PROGRAMMING OF INTERDEPENDENT ACTIVITIES II MATHEMATICAL MODEL[1]

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Activities (or production processes) are considered as building blocks out of which a technology is constructed. Postulates are developed by which activities may be combined. The main part of the paper is concerned with the discrete type model and the use of a linear maximization function for finding the "optimum" program. The mathematical problem associated with this approach is developed first in general notation and then in terms of a dynamic system of equations expressed in matrix notation. Typical problems from the fields of inter-industry relations, transportation, nutrition, warehouse storage, and air transport are given in the last section.

INTRODUCTION

The multitude of activities in which a large organization or a nation engages can be viewed not only as fixed objects but as representative building blocks of different kinds that might be recombined in varying amounts to form new blocks. If a structure can be reared of these blocks that is mutually self-supporting, the resulting edifice can be thought of as a technology. Usually the very elementary blocks have a wide variety of forms and quite irregular characteristics over time. Often they are combined with other blocks so that they will have "nicer" characteristics when used to build a complete system. Thus the science of programming, if it may be called a science, is concerned with the adjustment of the levels of a set of given activities (production processes) so that they remain mutually consistent and satisfy certain optimum properties.

It is highly desirable to have formal rules by which activities can be combined to form composite activities and an economy. These rules are set forth here as a set of postulates regarding reality. Naturally other postulates are possible; those selected have been chosen with a wide class of applications in mind and with regard to the limitations of present day computational techniques. The reader's attention is drawn to the last section of this report where a number of applications of the mathematical model are discussed. These are believed to be of sufficient interest in themselves, and may lend concreteness to the development which follows:

POSTULATES OF A LINEAR TECHNOLOGY

Postulate I: *There exists a set* $\{A\}$ *of activities.*

Postulate II: *All activities take place within a time span* 0 *to* t_0.

[1] A revision of a paper presented before the Madison Meeting of the Econometric Society on September 9, 1948. This is the second of two papers on this subject, both appearing in this issue. The first paper, with sub-title "General Discussion," will be referred to by Roman numeral I.

POSTULATE III: *There exists a finite set of commodity types, denoted by the subscript i,* $(i = 1, \cdots, m)$.

POSTULATE IV: *Each activity has a characteristic flow of commodities; the cumulative quantity of flow*[2] *of the i-th commodity up to time t is denoted by the function*

$$C_i(t \mid A), \qquad (i = 1, \cdots, m). \tag{1}$$

It will be noted that Postulate IV is given in terms of *cumulative* quantity of flow. This is convenient in the same sense that it is convenient to use the cumulative distribution function in probability theory. Thus at a time t there may be a discrete quantity of a commodity required by (or produced by) an activity or it may occur continuously over time. Either of these two cases is taken care of by the cumulative flow functions.

POSTULATE V: *The sum of any two activities, denoted by* $A_1 + A_2$, *is an activity whose flow function (for each commodity type) is the sum of the corresponding flow functions of* A_1 *and* A_2 :

$$C_i(t \mid A_1 + A_2) = C_i(t \mid A_1) + C_i(t \mid A_2), \qquad (i = 1, \cdots, m). \tag{2}$$

POSTULATE VI: *Any subset of activities,* $(A_1, A_2, \cdots, A_n)$, *constitutes a possible self-supporting system with respect to* A_0 *if the sum of the flow coefficients of* $(A_0; A_1, \cdots, A_n)$ *for each commodity type vanishes:*

$$\sum_{j=0}^{n} C_i(t \mid A_j) = 0, \qquad (i = 1, 2, \cdots, m), \tag{3}$$

where A_0 *(called the exogenous activity) is not necessarily an element of* $\{A\}$.

Scalar Multiplication of Activities: It should be noted that as far as any physical interpretations of the abstract model are concerned, activities are treated as though they have no common parts with each other. This forms the basis for Postulate V and permits the addition of corresponding flow functions of two activities. No distinction is made as to whether two activities are of the same or of different types. When $A_1 = A_2$ in (2) the concept of scalar multiplication of activities follows naturally. In general for any A there exists an activity, denoted by $x \cdot A$, whose flow functions are any integral multiple $x = 1, 2, 3, \cdots$, of the corresponding flow functions for A.

It is convenient mathematically to let x take on a continuous range of values,—in particular to permit infinite subdivision of an activity A.

[2] This paper will use the signs + and − to indicate *in* and *out*. Thus plus indicates the flow is in or towards the activity; while the sign minus indicates it is out or away from the activity. In equilibrium models particularly, where there are many inputs to one output, this convention will result in equations with coefficients whose signs are nearly all positive.

In actual practice, however, this is far from possible. Mass production activities often use, for example, special tools that cannot be constructed below a certain size. Accordingly, since an assumption of divisibility is made in Postulate VII, one must take care in real situations to discover significant indivisibilities and to make necessary adjustments in the results.

POSTULATE VII: *For all $x \geqslant 0$ and any A there exists an activity denoted by $x \cdot A$ whose flow function is given by:*

$$C_i(t \mid x \cdot A) = xC_i(t \mid A), \qquad (i = 1, \cdots, m). \tag{4}$$

The Null Activity: Setting $x = 0$, an activity is obtained whose flow functions vanish. This will be called the null activity and denoted by:

$$0 \cdot A = 0. \tag{5}$$

Exogenous Activity A_0: It is also convenient to single out one activity to express initial conditions at time $t = 0$ and flow from outside the system during the time span $0 \leqslant t \leqslant t_0$. There does not appear to be a meaningful interpretation of a scalar multiple of this activity. It is however desirable as in VI to combine A_0 with other activities so that their sum is the null activity.

POSTULATE VIII: *There exists a finite set of activities $A_1, A_2, \cdots, A_m$ such that any $A \neq A_0$ can be expressed as a nonnegative combination of this basic set of activities:*

$$A = y_1A_1 + y_2A_2 + \cdots + y_mA_m \tag{6}$$

where $y_j \geqslant 0$.

The existence of a finite basis implies that a nonvacuous linear economy (Postulate VI) is possible only if the flow functions are of a very special form relative to one another. Of special interest in this connection is the discrete model which we will discuss later. Recently W. W. Leontief and David Hawkins, [**8**], [**3**], have considered models which permit the continuous formation of capital. Such models satisfy Postulates I through VII, but fail to satisfy VIII in a rather unexpected way. Thus each A (entering into a solution of the model) can be shown to be representable in terms of a finite basis as in VIII. It is not possible however in this case to find a basis which requires only nonnegative weights.

LINEAR PROGRAM

It is now possible to express VI in terms of the basic set of activities. Thus a linear technology can be expressed as a set of activities satisfying

$$A_0 + x_1A_1 + x_2A_2 + \cdots + x_nA_n = 0 \tag{7}$$

where $x_j \geqslant 0$. Since there is a one to one correspondence between the addition and multiplication of activities and the corresponding operations on the vector function of A given in (8), we can identify A with this vector function of time and use the same symbol:

$$A = \{C_1(t \mid A), C_2(t \mid A), \cdots, C_m(t \mid A)\}. \tag{8}$$

Hence (7) implies a simultaneous system of linear equations covering m commodity types and a continuous range of values of t, $0 \leqslant t \leqslant t_0$. The value x_j will be referred to as the *level of the j-th activity* where (8) gives the flow functions for a unit level of activity. A set of values $x_j \geqslant 0$ satisfying (7) is called a *feasible program*. It is not always possible to find such a set of activities in which case no feasible program exists. Should, however, *one* feasible program exist, then usually many exist and the choice of program must depend on additional restrictions.

THE OBJECTIVE FUNCTION

The acceleration principle and the Keynesian multiplier are principles found in the economic literature which are presumed to be descriptive of actual behavior in an economy. From the viewpoint of the author these fall into the same class as the maximization of a welfare function, i.e., they are devices for driving an under-determined dynamic system in a direction that meets the "objectives" of an economy. In contrast, T. C. Koopmans' "efficient point" is broader in concept, [**4**]. Based on a moral principle underlying welfare economics, it has been summed up by P. A. Samuelson, [**9**], as follows: *that more of any one output, other commodities (or services) being constant, is desirable; similarly, less input for the same output, is desirable.* At an efficient point, it is not possible to increase output of one or a group of desirable commodities without causing the decrease of other commodities which are also deemed desirable. Each efficient point represents at least one feasible program. Since there are many such points, it is clear that this concept is more general than one based on the simple maximization of a welfare function, so that again for final determination of the system, additional restrictions must be imposed.

It is our purpose now to discuss the kinds of restrictions that fit naturally into linear programming. Usually objectives of an economy are expressed in terms of commodities such as a bill of goods for the final consumer, [**2**]. For example, the simplest possibility is that the consumption rate for each commodity type remains constant relative to other commodity types throughout the time span $0 \leqslant t \leqslant t_0$, and remains at a constant level. If we let the nth activity denote the activity of consuming a fixed pattern of commodities by the final consumer, we are assuming that a unit level activity A_n is of known form. The total amount of this activity in the economy is $x_n \cdot A_n$. *The objectives of the economy might be, then, to find values of* $x_1, x_2, \cdots, x_n$ *satisfying* (7) *such that*

$$x_n = \text{Max.} \tag{9}$$

A second possible type of restriction might fix the level of consumption at $x_n = x_n^0$. Then $A_0 + x_n^0 A_n$ can be treated as a new A_0. It is clear from this case that the concept of the exogenous activity A_0 should be broadened to include specified fixed activities, as well as specified outside flows and initial inventory. Having fixed $x_n = x_n^0$ the purpose of economy might be to achieve this with minimum cumulative labor force requirements during the time span $0 \leqslant t \leqslant t_0$. If cumulative labor force, considered as the mth commodity used by the jth activity, is given by $C_m(t \mid A_j)$ then

$$\sum_{j=0}^{n-1} x_j C_m(t_0 \mid A_j) = \text{Min.} \tag{10}$$

where the model in this case does not specify the labor force provided by the nth activity, but instead sets up the labor requirements for all other activities as part of the objective function and seeks a minimum of this function.

As a third example, suppose it is desirable to test whether a given feasible program $x_1^0, \cdots, x_n^0$ constitutes an "efficient point" [4]. By rearranging the subscripts on activities it is possible to let $x_1^0, x_2^0, \cdots, x_k^0$ specify the levels of consumer consumption of the set of desirable commodities. If there exists no solution $x_1 \geqslant x_1^0, x_2 \geqslant x_2^0, \cdots, x_k \geqslant x_k^0$ (except all equalities) of (7), then $x_1^0, x_2^0, \cdots, x_n^0$ is defined as an "efficient point." It will be noted that this definition expresses an efficient point in terms of activities. Setting $x_1 = x_1^0 + y_1, \cdots, x_k = x_k^0 + y_k$; $x_{k+1} = y_{k+1}, \cdots, x_n = y_n$ in (7), a solution $y_j \geqslant 0$ is sought to the system:

$$B_0 + y_1 A_1 + y_2 A_2 + \cdots + y_n A_n = 0 \tag{11}$$

$$y_1 + y_2 + \cdots + y_k = M = \text{Max.} \tag{12}$$

where B_0 is given by

$$B_0 = A_0 + x_1^0 A_1 + \cdots + x_k^0 A_k. \tag{13}$$

Because of the feasible set of values $x_1^0, \cdots, x_n^0$, there exists at least one solution to (11) with $y_1 = y_2 = y_k = 0$ so that $M \geqslant 0$. If $M = 0$, then $x_1^0, \cdots, x_n^0$ is an efficient point. If $M > 0$, then the new solution in terms of x_j constitutes such a point.

We shall formalize our observations regarding the objective function later.

THE DISCRETE TYPE MODEL

It is to be noted that (7) implies an infinite system of equations that must be satisfied for all t, $0 \leqslant t \leqslant t_0$. We shall restrict A to a class of activities having the property that a set of $x_j \geqslant 0$ will satisfy the system for the whole range of t providing it is satisfied for a certain finite set t_i. It will be assumed that, (a) activities are initiated at successive points in time $0, t_1, t_2, \cdots, t_T = t_0$ and are terminated at these discrete points in time and, (b) the rate of flow of a commodity between initiation and termination of an activity is constant.

Our purpose now is to develop the equations of the dynamic system using a discrete type model. We shall use equally spaced points in time $t = 0, 1, 2, \cdots, T$ for ease of notation; they may in fact correspond to unequal consecutive intervals of time.

Notation:

(a) $t = 0, 1, 2, \cdots, T$ denotes consecutive points in time.

(b) $A_j^{(t)}$ represents a basic (unit level) activity whose flow functions are zero up to time $t - 1$ and whose flow functions have no change after t where $j = 1, 2, \cdots, n_t$ and $t = 1, 2, \cdots, T$.

(c) $\alpha_{ij}^{(t)}$ = the discrete quantity added to the flow function of the ith commodity activity $A_j^{(t)}$ at time $t - 1$, (*input coefficient*).
$\bar{\alpha}_{ij}^{(t)}$ = the discrete quantity subtracted from the flow function of the ith commodity for $A_j^{(t)}$ at time t, (*output coefficient*).
$\beta_{ij}^{(t)}$ = constant rate of flow of the ith commodity to activity $A_j^{(t)}$ during the time period $t - 1$ to t, (*flow coefficient*).

(d) $x_j^{(t)}$ denotes the level of the jth type activity during the period $t - 1$ to t.

By our convention of signs + or − indicates *in* or *out*, see note after equation (1). Thus α and $\bar{\alpha}$ are usually positive while β is of either sign.

For the moment all activities are assumed to be of one time period duration. The cumulative flow functions for an activity, in the discrete model are easily obtained. For example, consider $A_j^{(2)}$

$$C_i(t \mid A_j^{(2)}) = \begin{cases} 0; & 0 \leqslant t \leqslant 1 \\ \alpha_{ij}^{(2)} + (t - 1)\beta_{ij}^{(2)}; & 1 \leqslant t \leqslant 2 \\ \alpha_{ij}^{(2)} + \beta_{ij}^{(2)} - \bar{\alpha}_{ij}^{(2)}; & 2 \leqslant t. \end{cases}$$

No use will be made of $C(t \mid A)$ in the above form. Instead equations (3) or (7) will be replaced by the corresponding equations relating to the discrete additions or subtractions to the flow functions at times $t = 0, 1, 2, \cdots, T$, and those relating the constant rate of flows between periods. Thus the equations of the dynamic system become

$$(14) \qquad \sum_{j=0}^{n_t} \alpha_{ij}^{(t)} x_j^{(t)} = \sum_{j=0}^{n_{t-1}} \bar{\alpha}_{ij}^{(t-1)} x_j^{(t-1)}, \qquad (i = 1, 2, \cdots, m)$$

$$(15) \qquad \sum_{j=0}^{n_t} \beta_{ij}^{(t)} x_j^{(t)} = 0, \qquad (i = 1, 2, \cdots, m)$$

where $t = 1, 2, \cdots, T$ and $\bar{\alpha}_{ij}^{(0)} = 0$.

Equations (14) may be interpreted as stating that the output (for each commodity) of all activities operating in the period $t - 1$ must equal the input of all activities operating during the tth period. Equation (15) states that the net rate of flow of any commodity, considered over all activities during a period, vanishes. It is interesting to note that if flow of the ith commodity is thought of as a new commodity type and given the index $m + i$, then we may consolidate equations (15) with (14) by making the substitutions

$$\alpha_{m+i,j}^{(t)} = \beta_{ij}^{(t)}; \quad \bar{\alpha}_{m+i,j}^{(t)} = 0.$$

J. von Neumann has considered a system of equations closely analogous to the above, [**12**]. There are, however, two important differences. First of all, the set of equations (15), relating rates of flow during a time period, is absent. This difference is not too important if the time periods are *short*, because the flows during the periods can usually be assumed to be either stored by the activity at the beginning of the period for use during the time interval or accumulated until the end of the period for use during the next time period. Letting corresponding Roman letters substitute for Greek letters we can make, in this case, the following adjustments in the coefficients:

$$(16) \qquad \begin{array}{cc} \text{If } \beta_{ij}^{(t)} \geqslant 0 & \text{If } \beta_{ij}^{(t)} < 0 \\ \begin{cases} a_{ij}^{(t)} = \alpha_{ij}^{(t)} + \beta_{ij}^{(t)} \\ \bar{a}_{ij}^{(t)} = \bar{\alpha}_{ij}^{(t)} \\ b_{ij}^{(t)} = 0 \end{cases} & \begin{cases} a_{ij}^{(t)} = \alpha_{ij}^{(t)} \\ \bar{a}_{ij}^{(t)} = \bar{\alpha}_{ij}^{(t)} - \beta_{ij}^{(t)} \\ b_{ij}^{(t)} = 0 \end{cases} \end{array}$$

The second important difference between the two systems is that in the von Neumann model, the set of equations (14) occur with a $\leqslant$ instead of $=$ symbol. In other words von Neumann requires only that there are *sufficient* quantities of commodities to carry out the activities; all surplus commodities are disposed of with no effort on the part of the system. They are also assumed nonrecoverable.

Storage and Inequality Relations: In only a few rare cases are these the kind of assumptions that one would like to make. If the commodity, for example, is *men*, "mustering out" pay is the usual rule. From a practical point of view it is best to introduce a specific storage and/or disposal activity and to replace the inequality by an equality relation. Storage activities, i.e., activities that hold over commodities for use in a subsequent time period, play an important role in any large scale system. In fact a large part of the efforts of a system may be tied up in such activities and *a solution which tries to reduce or eliminate storage or disposal activities may be a very inefficient one.* The nutrition problem which we shall discuss later is an excellent example of this inefficiency, [**10**].

If however, storage is to be introduced as an activity during tth period without "cost" or loss during storage, one may set $\alpha_{ij}^{(t)} = 1$, $\bar{\alpha}_{ij}^{(t)} = 1$ where i is the commodity in question and j the storage activity; for all other i the coefficients are zero.

Locked Activities: One may inquire how the model takes care of activities that occur over many time periods at a constant level. One way is to allow $A_j^{(t)}$ to have coefficients in other time periods. Thus $x_j^{(t)}$ will appear in an equation balancing out commodities of quite different time periods. In this form we are back to the more general case where the flow functions can extend over the entire time span $0 \leqslant t \leqslant t_0$. By a simple device, however, we can preserve the form of relations (14) and (15). To do this we subdivide an activity that extends over several time periods into a set of consecutive activities each extending for one time period and locked at the same level, i.e., set $x_j^{(t)} = x_j^{(t+1)}$, $x_j^{(t+1)} = x_j^{(t+2)}$, $\cdots$. We now invent a new commodity k which *only* these activities produce or use. Setting $\alpha_{kj}^{(t)} = 1$ and $\bar{\alpha}_{kj}^{(t)} = 1$, the necessary locked relations between $x_j^{(t)}$ and $x_j^{(t+1)}$ will result, providing that for the first time period for which the extended activity occurs, $\alpha_{kj}^{(t)} = 0$ and for the last time period, $\bar{\alpha}_{kj}^{(t)} = 0$.

LINEAR OBJECTIVE FUNCTION AND THE MATHEMATICAL PROBLEM

In a previous section we have shown how a wide variety of restrictions on programs can be put into the form of maximizing (or minimizing) a linear function. Based on the notation of the discrete model we shall assume in computing programs:

POSTULATE IX: *The optimum feasible program is that feasible program which maximizes a specified linear objective function:*

$$\sum_{t=1}^{T} \sum_{j=1}^{n_t} \gamma_j^{(t)} \cdot x_j^{(t)} = \text{Max.} \tag{17}$$

where $\gamma_j^{(t)}$ are constants.

The fact that the equations of the dynamic system impose additional linear restrictions on the unknown levels of activities (besides the condition that they must always remain nonnegative) leads to a very interesting mathematical problem that may be formulated in one of two ways: (1) *Maximize a linear function whose variables satisfy a system of linear* *inequalities;* (2) *Maximize a linear function of nonzero variables subject to a system of linear equalities.* These two problems are easily shown to be equivalent.

J. von Neumann has shown in this connection that the problem of finding the optimum mixed strategy of a zero sum two person game can be reduced to the above. The converse has been more difficult,—namely that of setting up a game problem which is equivalent to a linear programming problem. Professor A. W. Tucker and his group at Princeton have done some work along these lines. The author very recently established the equivalence of the converse.

Except for general properties of the solution very little can be found in the literature that helps to solve systems of equations involving many variables. One important property that is worth noting is the nonexistence of *local maxima.* Thus *any program which is not optimum can always be improved by making small changes.* A second property worth noting is that *the maximizing solution necessarily*[3] *involves as few activities as possible at positive levels and as many as possible at zero levels.*

It is proposed to solve linear programming problems which involve maximization of a linear form by means of large scale digital computers because even the simplest programming problems can involve a large number of calculations (e.g., see experience with the nutrition and transportation problem at end of this paper). Several computational procedures have been evolved so far and research is continuing actively in this field.

Matrix Notation: The essential form of the system of equations of the dynamic system is more clearly brought out by use of matrix notation: (a) Let

$$x^{(t)} = (x_1^{(t)}, x_2^{(t)}, \cdots, x_{n_t}^{(t)}) \tag{18}$$

be the vector of levels of activities in tth time period; (b) let

$$\alpha^{(t)} = [\alpha_{ij}^{(t)}]; \ \bar{\alpha}^{(t)} = [\bar{\alpha}_{ij}^{(t)}]; \ \beta^{(t)} = [\beta_{ij}^{(t)}] \tag{19}$$

represent the matrix of the input, output and flow coefficients respectively in the tth time period; and (c) let the vector of coefficients of the maximizing form associated with the activity levels occurring in the tth time period be denoted by

$$\gamma^{(t)} = (\gamma_1^{(t)}, \gamma_2^{(t)}, \cdots, \gamma_{n_t}^{(t)}). \tag{20}$$

The boundary conditions are given by the coefficients of the various

[3] In certain degenerate cases there may be more than one solution yielding the same maximum. If so, a unique solution could be obtained by the use of additional maximizing functions.

fixed activities $A_0^{(t)}$ which we also can express in vector notation; let

$$a^{(t)} = (\bar{\alpha}_{10}^{(t-1)} - \alpha_{10}^{(t)}, \bar{\alpha}_{20}^{(t-1)} - \alpha_{20}^{(t)}, \cdots, \bar{\alpha}_{mo}^{(t-1)} - \alpha_{mo}^{(t)}) \tag{21}$$

for $t = 2, \cdots, t$; for $t = 1$ set $\bar{\alpha}_{io}^{o} = 0$ above, for the $\beta_{io}^{(t)}$ coefficients set

$$b^{(t)} = (\beta_{10}^{(t)}, \beta_{20}^{(t)}, \cdots, \beta_{no}^{(t)}). \tag{22}$$

The equations of the dynamic system in matrix notation become

$$\begin{array}{lr} \alpha^{(1)}x^{(1)} & = a^{(1)} \\ \beta^{(1)}x^{(1)} & = b^{(1)} \\ -\bar{\alpha}^{(1)}x^{(1)} + \alpha^{(2)}x^{(2)} & = a^{(2)} \\ \qquad\qquad \beta^{(2)}x^{(2)} & = b^{(2)} \\ \qquad\qquad -\bar{\alpha}^{(2)}x^{(2)} + \alpha^{(3)}x^{(3)} & = a^{(3)} \\ \qquad\qquad\qquad\qquad \beta^{(3)}x^{(3)} & = b^{(3)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ -\bar{\alpha}^{(T-1)}x^{(T-1)} + \alpha^{(T)}x^{(T)} & = a^{(T)} \\ \beta^{(T)}x^{(T)} & = b^{(T)} \\ \gamma^{(1)}x^{(1)} + \gamma^{(2)}x^{(2)} + \gamma^{(3)}x^{(3)} + \cdots + \gamma^{(T)}x^{(T)} & = \text{Max.} \end{array} \tag{23}$$

where the $x^{(t)}$ are vectors of nonnegative elements. It should be noted that while the general mathematical problem is concerned with maximization of a linear form of nonnegative variables subject to a system of linear equalities, in the linear programming case one finds by observing the above system that the grand matrix of coefficients is composed mostly of blocks of zeros except for submatrices along and just off the "diagonal." Thus any good computational technique for solving programs would probably take advantage of this fact.

APPLICATIONS

(a) The inter-industry relations studies of W. W. Leontief and the Bureau of Labor Statistics are well known. Each activity is characterized by a steady flow of commodities over time so that the system of equations reduces to (15). Since the number of unspecified activities is equal to the number of different commodity types it is clear that once the bill of goods for the final consumer is specified a steady state solution can be determined. Because of the equality between the number of commodities and the number of activities, no maximization problem arises. There are no degrees of freedom left. It is interesting to note that the level of activities thus computed has been shown to be nonnegative. The Bureau of Labor Statistics has developed an iterative computational procedure based on this property for solving such a system that is rapidly convergent.

(b) T. C. Koopmans' transportation problem is an excellent example of a steady state solution that involves the minimization of a linear function. The problem may be stated as follows: A homogenous product[4] in the amounts of $q_1, q_2, \cdots, q_s$ respectively are to be shipped from s shipping point origins and amounts $r_1, r_2, \cdots, r_d$ respectively are to be received by d destinations; the cost to ship a unit amount of product from ith origin to jth destination is C_{ij}. The problem is to determine x_{ij}, the amount shipped from i to j, so as to minimize total transportation costs, i.e.,

$$\begin{aligned} &\sum_{j=1}^{d} x_{ij} = q_i, && (i = 1, 2, \cdots, s) \\ (24) \qquad &\sum_{i=1}^{s} x_{ij} = r_j, && (j = 1, 2, \cdots, d) \\ &\sum_{i=1}^{s}\sum_{j=1}^{d} C_{ij}x_{ij} = \text{Min.} \end{aligned}$$

Because of the special form of the equations, simplified computational procedures are possible. For example, a large scale problem involving about 25 origins and 60 destinations was solved recently in nine man days by hand computation techniques. Only simple additions and subtractions occurred in the process so that even the use of a desk calculator was not required.

(c) The minimum-cost adequate diet problem was formulated by Jerome Cornfeld in 1941 and by G. J. Stigler, [**10**]. It is assumed that (1) the composition in terms of dietary elements i.e. minerals, calories, vitamins, of a number of foods is known; (2) the prices of the foods are given and (3) the requirements in terms of dietary elements which will keep a person in good health are known. The problem is then to find a diet which will supply the requirements at minimum cost. Stigler found a solution to the problem by testing various combinations under the assumption that the body could dispose of any surplus of dietary elements. A solution to the problem which demanded that the requirements be met exactly cost nearly twice as much. This result illustrates the importance of disposal and storage activities. A problem involving 9 dietary elements and 77 foods took 120 man days to compute by hand. This may be contrasted with the above transportation problem.

[4] In Koopmans' case the homogenous product consisted of empty ships to be moved from ports of discharge to next ports of loading and the "cost" consisted of time spent by these ships in travel.

(d) A. Cahn [1] has proposed a warehouse problem which can be solved by linear programming techniques. An entrepreneur undertakes to operate a warehouse of fixed capacity by filling it with goods for which there is a seasonal production and consequently a seasonal price. When goods are in season he can purchase them at a low price and sell them later in the year at a higher price. Each month new goods become available, and the owner must make a decision as to the disposal or continued storage of his present holdings and as to the purchase of goods that have just become available to use up his idle capacity.

(e) In I an application is given of the discrete type model to a hypothetical air transport problem.

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