Multivariate Monte Carlo

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Problem

Given a 2×2 standardized variance-covariance matrix, also known as a correlation matrix, R, where ρ is the coefficient of correlation,

$$R = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

We start with uncorrelated variates x. Our job is to transform the uncorrelated variates to produce correlated variates z with the same expected variance-covariance matrix as R. This will put informational structure into the otherwise independently occurring x.

When we say transform we mean this mathematically.

$$z = Lx$$

such that

$$R = LL^T$$

Solution

We can decompose R into the product of upper and lower triangular symmetrical matrices. This is a standard trick of matrix algebra.

$$R = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \begin{bmatrix} \ell_{11} & 0 \\ \ell_{21} & \ell_{22} \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{21} \\ 0 & \ell_{22} \end{bmatrix}$$

When we multiply the upper and lower triangular matrices we get this interesting matrix.

$$R = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \begin{bmatrix} \ell_{11}^2 & \ell_{11}\ell_{21} \\ \ell_{11}\ell_{21} & \ell_{22}^2 \end{bmatrix}$$

Now it is a matter of matching elements of R and our new matrix. Here we go.

- $\ell_{11}^2 = 1$ implies $\ell_{11} = 1$
- $\ell_{11}\ell_{21} = \rho$ then implies $\ell_{21} = \rho$
- $\ell_{11}^2 + \ell_{22}^2 = 1$ then implies that $\ell_{22} = \sqrt{1 - \rho^2}$

That wasn't as back as we might have thought when we started. We then let L be the new matched to R matrix.

$$L = \begin{bmatrix} \ell_{11} & 0 \\ \ell_{21} & \ell_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix}$$

Now we can simulate

We generate correlated $z = Lx$ building on the uncorrelated x. We first generate a random x_1 and, independently, a random x_2. We can use the `=RAND()` function in Excel to perform this task.

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then we generate a z_1 and a z_2 using the x vector of random numbers, but transformed by pre-multiplying x with L.

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We do remember that By definition, if x_1 is not correlated with x_2, then $\rho_{12} = 0$. We can check our maths with this calculation.

$$xx^T = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1 x_2 \\ x_1 x_2 & x_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The multiplication of a column vector of independently drawn x with its transpose, the row vector of thos same x random numbers will always return the the identity matrix I. This shows that variates are perfectly correlated with themselves but not each other.

Back to the main business at hand. We now calculate

$$zz^T = (Lx)(Lx)^T = Lxx^T L^T = LIL^T = LL^T$$

But, $R = LL^T$ so that $R = zz^T$.

Thus we have sketched out these steps.

- Generate uncorrelated x.
- Generate $z = Lx$, where L reflects the desired correlation structure.
- Find that L generates the same correlation structure as the correlations in z.