ALGEBRAIC STACKS

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Overview

These notes roughly correspond to an attempt to learn stack theory that began in the winter of 2023. We will begin with the categorical preliminaries as laid out in [4] before developing the basic theory per the text of Olsson [2] and conclude with a sampling of the more advanced topics in [3, Part 7]. The standard texts are [1] and [2]. The compendium [3] is encyclopedic.

Contents

Grothendieck Topologies, Sites, and Fibered Categories

1. Grothendieck Topologies

Let us recall the following definitions.

Definition 1.1 (Presheaf)**.** Let X be a topological space. A presheaf of sets $\mathcal{F}$ on X is a functor $(X^{\mathsf{Opens}})^{\mathsf{Opp}} \to \mathsf{Sets}$.

A presheaf is a sheaf if it satisfies additional gluing axioms.

Definition 1.2 (Sheaf)**.** Let X be a topological space. A sheaf of sets $\mathcal{F}$ on X is a functor $(X^{\mathsf{Opens}})^{\mathsf{Opp}} \to \mathsf{Sets}$ such that the sequence

$$\mathcal{F}(X) \longrightarrow \prod_i \mathcal{F}(X_i) \rightrightarrows \prod_{i,j} \mathcal{F}(X_i \cap X_j)$$

is an equalizer for $\{X_i\}$ an open cover of X.

In this way, we say that a sheaf on X is a presheaf on X satisfying descent. Note here that we implicitly used the fact that X^{Opens} can be naturally endowed with the strucutre of a category with objects open sets of the topological space X and morphisms inclusions of such open sets. This begs the question if we can replace X^{Opens} with some other category C, allowing us to define a sheaf on an arbitrary category C. We do this via the construction of a Grothendieck topology by replacing open sets of a topological space with maps into this space.

Definition 1.3 (Grothendieck Topology)**.** Let C be a category. A Grothendieck topology on C is the data of a set $\{X_i \to X\}$ for each object $X \in \mathrm{Obj}(\mathsf{C})$ known as a covering of X such that the following conditions hold:

(a) If $Y \to X$ is an isomorphism then $\{Y \to X\}$ is a covering.
(b) If $\{X_i \to X\}$ is a covering and $Y \to X$ any morphism then $\{X_i \times_X Y\}$ exist and $\{X_i \times_X Y \to X\}$ is a covering.
(c) If $\{X_i \to X\}$ is a covering and for each i $\{X_{ij} \to X_i\}$ is a covering then the composites $\{X_{ij} \to X_i \to X\}$ is a covering.

This allows us to define a site.

Definition 1.4 (Site)**.** A site on a category C is the category C endowed with a Grothendieck topology.

Let us see some examples.

Example 1.5 (Site of a Topological Space)**.** Let X be a topological space and X^{Opens} the category of open sets of X with morphisms inclusions. One can endow X^{Opens} with a Grothendieck topology by associating to each $U \subseteq X$ open, the set of open covers of U. The fiber product is given by intersection of open sets, which agrees with our previous discussion of sheaves and presheaves.

Example 1.6. Consider the category of topological spaces Top. The category Top can be endowed with a Grothendieck topology by taking covers of a topological space $X \in \mathrm{Obj}(\mathsf{Top})$ to open, continuous, injective maps $X_i \to X$.

Example 1.7 (Étale Site of a Scheme)**.** Let X be a scheme. The étale site of X, denoted $X_{\text{ét}}$, is the full subcategory of Sch_X where covers are étale morphisms.

2. Fibered Categories

3. Descent

Algebraic Spaces

4. Algebraic Spaces

5. Quotients in Algebraic Spaces

6. Quasicoherent Sheaves on Algebraic Spaces

Algebraic Stacks

7. Algebraic Stacks

8. Quasicoherent Sheaves on Algebraic Stacks

The Geometry of Stacks

9. Geometric Properties of Stacks

10. Coarse Moduli Spaces

11. Gerbes

12. Cohomology of Stacks

13. Derived Categories of Stacks

End Material

Appendix A. Category Theory

References

[1] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*. French. Vol. 39. Ergeb. Math. Grenzgeb., 3. Folge. Berlin: Springer, 2000. ISBN: 3-540-65761-4.
[2] Martin Olsson. *Algebraic spaces and stacks*. English. Vol. 62. Colloq. Publ., Am. Math. Soc. Providence, RI: American Mathematical Society (AMS), 2016. ISBN: 978-1-4704-2798-6; 978-1-4704-2865-5.
[3] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2023.
[4] Angelo Vistoli. *Notes on Grothendieck topologies, fibered categories and descent theory*. 2007. arXiv: math/0412512 [math.AG].

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