (12th September)

End Matter

References

[CD+23] Baptiste Calmès et al. *Hermitian K-theory for stable ∞-categories II: Cobordism categories and additivity*. 2023. arXiv: 2009.07224 [math.KT]. URL: https://arxiv.org/abs/2009.07224.

[HA] Jacob Lurie. *Higher Algebra*. 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.

[HTT] Jacob Lurie. *Higher Topos Theory*. English. Vol. 170. Ann. Math. Stud. Princeton, NJ: Princeton University Press, 2009. ISBN: 978-0-691-14049-0; 978-0-691-14048-3. DOI: 10.1515/9781400830558.

[DC24] *Workshop on Dualizable Categories and Continuous K-Theory*. 2024. URL: https://archive.mpim-bonn.mpg.de/id/eprint/5092/.

Universität Bonn, Bonn, D-53111

Email address: wgabrielong@uni-bonn.de

URL: https://wgabrielong.github.io/