ALGEBRAIC STACKS

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Overview

These notes roughly correspond to an attempt to learn stack theory that began in the winter of 2023. We will begin with the categorical preliminaries as laid out in [5] before developing the basic theory per the text of Olsson [3] and conclude with a sampling of the more advanced topics in [4, Part 7]. The standard texts are [2] and [3]. The compendium [4] is encyclopedic.

Contents

Grothendieck Topologies, Sites, and Fibered Categories

1. Grothendieck Topologies

Let us recall the following definitions.

Definition 1.1 (Presheaf)**.** Let X be a topological space. A presheaf of sets $\mathcal{F}$ on X is a functor $(X^{\mathsf{Opens}})^{\mathsf{Opp}} \to \mathsf{Sets}$.

A presheaf is a sheaf if it satisfies additional gluing axioms.

Definition 1.2 (Sheaf)**.** Let X be a topological space. A sheaf of sets $\mathcal{F}$ on X is a functor $(X^{\mathsf{Opens}})^{\mathsf{Opp}} \to \mathsf{Sets}$ such that the sequence

$$\mathcal{F}(X) \longrightarrow \prod_i \mathcal{F}(X_i) \rightrightarrows \prod_{i,j} \mathcal{F}(X_i \cap X_j)$$

is an equalizer for $\{X_i\}$ an open cover of X.

In this way, we say that a sheaf on X is a presheaf on X satisfying descent. Note here that we implicitly used the fact that X^{Opens} can be naturally endowed with the strucutre of a category with objects open sets of the topological space X and morphisms inclusions of such open sets. This begs the question if we can replace X^{Opens} with some other category C, allowing us to define a sheaf on an arbitrary category C. We do this via the construction of a Grothendieck topology by replacing open sets of a topological space with maps into this space.

Definition 1.3 (Grothendieck Topology)**.** Let C be a category. A Grothendieck topology on C is the data of a set $\{X_i \to X\}$ for each object $X \in \mathrm{Obj}(\mathsf{C})$ known as a covering of X such that the following conditions hold:

(a) If $Y \to X$ is an isomorphism then $\{Y \to X\}$ is a covering.
(b) If $\{X_i \to X\}$ is a covering and $Y \to X$ any morphism then $\{X_i \times_X Y\}$ exist and $\{X_i \times_X Y \to X\}$ is a covering.
(c) If $\{X_i \to X\}$ is a covering and for each i $\{X_{ij} \to X_i\}$ is a covering then the composites $\{X_{ij} \to X_i \to X\}$ is a covering.

This allows us to define a site.

Definition 1.4 (Site)**.** A site on a category C is the category C endowed with a Grothendieck topology.

Let us see some examples.

Example 1.5 (Site of a Topological Space)**.** Let X be a topological space and X^{Opens} the category of open sets of X with morphisms inclusions. One can endow X^{Opens} with a Grothendieck topology by associating to each $U \subseteq X$ open, the set of open covers of U. The fiber product is given by intersection of open sets, which agrees with our previous discussion of sheaves and presheaves.

Example 1.6. Consider the category of topological spaces Top. The category Top can be endowed with a Grothendieck topology by taking covers of a topological space $X \in \mathrm{Obj}(\mathsf{Top})$ to open, continuous, injective maps $X_i \to X$.

Example 1.7 (Étale Site of a Scheme)**.** Let S be a scheme. The étale site of S, denoted $X_{\text{ét}}$, is the full subcategory of Sch_S where covers are étale morphisms.

It is sometimes useful to consider coverings that are not finitely presented. We do this by using the fpqc topology.

Definition 1.8 (fpqc Morphism)**.** Let $f : X \to Y$ be a surjective morphism of schemes. f is an fpqc morphism if for every affine open $V \subseteq Y$, $f^{-1}(V)$ is quasicompact in X.

Remark. The notation fpqc arises from the French, fidèlement plat et quasi-compact.

This allows us to define the fpqc topology on the category of schemes over a fixed scheme S.

Definition 1.9 (fpqc Site)**.** Let S be a scheme. The fpqc site on S-schemes $(\mathsf{Sch}_S)_{\mathsf{fpqc}}$ is the data of coverings $\{X_i \to X\}$ for each $X \in \mathrm{Obj}(\mathsf{Sch}_S)$ such that $\coprod_i X_i \to X$ is fpqc.

We can thus show the following proposition.

Proposition 1.10. *Let $X \to Y$ be a morphism of schemes and $\{Y_i \to Y\}$ an fpqc covering of Y. Suppose that for each i, the induced map $Y_i \times_Y X \to Y_i$ has one of the following properties:*

(a) separated,
(b) quasicompact,
(c) locally of finite presentation,
(d) proper,
(e) affine,
(f) finite,
(g) flat,
(h) smooth,
(i) unramified,
(j) étale,
(k) is an embedding,
(l) or is a closed embedding

then so does $X \to Y$.

Proof. All these properties and being fpqc are affine-local on target and the condition on the induced map $Y_i \times_Y X \to Y$ implies the condition on the map $X \to Y$ by [1, Prop. 2.7.1]. ∎

Having defined a topology, we can now define a sheaf theory on a site. Recall from earlier that sheaves on a topological space X is a presheaf that satisfies additional axioms. The sheaf condition can be generalized to any site, replacing intersections of open sets with fibered products.

Definition 1.11 (Separated Presheaf on a Site)**.** Let C be a site and $F : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ a presheaf of sets on C. F is a separated functor if there are $a, b \in F(X)$ whose pullbacks to $F(X_i)$ coincide then $a = b$ on $F(U)$.

Definition 1.12 (Sheaf on a Site)**.** Let C be a site and $F : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ a presheaf of sets on C. F is a sheaf if for all coverings $\{X_i \to X\}$ the sequence

$$F(X) \longrightarrow \prod_i F(X_i) \underset{\mathrm{pr}_2^*}{\overset{\mathrm{pr}_1^*}{\rightrightarrows}} \prod_{i,j} F(X_i \times_X X_j)$$

is an equalizer.

Remark. As in the case of sheaves on a topological space, the condition on equalizers is equivalent to the gluing of sections. Given a covering $\{X_i \to X\}$ function of sets $F(X) \to \prod_i F(X_i)$ is induced along restrictions and the natural projection maps

$$\mathrm{pr}_1 : X_i \times_X X_j \to X_i, \mathrm{pr}_2 : X_i \times_X X_j \to X_j$$

sends $a_i \in F(X_i)$ to $\mathrm{pr}_1^* a_i$ with pr_2^* defined similarly. The equalizer condition states that if pullbacks of sections $\mathrm{pr}_1^* a_i = \mathrm{pr}_2^* a_j$ as elements of $F(X_i \times_X X_j)$ then they are pulled back from some a on $F(X)$ restricting to the a_i on $F(X_i)$.

Given an object $X \in \mathrm{Obj}(\mathsf{C})$ we can define a sieve on X as follows.

Definition 1.13 (Sieve)**.** Let $X \in \mathrm{Obj}(\mathsf{C})$. A sieve on X is a subfunctor of the Yoneda functor $h_X : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$.

Given a subfunctor S of h_X, we get a collection $\mathcal{S}$ of arrows $T \to X$ by taking $\cup_{T \in \mathrm{Obj}(\mathsf{C})} S(T)$ such that for $T \to X$ in $\mathcal{S}$, every composite $T' \to T \to X$ is in $\mathcal{S}$. Conversely, given $\mathcal{S}$ we can restrict the Yoneda functor appropriately to get $S : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$.

For $\{X_i \to X\}$ a a collection of morphisms and $X \in \mathrm{Obj}(\mathsf{C})$ we can similarly define a subfunctor $h_{\{X_i \to X\}} : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ where $T \to X$ if and only if there exists a factorization $T \to X_i \to X$ for some i. In the case of the functor $h_{\{X_i \to X\}}$, the metaphor of the sieve becomes much more clear: the X_i are the holes of the sieve, and $T \to X$ is in $h_{\{X_i \to X\}}(T)$ if and only if it "goes through one of the holes".

For a collection $\{X_i \to X\}$ and $F : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ a functor, we can define $F(\{X_i \to X\})$ to be the set of elements of $\prod_i F(X_i)$ whose images in $\prod_{i,j} F(X_i \times_X X_j)$ are equal. Sections of $F(X)$ evidently give rise to sections of $\prod_i F(X_i)$ that agree on $\prod_{i,j} F(X_i \times_X X_j)$ inducing a map $F(X) \to F(\{X_i \to X\})$. Note here that if F is a sheaf the descent condition implies the map $F(X) \to F(\{X_i \to X\})$ is a bijection for all coverings.

We show that the set $F(\{X_i \to X\})$ can be defined in terms of sieves.

Proposition 1.14. There is a canonical bijection of sets

$$R : \mathrm{NatTrans}(h_{\{X_i \to X\}}, F) \to F(\{X_i \to X\})$$

such that the diagram

$$\begin{tikzcd}
\mathrm{NatTrans}(h_X, F) \arrow[r] \arrow[d] & F(X) \arrow[d] \\
\mathrm{NatTrans}(h_{\{X_i \to X\}}, F) \arrow[r, "R"'] & F(\{X_i \to X\})
\end{tikzcd}$$

commutes and R is universal with respect to this property.

2. Fibered Categories

3. Descent

Algebraic Spaces

4. Algebraic Spaces

5. Quotients in Algebraic Spaces

6. Quasicoherent Sheaves on Algebraic Spaces

Algebraic Stacks

7. Algebraic Stacks

8. Quasicoherent Sheaves on Algebraic Stacks

The Geometry of Stacks

9. Geometric Properties of Stacks

10. Coarse Moduli Spaces

11. Gerbes

12. Cohomology of Stacks

13. Derived Categories of Stacks

End Material

Appendix A. Category Theory

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