An Algebra of Probability as Binary Logic

Bill Foote

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Why this essay

The end we seek here is to discover a principled way to determine the validity, coherence, and plausibility of our statements. Our work takes the form of an argument which is just a sequence of statements with premises which offer a theory of the argument and also data to fit to the theory. Premises and data lead to a conclusion which essentially is all about the fit of data to the theory. We will work with basic algebra and arithmetic applied to natural language statements. But in effect, the premises, data, conclusions and the process itself of the argument must make practical sense to the arguer. We will call this requirement a canon of common sense. Theories are usually in some sort of conditional form, if-this, then-that. Observed (or at least stated as so) data relate to terms in the theory. The structure of the argument then infers a conclusion. Any statement here is either true or false. We will assign a 1 if true and a 0 if false to quantify our arguments in the binary integer field.

Why do we need a principled way to determine the validity of sequences of our statements? We will maintain a hypothesis we might call a canon of rationality. This canon will hold throughout all of our proceedings here. In a nutshell we will place the practice and science of logic in the only context we know, namely, ourselves. We are not machines or animals or plants or rocks, although we have aspects and attributes of each. We do however ask questions, rocks, plants, animals, machines (not even Artificial Intelligence) raises questions from experience, understanding, judgment, and decision. Yes, some machines may, if we switch them on, may have been programmed by a human to prompt strings of morphemes, terms, propositions, even arguments which have the same attributes as questions a human would ask. But of themselves, machines do not, on their own volition, prompt anything. Machines do not have a will. Machines can also be programmed to store information, instructions, and adapt responses to evolving instructions, They can also in that capacity write their own instructions, build and deploy, if programmed to, their own data bases and applications, and with great agility, even exceeding their human makers. Even a hammer exceeds human ability to break rocks which would otherwise break human hands if hands tried to break rocks. A technology, actually a technique, strictly, we will develop is an implementation of logic through the use of the rules of algebra. It is in this context that we need to situate our evolving technology of implementing ways to reason, judge, understand, the three common sense acts of the mind. Each of these acts of consciousness are quite necessarily experienced, observed, perhaps remembered and stored. Thus whenever we inquire about anything in our world, we generate data of what we observe, data of understanding, data of judging, and data of reasoning. We will call this polymorphic (many shaped) consciousness.

What is logic, still and up to this point using the procedure from high school algebra of "let X (logic) be an unknown variable for which we are seeking a solution"? We can begin through the practice logic, or what we might consider logical structures, by thinking about and through our reality. Thus using our thinking structures informed by our truly polymorphic human consciousness we can define logic. Through understanding we answer the question "what is it?". Within this question are the numerous questions of "what is it composed of?", questions of our experience of whatever it about which we inquire. By judgment we determine answers to the question "whether it is so?". By reason we mean finding valid conclusions which answer the question "why is it so?". Logic is the art and science of answering these questions. To enhance our understanding ("what is logic?") we can contrast logic with ethics which answers the question "what should we do about it?".

We can go one step further. Through each of these questions we can ferret out typical answers. To the questions for understanding, "what is it?", we can propose two answers about the clarity of our understanding (going binary on us right now): "it is clear what it is," or "it is not clear what it is." Similarly we can propose two answers to questions of judgment, "whether it is or not": "is it true," or "it is not true." Lastly, in the field of inquiry called logic, we can propose two answers about the validity of our reasoning: "it is valid," or "it is not valid." We must immediately realize we have mutually exclusive, and again in the realm of a particular inquiry, completely exhaustive answers, all with exclusive "or," that is, disjunctive. If we add up all the trues and all the falses we get all of the answers to all of the questions for judgment. So we suppose now we need some clarification about what we mean by clear, true, and valid.

We will consider our understanding to be clear when propositions are unambiguous, with terms well defined and not admitting of vague, hard to measure, and if not able to measure, hard to distinguish from other terms, and their role in propositions. Clarity is also achieved when the relationships among propositions are unambiguous.

Our judgments are true is easier if we follow Aristotle's formula in his Metaphysics 1011b25: "To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is, and of what is not that it is not, is true." This is correspondence theory of truth, the need for which is familiar perhaps to anyone who has dealt with two year olds.

Valid reasoning is even easier as this layer is all about arguments. An argument (terms in propositions, propositions in premises, reasoning in conclusions) is valid only when clear terms, true prepositions in premises, and the conclusion necessarily follows from the premises. As simple as this is, if terms are not well-defined, and/or propositions are not clearly true (or false), then an arguments are not valid. Even if terms are well-defined and unambiguous, propositions are clearly true or false, arguments are not valid if the conclusion follows a line of reason not supported by clear premises and unambiguous terms. Finally, and here we actually get practical, even if the line of reasoning formally is supported by clearly understood premises, an argument is not valid in the field of observable reality if the conclusion makes no sense, common or otherwise.

Let's use the cats and gods from the example in Peter Kreeft (2005) (32).

cats and gods	
A: if I am a cat, then B: I am a god	$A \rightarrow B$
but, A: I am a cat	A
therefore, B: I am a god	$\therefore B$

This argument makes mathematical sense in a way we will later demonstrate. If the premises are true, then we have a logically valid argument since the conclusion follows necessarily (meaning, always true, happens all the time everywhere) from the premises. And we will say mathematically true since, we will show, symbolically the logic robot we will build will always return a true value for the conclusion. But this is common non-sense! I am not a cat, and even if I were to be a cat with whiskers, talons, teeth, and a pretty furry face, I could not be a god no matter how hard I try! So, the premises are not possibly true in this reality we wake up to each day, and the argument fails, augers into the ground, burns down.

There exists right across from the typewriter producing a this text a cat named Waffles (ginger, about 17 years old, or so we think and he's asleep, again so we think). There exist other animals called dogs, and we know very well they are quite different beasts. Yes, we allow ourselves the luxury of begin able to abstract from individuals to existential groups of similar individuals, a pillar of the scientific method. We will now use this same logical argument, its structure, to inquire about Waffles.

cats and dogs	
A: if Waffles is a cat, then B: Waffles is not a dog	$A \rightarrow B$
but, A: Waffles is a cat	A
therefore, B: Waffles is not a dog	$\therefore B$

We have unambiguous terms, Waffles, cat, dog; true propositions; straight line of reasoning from true propositions to a necessary conclusion. We have a logically valid argument, which definitely makes sense in some way to Waffles.

Here is an example of a logically invalid argument.

cats and dogs	
A: if Waffles is a cat, then B: Waffles is not a dog	$A \to B$
but, A: Waffles is not a dog	A
therefore, B: Waffles is a cat	$\therefore B$

What we might not know is that there is also a hamster across the room also named Waffles. Hamster-Waffles is miffed at the suggestion that she is a cat. An now that we have a tad of clarity about validity, we can embark on a principled approach to a logic of probabilistic inference. This entails some further thought about inference in a setting with observed data and (un)observed conjectures about data. Our question is there, and if there is, how might we characterize the truthfulness of arguments in the face of sampling data relevant to the problem at hand. To do this we need to clarify further two approaches to probabilistic inference.

Two schools of thought

Cox (1946) describes two schools of probability. One is the frequency school which would sample events in ensembles indefinitely and across many universes. An ensemble is a combination of possible events. For example, the probability of extra-terrestrial life on one planet would require the counting of every possible combination of planetary existence in our universe and in any other universe as well. This would of course our universe is not a universe, nor are the other universes as well! Be that as it may, this notion of a long-run probability as a frequency requires, technically, an infinite amount of time across an infinite set of possible ensembles of events in a finite universe with finite time and space, let alone the annoying fact of a finite set of human observers with imperfect measurement technologies. The idea is thus a purely mental fabrication to aid comprehension of what is known to be incomplete (information about extra-terrestrial life), corruptible (measurement errors by human and their measurement technologies), and otherwise unimagined by humans who do not experience insight very often.

On the other hand there is the expectation school which thinks about probability as the logical consequence of how relatively likely one event or hypothesis is to another. In Keynes (2013) view, the theory of probability is the logic of probable inference. Probability is a syllogistic relation from a hypothesis to data, and on to a conclusion. Each hypothesis is more or less plausible than another as measured by the ways in which data are consistent with a particular hypothesis. Plausibilities are bounded by extremes of impossibility and certainty, Aristotelian deductive logic is a special case of plausibility. This means we need more than binary true-false deduction to make statements about the plausibility of hypotheses in the face of data. We also no longer need the certainty uncomfortably afforded by leaps of imaginative extension of samplings into an infinity of possible combinations of hypotheses and events, namely the count of incidents. Instead we do need the count of ways in which data are consistent with a particular hypothesis. We now start from plausible numbers of ways consistent with data instead of infinite counts of instances of data.

The expectation school's results build on an algebra of the logic of propositions. We will begin with tables of algebraic relationships among propositions. Once we establish these rules we can operate on any logical statement with the power of 9th grade elementary algebra. We need just three algebraic operations to work the algebra for pairs of propositions a and b: AND ($a \times b$), NOT ($1 + a$), XOR ($a + b$). We need to add something else to our analysis: common sense, both espoused by Jaynes (2003) and Kreeft (2005) in natural sciences and philosophy. Just because the math might work out logically (and usually will if we are very careful), does not mean we can interpret our results in a particular and realistic context. This is the problem of **existential import** which Boole (1848) and purely symbolic logicians seem to eschew. They would have us believe that the math is the math, without reference to anything real as surveyed by Wu (1969). At one

extreme end of this view any $a + b$, whatever a and b might represent objectively, are logically equivalent to any other a and b, whatever they might represent to the human being attempting to reason. They may indeed not even exist, even hypothetically. We will follow this convention analyzed by Read (2015) regarding Arristotle (1975) and Avicenna and Zabeeh (1971)'s commentaries.

"With an affirmation and negation one will always be false and the other true whether [Socrates] exists or not. For take 'Socrates is sick' and 'Socrates is not sick': if he exists it is clear that one or the other of them will be true or false, and equally if he does not; for if he does not exist 'he is sick' is false but 'he is not sick' true." (*De Interpretatione*, 13b26-31)

Here we will blithely consider propositions and their subject to at least probably exist so as to be able to discuss them with respect to the very real actions we undertake upon deliberation, reflection, and understanding of our conscious experience and held beliefs.

In what follows we develop together an algebraic approach to studying inference introduced by Boole (1848) and expanded upon by Peirce (1870), Keynes (2013), Jaynes (2003), and more recently Suleimenov et al. (2023). The logical system here relies on four fundamental logical forms developed by Aristotle, Striker, et al. (2009) and as explicated in practical logic by Kreeft (2005) and Toulmin (2003). Far deeper than this discussion, but notable in grounding these techniques is Carnielli and Matulovic (2015).

Rules

We clarify our approach with rules learnt from basic arithmetic. Addition is defined as addition of integers, but restricted to modulo 2, with only two possible values $\{0, 1\}$. These values are the terms and the propositions themselves, not to mention the possible outcomes of sequences of propositions we would call an argument. If we remember our basic maths and modulo arithmetic as a clock, the hands on this clock can sit at 0 or 1. If the hand is at 1 and we add 1 to this setting we end up at, yes, 0. A regular OR in many languages would often mean and/or, an inclusive OR, an either one or the other or both. The XOR excludes any possibility in common between two propositions. Propositions are mutually exclusive of one another. Here is a table to help us.

$+(XOR)$	0	1
0	0	1
1	1	0

Addition with boolean values $\{0, 1\}$ is the same as that of the logical exclusive OR operations, XOR. Since each element equals its opposite, subtraction is thus the same operation as addition, so that $0 - 0 = 0$, $0 - 1 = 1$ (remembering the clock with just two settings!), $1 - 0 = 1$, and $1 - 1 = 0$, anyway, all equivalent to the $+$ rules. In words, it is **one or the other, not both**.

The multiplication is multiplication modulo 2 (see the table below), and on boolean variables corresponds to the logical AND operation. Here is the **both** proposition with two terms.

$\times(AND)$	0	1
0	0	0
1	0	1

The NOT logic simply takes a 0 and flips it to 1, and if 1 flips it to zero.

$\neg(NOT)$	$1 + x$
0	1
1	0

With these three operations we can evaluate any logical statement. And we must remember that NOT is not the subtraction operation we are very familiar with from non-binary algebra. Some folks will write $\bar{x}$ instead of $\neg$.

Here is a table of several useful expressions based on AND, XOR, and NOT. For any two propositions x and y whose values can only take on a 0 or 1, so that when $x = 0$, we mean x is false, otherwise (that is, $x=1$) true.[1]

x	y	AND xy	XOR $x+y$	NOT $1+x$	$y \mid x$ $1+x+xy$	SUB $x-y = x+y$
0	0	0	0	1	1	0
0	1	0	1	1	1	1
1	0	0	1	0	0	1
1	1	1	0	0	1	0
x	y	$x \wedge y$	$x \oplus y$	$\neg x, \overline{x}$	$x \rightarrow y$	
x	y	$-$	$(x \vee y) \wedge \neg(x \wedge y)$	$\neg x \vee y$	$\neg(x \oplus (x \wedge y))$	

The last row describes the algebraic operations in term of propositional logic as applied to Boolean values of $\{0, 1\}$. I can easily get lost in that row. I tend to prefer 9th grade algebraic expressions.

The last column uses the negative to SUBtract y from x. We might be tempted to say that $x - y = x + (1 + y) = 1 + x + y = 1 + (x + y)$ so that the subtraction operator is "NOT the ADD" operator. Nice try! We must always go back to the most literal, most primitive operations, namely the addition and multiplication rules which govern these more derived expressions.

That next to the last column will deserves some important attention since it forms the foundation of conditional (reasonably expected) probability. But first some very helpful derived relationships will make our work going forward a lot smoother.

x	x	$xx = x^2$	$xxx = x^3$	$x + x = 2x$	$x + x + x = 3x$	$x - x = x + x$
0	0	0	0	0	0	0
0	0	0	0	0	0	0
1	1	1	1	0	1	0
1	1	1	1	0	1	0
	$=$	x	x	0	x	0

What happens here is that there are no monomial terms of any higher degree than 1; no quadratic or cubic terms at all. Two propositions literally cancel each other, as the Germans might say an *Aufhebung* event. But three return the single proposition. All of this is the result of the modulo 2 arithmetic to which we constrain ourselves.

We now study $y|x = 1 + x + xy$ in three moves. The first move is to realize that, at least for binary data, Aristotle discovered four logical forms, two in the affirmative, two negative; two universal, two contingent. Medieval logicians called the two affirmative Forms A and I for the first two vowels in *AffIrmo*, Latin for "I affirm," and the two negative forms E and O from the Latin *nEgO*, "I deny." Together they form the **Square of Opposition**. Here S is the subject and P the predicate. Any subject S signifies what it is we are talking about, say, rain. Any predicate P signifies what the subject is about, say, falling to the ground.

Equations and identities do not have a subject or a predicate and themselves might be the subject or predicate. But all propositions do have a subject and a predicate, just like in 3rd grade when we learned to write and speak in complete sentences, that is, in propositions, which contain a subject (usually a noun) and a predicate (usually a verb). We assume that when we apply the forms to concrete examples of propositions, the content of the form, that is, the S and the P, exist. For Thomas Aquinas signs are physical manifestations that allow us to understand something beyond their immediate appearance, like a footprint manifesting someone's presence or smoke manifesting fire. This something with an immediate appearance we will assume without further bother, that it somehow exists. Perhaps we append the particle *any* to S and P to get any rain and any falling (of rain).

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[1] Boole has the negation of x as 1-x. We follow Carnielli and Matulovic (2015) 1+x, given that all subtractions end up being addition in mod 2 arithmetic.

We build a table of logical forms in this first move. In the table, *decisions* is the subject S, what we are talking about, and *are rational* is the predicate P, what the subject is about.

Form	Proposition	Sentence	Algebra
A	All S is P	"All decisions are rational."	$a(1+b) = 0$
E	No S is P	"Not all decisions are rational."	$ab = 0$
I	Some S is P	"Some decisions are rational."	$ab \neq 0$
O	Some S is not P	"Some decisions are not rational."	$a(1+b) \neq 0$

The second move is to parse the Form A proposition "All decisions (a) are rational (b)." Logically a and not b is false (0) in Form A, that is, algebraicly, $a(1+b) = 0$. "Decisions" and "not-rational" is false, that is, inconsistent according to Form A. The obverse must be true, that "No-decisions are not-rational," as if we might be able to interpret this double negative. One more swipe at interpretation is called for. To say that "all of anything is something else" is, effectively to identify "anything" with "something else." If this statement is true, as we are positing here in the form, then it cannot be true that both "anything $= a$" and "not-something else $= (1 + b)$" can coexist. Thus the logical Form A seems to admit a very basic principle we would hold alongside all other principles (Aristotle called this *metaphysics*), namely, the *Principle of Non-Contradiction*. Yes, we cannot, so far it seems, to be in two places at the same time.

In a third move, we use the NOT operation on Form A. We recall we are using "+" as exclusive OR, XOR with this algebraic rearrangement.

statement	reason	
1	$a(1+b) = 0$	Form A definition
2	$1 + a(1+b) = 1$	Symmetric property
3	$1 + a + ab = 1$	Distribution of multiplication over addition property
4	$a \rightarrow b = b \mid a$	a gives some information to b

Simply negating (using 1+) the Form A definition reveals a conditional relationship between a and b. Saying that a AND $not - b$ is possible when we negate the Form A requirement, means a does share something with b. So-called "implication" means that a shares a's information with b. This is the primary meaning we ascribe to "a conditions b", $b \mid a$. We now have all the ingredients to use this framework to discover a principled path to the expectations approach to probability, that is, counting the ways in which data are consistent with hypotheses. But before we enter that particular room, we need to argue a bit more, that is, discover valid and invalid logical arguments.

Making a valid point

Let's begin with any two propositions or statements or messages A and B. Specifically, let's talk about the weather. We let

$$A \ = \text{You get an A.}$$
$$B \ = \text{I give you a lollipop.}$$

The implication is the statement *If A, then B*. We will label A the *antecedent* and B the *consequent* Some of us might even want to go further than the literal logic and interpret, allegorically, A as the *cause* and B the *effect*. This will also be our starting point with the coherence and validity of our reasoning as well as plausibility. But we must first begin even more strongly with a guaranteed statement from the following deduction that stems from the implicating condition, the if-then. And by guaranteed we will mean with certainty. Yes, we can still talk about maybe and plausibility if we grant that the proposition "this is plausible" is TRUE or FALSE. First no quibbling with "If it you get an A, then it *must* happen that I will give you a lollipop." But I can also quibble that, "If it you might get an A, then it *must* happen that"I might

give you a lollipop." Or even this (not so nice, perhaps passive aggressive approach) bit of wobbling that "If it you do get an A, then it *must* happen that"I might give you a lollipop."

The full name of structure of our first syllogism is **modus ponendo ponens** which is Latin for *the way that, by affirming, affirms* , also known as *affirming the antecedent* . And by *syllogism* we mean a linking together of at least two separate propositions, messages, symbols, tokens, and meanings.

	Modus Ponens	
if A is (plausibly) true, then B is (plausibly) true		$A \rightarrow B$
and A is (plausibly) true		A
therefore, B is (plausibly) true		$\therefore B$

With or without the qualifier *plausible* we will claim to have a deductive inference guaranteed to be consistent with the logic of implication.[2] We will demonstrate this claim shortly, actually prove it. For now let's just have a rather informal chat about TRUE and FALSE in this syllogism.

1. Suppose it is true that you get an A *and* it is true that I give you a lollipop. Since I kept my promise, the implication, the statement, is true. Doesn't this sound like a logical conjunction? Both A and B are true, that is $AB = 1 \times 1 = 1 = TRUE$ according to the hard work we did in building a binary algebra.

2. Suppose it's true that you get an A but it's false that I give you a lollipop. Since I didn't keep my promise, the implication is false. This truth values again act like a logical conjunction. If A is true and B is false, we have $AB = 1 \times 0 = 0 = FALSE$.

3. What if it is false that you get an A? Whether or not I give you a lollipop (Q is true or Q is false), I have not broken my promise. Thus, the implication cannot be false, so the statement must be true. This case is no longer a simple logical conjunction, but we can use the negation to help us out with the idea that If A is false it is not-A, when A would otherwise be true. The conjunction $\neg AB = 0 \times 1 = 0 = FALSE$ or equivalently $1 + AB = 1 + 1 \times 1 = 2 mod 2 = 0 = FALSE$.

Let's use our Algebra of Boole to check the reasoning in our informal chat.

1. We set out the structure. We let P : "you get an A" and Q : "I'll give you a lollipop." With this assignment of propositions we state the argument as $[\,P \text{ AND } (P \text{ IMPLIES } Q)\,] \text{ IMPLIES } Q$.

2. we translate this structure into algebra. For any A and B, $A \text{ AND } B = AB$, and $A \text{ IMPLIES } B = 1 + B + BA$. Also we know that $A^2 = AA = A$ and $A + A = 0$ in the mod 2 arithmetic of the algebra of Boole. This means that our argument can be represented algebraicly as this expression.

3. Then at last we reduce all expressions algebraicly. If we end up with a 1, then we have logical satisfaction.

4. We apply common sense and existential import.

$[\,P \text{ AND } (P \text{ IMPLIES } Q)\,] \text{ IMPLIES } Q$	[modus ponens]
$P(1 + P + PQ) \quad \text{IMPLIES } Q$	
$P + PP + PPQ \quad \text{IMPLIES } Q$	
$P + P + PQ \quad \text{IMPLIES } Q$	
$0 + PQ \quad \text{IMPLIES } Q$	
$1 + PQ + PQ(Q) \quad = 1 + PQ + PQ$	
$= 1 + 0$	
$= 1$	

[2]It is invalid to affirm the consequent, that is say B is true, so A must be true, a frequent and very misleading and logically illegal argument on talk shows, especially when the subject is political.

That wasn't so bad! We just proved, quite logically, that this argument returns a 1, to which we assign the value TRUE. Yes, it is valid logically.

Let's leap into the other mainstay of valid logic, the **modus tollens** argument structure. The full name is **modus tollendo tollens** which is Latin for *the way that, by denying, denies* , also known as *denying the consequent.* Grades and lollipops exist, and notwithstanding, the procedure is the same as for *modus ponens.* The structure looks like this.

Modus Tollens	
if A is true, then B is true	$A \to B$
and B is not true	$\neg B$
therefore, A is not true	$\therefore \neg A$

1. We set out the structure. We let P : "you get an A" and Q : "I'll give you a lollipop." With this assignment of propositions we state the *modus tollens* argument as $[\,\text{NOT } Q \text{ AND } (P \text{ IMPLIES } Q)\,]$ IMPLIES NOT P.

2. We translate this structure into algebra. Again for any A and B, $A \text{ AND } B = AB$, and $A \text{ IMPLIES } B = 1 + B + BA$. Also we use the operator NOT $A = 1 + A$ This means that our argument can be represented algebraicly as this expression.

3. We reduce all expressions algebraicly. If we end up with a 1, we have logical satisfaction: the argument is logically consistent.

4. We apply common sense and existential import.

$$
\begin{array}{rl}
[\,\text{NOT } Q \text{ AND } (P \text{ IMPLIES } Q)\,] \text{ IMPLIES NOT } P & [\text{modus tollens}] \\
(1+Q)(1+P+PQ) \ \text{ IMPLIES } 1+P & \\
1+P+PQ+Q+QP+PQQ \ \text{ IMPLIES } 1+P & \\
1+P+Q+PQ \ \text{ IMPLIES } 1+P & \\
= 1 + (1+P+Q+PQ) + (1+P+Q+PQ)(1+P) & \\
= 1 + (1+P+Q+PQ) + (1+P+Q+PQ+P+PP+PQ+PPQ) & \\
= 1 + 0 & \\
= 1 &
\end{array}
$$

TRUE again! We have yet another valid logical argument *structure* only if step 4, *common sense*, invokes a result. I have to be able to get a grade of A (it is possible at least) to get an actual lollipop (sugar-free and no sugar substitutes). Grades of A and lollipops exist and are at least somewhat accessible.

Nay and Yay

It turns out that much rhetoric, and some marketing campaigns and technology projects, use as least two invalid arguments: affirming the consequent and denying the antecedent. The first is a fallacy of *modus ponens*: Affirm the Consequent. The second is a fallacy of *modus tollens*: Deny the Antecedent. We shall focus our attention on splitting the hairs of the *modus tollens* fallacy. We might want to examine the symmetry of these arguments as well.

Using our grade-for-lollipop scheme *modus tollens* states that "P: If you get an A, then Q: you receive a lollipop (or other appropriate bribe/incentive)."; but "NOT Q: you did not get a lollipop"; so, "NOT P: you did not get an A." Common sense will arrive at lots of, reasons why you did not get a lollipop, for example, local lollipop supply chain disruptions. But there is only one reason you did not get a lollipop even admissible here (that is, in the premise).

Here is the fallacy portion of our program for *modus tollens*: Deny the Antecedent (P: a grade of A). This structure follows the line of thinking that "P: If you get an A, then Q: you receive a lollipop (or other

appropriate bribe/incentive).”; but “NOT P: you did not get an A”; so, “NOT Q: you did not get an lollipop.” Polling nine out of ten (on average) groups of board members, project teams, classrooms full of students, over a span of almost 50 years say “this makes sense. So what is the big deal?” At least let's examine, from our principled perch on the binary branch of the algebra of Boole, whether or not this is so (yet another binary choice: Latin, *an sit?*, “is it?”).

1. We set out the structure. We let P : “you get an A” and Q : “I'll give you a lollipop.” With this assignment of propositions we state the *modus tollens: denying the antecedent* argument as $[\text{NOT } P \text{ AND } (P \text{ IMPLIES } Q)]$ IMPLIES NOT Q.

2. We translate this structure into algebra. Again for any A and B, A AND $B = AB$, and A IMPLIES $B = 1 + B + BA$. Also we use the operator NOT $A = 1 + A$ This means that our argument can be represented algebraicly as this expression.

3. We reduce all expressions algebraicly. If we end up with a 1, we have logical satisfaction: the argument is logically consistent.

4. We apply common sense and our canon of existential import.

$$
\begin{array}{rl}
\overline{[\text{NOT } P \text{ AND } (P \text{ IMPLIES } Q)] \text{ IMPLIES NOT } Q} & \text{[denying the antecedent]} \\
(1+P)(1+P+PQ) \quad \text{IMPLIES } 1+Q \\
1+P+PQ+P+PP+PPQ \quad \text{IMPLIES } 1+Q \\
1+P \quad \text{IMPLIES } 1+Q \\
= 1+(1+P)+(1+P)(1+Q) \\
= 1+(1+P)+(1+Q+P+PQ) \\
= 1+P+PQ \\
\neq 1
\end{array}
$$

With our logical Form A $P + PQ = P(1+Q) = 0$ but someone arguing by denying the antecedent finds on the chalkboard that $1 + P + PQ = 1 + P(1+Q) \neq 1$, and then without any nudging from the instructor subtracts 1 from both sides to get $P(1+Q) \neq 0$ violating Form A's principle of non-contradiction. We can't be in two places at the same time, at least in Newtonian space-time. But look at Form O, “Some P is NOT-Q” $= P(1+Q) \neq 0$,” and we discover the reason why denying the antecedent is a fallacy: sometimes A's do not yield lollipops, in contradistinction to the premise of this whole argument. The only way the argument holds any water is to highly qualify the premise itself. But is that not, in philosophy class at least, putting “des-Cartes before the horse,” (all puns intended)?

Why belabor this point? First, in my casual sampling experience over 5 decades I find that my hypothesis that 9 out of 10 people with quite a bit of standardized and highly structured education believe that denying the antecedent is a valid piece of logic. Second, some of these folks, and nearly all who attend schools and colleges of business and engineering, have been drilled with Null Hypothesis Significance Testing as the only approach for practical statistical inference. Third, NHST uses a *modus tollens* valid piece of argumentation, but in many instances, these same structurally educated folk, often deploy arguments which deny the antecedent. [3]

In a regression of wages dependent on education, we suppose it is found that the impact of education is not consistent with being zero, somehow throughout most of the sample. We waive our rights to determine the thresholds inherent in the judgment of consistency, or even what a sample is and what *most of* entails.

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[3]To get an idea of the order of magnitude of the potential usage of NHST, according to recent enrollment trends published by AACSB about 1,000,000 students world-wide annually (on a very rough average over the decade spanning 2014-2024) and using U.S. Department of Education data for business school graduates with an estimate of 166,000 graduates in 1974, the 50 year linear average is about 600,000 graduates per year. Thus in a recent 10-year period approximately $(0.9)(0.8)(1,000,000)(10) = 7,200,000$ students and in a 50 year span a very rough order of magnitude estimate of $(0.9)(0.8)(600,000)(50) = 21,600,000$ students will have taken core curriculum statistics courses mostly (say 80% to be conservative) based on the null hypothesis / frequentist paradigm with my very rough posterior rate of 90% who identify the *modus tollens deny the antecedent* fallacy as logically virtuous.

Under the null hypothesis we suppose that impact is, for all intents and purposes, nil. Otherwise it is not. P: If nil, then Q: accept the null hypothesis of no impact. However, we find NOT-P: not nil. Therefore, NOT-Q: do not accept the null hypothesis as true. In fact, under null hypothesis lore, not to accept the null is to reject the null and accept the alternative, here that there is an impact (statistically significant, etc., usw, ktl, and so on). But, as we have already seen, this fallacious argument boils down to FORM O relief that "Some P: impacts are Q: not significant (null hypothesis)." [4]

There is a "yay" in the offing, namely, affirming the consequent, a fallacy of the logical abuse of *modus ponens*. Here is the fallacy portion of our program for *modus ponens*: Affirming the Consequent (Q: lollipop bribe).

1. We set out the structure. We let P : "you get an A" and Q : "I'll give you a lollipop." With this assignment of propositions we state the affirmation of the consequent argument as $[Q \text{ AND } (P \text{ IMPLIES } Q)] \text{ IMPLIES } P$.

2. we translate this structure into algebra. For any A and B, $A \text{ AND } B = AB$, and $A \text{ IMPLIES } B = 1 + B + BA$. Also we know that $A^2 = AA = A$ and $A + A = 0$ in the mod 2 arithmetic of the algebra of Boole. This means that our argument can be represented algebraicly as this expression.

3. Then at last we reduce all expressions algebraicly. If we end up with a 1, then we have logical satisfaction.

4. We apply common sense and existential import.

$$
\begin{array}{rl}
\hline
[Q \text{ AND } (P \text{ IMPLIES } Q)] \text{ IMPLIES } P & \text{[modus ponens]} \\
\hline
Q(1 + P + PQ) \text{ IMPLIES } P \\
Q + QP + PQQ \text{ IMPLIES } P \\
Q + PQ + PQ \text{ IMPLIES } P \\
Q + 0 \text{ IMPLIES } P \\
1 + Q + QP \neq 1
\end{array}
$$

This was a little easier to set up and subsequently also understand. We have one-way causality from getting an A to receiving a lollipop. That's the admissible rule we must abide by. In fact it is a constraint imposed on this argument, a requirement with which we must comply (at least here). The *modus ponens* affirmation of the consequent is yet another example of a logical inversion. The reply to this fallacy is that the fallacy titrates to making the consequent the antecedent. I'm pretty sure that if I gave my son a lollipop he might, or he might not, get an A on his mixed fractions test in 4th grade. In fact after I gave him the lollipop he broke out in a fever (not from the lollipop), but from yet another classroom plague of viruses. So he did not get his A.

Our rampant enchantment with connectivity technology is this affirmation of the consequent inversion with this structure. We let P : "humans" and Q : "technology" With this assignment of propositions we state the affirmation of the consequent argument as $[Q \text{ AND } (P \text{ IMPLIES } Q)] \text{ IMPLIES } P$. We have two subjects, humans and technology. The predicate which links these two subjects is the implicating action "govern." Thus $P\ IMPLIES\ Q =$ "humans govern technology." The affirmation of the consequent "technology" results in the inversion $Q\ IMPLIES\ P =$ "technology governs humans." Perhaps Čapek (1923) made a good point about the relationship between humans and their *roboti*.

---

[4]Recent work by Nakhlé et al. (n.d.) in clinical medical settings exploit this idea. While both null hypothesis testing and Bayesian approaches yielded the same binary result, clinicians benefit from a knowledge of the "strength of evidence" available only in the Bayesian approach. Mertens and Recker (2023) develop proposals for obviating known and emerging threats to interpretation of empirical research in information systems based on abuses of hypothesis testing approaches.

On the horns

That is, the *horns of the dilemma*, is what we ride on. We have dilemmas when we have options which conflict with one another, and each option has its own pros and cons. The pros and cons might be shared across options, but not completely, or we have literally, no dilemma to annoy us. Here are three examples, all conflicts.

- The infamous **Trolley** problem, for example, testing perceptions of harm in Cushman, Young, and Hauser (2006). We might imagine (that is, perform a thought experiment) whether to sacrifice one person to save a larger number.

- A variant of the trolley is the **conflict of interest** dilemma. Here we imagine we might impose our personal beliefs on others, sacrificing their interests for ourselves.

- Here's an **incentive conflict**. Conscientious workers will endeavor to meet their obligations without the threat of a loss in pay, and indifferent workers won't endeavor to meet their obligations study and learn even with the threat of an exam. So, incentives serve no purpose. The reasoning might be valid, but inconsistent with considering the existential import of what is a good versus a bad worker.

- **Option overload or aversion?** In this dilemma when presented with only a single option, really no option except the status quo, people may be unlikely to buy, even if it's an option they like, thus aversion. On the other hand, given too many, and possibly the double-barrel of too complex, options, can lead to *analysis paralysis*, no choice, and reliance, again, on the status quo.

All of the work we have done to help us relate various propositions, even test their validity and coherence, pull together the essential components of *dialectical reasoning*. In what follows we will build *constructive* (negative propositions not allowed), and *destructive*, (many, even double negatives abound to confuse coherence). These arguments will be *simple* (one consequent allowed) or (exclusive or) *complex* (more than one consequent allowed). We will construct a propositional logic table first, then translate to our algebraic formulation. $$

[propositional logic]	*simple*	*complex*
both true	$(p \rightarrow q) \wedge (r \rightarrow q)$	$(p \rightarrow q) \wedge (r \rightarrow s)$
yet, the horns	$p \vee r$	$p \vee r$
constructive	$\therefore q$	$\therefore q \vee s$
both true	$(p \rightarrow q) \wedge (r \rightarrow q)$	$(p \rightarrow q) \wedge (r \rightarrow s)$
yet, the horns	$\neg q \vee \neg r$	$\neg q \vee \neg s$
destructive	$\therefore \neg p$	$\therefore \neg p \vee \neg r$

[algebraic logic]	*simple*	*complex*
both true	$(1 + p + pq)(1 + r + rq)$	$(1 + p + pq)(1 + r + rs)$
yet, the horns	$p + r$	$p + r$
constructive	$\therefore q$	$\therefore q + s$
both true	$(1 + p + pq)(1 + r + rq)$	$(1 + p + pq)(1 + r + rs)$
yet, the horns	$(1 + q) + (1 + r)$	$(1 + q) + (1 + s)$
destructive	$\therefore 1 + p$	$\therefore (1 + p) + (1 + r)$

To illustrate our analysis of the logic, and import, of the *simple-constructive* argument, here is an example. Many of us might recognize this argument against skepticism.[5]

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[5]Plato's *Gorgias* displays an extreme skepticism is really nihilism, It works in three moves: I can't really know anything; and if I did I would not know if it is right or wrong; and if I was not ignorant of the value of knowing, I would not be able to share it anyway – thus, I can say or do anything. The argument in our dialectical reasoning example is squarely the opposite of the skeptical triple play. Is this not some version of "I think, therefore I am"? If you answer yes, then you might know of Rene DesCartes.

- If p: "I know anything at all with certainty," then q: "I certainly exist." *AND*

- If r: "I am ignorant of any certainty," then q: "I certainly exist."

- *BUT* "Either p: I know anything at all with certainty," *OR* " r: I am ignorant of any certainty."

- [*THEREFORE* q: "I certainly exist."]

We know we can translate these propositions, this argument, into our algebraic

$$
\begin{array}{rl}
(p \text{ IMPLIES } q) \text{ AND } (r \text{ IMPLIES } q) & \text{[simple-constructive]} \\
\text{AND } (p \text{ OR } r) & \text{IMPLIES } q \\
\hline
(1+p+pq)(1+r+rq)(p+r) & \text{IMPLIES } q \\
(1+r+rq+p+pr+prq+pq+pqr+pqqr)(p+r) & \text{IMPLIES } q \\
p+r+rp+rr+prq+rrq+pp+pr+ppr+prr+pprq+prrq & \text{IMPLIES } q \\
pq+rq & \text{IMPLIES } q \\
1+(pq+rq)+(pq+rq)q & = 1+pq+rq+pqq+rqq \\
& = 1+0+0 \\
& = 1
\end{array}
$$

Well worth the effort. All of the top-side propositions: both the two implications (conditionals) and the disjuction boil down to $(p+r)q = (p \lor r) \land q$. In words (and remembering we are using the exclusive either, or, not both) we have: "Both I certainly exist, and, either I know anything at all with certainty, or I do not." The two either-or conditions are linked both-and to the one conclusion.

Can we refute this argument, given its structure (always the maintained *hypothesis*: "if we believe this structure to be the horns of a dilemma, then we maintain this belief in all we do next with this structure.") Only by destroying the either-or conditions, or by denying the both-and linkage. Let's take the denial first. To deny the both-and linkage we would have to establish that any and all knowledge are incompatible with existence. And to do that we would have to deny the existence of a knower who can know either with or without certainty. And this is absurdly against common sense existential import. Someone is typing here! Now how can we possibly destroy the conditionals, the either "with" or "without" certitude. If we believe that nothing is certain, then the either-or statement is still true; similarly is at least one thing is certain, such as $1+1=0$ in our binary arithmetic, and on our binary clock too. Thus refutation seems absurd again. We may indeed have arrived at a logical immutable truth! This is called a *tautology*, a meaning that is true in its structural self (in Greek, *auton*). The only way we can then refute this tautology is to present alternative structure.

Ride with Socrates

We have three strategies with every dilemma: ride the horns (and risk getting gored), go between the horns (and risk getting thrown), and make another bovine to ride new horns (risk yet more horns, goring, and being thrown). On the upside, we have used logic to discover the logical consequences of our thinking. We will call all of this the intellectual process of *reasoning* goaded on by our incessant, undying (always a spark exists) drive to know all that there is (Being, that is), and we will maintain the practical circumstance of limited time and space, so that it is process, not perfection. We will use our process of reasoning to judge whether something is or is not. And we recall that probability, or any other qualifier such as, might, could, should, would, and so on, Can be part of any proposition

Just Suppose

We can suppose, for a minute or so, that there are three propositions A, B, and C. They have context in particular situations, for example, in decision making. In this context, we might suppose that A is the proposition "developers make reasonably successful decisions". We also suppose that B is the proposition

"product management makes reasonably successful decisions." And now we put proposition C into context as "Teams use data and experience consistent with prevailing recommended practices". Other than observations made by teams ad something that neither the development manager nor the product management manager might yet any specific idea about, but does have at the very least an intuitive feel about something which has yet to be revealed. Of course, we need at some point to define a bit more precisely what is meant by "reasonably successful" and "consistent with prevailing recommended practices" all for another time and deep conversation.

Let's put these pieces together and use the apparatus of our logical algebra to discover literally a new insight. To couch our analysis in terms of what we need, or wish for, in our analytical logic robot we require instructions for the robot must hold to these desiderata.

1. Plausibilities can be ordered as rational numbers. The so called irrational numbers such as the solution x to the equation $y = x^2$, can be approximated to any degree of accuracy the robot can determine within the finite bounds of the robot's memory, ability to process, and share with those who use the robot as a tool of discovery.

2. Whatever the robot helps us to infer must be subject to common sense in the real world of life and persons who use the robot as a tool.

3. Whatever unobserved hypothesis which the robot helps us find is more plausible than another must be consistent with observed data.

Now we can get to work.We ask "Is A and B true?" As managers of development and products we surely expect such a result. As usual there are two schools of thought on the matter.

1. [Case 1] B is true given C, that is, the plausibility B|C. Having decided that B is true given C, decide that A is true, that is, the plausibility A|BC.

2. [Case 2] A is true given C, that is, the plausibility A|C. Having decided that A is true given C, we, decide that B is true, that is, the plausibility B|AC.

We will show that Case 1 is equivalent to Case 2 logically even though they each represent two seemingly different perspectives. Case 1 is the perspective of product managers who launch products for a living, while Case 2 is the perspective of developers who, well, develop products for a living. Those of use you attempt to manage developers and product managers would very much like to know that AB is true.

Let's compute, with our robot's mechanisms, the joint proposition "A|BC and B|C". With our algebra of logic we have $x = A|BC$ and $y = B|C$ so that both-and amounts to multiplying $xy = (A|BC)(B|C)$. We also know through study of logical givenness, also known as implication, that for any propositions a and b $b|a = 1 + a + ab$. Now we can calculate.

For Case 1 we have this calculation.

1		$1 + BC + ABC$	definition of $A|BC$
2	$\times$	$1 + C + BC$	definition of $B|C$
3		$BC + BC + ABC$	since $X^2 = X$
4	$+$	$C + BC + ABC$	ditto
5	$+$	$1 + BC + ABC$	ditto
6	$=$	$1 + C + ABC$	since $X + X = 0$
7	$\therefore$	$AB|C = (A|BC)(B|C)$	definition of $1 + C + ABC$

For Case 2 we have this calculation.

1		$1 + AC + ABC$	definition of $B\|AC$
2	$\times$	$1 + C + AC$	definition of $A\|C$
3		$AC + AC + ABC$	since $X^2 = X$
4	$+$	$C + AC + ABC$	ditto
5	$+$	$1 + AC + ABC$	ditto
6	$=$	$1 + C + ABC$	since $X + X = 0$
7	$\therefore$	$BA\|C = (B\|AC)(A\|C)$	definition of $1 + C + ABC$

Since BA=AB by the commutativity of the both-and operation we now also have shown that Case 1 is logically equivalent to Case 2. Here is the new knowledge, an insight, from all of these deductions.

1		$AB\|C = (A\|BC)(B\|C)$	Case 1 is true
2	and	$BA\|C = (B\|AC)(A\|C)$	Case 2 is true
3	however	$AB = BA$	commutativity of AB
4	so that	$AB\|C = BA\|C$	substitution of equals
5	$\therefore$	$(A\|BC)(B\|C) = (B\|AC)(A\|C)$	transitivity of equality
			if a=c, and b=d, and a=b, then c=d

We have just also unveiled the vaunted product rule of probability. We have but one more step to apply this rule to the work of our managers, namely, the making of reasonable decisions, where reasonable at the very least means logically valid deductions about the consistency of data with hypotheses. Here a decision will take the form of a hypothesis, which, if true, will indicate a direction in which managers might reasonably act.

Torturing Data with Hypotheses

So line 5 of our conditional event proof is true. By desiderata 1, namely, that plausibilities can be ordered as rational numbers, we assign numbers to logical expressions which in turn represent events. For comparison purposes across wildly disparate events, plausibility scores will lie between 0 and 1 such that, for a mapping $P(x)$ of plausibility of the truth of a logical expression x to the interval of rational numbers (and approximations of so-called irrational numbers) between 0 and 1 inclusive, we have for rational number $0 < r \leq 1$

$$P(x) = \begin{cases} 0, & \text{if } x \text{ is false} \\ r, & \text{if } x \text{ is true} \end{cases}$$

Yes, we are optimistic, looking for the plausible truth value of a logical proposition.[6] Now it gets very serious since we now examine what exactly we will mean by reasonable, also known as rational.

We let $H = A$ for our ordered list of hypotheses H, $D = B$ for our list of data D, $X = C$ for our list of instances of prior experience, previous choices and data X. With these substitutions we now have this expression.

$$(H|DX)(D|X) = (D|HX)(H|X)$$

Since the logical expressions in the parentheses are mapped to a plausibility index $P(x)$, we can use the normal rules of arithmetic and algebraic to divide both sides by $(D|X)$.

$$\frac{(H|DX)(D|X)}{(D|X)} = \frac{(D|HX)(H|X)}{(D|X)}$$

Since $(D|X)/(D|X) = 1$ and anything times 1 is itself we arrive at our signal result with new notation $(x) = P(x)$.

$$P(H|DX) = \frac{P(D|HX)P(H|X)}{P(D|X)}$$

We can even calculate $(D|X)$ using some common sense and our logical algebra. First of all lets simple consider that X is true and calculate $P(D)$ by itself. Then we consider that (D) will be a mixture of $(D|H)$ and $(D|\overline{H})$ to account for all of the way D might occur. Thus

$$P(D) = P(D|H)P(H) + P(D|\overline{H})P(\overline{H})$$

The algebra works out to simplifying the RHS of this expression all by remembering that $1+1=0$ and $a+b = 0$ in mod 2 arithmetic and algebra. We just need to show that the RHS = LHS, other words, one side implies the other.

$$\begin{aligned} D \; &= (1 + H + DH)H + (1 + (1 + H) + (1 + H)D)(1 + H) \\ &= H + H + DH + 1 + 1 + H + D + DH + H + H + H + DH + DH \\ &= D \end{aligned}$$

We have a match. We have only now logically justified Bayes Rule. It would appear deductions can provide us with new knowledge and insights into our decisions and the data which consistently will help us guide our actions. We also have an alternative to the potentially misleading use of Null Hypothesis Significance testing.

How many ways?

We began this whole discussion with the two schools of thought proposed by Cox (1946). Here we implement the school of thought which states that inference is all about the expectation of an outcome. We condition the expectation with observed data and (un)observed hypotheses relevant to the question which gives rise to the inference. We then implement the computation of an expectation by simply counting the ways in which the expectation (data and hypotheses conditions) can occur. We effectively answer a question for reflection, of inference, that is, the question "whether it is?", with the frequency (the count) of the number of ways data are consistent with hypotheses. In this way we release ourselves both from the tyrannies of data alone and hypotheses alone. In a very realistic sense we have bridged the polar gaps, pun intended, of pure empiricism (*sola data*) and pure idealism (*sola hypotheses*).

Let's use a simple example. We have four voters in an upcoming election. They may be red or blue voters. Three of us go out and talk to three voters at random, that is, indiscriminately. One of us happens to come upon a blue voter, another of us, independently, happens to find a red voter, and the other separately finds a blue voter. This is the very definition of a random sample. Each of the finders does not know what the other is doing, all three do know that there are four voters out there and they happened to have independently talked to two blue and one red voter. How many red voters and how many blue voters are there?

Here are all of the possible conjectures we can make for $blue = \square$ and $red = \square$ voters.

hypotheses	1	2	3	4
1	$\square$	$\square$	$\square$	$\square$
2	$\square$	$\square$	$\square$	$\square$
3	$\square$	$\square$	$\square$	$\square$
4	$\square$	$\square$	$\square$	$\square$
5	$\square$	$\square$	$\square$	$\square$

Reading this we see that there are 4 voters, which we can call ways the voters are arranged, and 5 different voter compositions ranging from all red to all blue, each of which we call a hypothesis. Our sample is 2 blue voters and 1 red voter, so we can very safely eliminate the first and fifth conjectures from our analysis, but for the moment just keep them for completeness sake.

For each of the three remaining conjectures we may ask how many ways is the conjecture *consistent* with the collected data. For this task a tree is very helpful. Let's take the first realistic conjecture the □, □, □, □ hypothesis and check if, when we sample all of the four voters, what are all of the ways this conjecture fans out. So here we go.

1. We sampled a □ first. How many □'s are in this version of the composition of voters? only 1.

2. We then sampled independently a □. How many □'s are in this conjecture? Quite a few, 3.

3. Finally we sampled a □ at random. We know there is only one □ in this version of the truth.

So, it is just counting the ways: 1 □ way x 3 □ ways x 1 □ way $= 1 \times 3 \times 1 = 3$ ways altogether.

When asked, many surmise that the 2 blue and 3 red conjecture is the right one. Are they right? Here is a table of the ways each conjecture pans out. We then in a separate column compute the contribution of each conjecture to the total number of ways across the conjectures, which is $3 + 8 + 9 = 20$ ways. Also each of the conjecture propose a proportion p of the successes, that is, the blue voters in this context.

h	1	2	3	4	ways: $2\square 1\square$	relative ways
1	□	□	□	□	$0 \times 4 \times 0 = 0$	0.00
2	□	□	□	□	$1 \times 3 \times 1 = 3$	0.15
3	□	□	□	□	$2 \times 2 \times 2 = 8$	0.40
4	□	□	□	□	$3 \times 1 \times 3 = 9$	0.45
5	□	□	□	□	$0 \times 4 \times 0 = 0$	0.00

We cannot help but note that the proportion of ways for each conjecture can range from 0, perhaps to 1, since the proportions add up to 1. The number of ways also expresses the number of true consistencies of the data with the conjecture, an enumeration of the quality of the logical compatibility of conjectures with what we observe.

We might now revise our common sensical surmise that 2 blue and 2 red is the better conjecture. However, if we use the criterion that the conjecture with the most ways consistent with the data is the best choice for a conjecture, then clearly here we would say that there are 3 blues and 1 red. Perhaps we have a better criterion that would choose our equinanimous choice of 2 blues and 2 reds? It does not appear to be so.

Ways are the count, the frequencies of logical occurrence of a hypothesis *given* the data. The data includes the knowledge that there are possibly blues and reds, that there are 4 voters, and that we sampled 2 blues and 1 red. The relative frequency of the ways in which our conjectures are consistent with the data is what we will finally call *probability*.

1. The plausibility is a measure between and including 0 and 1.

2. The sum of all plausibilities is 1.

3. Common sense, again, rules the day, and the plausibility.

We have just quantified the *plausibility* of logical truth values. We have also found a very compelling criterion for the choice of a conjecture given the data and circumstances surrounding our inquiry.

How informative?

Now the piece de resistance we have been waiting for is at hand. We have the tools: information measurement, logic of plausible inference, probability as plausibility, a criterion for making a decision? If we were to say, pick the most plausible, now also the most probable inference, is this not really to say that it is the most informative and thus the most surprising inference?

We have $i = 5$ conjectures c_i, each with an assignment of the probability of finding the inference in data, thus 5 probabilities p_i. We also agree that each conjecture c_i is independent of the other, that is mutually exclusive, and also mutually exhaustive. In this way the probabilities indeed will add up to 1.

Each conjecture has $N = 4$ symbols in it. There are only 2 symbols to work with, $B = blue$ and $R = red$ in these messages. There are $N! = 4x3x2x1 = 24$ ways to arrange symbols. But these are constrained by two symbols in this pile so we have on average this many combinations:

$$\frac{(N + s - 1)!}{N!(s - 1)!} = \frac{5!}{4!1!} = \frac{120}{24} = 5 \; conjectures$$

In the numerator are the four voters, two kinds of voters, less one. This gives us $5! = 120$ total ways to arrange this set of conjectures. But there are $4!(2-1)! =$ ways to pick 2 kinds of voters from a population of 4 voters. That comes from the notion that there are 4! ways to choose 4 voters and having chosen the first voter from the voter urn, there are $(2 - 1)!$ more ways to choose voters, either blue or red, each and every time we randomly go out to the voting population, all with replacement. Phew!

Now that we have our sea legs in combinatorics, each conjecture of length $N = 4$ has two symbols $s_1 = blue$ and $s_2 = red$ with probabilities of occurrence in the message $p_1 = p(blue)$ and $p_2 = p(red)$ here 0.50 each. The portion of the messages that are blue are $Np_1 = 4 \times 0.5 = 2$ and red are $Np_2 = 4 \times 0.5 = 2$ as well. The ratio

$$\frac{N!}{(Np_1)!(Np_2)!} = \frac{4!}{2!2!} = \frac{24}{(2)(2)} = 6$$

in turn gives us the number of combinations of symbols and messages. This is the average number of ways a message can be formed. The expected amount of information in these messages, as conjectures, is

$$< I >= log_2 \left(\frac{N!}{(Np_1)!(Np_2)!} \right)$$

For large enough N we can use Stirling's Approximation[7]

$$< I >= N \sum_{i=1}^{2} (-p_i \, log_2 \, p_i)$$

So the amount of informativeness in each of the voter conjectures is

$$< I >= 4 \sum_{i=1}^{2} (-0.5 \, log_2 \, 0.5) = 4$$

This is intuitively correct: 4 bits of information in a population of 4 with 2 equally likely choices.

What about the 5 conjecture? Now $N = 5$ and we only need $p_2 = 0.15$, $p_3 = 0.40$, $p_4 = 0.45$ since the other probabilities are 0.

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[7]Named after James Stirling, this approximation was first stated by Abraham de Moivre without the use of π as $ln(n!) = n \, ln(n) - n + error$, very accurate even for small n.

$$< I >= 4 \sum_{i=1}^{3} (-p_i \, log_2 \, p_i) = 4(1.5) = 5.8$$

This is nearly 2 bits more informative than any single conjecture. The best conjecture contributes 2.1 bits of information on its own or 36% of the informativeness of the total distribution of conjectures we reviewed.

Further thoughts

1. The location and scale parameters μ and σ we often calculate are not the mean and standard deviation of data x in the expectations school of thought. They are not properties of the physical representation of events called data. These parameters do carry information about the probability distribution of the representation of physical reality in data. To say μ is the mean of the data is to ascribe a mind's eye idea to the physical reality, to invest in physical, objective reality, a property that only exists in the mind. This is an example of the *mind projection* or *reification* fallacy much in vogue in the circles of *fake*, or better yet, *chop logic*. In the same way, probabilities only exist in our minds: there is no physical reality that is a probability, just a mental construct that helps us think through potential outcomes abstracting similar and dissimilar properties as we observe from data.

2. When we say *A implies B* we mean that *A* and *AB* have the same truth value. In logic every true statement *implies* every other true statement. Just knowing that *A* and *B* are true does not imply a relationship exists between A and B, except that the propositions are true.

In the end

We can now study any arrangement of events in logical order and map them to plausibility values. The mapping, the logical order, the propositional space we have constructed from first principles all ground our notion and implementation of forming consistent, relevant, and necessary consistencies between what we expect and what we observe. Along the way we discovered a method, perhaps we can call this a technique, to satisfy our desire to determine the existence, and degree, of truthfulness of any set of propositions, encapsulating terms, and manifesting in an argument. We pulled together basic inferential structures and applied these to a first look at hypothesis testing, dialectical reasoning, and Bayesian inference. We need always remember that this approach requires at the outset a contextual matrix against which we analysts can interpret our results, examine their relevance, and inform the consumers of our analysis.

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