Group 5

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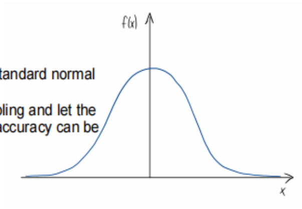
Exercise 1 (Markov Chain Monte Carlo)

a) rejection sampling

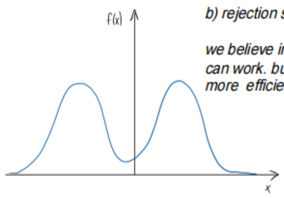
The density function matches the standard normal distribution.
So when we choose rejection sampling and let the g(x) be normal distribution, higher accuracy can be obtained.



(a)

b) rejection sampling

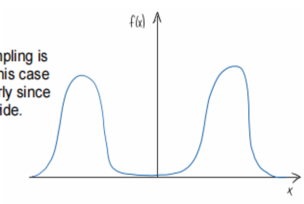
we believe in this case, both methods can work. but rejection sampling is more efficient.



(b)

c) Metropolis-Hastings

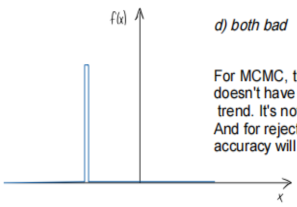
In general the g(x) for rejection sampling is uniform or normally distributed, in this case rejection sampling will perform poorly since the normal distribution will be too wide.



(c)

d) both bad

For MCMC, this density function doesn't have increasing or decreasing trend. It's not suiable to use MCMC. And for rejection sampling, the accuracy will be very low.



(d)

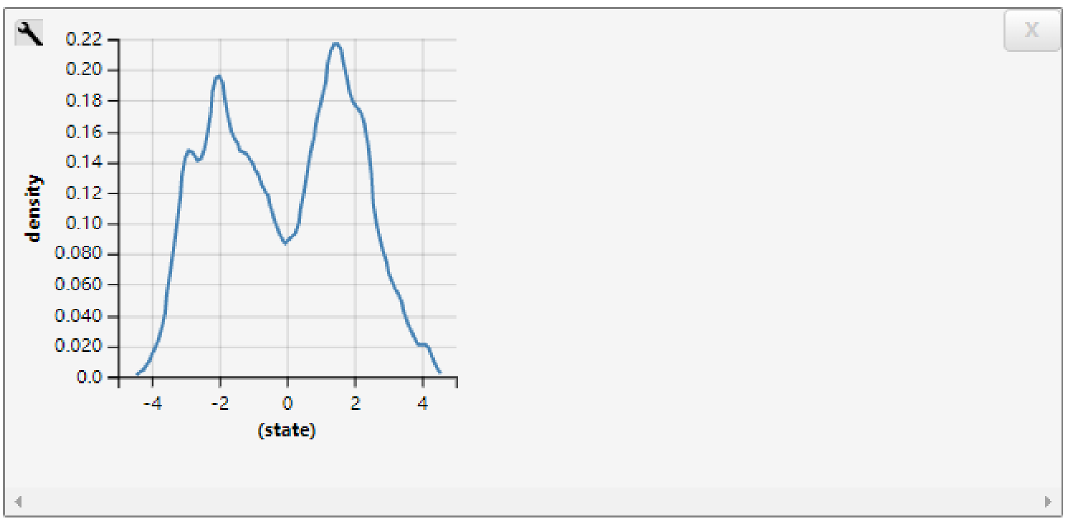
Exercise 2 (Metropolis Algorithm)

(a) & (b)

.wppl for code part, here we only show the visualization results.

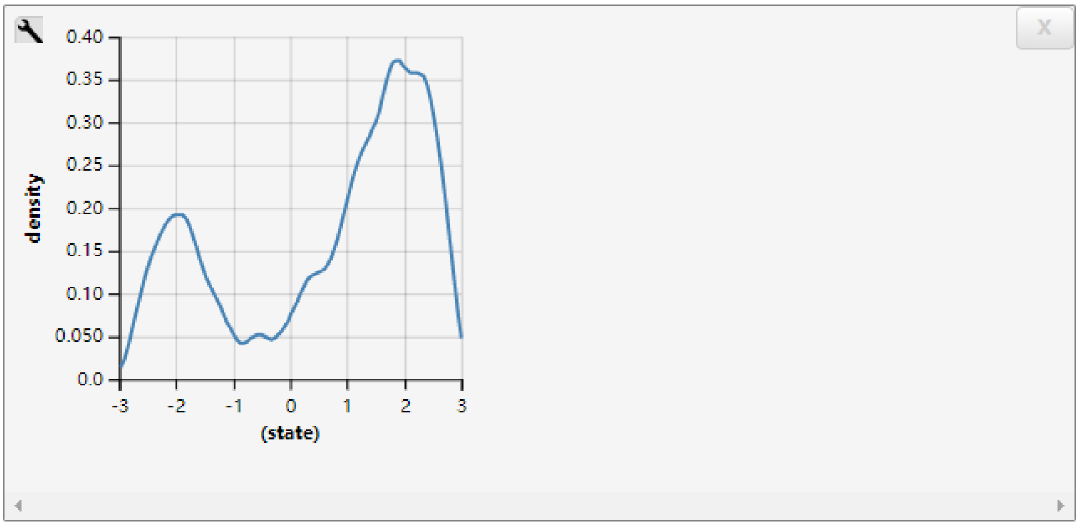
For function (1)

1. $f(x) = \frac{1}{2}e^{-\frac{(x-2)^2}{2}} + \frac{1}{2}e^{-\frac{(x+2)^2}{2}}$, with $g(x_{t+1}|x_t) \sim \mathcal{N}(x_t, 0.3)$.



For function (2)

$$2.\ f(x)=\begin{cases}-0.1319x^4+1.132x^2+0.5, & x\in(-3,3)\\ 0, & \text{o.w.}\end{cases}, \text{ with } g(x_{t+1}|x_t)\sim\mathcal{N}(x_t,0.3)$$



(c)

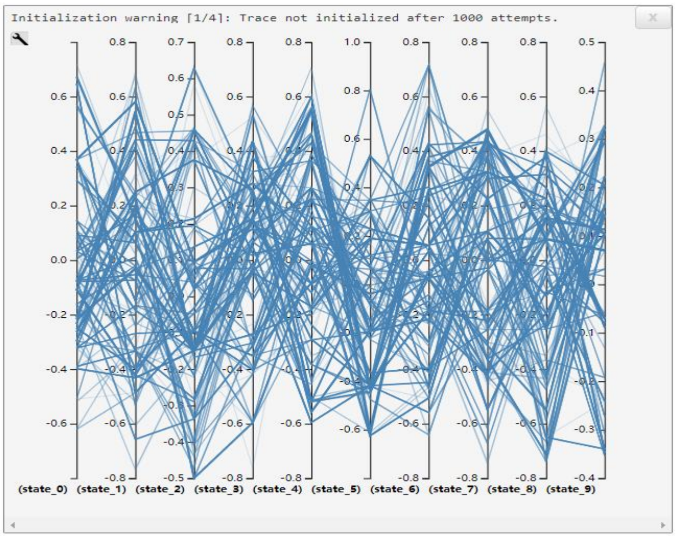
When σ^2 is larger than 0.3, which means that the step size is larger, the visualized graph will be wider than above.

When σ^2 is larger than 0.3, because the step size is too small, it will oscillate at the local optimum.

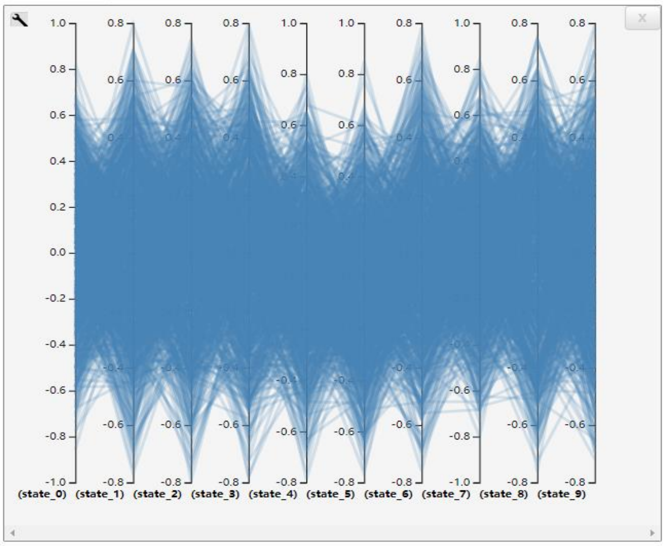
Exercise 3 (Rejection Sampling)

(a) & (b)

MCMC:



Rejection:



Since MH doesnot reject any point and adds a sample at each iterations, it would need 1000 steps to obtain 1000 samples.
On the other hand, as it is shown in c), for d=10 rejection sampling is going to reject with probability 0.0025 so we would need approximately 1000/0.0025 = 400,000 steps of the algorithm.
And this is the reason why we find MCMC is much quicker than Rejection.

c) $d = 10$

Volume $= \dfrac{2\pi^5}{10\,\Gamma(5)} \approx 2{,}55$

Probability: $\dfrac{2{,}55}{2^{10}} \approx 0{,}0025$

To obtain 1000 samples we would need $1000 : 0{,}0025$ steps

$d = 20$

Volume $= \dfrac{2\pi^{10}}{20\,\Gamma(10)} \approx 0{,}026$

Probability: $2{,}46 \cdot 10^{-8}$

To obtain 1000 samples we would need $\dfrac{10^3 \cdot 10^8}{2{,}46} \approx 4 \cdot 10^{10}$ steps

$d = 40$

Volume: $\dfrac{2\pi^{20}}{40\,\Gamma(20)} \approx 3{,}6 \cdot 10^{-9}$

Probability: $3{,}28 \cdot 10^{-21}$

To obtain 1000 accepted samples we would need around $3 \cdot 10^{23}$ steps

Exercise 4

a)
$$\sum_{i=1}^{n} \rho(i) P(i,j) \overset{\text{detailed balance condition}}{=} \sum_{i=1}^{n} \rho(j) \cdot P(j,i) = \rho(j) \underbrace{\sum_{i=1}^{n} P(j,i)}_{=1 \text{ by def. of MC}} = \rho(j) \quad \text{for all } j \in \underbrace{[m]}_{\{1,\dots,m\}}$$

in matrix form
$$\Rightarrow P^T_{(j)} \cdot \rho = \rho(j), \ \forall j \in [m] \Rightarrow P^T \rho = \rho \ \Rightarrow \rho \text{ is stationary distribution}$$

$P^T_{(j)}$: jth row of P^T

b) ii)
$$Q(i,j) = \mathbb{P}(Y_k = j \mid Y_{k+1} = i) = \frac{P(Y_{k+1} = i \mid Y_k = j) \cdot P(Y_k = j)}{P(Y_{k+1} = i)} = \frac{P(j,i) \cdot P(Y_0 = j)}{P(Y_0 = i)}$$

$$= \frac{P(j,i) \cdot \rho(j)}{\rho(i)}$$

c)
$$Q(i,j) \overset{b)}{=} \frac{P(j,i)\rho(j)}{\rho(i)} \overset{\text{time reversible}}{=} \frac{P(i,j)\rho(i)}{\rho(i)} = P(i,j) \ \square$$