MATH 292Z: FIRST STEPS IN INFINITY CATEGORIES

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Preliminaries

These notes roughly correspond to the course **MATH 292Z: First Steps in Infinity Categories** taught by Prof. Michael Hopkins at Harvard University in the Fall 2023 semester. These notes are LaTeX-ed after the fact with significant alteration and are subject to misinterpretation and mistranscription. Use with caution. Any errors are undoubtedly my own and any virtues that could be ascribed to these notes ought be attributed to the instructor and not the typist.

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1. Lecture 1 – 6th September 2023

Understanding infinity categories is a daunting task. Ideally, we will end this class with the student being conversant in the language of infinity categories, though this may be difficult to achieve. Let us start with the definition of a (1-)category.

Definition 1.1 (Category)**.** A category C is a class $\mathrm{Obj}(\mathsf{C})$ and for each $A, B \in \mathrm{Obj}(\mathsf{C})$ a set $\mathrm{Mor}_{\mathsf{C}}(A, B)$ of morphisms from A to B such that

(a) $\mathrm{id}_A \in \mathrm{Mor}_{\mathsf{C}}(A, A)$,
(b) A composition law $\mathrm{Mor}_{\mathsf{C}}(B, C) \times \mathrm{Mor}_{\mathsf{C}}(A, B) \to \mathrm{Mor}_{\mathsf{C}}(A, C)$ by $(g, f) \mapsto g \circ f$ that is unital $\mathrm{id}_A \circ f = f = f \circ \mathrm{id}_B$ and associating $(g \circ f) \circ h = g \circ (f \circ h)$.

Remark. We consider the underlying collection of objects of a category a class to avoid set-theoretic difficulties such as Russell's Paradox when dealing with the category of sets.

Let us consider some examples.

Example 1.2. Sets whose objects are sets and whose morphisms are set functions.

Example 1.3. Top whose objects are topological spaces and whose morphisms are continuous maps between them.

Example 1.4 (Representable Spaces of Groups)**.** Let G be a group. We can construct a category C_G whose object is a point $*$ and whose morphisms are given by $\mathrm{Mor}_{\mathsf{C}_G}(*, *) = G$ where the unital element is id_G and composition given by group multiplication.

Example 1.5 (Fundemental Groupoid)**.** Let X be a topological space. Consider $\Pi_{\leq 1}X$ the fundamental groupoid of X whose object is X itself and whose morphisms $\mathrm{Mor}_{\Pi_{\leq 1}X}(x, y)$ homotopy classes of paths from $x, y \in X$. Note that the unital and associative properties of path composition only hold after passing to homotopy equivalence.

Let us now define a functor.

Definition 1.6 (Functor)**.** Let C, D be categories. A functor $F : \mathsf{C} \to \mathsf{D}$ consists of a function $F : \mathrm{Obj}(\mathsf{C}) \to \mathrm{Obj}(\mathsf{D})$ such that for all $A, B \in \mathrm{Obj}(\mathsf{C})$, there is a function $\mathrm{Mor}_{\mathsf{C}}(A, B) \to \mathrm{Mor}_{\mathsf{D}}(F(A), F(B))$ satisfying $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(g \circ f) = F(g) \circ F(f)$.

A functor is a map between categories. Correspondingly, we can define a map between functors as a natural transformation.

Definition 1.7 (Natural Transformation)**.** Let $F, G : \mathsf{C} \to \mathsf{D}$ be functors. T is a natural transformation between F and G consists of a morphism $T(A) : F(A) \to G(A)$ in D for all $A \in \mathrm{Obj}(\mathsf{C})$ and the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "T(A)"] \arrow[d, "T(f)"'] & G(A) \arrow[d, "G(f)"'] \\
F(B) \arrow[r, "T(B)"] & G(B)
\end{tikzcd}$$

commutes for all $f \in \mathrm{Mor}_{\mathsf{C}}(A, B)$.

Example 1.8. Let X be a topological space. We can construct a covering space of X denoted $\widetilde{X}$ such that for the solid diagram,

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, "p"] \\
{[0,1]} \arrow[r] \arrow[ur, dashed, "\exists!"] & X
\end{tikzcd}$$

there is a unique dotted arrow lifting a path in X to $\widetilde{X}$. In other words, this is a functor $\Pi_{\leq 1}X \to \mathsf{Sets}$ by $x \mapsto p^{-1}(x)$.

In fact, and old theorem of Grothendieck states the following.

Theorem 1.9 (Grothendieck)**.** *Let X be a topological space. The category of covering spaces of X is equivalent ot the category of functors $\Pi_{\leq 1}X \to \mathsf{Sets}$.*

Within this framework, topological statements like Van-Kampen's theorem become much easier.

In topology, we have paths, and homotopies between paths. Indeed, we can construct homotopies between homotopies, and so on. This is the root of the concept of an infinity category.

And much of the topological information we wish to incorporate above is easiest done with simplices, leading the formalism of infinity categories to be built on the backbone of the rich combinatorial structure of simplical sets. Let's start with an easy example of the partially ordered set, or poset.

Definition 1.10 (Poset)**.** A poset $(S, \leq)$ is a set S together with a relation $\leq$ such that

(a) $a \leq a$ for all $a \in S$,
(b) if $a \leq b$ and $b \leq c$ then $a \leq c$,
(c) and if $a \leq b$ and $b \leq a$ then $a = b$.

A poset can be seen as a category whose objects are the elements of S and

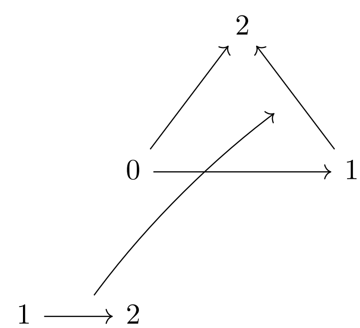
$$\mathrm{Mor}_S(a,b) = \begin{cases} \emptyset & a \not\leq b \\ * & a \leq b. \end{cases}$$

Indeed, one can show that any category with at most one morphism between any two objects is a poset under the relation induced by the morphisms. We can now introduce simplical sets.

Definition 1.11 (Simplices)**.** The category of simplices Δ has objects finite totally ordered sets $[n] = \{0, 1, 2, \ldots, n\}$ and morphisms order-preserving maps.

There is a functor $\Delta \to \mathsf{Top}$ taking $[n]$ to $|\Delta^n|$ the standard n-simplex, the convex hull of $\{e_0, \ldots, e_n\}$ in $\mathbb{R}^{n+1}$ and taking a morphism $[n] \to [m]$ to linear-maps $e_i \mapsto e_{f(i)}$. Such maps $f : [n] \to [m]$ are determined by their (finite) images $0 \leq f(1) \leq f(2) \leq \cdots \leq f(n) \leq m$ which we will write as $\langle f(1), \ldots, f(n) \rangle$.

Example 1.12. There are maps $\langle i, i+1 \rangle : [1] \to [n]$ for $0 \leq i \leq n-1$.



The above demonstrates the map $\langle 1, 2 \rangle : [1] \to [2]$.

Let X be an arbitrary topological space. We can get a contravariant functor $\Delta \to \mathsf{Sets}$ by taking $[n] \to \mathrm{Mor}_{\mathsf{Top}}(|\Delta^n|, X)$. We denote this functor $\mathrm{Sing}\, X : \Delta \to \mathsf{Sets}$. This allows us to define simplical sets.

Definition 1.13 (Simplicial Sets)**.** The category of simplicial sets SSets is the functor category whose objects are contravariant functors $\Delta \to \mathsf{Sets}$ and morphisms natural transformations between such functors.

Let us switch directions for a moment and consider nerves of categories. We have already seen how a poset can be made into a category. Analogously, a totally ordered set can be made into a category in a similar way, yielding a functor $\Delta \to \mathsf{Cat}$, the category whose objects are categories and whose morphisms are functors. Nerves help us understand how simplices live within categories.

Definition 1.14 (Nerve)**.** Let C be a category. The nerve of C is the simplicial set $N\mathsf{C}$ so that

$$(N\mathsf{C})_n = \mathrm{Mor}_{\mathsf{Cat}}([n], \mathsf{C})$$

the set of functors from $[n]$ to C such that the simplical operators $f : [n] \to [m]$ act by pre-composition where $a \mapsto a \circ f$ for $a : [n] \to \mathsf{C}$ in $(N\mathsf{C})_n$.

We can make this more concrete with a few examples.

Example 1.15. $(N\mathsf{C})_0$ is the simplicial set of maps from the point to C and hence corresponds to the objects of C, $\mathrm{Obj}(\mathsf{C})$.

$(N\mathsf{C})_1$ is the simplicial set of maps from $0 \to 1$ to C and hence corresponds to the morphisms of C.

$(N\mathsf{C})_2$ is the simplicial set of maps

$$\begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \arrow[d] \\ & 2 \end{tikzcd}$$

to C, that is, to pairs of composable morphisms in C.

More generally $(N\mathsf{C})_n$ is the set of composable n-tuples of morphisms in C.

Theorem 1.16. A simplicial set X is isomorphic to the nerve of some category C if and only if for all $n \geq 2$, the map

$$X_n \to \underbrace{X_1 \times_{X_0} X_1 \times_{X_0} \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ times}}$$

by

$$g \mapsto (\langle 0,1\rangle^* g, \langle 1,2\rangle^* g, \ldots, \langle n-1,n\rangle^* g)$$

is an isomorphism.

This proof follows that in [1, Proposition 1.4.8].

Proof. ($\Longrightarrow$) Suppose $X = N\mathsf{C}$ for some category C. A tuple of n composable morphisms in C is precisely the datum of the n-nerve $(N\mathsf{C})_n$.

($\Longleftarrow$) Suppose X is a simplicial set such that the function above is a bijection. We construct the category C whose objects $\mathrm{Obj}(\mathsf{C}) = X_0$ and whose morphisms are X_1. For $g \in X_1$, we have two functors $X_1 \to X_0$ taking $g \mapsto \langle 0\rangle^* g$ its source object and $g \to \langle 1\rangle^* g$ its target object. Since $0 \to 0$ is a morphism in $|\Delta^1|$, we trivially have identity morphisms on each object. Moreover, given a morphism $f \in X_1$, we can recover its source object $\langle 0\rangle^* f$ and its target object $\langle 1\rangle^* f$. And for any $(g,h) \in X_1 \times_{X_0} X_1$, the composite $h \circ g \in X_2$ is the unique morphism where $\langle 0,1\rangle^*(g \circ h) = h$ and $\langle 1,2\rangle^*(g \circ h) = g$ where uniqueness follows from bijectivity. We now claim that for any $g \in X_n$ and $0 \leq i \leq j \leq k \leq n$ we have $\langle i,k\rangle^* g = \langle j,k\rangle^* g \circ \langle j,i\rangle^* g$ where each of $\langle i,k\rangle^* g, \langle j,k\rangle^* g, \langle j,i\rangle^* g \in X_1$ as images of maps $[1] \to [n]$. Indeed any element of X_2 decomposes in this way. We can thus define maps $X_n \to (N\mathsf{C})_n$ by sending $g \in X_n$ to $[n] \to \mathsf{C}$ by $g \mapsto g$ and $(i \to j) \mapsto \langle j,i\rangle^* g$ which is a functor by the above. We can verify these maps are bijections since $((i-1) \to i) \mapsto \langle i, i-1\rangle^* g$. Furthermore, if $f : [m] \to [n]$ is a map of simplices, we compute

$$\langle j,i\rangle^*(g \circ f) = \langle f(j), f(i)\rangle^* g$$

giving an isomorphism $X \to N\mathsf{C}$ of simplical sets. ■

We now discuss the further significance of $\mathrm{Sing}\, X$. To do this, we first have to discuss horns and Kan complexes. Horns are subcomplexes of standard simplicies.

Definition 1.17 (Horn)**.** For each $n \geq 1$, there are subcomplexes $\Lambda^n_j \subseteq |\Delta^n|$ for each $0 \leq j \leq n$ defined by

$$(\Lambda^n_j)_k = \{f : [k] \to [n] | ([n] \setminus [j]) \not\subseteq f([k])\}.$$

Morally, we should think of a horn Λ^n_j is the union of faces of the n simplex other than the jth.

Example 1.18 (1-Horns)**.** The horns in $|\Delta^1|$ are the vertices $\Lambda^1_0 = \{0\}, \Lambda^1_1 = \{1\}$.

Example 1.19 (2-Horns)**.** There are three horns in the 2-simplex.

$$\begin{tikzcd} & 2 & \\ 0 \arrow[ur] \arrow[rr] & & 1 \end{tikzcd} \qquad \begin{tikzcd} & 2 & \\ 0 \arrow[rr] & & 1 \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & 2 & \\ 0 \arrow[ur] & & 1 \arrow[ul] \end{tikzcd}$$

$$\Lambda_0^2 \qquad\qquad \Lambda_1^2 \qquad\qquad \Lambda_2^2$$

We can now define Kan complexes.

Definition 1.20 (Kan Complex)**.** A simplicial set is a Kan complex if for all k, n the solid diagram

$$\begin{tikzcd} \Lambda_k^n \arrow[r] \arrow[d, hook] & X \\ |\Delta^n| \arrow[ur, dashed] & \end{tikzcd}$$

admits a dotted map making the diagram commute.

A theorem of Kan states the following:

Theorem 1.21 (Kan)**.** $\operatorname{Sing} X$ is a Kan complex.

This eventually led to the definition of quasicategories.

Definition 1.22 (Quasicategory; Boardman-Vogt, Joyal, Lurie)**.** A simplicial set is a quasicategory if every solid diagram

$$\begin{tikzcd} \Lambda_k^n \arrow[r] \arrow[d, hook] & X \\ |\Delta^n| \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

admits a dotted map making the diagram commute.

We will revisit these ideas in the following lecture.

2. Lecture 2 – 11th September 2023

We made a very quick pass of simplicial sets in the previous lecture. Let us go through it in more detail.

Let Δ be the category whose objects are finite ordered sets and whose morphisms are order-preserving maps. Following the notation in Rezk's text [1], for a morphism $[n] \to [k]$, we write it $\langle k_0, \ldots, k_n \rangle$ where $0 \leq k_0 \leq \cdots \leq k_n \leq k$. In the category Δ, there are two types of distinguished maps d_i and s_i which we now define.

Definition 2.1 (Face Maps)**.** In the category Δ, there are maps $d^k : [n-1] \to [n]$ by $\langle 0, 1, \ldots, \hat{k}, \ldots, n \rangle = \langle 0, 1, \ldots, k-1, k+1, \ldots, n \rangle \to [n]$.

Example 2.2. We should think of these as the inclusion of a face into the simplex.

$$\begin{tikzcd}
 & & 2 & & & 3 & \\
0 \arrow[urr] \arrow[drr] & & & {} \arrow[rr, bend right, "d^2"] & & & 2 \arrow[ul] \\
 & & 1 \arrow[uu] & & 0 \arrow[uur] \arrow[urr] \arrow[drr] & & \\
 & & & & & & 1 \arrow[uu] \arrow[uuul]
\end{tikzcd}$$

Here, we show the inclusion $d^2 : [2] \to [3]$ by $\langle 0, 1, 3 \rangle$, taking the 2-simplex to the face "opposite" the vertex 2, that is, the face bounded by the vertices 0, 1, and 3.

We can also define degeneracy maps as follows.

Definition 2.3 (Degeneracy Map)**.** In the category Δ, there are maps $s^k : [n] \to [n-1]$ by $[n] \mapsto \langle 0, 1, 2, \ldots, k, k, k+1, \ldots, n \rangle$.

Example 2.4. We should think of this as collapsing the kth to the $(k-1)$th vertex.

$$\begin{tikzcd}
1 \arrow[drr] & & \\
 & & 2 \\
0 \arrow[uu] \arrow[urr] & {} \arrow[dd, bend left, "s^0"] & \\
 & & \\
0 \arrow[rr] & {} & 1
\end{tikzcd}$$

Here we show $s^0 : [2] \to [1]$ by $\langle 0, 0, 1 \rangle$ collapsing the vertex 1 to the vertex 0.

With these operations on Δ, one can in fact show the following theorem.

Theorem 2.5. If f is a morphism in Δ, then f factors as the composition of face and degeneracy maps.

We omit the proof.

Now recall that a simplicial set X is a covariant functor $X : \Delta^{\mathsf{Opp}} \to \mathsf{Sets}$. We write

$$X_n = \mathrm{Mor}_{\mathsf{Cat}}([n], X).$$

For a simplicial set X, we have a diagram

$$\begin{tikzcd}
X_0 \arrow[r, "s^0" description] & X_1 \arrow[l, bend right, "d^0" description] \arrow[l, bend left, "d^1" description] \arrow[r, bend left=15, "s^0" description] \arrow[r, bend right=15, "s^1" description] & X_2 \arrow[l, bend right=40, "d^0" description] \arrow[l, "d^1" description] \arrow[l, bend left=40, "d^2" description] & \cdots & \cdots
\end{tikzcd}$$

induced by the degeneracy and face maps.

Definition 2.6 (n-Simplex)**.** The n-simplex is the functor $\Delta^n : \Delta^{\mathsf{Opp}} \to \mathsf{Sets}$ by $[k] \mapsto \mathrm{Mor}_\Delta([k], [n])$.

Remark. Note that Δ^n is the functor $\Delta^{\mathsf{Opp}} \to \mathsf{Sets}$, while the topological n-simplex $|\Delta^n| \simeq S^n \in \mathrm{Obj}(\mathsf{Top})$ is topological space homeomorphic (and homotopic) to the n-sphere.

In this way, we can think of Δ^n as the functor represented by $[n] \in \mathrm{Obj}(\Delta)$. More generally, one can define representable functors as follows.

Definition 2.7 (Representable Functor)**.** Let C be a category. A functor $F : \mathsf{C} \to \mathsf{Sets}$ is representable if there exists $A \in \mathrm{Obj}(\mathsf{C})$ such that there exists an isomorphism of functors $F \to \mathrm{Mor}_{\mathsf{C}}(-, A)$.

Generally, consider $F : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ a contravariant functor from an arbitrary category C to Sets. Let $f : B \to A$. $F(f) : F(B) \to F(A)$ is a map betewen sets. For $u \in F(A)$, u determines a natural transformation of functors $\mathrm{NatTrans}(\mathrm{Mor}_{\mathsf{C}}(-, A), F)$. This is Yoneda's lemma. We adapt the version from [2, Theorem 2.2.4].

Lemma 2.8 (Yoneda)**.** For a contravariant functor $F : \mathsf{C} \to \mathsf{Sets}$ and $A \in \mathrm{Obj}(\mathsf{C})$, there is a bijection

$$\mathrm{NatTrans}(\mathrm{Mor}_{\mathsf{C}}(-, A), F) \to F(A)$$

that associates a natural transformation of functors $\alpha : \mathrm{Mor}_{\mathsf{C}}(-, A) \Rightarrow F$ to the element $\alpha(\mathrm{id}_A) \in F(A)$, natural in both A and F.

Let us recall the definition of quasicategories as in Definition 1.22.

Definition 2.9 (Quasicategory)**.** A simplcial set X is a quasicategory if every inner horn Λ^n_k where $0 < k < n$ has a fill.

In other words, for every solid diagram,

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

there is a dotted map making the diagram commute.

Let us now fix some notation that we will use going forward. For X a simplicial set we denote $X_n = \mathrm{Mor}_{\mathsf{Cat}}([n], X)$ and $\alpha \in \mathrm{Mor}_{\Delta}([k], [n])$, we have $X_\alpha : X_n \to X_k$.

Theorem 2.10. A simplicial set is the nerve of some category if and only if every inner horn has a unique filler.

In other words, for every solid diagram,

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

there is a dotted map making the diagram commute.

We omit the proof of Theorem 2.10 which can be found in [1, §1.7.10].

We want to consider an analogue of Theorem 2.10 in the case of quasicategories. Let X be a quasicategory with objects X_0 and morphisms X_1. For $f \in X_0$, let $\langle 0\rangle^* f$ denote the source of the map f and $\langle 1\rangle^* f$ its target. Let

$$\begin{tikzcd} A \arrow[d, "f"'] & C \\ B \arrow[ur, "g"'] & \end{tikzcd}$$

be the image of a horn Λ^2_1 and $h = \tau\langle 0,2\rangle : A \to C$ be the fill.

$$\begin{tikzcd} A \arrow[r, "\tau\langle 0{,}2\rangle = h"] \arrow[d, "f"'] & C \\ B \arrow[ur, "g"'] & \end{tikzcd}$$

Since the fill is not unique, we say τ witnesses h as the composition of g with f. This begs the question of how we can perform some operation to make X into a category. This is done via the construction of the fundamental category, which can be done on simplicial sets in general.

Definition 2.11 (Fundamental Category)**.** Let X be a simplicial set. A category C is the fundamental category of X if every map $X \to \mathsf{D}$ for D factors through C and C is final with respect to that property.

Fundamental categories are in general hard to construct, but much easier in the case of quasicategories. Indeed if X is a quasicategory, its fundamental category coincides with its homotopy category hX. The homotopy category will have objects X_0 and morphisms those in X_1 up to "witnessing"-equivalence. We will have to discuss these notions of equivalence before defining the homotopy category rigorously.

Definition 2.12 (Left-Equivalence)**.** Let $f, g \in \mathrm{Mor}_X(A, B)$. We say that f is left-equivalent to g if there is $\tau \in X_2$ such that $\tau\langle 0,1\rangle = \mathrm{id}_A, \tau\langle 0,2\rangle = f, \tau\langle 1,2\rangle = g$.

$$\begin{tikzcd} A \arrow[d, "\mathrm{id}_A" description] \arrow[dr, "f" description] & \\ A \arrow[r, "g" description] & B \end{tikzcd}$$

Similarly, we have right-equivalence.

Definition 2.13 (Right-Equivalence)**.** Let $f, g \in \mathrm{Mor}_X(A, B)$. We say that f is right-equivalent to g if there is $\tau \in X_2$ such that $\tau\langle 1,2\rangle = \mathrm{id}_B, \tau\langle 0,1\rangle = f, \tau\langle 0,2\rangle = g$.

$$\begin{tikzcd} & B \\ A \arrow[ur, "g" description] \arrow[r, "f" description] & B \arrow[u, "\mathrm{id}_B" description] \end{tikzcd}$$

One then shows the following proposition before defining the homotopy category.

Proposition 2.14. The relations of left equivalence (Definition 2.12) and right equivalence (Definition 2.13) coincide.

We can now define the homotopy category.

Definition 2.15 (Homotopy Category)**.** Let X be a simplicial set. The homotopy category hX has objects those in X_0 and morphisms those in X_1 up to equivalence.

3. Lecture 3 – 13th September 2023

We will discuss some constructions in ordinary category theory. Thinking about these constructions will give us a better idea of analogous constructions in infinity category theory.

Definition 3.1 (Isomorphism)**.** Let C be a category. A morphism $f \in \mathrm{Mor}_{\mathsf{C}}(A, B)$ is an isomorphism if there exists $g \in \mathrm{Mor}_{\mathsf{C}}(B, A)$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

One can easily show that isomorphisms are preserved by functors.

Proposition 3.2. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. If $f \in \mathrm{Mor}_{\mathsf{C}}(A, B)$ is an isomorphism then $F(f) \in \mathrm{Mor}_{\mathsf{D}}(F(A), F(B))$ is an isomorphism.

Proof. The statement follows by the commutativity of the following diagram.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[d, "F"'] & B \arrow[l, shift left, "g"] \arrow[d, "F"'] \\ F(A) \arrow[r, shift left, "F(f)"] & F(B) \arrow[l, shift left, "F(g)"] \end{tikzcd}$$

More explicitly, there exists $g \in \mathrm{Mor}_{\mathsf{C}}(B, A)$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$ so by functoriality we have $F(g) \circ F(f) = \mathrm{id}_{F(A)}, F(f) \circ F(g) = \mathrm{id}_{F(B)}$ as desired. ■

The acute would percieve that Definition 3.1 may be slightly problematic since it involves some choice of $g \in \mathrm{Mor}_{\mathsf{C}}(B, A)$. The following proposition resolves this issue, showing that such a choice is unique.

Proposition 3.3. Let C be a category and $f \in \mathrm{Mor}_{\mathsf{C}}(A, B)$ is an isomorphism. If $g_1, g_2 \in \mathrm{Mor}_{\mathsf{C}}(B, A)$ satisfies $g_i \circ f = \mathrm{id}_A$ and $f \circ g_i = \mathrm{id}_B$ for $i \in \{1, 2\}$ then $g_1 = g_2$.

Proof. We have $g_1 = g_1 \circ \mathrm{id}_Y = g_1 \circ f \circ g_2 = \mathrm{id}_X \circ g_2 = g_2$. ■

In some settings, it can be difficult to find such g that satisfies the composition identities as set out in Definition 3.1. Fortunately, we can use the Yoneda functors from Lemma 2.8 to characterize isomorphisms.

Theorem 3.4 (Representable Functors Determine Isomorphism)**.** Let C be a category. A map $f : A \to B$ is an isomorphism if and only if there is an isomorphic natural transformation of the functors they represent $\mathrm{Mor}_{\mathsf{C}}(-, A) \to \mathrm{Mor}_{\mathsf{C}}(-, B)$.

Remark. It is equivalent to require that for all $Z \in \mathrm{Obj}(\mathsf{C})$, $\mathrm{Mor}_{\mathsf{C}}(Z, A) = \mathrm{Mor}_{\mathsf{C}}(Z, B)$.

Proof of Theorem 3.4. This follows from the Yoneda lemma, Lemma 2.8. ■

Note the increasing levels of abstraction we have encountered. We started with the "traditional" definition of isomorphism that one encounters in say a course on the theory of groups, followed by the characterization of isomorphisms via representable functors. Let us look at one more characterization of isomorphisms in the flavor of how we defined the objects of a category as the 0th nerve $(N\mathsf{C})_0$, we can define a category $I_{\cong}$ with two objects and two non-identity morphisms

$$\begin{tikzcd} * \arrow[r, shift left] & * \arrow[l, shift left] \end{tikzcd}$$

and define an isomorphism as follows:

Definition 3.5 (Isomorphism)**.** Let C be a category. An isomorphism in C is a functor $I_{\cong} \to \mathsf{C}$.

We will no doubt revisit isomorphisms in the infinity categorical setting later in the course. Now, let us recall the following definitions.

Definition 3.6 (Initial Object)**.** Let C be a category. An object $X \in \mathrm{Obj}(\mathsf{C})$ is initial if for all $Y \in \mathrm{Obj}(\mathsf{C})$, there is a unique map $X \to Y$.

Definition 3.7 (Final Object)**.** Let C be a category. An object $X \in \mathrm{Obj}(\mathsf{C})$ is final if for all $Y \in \mathrm{Obj}(\mathsf{C})$, there is a unique map $Y \to X$.

We now show that final objects (and initial objects, by taking the opposite category) are unique up to unique isomorphism. In other words, they satisfy a universal property. Universal properties are ubiquitous in many fields of mathematics and significantly streamlines one's thinking about mathematical constructions.

Proposition 3.8. If X_1, X_2 be final objects of a category C then X_1 is isomorphic to X_2.

Proof. Both X_1 and X_2 represent the same functor $\mathsf{C} \to \mathsf{Sets}$, taking any object of C to the one-point set $\{*\}$. ■

We have already seen how isomorphisms work in a category. We now define them in the setting of quasicategories.

Definition 3.9 (Isomorphism in Quasicategories)**.** Let X be a quasicategory. A morphism $f \in X_1$ is an isomorphism if and only if $[f] \in (hX)_1$ is an isomorphism.

Isomorphisms arising in homotopy categories will soon provide a tool for characterizing Kan complexes. We begin by defining the groupoid.

Definition 3.10 (Groupoid)**.** A category C is a groupoid if all morphisms in C are isomorphisms.

Example 3.11. The fundamental groupoid of a topological space $\Pi_{\leq 1}X$ is a groupoid. Morphisms correspond to paths, which are invertible by sending $\gamma(t) \to \gamma(1-t)$.

The following theorem outlines a correspondence between quasicategories and Kan complexes with the language of groupoids.

Theorem 3.12. A quasicategory X is a Kan complex if and only if its homotopy category hX is a groupoid.

Partial Proof of Theorem 3.12. ($\Longrightarrow$) We prove that if X is a Kan complex, its homotopy category is a groupoid.

$$\begin{tikzcd} A & \\ A \arrow[u, "\mathrm{id}_A"] \arrow[r, "f"'] & B \arrow[ul, dashed, "\exists"'] \end{tikzcd}$$

Let $f : A \to B$ be a morphism in X and since X is a Kan complex, the horn Λ^2_0 admits a fill by some $\tau \in X_2$. One verifies that for $g = \tau\langle 1, 2\rangle$ we have $g \circ f = \mathrm{id}_A$. To show $f \circ g = \mathrm{id}_B$ we construct the diagram

$$\begin{tikzcd} & B \\ A \arrow[ur, "f"] & B \arrow[u, "\mathrm{id}_B"'] \arrow[l, dashed, "\exists"] \end{tikzcd}$$

and by the fact that X is a Kan complex, the horn Λ^2_2 admits a fill by some $\tau' \in X_2$ such that $\tau'\langle 0, 1\rangle = h$ satisfying $f \circ h = \mathrm{id}_B$. But we have $g = g \circ \mathrm{id}_B = g \circ f \circ h = \mathrm{id}_A \circ h = h$ proving that there is a unique inverse in the homotopy category. ■

The converse, showing that a homotopy category in which all morphisms are isomorphisms arises as the homotopy category of a Kan complex is much more difficult requiring a significant combinatorial assertion. We will return to Joyal's proof of the converse later in this course.

Definition 3.13 (Quasigroupoid)**.** A quasicategory X is a quasigroupoid if hX is a groupoid.

As Theorem 3.12 shows, Kan complexes are examples of quasigroupoids.

Definition 3.14 (Small Category)**.** A category C is a small category if $\mathrm{Obj}(\mathsf{C})$ is a set as opposed to a class.

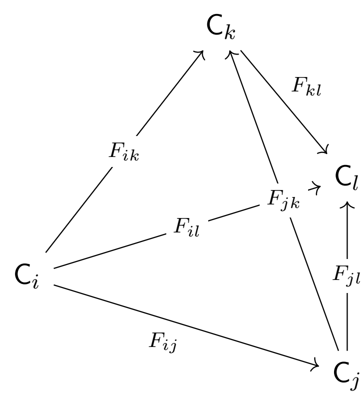
Having defined a small category, we can define the category of small categories.

Definition 3.15 (Category of Small Categories)**.** Denote Cat_1 the category of small categories whose objects are small categories and whose morphisms are functors betweeen them.

Analogously to nerves, we can consider $(\mathsf{Cat}_1)_n$ be the data of a category C_i for each $1 \leq i \leq n$, a functor $F_{ij} : \mathsf{C}_i \to \mathsf{C}_j$ for each $i \leq j$, and a natural isomorphism of functors $\varepsilon_{ijk} : F_{ik} \Rightarrow F_{jk} \circ F_{ij}$ making the following diagram commute.

$$\begin{tikzcd}
 & \mathsf{C}_j \arrow[dr, "F_{jk}"] & \\
\mathsf{C}_i \arrow[ur, "F_{ij}"] \arrow[rr, "F_{ik}"'] \arrow[rr, phantom, shift left=3ex, "\varepsilon_{ijk}\Uparrow"] & & \mathsf{C}_k
\end{tikzcd}$$

Additionally, we require the functors to satisfy the following "cocycle" conditons which we now explain. Given $(\mathsf{Cat}_1)_3$



we can "flatten" the tetrahedron to observe

$$\begin{tikzcd}
 & \mathsf{C}_l & \\
\mathsf{C}_i \arrow[ur, "F_{il}"] \arrow[dr, "F_{ij}"'] \arrow[r, phantom, "\varepsilon_{ijl}\Searrow"] & {} & \mathsf{C}_k \arrow[ul, "F_{kl}"'] \arrow[l, phantom, "\varepsilon_{jkl}\Searrow"] \\
 & \mathsf{C}_j \arrow[uu, "F_{jl}" description] \arrow[ur, "F_{jk}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathsf{C}_l & \\
\mathsf{C}_i \arrow[ur, "F_{il}"] \arrow[dr, "F_{ij}"'] \arrow[rr, "F_{ik}" description] \arrow[r, phantom, shift left=2ex, "\varepsilon_{ikl}\Searrow"] & {} \arrow[d, phantom, "\Downarrow\varepsilon_{ijk}"] & \mathsf{C}_k \arrow[ul, "F_{kl}"'] \\
 & \mathsf{C}_j \arrow[ur, "F_{jk}"'] &
\end{tikzcd}$$

that $\varepsilon_{ijk} \circ \varepsilon_{ikl} = \varepsilon_{jkl} \circ \varepsilon_{ikl}$. More explicitly, we have natural transformations

$$\begin{array}{ll} \varepsilon_{ijk} : F_{ik} \to F_{jk} \circ F_{ij} & \varepsilon_{ikl} : F_{il} \to F_{kl} \circ F_{ik} \\ \varepsilon_{jkl} : F_{jl} \to F_{kl} \circ F_{jk} & \varepsilon_{ijl} : F_{il} \to F_{jl} \circ F_{ij} \end{array}$$

where the "cocycle condition" asserts that the compositions of functors as they arise from different composite functors agree where composing the natural transformations to the "longest chain" in two different ways

$$\varepsilon_{jkl} \circ \varepsilon_{ikl} : F_{il} \to F_{kl} \circ F_{jk} \circ F_{ij}$$

$$\varepsilon_{jkl} \circ \varepsilon_{ikl} : F_{il} \to F_{kl} \circ F_{jk} \circ F_{ij}$$

agree, as one would naturally expect.

Remark. $h\mathsf{Cat}_1$ is the category whose objects are categories and whose morphisms are functors modulo isomorphic natural transformations.

This would imply that in $h\mathsf{Cat}_1$ the morphisms are isomorphisms of categories in some appropriate sense. We will characterize isomorphisms of categories using fullness, faithfullness, and essential surjectivity which we now define.

Definition 3.16 (Full Functor)**.** A functor $F : \mathsf{C} \to \mathsf{D}$ is full if for all $A, B \in \mathrm{Obj}(\mathsf{C})$ the map of sets $\mathrm{Mor}_{\mathsf{C}}(A, B) \to \mathrm{Mor}_{\mathsf{D}}(F(A), F(B))$ is surjective.

Definition 3.17 (Faithful Functor)**.** A functor $F : \mathsf{C} \to \mathsf{D}$ is faithful if for all $A, B \in \mathrm{Obj}(\mathsf{C})$ the map of sets $\mathrm{Mor}_{\mathsf{C}}(A, B) \to \mathrm{Mor}_{\mathsf{D}}(F(A), F(B))$ is injective.

Naturally, one defines a fully faithful functor as follows.

Definition 3.18 (Fully Faithful Functor)**.** A functor $F : \mathsf{C} \to \mathsf{D}$ is fully faithful if it is full and faithful, that is for all $A, B \in \mathrm{Obj}(\mathsf{C})$ the map of sets $\mathrm{Mor}_{\mathsf{C}}(A, B) \to \mathrm{Mor}_{\mathsf{D}}(F(A), F(B))$ is a bijection.

We now define essential surjectivity as follows.

Definition 3.19 (Essentially Surjective Functor)**.** A functor $F : \mathsf{C} \to \mathsf{D}$ is essentially surjective if for all $B \in \mathrm{Obj}(\mathsf{D})$ there is an isomorphism $F(X) \to B$ in D.

We then define an isomorphism of functors as follows.

Definition 3.20. A functor is an isomorphism if it is fully faithful and essentially surjective.

We have done a lot of work to think about basic notions such as isomorphisms in new ways. Now let us return to the hunble morphism. Recall that a morphism in a category C is a morphism from the free walking morphism $0 \to 1$ to C. More explicitly, we have a diagram

$$\begin{tikzcd}
\mathrm{Mor}_{\mathsf{C}}(A, B) \arrow[r] \arrow[d] & \mathrm{Mor}_{\mathsf{Cat}}([1], \mathsf{C}) \arrow[d] \\
* \arrow[r, "{(A,B)}"'] & \mathsf{C} \times \mathsf{C}
\end{tikzcd}$$

taking a morphism f to the tuple of objects in C, $(\langle 0\rangle^* f, \langle 1\rangle^* f) \in \mathrm{Obj}(\mathsf{C}) \times \mathrm{Obj}(\mathsf{C})$. A key notion here is the realization of regular morphisms as functors. This naturally generalizes to the functor category.

Definition 3.21. Let C, D be categories. The functor category $\mathrm{Mor}_{\mathsf{Cat}}(\mathsf{C}, \mathsf{D})$ has objects functors $\mathsf{C} \to \mathsf{D}$ and morphisms natural transformations between such functors.

In the case of simplicial sets, this admits an even nicer description. Let X, Y be simplicial sets. $\mathrm{Mor}_{\mathsf{SSets}}(X, Y) = Y^X$ is itself a simplicial set such that maps $Z \to Y^X$ can be identified with maps $X \times Z \to Y$. More generally, one can show that for C, D categories $\mathrm{Mor}_{\mathsf{Cat}}(N\mathsf{C}, N\mathsf{D}) = N\mathsf{D}^{N\mathsf{C}}$ is isomorphic to the nerve of the functor category $N\mathrm{Mor}_{\mathsf{Cat}}(\mathsf{C}, \mathsf{D})$. In the case of quasicategories, a theorem of Joyal states that we have the following.

Theorem 3.22 (Joyal)**.** If X is a quasicategory and A a simplical set, the functor category $\mathrm{Mor}_{\mathsf{Cat}}(A, X) = X^A$ is a quasicategory.

References

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