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PROGRAMMING OF INTERDEPENDENT ACTIVITIES
I GENERAL DISCUSSION[1]

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This paper is concerned with improved techniques of program planning, particularly as they apply to the scheduling of activities over time within an organization or economy in which the activities must share in the use of limited amounts of various commodities. The contemplated use of electronic computers for rapidly computing programs and the assumptions underlying the mathematical model are discussed. The paper is concluded by an illustrative example.

SUMMARY

THE MATHEMATICAL Model discussed here and in a subsequent paper is a generalization of the Leontief Inter-Industry Model. It is closely related to the one found in von Neumann's paper "A Model of General Economic Equilibrium."[2] Its chief points of difference lie in its emphasis on dynamic, rather than equilibrium or steady states. Its purpose is close control of an organization—hence it must be quite detailed; it is designed to handle highly dynamic problems—hence greater emphasis on time lags and capital equipment; it takes into consideration the many different ways of doing things—hence it explicitly introduces alternative activities; and it recognizes that any particular choice of a dynamic program depends on the "objectives" of the "economy,"—hence the selection and types of activities are made to depend on the maximization of an objective function.

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Programming, or program planning, may be defined as the construction of a schedule of actions by means of which an economy, organization, or other complex of activities, may move from one defined state to another

[1] A revision of a paper presented before the Cleveland Meeting of the Econometric Society on December 27, 1948. This paper is the first of two papers on this subject, both appearing in this issue. The second paper, with the sub-title "Mathematical Model," contains a more technical discussion of the mathematical formulation of the problem and will be referred to by Roman numeral II.

[2] *Review of Economic Studies*, Vol. 13 (1), No. 33, 1945–46, pp. 1–9. This was first published in German, under the title *Über ein Ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouweschen Fixpunktsatzes* in the volume entitled *Ergebuisse eines Mathematischen Seminars*, edited by K. Menger (Vienna, 1938).

or from a defined state toward some specifically defined objective. Such a schedule implies, and should explicitly prescribe, the resources and the goods and services utilized, consumed, or produced in the accomplishment of the programmed actions.

The economy or organization for which a program is to be constructed is here conceived of as comprising a finite number of discrete types of activities, each of whose magnitudes is to be specified over a certain time period. For convenience, the magnitudes (or levels) of each of the activities will be specified for each of a finite number of discrete time periods,[3] rather than continuously over the total time period involved. The resources and the goods and services utilized, consumed, or produced by the activities are hereafter referred to generically as "commodities," and are measured in terms of the quantities of specific types of commodities. The quantity of each commodity type used, consumed, or produced by each activity is assumed to be a function of the magnitude of the activity, usually proportional. Two activities are interdependent when they must share limited amounts of a commodity which they use in common; when one produces a commodity which is used by the other; or when they each produce a commodity used by a third activity.

These interdependencies arise because all practical programming problems are circumscribed by commodity limitations of one kind or another. The limited "commodity" may be raw materials, manpower, facilities, or funds. One or more of these is almost always limited in any type of program. To some extent, all of them are usually limited in most programming problems, since any program must start from a definitely prescribed initial status, at which point all commodities are limited. Generally, these limitations of initial status are felt over several succeeding time periods, because of the existence of limitations on the rates of growth of the activities producing the commodities.

There are two general formulations of the programming problem. In the first formulation, the quantities of each of several activities contributing directly to objectives (or "final demand") are specified for each time period; from this it is desired to determine the magnitudes of the required supporting activities, their total requirement for commodities from outside the system, and whether or not these total requirements are consistent with the initial status and subsequent limitations. Procedures for solution of this formulation have consisted generally of ordering the work in a series of stages. In the first stage, the input requirements of the specified "final demand" activities are computed. In the second stage, those supporting activities whose output is principally utilized by the "final demand" activities are computed. In the third stage, those supporting activities whose output is principally utilized

[3] The model described here is referred to in II as the "Discrete Type Model."

by both the "final demand" activitives and the activities whose resource requirements were computed in the second step are computed; and so on. To the extent that the conditions specified in the above arrangement can be met, this procedure yields consistent results. However, when one activity utilizes a commodity produced by another, and the other also utilizes a commodity produced by the first, a circular relationship exists which precludes satisfying the conditions of this arrangement, and a satisfactory solution can be produced only by successive iterations of the procedure. The procedure is also deficient in that it does not permit the consideration of alternative processes or activities.

In the second formulation, we seek to determine that program which will, in some sense, most nearly accomplish objectives without exceeding stated resource limitations. So far as is known, there is so far no satisfactory procedure for solution of this type of problem. At present, such problems can only be solved by successive iterations of the procedure described under the first formulation. Yet this second type of problem is precisely the one which we are constantly required to solve, often under conditions requiring an answer in days or hours.

To accomplish this, it is proposed to represent all the interrelationships in the organization or economy by a large system of simultaneous equations in which the variables are the quantities of the activities to be performed, the coefficients are the requirements of each activity for each commodity, and each equation expresses the sum of the requirements of all activities for a single commodity. To prepare a program it is necessary to insert into these equations a detailed specification of the initial status in terms of the quantities of each commodity on hand; any subsequent limitations (such as may be imposed by the capabilities of industries or other activities for expansion), and a statement of objectives.

To compute programs rapidly with such a mathematical model, it is proposed that all necessary information and instructions be systematically classified and stored on magnetized tapes in the "memory" of a large scale digital electronic computer. It will then be possible, we believe, through the use of mathematical techniques now being developed,[4] to determine the program which will maximize the accomplishment of our objectives within those stated resource limitations. Alternatively, it will be possible to determine the program which will minimize requirements, either in funds, or in amounts of any limiting commodity or group of commodities, required to accomplish any fixed objective.

The work being done on the mathematical model has clearly shown the necessity for a more precise formulation of objectives. Planners generally

[4] Some of these techniques are discussed in greater detail in II.

have been accustomed to state objectives in terms of means rather than ends; that is, they have been accustomed to state objectives in terms of specific operations whose relations to the accomplishment of basic ends could only be evaluated subjectively. Objectives must be stated in terms of basic ends, thus permitting the consideration of alternative means, if they are to be useful in programming operations designed to maximize objectives within resource limitations.

In military program planning, it is necessary to introduce quantitatively the various limitations of resources which restrict the capabilities of the military establishment during war time, as well as in peacetime. For the most part these may be traced to limitations in the industrial economy of the nation. It is necessary to know what part of the total national production can be made available for military purposes. This cannot be measured solely in terms of the productive capability of the aircraft industry or of the munitions industry, any more than the strength of an Air Force can be measured solely in terms of the number of groups.

It is necessary to know in detail the capacities of the steel, aluminum, electric power, transportation, mining, chemical, and a multitude of other industries supporting the aircraft, shipbuilding, and munitions industries, just as it is necessary to know the capacities of the training, transportation, maintenance and supply activities supporting the combat air groups. Further, it is necessary to determine whether these industries (or supporting activities) are balanced in the proper proportions to meet changing requirements.

Thus, since the determination of the "best" program necessarily starts with a consideration of limitations on resources, it must necessarily start with a consideration of the interrelationships of industries in the industrial economy of the nation.

The first steps toward the required analysis of inter-industry relationships have been taken by Professor Leontief and by the Bureau of Labor Statistics. These studies consider relationships in a static or equilibrium state. Theoretical work now underway by several groups will make it possible to handle these relationships dynamically and with due consideration of alternative procedures or processes as is done in the mathematical model we are now developing for the internal operations of the Air Force.

The basic equations of the proposed technique are best illustrated by an example. For this purpose a hypothetical "Airlift Model" was chosen (see accompanying table). In this model there are four exogenous activities and six activities within the "economy" being studied. The first exogenous activity specifies the commodities initially available and the "final demand" (airlift tonnage) for the first quarterly time period.

Thus we have available to carry out the operations of the first time period 85 aircraft and 210 active crews. The problem is to carry an airlift of 1.5 ($\times$ 100,000 tons) during the first quarter, 1.6 the second quarter, 1.8 the third quarter, and 2.0 the fourth quarter. These requirements for the second, third, and fourth time periods are specified by the corresponding exogenous activities. Activity 1 is the flying operation itself. Activity 2 stores aircraft (if there is a surplus). Activity 3 procures aircraft (if there is a deficit). Activities 4 and 5 play the same roles for crews. Activity 6 rests weary crews. (Activity 6 is an artificial operation which is included in the interest of making a complete model with the fewest possible activities. In actual practice, it is replaced by the utilization of crews in other productive activities.)

The objective function used, expression (5), is one which minimizes the total cost of all operations over all time periods. An alternative formulation of the problem might have specified the available funds, crews, or aircraft, instead of the required airlift tonnage, and used an objective function which would maximize the total airlift.

The body of the table contains the input and output coefficients of each activity for each commodity. The input coefficients specify the amounts of each commodity required at the beginning of any quarterly time period to support a unit level of the activity; the output coefficients specify the amounts of each commodity available at the end of a quarterly time period as a result of the operation of a unit quantity of an activity.

Equation (1) states that for each of the four quarterly time periods the quantity of Airlift flying (activity 1) must be equal to the quantity of airlift tonnage (commodity 1) shown under the four exogenous activities, respectively. Equation (2) states that the total aircraft required in each time period (represented by the sum of the products of activity quantities times input coefficients for aircraft, for activities 1 and 2) equals the total aircraft available from initial inventory plus the aircraft available as a result of the operations of the preceding time period (represented by the sum of the products of the output coefficients times activity quantities of the preceding time period for activities 1, 2, and 3). Equation (3) states that the total active crews required in each time period (represented by the sum of the products of activity quantities times input coefficients for active crews, for activities 1, 4, and 5) equals the total active crews available from initial inventory plus the active crews available as a result of the operations of the preceding time period (represented by the sum of the products of activity quantities for the preceding time period times output coefficients for active crews.) Equation (4) states that the sum of the nonactive or retired crews produced as a result of the airlift flying activity is equal

HYPOTHETICAL AIRLIFT MODEL SHOWING INPUT AND OUTPUT COEFFICIENTS AND EQUATIONS OF DYNAMIC SYSTEM

COMMODITY \ ACTIVITY	Level	Exogenous Activities $x_0^{(1)} = 1$	$x_0^{(2)} = 1$	$x_0^{(3)} = 1$	$x_0^{(4)} = 1$	Airlift Flying $x_1^{(t)}$	Storing Aircraft $x_2^{(t)}$	Procuring Aircraft $x_3^{(t)}$	Storing Crews $x_4^{(t)}$	Training Crews $x_5^{(t)}$	Resting Weary Crews $x_6^{(t)}$
1. Supply Shipped by Airlift (1 = 100,000 tons)	IN	+1.5	+1.6	+1.8	+2.0	−1					
	OUT										
2. Aircraft	IN	−85				50	1				
	OUT					49	1	1			
3. Active Crews	IN	−210				130			1	.05	
	OUT								1	1.00	1
4. Nonactive Crews	IN										1
	OUT					125					
5. Money (1 = $1000)	IN					9,000		200	7	10	5
	OUT										

EQUATIONS*

$$(1)\quad \alpha_{1,0}^{(t)} + x_1^{(t)} = 0$$

$$(2)\quad \alpha_{2,0}^{(t)} + 50x_1^{(t)} + x_2^{(t)} = 49x_1^{(t-1)} + x_2^{(t-1)} \quad x_3^{(t-1)}$$

$$(3)\quad \alpha_{3,0}^{(t)} + 130x_1^{(t)} + x_4^{(t)} + .05x_5^{(t)} = x_4^{(t-1)} + x_5^{(t-1)} + x_6^{(t-1)}$$

$$(4)\quad x_6^{(t)} = 125x_1^{(t-1)}$$

$$(5)\quad \sum_{t=1}^{4} (9000x_1^{(t)} + 200x_3^{(t)} + 7x_4^{(t)} + 10x_5^{(t)} + 5x_6^{(t)}) = \text{Min.}$$

* where $x_j^{(t)} \geqslant 0$ for $j = 1, \ldots, 6$; while for $t = 1, 2, 3, 4$, $\alpha_{1,0}^{(t)} = 1.5, 1.6, 1.8, 2.0$ respectively; $\alpha_{2,0}^{(t)} = -85, 0, 0, 0$, respectively, $\alpha_{3,0}^{(t)} = -210, 0, 0, 0$, respectively.

to the amount of nonactive crews used as input to activity 6, (resting weary crews).

For the sake of simplicity, the only alternatives introduced into this model are the trivial alternatives between production of aircraft and crews as required and production at an earlier time, with accompanying storage of the excess until needed. It will be seen, however, that real alternative activities, such as alternative methods of airlift flying, using different types of aircraft, can readily be introduced into the model.

A formal development of the mathematical model, together with a discussion of mathematical problems involved in using it, are discussed in the following paper (II).

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