Second Assignment: Linear, Integer, and Mixed-Integer Programming

Instructions

- Answer the conceptual questions for each section first.
- Then proceed to the modeling exercises.
- Clearly define decision variables and write constraints in proper linear, integer, or mixed-integer form.
- Use binary variables and Big-M technique when required.
- Focus on correct formulation; solving is not required.

1 Linear Programming Basics

1.1 Conceptual Questions

1. Explain what a linear programming (LP) problem is.
2. What are decision variables, objective function, and constraints in LP (I know it is repeated multiple times during the previous presentation)?
3. Why is it important to write all constraints in linear form for LP solvers (a solver for LP is a tool that takes the linear program of a problem and solve it, for now we do not yet care about the solution methods of linear programs)?
4. What is an auxiliary variable and when do we use it?

1.2 Modeling Exercise

A factory produces three products, P_1, P_2, P_3. Each has a profit per unit of \$40, \$30, \$50, respectively. To produce each of the products some resources are needed which is given in the following table:

	P_1	P_2	P_3
Labor (hrs/unit)	2	1	3
Material (units/unit)	3	2	4

Moreover, there is a limit for each of the resources; labor (150 hours) and material (200 units).

1. Define decision variables for the production quantities and explain their meaning.
2. Write the objective function and justify your formula.
3. Write the resource constraints in linear form and explain each constraint in plain text, i.e. what is it for, and how it works.

2 Integer and Binary Variables with Logical Constraints

2.1 Conceptual Questions

1. What is an integer programming (IP) problem and how does it differ from LP?
2. What is a binary variable and when do we use it?
3. Give an example of a real-world situation requiring integer variables.
4. Explain how logical conditions ("if-then", "either-or", "at least/at most r variables equal 1") can be modeled using linear inequalities with binary variables.

2.2 Logical Constraints Exercises

Let $x_1, x_2, \ldots, x_n$ be binary decision variables. For each of the following cases; a. write a logical constraint (where possible), and then b. its equivalent linear constraint:

1. Two variables x_1, x_2: If $x_1 = 0$, then $x_2 = 1$.
2. Two variables x_1, x_2: Either $x_1 = 1$ or $x_2 = 1$ (at least one is 1).
3. Two variables x_1, x_2: Exactly one of x_1 and x_2 equals 1.
4. Two variables x_1, x_2: Both x_1 and x_2 must be 1.
5. Three variables x_1, x_2, x_3: If $x_1 = 0$, then at least two of x_2 and x_3 must be 1.
6. n variables $x_1, \ldots, x_n$: At most one variable can be 1.
7. n variables $x_1, \ldots, x_n$: At most r variables can be 1.
8. n variables $x_1, \ldots, x_n$: At least r variables must be 1.
9. n variables $x_1, \ldots, x_n$: If $x_1 = 0$, then at least r of $x_2, \ldots, x_n$ must be 1.
10. n variables $x_1, \ldots, x_n$: If $x_1 = 1$, then at most r of $x_2, \ldots, x_n$ can be 1.

2.3 Modeling Exercise: Store Opening with Logical Constraints

A company can open up to 4 stores from 6 locations ($L_1, \ldots, L_6$). However, there are some rules that must be obeyed: - L_1 and L_2 cannot both open. - L_3 and L_4 must open together if either is opened. - Each open store must have at least 3 staff and at most 10 staff.

1. Define binary decision variables for store opening and integer variables for staff allocation.
2. Write linear constraints to enforce the logical rules and staff limits.
3. Include an objective function minimizing total fixed + staff costs.

3 Big-M and Conditional Constraints (MILP)

3.1 Conceptual Questions

1. What is the Big-M method in MILP?
2. Where and why is it used?
3. Give a simple illustrative example of Big-M usage.

3.2 Modeling Exercise

A machine can be turned on/off. If on, minimum production = 20 units; maximum = 100 units; if off, production = 0. Production cost = \$5/unit, fixed cost \$100.

1. Define a binary variable for machine operation and a continuous variable for production.
2. Formulate the conditional production constraint using Big-M.
3. Write the total cost objective function.

4 Linearizing Min/Max and Piecewise Functions

4.1 Conceptual Questions

1. Why is it important to linearize min and max operators in MILP?
2. How can an auxiliary variable be used to represent min or max in a linear model?
3. Explain how piecewise-linear cost functions can be handled in MILP.

4.2 Modeling Exercise

Two projects, A and B, require budget allocations x_A and x_B. Let $z = \min(x_A, x_B)$. Additionally, suppose each project has a bonus if the budget exceeds a threshold, modeled as a piecewise linear cost.

1. Introduce an auxiliary variable z and linearize the min operator.

2. Formulate linear constraints to model a simple piecewise bonus:

$$\text{bonus}_A = \begin{cases} 0 & x_A \leq 50 \\ 10 & x_A > 50 \end{cases}$$

3. Repeat for project B.

5 A MILP Example

5.1 Problem Description

A factory produces **3 products** (P_1, P_2, P_3) using **2 machines** (M_1, M_2). Each product can only be produced on one machine, and each machine has a minimum and maximum production capacity. The factory must allocate labor carefully because total available labor is limited. The goal is to **maximize profit** while respecting machine and labor constraints.

Product Data

Product	$ Profit/unit	Labor (hrs/unit)	Material (units/unit)	Min product	Max product
P_1	50	2	3	10	50
P_2	40	1	2	20	60
P_3	60	3	4	15	40

Machine Rules

- Each machine can be turned **on or off** (binary variable).
- If a machine is on, a **setup time of 10 hours** is required before production.
- Each product can only be assigned to **one machine**.
- Total labor (setup + production) $\leq$ 200 hours.

Tasks

1. Define decision variables:
   - Binary variables for machine on/off and product assignment.
   - Integer variables for units produced for each product.
2. Write linear constraints for:
   - Each product assigned to exactly one machine.
   - Minimum and maximum production conditional on machine being on (use Big-M).
   - Total labor constraint including setup time.
3. Write the objective function for **total profit**, including fixed setup costs for machines.

Hints

- Use Big-M to link production quantities to machine on/off status: if a machine is off, production = 0; if on, production $\geq$ minimum and $\leq$ maximum.
- Assignment variables ensure that each product is produced on only one machine.
- Total labor = setup hours of machines used + sum of production hours for all products.