A PRIMER ON PRISMATIC COHOMOLOGY

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Preliminaries

This document seeks to give a primer on the theory of prismatic cohomology first developed in Bhatt-Scholze's seminal article [BS22] based primarily on the lecture course of G. Bosco at the Universität Bonn in the Summer 2024 semester [Bos24] and of J. Anschütz at the Recent Advances in Algebraic K-Theory conference at the I'HES in Summer 2023 [Ans23]. In particular, no claims to completeness of proofs – if at all given – are made.

Contents

1. Motivations

Let X be a scheme smooth and proper over $\mathbb{Z}$. We can consider cohomology theories for schemes in characteristic p and coefficients in $\mathbb{Z}_\ell$:

- Singular cohomology, the cohomology on the underlying topological space of complex points $X(\mathbb{C})$.
- de Rham cohomology, the sheaf cohomology of the sheaf of (completed) differentials on X.
  - In the characteristic 0 setting over $\mathbb{C}$, de Rham cohomology agrees with singular cohomology due to results of Grothendieck and Serre giving an isomorphism

    $$H^i_{\mathsf{dR}}(X) = H^i(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$$

    giving rise to Hodge theory and Hodge cohomology.
- Crystalline cohomology, sheaf cohomology over the crystalline site which can be thought of as parametrizing infinitesmal thickenings of the scheme.
- Étale cohomology, sheaf cohomology on the étale site which refines the standard Zariski topology on the scheme.

These cohomology theories fit together in the following diagram.

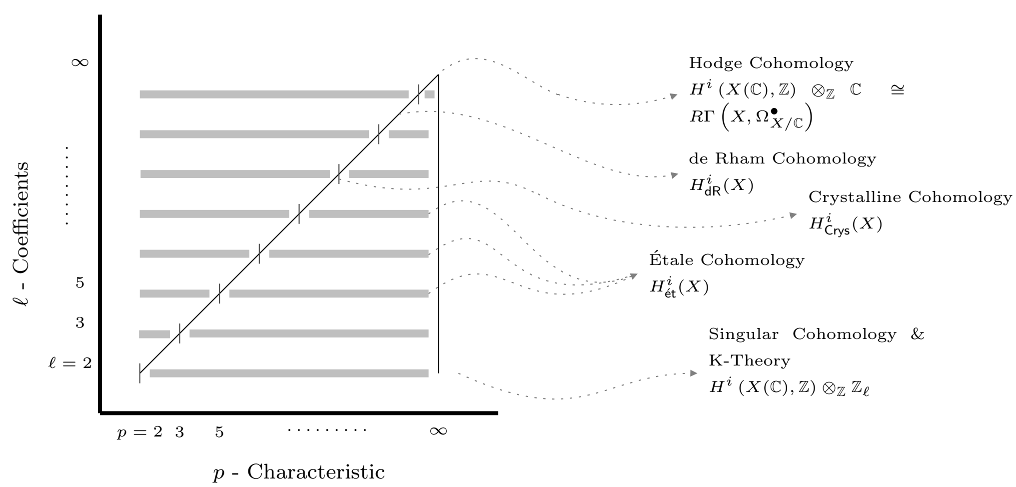

Figure 1. Cohomology theories for schemes.

For a scheme $X_{\mathbb{F}_p}$ over $\mathbb{F}_p$ we want to consider cohomology theories with coefficients in $\mathbb{Z}_p$. In this setting, étale cohomology is not defined due to the poor behavior of pth roots of unity in characteristic p. In particular, the Kummer sequence, key to defining étale cohomology is no longer exact.

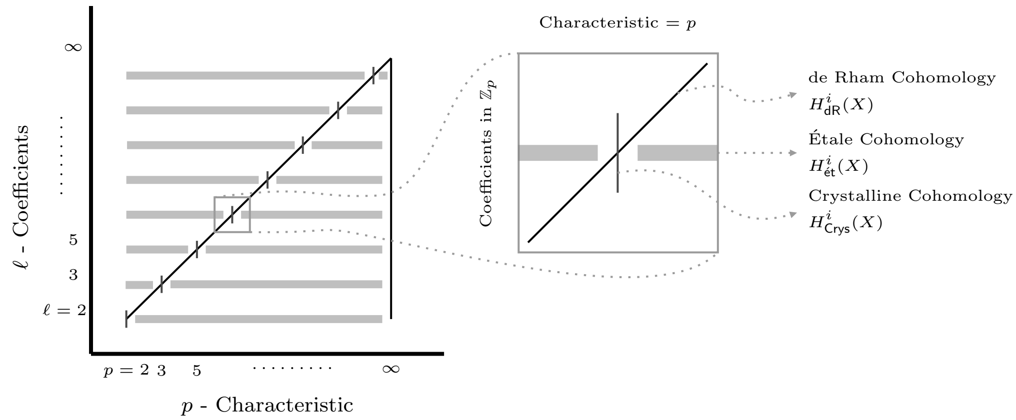

FIGURE 2. Cohomology of $\mathbb{F}_p$-schemes with coefficients in $\mathbb{Z}_p$.

Prismatic cohomology seeks to provide a cohomology theory that incorporates data from the various cohomology theories at the (p, p)-point.

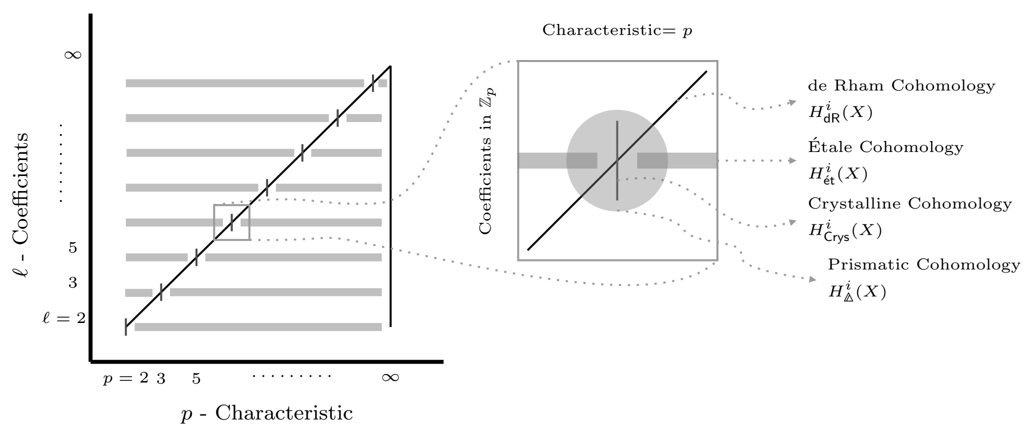

FIGURE 3. Prismatic cohomology at (p, p).

Prismatic cohomology can be thought of as a prism, refracting the beam of light into the various cohomology theories for formal schemes over $\mathbb{Z}_p$.

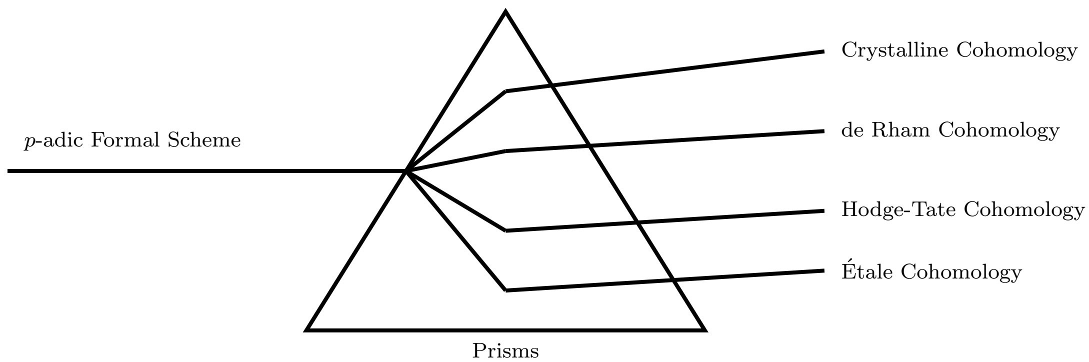

FIGURE 4. Refraction of cohomology theories.

2. Witt Vectors and Deformation Theory

Let p be a prime number and recall the following definition.

Definition 2.1 (Perfect $\mathbb{F}_p$-Algebra)**.** Let A be an $\mathbb{F}_p$-algebra. A is a perfect $\mathbb{F}_p$ algebra if the Frobenius map $\varphi_A : A \to A$ by $x \mapsto x^p$ is an isomorphism.

Perfect $\mathbb{F}_p$-algebras naturally form a category $\mathsf{Perf}_{\mathbb{F}_p}$ which is a full subcategory of all $\mathbb{F}_p$-algebras $\mathsf{Alg}_{\mathbb{F}_p}$. We can construct perfect $\mathbb{F}_p$ algebras from $\mathbb{F}_p$ algebras by taking either a limit or colimit along constant diagrams with transition maps given by Frobenii.

Definition 2.2 (Perfection)**.** Let A be an $\mathbb{F}_p$-algebra. The perfect $\mathbb{F}_p$-algebras A^{perf} (resp. A_{perf}) are given by

$$\lim \Big(A \xrightarrow{\varphi_A} A \longrightarrow \dots \Big) \qquad \Big(\text{resp.} \operatorname{colim} \Big(A \xrightarrow{\varphi_A} A \longrightarrow \dots \Big)\Big).$$

Remark 2.3. That $A^{\mathsf{perf}}, A_{\mathsf{perf}}$ are perfect $\mathbb{F}_p$ algebras is given in [Lep22, Prop. 1.4, Prop. 1.12], respectively.

Remark 2.4. $A \mapsto A_{\mathsf{perf}}$ and $A \mapsto A^{\mathsf{perf}}$ are the left and right adjoints of $\mathsf{Perf}_{\mathbb{F}_p} \to \mathsf{Alg}_{\mathbb{F}_p}$, respectively. Where unclear from the context we will refer to these as the limit and colimit perfections, respectively.

We can also construct perfect $\mathbb{F}_p$-algebras from p-adically complete, p-torsion free $\mathbb{Z}_p$-algebras – those $\mathbb{Z}_p$-algebras that are complete with respect to the p-adic norm and the kernel of the pth power endomorphism is trivial.

Theorem 2.5. *Let $\widehat{\mathsf{Alg}}_{\mathbb{Z}_p}$ be the category of p-adically complete, p-torsion free $\mathbb{Z}_p$-algebras. There is a functor $\widehat{\mathsf{Alg}}_{\mathbb{Z}_p} \to \mathsf{Perf}_{\mathbb{F}_p}$ by $B \mapsto B^\flat = (B/(p))^{\mathsf{perf}}$ which admits a left adjoint.*

The image of a perfect $\mathbb{F}_p$-algebra A under the left adjoint is the ring of p-typical Witt vectors of A.

Definition 2.6 (Witt Vectors)**.** Let A be a perfect $\mathbb{F}_p$-algebra. The ring of p-typical Witt-vectors $\mathrm{W}(A)$ of A is the p-adically complete, p-torsion free $\mathbb{Z}_p$-algebra that is the image of A under the left adjoint to $(-)^\flat : \widehat{\mathsf{Alg}}_{\mathbb{Z}_p} \to \mathsf{Perf}_{\mathbb{F}_p}$.

We now recall some results from deformation theory.

Let B be an A-algebra. We can construct the standard resolution $P^\bullet_{B/A}$ with terms

$$P^n_{B/A} = \underbrace{A\,[A[\dots A[B]]]}_{n-1 \text{ times}}$$

with the face and degeneracy maps given in [Stacks, Tag 09CB] which is an agumentation over B as in [Stacks, Tag 018G]. We define the cotangent complex as follows.

Definition 2.7 (Cotangent Complex)**.** Let $A \to B$ be a morphism of rings. The cotangent complex $\mathbb{L}_{B/A}$ of $A \to B$ is the complex of B-modules $\Omega_{P^\bullet_{B/A}/A} \otimes_{P^\bullet_{B/A},\varepsilon} B$ with augumentation $\varepsilon : P^\bullet_{B/A} \to B$.

Remark 2.8. The simplicial B-module $\Omega_{P^\bullet_{B/A}/A} \otimes_{P^\bullet_{B/A},\varepsilon} B$ can be regarded as a chain complex by formation of the Moore complex as in [Stacks, Tag 0194].

The following proposition allows us to see why the cotangent complex is an enhancement of the the classical theory of Kähler differentials.

Proposition 2.9 ([Stacks, Tag 08QF])**.** If $A \to B$ is a morphism of rings and $\mathbb{L}_{B/A}$ the cotangent complex then $H^0(\mathbb{L}_{B/A}) = \Omega_{B/A}$.

In this light, we have the following analogue of the relative affine cotangent sequence.

Proposition 2.10 ([Stacks, Tag 08SA])**.** If $A \to B \to C$ are morphisms of rings with cotangent complexes $\mathbb{L}_{B/A}, \mathbb{L}_{C/B}$ and $\mathbb{L}_{C/A}$ the cotangent complex of the composite then there is a canonical distinguished triangle

$$\mathbb{L}_{B/A} \otimes^L_B C \longrightarrow \mathbb{L}_{C/A} \longrightarrow \mathbb{L}_{C/B} \longrightarrow \mathbb{L}_{B/A} \otimes^L_B C[1]$$

in $D(C)$.

The main result of deformation theory is as follows.

Theorem 2.11. Let A be a ring and C_A be the category of flat A-algebras such that $\mathbb{L}_{(-)/A} = 0$.

(i) Then for any infinitesmal thickening $B \to A$ base change to B induces an equivalence of categories $\mathsf{C}_A \to \mathsf{C}_B$ hence inducing a lift of $A \hookrightarrow R$ for a flat A-algebra R with $\mathbb{L}_{R/A} = 0$ to $B \hookrightarrow \widetilde{R}$.

(ii) Furthermore, for all morphisms $A \hookrightarrow R$ in C_A and surjective A-algebra maps $B' \to B$ with nilpotent kernel, each A-algebra map $R \to B$ lifts uniquely to an A-algebra map $A \to B'$.

Remark 2.12. In particular, Theorem 2.11 (i) holds for infinitesmal thickenings of a ring A: surjections $B \to A$ such that $\ker(B \to A)^n = 0$ for some $n \geq 0$.

Remark 2.13. The proof of Theorem 2.5 in fact uses Theorem 2.11 by considering infinitesmal thickenings of $\mathbb{F}_p$.

Note that if $A \in \mathsf{Perf}_{\mathbb{F}_p}$ then $\mathbb{L}_{A/\mathbb{F}_p} = 0$ since the Frobenius map induces an isomorphism $\mathbb{L}_{A/\mathbb{F}_p} \to \mathbb{L}_{A/\mathbb{F}_p}$ by functoriality which is the zero map since the Kähler differentials of A over $\mathbb{F}_p$ vanish. Similarly, one can easily verify that A is flat as an $\mathbb{F}_p$-algebra (for example via [Stacks, Tag 00HD (3)]) so Theorem 2.11 applies. As such, for the infinitesmal deformation $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{F}_p$ we can lift the map $\mathbb{F}_p \hookrightarrow A$ to a map with target some $\mathbb{Z}/p^n\mathbb{Z}$-algebra which we define to be the n-truncated Witt vectors of A.

Definition 2.14 (n-Truncated Witt Vectors)**.** Let $A \in \mathsf{Perf}_{\mathbb{F}_p}$. The ring of n-truncated Witt vectors $\mathrm{W}_n(A)$ of A is the target of the unique lift of $\mathbb{F}_p \hookrightarrow A$ in flat $\mathbb{Z}/p^n\mathbb{Z}$-algebras with trivial cotangent complex under the equivalence $\mathsf{C}_{\mathbb{F}_p} \to \mathsf{C}_{\mathbb{Z}/p^n\mathbb{Z}}$ of Theorem 2.11 (i).

Remark 2.15. There is an isomorphism of rings $\lim_n \mathrm{W}_n(A) \cong \mathrm{W}(A)$.

These rings of n-truncated Witt vectors allow us to develop a theory of Teichmüller lifts[1], extending maps in $\mathsf{Perf}_{\mathbb{F}_p}$ to n-truncated Witt vectors and thus to the ring of Witt vectors.

Under the identification of the ring of p-typical Witt-vectors with sequences of elements of A [Ked21, Def. 3.1.6, 3.2.4], we can define the Teichmüller lift $A \to \mathrm{W}(A)$ as follows.

Definition 2.16 (Teichmüller Lift)**.** Let $A \in \mathsf{Perf}_{\mathbb{F}_p}$. The Teichmüller lift of A is the morphism $A \to \mathrm{W}(A)$ by $a \mapsto (a, 0, \dots)$.

Remark 2.17. By functoriality, any $A \in \mathsf{Perf}_{\mathbb{F}_p}$ functoriality of the lift gives a Frobenius automorphism on the Witt ring.

Let B be a p-complete $\mathbb{Z}_p$ algebra and A a perfect $\mathbb{F}_p$-algebra with a map $A \to B/(p)$ and the surjection $B/(p^n) \to B/(p)$. By Definition 2.14, we have an inclusion $\mathbb{Z}/p^n\mathbb{Z} \hookrightarrow \mathrm{W}_n(A)$ so applying Theorem 2.11 (ii) we have a unique $\mathbb{Z}/p^n\mathbb{Z}$-algebra map extending $\mathrm{W}_n(A) \to A \to B/(p)$ to $\mathrm{W}_n(A) \to B/(p^n)$. By the universal property of the limit, this extends uniquely to a map $\mathrm{W}(A) \to B/(p^n)$.

We now consider a further specialization of the scenario above. For $B \in \widehat{\mathsf{Alg}}_{\mathbb{Z}_p}$ we have a surjection $B \to B/(p)$ and a canonical map $\overline{\theta} : B^\flat \to B/(p)$. By the discussion above, we can construct a map $\theta : \mathrm{W}(B^\flat) \to B$. This lift in fact introduces the construction of $\mathbb{A}_{\mathrm{inf}}$ as we now define.

Definition 2.18 ($\mathbb{A}_{\mathrm{inf}}$)**.** Let $A \in \widehat{\mathsf{Alg}}_{\mathbb{Z}_p}$. $\mathbb{A}_{\mathrm{inf}}(A)$ is defined to be $\mathrm{W}(A^\flat)$

[1]Oswald Teichmüller (1913-1943) was a German mathematician who was involved in the National Socialist Party. See `mathshistory.st-andrews.ac.uk/Biographies/Teichmuller/` for more information.

3. δ-Rings

δ-rings are rings with a lift of Frobenius modulo p.

Definition 3.1 (δ-Ring)**.** A delta ring (A, δ) is a pair consisting of a ring R and a set map $\delta : A \to A$ such that:

(i) $\delta(0) = \delta(1) = 0$.
(ii) $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$.
(iii) $\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p+y^p-(x+y)^p}{p}$.

δ-rings naturally form a category Ring_δ with objects δ-rings and morphisms ring maps preserving the δ-structures.

The lifting condition alluded to above is made clear in the following proposition.

Proposition 3.2. If (R, δ) is a δ-ring then the map $\widetilde{\varphi}_A : A \to A$ by $x \mapsto x^p + p\delta(x)$ is an endomorphism of R that lifts the Frobenius on $A/(p)$.

Proof. The map clearly lifts the Frobenius on $A/(p)$ and that this is an endomorphism can be verified by the following direct computations:

$$\begin{aligned}
\widetilde{\varphi}_A(x + y) &= (x + y)^p + p\delta(f + g) \\
&= (x + y)^p + p\delta(x) + p\delta(y) + x^p + y^p - (x + y)^p && \text{Definition 3.1 (iii)} \\
&= x^p + p\delta(x) + y^p + p\delta(y) \\
&= \widetilde{\varphi}_A(x) + \widetilde{\varphi}_A(y) \\
\widetilde{\varphi}_A(xy) &= x^py^p + p\delta(xy) \\
&= x^py^p + px^p\delta(y) + py^p\delta(x) + p^2\delta(x)\delta(y) && \text{Definition 3.1 (ii)} \\
&= (x^p + p\delta(x))(y^p + p\delta(y)) \\
&= \widetilde{\varphi}_A(x)\widetilde{\varphi}_A(y)
\end{aligned}$$

■

Indeed, for lifts of Frobeneii $\widetilde{\varphi}_A : A \to A$ on $A/(p)$ we can construct a δ-structure on A for A p-torsion free mutually inverse to the construction Proposition 3.2.

Proposition 3.3. Let A be a p-torsion free ring. Each δ-structure on A arises uniquely from an endomorphism of A lifting the Frobenius on $A/(p)$.

Proof. Suppose $\widetilde{\varphi} : A \to A$ is a lift of Frobenius on $A/(p)$. Note that we have $\widetilde{\varphi}(x) - x^p \in (p)$ so $\frac{\widetilde{\varphi}(x)-x^p}{p}$ is well-defined and setting $\delta(x) = \frac{\widetilde{\varphi}(x)-x^p}{p}$ we can verify by explicit computation that such a map δ defines a δ-structure on A which is inverse to the construction of Proposition 3.2 since A is p-torsion free. ■

We have already encountered an important example of δ-rings: the ring of p-typical Witt vectors over a perfect $\mathbb{F}_p$-algebra A.

Proposition 3.4. Let $A \in \mathsf{Perf}_{\mathbb{F}_p}$ and $\mathrm{W}(A)$ its ring of p-typical Witt vectors. Then $\mathrm{W}(A)$ is a δ-ring.

Proof. We have a lift of the Frobenius on $A \cong \mathrm{W}(A)/(p)$ to $\mathrm{W}(A)$ by Remark 2.17 inducing a δ-structure on $\mathrm{W}(A)$ by Proposition 3.3 since $\mathrm{W}(A)$ is p-torsion free. ∎

The category Ring_δ possesses nice categorical properties, in particular, it admits all limits and colimits which can be computed in the category of rings.

Lemma 3.5 ([Ked21, Lem. 2.4.3])**.** Ring_δ admits all colimits and limits. Furthermore, these limtis and colimits commute with the forgetful functor to the category of rings Ring.

In particular, by the adjoint functor theorem, the inclusion $\mathsf{Ring}_\delta \hookrightarrow \mathsf{Ring}$ admits both a left and right adjoint.

We now relate $\mathsf{Perf}_{\mathbb{F}_p}$ by introducing the notion of a perfect δ-ring.

Definition 3.6 (Perfect δ-Ring)**.** Let (A, δ) be a δ-ring. (A, δ) is a perfect δ-ring if $\widetilde{\varphi}_A : A \to A$ by $x \mapsto x^p + p\delta(x)$ is an isomorphism of rings.

The category of perfect δ-rings Perf_δ forms a full subcategory of Ring_δ. Analogous to the case of $\mathbb{F}_p$-algebras in Definition 2.2 we can take (co)limits along the endomorphisms $\widetilde{\varphi}_A$ defining for a $R \in \mathsf{Ring}_\delta$

$$R^{\mathsf{perf}} = \lim \left(R \xrightarrow{\widetilde{\varphi}_A} R \longrightarrow \dots \right) \qquad R_{\mathsf{perf}} = \operatorname{colim} \left(A \xrightarrow{\widetilde{\varphi}_A} A \longrightarrow \dots \right)$$

whih are the right and left adjoints of the inclusion $\mathsf{Perf}_\delta \to \mathsf{Ring}_\delta$.

The relationship between $\mathsf{Perf}_{\mathbb{F}_p}$ and Perf_δ is given explicitly by the following proposition.

Proposition 3.7 ([Ked21, Prop. 3.3.6])**.** The following categories are equivalent:

- $\mathsf{Perf}_{\mathbb{F}_p}$, the category of perfect $\mathbb{F}_p$-algebras (Definition 2.1).
- The subcategory of p-complete objects of Perf_δ, those perfect δ-rings that are p-adically complete.
- The category of p-complete, p-torsion free rings with perfect quotient modulo (p).

Remark 3.8. The functor from $\mathsf{Perf}_{\mathbb{F}_p}$ to p-complete perfect δ-rings proceeds by formation of the Witt ring, from p-complete perfect δ-rings to p-complete, p-torsion free rings with perfect quotient modulo (p) is forgetful, and from p-complete, p-torsion free rings with perfect quotient modulo (p) to $\mathsf{Perf}_{\mathbb{F}_p}$ by taking the quotient modulo (p).

We conclude by recalling further structural results on δ-rings.

Lemma 3.9 ([Ked21, Lem. 2.4.2])**.** Let (A, δ) be a δ-ring and $I \subseteq A$ an ideal such that $\delta(I) \subseteq I$ as sets. Then there exists a unique δ-structure on A/I such that the quotient $A \to A/I$ is a morphism of δ-rings.

Lemma 3.10 ([Ked21, Lem 2.4.8])**.** Let (A, δ) be a δ-ring and $S \subseteq A$ a multiplicative subset such that $\widetilde{\varphi}_A(S) \subseteq S$ as sets. Then there exists a unique δ-structure on $S^{-1}A$ such that the localization $A \to S^{-1}A$ is a morphism of δ-rings.

4. Prisms and Perfectoid Rings

We consider a subset of elements of a delta ring (A, δ) which play a crucial role in existing p-adic cohomology theories.

Definition 4.1 (Distinguished Element)**.** Let (A, δ) be a δ-ring. An element $d \in A$ is a distniguished element if $\delta(d) \in A^\times$.

Distinguished elements are closely related to crystalline cohomology, q-de Rham cohomology, Breuil-Kisin cohomology, and $\mathbb{A}_{\mathrm{inf}}$-cohomology. See [Ked21, Ex. 5.13-16] for further discussion.

We recover the following structural results for distinguished elements.

Lemma 4.2 ([Ked21, Lem. 5.2.1])**.** Let (A, δ) be a δ-ring and $a, b \in A$. ab is distinguished if and only if both a and b are distinguished and $(p, a, b) = A$.

Lemma 4.3 ([Ked21, Lem. 5.2.3])**.** Let (A, δ) be a δ-ring and $a \in A$. a is distinguished if and only if $p \in (p^2, a, \widetilde{\varphi}_A)$.

We will define prisms to be derived (p, I)-adically complete δ-rings on a δ-ring A and where I is an invertible A-module. Let A be a commutative ring, M an A-module, and $I \subseteq A$ an ideal. Recall that A is I-adically complete (resp. M is I-adically complete) if there is an isomorphism $A \cong \lim_n A/I^n$ (resp. $M \cong \lim_n M/I^n M$). However, I-adic completeness does not often interact well with categorical constructions – see [Stacks, Tag 05JD] for an example. This poor categorical behavior, however, can be resolved by considering derived completions.

Definition 4.4 (Derived I-Adically Complete)**.** Let A be a ring (resp. M an A-module). A is derived I-adically complete (resp. M is derived I-adically complete) if $\mathrm{Ext}^n_A(A_f, A) = 0$ (resp. $\mathrm{Ext}^n_A(A_f, M) = 0$) for all n and all $f \in I$.

We can now define prisms.

Definition 4.5 (Prism)**.** A prism (A, I) consists of a δ-ring (A, δ) and an invertible ideal $I \subseteq A$ such that A is derived (p, I)-adically complete and $p \in (I, \widetilde{\varphi}_A(I))$.

There is a category Prism with objects prisms and morphisms $f : (A, I) \to (B, J)$ given by a morphism of δ-rings $f : A \to B$ such that $f(I) \subseteq J$.

Under additional conditions on A and I, we can define additional types of prisms as follows.

Definition 4.6 (Perfect Prism)**.** A prism (A, I) is perfect if $\widetilde{\varphi}_A : A \to A$ is an automorphism of A.

Definition 4.7 (Oriented Prism)**.** A prism (A, I) is oriented if I is a principal ideal.

Remark 4.8. In this situation, a choice of generator of I gives an orientation for the prism.

Definition 4.9 (Bounded Prism)**.** A prism (A, I) is bounded if A/I has bounded p^∞-torsion: there exists $n \in \mathbb{N}$ such that $(A/I)[p^n] = (A/I)[p^\infty]$.

Definition 4.10 (Crystalline Prism)**.** A prism (A, I) is crystalline if $I = (p)$.

Perfect prisms are the best behaved, and perfection in fact implies orientability and boundedness.

Proposition 4.11 ([Ked21, Thm. 7.2.2])**.** Let (A, I) be a perfect prism. Then:

(i) (A, I) is an oriented prism and each orientation of (A, I) is given by a distinguished element.
(ii) (A, I) is a bounded prism.

Moreover, to any prism (A, I), we can canonically produce a perfect prism in analogy to the colimit perfection of Definition 2.2.

Definition 4.12 (Perfection of a Prism)**.** Let (A, I) be a prism. The perfection of (A, I) is a perfect prism $(A, I)_{\mathsf{perf}}$ such that all morphisms of prisms $(B, J) \to (A, I)$ with (B, J) perfect, there exists a unique factorization of the map through $(A, I)_{\mathsf{perf}}$.

By Yoneda's lemma, $(A, I)_{\mathsf{perf}}$ is determined uniquely up to unique isomorphism. The construction can also be made explicit as follows.

Proposition 4.13 ([Ked21, Prop. 7.2.3])**.** Let (A, I) be a prism and A_{perf} the colimit perfection of the δ-ring A. Then:

(i) The derived (p, I)-adic completion of A_{perf} agrees with the classical (p, I)-adic completion of A_{perf}.
(ii) For A_∞ the derived (p, I)-adic completion of A_{perf}, (A_∞, IA_∞) is the perfection $(A, I)_{\mathsf{perf}}$ of (A, I).

Perfect prisms are closely related to the theory of perfectoid rings which play a key role in contemporary arithmetic geometry. We recall the following definition.

Definition 4.14 (Integral Perfectoid Ring)**.** Let R be a commutative ring. R is an integral perfectoid ring if $R \cong A/I$ for some perfect prism (A, I).

We can show that the functor $(A, I) \mapsto A/I$ is fully faithful on perfect prisms where in conjunction with Definition 4.14, we can deduce the following.

Proposition 4.15 ([Ked21, Thm. 7.3.5])**.** The following categories are equivalent:

- The category of perfect prisms.
- The category of integral perfectoid rings.

Remark 4.16. The proof here does not necessitate an explicit description of the inverse functor due to the description of the essential image of $(A, I) \mapsto A/I$ as integral perfectoid rings Definition 4.14.

Alternatively, integral perfectoid rings can be characterized as follows.

Proposition 4.17 ([Ked21, Prop. 8.2.5])**.** Let R be a commutative ring. R is an integral perfectoid ring if and only if all of the following conditions hold:

(i) R is classically p-adically complete.
(ii) $\varphi_R : R/(p) \to R/(p)$ is surjective.
(iii) The kernel of $\mathbb{A}_{\mathrm{inf}}(R) \to R$ is principal.
(iv) There exists $\varpi \in R$ such that $\varpi^p = pu$ for some $u \in R^\times$.

We can provide a more explicit characterization of the functors involved in Proposition 4.15 via the tilting and untilting constructions.

Definition 4.18 (Tilt)**.** Let R be an integral perfectoid ring. The tilt $R^\flat$ of R is the limit perfection $(R/(p))^{\mathsf{perf}}$ of the quotient $R/(p)$.

Remark 4.19. Tilts can also be defined on prisms (A, I) by taking $(\overline{A}/(p))^{\mathsf{perf}}$ of $\overline{A}/(p)$ where $\overline{A} = A/I$. If (A, I) is a perfect prism, this tilt agrees with the tilt of the corresponding integral perfectoid ring A/I.

Definition 4.20 (Untilt)**.** Let A be a perfect $\mathbb{F}_p$-algebra. An untilt of A is an integral perfectoid ring R such that $R^\flat \cong A$.

The Witt vector construction recovers this property.

Proposition 4.21 ([Ked21, Prop. 7.3.3])**.** Let A be a perfect $\mathbb{F}_p$-algebra. Then $\mathrm{W}(A)$ is such that $\mathrm{W}(A)/(p) \cong A$

We can extend the untilting construction by taking the prism

$$(\mathrm{W}(R^\flat), \ker(\mathrm{W}(R^\flat) \to R))$$

which describes the inverse functor of Proposition 4.15 for an integral perfectoid ring R.

Furthermore, tilting induces an equivalence between perfectoid algebras over a perfectoid ring and perfectoid algebras over its tilt.

Proposition 4.22 ([Mor17, Thm. 1.7])**.** Let R be an integral perfectoid ring. The tilting construction results in an equivalence between the following categories:

- The category of integral perfectoid R-algebras.
- The category of integral perfectoid $R^\flat$-algebras.

Having developed the necessary language of prisms, prismatic cohomology can be defined via the prismatic site.

5. Prismatic Cohomology

Prismatic cohomology will be defined on the prismatic site associated to a formal scheme. For preliminaries on the latter, consult [Stacks, Tag 0AHY]. The key result of the theory of formal schemes is that an $\mathbb{F}_p$-scheme X, the formal completion functor produces a p-adic formal scheme isomorphic to X allowing for cohomology computations to be done in the category of p-adic formal schemes.

Covers in the prismatic site will be given by (p, I)-completely faithfully flat prisms over the base as we now define.

Definition 5.1 (I-Completely Faithfully Flat Morphism)**.** Let A be a commutative ring and $I \subseteq A$ an ideal. An A-module M is I-completely faithfully flat if:

(i) For all I-torsion modules N, $M \otimes_A^L N$ is a complex concentrated in degree 0 in $D(A)$.
(ii) $M \otimes_A^L (A/I)$ is a complex concentrated in degree 0 in $D(A)$ and faithfully flat as an A/I-module.

Given a X smooth p-adic formal scheme over $\mathrm{Spf}(A/I)$ with (A, I) a bounded prism, we can construct the prismatic site $(X/A)_{\mathbb{\Delta}}$ as follows.

Definition 5.2 (Prismatic Site)**.** Let X be a smooth p-adic formal scheme over $\mathrm{Spf}(A/I)$ for some bounded prism (A, I). The prismatic site $(X/A)_{\mathbb{\Delta}}$ is given by:

- Objects $\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X$ where $(A, I) \to (B, IB)$ is a map of bounded prisms and $\mathrm{Spf}(B/IB) \to X$ a map of p-adic formal schemes over $\mathrm{Spf}(A/I)$.
- Morphisms $(\mathrm{Spf}(C) \to \mathrm{Spf}(C/IC) \to X) \to (\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X)$ where $(B, IB) \to (C, IC)$ is a map of bounded prisms under (A, I) and $\mathrm{Spf}(C/IC) \to \mathrm{Spf}(B/IB)$ is a map of formal schemes over X.
- Covers $(\mathrm{Spf}(C) \to \mathrm{Spf}(C/IC) \to X) \to (\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X)$ where $(B, IB) \to (C, IC)$ is a morphism of prisms such that C is a (p, IB)-completely faithfully flat B-module.

Remark 5.3. If $X = \mathrm{Spf}(R)$ then we denote the prismatic site $(R/A)_{\mathbb{\Delta}}$.

We can define presheaves $\mathcal{O}_{\mathbb{\Delta}}$ and $\overline{\mathcal{O}}_{\mathbb{\Delta}}$ by

$$\begin{aligned} \mathcal{O}_{\mathbb{\Delta}}(\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X) &= B \\ \overline{\mathcal{O}}_{\mathbb{\Delta}}(\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X) &= B/IB \end{aligned} \tag{5.1}$$

which are in fact sheaves.

Proposition 5.4 ([BS22, Corr. 3.12])**.** Let X be a p-adic formal scheme over $\mathrm{Spf}(A/I)$ for some bounded prism (A, I). The presheaves $\mathcal{O}_{\mathbb{\Delta}}$ and $\overline{\mathcal{O}}_{\mathbb{\Delta}}$ on $(X/A)_{\mathbb{\Delta}}$ in (5.1) are sheaves.

These sheaves are known as the structure sheaf and reduced structure sheaf on the prismatic site.

Definition 5.5 (Structure Sheaves on the Prismatic Site)**.** Let X be a p-adic formal scheme over $\mathrm{Spf}(A/I)$ for some bounded prism (A, I). The sheaf $\mathcal{O}_{\mathbb{\Delta}}$ (resp. $\overline{\mathcal{O}}_{\mathbb{\Delta}}$) is the structure sheaf (resp. reduced structure sheaf) on the prismatic site.

Prismatic cohomology of a p-adic formal scheme is taken to be the derived functor cohomology on the prismatic site.

Definition 5.6 (Prismatic Cohomology)**.** Let X be a smooth formal scheme over $\mathrm{Spf}(A/I)$ for some bounded prism (A, I). The prismatic cohomology of X/A is

$$R\Gamma_{\Delta}(X/A) = R\Gamma((X/A)_{\Delta}, \mathcal{O}_{\Delta}) \in D(A).$$

Remark 5.7. We define the analogous ringed site with reduced structure sheaf $\overline{\mathcal{O}}_{\Delta}$ the reduced prismatic site, which we denote $(X/A)_{\overline{\Delta}}$.

Remark 5.8. This complex of A-modules is equipped with the $\widetilde{\varphi}_A$-linear endomorphism arising from the Frobenius action on $\mathcal{O}_{\Delta}$.

6. Comparisons

We first state the following affinoid base change theorem that plays a key role in the various comparison theorems.

Theorem 6.1 (Affinoid Base Change; [Ked21, Lem. 15.1.3])**.** Let R be a p-completely smooth A/I-algebra for some bounded prism (A, I) and $(A, I) \to (B, IB)$ a map of bounded prisms of finite (p, I)-complete Tor amplitude. For $\widetilde{R} = R\widehat{\otimes}_A^L B$ there is an equivalence of sites $(\widetilde{R}/B)_{\Delta} \cong (R/A)_{\Delta}\widehat{\otimes}_A^L B$ (resp. $(\widetilde{R}/B)_{\overline{\Delta}} = (R/A)_{\overline{\Delta}}\widehat{\otimes}_A^L \widetilde{R}$).

6.1. **Crystalline Comparison.** Let (A, I) be a crystalline prism as in Definition 4.10 so $I = (p)$. Further preliminearies on crystalline cohomology are given in Appendix A.1.

By the construction of crystalline cohomology, it is reasonable to expect that there is a comparison map with prismatic cohomology. The precise relationship is given as follows.

Theorem 6.2 (Affinoid Crystalline Comparison)**.** Let (A, I) be a crystalline prism, J a divided power ideal of A, and $\psi : A/J \to A/I$ the morphism induced by the Frobenius $\varphi_A : A/I \to A/I$. If B is a smooth A/J-algebra then there is a canonical isomorphism $\Delta_{(B\otimes_{A/I}\psi(A/I))/A} \cong R\Gamma_{\mathsf{Crys}}(B/A)$.

By gluing we yield the following result on formal schemes.

Theorem 6.3 (Global Crystalline Comparison)**.** If X is a smooth p-adic formal scheme over a crystalline prism (A, I) then there is a canonical $\widetilde{\varphi}_A$-equivariant isomorphism $R\Gamma_{\Delta}(X/A)\widehat{\otimes}_A^L\widetilde{\varphi}_A(A) \to R\Gamma_{\mathsf{Crys}}(X/A)$.

6.2. **Algebraic de Rham Cohomology.** Recall that the algebraic de Rham complex is defined as the complex by taking successive differentials of a ring map

$$\mathcal{O}_X \xrightarrow{\quad d \quad} \Omega^1_{X/S} \xrightarrow{\quad d \quad} \Omega^2_{X/S} \xrightarrow{\quad d \quad} \dots. \tag{6.1}$$

It can be shown that the following holds.

Theorem 6.4 ([Ked21, Thm. 15.4.3])**.** Let (A, I) be a bounded prism such that $\mathrm{W}(A/I)$ is p-torsion free. If B is a p-completely smooth A/I-algebra, there is a natural isomorphism $\Delta_{B/A}\widehat{\otimes}_A^L(A/I) \cong \Omega^\bullet_{B/(A/I)}$.

In the setting of Hodge-Tate cohomology where the maps between sheaves in (6.1) are given by the trivial maps, the comparison results can be extended as we now discuss.

6.3. **Hodge-Tate Cohomology.** We make the following preliminar constructions.

Definition 6.5 (Breuil-Kisin Twist)**.** Let (A, I) be a prism, B an A/I-algebra, and M an A/I-module. The Breuil-Kisin twist $M\{n\}$ of M is defined to be $M \otimes_{A/I} (I/I^2)^{\otimes n}$.

Breuil-Kisin twists give rise to the Bockstein differential.

Definition 6.6 (Bockstein Differential)**.** Let (A, I) be a prism and B an A/I-algebra. The Bockstein differential is defined to be the boundary map $H^i(\overline{\Delta}_{B/A})\{i\} \to H^{i+1}(\overline{\Delta}_{B/(A/I)})\{i+1\}$ induced by the complex

$$0 \to \overline{\mathcal{O}}_{\Delta_{B/A}}\{i+1\} \to \mathcal{O}_{\Delta_{B/(A/I)}} \otimes_A (I^i/I^{i+2}) \to \overline{\mathcal{O}}_{\Delta_{B/A}}\{i\} \to 0.$$

By unwinding the definitions, one can see that $\bigoplus_{n\geq 0} H^n(\overline{\Delta}_{B/A})\{n\}$ is a commutative differential graded algebra over A/I. By the universal property of the de Rham complex, there is a canonical morphism $\Omega^\bullet_{B/(A/I)} \to H^\bullet(\overline{\Delta}_{B/A})\{\bullet\}$ which can be shown to be an isomorphism in both the local and global settings.

Theorem 6.7 (Hodge-Tate Comparison)**.** If X is a smooth p-adic formal scheme over a prism (A, I) then $\Omega^\bullet_{X/(A/I)} \to H^\bullet(\overline{\Delta}_{X/A})\{\bullet\}$ is an isomorphism.

6.4. **Étale Comparison.** Prismatic cohomology also admits a comparison map to étale cohomology. We state the result which is as follows.

Theorem 6.8 (Étale Comparison)**.** If X is a smooth p-adic formal scheme over a perfect prism (A, I) with generic fiber X_η there is a canonical isomorphism

$$\mathrm{fib}\left((R\Gamma_{\Delta}(X/A)/p^n)\left[\tfrac{1}{I}\right] \xrightarrow{\widetilde{\varphi}_A - 1} (R\Gamma_{\Delta}(X/A)/p^n)\left[\tfrac{1}{I}\right] \right) \cong R\Gamma_{\text{ét}}(X_\eta, \mathbb{Z}/p^n\mathbb{Z}).$$

End Matter

Appendix A. Cohomology of Arithmetic Schemes

In this section, we recall the definitions and constructions of cohomology theories on arithmetic schemes.

A.1. **Crystalline Cohomology.**

Crystalline cohomology seeks to produce a cohomology theory for schemes X over a perfect field k of characteristic p by lifting X to a smooth proper scheme $\widetilde{X}$ over the Witt vectors – that is, $X = \widetilde{X} \times_{\mathrm{Spec}(\mathrm{W}(k))} \mathrm{Spec}(k)$ – and computing the algebraic de Rham cohomology $R\Gamma\left(\widetilde{X}, \Omega^{\bullet}_{\widetilde{X},\mathrm{W}(k)}\right)$.

The definition of crystalline cohomology proceeds via divided power rings as we now define.

Definition A.1 (Divided Power Structure)**.** Let A be a commutative ring and $I \subseteq A$ an ideal. A divided power structure on I is a collection of $\mathbb{Z}_{\geq 0}$-indexed maps $\{\gamma_n : I \to I\}_{n \in \mathbb{Z}_{\geq 0}}$ such that for all $m \in \mathbb{N}$ and all $x, y \in I$:

(i) $\gamma_0(x) = 1$.
(ii) $\gamma_1(x) = x$.
(iii) $\gamma_n(x) \cdot \gamma_m(x) = \frac{(n+m)!}{n! \cdot m!} \gamma_{n+m}(x)$.
(iv) $\gamma_n(ax) = a^n \gamma_n(x)$ for all $a \in A$.
(v) $\gamma_n(x+y) = \sum_{i=0}^{n} \gamma_i(x) \gamma_{n-i}(y)$.
(vi) $\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n} \gamma_{mn}(x)$.

Divided power rings are triples (A, I, γ) where γ defines a divided power structure on I.

Definition A.2 (Divided Power Ring)**.** A divided power ring is a triple (A, I, γ) where A is a commutative ring, $I \subseteq A$ an ideal, and γ a divided power structure on I.

Remark A.3 ([Ked21, Rmk 14.2.5])**.** We will at times use the shorthand PD for "divided powers," following the French "puissances diviées."

A morphism of divided power rings $f : (A, I, \gamma_A) \to (B, J, \gamma_B)$ is a morphism of rings $f : A \to B$ such that $f(I) \subseteq J$ and $\gamma_{B,m}(f(x)) = f(\gamma_{A,m}(x))$ for all $x \in I$ and $m \in \mathbb{N}$.

Specializing to $\mathbb{Z}_{(p)}$-algebras, we have the following result.

Lemma A.4 ([Stacks, Tag 07GN])**.** Let A be a a $\mathbb{Z}_{(p)}$-algebra. The ideal $(p) \subseteq A$ hsa a canonical divivided power structure where for $x = pa \in I$, $\gamma_n(x) = \frac{p^n}{n!} \cdot a^n$.

We are most concerned with the setting of p-complete p-torsion free $\mathbb{Z}_p$-algebras as obtained by completion of some $\mathbb{Z}_{(p)}$-algebra. We deduce the following as a corollary.

Corollary A.5. If A is a p-complete p-torsion free $\mathbb{Z}_p$-algebra then A admits a canonical divided power structure.

We can now define the (big) crystalline site as follows.

Definition A.6 (Crystalline Site)**.** Let A be a $\mathbb{Z}_p$-algebra and X a scheme over $A/(p)$. The crystalline site $(X/A)_{\mathsf{Crys}}$ is given by:

- Objects (B, I, γ_B, f_B) where (B, I, γ_B) is a divided power algebras over $(A, (p), \gamma_A)$ with B p-complete as an A-module and $f_B : \operatorname{Spec}(B/J) \to X$ is a morphism of schemes over $\operatorname{Spec}(A/(p))$.
- Morphisms $(B, I, \gamma_B, f_B) \to (C, J, \gamma_C, f_C)$ where $(C, J, \gamma_C) \to (B, I, \gamma_B)$ is a morphism of divided power rings inducing a morphism $\operatorname{Spec}(B/I) \to \operatorname{Spec}(C/J)$ over X.
- Covers morphisms $(B, I, \gamma_B, f_B) \to (C, J, \gamma_C, f_C)$ where $(C, J, \gamma_C) \to (B, I, \gamma_B)$ is p-completely faithfully flat morphism of divided power algebras over $(A, (p), \gamma_A)$ as in Definition 5.1 and $I = JC$.

One can naturally associate a presheaf on the crystalline site

$$\mathcal{O}_{\mathsf{Crys}}\left((B, I, \gamma_B, f_B)\right) \mapsto B \tag{A.1}$$

which is in fact a sheaf.

Proposition A.7 ([Stacks, Tag 07I5])**.** Let A be a $\mathbb{Z}_p$-algebra and X a scheme over $A/(p)$. The presheaf of rings $\mathcal{O}_{\mathsf{Crys}}$ as in (A.1) is a sheaf.

Crystalline cohomology computes cohomology of the site with respect to this structure sheaf.

Definition A.8 (Crystalline Cohomology)**.** Let A be a $\mathbb{Z}_p$-algebra and X a scheme over $A/(p)$. The crystalline cohomology of X/A is

$$R\Gamma_{\mathsf{Crys}}(X/A) = R\Gamma\left((X/A)_{\mathsf{Crys}}, \mathcal{O}_{\mathsf{Crys}}\right).$$

A.2. **Algebraic de Rham Cohomology.** Recall the following algebraic preliminary.

Definition A.9 (Differential Graded Algebra)**.** Let R be a commutative ring. A differential graded algebra over R is a cochain complex of R-modules

$$\dots \xrightarrow{d^{n-2}} E^{n-1} \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} E^{n+1} \xrightarrow{d^{n+1}} \dots$$

with R-bilinear maps $E^n \times E^m \to E^{n+m}$ by $(a, b) \mapsto ab$ such that $d^{n+m}(ab) = d^n(a) \cdot b + (-1)^n d^m(b) \cdot a$ and $\bigoplus_n E^n$ is an associative unital R-algebra.

Further objects of this type are defined as follows.

Definition A.10 (Commutative DGA)**.** Let R be a commutative ring and $E^\bullet$ a differential graded algebra over R. $E^\bullet$ is a commutative differential graded algebra if $ab = (-1)^{\deg(a)\deg(b)} ba$.

Definition A.11 (Strictly Commutative DGA)**.** Let R be a commutative ring and $E^\bullet$ a differential graded algebra over R. $E^\bullet$ is a strictly commutative differential graded algebra if it is commutative and $a^2 = 0$ if $\deg(a) \equiv 1 \pmod 2$.

The algebraic de Rham complex can be shown to be the universal strictly commutative differential graded algebra in the following sense.

Proposition A.12 ([Ked21, Lem 12.2.3])**.** If $(E^\bullet, d)$ is a strictly commutative differential graded algebra and $\eta : B \to E^0$ is a map of A-algebras then η extends uniquely to a map $\Omega^\bullet_{B/A} \to E^\bullet$ of differential graded algebras.

References

[Ans23] Johannes Anschütz. *Introduction to Prismatic Cohomology.* 2023. URL: https://youtube.com/playlist?list=PLx5f8IelFRgFu9g5CMEUI6PjsLrN_Nhxy&si=URo38WNGvq3KTk37.

[BS22] Bhargav Bhatt and Peter Scholze. "Prisms and prismatic cohomology". English. In: *Ann. Math. (2)* 196.3 (2022), pp. 1135–1275. ISSN: 0003-486X. DOI: 10.4007/annals.2022.196.3.5.

[Bos24] Guido Bosco. *V5A4 - Prismatic Cohomology.* Notes Availible on eCampus. 2024.

[Ked21] Kiran Kedlaya. *Notes on Prismatic Cohomology.* 2021. URL: https://kskedlaya.org/prismatic/frontmatter-1.html.

[Lep22] Florian Leptien. *Perfect Closures of Rings and Schemes.* 2022. URL: https://www.esaga.uni-due.de/f/jan.kohlhaase/Leptien_Bachelorarbeit.pdf.

[Mor17] Matthew Morrow. *Foundations of Perfectoid Spaces.* Notes for the Fargues-Fontaine Curve Seminar. 2017. URL: https://www.math.ias.edu/~lurie/ffcurve/Lecture6-8-Perfectoid.pdf.

[Stacks] The Stacks project authors. *The Stacks project.* https://stacks.math.columbia.edu. 2024.

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