MATH 292Z: FIRST STEPS IN INFINITY CATEGORIES

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Preliminaries

These notes roughly correspond to the course **MATH 292Z: First Steps in Infinity Categories** taught by Prof. Michael Hopkins at Harvard University in the Fall 2023 semester. These notes are LaTeX-ed after the fact with significant alteration and are subject to misinterpretation and mistranscription. Use with caution. Any errors are undoubtedly my own and any virtues that could be ascribed to these notes ought be attributed to the instructor and not the typist.

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1. Lecture 1 – 6th September 2023

Understanding infinity categories is a daunting task. Ideally, we will end this class with the student being conversant in the language of infinity categories, though this may be difficult to achieve. Let us start with the definition of a (1-)category.

Definition 1.1 (Category). A category C is a class $\mathrm{Obj}(\mathsf{C})$ and for each $A, B \in \mathrm{Obj}(\mathsf{C})$ a set $\mathrm{Mor}_{\mathsf{C}}(A, B)$ of morphisms from A to B such that

(a) $\mathrm{id}_A \in \mathrm{Mor}_{\mathsf{C}}(A, A)$,
(b) A composition law $\mathrm{Mor}_{\mathsf{C}}(B, C) \times \mathrm{Mor}_{\mathsf{C}}(A, B) \to \mathrm{Mor}_{\mathsf{C}}(A, C)$ by $(g, f) \mapsto g \circ f$ that is unital $\mathrm{id}_A \circ f = f = f \circ \mathrm{id}_B$ and associating $(g \circ f) \circ h = g \circ (f \circ h)$.

Remark. We consider the underlying collection of objects of a category a class to avoid set-theoretic difficulties such as Russell's Paradox when dealing with the category of sets.

Let us consider some examples.

Example 1.2. Sets whose objects are sets and whose morphisms are set functions.

Example 1.3. Top whose objects are topological spaces and whose morphisms are continuous maps between them.

Example 1.4 (Representable Spaces of Groups). Let G be a group. We can construct a category C_G whose object is a point $*$ and whose morphisms are given by $\mathrm{Mor}_{\mathsf{C}_G}(*, *) = G$ where the unital element is id_G and composition given by group multiplication.

Example 1.5 (Fundemental Groupoid). Let X be a topological space. Consider $\Pi_{\leq 1} X$ the fundamental groupoid of X whose object is X itself and whose morphisms $\mathrm{Mor}_{\Pi_{\leq 1} X}(x, y)$ homotopy classes of paths from $x, y \in X$. Note that the unital and associative properties of path composition only hold after passing to homotopy equivalence.

Let us now define a functor.

Definition 1.6 (Functor)**.** Let C, D be categories. A functor $F : \mathsf{C} \to \mathsf{D}$ consists of a function $F : \mathrm{Obj}(\mathsf{C}) \to \mathrm{Obj}(\mathsf{D})$ such that for all $A, B \in \mathrm{Obj}(\mathsf{C})$, there is a function $\mathrm{Mor}_{\mathsf{C}}(A, B) \to \mathrm{Mor}_{\mathsf{D}}(F(A), F(B))$ satisfying $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(g \circ f) = F(g) \circ F(f)$.

A functor is a map between categories. Correspondingly, we can define a map between functors as a natural transformation.

Definition 1.7 (Natural Transformation)**.** Let $F, G : \mathsf{C} \to \mathsf{D}$ be functors. T is a natural transformation between F and G consists of a morphism $T(A) : F(A) \to G(A)$ in D for all $A \in \mathrm{Obj}(\mathsf{C})$ and the diagram

$$\begin{tikzcd} F(A) \arrow[r, "T(A)"] \arrow[d, "T(f)"'] & G(A) \arrow[d, "G(f)"'] \\ F(B) \arrow[r, "T(B)"] & G(B) \end{tikzcd}$$

commutes for all $f \in \mathrm{Mor}_{\mathsf{C}}(A, B)$.

Example 1.8. Let X be a topological space. We can construct a covering space of X denoted $\widetilde{X}$ such that for the solid diagram,

$$\begin{tikzcd} * \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, "p"] \\ {[0,1]} \arrow[r] \arrow[ur, dashed, "\exists!"] & X \end{tikzcd}$$

there is a unique dotted arrow lifting a path in X to $\widetilde{X}$. In other words, this is a functor $\Pi_{\leq 1} X \to \mathsf{Sets}$ by $x \mapsto p^{-1}(x)$.

In fact, and old theorem of Grothendieck states the following.

Theorem 1.9 (Grothendieck)**.** Let X be a topological space. The category of covering spaces of X is equivalent ot the category of functors $\Pi_{\leq 1} X \to \mathsf{Sets}$.

Within this framework, topological statements like Van-Kampen's theorem become much easier.

In topology, we have paths, and homotopies between paths. Indeed, we can construct homotopies between homotopies, and so on. This is the root of the concept of an infinity category.

And much of the topological information we wish to incorporate above is easiest done with simplices, leading the formalism of infinity categories to be built on the backbone of the rich combinatorial structure of simplical sets. Let's start with an easy example of the partially ordered set, or poset.

Definition 1.10 (Poset)**.** A poset $(S, \leq)$ is a set S together with a relation $\leq$ such that

(a) $a \leq a$ for all $a \in S$,
(b) if $a \leq b$ and $b \leq c$ then $a \leq c$,
(c) and if $a \leq b$ and $b \leq a$ then $a = b$.

A poset can be seen as a category whose objects are the elements of S and

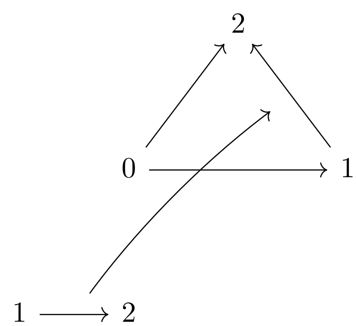
$$\mathrm{Mor}_S(a, b) = \begin{cases} \emptyset & a \not\leq b \\ * & a \leq b. \end{cases}$$

Indeed, one can show that any category with at most one morphism between any two objects is a poset under the relation induced by the morphisms. We can now introduce simplical sets.

Definition 1.11 (Simplices)**.** The category of simplices Δ has objects finite totally ordered sets $[n] = \{0, 1, 2, \ldots, n\}$ and morphisms order-preserving maps.

There is a functor $\Delta \to \mathsf{Top}$ taking $[n]$ to $|\Delta^n|$ the standard n-simplex, the convex hull of $\{e_0, \ldots, e_n\}$ in $\mathbb{R}^{n+1}$ and taking a morphism $[n] \to [m]$ to linear-maps $e_i \mapsto e_{f(i)}$. Such maps $f : [n] \to [m]$ are determined by their (finite) images $0 \leq f(1) \leq f(2) \leq \cdots \leq f(n) \leq m$ which we will write as $\langle f(1), \ldots, f(n) \rangle$.

Example 1.12. There are maps $\langle i, i+1 \rangle : [1] \to [n]$ for $0 \leq i \leq n-1$.



The above demonstrates the map $\langle 1, 2 \rangle : [1] \to [2]$.

Let X be an arbitrary topological space. We can get a contravariant functor $\Delta \to \mathsf{Sets}$ by taking $[n] \to \mathrm{Mor}_{\mathsf{Top}}(|\Delta^n|, X)$. We denote this functor $\mathrm{Sing}\, X : \Delta \to \mathsf{Sets}$. This allows us to define simplical sets.

Definition 1.13 (Simplicial Sets)**.** The category of simplicial sets SSets is the functor category whose objects are contravariant functors $\Delta \to \mathsf{Sets}$ and morphisms natural transformations between such functors.

Let us switch directions for a moment and consider nerves of categories. We have already seen how a poset can be made into a category. Analogously, a totally ordered set can be made into a category in a similar way, yielding a functor $\Delta \to \mathsf{Cat}$, the category whose objects are categories and whose morphisms are functors. Nerves help us understand how simplices live within categories.

Definition 1.14 (Nerve)**.** Let C be a category. The nerve of C is the simplicial set $N\mathsf{C}$ so that

$$(N\mathsf{C})_n = \mathrm{Mor}_{\mathsf{Cat}}([n], \mathsf{C})$$

the set of functors from $[n]$ to C such that the simplical operators $f : [n] \to [m]$ act by pre-composition where $a \mapsto a \circ f$ for $a : [n] \to \mathsf{C}$ in $(N\mathsf{C})_n$.

We can make this more concrete with a few examples.

Example 1.15. $(N\mathsf{C})_0$ is the simplicial set of maps from the point to C and hence corresponds to the objects of C, $\mathrm{Obj}(\mathsf{C})$.

$(N\mathsf{C})_1$ is the simplicial set of maps from $0 \to 1$ to C and hence corresponds to the morphisms of C.

$(N\mathsf{C})_2$ is the simplicial set of maps

$$\begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \arrow[d] \\ & 2 \end{tikzcd}$$

to C, that is, to pairs of composable morphisms in C.

More generally $(N\mathsf{C})_n$ is the set of composable n-tuples of morphisms in C.

Theorem 1.16. A simplicial set X is isomorphic to the nerve of some category C if and only if for all $n \geq 2$, the map

$$X_n \to \underbrace{X_1 \times_{X_0} X_1 \times_{X_0} \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ times}}$$

by

$$g \mapsto (\langle 0,1\rangle^* g, \langle 1,2\rangle^* g, \ldots, \langle n-1,n\rangle^* g)$$

is an isomorphism.

This proof follows that in [1, Proposition 1.4.8].

Proof. ($\Longrightarrow$) Suppose $X = N\mathsf{C}$ for some category C. A tuple of n composable morphisms in C is precisely the datum of the n-nerve $(N\mathsf{C})_n$.

($\Longleftarrow$) Suppose X is a simplicial set such that the function above is a bijection. We construct the category C whose objects $\mathrm{Obj}(\mathsf{C}) = X_0$ and whose morphisms are X_1. For $g \in X_1$, we have two functors $X_1 \to X_0$ taking $g \mapsto \langle 0\rangle^* g$ its source object and $g \to \langle 1\rangle^* g$ its target object. Since $0 \to 0$ is a morphism in $|\Delta^1|$, we trivially have identity morphisms on each object. Moreover, given a morphism $f \in X_1$, we can recover its source object $\langle 0\rangle^* f$ and its target object $\langle 1\rangle^* f$. And for any $(g,h) \in X_1 \times_{X_0} X_1$, the composite $h \circ g \in X_2$ is the unique morphism where $\langle 0,1\rangle^*(g \circ h) = h$ and $\langle 1,2\rangle^*(g \circ h) = g$ where uniqueness follows from bijectivity. We now claim that for any $g \in X_n$ and $0 \leq i \leq j \leq k \leq n$ we have $\langle i,k\rangle^* g = \langle j,k\rangle^* g \circ \langle j,i\rangle^* g$ where each of $\langle i,k\rangle^* g, \langle j,k\rangle^* g, \langle j,i\rangle^* g \in X_1$ as images of maps $[1] \to [n]$. Indeed any element of X_2 decomposes in this way. We can thus define maps $X_n \to (N\mathsf{C})_n$ by sending $g \in X_n$ to $[n] \to \mathsf{C}$ by $g \mapsto g$ and $(i \to j) \mapsto \langle j,i\rangle^* g$ which is a functor by the above. We can verify these maps are bijections since $((i-1) \to i) \mapsto \langle i, i-1\rangle^* g$. Furthermore, if $f : [m] \to [n]$ is a map of simplices, we compute

$$\langle j,i\rangle^*(g \circ f) = \langle f(j), f(i)\rangle^* g$$

giving an isomorphism $X \to N\mathsf{C}$ of simplical sets. ■

We now discuss the further significance of $\mathrm{Sing}\, X$. To do this, we first have to discuss horns and Kan complexes. Horns are subcomplexes of standard simplicies.

Definition 1.17 (Horn)**.** For each $n \geq 1$, there are subcomplexes $\Lambda^n_j \subseteq |\Delta^n|$ for each $0 \leq j \leq n$ defined by

$$(\Lambda^n_j)_k = \{f : [k] \to [n] | ([n] \setminus [j]) \not\subseteq f([k])\}.$$

Morally, we should think of a horn Λ^n_j is the union of faces of the n simplex other than the jth.

Example 1.18 (1-Horns)**.** The horns in $|\Delta^1|$ are the vertices $\Lambda^1_0 = \{0\}, \Lambda^1_1 = \{1\}$.

Example 1.19 (2-Horns)**.** There are three horns in the 2-simplex.

$$\begin{tikzcd} & 2 & \\ 0 \arrow[ur] \arrow[rr] & & 1 \end{tikzcd} \qquad \begin{tikzcd} & 2 & \\ 0 \arrow[rr] & & 1 \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & 2 & \\ 0 \arrow[ur] & & 1 \arrow[ul] \end{tikzcd}$$

$$\Lambda^2_0 \qquad\qquad \Lambda^2_1 \qquad\qquad \Lambda^2_2$$

We can now define Kan complexes.

Definition 1.20 (Kan Complex)**.** A simplicial set is a Kan complex if for all k, n the solid diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ {|\Delta^n|} \arrow[ur, dashed] & \end{tikzcd}$$

admits a dotted map making the diagram commute.

A theorem of Kan states the following:

Theorem 1.21 (Kan)**.** $\operatorname{Sing} X$ is a Kan complex.

This eventually led to the definition of quasicategories.

Definition 1.22 (Quasicategory; Boardman-Vogt, Joyal, Lurie)**.** A simplicial set is a quasicategory if every solid diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ {|\Delta^n|} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

admits a dotted map making the diagram commute.

We will revisit these ideas in the following lecture.

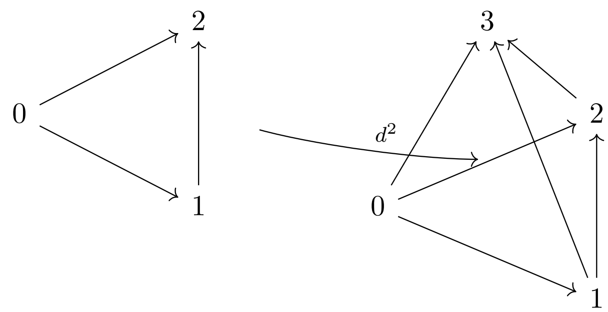
2. Lecture 2 – 11th September 2023

We made a very quick pass of simplicial sets in the previous lecture. Let us go through it in more detail.

Let Δ be the category whose objects are finite ordered sets and whose morphisms are order-preserving maps. Following the notation in Rezk's text [1], for a morphism $[n] \to [k]$, we write it $\langle k_0, \dots, k_n \rangle$ where $0 \le k_0 \le \cdots \le k_n \le k$. In the category Δ, there are two types of distinguished maps d_i and s_i which we now define.

Definition 2.1 (Face Maps)**.** In the category Δ, there are maps $d^i : [n-1] \to [n]$ by $\langle 0, 1, \dots, \hat{k}, \dots, n \rangle = \langle 0, 1, \dots, k-1, k+1, \dots, n \rangle \to [n]$.

Example 2.2. We should think of these as the inclusion of a face into the simplex.

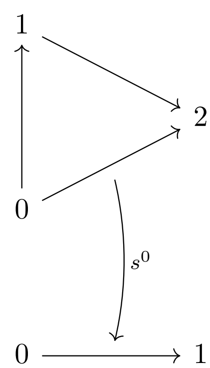


Here, we show the inclusion $d^2 : [2] \to [3]$ by $\langle 0, 1, 3\rangle$, taking the 2-simplex to the face "opposite" the vertex 2, that is, the face bounded by the vertices 0, 1, and 3.

We can also define degeneracy maps as follows.

Definition 2.3 (Degeneracy Map)**.** In the category Δ, there are maps $s^i : [n] \to [n-1]$ by $[n] \mapsto \langle 0, 1, 2, \ldots, k, k, k+1, \ldots, n\rangle$.

Example 2.4. We should think of this as collapsing the kth to the $(k-1)$th vertex.



Here we show $s^0 : [2] \to [1]$ by $\langle 0, 0, 1\rangle$ collapsing the vertex 1 to the vertex 0.

With these operations on Δ, one can in fact show the following theorem.

Theorem 2.5. If f is a morphism in Δ, then f factors as the composition of face and degeneracy maps.

We omit the proof.

Now recall that a simplicial set X is a covariant functor $X : \Delta^{\mathsf{Opp}} \to \mathsf{Sets}$. We write

$$X_n = \mathrm{Mor}_{\mathsf{Cat}}([n], X).$$

For a simplicial set X, we have a diagram

$$\begin{tikzcd}
X_0 \arrow[r, "s^0" description] & X_1 \arrow[l, bend right, "d^0" description] \arrow[l, bend left, "d^1" description] \arrow[r, bend left=15, "s^0" description] \arrow[r, bend right=15, "s^1" description] & X_2 \arrow[l, "d^1" description] \arrow[l, bend right=40, "d^0" description] \arrow[l, bend left=40, "d^2" description] & \cdots & \cdots
\end{tikzcd}$$

induced by the degeneracy and face maps.

Definition 2.6 (n-Simplex)**.** The n-simplex is the functor $\Delta^n : \Delta^{\mathsf{Opp}} \to \mathsf{Sets}$ by $[k] \mapsto \mathrm{Mor}_{\Delta}([k], [n])$.

Remark. Note that Δ^n is the functor $\Delta^{\mathsf{Opp}} \to \mathsf{Sets}$, while the topological n-simplex $|\Delta^n| \simeq S^n \in \mathrm{Obj}(\mathsf{Top})$ is topological space homeomorphic (and homotopic) to the n-sphere.

In this way, we can think of Δ^n as the functor represented by $[n] \in \mathrm{Obj}(\Delta)$. More generally, one can define representable functors as follows.

Definition 2.7 (Representable Functor)**.** Let C be a category. A functor $F : \mathsf{C} \to \mathsf{Sets}$ is representable if there exists $A \in \mathrm{Obj}(\mathsf{C})$ such that there exists an isomorphism of functors $F \to \mathrm{Mor}_{\mathsf{C}}(-, A)$.

Generally, consider $F : \mathsf{C}^{\mathsf{Opp}} \to \mathsf{Sets}$ a contravariant functor from an arbitrary category C to Sets. Let $f : B \to A$. $F(f) : F(B) \to F(A)$ is a map betewen sets. For $u \in F(A)$, u determines a natural transformation of functors $\mathrm{NatTrans}(\mathrm{Mor}_{\mathsf{C}}(-, A), F)$. This is Yoneda's lemma. We adapt the version from [2, Theorem 2.2.4].

Lemma 2.8 (Yoneda)**.** For a contravariant functor $F : \mathsf{C} \to \mathsf{Sets}$ and $A \in \mathrm{Obj}(\mathsf{C})$, there is a bijection

$$\mathrm{NatTrans}(\mathrm{Mor}_{\mathsf{C}}(-, A), F) \to F(A)$$

that associates a natural transformation of functors $\alpha : \mathrm{Mor}_{\mathsf{C}}(-, A) \Rightarrow F$ to the element $\alpha(\mathrm{id}_A) \in F(A)$, natural in both A and F.

Let us recall the definition of quasicategories as in Definition 1.22.

Definition 2.9 (Quasicategory)**.** A simplcial set X is a quasicategory if every inner horn Λ^n_k where $0 < k < n$ has a fill.

In other words, for every solid diagram,

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

there is a dotted map making the diagram commute.

Let us now fix some notation that we will use going forward. For X a simplicial set we denote $X_n = \mathrm{Mor}_{\mathsf{Cat}}([n], X)$ and $\alpha \in \mathrm{Mor}_{\Delta}([k], [n])$, we have $X_\alpha : X_n \to X_k$.

Theorem 2.10. A simplicial set is the nerve of some category if and only if every inner horn has a unique filler.

In other words, for every solid diagram,

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

there is a dotted map making the diagram commute.

We omit the proof of Theorem 2.10 which can be found in [1, §1.7.10].

We want to consider an analogue of Theorem 2.10 in the case of quasicategories. Let X be a quasicategory with objects X_0 and morphisms X_1. For $f \in X_0$, let $\langle 0 \rangle^* f$ denote the source of the map f and $\langle 1 \rangle^* f$ its target. Let

$$\begin{tikzcd} A \arrow[d, "f"'] & C \\ B \arrow[ur, "g"'] & \end{tikzcd}$$

be the image of a horn Λ_1^2 and $h = \tau\langle 0, 2\rangle : A \to C$ be the fill.

$$\begin{tikzcd} A \arrow[r, "\tau\langle 0,2\rangle = h"] \arrow[d, "f"'] & C \\ B \arrow[ur, "g"'] & \end{tikzcd}$$

Since the fill is not unique, we say τ witnesses h as the composition of g with f. This begs the question of how we can perform some operation to make X into a category. This is done via the construction of the fundamental category, which can be done on simplicial sets in general.

Definition 2.11 (Fundamental Category)**.** Let X be a simplicial set. A category C is the fundamental category of X if every map $X \to \mathsf{D}$ for D factors through C and C is final with respect to that property.

Fundamental categories are in general hard to construct, but much easier in the case of quasicategories. Indeed if X is a quasicategory, its fundamental category coincides with its homotopy category hX. The homotopy category will have objects X_0 and morphisms those in X_1 up to "witnessing"-equivalence. We will have to discuss these notions of equivalence before defining the homotopy category rigorously.

Definition 2.12 (Left-Equivalence)**.** Let $f, g \in \mathrm{Mor}_X(A, B)$. We say that f is left-equivalent to g if there is $\tau \in X_2$ such that $\tau\langle 0, 1\rangle = \mathrm{id}_A, \tau\langle 0, 2\rangle = f, \tau\langle 1, 2\rangle = g$.

$$\begin{tikzcd} A \arrow[d, "\mathrm{id}_A" description] \arrow[dr, "f" description] & \\ A \arrow[r, "g" description] & B \end{tikzcd}$$

Similarly, we have right-equivalence.

Definition 2.13 (Right-Equivalence)**.** Let $f, g \in \mathrm{Mor}_X(A, B)$. We say that f is right-equivalent to g if there is $\tau \in X_2$ such that $\tau\langle 1, 2\rangle = \mathrm{id}_B, \tau\langle 0, 1\rangle = f, \tau\langle 0, 2\rangle = g$.

$$\begin{tikzcd} & B \\ A \arrow[ur, "g" description] \arrow[r, "f" description] & B \arrow[u, "\mathrm{id}_B" description] \end{tikzcd}$$

One then shows the following proposition before defining the homotopy category.

Proposition 2.14. The relations of left equivalence (Definition 2.12) and right equivalence (Definition 2.13) coincide.

We can now define the homotopy category.

Definition 2.15 (Homotopy Category)**.** Let X be a simplicial set. The homotopy category hX has objects those in X_0 and morphisms those in X_1 up to equivalence.

References

[1] Charles Rezk. *Introduction to Quasicategories*. URL: https://people.math.rochester.edu/faculty/doug/otherpapers/Rezk-quasicats-intro.pdf.

[2] Emily Riehl. *Category theory in context*. English. Mineola, NY: Dover Publications, 2016. ISBN: 978-0-486-80903-8.

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