ALGEBRAIC STACKS

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Overview

These notes roughly correspond to an attempt to learn stack theory that began in the winter of 2023. We will begin with the categorical preliminaries as laid out in [5] before developing the basic theory per the text of Olsson [3] and conclude with a sampling of the more advanced topics in [4, Part 7]. The standard texts are [2] and [3]. The compendium [4] is encyclopedic.

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Grothendieck Topologies, Sites, and Fibered Categories

1. Grothendieck Topologies and Sites

Let us recall the following definitions.

Definition 1.1 (Presheaf)**.** Let X be a topological space. A presheaf of sets $\mathcal{F}$ on X is a functor $(X^{\mathsf{Opens}})^{\mathsf{Opp}} \to \mathsf{Sets}$.

A presheaf is a sheaf if it satisfies additional gluing axioms.

Definition 1.2 (Sheaf)**.** Let X be a topological space. A sheaf of sets $\mathcal{F}$ on X is a functor $(X^{\mathsf{Opens}})^{\mathsf{Opp}} \to \mathsf{Sets}$ such that the sequence

$$\mathcal{F}(X) \longrightarrow \prod_i \mathcal{F}(X_i) \rightrightarrows \prod_{i,j} \mathcal{F}(X_i \cap X_j)$$

is an equalizer for $\{X_i\}$ an open cover of X.

In this way, we say that a sheaf on X is a presheaf on X satisfying descent. Note here that we implicitly used the fact that X^{Opens} can be naturally endowed with the strucutre of a category with objects open sets of the topological space X and morphisms inclusions of such open sets. This begs the question if we can replace X^{Opens} with some other category C, allowing us to define a sheaf on an arbitrary category C. We do this via the construction of a Grothendieck topology by replacing open sets of a topological space with maps into this space.

Definition 1.3 (Covering)**.** Let C be a category. A covering on $X \in \mathrm{Obj}(\mathsf{C})$ is a set of collections of morphisms $\{X_i \to X\}$ such that the following conditions hold:

(a) If $Y \to X$ is an isomorphism then $\{Y \to X\}$ is a covering.
(b) If $\{X_i \to X\}$ is a covering and $Y \to X$ any morphism then $\{X_i \times_X Y\}$ exist and $\{X_i \times_X Y \to X\}$ is a covering.
(c) If $\{X_i \to X\}$ is a covering and for each i $\{X_{ij} \to X_i\}$ is a covering then the composites $\{X_{ij} \to X_i \to X\}$ is a covering.

This allows us to define Grothendieck topologies and sites.

Definition 1.4 (Grothendieck Topology)**.** Let C be a category. A Grothendieck topology on C is the data of a covering for each $X \in \mathrm{Obj}(\mathsf{C})$.

Definition 1.5 (Site)**.** A site on a category C is the category C endowed with a Grothendieck topology.

Let us see some examples.

Example 1.6 (Site of a Topological Space)**.** Let X be a topological space and X^{Opens} the category of open sets of X with morphisms inclusions. One can endow X^{Opens} with a Grothendieck topology by associating to each $U \subseteq X$ open, the set of open coverings of U. The fiber product is given by intersection of open sets, which agrees with our previous discussion of sheaves and presheaves.

Example 1.7. Consider the category of topological spaces Top. The category Top can be endowed with a Grothendieck topology by taking coverings of a topological space $X \in \mathrm{Obj}(\mathsf{Top})$ to open, continuous, injective maps $X_i \to X$.

1.1. Sites in Algebraic Geometry.

In algebraic geometry, is sometimes useful to consider coverings that are not finitely presented, and more generally topologies that are finer – containing more open sets – than the Zariski topology. We introduce some of these structures here.

Definition 1.8 (fpqc Morphism)**.** Let $f : X \to Y$ be a surjective morphism of schemes. f is an fpqc morphism if for every affine open $V \subseteq Y$, $f^{-1}(V)$ is quasicompact in X.

Remark. The notation fpqc arises from the French, fidèlement plat et quasi-compact, or faithfully flat and quasicompact.

This allows us to define the various Grothendieck topologies on the category of schemes over a fixed scheme S.

Definition 1.9 (Small Zariski Site)**.** Let X be a scheme. The small Zariski site on X, X_{Zar} is the category with objects open subschemes $U \subseteq X$ and morphisms inclusions. The Grothendieck topology is given by coverings of collections of open embeddings $\{U_i \to U\}$ such that $\cup_i U_i = U$.

Definition 1.10 (Big Zariski Site)**.** Let S be a scheme and Sch_S the category of S-schemes. The big Zariski site on S-schemes $(\mathsf{Sch}_S)_{\mathsf{Zar}}$ has coverings given by sets of S-morphisms $\{X_i \to X\}$ for which $X_i \to X$ is an open embedding and $\cup_i X_i = X$.

Definition 1.11 (Smooth Site)**.** Let S be a scheme and Sch_S the category of X-schemes. The smooth site on S-schemes $(Sch_S)_{\mathsf{Sm}}$ is the full subcategory of the slice category Sch_S with objects smooth morphisms $(X \to S)$ and morphisms smooth commuting triangles. The Grothendieck topology is given by collections of smooth morphisms $\{X_i \to X\}$ such that $\coprod_i X_i \to X$ is surjective.

Definition 1.12 (Small Étale Site)**.** Let X be a scheme. The small étale site on X, $X_{\mathsf{ét}}$ is the full subcategory of the slice category Sch_X with objects étale morphisms $(U \to X)$ and morphisms étale commuting triangles. The Grothendieck topology is given by collections of étale morphisms $\{U_i \to U\}$ such that $\coprod_i U_i \to U$ is surjective.

Definition 1.13 (Big Étale Site)**.** Let S be a scheme and Sch_S the category of S-schemes. The big étale site on S-schemes $(\mathsf{Sch}_S)_{\mathsf{ét}}$ has coverings given by collections of S-morphisms $\{X_i \to X\}$ for which $X_i \to X$ is étale and $\coprod_i X_i \to X$ is surjective.

Definition 1.14 (Lisse-Étale Site)**.** Let X be a scheme. The lisse-étale site on X, $X_{\mathsf{LisÉt}}$ is the full subcategory of the slice category Sch_X with objects smooth morphisms $(U \to X)$ and morphisms smooth commuting triangles. The Grothendieck topology is given by collections of smooth morphisms $\{U_i \to U\}$ such that $\coprod_i U_i \to U$ is surjective.

Definition 1.15 (fppf Site)**.** Let S be a scheme and Sch_S the category of S-schemes. The fppf site on S-schemes $(\mathsf{Sch}_S)_{\mathsf{fppf}}$ has coverings given by collections $\{X_i \to X\}$ for which $X_i \to X$ is flat and locally of finite presentation and $\coprod_i X_i \to X$ is surjective.

Definition 1.16 (fpqc Site)**.** Let S be a scheme and Sch_S the category of S-schemes. The fpqc site on S-schemes $(\mathsf{Sch}_S)_{\mathsf{fpqc}}$ has coverings given by collections $\{X_i \to X\}$ for each $X \in \mathrm{Obj}(\mathsf{Sch}_S)$ such that $\coprod_i X_i \to X$ is fpqc.

The following proposition shows that the various types of morphisms play well with the definition of coverings.

Proposition 1.17. Let $X \to Y$ be a morphism of schemes and $\{Y_i \to Y\}$ an fpqc covering of Y. Suppose that for each i, the induced map $Y_i \times_Y X \to Y_i$ has one of the following properties:

(a) separated,
(b) quasicompact,
(c) locally of finite presentation,
(d) proper,
(e) affine,

(f) finite,
(g) flat,
(h) smooth,
(i) unramified,
(j) étale,
(k) is an embedding,
(l) or is a closed embedding

then so does $X \to Y$.

Proof. All these properties and being fpqc are affine-local on target and the condition on the induced map $Y_i \times_Y X \to Y$ implies the condition on the map $X \to Y$ by [1, Prop. 2.7.1]. ■

The the topologies can be classified according to coarseness, with the Zariski topology being the coarsest – having the fewest open sets.

Theorem 1.18. In increasing level of coarseness,

$$\text{fpqc} \leq \text{fppf} \leq \text{étale} \leq \text{Zariski}.$$

In particular any sheaf in a topology on the left is a sheaf in the topology on the right.

Remark. We will soon clarify how sheaves in a finer topology are sheaves in a coarser one.

Proof. This can be directly checked by considering the properties of the corresponding types of morphisms. ■

1.2. **Sheaves on Sites.** Having defined a topology, we can now define a sheaf theory on a site. Recall from earlier that sheaves on a topological space X is a presheaf that satisfies additional axioms. The sheaf condition can be generalized to any site, replacing intersections of open sets with fibered products.

Definition 1.19 (Separated Presheaf on a Site)**.** Let C be a site and $F : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ a presheaf of sets on C. F is a separated functor if there are $a, b \in F(X)$ whose pullbacks to $F(X_i)$ coincide then $a = b$ on $F(U)$.

Definition 1.20 (Sheaf on a Site)**.** Let C be a site and $F : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ a presheaf of sets on C. F is a sheaf if for all coverings $\{X_i \to X\}$ the sequence

$$F(X) \longrightarrow \prod_i F(X_i) \overset{\mathrm{pr}_1^*}{\underset{\mathrm{pr}_2^*}{\rightrightarrows}} \prod_{i,j} F(X_i \times_X X_j)$$

is an equalizer.

Remark. As in the case of sheaves on a topological space, the condition on equalizers is equivalent to the gluing of sections. Given a covering $\{X_i \to X\}$ function of sets $F(X) \to \prod_i F(X_i)$ is induced along restrictions and the natural projection maps

$$\mathrm{pr}_1 : X_i \times_X X_j \to X_i, \mathrm{pr}_2 : X_i \times_X X_j \to X_j$$

sends $a_i \in F(X_i)$ to $\mathrm{pr}_1^* a_i$ with pr_2^* defined similarly. The equalizer condition states that if pullbacks of sections $\mathrm{pr}_1^* a_i = \mathrm{pr}_2^* a_j$ as elements of $F(X_i \times_X X_j)$ then they are pulled back from some a on $F(X)$ restricting to the a_i on $F(X_i)$.

Given an object $X \in \mathrm{Obj}(\mathsf{C})$ we can define a sieve on X as follows.

Definition 1.21 (Sieve)**.** Let $X \in \mathrm{Obj}(\mathsf{C})$. A sieve on X is a subfunctor of the Yoneda functor $h_X : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$.

Given a subfunctor S of h_X, we get a collection $\mathcal{S}$ of arrows $T \to X$ by taking $\cup_{T \in \mathrm{Obj}(\mathsf{C})} S(T)$ such that for $T \to X$ in $\mathcal{S}$, every composite $T' \to T \to X$ is in $\mathcal{S}$. Conversely, given $\mathcal{S}$ we can restrict the Yoneda functor appropriately to get $S : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$.

For $\{X_i \to X\}$ a a collection of morphisms and $X \in \mathrm{Obj}(\mathsf{C})$ we can similarly define a subfunctor $h_{\{X_i \to X\}} : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ where $T \to X$ if and only if there exists a factorization $T \to X_i \to X$ for some i. In the case of the functor $h_{\{X_i \to X\}}$, the metaphor of the sieve becomes much more clear: the X_i are the holes of the sieve, and $T \to X$ is in $h_{\{X_i \to X\}}(T)$ if and only if it "goes through one of the holes".

For a collection $\{X_i \to X\}$ and $F : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ a functor, we can define $F(\{X_i \to X\})$ to be the set of elements of $\prod_i F(X_i)$ whose images in $\prod_{i,j} F(X_i \times_X X_j)$ are equal. Sections of $F(X)$ evidently give rise to sections of $\prod_i F(X_i)$ that agree on $\prod_{i,j} F(X_i \times_X X_j)$ inducing a map $F(X) \to F(\{X_i \to X\})$. Note here that if F is a sheaf the descent condition implies the map $F(X) \to F(\{X_i \to X\})$ is a bijection for all coverings.

We show that the set $F(\{X_i \to X\})$ can be defined in terms of sieves.

Proposition 1.22. There is a canonical bijection of sets

$$R : \mathrm{NatTrans}(h_{\{X_i \to X\}}, F) \to F(\{X_i \to X\})$$

such that the diagram

$$\begin{tikzcd}
\mathrm{NatTrans}(h_X, F) \arrow[r] \arrow[d] & F(X) \arrow[d] \\
\mathrm{NatTrans}(h_{\{X_i \to X\}}, F) \arrow[r, "R"'] & F(\{X_i \to X\})
\end{tikzcd}$$

commutes and R is universal with respect to this property.

Evidently $\mathrm{NatTrans}(h_X, F) \to F(X)$ is the bijection induced by the Yoneda embedding and the vertical maps given by restriction h_X to $h_{\{X_i \to X\}}$ and $F(X)$ to $F(\{X_i \to X\})$, respectively.

Proof. Suppose $\phi : h_{\{X_i \to X\}} \Rightarrow F$. For each i, $X_i \to X$ is in $h_{\{X_i \to X\}}(X_i)$ where we define $R\phi = \phi(X_i \to X) \in \prod_i F(X_i)$. By the definition of a sieve, the pullbacks $\mathrm{pr}_1^* \phi(X_i \to X)$ and $\mathrm{pr}_2^*(X_i \to X)$ agree on $X_i \times_X X_j$ so $R\phi \in F(\{X_i \to X\})$, defining a function $R : \mathrm{NatTrans}(h_{\{X_i \to X\}}, F) \to F(\{X_i \to X\})$ and giving the commutativity of the diagram.

We now show R is a bijection. Suppose there are two natural transformations $\phi, \psi : h_{\{X_i \to X\}} \Rightarrow F(\{X_i \to X\})$ such that $R\phi = R\psi$. For $T \to X$ of some $h_{\{X_i \to X\}}(T)$ there exists $f : T \to X_i$ so by definition of natural transformations we have

$$\phi(T \to X) = f^* \phi(X_i \to X) = f^* \psi(X_i \to X) = \psi(T \to X)$$

so $\phi = \psi$ proving injectivity. For surjectivity suppose we have some $\xi_i \in F(\{X_i \to X\})$ we show that ξ_i defines a natural transformation $h_{\{X_i \to X\}} \to F$. For some element of $h_{\{X_i \to X\}}(T)$ choose a factorization with $f : T \to X_i$ defining a pullback $f^* \xi_i$ of $F(X)$. Given some other factorization $g : T \to X_j$ we get a morphism $T \to X_i \times_X X_j$ whose composites with $\mathrm{pr}_1, \mathrm{pr}_2$ are f, g, respectively. Since $\mathrm{pr}_1^* \xi_i = \mathrm{pr}_2^* \xi_j$, we have $f^* \xi_i = g^* \xi_j$ showing surjectivity defining a natural transformation where $R\phi = \xi_i$. ∎

This gives us a characterization of sheaves as follows.

Corollary 1.23. A functor $\mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ is a sheaf if and only if for any covering $\{X_i \to X\}$ in C the induced function

$$F(X) = \mathrm{NatTrans}(h_X, F) \to \mathrm{NatTrans}(h_{\{X_i \to X\}}, F)$$

is a bijection.

We can in fact sharpen this characterization.

Definition 1.24 (Belong To)**.** Let T be a Grothendieck topology on a category C. A sieve $S \subseteq h_X$ of an object $X \in \mathrm{Obj}(\mathsf{C})$ belongs in T if there is a covering $\{X_i \to X\}$ in T such that $h_{\{X_i \to X\}} \subseteq S$.

Remark. If C is a site, the sieves of C are the sieves belonging to the Grothendieck topology of C.

The importance of this characterization is crucial to the characterization of sheaves on a Grothendieck topology.

Proposition 1.25. Let $F : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ be a presheaf of sets in a Grothendieck topology T. F is a sheaf if and only if for all sieves $S \in \mathsf{T}$ the induced map

$$F(X) = \mathrm{Mor}_{\mathsf{PSh}(\mathsf{C})}(h_X, F) \to \mathrm{Mor}_{\mathsf{PSh}(\mathsf{C})}(S, F)$$

is a bijection.

Proof. ($\Longrightarrow$) This is immediate from Corollary 1.23 as a natural transformation of functors $\mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ are exactly morphsims in $\mathsf{PSh}(\mathsf{C})$.

($\Longleftarrow$) Suppose F is a sheaf and for some $S \subseteq h_X$ belonging to the Grothendieck topology T of the category C. Let $\{X_i \to X\}$ be a covering of X with $h_{\{X_i \to X\}} \subseteq S$. Once again by Corollary 1.23 we have that

$$\mathrm{Mor}_{\mathsf{PSh}(\mathsf{C})}(h_X, F) \to \mathrm{Mor}_{\mathsf{PSh}(\mathsf{C})}(S, F) \to \mathrm{Mor}_{\mathsf{PSh}(\mathsf{C})}\left(h_{\{X_i \to X\}}, F\right)$$

is a bijection. To show $\mathrm{Mor}_{\mathsf{PSh}(\mathsf{C})}(h_X, F) \to \mathrm{Mor}_{\mathsf{PSh}(\mathsf{C})}(S, F)$ is a bijection also, it suffices to show that $\mathrm{Mor}_{\mathsf{PSh}(\mathsf{C})}(S, F) \to \mathrm{Mor}_{\mathsf{PSh}(\mathsf{C})}\left(h_{\{X_i \to X\}}, F\right)$ is an injection. Let $\phi, \psi : S \to F$ be two natural transformations with the same image in $\mathrm{Mor}_{\mathsf{PSh}(\mathsf{C})}\left(h_{\{X_i \to X\}}, F\right)$, $(Y \to X) \in S(Y)$. Taking pullbacks, we have a covering $\{X_i \times_X Y \to X\}$ with each map in $h_{\{X_i \to X\}}(X_i \times_X Y)$ by the sieve axioms and projections $\mathrm{pr}_{2,i} : X_i \times_X Y \to Y$. We thus have equalities

$$\mathrm{pr}_{2,i}^* \phi(Y \to X) = \phi(X_i \times_X Y) = \psi(X_i \times_X Y) = \mathrm{pr}_{2,i}^* \psi(Y \to X)$$

but since F is a sheaf, it satisfies the identity axiom so the equality above in fact shows

$$\phi(Y \to X) = \psi(Y \to X)$$

as desired. ∎

To further elaborate on an important idea used in the preceding proof, consider $\{X_i \to X\}$ and $\{Y_j \to X\}$ two coverings of $X \in \mathrm{Obj}(\mathsf{C})$ with C some category with a Grothendieck topology. We can define a new covering $\{X_i \times_X Y_j \to X\}$ another covering. The behavior of a map $Z \to X$ and its factorizations through the various covers are governed by the following proposition.

Proposition 1.26. Let C be a category and $X \in \mathrm{Obj}(\mathsf{C})$.

(a) If $\{X_i \to X\}$ and $\{Y_j \to X\}$ are two coverings and $\{X_i \times_X Y_j \to X\}$ the covering associated to the fibered product then
$$h_{\{X_i \times_X Y_j \to X\}} = h_{\{X_i \to X\}} \cap h_{\{Y_j \to X\}} \subseteq h_X.$$

(b) If S_1, S_2 are sieves on X in the Grothendieck topology T, the intersection $S_1 \cap S_2 \subseteq h_X$ is also a sieve in the Grothendieck topology T.

Proof. Suppose $Z \to X$ is a morphism factoring through the covering $\{X_i \times_X Y_j \to X\}$ then by the universal property of fibered products, there is a commuting diagram of the following form:

$$\begin{tikzcd}
Z \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X_i \times_X Y_j \arrow[r] \arrow[d] & X_i \arrow[d] \\
& Y_j \arrow[r] & X
\end{tikzcd}$$

showing that Z factors through both the coverings $\{X_i \to X\}$ and $\{Y_j \to X\}$, that is, $h_{\{X_i \times_X Y_j \to X\}} \subseteq h_{\{X_i \to X\}} \cap h_{\{Y_j \to X\}}$ where the maps $Z \to X_i, Z \to Y_j$ arise from composing $Z \to X$ with the projection maps. The reverse converse containment $h_{\{X_i \to X\}} \cap h_{\{Y_j \to Y\}} \subseteq h_{\{X_i \times_X Y_j \to X\}}$ evident by the universal property of fibered products. This shows (a) which immediately implies (b). ∎

We now consider how the varying Grothendieck topologies on a site can give rise to different sheaves.

Definition 1.27 (Refinement)**.** Let C be a category and $\{X_i \to X\}_{i \in I}, \{Y_j \to X\}_{j \in J}$ two collections of morphisms. The collection $\{Y_j \to X\}_{j \in J}$ is a refinement of $\{X_i \to X\}_{i \in I}$ if for all $j \in J$ there exists $i \in I$ such that the map $Y_j \to X$ factors as
$$Y_j \to X_i \to X.$$

Remark. Note that the data of factorizations is not part of the data of a refinement. The definition solely requires their existence.

This condition of refinement is best phrased in terms of sieves.

Proposition 1.28. Let C be a category and $\{X_i \to X\}_{i \in I}, \{Y_j \to X\}_{j \in J}$ two collections of morphisms. The collection $\{Y_j \to X\}_{j \in J}$ is a refinement of $\{X_i \to X\}_{i \in I}$ if and only if $h_{\{Y_j \to X\}}$ is a subfunctor of $h_{\{X_i \to X\}}$.

Proof. Immediate from Definition 1.27. ∎

A refinement of a refinement is evidently a refinement. Furthermore, any covering is tautologically a refinement of itself: given a cover $\{X_i \to X\}$ it is a refinement of $\{X_i \to X\}$ in the sense that for all i there exists a factorization
$$X_i \xrightarrow{\mathrm{id}_{X_i}} X_i \longrightarrow X$$

Thus the relation of being a refinement provides an order between coverings of an object $X \in \mathrm{Obj}(\mathsf{C})$.

Definition 1.29 (Subordinate Topologies)**.** Let C be a category and T, T' two Grothendieck topologies on C. The Grothendieck topology T is subordinate to the Grothendieck topology T' if every covering in T has a refinement that is a covering in T'.

Remark. For T subordinate to T' we write $T \preccurlyeq \mathsf{T}'$.

Definition 1.30 (Equivalent Topologies)**.** Let C be a category and T, T' be two Grothendieck topologies on C. If $\mathsf{T} \preccurlyeq \mathsf{T}'$ and $\mathsf{T}' \preccurlyeq \mathsf{T}$ then the Grothendieck topologies T, T' are equivalent.

We previously mentioned that being a refinement was a transitive and reflexive between sets of morphisms to an object. Globally this is incarnated as the relation of being subordinate to, which is a transitive and reflexive relation between Grothendieck topologies on a category C. Equivalence of Grothendieck topologies as defined in Definition 1.30 is thus an equivalence relation. Once again, this is most clearly stated in terms of sieves.

Proposition 1.31. Let C be a category and T, T' two Grothendieck topologies on C. $\mathsf{T} \preccurlyeq \mathsf{T}'$ if and only if every sieve of T belongs to T'.

Proof. This is checked elementwise, from which the result follows by Proposition 1.28. ■

This naturally leads to the following corollary.

Corollary 1.32. Two Grothendieck topologies T, T' on a category C are equivalent if and only if they have the same sieves.

These statements on subordinate topologies immediately lead to the following statements about sheaves.

Proposition 1.33. Let C be a category and T, T' two Grothendieck topologies on C. If $\mathsf{T} \preccurlyeq \mathsf{T}'$ then every sheaf in T' is a sheaf on T.

Proof. Denote $\mathsf{C}_{\mathsf{T}}, \mathsf{C}_{\mathsf{T}'}$ the sites with Grothendieck topologies T, T', respectively. From Proposition 1.25, we know that if F is a sheaf on $\mathsf{C}_{\mathsf{T}'}$ we have a bijective map $\mathrm{Mor}_{\mathsf{PSh}(\mathsf{C}_{\mathsf{T}'})}(h_X, F) \to \mathrm{Mor}_{\mathsf{PSh}(\mathsf{C}_{\mathsf{T}'})}(S, F)$ for all $X \in \mathrm{Obj}(\mathsf{C})$ and all sieves $S \in \mathsf{T}'$. But all sieves in T' are in T too by Proposition 1.31 so the map $\mathrm{Mor}_{\mathsf{PSh}(\mathsf{C}_{\mathsf{T}})}(h_X, F) \to \mathrm{Mor}_{\mathsf{PSh}(\mathsf{C}_{\mathsf{T}})}(S, F)$ is also a bijection, showing that F is a sheaf in C_{T}. ■

Once again, we have the following corollary.

Corollary 1.34. Let C be a category T, T' two Grothendieck topologies on C. If T and T' are equivalent as Grothendieck topologies then the sites $\mathsf{C}_{\mathsf{T}}, \mathsf{C}_{\mathsf{T}'}$ have the same sheaves.

Proof. Immediate from Corollary 1.32 and Proposition 1.33. ■

Remark. Note that in the preceding discussion we used the language of topologies, which contrasts with that of Grothendieck's construction using pretopologies.

We introduce a few final notions about Grothendieck topologies before turning to a discussion of representable functors.

Definition 1.35 (Saturated Topology)**.** Let C be a category and T a Grothendieck topology on the category C. T is a saturated Grothendieck topology if every collection of morphisms having a refinement in T is in T.

Definition 1.36 (Saturation)**.** Let C be a category and T a Grothendieck topology on the category C. The saturation $\overline{\mathsf{T}}$ of T is the smallest collection of coverings that is saturated.

Remark. Equivalently $\overline{\mathsf{T}}$ is the collection of sets of morphisms which have a refinement in T.

This leads to the following proposition.

Proposition 1.37. Let T be a Grothendieck topology on a category C.

(a) T is a subcollection of sets of morphisms in $\overline{\mathsf{T}}$,
(b) the Grothendieck topologies T and $\overline{\mathsf{T}}$ agree,
(c) T is a saturated Grothendieck topology if and only if the collections of sets of morphisms in $\mathsf{T}, \overline{\mathsf{T}}$ agree,
(d) a Grothendieck topology T' on C is subordinate to T if and only if the collection of sets of morphisms in T' is a subcollection of sets of morphisms in T,
(e) two Grothendieck topologies T, T' on a category C are equivalent if and only if the collections of sets of morphisms in $\overline{\mathsf{T}}, \overline{\mathsf{T}'}$ are equal,
(f) a Grothendieck topology on C is equivalent to a unique saturated topology.

The statement and proof of Proposition 1.33 is suggestive of the connection between sheaves on sites and representable functors, which we now turn to.

1.3. **Sheaves on the Site of Schemes.** In the case of topological spaces, one can observe that representable functors $\mathsf{Top}^{\mathsf{Opp}} \to \mathsf{Sets}$ is a sheaf in the global topology in the sense of Example 1.7. Indeed, given X, Y topological spaces and an open covering $\{Y_i\}_{i \in I}$ of Y (in the usual sense), and continuous functions $f_i : Y_i \to X$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ agree for all i, j, there exists a unique continuous function $f : U \to X$ such that $f_i = f|_{U_i}$ for all i – this is the content of the gluability axiom. In other words, since the set $\mathrm{Mor}_{\mathsf{Top}}(Y, X)$ can be reconstructed from local data, so to can the functor $\mathrm{Mor}_{\mathsf{Top}}(-, X)$.

However, this is no longer obvious the case in the category of schemes. The remainder of this section will be dedicated to proving the following difficult theorem of Grothendieck.

Theorem 1.38 (Grothendieck)**.** Let S be a scheme and consider the category Sch_S. A representable functor on Sch_S is a sheaf in the fpqc topology.

By the refinement of Grothendieck topologies on the category of schemes, a representable functor on Sch_S is also a sheaf in the étale and fppf topologies.

2. Fibered Categories

Fix a base category S. We consider categories over S. That is, those categories F with a functor $p : \mathsf{F} \to \mathsf{S}$. More explicitly, we have the following definition.

Definition 2.1 (Category Over)**.** We say that F is a category over S if there exists a functor $p : \mathsf{F} \to \mathsf{S}$.

The relationship between the categories F and S are given by "lying over" in the following sense.

Definition 2.2 (Lying Over)**.** Consider F over S with $p : \mathsf{F} \to \mathsf{S}$.

(a) (Objects) An object $\alpha \in \mathrm{Obj}(\mathsf{F})$ lies over $A \in \mathrm{Obj}(\mathsf{S})$ if $p(\alpha) = A$.
(b) (Morphisms) An morphism $\phi : \alpha \to \beta$ in F lies over $f : A \to B$ in S if the diagram

$$\begin{tikzcd} \alpha \arrow[r, "\phi"] \arrow[d, "p(-)"'] & \beta \arrow[d, "p(-)"] \\ A \arrow[r, "p(\phi)=f"'] & B \end{tikzcd}$$

commutes.

This allows us to define the notion of a Cartesian morphism in F.

Definition 2.3 (Cartesian Morphism)**.** Let F be a category over S. A morphism $\phi : \alpha \to \beta$ in F is Cartesian if for any $\psi : \eta \to \beta$ in F and $g : p(\eta) \to p(\alpha)$ in S with $p(\phi) \circ g = p(\psi)$ in S there exists a unique $\rho : \eta \to \alpha$ lying over g making the diagram

$$\begin{tikzcd} \eta \arrow[d] \arrow[dr, dashed, "\exists!\rho"] \arrow[drr, "\psi"] & & \\ p(\eta) \arrow[dr, "g"] \arrow[drr] & \alpha \arrow[r, "\phi"] \arrow[d] & \beta \arrow[d] \\ & p(\alpha) \arrow[r] & p(\beta) \end{tikzcd}$$

commute, and universal with respect to that property.

One can show that Cartesian morphisms satisfy the following properties.

Proposition 2.4. Let F be a category over S.

(a) The composite of Cartesian arrows in F is Cartesian.
(b) If $\alpha \to \beta$ and $\beta \to \eta$ are morphisms in F and $\beta \to \eta$ is Cartesian then $\alpha \to \beta$ is Cartesian if and only if the composite $\alpha \to \beta \to \eta$ is Cartesian.
(c) A morphism ϕ in F such that $p(\phi)$ is an isomorphism in S is Cartesian if and only if ϕ is an isomorphism in F.
(d) Let $p : \mathsf{C} \to \mathsf{S}$ and $F : \mathsf{F} \to \mathsf{C}$ be functors and $\phi : \alpha \to \beta$ a morphism in F. If ϕ is Cartesian over $F(\phi) : F(\alpha) \to F(\beta)$ in C and $F(\phi)$ is Cartesian over $p(F(\phi)) : p(F(\alpha)) \to p(F(\beta))$ in S then ϕ is Cartesian over $p(F(\phi)) : p(F(\alpha)) \to p(F(\beta))$ in S.

This in turn allows us to define fibered categories.

Definition 2.5 (Fibered Category)**.** Let F be a category over S. F is a fibered category over S if for all $f : A \to B$ in S and β lying over B there is a Cartesian morphism $\phi : \alpha \to \beta$ in F lying over f in S.

This means that for $p : \mathsf{F} \to \mathsf{S}$ a fibered category, objects of F can be pulled back along any morphism in S and that these pullbacks are unique up to unique isomorphism. Naturally we want these fibered functors to amalgamate into the data of a (possibly higher) category. To that end, we define morphisms of fibered categories as follows.

Definition 2.6 (Morphism of Fibered Categories)**.** Let $p : \mathsf{F} \to \mathsf{S}, q : \mathsf{G} \to \mathsf{S}$ be categories fibered over S. A morphism of fibered categories is a functor $F : \mathsf{F} \to \mathsf{G}$ such that $q \circ F = p$ and F takes Cartesian morphisms in F to Cartesian morphisms in G.

The language of fibered categories should be quite suggestive of analogous concepts in algebraic topology. Given a topological space B and $E \to B$ a fibration, one can consider E_b the fiber over $b \in B$. In the context of fibered categories, we define the following.

Definition 2.7 (Fiber)**.** Let $p : \mathsf{F} \to \mathsf{S}$ be a fibered category. For $X \in \mathrm{Obj}(\mathsf{S})$ the fiber F_X is the subcategory of F whose objects are those lying over X and whose morphisms are those lying over id_X.

For a morphism of fibered categories F as in Definition 2.6, the functor F sends F_X to G_X and hence restricts to a subfunctor $F_X : \mathsf{F}_X \to \mathsf{G}_X$. Let us consider how this definition of a fiber is reasonably compatible with the pullback structure we have previously discussed.

Let F be fibered over S and $f : A \to B$ a morphism in S. For each β lying over B in F we choose a pullback $\phi_\beta : f^*\beta \to \beta$ with ϕ_β lying over f and $f^*\beta$ lying over A we can define a functor on the fiber categories $f^* : \mathsf{F}_B \to \mathsf{F}_A$ by defining a map on objects $\beta \mapsto f^*\beta$ and for $\tau : \beta \to \beta'$ over id_B in F_B the unique map $f^*\tau : f^*\beta \to f^*\beta'$ that makes the diagram

$$\begin{tikzcd} f^*\beta \arrow[r, "\phi_\beta"] \arrow[d, dashed, "f^*\tau"'] & \beta \arrow[d, "\tau"] \\ f^*\beta' \arrow[r, "\phi_{\beta'}"'] & \beta' \end{tikzcd}$$

commute.

Definition 2.8 (Cleavage)**.** Let $p : \mathsf{F} \to \mathsf{S}$ be a fibered category. A cleavage of F is a collection of Cartesian morphisms K of F such that for each $f : A \to B$ in S and $\beta \in \mathrm{Obj}(\mathsf{F}_B)$ there is $\alpha \in \mathsf{F}_A$ such that $p(\alpha) = A$ and a unique Cartesian morphism $\phi : \alpha \to \beta$ in K lying over f.

Visually we have the following diagram

$$\begin{tikzcd} \alpha \arrow[r, "\phi"] \arrow[d, "p"'] & \beta \arrow[d, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

where ϕ is a Cartesian morphism in K.

2.1. **Higher Functors and the 2-Category** Cat**.** By the axiom of choice, every fibered category has a cleavage. For $p : \mathsf{F} \to \mathsf{S}$ a fibered category and each $f : A \to B$ there is a functor $f^* : \mathsf{F}_B \to \mathsf{F}_A$ between the fibers. However, there are several fundamental issues. Firstly, the pullback along identities $\mathrm{id}_A^* : \mathsf{F}_A \to \mathsf{F}_A$ need not be identities; secondly, the category of categories does not form a category itself, but instead a 2-category Cat – the 2-category Cat then would have objects categories, morphisms functors between these categories, and 2-morphisms natural transformations between functors. Instead of a functor, we get a lax 2-functor, which we now define.

Definition 2.9 (Lax 2-Functor)**.** Let S be a category. A lax 2-functor Φ on S consists of the following data.

(a) For each $X \in \mathrm{Obj}(\mathsf{S})$ a category $\Phi(X)$.
(b) For each $(f : X \to Y) \in \mathrm{Mor}_{\mathsf{S}}$ a functor $f^* : \Phi(Y) \to \Phi(X)$.
(c) For each $X \in \mathrm{Obj}(\mathsf{C})$ a natural isomorphism $\varepsilon_X : \mathrm{id}_X^* \Longrightarrow \mathrm{id}_{\Phi(X)}$ between functors $\Phi(X) \to \Phi(X)$.
(d) For $(f : X \to Y), (g : Y \to Z) \in \mathrm{Mor}_{\mathsf{S}}$ a natural isomorphism $\alpha_{f,g} : f^*g^* \Longrightarrow (fg)^*$ between functors $\Phi(X) \to \Phi(Z)$ such that the following conditions hold:
   (i) For $\beta \in \mathrm{Obj}(\Phi(Y))$ we have
$$\alpha_{\mathrm{id}_X,f}(\beta) = \varepsilon_X(f^*\beta) : \mathrm{id}_X^* f^*\beta \to f^*\beta$$
   and
$$\alpha_{f,\mathrm{id}_Y}(\beta) = f^*\varepsilon_Y : f^*\mathrm{id}_Y^*\beta \to f^*\beta.$$
   (ii) If also we have $(h : Z \to W) \in \mathrm{Mor}_{\mathsf{S}}$ and $\gamma \in \mathrm{Obj}(\Phi(W))$ the diagram
$$\begin{tikzcd} f^*g^*h^*\gamma \arrow[r, "\alpha_{f,g}(h^*\gamma)"] \arrow[d, "f^*\alpha_{g,h}(\gamma)"'] & (gf)^*h^*\gamma \arrow[d, "\alpha_{gf,h}(\gamma)"] \\ f^*(gh)^*\gamma \arrow[r, "\alpha_{f,hg}(\gamma)"'] & (hgf)^*\gamma \end{tikzcd}$$
   commutes.

This construction of a lax 2-functor rigidifies the construction of a (higher) functor into Cat, the category of categories. Indeed, *a priori*, even if it is natural to do so, there is neither a reason why the functor induced by pullback along $(\mathrm{id}_X : X \to X) \in \mathrm{Mor}_{\mathsf{S}}$ induces an equivalence of categories $\mathrm{id}_{\Phi(X)} : \Phi(X) \to \Phi(X)$ nor is there a reason why for $(f : X \to Y), (g : Y \to Z) \in \mathrm{Mor}_{\mathsf{S}}$ the functors
$$\Phi(Z) \xrightarrow{g^*} \Phi(Y) \xrightarrow{f^*} \Phi(Z)$$
necessarily agrees with the composition
$$\Phi(Z) \xrightarrow{(fg)^*} \Phi(Z).$$
This makes the diagram
$$\begin{tikzcd} \Phi(X) & & \Phi(Z) \arrow[ll, "(fg)^*"'] \arrow[dl, "g^*"] \\ & \Phi(Y) \arrow[ul, "f^*"] \arrow[u, phantom, "\overset{\Uparrow}{\alpha_{f,g}}"] & \end{tikzcd}$$
rigid in the sense that the composition is unique.

Remark. The astute would note this is the definition of a contravariant lax 2-functor. We omit the more general definitions for ease of exposition.

Remark. Those familiar with higher categorical language will note that the exposition above is equivalent to the 2-category of categories Cat being a strict 2-category, in the sense that all coherence data above the level of 2-morphisms are rigid.

One would expect a fibered category $p : \mathsf{F} \to \mathsf{S}$ equipped with a cleavage gives rise to a lax 2-functor – the cleavage here gives uniqueness of the functor up to a choice of preimage. This is indeed the case as we now show.

Lemma 2.10. Let $p : \mathsf{F} \to \mathsf{S}$ be a fibered category. p defines a lax 2-functor to the category of categories associating to each object $A \in \mathrm{Obj}(\mathsf{S})$ a category

$$A \mapsto \mathsf{F}_A$$

and to each morphism a functor

$$(f : A \to B) \mapsto (f^* : \mathsf{F}_B \to \mathsf{F}_A).$$

Proof. Evidently (a) and (b) are fulfilled. (c) follows from the fact that pullbacks are unique up to unique isomorphism so $\mathrm{id}_A^* : \mathsf{F}_A \to \mathsf{F}_A$ is naturally isomorphic to the identity functor. For (d) any functor completing the solid diagram

$$\begin{tikzcd}
\mathsf{F}_A & & \mathsf{F}_C \arrow[ll, dashed] \arrow[dl, "g^*"] \\
& \mathsf{F}_B \arrow[ul, "f^*"] &
\end{tikzcd}$$

it is unique since $\mathrm{id}_{\mathsf{F}_A} : \mathsf{F}_A \to \mathsf{F}_A$ is Cartesian.

(d)(i) is given by the commutativity of the following diagram

$$\begin{tikzcd}
\mathrm{id}_A^* f^*\beta \arrow[d] & & & \\
A \arrow[rrr, "f"] \arrow[drr, "\mathrm{id}_A"'] & & & \beta \arrow[lllu, bend right, "\mathrm{id}_A^* f^*"'] \arrow[dl, "f^*"] \arrow[d] \\
& & f^*\beta \arrow[uull, "\mathrm{id}_A^*\cong\mathrm{id}_{\mathsf{F}_A}"] \arrow[d] & B \\
& & A \arrow[ur, "f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*\mathrm{id}_B^*\beta \arrow[d] & & & \\
A \arrow[rrrd, "f"] \arrow[drr, "f"'] & & & \beta \arrow[lllu, bend right, "f^*\mathrm{id}_B^*"'] \arrow[dl, "\mathrm{id}_B^*\cong\mathrm{id}_{\mathsf{F}_B}"] \arrow[d] \\
& & \mathrm{id}_B^*\beta \arrow[uull, "f^*"] \arrow[d] & B \\
& & B \arrow[ur, "\mathrm{id}_B"'] &
\end{tikzcd}$$

To verify (d)(ii) with the setup $f : A \to B, g : B \to C, h : C \to D$ we have $f^*g^*h^*\delta$ and $(hgf)^*\delta$ both pullbacks of some $\delta \in \mathrm{Obj}\,(\mathsf{F}_D)$ but there is a unique morphism in F_A lying over id_A such that the diagram

$$\begin{tikzcd}
f^*g^*h^*\delta \arrow[r] \arrow[d, "p"'] & (hgf)^*\delta \arrow[d, "p"] \\
A \arrow[r, "\mathrm{id}_A"'] & A
\end{tikzcd}$$

commutes by the definition of Cartesian arrows and the natural isomorphisms in the following diagram

$$\begin{tikzcd}
f^*g^*h^*\delta \arrow[r, "\alpha_{f,g}(h^*\delta)"] \arrow[d, "f^*\alpha_{g,h}(\delta)"'] \arrow[dr] & (gf)^*h^*\delta \arrow[d, "\alpha_{gf,h}(\delta)"] \arrow[dl, phantom, "\Swarrow"] \\
f^*(hg)^*\delta \arrow[r, "\alpha_{f,hg}(\delta)"'] \arrow[ur, phantom, "\Nearrow"] & (hgf)^*\delta
\end{tikzcd}$$

as desired. ∎

Under special circumstances, a lax 2-functor from a category S to the category of categories can be made into a functor.

Definition 2.11 (Splitting Cleavage)**.** Let $p : \mathsf{F} \to \mathsf{S}$ be a fibered category with cleavage K. The cleavage K is splitting if it contains all identities and is closed under composition.

One then shows the following.

Lemma 2.12. The lax 2-functor associated to the cleavage is a functor if and only if the cleavage is splitting.

Proof. If the cleavage is splitting then the composition law is rigid with unitality and associativity by Definition 2.9. ∎

Remark. Generally, fibered categories do not admit a splitting.

The converse to Lemma 2.10 also holds. That is, given a fibered category with a lax 2-functor, one can construct a cleavage.

Lemma 2.13. Suppose $p : \mathsf{F} \to \mathsf{S}$ is a fibered category and Φ some lax 2-functor on S given on objects by

$$A \mapsto \mathsf{F}_A$$

and on morphisms by

$$(f : A \to B) \mapsto (f^* : \mathsf{F}_B \to \mathsf{F}_A).$$

There exists a cleavage K realizing Φ.

Evidently this results in the following theorem.

Theorem 2.14. Let $p : \mathsf{F} \to \mathsf{C}$ be a fibered category. The following are equivalent.

(a) There is a lax 2-functor $\Phi : \mathsf{S} \to \mathsf{Cat}$.
(b) The category S has a cleavage K.

Proof. Immediate from Lemma 2.13 and Lemma 2.10. ∎

We list an additional property of fibered categories.

Definition 2.15 (Stable Property of Morphisms)**.** A collection of morphisms $\mathsf{P} \subseteq \mathrm{Mor}_{\mathsf{C}}$ of a category C is stable if the following two conditions hold:

(a) If $f : A \to B$ is in P and $\phi : A' \to A, \psi : B' \to B$ are isomorphisms then the composition $(\psi \circ f \circ \phi) : A' \to B'$ is in P as well.
(b) Given $A \to B$ in P and any map $C \to B$ the fibered product

$$\begin{tikzcd} A \times_B C \arrow[r] \arrow[d] & C \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

exists and $A \times_B C \to C$ is in P.

2.2. **Categories Fibered in Groupoids.** We now turn to a very important example of fibered categories. Categories fibered in groupoids. Recall here the following definition from category theory.

Definition 2.16 (Groupoid)**.** Let C be a category. C is a groupoid if and only if every morphism in C is an isomorphism.

Naturally, we define a category fibered in groupoids as follows.

Definition 2.17 (Category Fibered in Groupoids)**.** Let $p : \mathsf{F} \to \mathsf{S}$ be a fibered category. $p : \mathsf{F} \to \mathsf{S}$ is a category fibered in groupoids if for all $A \in \mathrm{Obj}(\mathsf{S})$ the category F_A is a groupoid.

One can alternatively characterize categories fibered in groupoids as follows.

Lemma 2.18. Let $p : \mathsf{F} \to \mathsf{S}$ be a fibered category. $p : \mathsf{F} \to \mathsf{S}$ is fibered in groupoids if and only if the following two properties hold:

(a) Every arrow in F is Cartesian.
(b) Given $\beta \in \mathrm{Obj}(\mathsf{F})$ and $f : A \to p(\beta)$ in S, there exists $\phi : \alpha \to \beta$ in F such that $p(\phi) = f$.

Proof. ($\Longrightarrow$) Suppose $p : \mathsf{F} \to \mathsf{S}$ is a category fibered in groupoids. Lifts exist and are unique by $\mathrm{id}_{\mathsf{F}_B}$ Cartesian satisfying (ii). For (i), let $\phi : \alpha \to \beta$ lying over $f : A \to B$ and $\phi' : \alpha' \to \beta$ a pullback of $\beta \in \mathrm{Obj}(\mathsf{F}_B)$ to F_A, there is a map $\eta : \alpha \to \alpha'$ in F_A which is an isomorphism since $p : \mathsf{F} \to \mathsf{S}$ is a category fibered in groupoids. But isomorphisms are unique so ϕ is Cartesian.

($\Longleftarrow$) Suppose conditions (a) and (b) hold and let $\phi : \alpha' \to \alpha$ be a morphism F_A over $A \in \mathrm{Obj}(\mathsf{S})$. Since this morphism ϕ is Cartesian there is $\psi : \alpha \to \alpha'$ filling in the following diagram.

$$\begin{tikzcd}
\alpha \arrow[d] \arrow[dr, dashed, "\psi"] \arrow[drr, "\mathrm{id}_\alpha"] & & \\
A \arrow[dr, "\mathrm{id}_A"] \arrow[drr, "\mathrm{id}_A"] & \alpha' \arrow[r, "\phi"] \arrow[d] & \alpha \arrow[d] \\
& A \arrow[r, "\mathrm{id}_A"] & A
\end{tikzcd}$$

By the commutativity of the upper triangle, $\phi \circ \psi \Longrightarrow \mathrm{id}_\alpha$ so ψ is a right inverse of ϕ and by the diagram

$$\begin{tikzcd}
\alpha' \arrow[d] \arrow[dr, dashed, "\phi"] \arrow[drr, "\mathrm{id}_{\alpha'}"] & & \\
A \arrow[dr, "\mathrm{id}_A"] \arrow[drr, "\mathrm{id}_A"] & \alpha \arrow[r, "\psi"] \arrow[d] & \alpha' \arrow[d] \\
& A \arrow[r, "\mathrm{id}_A"] & A
\end{tikzcd}$$

so too is ϕ the right inverse of ψ, showing ϕ is an isomorphism. ■

We now consider the higher categorical variant of Yoneda's lemma. Recall that we have seen how a category C can be embedded into the functor category $\mathrm{Fun}(\mathsf{C}^{\mathsf{Opp}}, \mathsf{Sets})$, equivalently described as the category of set-valued presheaves on C. One can similarly define for a fibered category $p : \mathsf{F} \to \mathsf{S}$ an embedding $\mathrm{Fun}(\mathsf{S}^{\mathsf{Opp}}, \mathsf{Sets})$ into the 2-category of fibered categories over S denoted $\mathsf{FibCat}(\mathsf{S})$ by taking for $F \in \mathrm{Obj}\left(\mathrm{Fun}(\mathsf{S}^{\mathsf{Opp}}, \mathsf{Sets})\right)$ defining the fiber to be the collection of objects

$$\mathsf{F}_A = \{(A, \alpha) \in \mathrm{Obj}(\mathsf{S}) \times F(A)\}$$

and morphisms $f^* : \mathsf{F}_B \to \mathsf{F}_A$ such that

$$f^*\left((B, F(f(\alpha)))\right) = (A, \alpha).$$

By composing these embeddings, we have an embedding of a category C to the 2-category of categories fibered over C, $\mathsf{FibCat}(\mathsf{C})$. For a category C, the map on objects takes $Z \in \mathrm{Obj}(\mathsf{C})$ to the slice category $\mathsf{C}_{(-/Z)}$ whose objects are morphisms $X \to Z$ and whose morphisms are commutative triangles

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\
& Z &
\end{tikzcd}$$

which we denote f, taking the morphisms to Z, known as the structure morphisms, as implicit. One yields a map $\mathsf{C}_{(-/Z)} \to \mathsf{C}$ by $(X \to Z) \mapsto X$, compatible with morphisms in the obvious way.

Definition 2.19 (Representable Fibered Category)**.** A fibered category over C is representable if it is equivalent as a category to a slice category $\mathsf{C}_{(-/Z)}$ for some $Z \in \mathrm{Obj}(\mathsf{C})$.

We now state and prove the 2-categorical Yoneda lemma.

Lemma 2.20 (2-Categorical Yoneda)**.** Let $p : \mathsf{F} \to \mathsf{C}$ be a fibered category and $Z \in \mathrm{Obj}(\mathsf{C})$. There is an equivalence of categories

$$\mathrm{Fun}\left(\mathsf{C}_{(-/Z)}, \mathsf{F}\right) \to \mathsf{F}_Z.$$

3. Descent

Descent generalizes the identity and gluing axioms for sheaves on topological spaces to the setting of sites. We have already seen that for S a site, we should think of a fibered category over S as a lax 2-functor in the sense of Definition 2.9, that is, as a presheaf of categories on S. A stack, which we will soon encounter, is a sheaf of categories over S.

3.1. **Descent in Fibered Categories.** Let $p : \mathsf{F} \to \mathsf{S}$ be a category fibered over a site S with a fixed cleavage K. For a covering $\{X_i \to X\}$, denote $X_{ij} = X_i \times_X X_j$ and $X_{ijk} = X_i \times_X X_j \times_X X_k$. We now define an object with descent data.

Definition 3.1 (Object With Descent Data)**.** Let $\{X_i \to X\}$ be a covering in S. An object with descent data on $\{X_i \to X\}$ is the tuple $(\{\xi_i\}, \{\phi_{ij}\})$ with objects $\xi_i \in \mathsf{F}_{X_i}$ and isomorphisms $\phi_{ij} : \mathrm{pr}_2^*\xi_j \to \mathrm{pr}_1^*\xi_i$ in $\mathsf{F}_{X_{ij}}$ such that

$$\mathrm{pr}_{1,3}^*\phi_{ik} = \mathrm{pr}_{1,2}^*\phi_{ij} \circ \mathrm{pr}_{2,3}^*\phi_{jk} : \mathrm{pr}_3^*\xi_k \to \mathrm{pr}_1^*\xi_i.$$

Remark. The maps ϕ_{ij} are known as transition isomorphisms of the object X with descent data.

One naturally defines morphisms of objects with descent data as follows.

Definition 3.2 (Morphisms of Objects with Descent Data)**.** A morphism between objects with descent data on $\{X_i \to X\}$, $(\{\xi_i\}, \{\phi_{ij}\})$ and $(\{\upsilon_i\}, \{\psi_{i,j}\})$, is a tuple $\{\alpha_i\}$ with $\alpha_i : \mathsf{F}_{X_i} \to \mathsf{F}_{X_i}$ where $\xi_i \mapsto \upsilon_i$ such that for each pair i, j the diagram

$$\begin{tikzcd}
\mathrm{pr}_2^*\xi_j \arrow[r, "\mathrm{pr}_2^*\alpha_j"] \arrow[d, "\phi_{ij}"'] & \mathrm{pr}_2^*\upsilon_j \arrow[d, "\psi_{ij}"] \\
\mathrm{pr}_1^*\xi_i \arrow[r, "\mathrm{pr}_1^*\alpha_i"'] & \mathrm{pr}_1^*\upsilon_i
\end{tikzcd}$$

commutes.

The morphisms α_i compose in the obvious way making objects with descent data the objects of a category $\mathsf{F}_{\{X_i \to X\}}$.

More explicitly, let $\xi \in \mathsf{F}_X$ and $\{\sigma_i : X_i \to X\}$ a covering. We can construct an object with descent data as follows. Set $\xi_i = \sigma_i^*\xi$ and transition isomorphisms the identity after identifying $\mathrm{pr}_2^*\sigma_j^*\xi$ and $\mathrm{pr}_1^*\sigma_i^*\xi$, the pullbacks of ξ to X_{ij}. For some $\alpha : \xi \to \upsilon$ in F_X we get $\alpha_i = \sigma_i^*\alpha : \sigma_i^*\xi \to \sigma_i^*\upsilon$, yielding a morphism of objects with descent data $\{\alpha_i\}$ from $(\{\xi_i\}, \{\phi_{ij}\})$ to $(\{\upsilon_i\}, \{\psi_{ij}\})$. Evidently this is a functor $\mathsf{F}_X \to \mathsf{F}_{\{X_i \to X\}}$ where on objects is given by

$$\xi \mapsto (\{\xi_i\}, \{\phi_{ij}\})$$

and on morphisms by

$$(\alpha : \xi \to \upsilon) \mapsto \{\alpha_i : \xi_i \to \upsilon_i\}$$

which are compatible with the transition isomorphisms ϕ_{ij}, ψ_{ij}.

Remark. This construction does not depend on the cleavage in the sense that the categories $\mathsf{F}_{\{X_i \to X\}}$ are equivalent regardless of the choice of cleavage.

We can now define some long-awaited notions.

Definition 3.3 (Prestack)**.** Let $p : \mathsf{F} \to \mathsf{S}$ be a category fibered over a site. F is a prestack over S if for each covering $\{X_i \to X\}$ in S the functor $\mathsf{F}_X \to \mathsf{F}_{\{X_i \to X\}}$ is fully faithful.

Definition 3.4 (Stack)**.** Let $p : \mathsf{F} \to \mathsf{S}$ be a category fibered over a site. F is a stack over S if for each covering $\{X_i \to X\}$ in S the functor $\mathsf{F}_X \to \mathsf{F}_{\{X_i \to X\}}$ is an equivalence of categories.

Unpacking the definition, to be a prestack means that for $X \in \mathrm{Obj}(\mathsf{S})$; $\xi, \upsilon \in \mathrm{Obj}(\mathsf{F}_X)$; $\{X_i \to X\}$ a covering; ξ_i, υ_i pullbacks of ξ, υ to the X_i; ξ_{ij}, υ_{ij} pullbacks to X_{ij} and there was some $a_i : \xi_i \to \upsilon_i$ in F_{X_i} such that $\mathrm{pr}_1^* \alpha_i = \mathrm{pr}_2^* \alpha_j : \xi_{ij} \to \upsilon_{ij}$ then there is a unique $\alpha : \xi \to \upsilon$ in F_X whose pullback is $\alpha_i : \xi_i \to \upsilon_i$ for all i. Clarifying what happens if $p : \mathsf{F} \to \mathsf{S}$ is a stack will require introducing the following notion.

Definition 3.5 (Effective Descent Data)**.** An object with descent data $(\{\xi_i\}, \{\phi_{ij}\})$ in $\mathsf{F}_{\{X_i \to X\}}$ is effective if it is isomorphic to the image of an object in F_X.

In other words an object with descent data $(\{\xi_i\}, \{\phi_{ij}\})$ in $\mathsf{F}_{\{X_i \to X\}}$ is effective if there exists $\xi \in \mathrm{Obj}(\mathsf{F}_X)$ and Cartesian arrows $\xi_i \to \xi$ over $\sigma_i : X_i \to X$ such that the pentagonal diagram

$$\begin{tikzcd}
 & & \xi & & \\
\xi_i \arrow[urr] & & & & \xi_j \arrow[ull] \\
 & \mathrm{pr}_1^* \xi_i \arrow[ul] & & \mathrm{pr}_2^* \xi_j \arrow[ur] \arrow[ll, "\phi_{ij}"] &
\end{tikzcd}$$

commutes for all i, j.

If $p : \mathsf{F} \to \mathsf{S}$ is a stack as in Definition 3.4, we know the functor $\mathsf{F}_X \to \mathsf{F}_{\{X_i \to X\}}$ is fully faithful and essentially surjective. So all $(\{\xi_i\}, \{\phi_{ij}\})$ are isomorphic to the image of some object in F_X, that is, all descent data is effective.

3.2. **Descent for Torsors.** Torsors are closely related to principal G-bundles in algebraic topology, and are central objects in the study of both algebraic and arithmetic geometry.

Definition 3.6 (Torsor)**.** Let S be a site and $\boldsymbol{\mu}$ a sheaf of groups on S. A $\boldsymbol{\mu}$-torsor on S is a pair $(\mathcal{P}, \rho)$ where $\mathcal{P}$ is a sheaf on S with a left action

$$\rho : \boldsymbol{\mu} \times \mathcal{P} \longrightarrow \mathcal{P}$$

such that the following conditions hold:

(1) For all $X \in \mathrm{Obj}(\mathsf{S})$ there is a covering $\{X_i \to X\}_{i \in I}$ such that $\mathcal{P}(X_i) \neq \emptyset$ for all $i \in I$.
(2) The morphism of sheaves

$$\boldsymbol{\mu} \times \mathcal{P} \to \mathcal{P} \times \mathcal{P}; (g, p) \mapsto (p, g \cdot p)$$

is an isomorphism.

Remark. The sheaf of groups $\boldsymbol{\mu}$ need not be a sheaf of Abelian groups.

Remark. The condition (b) in Definition 3.6 says that for $\mathcal{P}(X)$ the action of the group $\boldsymbol{\mu}(X)$ on $\mathcal{P}(X)$ is simply transitive: that it is both free – the identity is the only element fixing any point – and for any two sections $p, p' \in \mathcal{P}(X)$ there is a unique $g \in \mu(X)$ such that $g \cdot p = p'$.

Definition 3.7 (Trivial Torsor)**.** Let S be a site, $\boldsymbol{\mu}$ a sheaf of groups on S, and $(\mathcal{P}, \rho)$ a $\boldsymbol{\mu}$-torsor on S if $\mathcal{P}$ has a global section.

If $(\mathcal{P}, \rho)$ is a trivial $\boldsymbol{\mu}$-torsor on a site S, then for a global section $p \in \mathcal{P}(X)$ we have an isomorphism

$$\boldsymbol{\mu} \to \mathcal{P}; g \mapsto g \cdot p$$

identifying $\mathcal{P}$ with $\boldsymbol{\mu}$ and the action ρ with the endomorphism by left multiplication. Morphisms of torsors are defined in the natural way as follows.

Definition 3.8 (Morphism of Torsors)**.** Let S be a site, $\boldsymbol{\mu}$ a sheaf of groups on S, and $(\mathcal{P}, \rho), (\mathcal{P}, \rho')$ two $\boldsymbol{\mu}$-torsors on S. A morphism of torsors $f : (\mathcal{P}', \rho') \to (\mathcal{P}, \rho)$ is the data of a morphism of sheaves $f : \mathcal{P}' \to \mathcal{P}$ such that the diagram

$$\begin{tikzcd} \boldsymbol{\mu} \times \mathcal{P}' \arrow[r, "\mathrm{id}_{\boldsymbol{\mu}} \times f"] \arrow[d, "\rho'"'] & \boldsymbol{\mu} \times \mathcal{P} \arrow[d, "\rho"] \\ \mathcal{P}' \arrow[r, "f"'] & \mathcal{P} \end{tikzcd}$$

commutes.

Torsors are intimately connected to principal G-bundles in algebraic geometry and algebraic topology.

Definition 3.9 (Principal G-Bundle)**.** Let $(\mathsf{Sch}_X)_{\mathsf{fppf}}$ denote the fppf site of X-schemes for X a base scheme and $\boldsymbol{\mu}$ a sheaf of groups on $(\mathsf{Sch}_X)_{\mathsf{fppf}}$ representable by a flat locally finitely presented X-group scheme G. A principal G-bundle on X is a pair $(\pi : P \to X, \rho)$ where $\pi : P \to X$ is a flat, locally finitely presented, surjective morphism of schemes and

$$\rho : G \times_X P \to P$$

a morphism such that the following conditions hold:

(a) The diagram

$$\begin{tikzcd} G \times_X G \times_X P \arrow[r, "\mathrm{id}_G \times \rho"] \arrow[d, "m \times \mathrm{id}_P"'] & G \times_X P \arrow[d, "\rho"] \\ G \times_X P \arrow[r, "\rho"'] & P \end{tikzcd}$$

commutes, here denoting $m : G \times_X G \to G$ the group operation on the group scheme G.

(b) If $e : X \to G$ is the identity action, the composition

$$P \xrightarrow{e \circ \pi \times \mathrm{id}_P} G \times_X P \xrightarrow{\rho} P$$

is the identity on P.

(c) The map $(\rho, \mathrm{pr}_2) : G \times_X P \to P \times_X P$ is an isomorphism

The morphisms of principal G-bundles are given in the obvious way.

Definition 3.10 (Morphisms of Principal G-Bundles)**.** Let $(\mathsf{Sch}_X)_{\mathsf{fppf}}$ denote the fppf site of X-schemes for X a base scheme and $\boldsymbol{\mu}$ a sheaf of groups on $(\mathsf{Sch}_X)_{\mathsf{fppf}}$ representable by a flat locally finitely presented X-group scheme G. A morphism of principal G-bundles

$$f : (\pi' : P' \to X, \rho') \to (\pi : P \to X, \rho)$$

is a morphism of X-schemes $f : P' \to P$ such that the diagram

$$\begin{tikzcd} G \times_X P' \arrow[r, "\mathrm{id}_G \times f"] \arrow[d, "\rho'"'] & G \times_X P \arrow[d, "\rho"] \\ P' \arrow[r, "f"'] & P \end{tikzcd}$$

commutes.

Given a principal G-bundle $(\pi : P \to X, \rho)$ on X, we get a μ-torsor bytaking $\mathcal{P}$ to be the sheaf on $(\mathsf{Sch}_X)_{\mathsf{fppf}}$ represented by the scheme P with action that induced by ρ. The conditions (a) and (b) of Definition 3.9 imply that the map $\boldsymbol{\mu} \times \mathcal{P} \to \mathcal{P}$ is an action while condition (c) imposes that the action is simply transitive. Furthermore, since $\pi : P \to X$ is flat, locally finitely presented, and surjective as a map of schemes, there exists an fppf cover $\{X_i \to X\}$ such that $\mathcal{P}(X_i) \neq \emptyset$ for all $i \in I$.

Furthermore, one can observe that these constructions are functorial, yielding a categories of principal G-bundles on a scheme X and $\boldsymbol{\mu}$-torsors on the fppf site of X-schemes $(\mathsf{Sch}_X)_{\mathsf{fppf}}$. Indeed, Yoneda's lemma tells us that this functor is fully faithful that is essentially surjective on the condition that the structure morphism of the algebraic group G is affine.

Proposition 3.11. Let $(\mathsf{Sch}_X)_{\mathsf{fppf}}$ denote the fppf site of X-schemes for X a base scheme and $\boldsymbol{\mu}$ a sheaf of groups on $(\mathsf{Sch}_X)_{\mathsf{fppf}}$ representable by a flat locally finitely presented X-group scheme G. If the structure morphism $G \to X$ is an affine morphism then there is a equivalence between the category of principal G-bundles on X and the category $\boldsymbol{\mu}$-torsors on the fppf site of X-schemes $(\mathsf{Sch}_X)_{\mathsf{fppf}}$.

In some simple cases, we can describe the category of torsors in a relatively concrete manner.

Consider the scheme X and n an integer invertible on the scheme X. Denote $\boldsymbol{\mu}_n$ the group scheme such that

$$\boldsymbol{\mu}_n(S) = \left\{ f \in \mathcal{O}_X^\times : f^n = 1 \right\}.$$

The category of $\boldsymbol{\mu}_n$-torsors $\mathsf{Tors}(\boldsymbol{\mu}_n)$ on the small étale site of X $X_{\text{ét}}$ can be described as follows: consider Σ_n the category with objects pairs (L, σ) where L is an invertible sheaf on the scheme X and $\sigma : L^{\otimes n} \to \mathcal{O}_X$ a trivialization of the n-th power of L, considering L as a sheaf in the étale topology. The morphisms in Σ_n between two objects (L', σ') and (L, σ) are morphisms of line bundles $\rho : L' \to L$ such that the diagram

$$\begin{tikzcd} L'^{\otimes n} \arrow[rr, "\rho^{\otimes n}"] \arrow[dr, "\sigma'"'] & & L^{\otimes n} \arrow[dl, "\sigma"] \\ & \mathcal{O}_X & \end{tikzcd}$$

commutes. One can construct a functor

$$F : \Sigma_n \longrightarrow \mathsf{Tors}(\boldsymbol{\mu}_n)$$

by associating to each $(L, \sigma) \in \Sigma_n$ a sheaf $\mathcal{P}_{(L,\sigma)}$ a sheaf on the étale site $X_{\text{ét}}$ such that for any $U \to X$ étale $\mathcal{P}_{(L,\sigma)}(U)$ is the set of trivializations $\lambda : \mathcal{O}_U \to L|_U$ such that the composite

$$\mathcal{O}_U \xrightarrow{\lambda^{\otimes n}} L^{\otimes n}|_U \xrightarrow{\sigma|_U} \mathcal{O}_U$$

is the identity on the sheaf $\mathcal{O}_U$. There is an action of $\boldsymbol{\mu}_n(U)$ on $\mathcal{P}_{(L,\sigma)}(U)$ for which $\zeta \in \boldsymbol{\mu}_n(U)$ acts by $\lambda \mapsto \zeta \cdot \lambda$ which is simply transitive, endowing $\mathcal{P}$ with the structure of a $\boldsymbol{\mu}_n$-torsor restricting on fibers in the previously described way.

Remark. It is necessary to work in the étale topology here, since it is not always possible to find a trivialization of the line bundle Zariski-locally.

Algebraic Spaces

4. Algebraic Spaces

We begin with a discussion of various algebro-geometric and categorical preliminaries required to define algebraic spaces.

4.1. **The Category of Schemes.** First recall that we have defined stable properties of morphisms in Definition 2.15. We now define a stable property of objects in a category.

Definition 4.1 (Stable Class of Objects)**.** Let S be a site such that all representable presheaves are sheaves. A subclass of objects $\mathrm{Obj}(\mathsf{C}) \subseteq \mathrm{Obj}(\mathsf{S})$ is stable with respect to a property P if for all coverings $\{X_i \to X\}$, X has property P if and only if X_i have property P for all i.

Definition 4.2 (Stable Property of Objects)**.** A property P is stable if the class of objects containing P is stable.

Turning to some categorical notions, we define the following.

Definition 4.3 (Closed Subcategory)**.** Let C be a category. A closed subcategory D of C is a subcategory such that the following conditions hold:

(a) D contains all isomorphisms in C.
(b) For all Cartesian diagrams in C

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

if $f \in \mathrm{Mor}_{\mathsf{D}}$ then $f' \in \mathrm{Mor}_{\mathsf{D}}$.

Definition 4.4 (Stable Subcategory)**.** Let D be a closed subcategory of a category C. D is stable if for all $(f : X \to Y) \in \mathrm{Mor}_{\mathsf{C}}$ and all coverings $\{Y_i \to Y\}$, f is a morphism in D if and only if all morphisms $f_i : X \times_Y XY_i \to Y_i$ are in D.

Definition 4.5 (Local On Domain)**.** A stable closed subcategory D of a category C is local on domain if for all $f : X \to Y$ and coverings $\{X_i \to X\}$, f is a morphism in D if and only if $X_i \to X \to Y$ is in D.

We alternatively define stable properties of morphisms Definition 2.15 as follows.

Definition 4.6 (Stable Property of Morphisms)**.** Let P be some property of morphisms in C satisfied by isomorphisms and closed under compositions and D_{P} the subcategory of C with the same objects but morphisms those satisfying property P. P is a stable property of morphisms if D_{P} is a stable subcategory.

Similarly, we define locality on domain as follows.

Definition 4.7 (Morphisms Local on Domain)**.** Let P be some property of morphisms in C satisfied by isomorphisms and closed under compositions and D_{P} the subcategory of C with the same objects but morphisms those satisfying property P. P is a stable property of morphisms if D_{P} is local on domain as a subcategory.

One can show the following fact about morphisms of schemes.

Proposition 4.8. Let S be a scheme and consider the big étale site $(\mathsf{Sch}_S)_{\text{ét}}$.

(a) The following properties of morphisms in $(\mathsf{Sch}_S)_{\text{ét}}$ are stable: proper, separated, surjective, and quasi-compact.
(b) The following properties of morphisms in $(\mathsf{Sch}_S)_{\text{ét}}$ are stable and local on domain: locally of finite type, locally of finite presentation, flat, étale, universally open, locally quasi-finite, and smooth.

4.2. **Representability and Sheaves on the Étale Site.** We now introduce some language to describe morphisms of sheaves on the étale site.

Definition 4.9 (Representable by Schemes)**.** Let S be a scheme and let F, G be sheaves on the site $(\mathsf{Sch}_S)_{\text{ét}}$ and $f : \mathsf{F} \to \mathsf{G}$ a morphism of sheaves. The morphism f is representable by schemes if for every S-scheme T and morphism $T \to \mathsf{G}$ the fibered product functor $\mathsf{F} \times_{\mathsf{G}} T$ is representable by a scheme.

Definition 4.10 (Properties of Morphisms of Étale Sheaves)**.** Let S be a scheme and let F, G be sheaves on the site $(\mathsf{Sch}_S)_{\text{ét}}$, $f : \mathsf{F} \to \mathsf{G}$ a morphism of sheaves, and P a stable property of morphisms of schemes. If f is representable by schemes, then f has property P if for every S-scheme T the morphism $\mathrm{pr}_2 : \mathsf{F} \times_{\mathsf{G}} T \to T$ has P.

Example 4.11. Let F, G be representable sheaves, say by X, Y, respectively. Any morphism $f : \mathsf{F} \to \mathsf{G}$ is representable by sheaves, induced by the corresponding morphism of schemes $X \to Y$. For any S-scheme T with a map $T \to Y$ we have that $\mathsf{F} \times_{\mathsf{G}} T$ is the scheme $T \times_Y X$.

Lemma 4.12. Let S be a scheme and $f : X \to Y$ be a S-morphism of schemes and P a stable property of morphisms of schemes. The morphism f has property P if and only if the morphism of representable sheaves $h_f : h_X \to Y$ has property P.

Proof. Let T be an S-scheme and $g : T \to Y$ with $h_g : h_T \to h_Y$ where we have a natural isomorphism $h_T \times_{h_Y} h_X \Longrightarrow h_{X \times_Y T}$.

$$\begin{array}{ccc} X \times_Y T & \longrightarrow & T \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array} \qquad\qquad \begin{array}{ccc} h_T \times_{h_Y} h_X \Rightarrow h_{X \times_Y T} & \longrightarrow & h_T \\ \downarrow & & \downarrow \\ h_X & \longrightarrow & h_Y \end{array}$$

We have that f is P if and only if $X \times_Y T \to T$ is P if and only if $h_{X \times_Y T} \to h_T$ is P if and only if h_f is P upon repeated application of the stable property of morphisms. ∎

We prove an additional lemma before defining algebraic spaces.

Lemma 4.13 (Representable Diagonal Implies Representability)**.** Let S be a scheme and F a sheaf on $(\mathsf{Sch}_S)_{\text{ét}}$. If $\delta_{\mathsf{F}} : \mathsf{F} \to \mathsf{F} \times \mathsf{F}$ is representable by schemes and T any S-scheme then the morphism of sheaves $f : T \to \mathsf{F}$ is representable by schemes.

Proof. This follows from general abstract nonsense. See [4, Lemma 0022] and [4, Lemma 0024]. ∎

4.3. **Algebraic Spaces.** We are now able to define algebraic spaces, an important, slightly more restricted notion of an algebraic stack.

Definition 4.14 (Algebraic Space)**.** Let S be a scheme. An algebraic space over S is a functor $X : \mathsf{Sch}_S^{\mathsf{Opp}} \to \mathsf{Sets}$ such that the following hold:

(a) X is a sheaf on the big étale site $(\mathsf{Sch}_S)_{\text{ét}}$.
(b) $\delta_{\mathsf{F}} : \mathsf{F} \to \mathsf{F} \times_S \mathsf{F}$ is representable by schemes.
(c) There is an S-scheme U and a surjective étale morphism $U \to X$.

Remark. Let S be a scheme. Objects of the category of S-schemes Sch_S are tautologically algebraic spaces.

Given a base scheme S, we can consider the category of algebraic spaces Spaces_S with objects sheaves of sets on the big étale site $(\mathsf{Sch}_S)_{\text{ét}}$ and morphisms base-preserving 2-functors, or morphisms of sheaves on the site.

4.4. **Sheaf Quotients.** We begin with the definition of an étale equivalence relation.

Definition 4.15 (Étale Equivalence Relation)**.** Let S be a scheme. An étale equivalence relation on an S-scheme X is a monomorphism of schemes $R \hookrightarrow X \times_S X$ such that

(a) For all S-schemes T, the T-points $R(T) \subseteq X(T) \times X(T)$ is an equivalence relation on $X(T)$.
(b) The maps $s, t : R \to X$ induced by the projections from $X \times_S X$ are étale morphisms.

$$\begin{tikzcd}
R \arrow[dr, hook] \arrow[drr, bend left, "s"] \arrow[ddr, bend right, "t"'] & & \\
& X \times_S X \arrow[r] \arrow[d] & X \\
& X &
\end{tikzcd}$$

Remark. If $S = \operatorname{Spec} \mathbb{Z}$ then Definition 4.15 defines an "absolute" étale equivalence relation, as opposed to the S-relative notion we just discussed. We omit these technicalities in our discussion going forward.

Taking the quotient by R, we have a functor

$$(\mathsf{Sch}_S)^{\mathsf{Opp}} \longrightarrow \mathsf{Sets}$$

by

$$T \mapsto X(T)/R(T).$$

We denote the étale sheaf X/R which is in fact an algebraic space.

Proposition 4.16. (a) Let S be a scheme. If R is an étale equivalence relation on an S-scheme X, then X/R is an algebraic space.
(b) If further Y is an S-algebraic space and $X \to Y$ an étale surjective morphism, then R is an étale equivalence relation and $X/R \to Y$ is an isomorphism.

4.5. **Properties of Algebraic Spaces.** We consider various properties of algebraic spaces, and how they relate to to properties of schemes.

Definition 4.17 (Property of Algebraic Space)**.** Let P be a property of schemes stable on the étale site. An algebraic space X has the property P if there exists a scheme U with property P and an étale surjection $U \to X$.

One can define a property of a morphism of algebraic spaces in the following way.

Definition 4.18 (Properties of Morphisms of Algebraic Spaces)**.** Let $f : X \to Y$ be a morphism of S-algebraic spaces and P a property of morphisms stable on the étale site. The morphism f has the property P if there exists a scheme V and an étale surjection $V \to Y$ such that

$$\begin{tikzcd} V \times_Y X \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

the morphism $V \times_Y X \to V$ has P.

One particularly nice property of the category of S-algebraic spaces is the following.

Proposition 4.19. The category of S-algebraic spaces Spaces_S has all finite limits.

Building on Definitions 4.17 and 4.18, we can define separatedness of morphisms of algebraic spaces and of algebraic spaces themselves as follows.

Definition 4.20 (Quasiseparated Morphism of Spaces)**.** Let $f : X \to Y$ be a morphism of S-algebraic spaces. f is quasiseparated if the relative diagonal $\delta_{X/Y} : X \to X \times_Y X$ is quasi-compact.

Definition 4.21 (Locally Separated Morphism of Spaces)**.** Let $f : X \to Y$ be a morphism of S-algebraic spaces. f is locally separated if the relative diagonal $\delta_{X/Y} : X \to X \times_Y X$ is an embedding.

Definition 4.22 (Separated Morphism of Spaces)**.** Let $f : X \to Y$ be a morphism of S-algebraic spaces. f is separated if the relative diagonal $\delta_{X/Y} : X \to X \times_Y X$ is a closed embedding.

The notions for schemes are defined by the structure morphism to S having that particular property. More explicitly, we have the following.

Definition 4.23 (Quasiseparated Algebraic Space)**.** Let X be an S algebraic space. X is quasiseparated if the morphism $X \to S$ is quasiseparated.

Definition 4.24 (Locally Separated Algebraic Space)**.** Let X be an S algebraic space. X is locally separated if the morphism $X \to S$ is locally separated.

Definition 4.25 (Separated Algebraic Space)**.** Let X be an S algebraic space. X is separated if the morphism $X \to S$ is separated.

Remark. Note here that everything can be checked in the category of schemes since the diagonals are representable by schemes.

Remark. This style of definition will recur in considering algebraic stacks, Definitions 6.20 to 6.23.

Another useful definition of morphisms of spaces is as follows.

Definition 4.26 (Properties of Morphisms of Algebraic Spaces)**.** Let $f : X \to Y$ be a morphism of S-algebraic spaces and P a property of morphisms stable on the étale site and local on domain. The morphism f has the property P if there are schemes U, V and étale maps $u : U \to X, v : V \to Y$ such that

$$\begin{tikzcd} V \times_Y U \arrow[r] \arrow[d] & V \arrow[d, "v"] \\ U \arrow[r, "f \circ u"'] & Y \end{tikzcd}$$

the morphism of schemes $V \times_Y U \to V$ has P.

We can refine the topology under consideration and think of algebraic spaces as fppf sheaves as well.

Theorem 4.27. If S is a scheme and X an S-algebraic space with quasicompact diagonal then X is an fppf sheaf on the site $(\mathsf{Sch}_S)_{\mathsf{fppf}}$.

5. Quotients in Algebraic Spaces

In the category of affine schemes and G a finite group acting on an affine scheme $\operatorname{Spec} A$, the quotient of the affine scheme by the group action is the affine scheme $\operatorname{Spec} A^G$, where A^G is the ring of G-invariant elements of A. We consider more general quotients using the tools of algebraic spaces.

Algebraic Stacks

6. Algebraic Stacks

Recall the definition of a stack (in the category-theoretic sense) from Definition 3.4, here taking it as a sheaf over the big étale site with all descent data effective.

Definition 6.1 (Representable Morphism of Stacks)**.** Let $\mathcal{X}, \mathcal{Y}$ be stacks. A morphism of stacks $f : \mathcal{X} \to \mathcal{Y}$ is representable if for every scheme U and morphism $y : h_U \to \mathcal{Y}$ the fibered product $\mathcal{X} \times_{\mathcal{Y},y} h_U$ is representable by an algebraic space.

Note how this is different from the notion of being representable by schemes Definition 4.9 as we now explain.

Lemma 6.2. Let $\mathcal{X}, \mathcal{Y}$ be stacks and $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks. If $f : \mathcal{X} \to \mathcal{Y}$ is representable then for all algebraic spaces V and morphsims $y : V \to \mathcal{Y}$ the fibered product $V \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space.

We can now define an algebraic stack, also known as an Artin stack. We will soon define Deligne-Mumford stacks, another type of algebraic stack.

Definition 6.3 (Algebraic Stack)**.** Let $\mathcal{X}$ be a stack over the big étale site $(\mathsf{Sch}_S)_{\text{ét}}$. $\mathcal{X}$ is an algebraic stack if the following conditions hold:

(a) The diagonal morphism $\delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by algebraic spaces.
(b) There is a smooth surjective morphism $X \to \mathcal{X}$ for X a scheme.

Naturally one defines a morphism of algebraic stacks as a stack morphism between algebraic stacks. We should also be careful to note that being a stack is a property of fibered categories over the site $(\mathsf{Sch}_S)_{\text{ét}}$, with their morphisms being functors between fibered categories. In particular, $\mathrm{Mor}_{\mathsf{Stacks}}(\mathcal{X}, \mathcal{Y})$ forms a category and not a set.

Lemma 6.4. Let S be a scheme and $\mathcal{X}$ be a stack on the big étale site of S-schemes $(\mathsf{Sch}_S)_{\text{ét}}$. The diagonal $\delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable if and only if for every S-scheme U and $u_1, u_2 \in \mathcal{X}(U)$ the sheaf $\underline{\mathrm{Isom}}(u_1, u_2)$ on Sch_U is an algebraic space.

Proof. This follows by the square

$$\begin{tikzcd} \underline{\mathrm{Isom}}(u_1, u_2) \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_S \mathcal{X} \end{tikzcd}$$

being Cartesian. ■

In fact, one can show that morphisms from an algebraic space to a stack are representable.

Proposition 6.5. If $\mathcal{X}$ is an S-algebraic stack and $x : X \to \mathcal{X}, y : Y \to \mathcal{X}$ maps from S-algebraic spaces X, Y to $\mathcal{X}$

$$\begin{tikzcd} Y \times_{\mathcal{X}} X \arrow[r] \arrow[d] & Y \arrow[d, "y"] \\ X \arrow[r, "x"'] & \mathcal{X} \end{tikzcd}$$

then the fibered product $Y \times_{\mathcal{X}} X$ is an algebraic space.

From which one can deduce the following corollary.

Corollary 6.6. If $x : X \to \mathcal{X}$ is a morphism from an algebraic space X to an algebraic stack $\mathcal{X}$ then $x : X \to \mathcal{X}$ is representable.

6.1. **The Classifying Stack of a Group.** Perhaps the most evident way algebraic stacks arise is as quotients of schemes by a group action. We first construct such a quotient as a fibered category, before showing it is a stack.

Definition 6.7 (The Fibered Category $[X/G]$)**.** Let X be an S-algebraic space and G a smooth group scheme over a base scheme S with an action on X. Let $[X/G]$ be the category fibered over the big étale site $(\mathsf{Sch}_S)_{\text{ét}}$ with

(a) Objects triples $(T, \mathcal{P}, \pi)$ where T is an S-scheme, $\mathcal{P}$ is a $(G \times_S T)$-torsor on the big étale site $(\mathsf{Sch}_T)_{\text{ét}}$, and $\pi : \mathcal{P} \to X \times_S T$ is a $(G \times_S T)$-equivariant morphism of T-schemes.

(b) Morphisms

$$(f, f^b) : (T', \mathcal{P}', \pi') \longrightarrow (T, \mathcal{P}, \pi)$$

where $f : T' \to T$ is a morphism of S-schemes and $f^b : \mathcal{P}' \to f^*\mathcal{P}$ an isomorphism of $(G \times_S T')$ torsors in $\mathsf{Sch}_{T'}$ such that the diagram

$$\begin{tikzcd} \mathcal{P}' \arrow[rr, "f^b"] \arrow[dr, "\pi'"'] & & \mathcal{P} \arrow[dl, "f^*\circ\pi"] \\ & X \times_S T' & \end{tikzcd}$$

commutes.

Remark. For condition (b) in Definition 6.7 above, we are considering the diagram obtained by repeated pullback

$$\begin{tikzcd} (G \times_S T) \times_T T' \arrow[r] \arrow[d] & G \times_S T \arrow[r] \arrow[d] & G \arrow[d] \\ T' \arrow[r, "f"'] & T \arrow[r] & S \end{tikzcd}$$

where it is a standard categorical fact (vis. eg. [4, Tag 001U]) that the outer square with horizontal maps compositions is Cartesian as well yielding a cannonical isomorphism

$$(G \times_S T) \times_T T' \cong G \times_S T'.$$

One can show that this construction in fact yields a stack.

Proposition 6.8. If X is an S-algebraic space and G a smooth group scheme over a base scheme S with an action on X then fibered category $[X/G]$ is an S-algebraic stack.

At the expense of being pedantic, we make the following definition.

Definition 6.9 (Quotient Stack)**.** Let X be an S-algebraic space and G a smooth group scheme over a base scheme S with an action on X. The quotient stack of X by G is the S-algebraic stack $[X/G]$.

This yields the important construction of the classifying stack as follows.

Definition 6.10 (Classifying Stack)**.** Let G be a smooth group scheme over S. The classifying stack BG is the stack quotient $[S/G]$.

6.2. **Properties of Stacks and Their Morphisms.**

Definition 6.11 (Inertia Stack)**.** Let $\mathcal{X}$ be an algebraic stack over a base scheme S. The inertia stack $\mathcal{I}_{\mathcal{X}}$ is the fibered product of the following diagram:

$$\begin{tikzcd} & \mathcal{X} \arrow[d, "\delta_{\mathcal{X}}"] \\ \mathcal{X} \arrow[r, "\delta_{\mathcal{X}}"'] & \mathcal{X} \times_S \mathcal{X} \end{tikzcd}$$

More explicitly, we can understand the inertia stack as follows: it is a category whose objects are (x, g) with $x \in \mathcal{X}$ and g an automorphism of $x \in \mathcal{X}(T)$. A morphism $(x', g') \to (x, g)$ in the inertia stack $\mathcal{I}_{\mathcal{X}}$ is a morphism $f : x' \to x$ in $\mathcal{X}$ such that the diagram

$$\begin{tikzcd} x' \arrow[r, "f"] \arrow[d, "g'"'] & x \arrow[d, "g"] \\ x' \arrow[r, "f"'] & x \end{tikzcd}$$

commutes. There is naturally a forgetful functor $p : \mathcal{I}_X \to \mathcal{X}$ by $(x, g) \mapsto x$.

Remark. For any scheme T and object $x \in \mathcal{X}(T)$ the fibered product of the diagram

$$\begin{tikzcd} & T \arrow[d, "t"] \\ \mathcal{I}_{\mathcal{X}} \arrow[r] & \mathcal{X} \end{tikzcd}$$

is the algebraic space $\underline{\mathrm{Aut}}_t$ – a functor on the category of T-schemes.

Definition 6.12 (Property of Stack)**.** Let P be a property of S-schemes stable on the smooth site $(\mathsf{Sch}_S)_{\mathsf{Sm}}$ and $\mathcal{X}$ an algebraic stack over S. The algebraic stack $\mathcal{X}$ has the a property P if there is a smooth surjective morphism $X \to \mathcal{X}$ with X an S-scheme with property P.

Lemma 6.13. Let P be a property of S-schemes stable on the smooth site $(\mathsf{Sch}_S)_{\mathsf{Sm}}$ and $\mathcal{X}$ an algebraic stack over S with property P. For any smooth morphism $y : Y \to \mathcal{X}$ with Y an algebraic space, the algebraic space Y has property P.

Proof. Let $\pi : X \to \mathcal{X}$ be a smooth surjective morphism with X a scheme with property P. In the fibered product

$$\begin{tikzcd} X \times_{\mathcal{X}} Y \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "y"'] & \mathcal{X} \end{tikzcd}$$

we have $\mathrm{pr}_1 : X \times_{\mathcal{X}} Y \to X$ smooth and $\mathrm{pr}_2 : X \times_{\mathcal{X}} Y \to Y$ smooth surjective. So since X has property P, the fibered product $X \times_{\mathcal{X}} Y$ has property P, and thus Y has property P as well by stability on the smooth site. ∎

As we did with schemes, we also want to define properties of morphisms of stacks. We define the following preliminary notions.

Definition 6.14 (Chart)**.** Let $f : \mathcal{X} \to \mathcal{Y}$ a morphism of algebraic stacks over a scheme S. A chart for f is the data of a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[rr, bend left, "h"] \arrow[dr, "q"'] & Y \times_{\mathcal{Y}} \mathcal{X} \arrow[r, "f'"] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ & \mathcal{X} \arrow[r, "f"'] & \mathcal{Y} \end{tikzcd}$$

such that

(a) X, Y are algebraic spaces
(b) and g, p are smooth surjective morphisms.

Definition 6.15 (Chart by Schemes)**.** Let $f : \mathcal{X} \to \mathcal{Y}$ a morphism of algebraic stacks over a scheme S. A chart for f by schemes is the data of a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[rr, bend left, "h"] \arrow[dr, "q"'] & Y \times_{\mathcal{Y}} \mathcal{X} \arrow[r, "f'"] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ & \mathcal{X} \arrow[r, "f"'] & \mathcal{Y} \end{tikzcd}$$

such that

(a) X, Y are schemes
(b) and g, p are smooth surjective morphisms.

This allows us to define properties of morphisms of stacks in the following way.

Definition 6.16 (Property of Stack Morphism)**.** Let P be a property of schemes stable and local on domain on the smooth site on S-schemes $(\mathsf{Sch}_S)_{\mathsf{Sm}}$. A morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ over S has property P if there exists a chart of f by schemes

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[rr, bend left, "h"] \arrow[dr, "q"'] & Y \times_{\mathcal{Y}} \mathcal{X} \arrow[r, "f'"] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ & \mathcal{X} \arrow[r, "f"'] & \mathcal{Y} \end{tikzcd}$$

such that the map of schemes $h : X \to Y$ has property P.

This makes sense as the definition in Definition 6.16 is in fact independent of chart.

Lemma 6.17 (Independence of Chart)**.** Let P be a property of schemes stable and local on domain on the smooth site of S-schemes $(\mathsf{Sch}_S)_{\mathsf{Sm}}$. A morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ has property P if and only if for every chart of f

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[rr, bend left, "h"] \arrow[dr, "q"'] & Y \times_{\mathcal{Y}} \mathcal{X} \arrow[r, "f'"] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ & \mathcal{X} \arrow[r, "f"'] & \mathcal{Y} \end{tikzcd}$$

the morphism of algebraic spaces $h : X \to Y$ has the property P.

Proof. With the chart as above and $Y' \to Y$ a smooth surjective morphism of S-algebraic spaces, we get a commutative diagram

$$\begin{tikzcd}
(Y' \times_Y (Y' \times_{\mathcal{Y}} \mathcal{X})) \times_{(Y \times_{\mathcal{Y}} \mathcal{X})} X \arrow[r, "\mathrm{pr}_1'"] \arrow[d, "\mathrm{pr}_2'"'] \arrow[rr, bend left, "h'"] & Y' \times_Y (Y \times_{\mathcal{Y}} \mathcal{X}) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & Y' \arrow[d] \\
X \arrow[r, "g"] \arrow[dr, "q"'] & Y \times_{\mathcal{Y}} \mathcal{X} \arrow[r, "f'"] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\
 & \mathcal{X} \arrow[r, "f"'] & \mathcal{Y}
\end{tikzcd}$$

where there are isomorphisms

$$Y' \times_{\mathcal{Y}} \mathcal{X} \cong Y' \times_Y (Y \times_{\mathcal{Y}} \mathcal{X})$$

and

$$Y' \times_Y X \cong \left(Y' \times_{\mathcal{Y}} \mathcal{X}\right) \times_{(Y \times_{\mathcal{Y}} \mathcal{X})} X \cong \left(Y' \times_Y \left(Y' \times_{\mathcal{Y}} \mathcal{X}\right)\right) \times_{(Y \times_{\mathcal{Y}} \mathcal{X})} X$$

as the three squares being Cartesian imply that the outer top horizontal rectangle and outer right vertical rectangle are Cartesian as well. This yields the following chart of $f : \mathcal{X} \to \mathcal{Y}$.

$$\begin{tikzcd}
X \times_Y Y' \arrow[r, "\mathrm{pr}_1'"] \arrow[dr, "q \circ \mathrm{pr}_2'"'] \arrow[rr, bend left, "h'"] & Y' \times_{\mathcal{Y}} \mathcal{X} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "p' \circ \mathrm{pr}_2"'] & Y' \arrow[d] \\
 & \mathcal{X} \arrow[r, "f"'] & \mathcal{Y}
\end{tikzcd}$$

Since $\mathrm{pr}_2' : X \times_Y Y' \to X$ is smooth surjective as a map of spaces, h has property P if and only if $h' : X \times_Y Y' \to Y'$ has property P.

For any two charts

$$\begin{tikzcd}
X_1 \arrow[r, "g_1"] \arrow[dr, "q_1"'] \arrow[rr, bend left, "h_1"] & Y_1 \times_{\mathcal{Y}} \mathcal{X} \arrow[r, "f_1'"] \arrow[d, "p_1'"'] & Y_1 \arrow[d, "p_1"] \\
 & \mathcal{X} \arrow[r, "f"'] & \mathcal{Y}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_2 \arrow[r, "g_2"] \arrow[dr, "q_2"'] \arrow[rr, bend left, "h_2"] & Y_2 \times_{\mathcal{Y}} \mathcal{X} \arrow[r, "f_2'"] \arrow[d, "p_2'"'] & Y_2 \arrow[d, "p_2"] \\
 & \mathcal{X} \arrow[r, "f"'] & \mathcal{Y}
\end{tikzcd}$$

we show that h_1 has property P if and only if h_2 does also. Taking $Y' = Y_1 \times_{\mathcal{Y}} Y_2$

$$\begin{tikzcd}
(Y_1 \times_{\mathcal{Y}} Y_2) \times_{Y_i} X \arrow[r, "\mathrm{pr}_1'"] \arrow[dr, "q \circ \mathrm{pr}_2'"'] \arrow[rr, bend left, "h_i'"] & (Y_1 \times_{\mathcal{Y}} Y_2) \times_{\mathcal{Y}} \mathcal{X} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "p' \circ \mathrm{pr}_2"'] & Y_1 \times_{\mathcal{Y}} Y_2 \arrow[d] \\
 & \mathcal{X} \arrow[r, "f"'] & \mathcal{Y}
\end{tikzcd}$$

we see that h_1, h_2 have the property P if and only if $h_i' : Y' \times_{\mathcal{Y}} X \to \mathcal{Y}$ does for both $i \in \{1, 2\}$. In particular, this is symmetric in the Y_i's, so without loss of generality take $Y_1 = Y_2$ where we have the diagram

$$\begin{tikzcd}
& X_1 \arrow[ddl, "g_1"'] \arrow[ddr, "h_1"] & \\
& X_2 \arrow[dl, "g_2"'] \arrow[dr, "h_2"] & \\
(Y_1 \times_{\mathcal{Y}} \mathcal{X}) = (Y_2 \times_{\mathcal{Y}} \mathcal{X}) \arrow[rr, "f_1'=f_2'"] \arrow[d, "p_1'=p_2'"'] & & Y_1 = Y_2 \arrow[d, "p_1=p_2"] \\
\mathcal{X} \arrow[rr, "f"'] & & \mathcal{Y}
\end{tikzcd}$$

with g_1, g_2 smooth surjections. Taking the diagram

$$\begin{tikzcd}
& X_1 \times_{\mathcal{X}} X_2 \arrow[dl, "\pi_1"'] \arrow[ddl, "\pi_2"] \arrow[ddd, "h_1 \circ \pi_1 = h_2 \circ \pi_2"] \\
X_1 \arrow[ddr, "h_1"] & \\
X_2 \arrow[dr, "h_2"'] & \\
& Y_1 = Y_2
\end{tikzcd}$$

we see that h_1 or h_2 have property P if and only

$$h_1 \circ \pi_1 = h_2 \circ \pi_2 : X_1 \times_{\mathcal{X}} X_2 \to Y_1$$

with $Y_1 = Y_2$ has P. But since π_1, π_2 are smooth surjective and P stable on the smooth site and local on domain, h_1 has property P if and only if h_2 does too. ■

We now state an alternative way to define properties of morphisms of algebraic stacks via algebraic spaces.

Definition 6.18 (Property of Stack Morphism)**.** Let P be a property of S-algebraic spaces stable on the smooth site $(\mathsf{Spaces}_S)_{\mathsf{Sm}}$. A representable morphism of S-algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ has property P if for all morphisms $Y \to \mathcal{Y}$ for Y an algebraic space

$$\begin{tikzcd}
Y \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & Y \arrow[d] \\
\mathcal{X} \arrow[r, "f"'] & \mathcal{Y}
\end{tikzcd}$$

the morphism $Y \times_{\mathcal{Y}} \mathcal{X} \to Y$ has property P.

Properties such as étaleness, smoothness of relative dimension d, separatedness, properness, affineness, finiteness, unramifiedness, being a closed embedding, being an open embedding, and being an embedding, are all stable on the smooth site of S-spaces for a base scheme S and thus can be used to describe morphisms of stacks.

Definition 6.19 (Relative Stack Diagonal)**.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over S. The relative stack diagonal is the stack morphism

$$\delta_{\mathcal{X}/\mathcal{Y}} : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}.$$

This allows us to understand separatedness of stacks.

Definition 6.20 (Quasiseparated Stack Morphism)**.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over a base scheme S. If $\delta_{\mathcal{X}/\mathcal{Y}}$ representable by a quasiseparated and quasicompact algebraic space then f is a quasiseparated morphism of stacks.

Definition 6.21 (Separated Stack Morphism)**.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over a base scheme S. If $\delta_{\mathcal{X}/\mathcal{Y}}$ representable by a proper algebraic space then f is a separated morphism of stacks.

When the target stack is the base scheme S, that is $\mathcal{Y} = S$, then we say that the stack has such a property.

Definition 6.22 (Quasiseparated Stack)**.** Let $f : \mathcal{X} \to S$ be a morphism of algebraic stacks over a base scheme S. If $\delta_{\mathcal{X}/S}$ representable by a quasiseparated and quasicompact algebraic space then $\mathcal{X}$ is a quasiseparated algebraic stack.

Definition 6.23 (Separated Stack)**.** Let $f : \mathcal{X} \to S$ be a morphism of algebraic stacks over a base scheme S. If $\delta_{\mathcal{X}/S}$ representable by a proper algebraic space then $\mathcal{X}$ is a separated algebraic stack.

6.3. **Deligne-Mumford Stacks.** We now define Deligne-Mumford stacks.

Definition 6.24 (Deligne-Mumford Stack)**.** An S-algebraic stack $\mathcal{X}$ is Deligne-Mumford if there is a scheme X and an étale surjection $X \to \mathcal{X}$.

Recall that a map of schemes $X \to Y$ is formally unramified if the sheaf of differentials is trivial, that is $\Omega^1_{X/Y} = 0$. In particular, formal unramifiedness is stable and local on domain on the smooth site for S-schemes $(\mathsf{Sch}_S)_{\mathsf{Sm}}$ and stable on the étale site for S-schemes $(\mathsf{Sch}_S)_{\mathsf{\acute{e}t}}$. It thus makes sense for representable morphisms of stacks to be formally unramified.

This allows us to state an alternative characterization of Deligne-Mumford stacks.

Theorem 6.25. Let $\mathcal{X}$ be an S-algebraic stack. The algebraic stack $\mathcal{X}$ is a Deligne-Mumford stack if and only if the diagonal

$$\delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$$

is formally unramified.

One can then deduce the following corollary.

Corollary 6.26. Let $\mathcal{X}$ be an S-algebraic stack. If for every S-scheme U and all $x \in \mathcal{X}(U)$ satisfies $\underline{\mathrm{Aut}}_x = 0$ then $\mathcal{X}$ is an algebraic space.

Proof. Since automorphism groups are trivial on all scheme-valued sections, $\delta_{\mathcal{X}}$ is representable by monomorphisms and thus $\mathcal{X}$ is a Deligne-Mumford stack. Let $X \to \mathcal{X}$ be the étale surjection with X a scheme. The map $X \times_{\mathcal{X}} X \to X \times_S X$ an étale equivalence relation and thus $\mathcal{X} \cong [X/X \times_{\mathcal{X}} X]$, that is, $\mathcal{X}$ is an algebraic space. ■

7. Quasicoherent Sheaves on Algebraic Stacks

We work over a fixed base scheme S and begin by defining the category $\mathsf{Spaces}_{\mathcal{X}}$ of $\mathcal{X}$ algebraic spaces for $\mathcal{X}$ an S-algebraic stack.

Definition 7.1 ($\mathsf{Spaces}_{\mathcal{X}}$)**.** Let $\mathcal{X}$ be an S-algebraic stack. The category of $\mathcal{X}$-spaces, $\mathsf{Spaces}_{\mathcal{X}}$

(a) has objects (T,t) where T is an S-algebraic space and $t : T \to \mathcal{X}$ an S-morphism
(b) has morphisms (f, f^b) where $f : T' \to T$ is an S-morphism of algebraic spaces and $f^b : t' \to t \circ f$ is a natural isomorphism of S-morphisms $T' \to \mathcal{X}$ such that for

$$(T'', t'') \xrightarrow{(g,g^b)} (T', t') \xrightarrow{(f,f^b)} (T, t)$$

(i) $f \circ g : T'' \to T$ is an S-morphism of algebraic spaces
(ii) and

$$t'' \xrightarrow{g^b} t' \circ g \xrightarrow{g(f^b)} t \circ f \circ g$$

are natural isomorphisms of S-morphisms $T'' \to \mathcal{X}$.

Remark. For $(f, f^b) : (T', t') \to (T, t)$ a morphism in $\mathsf{Spaces}_{\mathcal{X}}$, we think of $f^b : t \to t \circ f$ a an isomorphism in $\mathcal{X}(T')$ using the 2-Yoneda lemma.

Remark. If the algebraic stack $\mathcal{X}$ is in fact an algebraic space, the category $\mathsf{Spaces}_{\mathcal{X}}$ is equivalent to the slice category $\mathsf{Spaces}_{(-/\mathcal{X})}$.

One can naturally restrict the category $\mathsf{Spaces}_{\mathcal{X}}$ to get a category of $\mathcal{X}$-schemes $\mathsf{Sch}_{\mathcal{X}}$.

Definition 7.2 ($\mathsf{Sch}_{\mathcal{X}}$)**.** Let $\mathcal{X}$ be an S-algebraic stack. The category of $\mathcal{X}$-schemes, $\mathsf{Sch}_{\mathcal{X}}$ is the full subcategory of $\mathsf{Spaces}_{\mathcal{X}}$ with objects schemes.

We can in fact make a similar construction in the category of stacks.

Definition 7.3 ($\mathcal{Z}$-Morphism of Stacks)**.** Let $\mathcal{Z}$ be an S-algebraic stack and $f : \mathcal{X} \to \mathcal{Z}, g : \mathcal{Y} \to \mathcal{Z}$ be two morphisms of stacks. A $\mathcal{Z}$-morphism of stacks $\mathcal{X} \to \mathcal{Y}$ is a pair (h, σ) with $h : \mathcal{X} \to \mathcal{Y}$ a morphism of stacks and a natural isomorphism of stack morphisms $\sigma : f \to g \circ h$.

This is perhaps more easily seen as the data of the following commutative diagram.

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "h"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Leftarrow\sigma\Rightarrow"] & \mathcal{Y} \arrow[dl, "g"] \\ & \mathcal{Z} & \end{tikzcd}$$

Note here that in general stacks are 2-categories so their Hom-sets are in fact 1-categories, giving us a natural notion of the category of morphisms between two stacks over a base stack.

Definition 7.4 (The Category $\mathrm{Hom}_{\mathcal{Z}}(\mathcal{X}, \mathcal{Y})$)**.** Let $\mathcal{Z}$ be an S-algebraic stack and $f : \mathcal{X} \to \mathcal{Z}, g : \mathcal{Y} \to \mathcal{Z}$ be two morphisms of stacks. The category of $\mathcal{Z}$-morphisms $\mathrm{Hom}_{\mathcal{Z}}(\mathcal{X}, \mathcal{Y})$ has objects (h, σ) with morphisms between two objects $\lambda : (h, \sigma) \to (h', \sigma')$ a natural isomorphism of S-morphisms $\lambda : h \to h'$ such that the diagram of stack morphisms $\mathcal{X} \to \mathcal{Z}$

$$\begin{tikzcd} f \arrow[r, "\sigma"] \arrow[dr, "\sigma'"'] & g \circ h \arrow[d, "\lambda"] \\ & g \circ h' \end{tikzcd}$$

commutes.

However, if in in Definition 7.4, the morphisms $f : \mathcal{X} \to \mathcal{Z}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are representable, the isomorphism λ is determined uniquely giving us the construction of a 1-category of relative spaces over $\mathcal{Z}$. More explicitly, if λ is unique then for a triplet given by the additional data of $\lambda : (h, \sigma) \to (h', \sigma')$ and $\lambda' : (h', \sigma') \to (h'', \sigma'')$ we have

$$\begin{tikzcd}
f \arrow[r, "\sigma"] \arrow[dr, "\sigma'"'] \arrow[ddr, "\sigma''"'] & g \circ h \arrow[d, "\lambda"] \\
& g \circ h' \arrow[d, "\lambda'"] \\
& g \circ h''
\end{tikzcd}$$

with λ' unique as well so looking within $\mathrm{Hom}_{\mathcal{Z}}(\mathcal{X}, \mathcal{Y})$ we have a diagram

$$\begin{tikzcd}
(h, \sigma) \arrow[rr, "\lambda"] \arrow[dr, "\widetilde{\lambda}"'] & & (h', \sigma') \arrow[dl, "\lambda'"] \\
& (h'', \sigma'') &
\end{tikzcd}$$

where the map $(h, \sigma) \to (h'', \sigma'')$ is uniquely determined as

$$\lambda' \circ \lambda \circ \sigma = \widetilde{\lambda} \circ \sigma$$

is a natural isomorphism with σ a natural isomorphism, in particular an epimorphism, so $\lambda' \circ \lambda = \widetilde{\lambda}$ showing compositions are unique and $\mathrm{Hom}_{\mathcal{Z}}(\mathcal{X}, \mathcal{Y})$ is a 1-category.

We then define the category of relative spaces as follows.

Definition 7.5 ($\mathsf{RelSpaces}_{\mathcal{Z}}$)**.** Let $\mathcal{Z}$ be an S-algebraic stack. The category $\mathsf{RelSpaces}_{\mathcal{Z}}$ has objects representable morphisms of stacks $(\mathcal{X} \to \mathcal{Z})$ and whose morphisms between $(\mathcal{X} \to \mathcal{Z})$ and $(\mathcal{Y} \to \mathcal{Z})$ are $\mathcal{Z}$-morphisms of stacks $\mathcal{X} \to \mathcal{Y}$.

Remark. One could equivalently define the morphisms in $\mathsf{RelSpaces}_{\mathcal{Z}}$ between representable morphisms $(\mathcal{X} \to \mathcal{Z})$ and $(\mathcal{Y} \to \mathcal{Z})$ as commuting triangles of the following type:

$$\begin{tikzcd}
\mathcal{X} \arrow[rr] \arrow[dr] & & \mathcal{Y} \arrow[dl] \\
& \mathcal{Z} &
\end{tikzcd}$$

Take caution that while the category of relative spaces $\mathsf{RelSpaces}_{\mathcal{Z}}$ has "spaces" in the name, it may have objects that are stacks. In particular those stacks $\mathcal{X}$ such that $\mathcal{X} \to \mathcal{Z}$ is representable. In particular for $\mathsf{Spaces}_{\mathcal{Z}}$ as in Definition 7.1, there is an inclusion of 1-categories $\mathsf{Spaces}_{\mathcal{Z}} \to \mathsf{RelSpaces}_{\mathcal{Z}}$.

For $f : \mathcal{X} \to \mathcal{Y}$ a morphism of stacks, one can define a functor $\mathsf{Spaces}_{\mathcal{X}} \to \mathsf{Spaces}_{\mathcal{Y}}$ given on objects by

$$(T, t) \mapsto (T, f \circ t)$$

and on morphisms by

$$\Big((g, g^b) : (T', t') \to (T, t)\Big) \mapsto \Big((g, f \circ g^b) : (T', f \circ t') \to (T, f \circ t)\Big).$$

This is evidently compatible with the structure on $\mathsf{Spaces}_{\mathcal{Y}}$ since the composition $f \circ t : T \to \mathcal{Y}$ is an S-morphism from an S-algebraic space T to the stack $\mathcal{Y}$ and

$$\begin{tikzcd} T' \arrow[d, "g"'] \arrow[dr, "t'"] \arrow[drr, bend left, "f \circ t'"] & & \\ & \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \\ T \arrow[ur, "t"'] \arrow[urr, bend right, "f \circ t"'] \arrow[u, phantom, "\Updownarrow g^b"] & & \end{tikzcd} \qquad \begin{tikzcd} T' \arrow[d, "g"'] \arrow[dr, "f \circ t'"] & \\ & \mathcal{Y} \\ T \arrow[ur, "f \circ t"'] \arrow[u, phantom, "\Updownarrow f(g^b)"] & \end{tikzcd}$$

we have natural isomorphisms of $T' \to \mathcal{Y}$

$$f \circ g^b : f \circ t \circ g \to f \circ t'.$$

This restricts to a functor $\mathsf{Sch}_{\mathcal{X}} \to \mathsf{Sch}_{\mathcal{Y}}$.

7.1. **The Lisse-Étale Site of a Stack.** We can now construct the lisse-étale site of an S-stack $\mathcal{X}$.

Definition 7.6 (The Lisse-Étale Site of a Stack)**.** Let $\mathcal{X}$ be an S-algebraic stack. The lisse-étale site $\mathsf{Lis\acute{E}t}(\mathcal{X})$ on $\mathcal{X}$ is the full subcategory of $\mathsf{Sch}_{\mathcal{X}}$ with objects (T, t) such that $t : T \to \mathcal{X}$ is a smooth morphism. The Grothendieck topology of $\mathsf{Lis\acute{E}t}(\mathcal{X})$ is given by coverings

$$\Big\{(f_i, f_i^b) : (T_i, t_i) \to (T, t)\Big\}$$

where $\{f_i : T_i \to T\}$ a covering in the small étale site on T, $T_{\text{ét}}$.

Remark. Here f_i^b and t_i defined in the obvious way, with $t_i = t \circ f_i$ and $f_i^b : t_i \to t \circ f_i$ a natural isomorphism such that the diagram

$$\begin{tikzcd} T_i \arrow[rr, "t_i"] \arrow[dr, "f_i"'] & \arrow[d, phantom, "\Updownarrow f_i^b"] & \mathcal{X} \\ & T \arrow[ur, "t"'] & \end{tikzcd}$$

commutes.

A Grothendieck topology allows us to define presheaves and sheaves on a site so we can define the lisse-étale topos as follows.

Definition 7.7 (The Lisse-Étale Topos of a Stack)**.** Let $\mathcal{X}$ be an S-stack. The lisse-étale topos on $\mathcal{X}$, $\mathcal{X}_{\mathsf{Lis\acute{E}t}}$ is the category of sheaves on the lisse-étale site $\mathsf{Lis\acute{E}t}(\mathcal{X})$.

Note that for any object $(T, t) \in \operatorname{Obj}\Big(\mathsf{Lis\acute{E}t}(\mathcal{X})\Big)$ there is an inclusion of categories from the small étale site of T to the lisse-étale site $\iota_T : T_{\text{ét}} \to \mathsf{Lis\acute{E}t}(\mathcal{X})$ given on objects by

$$(h : T' \to T) \mapsto (T', t \circ h)$$

as $t \circ h : T' \to \mathcal{X}$ is a smooth morphism by h étale and on morphisms $q : T'' \to T'$ over $(h' : T'' \to T) \to (h' : T' \to T)$ by

$$\big(q : T'' \to T'\big) \mapsto \Big(\big(\iota_T(q), \iota_T(q)^b\big) : (T'', t \circ h \circ q) \to (T', t \circ h)\Big)$$

where $\iota_T(q)^b$ is the natural transformation making the diagram on the right commute

$$\begin{tikzcd}
T'' \arrow[dr, "h'"] \arrow[d, "q"'] \arrow[drr, bend left, "t\circ h'"] & & \\
T' \arrow[r, "h"'] \arrow[drr, phantom, ""] \arrow[rr, bend right, "t\circ h"'] & T \arrow[r, "t"] & \mathcal{X}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
T'' \arrow[dr, "t\circ h'"] \arrow[dd, "\iota_T(q)"'] & \\
 & \mathcal{X} \arrow[l, phantom, "\iota_T(q)^b"] \\
T \arrow[ur, "t\circ h"'] &
\end{tikzcd}$$

induced by the commutativity of the triangle

$$\begin{tikzcd}
T'' \arrow[rr, "q"] \arrow[dr, "h'"'] & & T' \arrow[dl, "h"] \\
 & T &
\end{tikzcd}$$

in the small étale site $T_{\text{ét}}$. This is compatible with the structure of coverings as coverings in $T_{\text{ét}}$ are by étale morphisms so for a covering $\{\phi_i : T'_i \to T'\}$ of $(h : T' \to T) \in \operatorname{Obj}(T_{\text{ét}})$, we have

$$\iota_T\left(\{\phi_i : T'_i \to T'\}\right) = \left\{(\phi_i, \phi_i^b) : \left(T'_i, t\circ h\circ\phi_i\right) \to (T', t\circ h)\right\}$$

where the compositions

$$T'_i \xrightarrow{\quad\phi_i\quad} T' \xrightarrow{\quad h\quad} T \xrightarrow{\quad t\quad} \mathcal{X}$$

are compositions of étale morphisms ϕ_i, h and a smooth morphism t and hence $t\circ h\circ\phi_i$ is smooth with the natural transformations ϕ_i^b are such that the diagram

$$\begin{tikzcd}
T'_i \arrow[dr, "t\circ h\circ\phi_i"] \arrow[dd, "\phi_i"'] & \\
 & \mathcal{X} \arrow[l, phantom, "\phi_i^b"] \\
T' \arrow[ur, "t\circ h"'] &
\end{tikzcd}$$

commutes. One can show that the condition of being a sheaf on $\mathsf{Lis\acute{E}t}(\mathcal{X})$ is can be checked locally on these inclusions of small étale sites.

Lemma 7.8. Let $\mathcal{X}$ be an S-algebraic stack, $\mathsf{Lis\acute{E}t}(\mathcal{X})$ the lisse-étale site on $\mathcal{X}$,

$$\iota_T : T_{\text{ét}} \to \mathsf{Lis\acute{E}t}(\mathcal{X})$$

the inclusion functor of the etale site $T_{\text{ét}}$ for $(T,t) \in \operatorname{Obj}\left(\mathsf{Lis\acute{E}t}(\mathcal{X})\right)$, and $F : \mathsf{Lis\acute{E}t}(\mathcal{X}) \to \mathsf{Sets}$ a set-valued functor on $\mathsf{Lis\acute{E}t}(\mathcal{X})$. F is a sheaf of sets on the lisse-étale site $\mathsf{Lis\acute{E}t}(\mathcal{X})$ if and only if $F|_{\iota_T(T_{\text{ét}})}$ is a sheaf of sets for all $(T,t) \in \operatorname{Obj}\left(\mathsf{Lis\acute{E}t}(\mathcal{X})\right)$.

7.2. **Sheaves on Stacks.** In algebraic geometry, we saw how each scheme X has a special sheaf $\mathcal{O}_X$, its structure sheaf, which we then use to define more complex structures such as quasicoherent sheaves. Developing stack theory in a similar way, we define the structure sheaf of an S-algebraic stack $\mathcal{X}$ as follows.

Definition 7.9 (Structure Sheaf of a Stack)**.** Let $\mathcal{X}$ be an S-algebraic stack. The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is a functor

$$\mathcal{O}_{\mathcal{X}} : \mathsf{Lis\acute{E}t}(\mathcal{X})^{\mathsf{Opp}} \to \mathsf{Rings}$$

such that $(T,t) \mapsto \Gamma(T, \mathcal{O}_T)$.

This is evidently compatible with morphisms: for $(f, f^b) : (T', t') \to (T, t)$ the diagram

$$\begin{tikzcd} (T',t') \arrow[r, "(f,f^b)"] \arrow[d] & (T,t) \arrow[d] \\ \Gamma(T', \mathcal{O}_{T'}) & \Gamma(T, \mathcal{O}_T) \arrow[l] \end{tikzcd}$$

commutes taking $\Gamma(T, \mathcal{O}_T)$ to $\Gamma(f(T), \mathcal{O}_{T'}) \subseteq \Gamma(T', \mathcal{O}_{T'})$. Herein let $\mathcal{F} \in \mathcal{X}_{\mathsf{Lis\acute{E}t}}$, $(T,t) \in \mathrm{Obj}\left(\mathsf{Lis\acute{E}t}(\mathcal{X})\right)$, and $\mathcal{F}_{(T,t)}$ the restriction of the sheaf $\mathcal{F}$ to $(T,t) \in \mathrm{Obj}\left(\mathsf{Lis\acute{E}t}(\mathcal{X})\right)$, that is, $\mathcal{F}\left((T,t)\right) \in \mathrm{Obj}(\mathsf{Rings})$ as a functor.

Definition 7.10 (Cartesian Sheaf of Λ-Modules)**.** Let $\mathcal{X}$ be an S-algebraic stack, $\mathsf{Lis\acute{E}t}(\mathcal{X})$ the lisse-étale site on $\mathcal{X}$,

$$\Lambda : \mathsf{Lis\acute{E}t}(\mathcal{X})^{\mathsf{Opp}} \to \mathsf{Rings}$$

a sheaf of rings on $\mathsf{Lis\acute{E}t}(\mathcal{X})$, and $\mathcal{F}$ of Λ-modules on $\mathsf{Lis\acute{E}t}(\mathcal{X})$. The sheaf of Λ-modules $\mathcal{F}$ is Cartesian if for all $(f, f^b) : (T', t') \to (T, t)$ the morphism of $\Lambda_{(T',t')}$-modules

$$f^*\mathcal{F}_{(T,t)} : f^{-1}\mathcal{F}_{(T,t)} \otimes_{(f^{-1}\Lambda_{(T,t)})} \Lambda_{(T',t')} \longrightarrow \mathcal{F}_{(T',t')}$$

is an isomorphism.

A quasicoherent sheaf on a stack is defined to be a Cartesian sheaf in the sense of Definition 7.10 such that it is a quasicoherent sheaf of $\mathcal{O}_T$-modules for all $(T,t) \in \mathrm{Obj}\left(\mathsf{Lis\acute{E}t}(\mathcal{X})\right)$.

Definition 7.11 (Quasicoherent Sheaf on a Stack)**.** Let $\mathcal{X}$ be an S-algebraic stack, $\mathsf{Lis\acute{E}t}(\mathcal{X})$ the lisse-étale site on $\mathcal{X}$,

$$\Lambda : \mathsf{Lis\acute{E}t}(\mathcal{X})^{\mathsf{Opp}} \to \mathsf{Rings}$$

a sheaf of rings on $\mathsf{Lis\acute{E}t}(\mathcal{X})$, and $\mathcal{F}$ of Λ-modules on $\mathsf{Lis\acute{E}t}(\mathcal{X})$. The sheaf of Λ-modules $\mathcal{F}$ is a quasicoherent sheaf if $\mathcal{F}$ is a Cartesian sheaf of Λ-modules and $\mathcal{F}_{(T,t)}$ is a quasicoherent sheaf of $\mathcal{O}_T$-modules for all $(T,t) \in \mathrm{Obj}\left(\mathsf{Lis\acute{E}t}(\mathcal{X})\right)$.

As in the case of schemes, coherent sheaves require an additional Noetherian hypothesis.

Definition 7.12 (Locally Noetherian Stack)**.** Let $\mathcal{X}$ be an S-algebraic stack, $\mathsf{Lis\acute{E}t}(\mathcal{X})$ the lisse-étale site on $\mathcal{X}$. $\mathcal{X}$ is a locally Noetherian algebraic stack if the scheme T is locally Noetherian for all $(T,t) \in \mathrm{Obj}\left(\mathsf{Lis\acute{E}t}(\mathcal{X})\right)$.

We can now define coherent sheaves on algebraic stacks.

Definition 7.13 (Coherent Sheaf on a Stack)**.** Let $\mathcal{X}$ be a locally Noetherian S-algebraic stack, $\mathsf{Lis\acute{E}t}(\mathcal{X})$ the lisse-étale site on $\mathcal{X}$,

$$\Lambda : \mathsf{Lis\acute{E}t}(\mathcal{X})^{\mathsf{Opp}} \to \mathsf{Rings}$$

a sheaf of rings on $\mathsf{Lis\acute{E}t}(\mathcal{X})$, and $\mathcal{F}$ of Λ-modules on $\mathsf{Lis\acute{E}t}(\mathcal{X})$. The sheaf of Λ-modules $\mathcal{F}$ is a coherent sheaf if $\mathcal{F}$ is a quasicoherent sheaf of Λ-modules on $\mathsf{Lis\acute{E}t}(\mathcal{X})$ and $\mathcal{F}_{(T,t)}$ is a finitely generated $\Lambda_{(T,t)}$-module for all $(T,t) \in \mathrm{Obj}\left(\mathsf{Lis\acute{E}t}(\mathcal{X})\right)$.

We can then construct the category of quasicoherent sheaves on the S-algebraic stack $\mathcal{X}$ which we denote $\mathsf{QCoh}(\mathcal{X})$ with objects quasicoherent sheaves on $\mathsf{Lis\acute{E}t}(\mathcal{X})$ and morphisms $\mathsf{Lis\acute{E}t}(\mathcal{X})$-preserving 2-functors $\mathcal{F} \to \mathcal{G}$ such that the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[rr] \arrow[dr] & & \mathcal{G} \arrow[dl] \\ & \mathsf{Lis\acute{E}t}(\mathcal{X}) & \end{tikzcd}$$

commutes. There is a natural embedding of topoi into the lisse-étale topos as in Definition 7.7 $\mathcal{X}_{\mathsf{Lis\acute{E}t}}$

$$\mathsf{QCoh}(\mathcal{X}) \hookrightarrow \mathcal{X}_{\mathsf{Lis\acute{E}t}}.$$

Proposition 7.14. Let $\mathcal{X}$ be an S-algebraic stack (resp. locally Noetherian S-algebraic stack), $\mathcal{F}$ a Cartesian sheaf of $\mathcal{O}_{\mathcal{X}}$-modules, and $x : X \to \mathcal{X}$ a smooth surjective map with X a scheme. $\mathcal{F}$ is a quasicoherent sheaf of $\mathcal{O}_{\mathcal{X}}$-modules (resp. coherent sheaf of $\mathcal{O}_{\mathcal{X}}$-modules) if and only if $\mathcal{F}_{(X,x)}$ is a quasicoherent sheaf of $\mathcal{O}_X$-modules (resp. is a coherent sheaf of of $\mathcal{O}_X$-modules).

Remark. The use of notation $\mathcal{O}_X$-modules is justified as we have isomorphisms

$$(\mathcal{O}_{\mathcal{X}})_{(X,x)} = \Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X).$$

Proof. Let $(Z, z) \in \mathrm{Obj}\left(\mathsf{Lis\acute{E}t}(\mathcal{X})\right)$ be another object and consider the following Cartesian square.

$$\begin{tikzcd} X \times_{\mathcal{X}} Z \arrow[r, "q"] \arrow[d, "p"'] & X \arrow[d, "x"] \\ Z \arrow[r, "z"'] & \mathcal{X} \end{tikzcd}$$

Note that p is smooth as it is obtained from $x : X \to \mathcal{X}$ smooth by base change along $z : Z \to \mathcal{X}$. Arguing by restricting to quasicoherent sheaves on schemes [4, Tag 05JU] there is an étale surjection $\pi : Z' \to Z$ and $s : Z' \to X \times_{\mathcal{X}} Z$ such that the diagram

$$\begin{tikzcd} Z' \arrow[r, "s"] \arrow[dr, "\pi"'] & X \times_{\mathcal{X}} Z \arrow[r, "q"] \arrow[d, "p"'] & X \arrow[d, "x"] \\ & Z \arrow[r, "z"'] & \mathcal{X} \end{tikzcd}$$

commutes. Thus we have that $\mathcal{F}_{(Z,z)}$ is quasicoherent (resp. coherent) if and only if $\pi^*\mathcal{F}_{(Z,z)} = \mathcal{F}_{(Z',z\circ\pi)}$ is quasicoherent (resp. coherent). It thus suffices to verify that $\mathcal{F}_{(Z,z)}$ is quasicoherent when z factors in the following way

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[rr, bend right, "z"'] & X \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

in which case we have $\mathcal{F}_{(Z,z)} \cong g^*\mathcal{F}_{(X,x)}$ since $\mathcal{F}$ is Cartesian giving the claim. ■

7.3. **The Étale Site of a Stack.** We defined the lisse-étale site on an S-algebraic stack $\mathcal{X}$ in Definition 7.6 as the category of schemes admitting a smooth morphism to $\mathcal{X}$ with the data of coverings roughly those of the small étale site on each scheme. We can further restrict this site to those schemes with an étale morphism to the stack $\mathcal{X}$ to define the étale site on the stack.

Definition 7.15 (Étale Site on a Stack)**.** Let $\mathcal{X}$ be an S-algebraic stack. The étale site $\mathsf{Ét}(\mathcal{X})$ on $\mathcal{X}$ is the full subcategory of $\mathsf{Sch}_{\mathcal{X}}$ with objects (T,t) such that $t : T \to \mathcal{X}$ is an étale morphism. The Grothendieck topology of $\mathsf{Ét}(\mathcal{X})$ is given by coverings

$$\left\{(f_i, f_i^b) : (T_i, t_i) \to (T,t)\right\}$$

where $\{f_i : T_i \to T\}$ is a covering in the small étale site on T, $T_{\mathsf{ét}}$.

As in the case of the lisse-étale site, the étale topos $\mathcal{X}_{\mathsf{Ét}}$ is the category of sheaves on the site $\mathsf{Ét}(\mathcal{X})$ and $\mathcal{O}_{\mathcal{X}_{\mathsf{Ét}}}$ the subfunctor of the structure sheaf on the lisse-étale site restricted to the full subcategory $\mathsf{Ét}(\mathcal{X})$ of $\mathsf{LisÉt}(\mathcal{X})$ as in Definition 7.9. Naturally, a sheaf of $\mathcal{O}_{\mathcal{X}_{\mathsf{Ét}}}$-modules $\mathcal{F}$ is quasicoherent (resp. coherent) if $\mathcal{F}_{(T,t)}$ is a sheaf of quasicoherent $\mathcal{O}_T$-modules (resp. sheaf of coherent $\mathcal{O}_T$-modules) for all $(T,t) \in \mathrm{Obj}\left(\mathsf{Ét}(\mathcal{X})\right)$.

Having defined the étale site $\mathsf{Ét}(\mathcal{X})$ as a site whose underlying category is a full subcategory of $\mathsf{LisÉt}(\mathcal{X})$ with the same coverings, there is a natural inclusion of categories and thus sites

$$\mathsf{Ét}(\mathcal{X}) \hookrightarrow \mathsf{LisÉt}(\mathcal{X}).$$

However, as previously described, $\mathcal{O}_{\mathcal{X}_{\mathsf{Ét}}}$ is a subfunctor and thus subsheaf of $\mathcal{O}_{\mathcal{X}}$ inducing a morphism of ringed topoi

$$r : \left(\mathcal{X}_{\mathsf{LisÉt}}, \mathcal{O}_{\mathcal{X}}\right) \to \left(\mathcal{X}_{\mathsf{Ét}}, \mathcal{O}_{\mathcal{X}_{\mathsf{Ét}}}\right).$$

More explicitly, let $\mathsf{RelSpaces}_{\mathcal{X}}$ be category of relative spaces over the stack $\mathcal{X}$ as in Definition 7.5 and $\mathsf{Ét}\left(\mathsf{RelSpaces}_{\mathcal{X}}\right)$ the full subcategory of relative spaces $\mathsf{RelSpaces}_{\mathcal{X}}$ with objects objects étale representable morphisms of stacks $(\mathcal{Y} \to \mathcal{X})$ – here taking algebraic spaces and schemes to be a stack in the tautological way. Similarly define $\mathsf{LisÉt}\left(\mathsf{RelSpaces}_{\mathcal{X}}\right)$ following Definition 7.6 as the full subcategory of relative spaces $\mathsf{RelSpaces}_{\mathcal{X}}$ with objects smooth representable morphisms $(\mathcal{Y} \to \mathcal{X})$ for which there is surjective étale morphism $y : Y \to \mathcal{Y}$ for Y a scheme and coverings given by jointly surjective étale morphisms. There is an inclusion

$$\mathsf{Ét}\left(\mathsf{RelSpaces}_{\mathcal{X}}\right) \hookrightarrow \mathsf{LisÉt}\left(\mathsf{RelSpaces}_{\mathcal{X}}\right)$$

which induces a continuous morphism of sites that lifts to a morphism of topoi under certain limit and colimit preserving hypotheses satisfied by this functor (vis. eg. [4, Tag 039Z]).

On Deligne-Mumford stacks, however, quasicoherent sheaves on the lisse-étale site and the étale site coincide.

Proposition 7.16. Let $\mathcal{X}$ be an S-algebraic stack. If $\mathcal{X}$ is a Deligne-Mumford stack then the restriction functor on quasicoherent sheaves

$$r_* : \mathsf{QCoh}\left(\mathcal{X}\right)_{\mathsf{LisÉt}} \longrightarrow \mathsf{QCoh}\left(\mathcal{X}\right)_{\mathsf{Ét}}$$

is an equivalence of categories.

The Geometry of Stacks

8. Geometric Properties of Stacks

We now consider some further properties of morphisms of stacks.

Definition 8.1 (Embedding)**.** Let $f : \mathcal{Z} \to \mathcal{X}$ be a morphism of S-algebraic stacks. f is an embedding if f is a representable morphism of stacks and for all $Y \to \mathcal{Y}$ for Y an algebraic space the map $Y \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}$ is an embedding of algebraic spaces.

Definition 8.2 (Open Embedding)**.** Let $f : \mathcal{Z} \to \mathcal{X}$ be a morphism of S-algebraic stacks. f is an open embedding if f is a representable morphism of stacks and for all $Y \to \mathcal{Y}$ for Y an algebraic space the map $Y \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}$ is an embedding of algebraic spaces.

Definition 8.3 (Closed Embedding)**.** Let $f : \mathcal{Z} \to \mathcal{X}$ be a morphism of S-algebraic stacks. f is a closed embedding if f is a representable morphism of stacks and for all $Y \to \mathcal{Y}$ for Y an algebraic space the map $Y \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}$ is an embedding of algebraic spaces.

Here we used Definition 6.18 and the following Cartesian diagram

$$\begin{tikzcd} Y \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathcal{X} \arrow[r, "f"'] & \mathcal{Y} \end{tikzcd}$$

for $Y \to \mathcal{Y}$ a map from an algebraic space Y to the stack $\mathcal{Y}$.

Definition 8.4 (Closed Substack)**.** Let $\mathcal{X}$ be an S-algebraic stack. A closed substack of $\mathcal{X}$ is an equivalence class of closed embeddings $\mathcal{Z} \to \mathcal{X}$ such that

$$(f_1 : \mathcal{Z}_1 \to \mathcal{X}) \sim (f_2 : \mathcal{Z} \to \mathcal{X})$$

if and only if there is a morphism of stacks $g : \mathcal{Z}_1 \to \mathcal{Z}_2$ and a S-natural isomorphism of stack morphisms $\sigma : f_2 \circ g \to f_1$.

Remark. In the setup of Definition 8.3 that f is a representable morphism of stacks so in the definition of closed substacks (8.4) the pair (g, σ) is unique up to unique isomorphism.

We now define the property closedness and universal closedness of morphisms of stacks.

Definition 8.5 (Closed Morphism of Stacks)**.** Let $\mathcal{X}$ be an S-algebraic stack and $f : \mathcal{X} \to Y$ a morphism from $\mathcal{X}$ to a scheme Y. f is a closed morphism of stacks if for all closed substacks $\mathcal{Z} \subseteq \mathcal{X}$, $f(\mathcal{Z}) \subseteq Y$ is a closed subscheme.

Definition 8.6 (Universally Closed Morphism of Stacks)**.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of S-algebraic stacks. f is a universally closed morphism of stacks if for all schemes Y and $Y \to \mathcal{Y}$

$$\begin{tikzcd} Y \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathcal{X} \arrow[r, "f"'] & \mathcal{Y} \end{tikzcd}$$

the morphism $Y \times_{\mathcal{Y}} \mathcal{X} \to Y$ is a closed morphism of stacks.

This leads us to the definition of a proper morphism of algebraic stacks, building on the notion of separatedness of stack morphisms as introduced in Definition 6.21.

Definition 8.7 (Proper Morphism of Stacks)**.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of S-algebraic stacks. f is a proper morphism if it is separated, of finite type, and universally closed.

We can characterize universal closedness for representable separated morphisms of finite type in terms of properness.

Proposition 8.8. Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable separated morphism of finite type of S-algebraic stacks. f is universally closed if and only if f is proper.

8.1. **The Functors $\underline{\mathrm{Spec}}$ and $\underline{\mathrm{Proj}}$.** Once again drawing parallels to the constructions in the category of schemes, we can construct stacks from sheaves using relative spec $\underline{\mathrm{Spec}}$ and relative proj $\underline{\mathrm{Proj}}$.

Definition 8.9 (Relative Spec for Stacks)**.** Let $\mathcal{X}$ be an S-algebraic stack and

$$\mathcal{A} : \mathsf{Lis\acute{E}t}\,(\mathcal{X})^{\mathsf{Opp}} \longrightarrow \mathsf{Alg}_{\mathcal{O}_{\mathcal{X}}}$$

be a quasicoherent sheaf of algebras on $\mathcal{X}$. The stack $\underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{A})$ has

(a) objects triples (T, x, ρ) where T is an S-scheme, $x \in \mathcal{X}(T)$, and $\rho : x^*\mathcal{A} \to \mathcal{O}_T$ is a morphism of sheaves of algebras on the scheme T;
(b) morphisms $(g, g^b) : (T', x', \rho') \to (T, x, \rho)$ such that $g : T' \to T$ a morphism of S-schemes and $g^b : x' \to x$ a morphism in $\mathcal{X}$ over g such that the diagram

$$\begin{tikzcd} x'^*\mathcal{A} \arrow[rr, "g^b"] \arrow[dr, "\rho'"'] & & g^*x^*\mathcal{A} \arrow[dl, "g^*\rho"] \\ & \mathcal{O}_{T'} & \end{tikzcd}$$

commutes.

Recall that we have descent for quasicoherent sheaves and thus $\underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{A})$ is a stack in the étale topology. There there is a natural forgetful map of stacks

$$\underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{A}) \longrightarrow \mathcal{X}$$

by $(T, x, \rho) \mapsto (T, x)$.

Recall that by specializing Definition 6.18, a representable morphism of S-algebraic stacks $\mathcal{X} \to \mathcal{Y}$ is affine if for all morphisms $y : Y \to \mathcal{X}$ and Y an algebraic space the map $Y \times_{\mathcal{Y}} \mathcal{X} \to Y$ arising from the Cartesian square

$$\begin{tikzcd} Y \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathcal{X} \arrow[r, "f"'] & \mathcal{Y} \end{tikzcd}$$

is an affine morphism. In particular, for $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ two affine morphisms, the $\mathcal{Z}$-morphisms $\mathrm{Mor}_{\mathcal{Z}}(\mathcal{X}, \mathcal{Y})$ are a set, and hence the full subacategory of the slice category of stacks over $\mathcal{Z}$ with structure morphisms affine morphisms form a 1-category. Relative spec then is a functor from quasicoherent sheaves on a stack to the slice category.

Proposition 8.10. Let $\mathsf{Aff}_{\mathcal{X}}$ be the 1-category of the full subcategory of the slice category of S-algebraic stacks over $\mathcal{X}$ with structure morphisms affine morphisms. The functor

$$\underline{\mathrm{Spec}}_{\mathcal{X}}(-) : \mathsf{QCoh}(\mathcal{X})_{\mathsf{Lis\acute{E}t}} \longrightarrow \mathsf{Aff}_{\mathcal{X}}$$

by $\mathcal{A} \mapsto \underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{A})$ is an equivalence of categories.

The construction of relative proj constructs a stack in a similar way. Recall the setup for schemes. Let T be an S-scheme and $\mathcal{A} = \bigoplus_{d\geq 0} \mathcal{A}_d$ a quasicoherent sheaf of graded $\mathcal{O}_T$-algebras we have a scheme $\underline{\mathrm{Proj}}_T(\mathcal{A})$ constructed by gluing $\mathrm{Proj}(\Gamma(U,\mathcal{A}))$ over all $U \subseteq T$ affine open. There is a natural map $\pi : \underline{\mathrm{Proj}}_T(\mathcal{A}) \to T$ that factors as

$$\underline{\mathrm{Proj}}_T(\mathcal{A})|_{\pi^{-1}(U)} \xrightarrow{\sim} \mathrm{Proj}(\mathcal{A}(U)) = \mathrm{Proj}(\Gamma(U,\mathcal{A})) \longrightarrow U \xrightarrow{\iota_U} T$$

on all restrictions

$$\pi|_{\pi^{-1}(U)} : \underline{\mathrm{Proj}}_T(\mathcal{A})|_{\pi^{-1}(U)} \to T$$

for $U \subseteq T$ open and inclusion $\iota_U : U \to T$. One can think of $\pi : \underline{\mathrm{Proj}}_T(\mathcal{A}) \to T$ as a fibration on T by projective schemes. Naturally a section ρ is a map $\rho : T \to \underline{\mathrm{Proj}}_T(\mathcal{A})$

$$\begin{tikzcd} \underline{\mathrm{Proj}}_T(\mathcal{A}) \arrow[d, "\pi"] \\ T \arrow[u, bend left, "\rho"] \end{tikzcd}$$

such that $\pi \circ \rho = \mathrm{id}_T$.

Definition 8.11 (Relative Proj for Stacks)**.** Let $\mathcal{X}$ be an S-algebraic stack and $\mathcal{A} = \bigoplus_{d\geq 0} \mathcal{A}_d$ a quasicoherent sheaf of graded $\mathcal{O}_{\mathcal{X}}$-algebras on $\mathcal{X}$. The stack $\underline{\mathrm{Proj}}_{\mathcal{X}}(\mathcal{A})$ has

(a) objects triples (T, x, ρ) with T an S-scheme, $x \in \mathcal{X}(T)$, and $\rho : T \to \underline{\mathrm{Proj}}_T(x^*\mathcal{A})$ a section of the T-scheme;
(b) morphisms $(g, \widetilde{g}) : (T', x', \rho') \to (T, x, \rho)$ such that $g : T' \to T$ is a morphism of S-schemes and $\widetilde{g} : x' \to x$ a morphism in $\mathcal{X}$ over g such that the diagram

$$\begin{tikzcd} T' \arrow[r, "g"] \arrow[d, "\rho'"'] & T \arrow[d, "\rho"] \\ \underline{\mathrm{Proj}}_{T'}(x'^*\mathcal{A}) \arrow[r, "\widetilde{g}"'] & \underline{\mathrm{Proj}}_T(x^*\mathcal{A}) \end{tikzcd}$$

commutes.

Remark. For

$$\mathcal{A} = \bigoplus_{d\geq 0} \mathcal{A}_d$$

a quasicoherent sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras for all $(T,t) \in \mathrm{Obj}\left(\mathsf{Lis\acute{E}t}(\mathcal{X})\right)$, $\mathcal{A}_{(T,t)}$ is a quasicoherent sheaf of graded $\mathcal{O}_T$-algebras with equivalence $(\mathcal{O}_{\mathcal{X}})_{(T,t)} = \Gamma(T, \mathcal{O}_T) = \mathcal{O}_T(T)$ by definition.

8.2. **Root Stacks.** One important construction one encounters is that of root stacks. Let X be a scheme and D an effective Cartier divisor on X. One might want to find an effective Cartier divisor E and integer n such that $nE \sim D$. This is not possible in general, but one could find a morphism of schemes $f : Y \to X$ such that there exists an effective Cartier divisor E on Y and an integer n giving the following equivalence of divisors $nE \sim f^*D$. The root stack construction attempts to give a solution to this problem by finding a "universal" such (Y, E). Recall that divisors do not pull back along arbitrary morphisms, necessitating the introduction of the following more general object.

Definition 8.12 (Generalized Effective Cartier Divisor)**.** Let X be a scheme. A generalized Cartier divisor on X is a pair (L, ρ) such that L is an invertible sheaf on X and $\rho : L \to \mathcal{O}_X$ a morphism of $\mathcal{O}_X$-modules.

An isomorphism of generalized Cartier divisors (L',ρ') and (L,ρ) is an isomorphism of line bundles $\sigma : L' \to L$ such that the diagram

$$\begin{tikzcd} L' \arrow[rr, "\sigma"] \arrow[dr, "\rho'"'] & & L \arrow[dl, "\rho"] \\ & \mathcal{O}_X & \end{tikzcd}$$

commutes. Indeed for $D \subset X$ an effective Cartier divisor and $\mathcal{I}_D$ its ideal sheaf, the inclusion $D \hookrightarrow X$ induces a morphism of $\mathcal{O}_X$-modules $j_D : \mathcal{I}_D \to \mathcal{O}_X$ where for D, D' two effective Cartier divisors $(\mathcal{I}_D, j_D)$ and $(\mathcal{I}_{D'}, j_{D'})$ are isomorphic if and only if $D \sim D'$ if and only if $\mathcal{I}_D \cong \mathcal{I}_{D'}$ as $\mathcal{O}_X$-modules. One can define the product of two generalized Cartier divisors (L,ρ) and (L',ρ')

$$(L,\rho)\cdot(L',\rho') = (L\otimes L', \rho\otimes\rho')$$

with

$$\rho\otimes\rho' : L\otimes L' \longrightarrow \mathcal{O}_X \cong \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X.$$

For $n \geq 0$ one can define the effective Cartier divisor $(L^{\otimes n}, \rho^{\otimes n})$ to be the N-fold product of (L,ρ) with itself in the abovementioned way. Let $\mathcal{D}iv^+(X)$ to be the set of isomorphism classes of generalized effective Cartier divisors. The tensor product endows $\mathcal{D}iv^+(X)$ with the structure of a commutative monoid.

The advantage of generalized Cartier divisors is that they can be pulled back along morphisms of schemes as for $f : Y \to X$ and (L,ρ) a generalized Cartier divisor on X, g^*L is a generalized Cartier divisor on Y with morphism to $\mathcal{O}_Y$ given by $g^*\rho : g^*L \to g^*\mathcal{O}_X = \mathcal{O}_Y$. The construction of $\mathcal{D}iv^+(X)$ is thus functorial on schemes, allowing us to make the following construction.

Definition 8.13 (Div)**.** Let Div be the category with objects pairs $(T,(L,\rho))$ with T a scheme and $(L,\rho) \in \mathcal{D}iv^+(T)$ and morphisms

$$(g, g^b) : \big(T', (L',\rho')\big) \longrightarrow (T,(L,\rho))$$

with $g : T' \to T$ a morphism of schemes and $g^b : (L',\rho') \to (g^*L, g^*\rho)$ an isomorphism of effective Cartier divisors on T'.

By descent for sheaves, Div is a stack on the category of schemes. In fact, this stack has an especially nice description.

Proposition 8.14. There is an isomorphism of stacks $\mathsf{Div} \to [\mathbb{A}^1/\mathbb{G}_m]$.

9. Coarse Moduli Spaces

An important result in stack theory is the Keel-Mori theorem, showing the existence of coarse moduli spaces of algebraic stacks with finite diagonal, which in turn allows us to understand a number of important properties: the local structure of Deligne-Mumford stacks, a variant of Chow's lemma, and finiteness of cohomology of sheaves on Deligne-Mumford stacks.

Definition 9.1 (Coarse Moduli Space)**.** Let $\mathcal{X}$ be an S-algebraic stack for a base scheme S. A coarse moduli space for the stack $\mathcal{X}$ is a morphism $\pi : \mathcal{X} \to X$ to a scheme X such that the following conditions hold:

(a) π is initial for maps to S-algebraic spaces.
(b) For an algebraically closed field k, there is a bijection between isomorphism classes of $\mathcal{X}(k)$ to the k-rational points of X

$$|\mathcal{X}(k)| \to X(k).$$

Remark. π being initial means that the scheme X is the scheme X is the initial object of the slice-over category $\mathsf{Spaces}_{(\mathcal{X}/-)}$ with objects morphisms $(\mathcal{X} \to Y)$ and morphisms between two objects $(\mathcal{X} \to Y), (\mathcal{X} \to Z)$ commuting triangles of the following type.

$$\begin{tikzcd} & \mathcal{X} \arrow[dl] \arrow[dr] & \\ Y \arrow[rr] & & Z \end{tikzcd}$$

More explicitly, for $g : \mathcal{X} \to Z$ with Z an algebraic space,

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\pi"] \arrow[dr, "g"'] & X \arrow[d, dashed, "\exists! f"] \\ & Z \end{tikzcd}$$

there is a unique morphism of stacks such that $g = f \circ \pi$.

9.1. **The Theorem of Keel and Mori.**

We now state the Keel-Mori theorem.

Theorem 9.2 (Keel-Mori)**.** Let $\mathcal{X}$ be an S-algebraic stack locally of finite presentation over S with a finite diagonal over a locally Noetherian base scheme S. The algebraic stack $\mathcal{X}$ admits a coarse moduli space such that:

(a) X is an S-scheme locally of finite type. Furthermore if $\mathcal{X}$ is a separated algebraic stack, then X is a separated S-scheme.
(b) π is a proper morphism and $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism.
(c) If $X' \to X$ is a flat morphism with X' an algebraic space,

$$\begin{tikzcd} X' \times_X \mathcal{X} \arrow[r] \arrow[d] & X' \arrow[d] \\ \mathcal{X} \arrow[r, "\pi"'] & X \end{tikzcd}$$

then X' is a coarse moduli space for $X' \times_X \mathcal{X}$.

The Keel-Mori theorem allows us to connect Deligne-Mumford stacks (6.24) with (topological) orbifolds.

9.2. **Local Structure of Deligne-Mumford Stacks.**

Theorem 9.3 (Local Structure of Deligne-Mumford Stacks)**.** Let $\mathcal{X}$ be a Deligne-Mumford stack locally of finite type with finite diagonal over a locally Noetherian base scheme S, and $\pi : \mathcal{X} \to X$ its coarse moduli space. Let $\widetilde{x}$ be a geometric point of $\mathcal{X}$ with image $\pi(\widetilde{x}) = \overline{x}$ and $G_{\widetilde{x}}$ the automorphism group of $\widetilde{x}$. There exists $\overline{x} \in U \subseteq X$ an étale neighborhood of $\overline{x}$ and a finite U-scheme V with action of $G_{\widetilde{x}}$ such that in the Cartesian square

$$\begin{tikzcd} U \times_X \mathcal{X} \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{X} \arrow[r, "\pi"'] & X \end{tikzcd}$$

we have an isomorphism of stacks

$$U \times_X \mathcal{X} \cong [V/G_{\widetilde{x}}] .$$

Remark. Since $\mathcal{X}$ is a Deligne-Mumford stack with $\widetilde{x}$ a geometric point, $G_{\widetilde{x}}$ is a finite group.

The finiteness of the automorphism group $G_{\widetilde{x}}$ allows us to define another property of Deligne-Mumford stacks.

Definition 9.4 (Tame Deligne-Mumford Stack)**.** Let $\mathcal{X}$ be an algebraic stack locally of finite type over a locally Noetherian base scheme S. $\mathcal{X}$ is a tame stack if for every geometric point $\widetilde{x} : \mathrm{Spec}(k) \to \mathcal{X}$, the order of the automorphism group $|G_{\widetilde{x}}|$ is an invertible element in k.

In the case of sufficiently nice Deligne-Mumford stacks, its sheaves are characterized by the sheaves on the coarse space.

Proposition 9.5. Let $\mathcal{X}$ be Deligne-Mumford stack locally of finite type with finite diagonal over a locally Noetherian base scheme S and coarse space $\pi : \mathcal{X} \to X$. If $\mathcal{X}$ is a tame stack, then the functor

$$\pi_* : \mathsf{QCoh}\,(\mathcal{X})_{\mathsf{Lis\acute{E}t}} \longrightarrow \mathsf{QCoh}\,(X)$$

is exact.

The following theorem characterizes the behavior of coarse spaces under base change.

Theorem 9.6. Let $\mathcal{X}$ be a separated Deligne-Mumford stack of finite type over a locally Noetherian base scheme S and coarse space $\pi : \mathcal{X} \to X$. For $S' \to S$, $\tau : \mathcal{X} \times_S S' \to Y$ the coarse space of the base change of the stack $\mathcal{X}$ and $p : Y \to X \times_S S'$ the morphism induced by the universal property of the coarse space, p is a universal homeomorphism. If further $S' \to S$ is flat or $\mathcal{X}$ is a tame stack, then p is an isomorphism.

Remark. The properties of local finite typeness and separatedness are preserved by base change so the stack $\mathcal{X} \times_S S'$ and the scheme $X \times_S S'$ obtained by base changes

$$\begin{tikzcd} \mathcal{X} \times_S S' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ S' \arrow[r] & S \end{tikzcd} \qquad \begin{tikzcd} X \times_S S' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

is also locally of finite type and separated, hence admitting a smooth moduli space by the Keel-Mori theorem Theorem 9.2.

9.3. Chow's Lemma for Deligne-Mumford Stacks.

Theorem 9.7 (Chow's Lemma for Stacks)**.** Let $\mathcal{X}$ be a Deligne-Mumford stack of finite type with finite diagonal over a Noetherian base-scheme S. There exists a proper surjective morphism $X' \to \mathcal{X}$ with X' an S-scheme finite over a dense open substack of $\mathcal{X}$ such that the composition

$$X' \longrightarrow \mathcal{X} \longrightarrow S$$

is a projective morphism of schemes.

10. Gerbes

Recall that for a scheme X and $\boldsymbol{\mu}$ a sheaf of Abelian groups on the small étale site of X, the first sheaf cohomology $H^1(X, \boldsymbol{\mu})$ is isomorphic to the group of $\boldsymbol{\mu}$-torsors on X. The interpretation can be extended to $H^2(X, \boldsymbol{\mu})$ as the set of $\boldsymbol{\mu}$-gerbes which can be roughly thought of as a stack over X which is a twisted form of the classifying stack $B\boldsymbol{\mu}$.

10.1. **Torsors and First Cohomology.** Let $\mathcal{T}$ be a topos on a site S and $\boldsymbol{\mu} \in \mathrm{Obj}(\mathcal{T})$ a sheaf of Abelian groups on S. We describe a bijection between the set of $\boldsymbol{\mu}$-torsors and $H^1(\mathcal{T}, \boldsymbol{\mu})$.

Definition 10.1 (Wedge of Torsors)**.** Let $(\mathcal{P}, \rho), (\mathcal{P}', \rho')$ be two $\boldsymbol{\mu}$-torsors on a site S with a final object. The wedge $(\mathcal{P} \wedge \mathcal{P}', \rho \wedge \rho')$ is the quotient of $\mathcal{P} \times \mathcal{P}'$ by the action

$$g \cdot (p, p') \mapsto (gp, g^{-1}p').$$

This easily implies the following.

Proposition 10.2 (Magma Structure on Tors)**.** Let $(\mathcal{P}, \rho), (\mathcal{P}', \rho')$ be two $\boldsymbol{\mu}$-torsors on a site S with a final object. The operation

$$\wedge : \mathsf{Tors}(\boldsymbol{\mu}) \times \mathsf{Tors}(\boldsymbol{\mu}) \longrightarrow \mathsf{Tors}(\boldsymbol{\mu})$$

$$\big((\mathcal{P}, \rho), (\mathcal{P}', \rho')\big) \mapsto \big(\mathcal{P} \wedge \mathcal{P}', \rho \wedge \rho'\big)$$

endows $\mathsf{Tors}(\boldsymbol{\mu})$ with the structure of a magma.

Remark. Recall here that a magma is a set closed under a binary operation.

Proof. The action

$$\rho \wedge \rho' : \boldsymbol{\mu} \times \mathcal{P} \times \mathcal{P}' \longrightarrow \mathcal{P}$$

by

$$g \cdot (p, p') \mapsto (gp, p')$$

descends to $\mathcal{P} \wedge \mathcal{P}'$ endowing it with the structure of a $\boldsymbol{\mu}$-torsor. ■

Denote $\underline{\mathbb{Z}}$ the constant sheaf on the integers $\mathbb{Z}$. We can define a category $\mathsf{Ext}(\underline{\mathbb{Z}}, \boldsymbol{\mu})$ in the following way.

Definition 10.3 ($\mathsf{Ext}(\underline{\mathbb{Z}}, \boldsymbol{\mu})$)**.** Let S be a site with a final object and $\boldsymbol{\mu}$ a sheaf of Abelian groups on S. Define $\mathsf{Ext}(\underline{\mathbb{Z}}, \boldsymbol{\mu})$ be the category with:

(a) Objects short exact sequences of sheaves of Abelian groups on S, $(0 \to \boldsymbol{\mu} \to \mathcal{E} \to \underline{\mathbb{Z}} \to 0)$.
(b) Morphisms between

$$(0 \to \boldsymbol{\mu} \to \mathcal{E}' \to \underline{\mathbb{Z}} \to 0), (0 \to \boldsymbol{\mu} \to \mathcal{E} \to \underline{\mathbb{Z}} \to 0) \in \mathrm{Obj}\left(\mathsf{Ext}(\underline{\mathbb{Z}}, \boldsymbol{\mu})\right)$$

commutative diagrams of the following type:

$$\begin{tikzcd}
0 \arrow[r] & \boldsymbol{\mu} \arrow[r] \arrow[d, "\wr"'] & \mathcal{E}' \arrow[r] \arrow[d] & \underline{\mathbb{Z}} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \boldsymbol{\mu} \arrow[r] & \mathcal{E} \arrow[r] & \underline{\mathbb{Z}} \arrow[r] & 0
\end{tikzcd}$$

We can check locally to show that for any such morphism in Definition 10.3 (b), the morphism $\mathcal{E}' \to \mathcal{E}$ is in fact an isomorphism. One can then define a functor from $\mathsf{Ext}(\underline{\mathbb{Z}}, \boldsymbol{\mu})$ to the category of $\boldsymbol{\mu}$-torsors $\mathsf{Tors}(\boldsymbol{\mu})$ and show that it is an equivalence of categories.

Proposition 10.4. The functor

$$\pi : \mathsf{Ext}(\underline{\mathbb{Z}}, \boldsymbol{\mu}) \longrightarrow \mathsf{Tors}(\boldsymbol{\mu})$$

by

$$(0 \to \boldsymbol{\mu} \to \mathcal{E} \to \underline{\mathbb{Z}} \to 0) \mapsto \pi^{-1}(1) \in \mathcal{E}; 1 \in \Gamma(\mathsf{S}, \boldsymbol{\mu})$$

is an equivalence of categories.

This allows us to deduce the desired result.

Corollary 10.5. There is a bijection between $H^1(\mathsf{S}, \boldsymbol{\mu})$ and the set of isomorphism classes of $\boldsymbol{\mu}$-torsors.

10.2. **Gerbes and Second Cohomology.** Recall for $p : \mathsf{F} \to \mathsf{S}$ a (categorical) stack over a site, for any $x \in \mathrm{Obj}(\mathsf{F}_X)$ the fiber over $X \in \mathrm{Obj}(\mathsf{S})$ there is a sheaf $\underline{\mathrm{Aut}}_x$ over the slice category $\mathsf{S}_{(-/X)}$. For $\boldsymbol{\mu}$ a sheaf of Abelian groups on S, a gerbe is a stack constructed such that the restriction of the sheaf $\boldsymbol{\mu}$ to the slice category $\mathsf{S}_{(-/X)}$ is isomorphic to the automorpihsm group $\underline{\mathrm{Aut}}_x$. More explicitly, we have the following.

Definition 10.6 (Gerbe)**.** Let S be a site and $\boldsymbol{\mu}$ a sheaf of Abelian groups on S. A $\boldsymbol{\mu}$-gerbe over S is a stack $p : \mathsf{F} \to \mathsf{S}$ where F is the fibered category with objects (x, ι_x) where for all $x \in \mathrm{Obj}(\mathsf{F})$ an isomorphism of sheaves of groups

$$\iota_x : \boldsymbol{\mu}|_{\mathsf{S}_{(-/p(x))}} \longrightarrow \underline{\mathrm{Aut}}_x$$

such that:

(a) For any $X \in \mathrm{Obj}(\mathsf{S})$ there is a covering $\{\phi_i : X_i \to X\}_{i \in I}$ with $\mathsf{F}_{X_i} \neq \emptyset$ for all $i \in I$.
(b) For any $x, x' \in \mathsf{F}_X$ there is a covering $\{\phi_i : X_i \to X\}_{i \in I}$ with $\phi_i^* x = \phi_i^* x'$ in F_{X_i} for all $i \in I$.
(c) For all $X \in \mathrm{Obj}(\mathsf{S})$ and isomorphism $\sigma : x \to x'$ in F_X a commutative diagram of the following type:

$$\begin{tikzcd} & \boldsymbol{\mu} \arrow[dl, "\iota_x"'] \arrow[dr, "\iota_{x'}"] & \\ \underline{\mathrm{Aut}}_x \arrow[rr, "\sigma"'] & & \underline{\mathrm{Aut}}_{x'} \end{tikzcd}$$

Remark. Morphisms between objects in fibers of F are induced along morphisms in S. For $(g : Y \to X) \in \mathrm{Mor}_{\mathsf{S}}$ there is a functor of categories $g^* : \mathsf{F}_Y \to g^* \mathsf{F}_X$ that for $p(y) = Y$ with $y = g^* x$ an isomorphism of groups $\iota_y : \boldsymbol{\mu}|_{(-/p(y))} \to \underline{\mathrm{Aut}}_y$ satisfying the additional conditions (a), (b), and (c) in Definition 10.6 such that the following diagram commutes.

$$\begin{tikzcd} & \boldsymbol{\mu} \arrow[dl, "\iota_y"'] \arrow[dr, "\iota_x"] & \\ \underline{\mathrm{Aut}}_y \arrow[rr, "(f^*)_*"'] & & \underline{\mathrm{Aut}}_x \end{tikzcd}$$

Note further that the morphism of sheaves of groups $(f^*)_*$ need not be an isomorphism. This is not crucial for the definition of the gerbe, and we hence omit it from the definition.

A morphism of $\boldsymbol{\mu}$-gerbes can be defined in the natural way.

Definition 10.7 (Morphism of Gerbes)**.** Let S be a site and $\boldsymbol{\mu}$ a sheaf of Abelian groups on S with $p : \mathsf{F} \to \mathsf{S}$ and $p' : \mathsf{F}' \to \mathsf{S}$ to $\boldsymbol{\mu}$-gerbes on S. A morphism of $\boldsymbol{\mu}$-gerbes is a morphism of stacks $f : \mathsf{F}' \to \mathsf{F}$ such that for $f(x') = x$ the diagram

$$\begin{tikzcd} & \boldsymbol{\mu} \arrow[dl, "\iota_{x'}"'] \arrow[dr, "\iota_x"] & \\ \underline{\mathrm{Aut}}_{x'} \arrow[rr, "f_*"'] & & \underline{\mathrm{Aut}}_x \end{tikzcd}$$

commutes.

We can in fact refine this characterization of morphisms of $\boldsymbol{\mu}$-gerbes as we now show.

Proposition 10.8. Let S be a site and $\boldsymbol{\mu}$ a sheaf of Abelian groups on S with $p : \mathsf{F} \to \mathsf{S}$ and $p' : \mathsf{F}' \to \mathsf{S}$ to $\boldsymbol{\mu}$-gerbes on S. If $f : \mathsf{F}' \to \mathsf{F}$ is a morphism of gerbes then $f : \mathsf{F}' \to \mathsf{F}$ is an isomorphism of gerbes.

We now show the correspondence with second cohomology.

Consider the following construction. Let $\boldsymbol{\mu}$ be a sheaf of Abelian groups on a site S and

$$1 \longrightarrow \boldsymbol{\mu} \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be an exact sequence of sheaves of groups where G, Q need not be sheaves of Abelian groups. Given a Q-torsor $\mathcal{P}$ as in Definition 3.6 we can construct a fibered category over the site S.

Remark. We use multiplicative notation for the sequence above, since the construction in practice often uses multiplicative groups such as $\mathbb{G}_m$.

Definition 10.9 (The Category $\mathcal{G}_{\mathcal{P}}$)**.** Consider the following construction. Let $\boldsymbol{\mu}$ be a sheaf of Abelian groups on a site S and

$$1 \longrightarrow \boldsymbol{\mu} \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be an exact sequence of sheaves of groups and $\mathcal{P}$ a Q-torsor. Let $\mathcal{G}_{\mathcal{P}}$ be the fibered category over S with

(a) Objects triples $\left(X, \widetilde{\mathcal{P}}, \epsilon\right)$ with $X \in \mathrm{Obj}(\mathsf{S})$, $\widetilde{\mathcal{P}}$ a $G|_{\mathsf{S}_{(-/X)}}$-torsor on the slice category $\mathsf{S}_{(-/X)}$, and $\epsilon : b_*\widetilde{\mathcal{P}} \to \mathcal{P}|_{\mathsf{S}_{(-/X)}}$ an isomorphism of $Q|_{\mathsf{S}_{(-/X)}}$-torsors.
(b) Morphsims
$$(f, f^b) : \left(X', \widetilde{\mathcal{P}'}, \epsilon'\right) \longrightarrow \left(X, \widetilde{\mathcal{P}}, \epsilon\right)$$
where $f : X' \to X$ is a morphism in S and $f^b : f^*\widetilde{\mathcal{P}} \to \widetilde{\mathcal{P}'}$ an isomorphism of $G|_{\mathsf{S}_{(-/X')}}$-torsors such that the pentagon

$$\begin{tikzcd} & & \mathcal{P}|_{\mathsf{S}_{(-/X')}} & & \\ b_*\widetilde{\mathcal{P}'} \arrow[urr, "\epsilon'"] & & & & f^*\mathcal{P}|_{\mathsf{S}_{(-/X)}} \arrow[ull, "\sim"'] \\ & b_*f^*\widetilde{\mathcal{P}} \arrow[ul, "b_*f^b"] \arrow[rr, "\sim"'] & & f^*b_*\widetilde{\mathcal{P}} \arrow[ur, "f^*\epsilon"'] & \end{tikzcd}$$

commutes.

Remark. $p : \mathcal{G}_{\mathcal{P}} \to \mathsf{S}$ is defined by the functorial assignment

$$\left(X, \widetilde{\mathcal{P}}, \epsilon\right) \mapsto X.$$

In particular, the fibered category $\mathcal{G}_{\mathcal{P}}$ is a category fibered in groupoids over the site S.

Note that for any $\left(X, \widetilde{\mathcal{P}}, \epsilon\right) \in \mathrm{Obj}\left(\mathcal{G}_{\mathcal{P}}\right)$ the automorphism presheaf $\underline{\mathrm{Aut}}_{\left(X, \widetilde{\mathcal{P}}, \epsilon\right)}$ is the presheaf of groups such that for any $g : X' \to X$ associates the automorphism group of the $G|_{\mathsf{S}_{(-/X')}}$-torsor $\widetilde{\mathcal{P}}|_{\mathsf{S}_{(-/X')}}$ such that the pushout

$$\begin{tikzcd}
\mathcal{P}|_{\mathsf{S}_{(-/X')}} \arrow[r, "b_*"] \arrow[d, dash] & \mathcal{P}|_{\mathsf{S}_{(-/X)}} \arrow[d, dash] \\
X' \arrow[r, "g"'] & X
\end{tikzcd}$$

along b_* is the identity automorphism. In particular this group of automorphisms is $\boldsymbol{\mu}(X')$ giving a cannonical isomorphism

$$\iota_x : \boldsymbol{\mu}|_{\mathsf{S}_{(-/X)}} \longrightarrow \underline{\mathrm{Aut}}_{\left(X, \widetilde{\mathcal{P}}, \epsilon\right)}.$$

Proposition 10.10. Consider the following construction. Let $\boldsymbol{\mu}$ be a sheaf of Abelian groups on a site S and

$$1 \longrightarrow \boldsymbol{\mu} \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be an exact sequence of sheaves of groups and $\mathcal{P}$ a Q-torsor. The fibered category $p : \mathcal{G}_{\mathcal{P}}$ is a $\boldsymbol{\mu}$-gerbe. Furthermore, the construction is functorial with respect to Q-torsors.

We now describe the correspondence between second cohomology and $\boldsymbol{\mu}$-torsors.

For $\alpha \in H^2(\mathsf{S}, \boldsymbol{\mu})$ we can construct a $\boldsymbol{\mu}$-gerbe $\mathcal{G}_\alpha$ as follows: choose some inclusion of sheaves of Abelian groups $i : \boldsymbol{\mu} \to I$ and since i is injective, this yields an exact sequence of sheaves of Abelian groups

$$0 \longrightarrow \boldsymbol{\mu} \longrightarrow I \longrightarrow I/\boldsymbol{\mu} \longrightarrow 0$$

and once again with i injective, the boundary map

$$\partial : H^1(\mathsf{S}, I/\boldsymbol{\mu}) \longrightarrow H^2(\mathsf{S}, \boldsymbol{\mu})$$

is an isomorphism. Thus our previous construction from Q-torsors as described in Definition 10.9 suffices and is a $\boldsymbol{\mu}$-gerbe by Proposition 10.10 with the correspondence described by

$$\alpha \mapsto \mathcal{G}_{\partial^{-1}(\alpha)}.$$

This is functorial on exact sequences: for $i' : \boldsymbol{\mu} \to I'$ another inclusion and for a choice $\rho : I \to I'$ descending to the quotient $\overline{\rho} : I/\boldsymbol{\mu} \to I'/\boldsymbol{\mu}$ we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \boldsymbol{\mu} \arrow[r] \arrow[d, "\wr"'] & I \arrow[r] \arrow[d, "\rho"'] & I/\boldsymbol{\mu} \arrow[r] \arrow[d, "\overline{\rho}"'] & 0 \\
0 \arrow[r] & \boldsymbol{\mu} \arrow[r] & I' \arrow[r] & I'/\boldsymbol{\mu} \arrow[r] & 0
\end{tikzcd}$$

where $i' = \rho \circ i$ that induces the commutative triangle

$$\begin{tikzcd} H^1(\mathsf{S}, I/\boldsymbol{\mu}) \arrow[rr, "\overline{\rho}"] \arrow[dr, "\partial"'] & & H^1(\mathsf{S}, I'/\boldsymbol{\mu}) \arrow[dl, "\partial'"] \\ & H^2(\mathsf{S}, \boldsymbol{\mu}) & \end{tikzcd}$$

but taking $\gamma \in H^1(\mathsf{S}, I/\boldsymbol{\mu})$ such that $\overline{\rho}(\gamma) = \gamma' \in H^1(\mathsf{S}, I'/\boldsymbol{\mu})$ with $\boldsymbol{\mu}$-gerbes $\mathcal{G}_\gamma, \mathcal{G}_{\gamma'}$, respectively, and $\mathcal{P}$ the $I/\boldsymbol{\mu}$-torsor corresponding to $\mathcal{G}_\gamma$ we can pushout along $\overline{\rho}$ where for $\left(X, \widetilde{\mathcal{P}}, \epsilon\right) \in \mathcal{G}_{\mathcal{P}}(X)$ we have $\left(X, \rho_* \widetilde{\mathcal{P}}, \overline{\rho}_* \epsilon\right) \in\in \mathcal{G}_{\mathcal{P}'}(X)$ inducing

$$\mathcal{G}_{\mathcal{P}} \longrightarrow \mathcal{G}_{\mathcal{P}'}$$

but such morphisms are isomorphisms by Proposition 10.8 giving the claim.

Theorem 10.11. Let S be a site and $\boldsymbol{\mu}$ a sheaf of Abelian groups on S. There is a bijection between the cohomology classes in $H^2(\mathsf{S}, \boldsymbol{\mu})$ and isomorphism classes of $\boldsymbol{\mu}$-gerbes.

10.3. **Twisted Sheaves.** One application the theory of gerbes and torsors have is to twisted sheaves. Following Section 10.1, let $\underline{\mathbb{G}_m}$ be the sheaf

$$\underline{\mathbb{G}_m} : (\mathsf{Sch}_S)^{\mathsf{Opp}}_{\text{ét}} \longrightarrow \mathsf{Grp}$$

given on objects by

$$(T \to S) \mapsto \Gamma\left(T, \mathcal{O}_T^\times\right)$$

and on S-morphisms

$$(f : T' \to T) \mapsto \left(f^\sharp : \Gamma(T, f_* \mathcal{O}_{T'}^\times) \longrightarrow \Gamma(T', \mathcal{O}_{T'}^\times)\right).$$

One can show that an $\underline{\mathbb{G}_m}$-gerbe is an algebraic stack. Let $\mathcal{G}$ be an $\underline{\mathbb{G}_m}$-gerbe and E a locally free sheaf on $\mathcal{G}$ of finite rank. Given a field k and a geometric point $x : \operatorname{Spec}(k) \to \mathcal{G}$ we have an action of $\operatorname{Aut}_x \cong \mathbb{G}_m(k) \cong k^\times$ on the k-vector space $E(x) = x^* E$. Any representation of $\mathbb{G}_m(k)$ V admits a decomposition as graded vector spaces

$$V \cong \bigoplus_{i \in \mathbb{Z}} V_i$$

where the action of $\mathbb{G}_m(k)$ acts by $u \cdot v \mapsto u^i v$ for $v \in V_i$. The set of integers $i \in \mathbb{Z}$ such that $V_i \neq 0$ are known as the weights of the representation.

Definition 10.12 (Twisted Sheaf)**.** Let $\mathcal{G}$ be a $\underline{\mathbb{G}_m}$-gerbe on $(\mathsf{Sch}_S)_{\text{ét}}$, E a locally free sheaf of finite rank on $\mathcal{G}$, and $n \in \mathbb{Z}$. E is an n-twisted sheaf if for all fields k and all geometric points $x : \operatorname{Spec}(k) \to \mathcal{G}$ the representation is of weight n, that is $E(x) = E(x)_n$.

Twisted sheaves satisfy several nice properties.

Lemma 10.13. Let $\mathcal{G}$ be a $\underline{\mathbb{G}_m}$-gerbe on $(\mathsf{Sch}_S)_{\text{ét}}$ with structure morphism $\pi : \mathcal{G} \to S$.

(a) If E_1 is an n_1-twisted sheaf and E_2 is an n_2-twisted sheaf on $\mathcal{G}$ for $n_1, n_2 \in \mathbb{Z}$ then $E_1 \otimes E_2$ is an $(n_1 + n_2)$-twisted sheaf.

(b) If M is a locally free sheaf of finite rank on S, then $\pi^* M$ is a 0-twisted sheaf on $\mathcal{G}$ and defines an equivalence between the category of locally free sheaves of finite rank on S and 0-twisted sheaves on $\mathcal{G}$.

(c) If E is an n-twisted sheaf on $\mathcal{G}$, then the dual $E^\vee$ is a $(-n)$-twisted sheaf on $\mathcal{G}$.

Note that the correspondence in Lemma 10.13 (b) is compatible with tensor products of sheaves both on $\mathcal{G}$ and S. In particular, for E an n-twisted sheaf on the gerbe $\mathcal{G}$, we can construct a sheaf of non-commutative algebras by taking endomorphisms

$$\mathcal{E}nd(E) = \mathrm{Hom}_{\mathsf{Sh}(\mathcal{G})}(E, E) = E^\vee \otimes E$$

is 0-twisted by Lemma 10.13 (a) and (c). In particular, it is isomorphic to a pullback $\pi^*\mathcal{A}_E$ of a sheaf of lcoally free algebras $\mathcal{A}_E$ on S. This connects the study of gerbes and Brauer groups.

Definition 10.14 (Azumaya Algebra)**.** Let S be a scheme. An Azumaya algebra is a sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ such that étale locally is isomorphic to $\mathcal{E}nd(E)$ for a locally free sheaf E of finite rank.

Remark. For $\mathcal{A}$ an Azumaya algebra on S the ring of global sections $\Gamma(S, \mathcal{O}_S)$ is isomorphic to the center of $\mathcal{A}$.

If E is a locally free sheaf of finite rank on S and L a line bundle, we can twist E by the line bundle L yielding a canonical isomorphism of algebras

$$\mathcal{E}nd(E) \cong \mathcal{E}nd(E \otimes L)$$

and thus the locally free sheaf of finite rank E is not unique. We can attempt to refine this construction in the following way.

Definition 10.15 (The Stack $\mathcal{G}_\mathcal{A}$)**.** Let $\mathcal{A}$ be an Azumaya algebra on a scheme S and define $\mathcal{G}_\mathcal{A}$ to be the stack over the big étale site of S-schemes $(\mathsf{Sch}_S)_{\text{ét}}$ with:

(a) Objects $(f : T \to S, E, \sigma)$ where f is a morphism of S-schemes, E a locally free sheaf of finite rank on T, and $\sigma : \mathcal{E}nd(E) \to f^*\mathcal{A}$ an isomorphism of $\mathcal{O}_T$-algebras.

(b) Morphisms
$$(g, g^b) : (f' : T' \to S, E', \sigma') \longrightarrow (f : T \to S, E, \sigma)$$
where $g : T' \to T$ is an S-morphism and $g^b : g^*E \to E'$ an isomorphism of locally free sheaves of finite rank on T' such that the diagram

$$\begin{tikzcd} \mathcal{E}nd(g^*E) \arrow[rr, "g^b"] \arrow[dr, "\sigma"'] & & \mathcal{E}nd(E') \arrow[dl, "\sigma'"] \\ & f'^*\mathcal{A} & \end{tikzcd}$$

commutes.

Remark. For any $(f : T \to S, E, \sigma) \in \mathrm{Obj}(\mathcal{G}_\mathcal{A})$ there is a natural map

$$\underline{\mathbb{G}_m}(T) \to \mathrm{Aut}((f : T \to S, E, \sigma))\,; u \mapsto (\mathrm{id}, u).$$

We can in fact show that this stack is a $\underline{\mathbb{G}_m}$-gerbe.

Proposition 10.16. Let $\mathcal{A}$ be an Azumaya algebra on the scheme S. The algebraic stack $\mathcal{G}_\mathcal{A}$ on the big étale site of S-schemes $(\mathsf{Sch}_S)_{\text{ét}}$ is a $\underline{\mathbb{G}_m}$-gerbe.

The construction of the stack $\mathcal{G}_\mathcal{A}$ admits a refinement by the stack $\widetilde{\mathcal{G}_\mathcal{A}}$.

Definition 10.17 (The Stack $\widetilde{\mathcal{G}_{\mathcal{A}}}$)**.** Let $\mathcal{A}$ be an Azumaya algebra on a scheme S and define $\widetilde{\mathcal{G}_{\mathcal{A}}}$ to be the stack over the big étale site of S-schemes $(\mathsf{Sch}_S)_{\text{ét}}$ with:

(a) Objects quadruples $(f : T \to S, E, \sigma, \lambda)$ where $f : T \to S$ is a morphism of S-schemes, E a locally free sheaf of finite rank on T, $\sigma : \mathcal{E}nd(E) \to f^*\mathcal{A}$ an isomorphism of $\mathcal{O}_T$-algebras, and $\lambda : \mathcal{O}_T \to \bigwedge^{\sqrt{\operatorname{rank}(\mathcal{A})}} E$ a trivialization of the line bundle $\det E$.

(b) Morphisms

$$(g, g^b) : (f' : T' \to S, E', \sigma', \lambda') \longrightarrow (f : T \to S, E, \sigma, \lambda)$$

where $g : T' \to T$ is an S-morphism, $g^b : g^*E \to E'$ an isomorphism of locally free sheaves of finite rank on T' such that the diagrams

$$\begin{tikzcd} \mathcal{E}nd(g^*E) \arrow[rr, "g^b"] \arrow[dr, "\sigma"'] & & \mathcal{E}nd(E') \arrow[dl, "\sigma'"] \\ & f'^*\mathcal{A} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_T \arrow[r, "\lambda'"] \arrow[dr, "g^*\lambda"'] & \bigwedge^{\sqrt{\operatorname{rank}(\mathcal{A})}} E' \arrow[d, "g^b"] \\ & \bigwedge^{\sqrt{\operatorname{rank}(\mathcal{A})}} g^*E \end{tikzcd}$$

commute.

For $(f : T \to S, E, \sigma, \lambda) \in \operatorname{Obj}\left(\widetilde{\mathcal{G}_{\mathcal{A}}}\right)$, the subgroup $\boldsymbol{\mu}_{\sqrt{\operatorname{rank}(\mathcal{A})}} \hookrightarrow \underline{\mathbb{G}_m}$ of $(\sqrt{\operatorname{rank}(\mathcal{A})})$-th roots of unity preserves λ. In particular, one can show the following.

Proposition 10.18. Let $\mathcal{A}$ be an Azumaya algebra on the scheme S. The algebraic stack $\widetilde{\mathcal{G}_{\mathcal{A}}}$ on the big étale site of S-schemes is a $\boldsymbol{\mu}_r$-gerbe where $r = \sqrt{\operatorname{rank}(\mathcal{A})}$. Furthermore, the pushout along the inclusion $\boldsymbol{\mu}_r \hookrightarrow \underline{\mathbb{G}_m}$ yields an equivalence of categories from the pushout to the $\underline{\mathbb{G}_m}$-gerbe $\mathcal{G}_{\mathcal{A}}$.

Theorem 10.11 suggests that there is a bijection between classes of the second cohomology on the big étale site $H^2\left((\mathsf{Sch}_S)_{\text{ét}}, \underline{\mathbb{G}_m}\right)$ and $\underline{\mathbb{G}_m}$-torsors up to isomorphism. We describe the cohomology class

$$[\mathcal{G}_{\mathcal{A}}] \in H^2\left((\mathsf{Sch}_S)_{\text{ét}}, \underline{\mathbb{G}_m}\right)$$

as follows. Let $r = \sqrt{\operatorname{rank}(\mathcal{A})}$ and consider

$$\underline{\mathrm{GL}_r} : (\mathsf{Sch}_S)_{\text{ét}}^{\mathsf{Opp}} \to \mathsf{Grp}$$

by

$$(T \to S) \mapsto \mathrm{GL}_r(\mathcal{O}_T)$$

and

$$\underline{\mathrm{PGL}_r} : (\mathsf{Sch}_S)_{\text{ét}}^{\mathsf{Opp}} \to \mathsf{Grp}$$

by

$$(T \to S) \mapsto \mathrm{PGL}_r(\mathcal{O}_T)$$

sheaves of groups on the big étale site $(\mathsf{Sch}_S)_{\text{ét}}$ sending a scheme $(T \to S)$ to invertible matrices with entries in the ring of global sections and equivalence classes of such matrices modulo the diagonal action, respectively.

Remark. One can easily show compatibility with morphisms in the standard way. We omit the verification for ease of exposition.

There is a short exact sequence of sheaves on the big étale site

$$1 \longrightarrow \underline{\mathbb{G}_m} \longrightarrow \underline{\mathrm{GL}_r} \longrightarrow \underline{\mathrm{PGL}_r} \longrightarrow 1$$

and denote $P_{\mathcal{A}}$ the functor

$$P_{\mathcal{A}} : (\mathsf{Sch}_S)_{\text{ét}} \longrightarrow \mathsf{Sets}$$

by

$$(T \to S) \mapsto \operatorname{Isom}\left(\operatorname{Mat}_r(\mathcal{O}_T), f^*\mathcal{A}\right).$$

Any automorphism of $\operatorname{Mat}_r(\mathcal{O}_T)$ is locally inner and conjugation by an invertible matrix in $\operatorname{GL}_r(\mathcal{O}_T)$ is trivial if and only if the element is a scalar in $\operatorname{GL}_r(\mathcal{O}_T)$. In particular, $P_{\mathcal{A}}$ is a $\underline{\operatorname{PGL}_r}$-torsor and representable by an S-scheme.

Lemma 10.19. Let $\mathcal{A}$ be an Azumaya algebra on S and $\mathcal{G}_{\mathcal{A}}$ the corresponding $\underline{\mathbb{G}_m}$-gerbe on the big étale site $(\mathsf{Sch}_S)_{\text{ét}}^{\mathsf{Opp}}$ and $P_{\mathcal{A}} : (\mathsf{Sch}_S)_{\text{ét}} \to \mathsf{Sets}$ be the $\underline{\operatorname{PGL}_r}$-torsor taking an S-scheme T to isomorphisms between $\operatorname{Mat}_r(\mathcal{O}_T)$ and $f^*\mathcal{A}$ with $f : T \to S$. The cohomology class

$$[\mathcal{G}_{\mathcal{A}}] \in H^2\left((\mathsf{Sch}_S)_{\text{ét}}, \underline{\mathbb{G}_m}\right)$$

is the image of $[P_{\mathcal{A}}]$ under the boundary map

$$\partial : H^1\left((\mathsf{Sch}_S)_{\text{ét}}, \underline{\operatorname{PGL}_r}\right) \longrightarrow H^2\left((\mathsf{Sch}_S)_{\text{ét}}, \underline{\mathbb{G}_m}\right).$$

Let $\pi : \mathcal{G}_{\mathcal{A}} \to S$ be the structure morphism. On the $\underline{\mathbb{G}_m}$-gerbe there is a tautological locally free sheaf $\mathcal{E}$ and an isomorphism

$$\sigma : \mathcal{E}nd(\mathcal{E}) \longrightarrow \pi^*\mathcal{A}$$

of algebras over $\mathcal{G}_{\mathcal{A}}$ and we can recover the Azumaya algebra $\mathcal{A}$ on S as the pushforward $\mathcal{A} = \pi_* \mathcal{E}nd(\mathcal{E})$ where $\mathcal{E}$ is a 1-twisted sheaf on $\mathcal{G}_{\mathcal{A}}$. A converse is given as follows.

Proposition 10.20. Let $\pi : \mathcal{G} \to S$ be a $\underline{\mathbb{G}_m}$-gerbe and suppose there is a 1-twisted locally free sheaf of finite rank $\mathcal{E}$ on $\mathcal{G}$. $\pi_* \mathcal{E}nd(\mathcal{E})$ is an Azumaya algebra on S, $\pi^*\pi_* \mathcal{E}nd(\mathcal{E}) \to \mathcal{E}nd(\mathcal{E})$ is an isomorphism of sheaves on $\mathcal{G}$, and $\mathcal{G} \cong \mathcal{G}_{\pi_* \mathcal{E}nd(\mathcal{E})}$.

11. Cohomology of Stacks

12. Derived Categories of Stacks

End Material

Appendix A. Representable Functors

Let C be a category and consider functors from $\mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$. These are objects of the functor category $\mathrm{Fun}(\mathsf{C}^{\mathsf{Opp}}, \mathsf{Sets})$ whose objects are functors $\mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ and whose morphisms are natural transformations between such functors. For any object $A \in \mathrm{Obj}(\mathsf{C})$, we can define a set-valued functor h_A as follows:

$$h_A : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets} \text{ by } A \mapsto \mathrm{Mor}_{\mathsf{C}}(-, A).$$

For an object $B \in \mathrm{Obj}(\mathsf{C})$, our functor h_A sends B to the set $\mathrm{Mor}_{\mathsf{C}}(B, A)$ the set of morphisms from B to A in the category C. Given $f \in \mathrm{Mor}_{\mathsf{C}}(B, C)$ we get a map of sets $h_A(f) : h_A(C) \to h_A(B)$ defined by composition with f. For another $g \in \mathrm{Mor}_{\mathsf{C}}(A, D)$, we get a natural transformation of functors $h_f : h_A \to h_D$ where for any $f \in \mathrm{Mor}_{\mathsf{C}}(B, C)$ we have the following commutative diagram.

$$\begin{tikzcd} h_A(C) \arrow[r, "h_g(C)"] \arrow[d, "h_A(f)"'] & h_D(C) \arrow[d, "h_D(f)"] \\ h_A(B) \arrow[r, "h_g(B)"'] & h_D(B) \end{tikzcd}$$

The map $h_{(-)} : \mathsf{C} \to \mathrm{Fun}(\mathsf{C}^{\mathsf{Opp}}, \mathsf{Sets})$ sends each object of C to a set-valued functor, the functor it represents, which we now define.

Definition A.1 (Representable Functor)**.** A functor $F : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ is representable if there exists $A \in \mathrm{Obj}(\mathsf{C})$ and a natural isomorphism $F \Longrightarrow h_A$.

Yoneda's lemma shows that the functor $h_{(-)} : \mathsf{C} \to \mathrm{Fun}(\mathsf{C}^{\mathsf{Opp}}, \mathsf{Sets})$ is fully faithful.

Lemma A.2 (Yoneda)**.** Let $F : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ be a functor. For any $A \in \mathrm{Obj}(\mathsf{C})$, there is a bijection

$$\mathrm{NatTrans}(h_A, F) \to F(A).$$

Yoneda's lemma tells us that C is embedded in the functor category $\mathrm{Fun}(\mathsf{C}^{\mathsf{Opp}}, \mathsf{Sets})$. Further, we know that for a functor $F : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ we get a functor $h_F : \mathrm{Fun}(\mathsf{C}^{\mathsf{Opp}}, \mathsf{Sets})^{\mathsf{Opp}} \to \mathsf{Sets}$ taking $G \in \mathrm{Obj}(\mathrm{Fun}(\mathsf{C}^{\mathsf{Opp}}, \mathsf{Sets}))$ to the set of natural transformations between G and F $\mathrm{NatTrans}(G, F)$. This suggests that every functor $\mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ can be extended to a representable functor. This can be done via a process known as Kan extension, referring to

Note, however, that the functor $h_{(-)} : \mathsf{C} \to \mathrm{Fun}(\mathsf{C}^{\mathsf{Opp}}, \mathsf{Sets})$ is not essentially surjective, and hence would not define an equivalence of categories. This means that not every functor $\mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ is representable by some object $A \in \mathrm{Obj}(\mathsf{C})$. However, if we restrict to the full subcategory of representable functors in the functor category $\mathrm{Fun}(\mathsf{C}^{\mathsf{Opp}}, \mathsf{Sets})$, $h_{(-)}$ would indeed define a categorical equivalence between C and the subcategory of $\mathrm{Fun}(\mathsf{C}^{\mathsf{Opp}}, \mathsf{Sets})$ of representable functors.

From the proof of Lemma A.2, we know that given a natural transformation of functors $T : h_A \to F$ on evaluation we get $T(A) : h_A(A) \to F(A)$ yielding an element $T(A)(\mathrm{id}_A) = \alpha \in F(A)$ for $\mathrm{id}_A \in h_A(A) = \mathrm{Mor}_{\mathsf{C}}(A, A)$ the identity morphism on A. This defined a set function $\mathrm{NatTrans}(h_A, F) \to F(A)$. Conversely, given some $\alpha \in F(A)$, we can define a natural transformation $T : h_A \longrightarrow F$ as follows: for any $D \in \mathrm{Obj}(\mathsf{C})$ an element of the set $h_A(D)$ is a morphism $g : D \to A$ which defines a map of sets $F(g) : F(A) \to F(D)$. We define $T(D) : h_A(D) \to F(D)$ by $g \mapsto F(g)(\alpha)$ where the latter lies in $F(D)$ by the definition of $T(D)$ above. This coheres into the data of a natural transformation by the appropriate pointwise verifications. This suggests that elements of $F(A)$ for an object $A \in \mathrm{Obj}(\mathsf{C})$, can exhibit control over the functor F. This motivates the following discussion.

Definition A.3 (Universal Object)**.** Let $F : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ be a functor. A universal object for F is a pair $(A, \alpha) \in \mathrm{Obj}(\mathsf{C}) \times F(A)$ such that for each $B \in \mathrm{Obj}(\mathsf{C})$ and each $\beta \in F(B)$ there is a unique map $f \in \mathrm{Mor}_{\mathsf{C}}(B, A)$ such that $F(f)(\alpha) = \beta$.

From the exposition above, we can see that (A, α) is a universal object if the natural transformation $h_A \Longrightarrow F$ defined by α is a natural isomorphism. Since every natural transformation $h_A \Longrightarrow F$ is defined by some object $\alpha \in F(A)$, we can conclude the following.

Proposition A.4. A functor $F : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ is representable if and only if it has a universal object.

Let us now consider some examples in the context of algebraic geometry.

Example A.5. Let $S = \operatorname{Spec} A$ and consider the affine line $\mathbb{A}^1_S = \operatorname{Spec} A[t]$. We have a functor $\mathcal{O} : \mathsf{Sch}_S^{\mathsf{Opp}} \to \mathsf{Rings}$ taking an S-scheme X to $\mathcal{O}(X)$ its ring of global sections. For $f : X \to Y$ a map of schemes, we get a ring homomorphism $\mathcal{O}(Y) \to \mathcal{O}(X)$ induced by the taking global sections of the pullback of sheaves $f^\sharp : \mathcal{O}_Y \to f^*\mathcal{O}_X$. Since $\mathbb{A}^1_S$ is an affine scheme, we have $t \in \mathcal{O}(\mathbb{A}^1_S) = A[t]$ and for any S-scheme X and $g \in \mathcal{O}(X)$ there is a unique morphism $X \to \mathbb{A}^1_S$ such that the pullback of t to x is exactly g. So $(\mathbb{A}^1_S, t)$ is the universal object for and $\mathbb{A}^1_S$ represents the functor $\mathcal{O} : \mathsf{Sch}_S^{\mathsf{Opp}} \to \mathsf{Rings}$.

Example A.6. Let $S = \operatorname{Spec} A$ and consider $\mathbb{G}_{m,S} = \mathbb{A}^1_S \setminus \{0_S\} = \operatorname{Spec} A[t, t^{-1}]$. A morphism of S-schemes $X \to \mathbb{G}_{m,S}$ is determined by the image of $t \in \mathcal{O}(\mathbb{G}_{m,S}) = A[t, t^{-1}]$ in $\mathcal{O}(S) = A$. So $\mathbb{G}_{m,S}$ represents $\mathcal{O}(X)^\times$ the group of invertible sections of the structure sheaf.

Appendix B. Group Objects in Categories

Let us recall the definition of group objects in a category. These are important examples in arithmetic and in geometry. For example, elliptic curves are group objects in the category of schemes Sch and topological groups are group objects in the category of topological spaces Top.

Definition B.1. Let C be a category with finite products and a final object Z. A group object is an object $X \in \mathrm{Obj}(\mathsf{C})$ together with maps $m : X \times X \to X, i : X \to X, e : Z \to X$ satisfying the following axioms:

(a) (Associativity Axiom) The diagram

$$\begin{tikzcd} X \times X \times X \arrow[r, "{(m,\mathrm{id}_X)}"] \arrow[d, "{(\mathrm{id}_X,m)}"'] & X \times X \arrow[d, "m"] \\ X \times X \arrow[r, "m"'] & X \end{tikzcd}$$

commutes.

(b) (Identity Axiom) The two maps

$$X \xrightarrow{\sim} Z \times X \xrightarrow{(e,\mathrm{id}_X)} X \times X \xrightarrow{m} X$$

$$X \xrightarrow{\sim} X \times Z \xrightarrow{(\mathrm{id}_X,e)} X \times X \xrightarrow{m} X$$

are both the identity map on X.

(c) (Inverse Axiom) The maps

$$X \xrightarrow{(i,\mathrm{id}_X)} X \times X \xrightarrow{m} X$$

$$X \xrightarrow{(\mathrm{id}_X,i)} X \times X \xrightarrow{m} X$$

are both the composition

$$X \longrightarrow Z \xrightarrow{e} X.$$

One can equivalently define a group object as an object $X \in \mathrm{Obj}(\mathsf{C})$ with a functor $\mathsf{C}^{\mathsf{Opp}} \to \mathsf{Grp}$ such that the composition with the forgetful functor to Sets is h_X, or as an object $X \in \mathrm{Obj}(\mathsf{C})$ and for all $Y \in \mathrm{Obj}(\mathsf{C})$ a group structure on $\mathrm{Mor}_{\mathsf{C}}(Y, X)$ such that for all $Y \to W$ there is an induced homomorphism of groups $\mathrm{Mor}_{\mathsf{C}}(W, X) \to \mathrm{Mor}_{\mathsf{C}}(Y, X)$.

Example B.2. Let $S = \operatorname{Spec} A$ and consider $\mathbb{A}^1_S = \operatorname{Spec} A[t] = \mathbb{G}_{a,S}$. This is a group scheme since for any $X \in \mathrm{Obj}(\mathsf{Sch}_S)$ there is a group structure on $\mathrm{Mor}_{\mathsf{C}}(X, \mathbb{A}^1_S) = \mathcal{O}(X)$ the global sections of the structure sheaf and for any $X \to Y$ there is a natural homomorphism of groups $\mathcal{O}(Y) \to \mathcal{O}(X)$ by the pullback of sheaves $f^\sharp : \mathcal{O}_Y \to f^*\mathcal{O}_X$.

Example B.3. Similarly for $\mathbb{G}_{m,S} = \operatorname{Spec} A[t, t^{-1}]$ we know it represents the functor sending a scheme X to its invertible sections $\mathcal{O}(X)^\times$. For $X \to Y$, the homomorphism $\mathcal{O}(Y) \to \mathcal{O}_X$ by $\mathcal{O}_Y \to f^*\mathcal{O}_X$ restricts to a homomorphism of multiplicative groups $\mathcal{O}(Y)^\times \to \mathcal{O}(X)^\times$ as homomorphisms of rings send 0 to 0 and 1 to 1.

Example B.4. Consider a functor $\mathsf{Sch}_S^{\mathsf{Opp}} \to \mathsf{Sets}$ by $X \mapsto M_n(\mathcal{O}(X))$, $n \times n$ matrices with entries in $\mathcal{O}(X)$ represented by the scheme $\mathbb{A}^{n\times n}_S$. One can apply the determinant map on matrices $\det : \mathbb{A}^{n\times n}_S \to \mathbb{A}^1_S$ and consider the preimage of $\mathbb{G}_{m,S} \subseteq \mathbb{A}^1_S$, those matrices whose determinant is nonvanishing. This is an open subscheme of invertible $n \times n$ matrices $\mathrm{GL}_{n,S} \subseteq \mathbb{A}_S^{n\times n}$ which has a group structure by matrix multiplication.

Definition B.5 (Homomorphism of Group Objects)**.** Let C be a category with finite products and final object Z. Let $X, Y \in \mathrm{Obj}(\mathsf{C})$ be group objects. A homomorphism of group objects is a morphism $f : X \to Y$ in C such that the diagram

$$\begin{tikzcd} X \times X \arrow[r, "m_X"] \arrow[d, "f\times f"'] & X \arrow[d, "f"] \\ Y \times Y \arrow[r, "m_Y"'] & Y \end{tikzcd}$$

commutes.

Group objects are inherently interesting objects of study in algebraic geometry, such as in the study of Abelian varieties. However, group objects also act on other objects in the category, creating a new direction of study.

Definition B.6 (Action of Group Object)**.** Let C be a category. A left action α of a functor $G : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Grp}$ on a functor $F : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ is a natural transformation $\alpha : G \times F \to F$ such that for all $Y \in \mathrm{Obj}(\mathsf{C})$ the induced map $\alpha(Y) : G(Y) \times F(Y) \to F(Y)$ is an action of the group $G(Y)$ on the set $F(Y)$.

For a group object in a category, its action on another object is given by the action of the functor the group object represents to the functor that the other object represents.

Definition B.7 (Equivariant)**.** Let C be a category and $X, Y \in \mathrm{Obj}(\mathsf{C})$ objects with an action by a group object G. A morphism $f : X \to Y$ is G-equivariant if the diagram

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d, "\mathrm{id}_G\times f"'] & X \arrow[d, "f"] \\ G \times Y \arrow[r] & Y \end{tikzcd}$$

with horizontal maps given by the action commutes.

This is equivalent to the data of the data of $X(W) \to Y(W)$ being $G(W)$-equivariant for all $W \in \mathrm{Obj}(\mathsf{C})$.

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