Diversification and portfolio optimization

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Risk and return

The major questions for the rest of the course are:

- How we should define and measure investment risk?
- What kind of return do we have to forecast on an investment, in order for it to be worthwhile given its risk?

These are difficult questions, and we will not claim to “answer” them.

We will simply explore the way people have thought about them.

The approach of modern portfolio theory (MPT)

1. **What goal is an investor trying to accomplish?** Roughly speaking, maximize the average return on her wealth, minus a penalty for the variance of this return.

2. **Then, what risky portfolio does she want to hold?** The one with the highest possible Sharpe ratio.

3. **Then, how will she assess an individual investment?** Based on its alpha (α) compared to her current portfolio. If $\alpha > 0$, she wants to hold more of it. If $\alpha < 0$, less.

4. **Then, how will the average investor think?** The average investor must hold the market portfolio. Then the most natural "equilibrium" is given by the CAPM: All α values are zero, when compared with the market portfolio.

Useful approximation of geometric mean

- Geometric average return is the true measure of performance, because you can convert it directly to a cumulative return.
    - Geometric average is always less than arithmetic average, except in the special case that the return is constant over time.
    - A high arithmetic average return can be misleading!
- The geometric average return of a portfolio is hard to forecast, but we can apply a useful approximation:

$$\text{Geometric mean return} \approx \mu - \frac{1}{2}\sigma^2$$

(textbook formula 5.15)

Here σ is the volatility (standard deviation) of the return.

Tradeoff between average and volatility of portfolio returns

- From $\mu - \frac{1}{2}\sigma^2$ we see that **volatility hurts performance.**
  - This effect is often called “volatility drag” or the “volatility tax”.
- Diversification offers one way to lower the σ of a portfolio, and this can sometimes be worthwhile even if it lowers μ.
- This is true even for a patient and long-term investor!

Possible investor objectives

A **growth-optimal strategy** seeks to maximize average growth rate, so their "objective function" is

$$\mu - \frac{1}{2}\sigma^2$$

This apoproach is also called the “log-optimal” strategy or the “Kelly criterion.”

A more flexible objective is a *mean-variance utility function,*

$$\mu - A\frac{1}{2}\sigma^2$$

where A > 0 is potentially different for every individual. With $A = 1$ this is the growth-optimal strategy from above.

In principle, one could imagine many other objectives as well.

What we will assume about the investor's objective

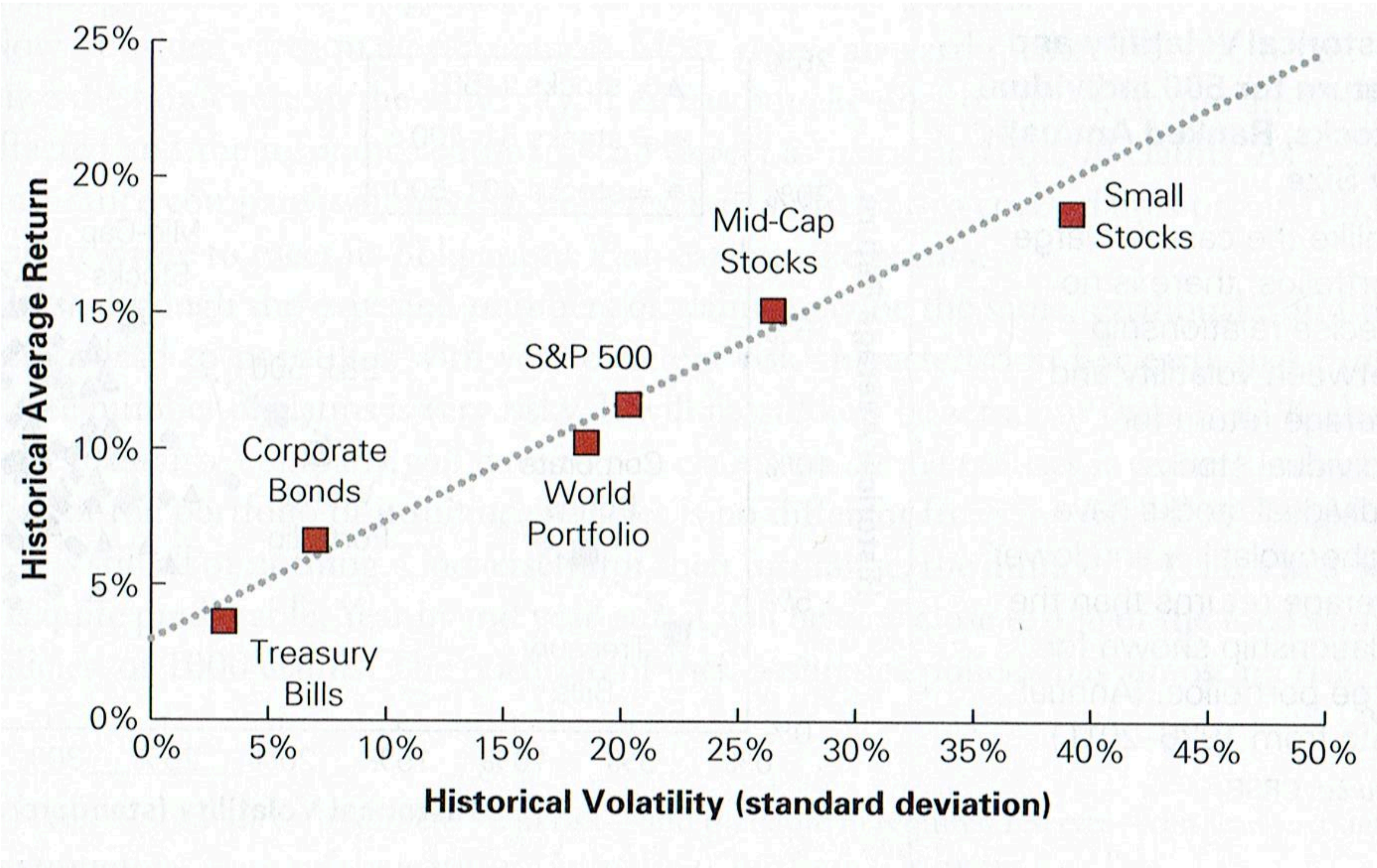
We will usually only assume that investors are **risk-averse:**
For a given level of μ, they prefer portfolios with lower σ.

- This assumption was true for both of the objectives above.
- Again, it seems reasonable even for a patient long-term investor.

We will also assume that σ is the investor's *only* risk measure.

- This may be simplistic, compared to how investors *think*. But it seems to capture most of how they *behave*.
- To be clear, we are assuming that investors measure the risk of their *entire wealth portfolio* only through its volatility. This does not mean that they evaluate individual investments based on volatility. In fact, we will see that they should not.

Volatility and average return across asset classes



Math note 1: μ and σ for a portfolio of two investments

A portfolio's return is the weighted average of its ingredients' returns.

For example, with just two investments A and B,

$$r_p = w_A \times r_A + w_B \times r_B$$

The weights w_A and w_B are the percentage of the portfolio's dollar allocated to A and B.

The portfolio's arithmetic average return μ works in the same way:

$$\mu_p = w_A \times \mu_A + w_B \times \mu_B$$

The portfolio's return volatility is more complicated (textbook formula 7.7): $\sigma_p = \sqrt{\sigma_p^2}$, where

$$\sigma_p^2 = w_A^2 \times \sigma_A^2 + w_B^2 \times \sigma_B^2 + 2 \times w_A \times w_B \times \sigma_{AB}$$

$$\sigma_{AB} \equiv \sigma_A \times \sigma_B \times \rho_{AB}$$

(These formulas all work with either raw returns or excess returns.)

Math note 2: μ and σ for a portfolio of many investments

Matrix formulas are cleaner, and work for any number of portfolio ingredients.

Suppose we have N investments with

$$r_t \equiv \begin{bmatrix} r_{1t} \\ r_{2t} \\ \vdots \\ r_{Nt} \end{bmatrix} \quad , \quad \mu \equiv \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix} \quad , \quad \Sigma \equiv \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \dots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \dots & \sigma_N^2 \end{bmatrix}$$

If we build a portfolio with weights $w = [w_1, w_2, \dots, w_N]$ then

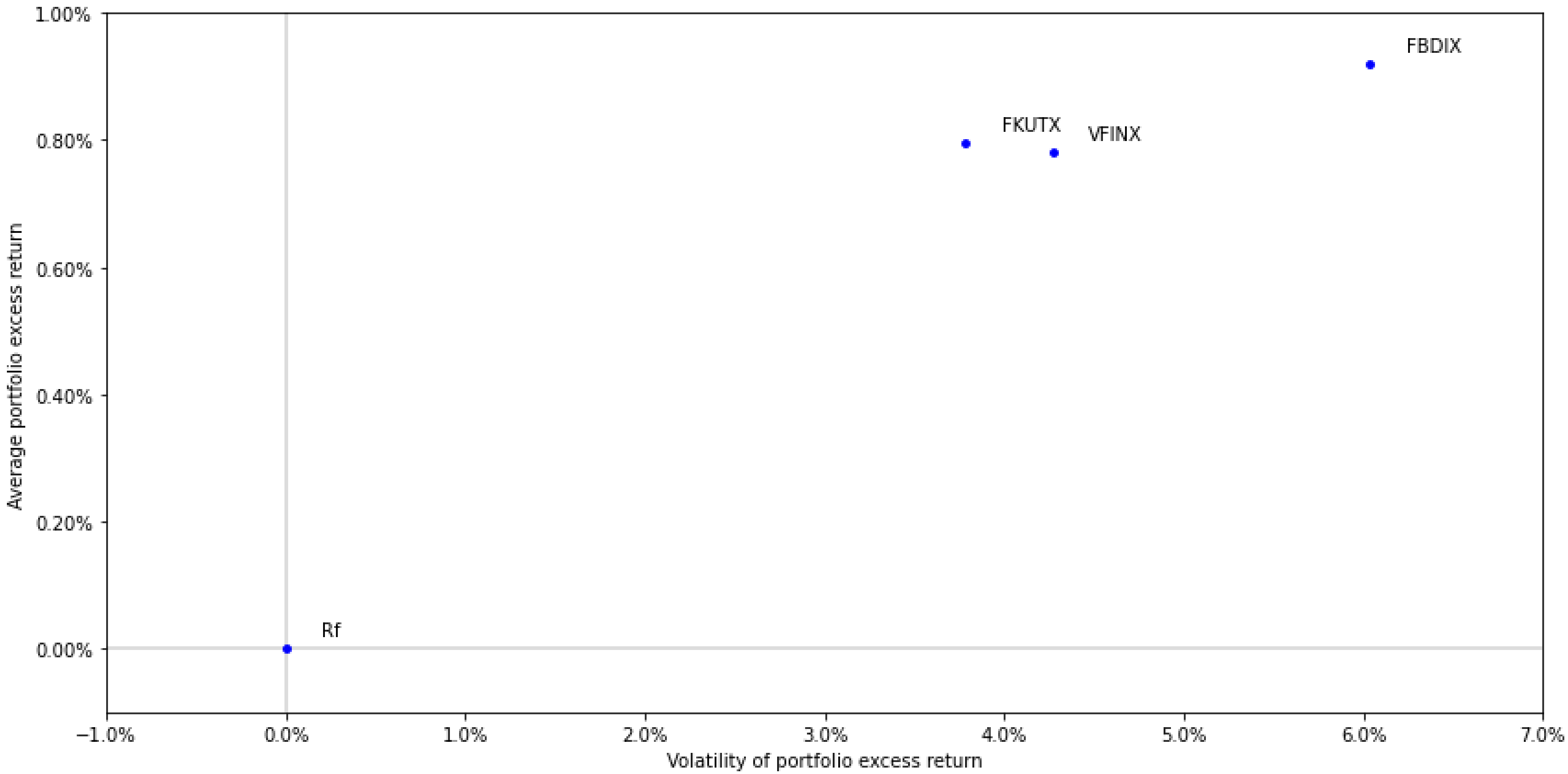
$$\mu_p = w'\mu \quad , \quad \sigma_p^2 = w'\Sigma w$$

You can check that $N = 2$ gives the formulas on the previous slide.

I will not test you on these formulas, but will use them at times.

Determining the investor’s optimal risky portfolios.

Suppose the investor has just three funds to work with, plus risk-free saving. Let's plot the average and volatility of their excess returns.



Allocation between risk-free saving and a risky portfolio

We separate the investor's decision into two questions:

- Which portfolio of risky investments to hold?
- How much to invest in it, and how much to save risk-free?

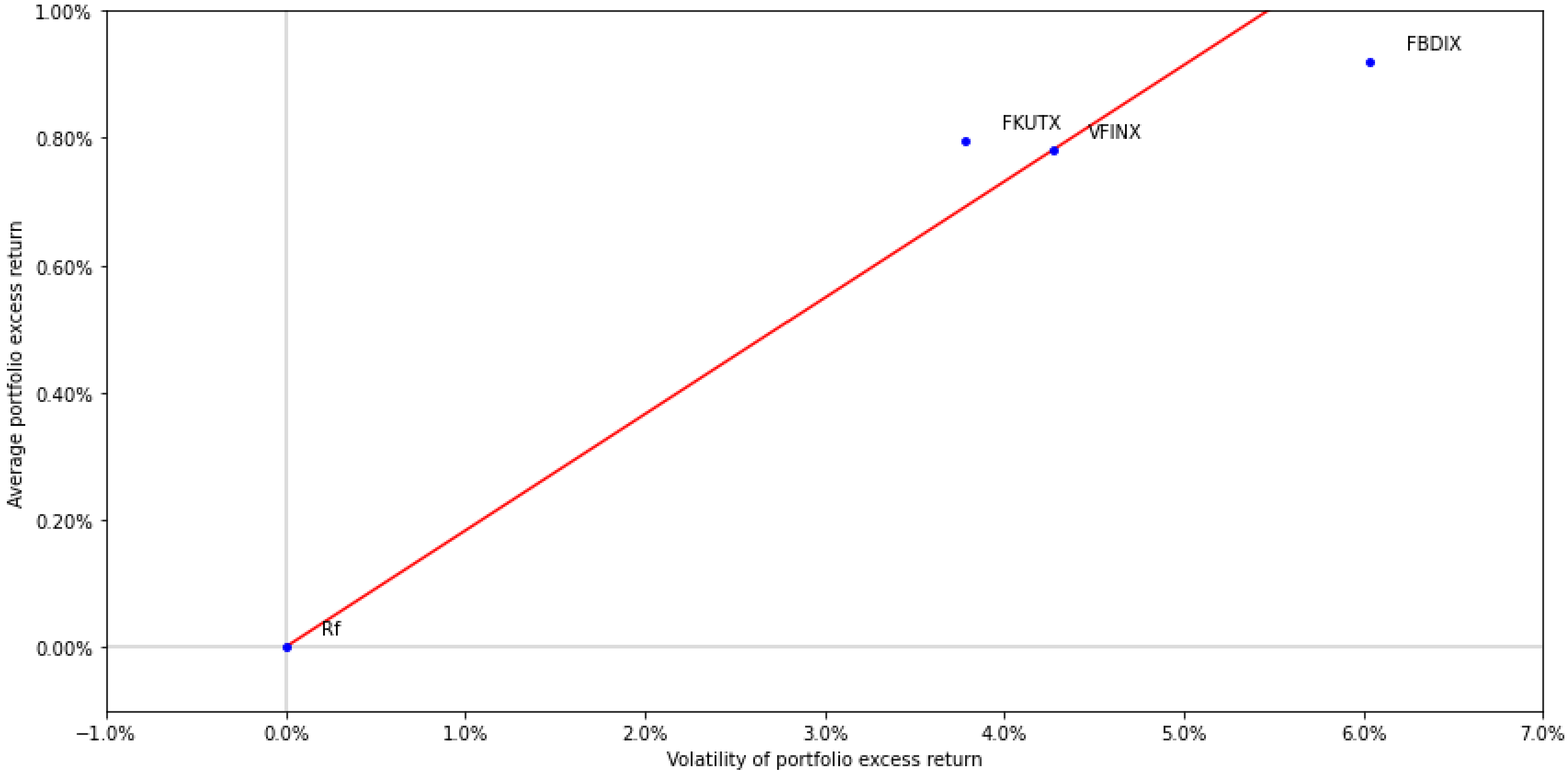
To understand the first question, we think ahead to the second. that has average excess return μ_p with volatility σ_p,

- Suppose an investor allocates weight w to a risky portfolio p,

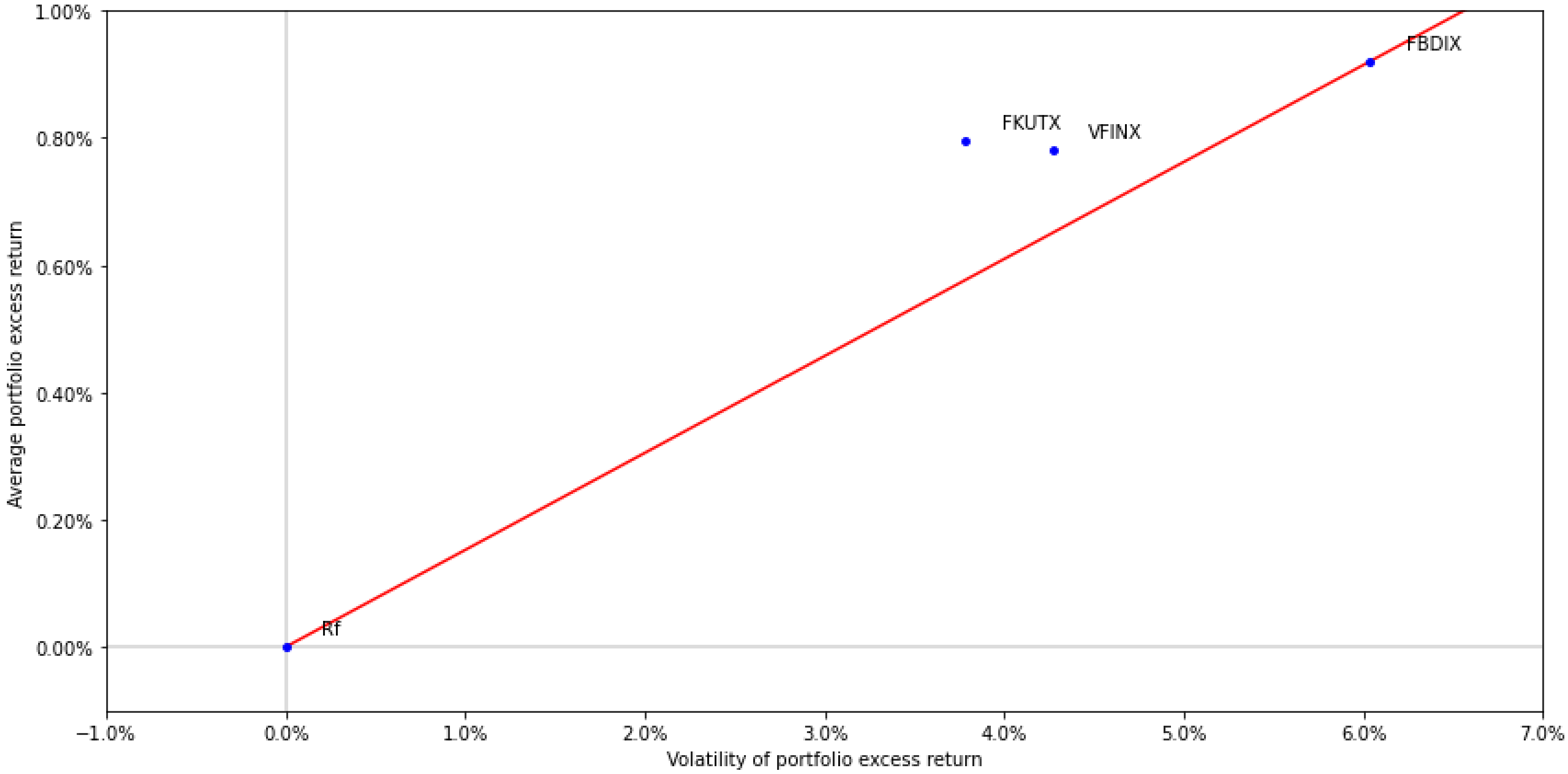
and allocates the rest $1 - w$ to the risk-free investment.

- A risk-free investment has zero excess return and volatility.
  That makes the earlier formulas much simpler:
  The average excess return is $w \times \mu_p$, with volatility $w \times \sigma_p$. In the figure, different choices of w trace out a straight line.

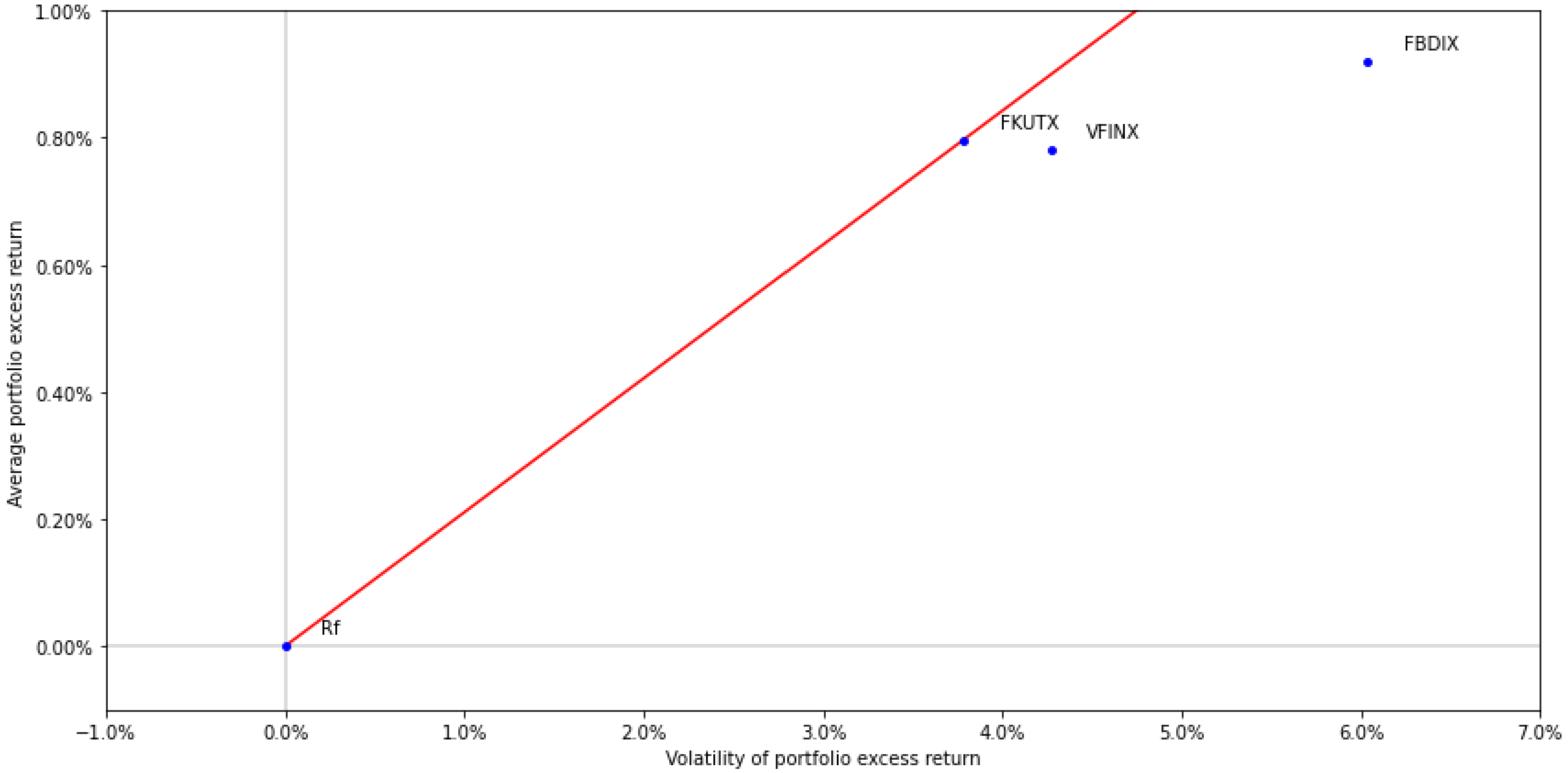
Allocation between risk-free and VFINX



Allocation between risk-free and FBDIX



Allocation between risk-free and FKUTX



The investor wants the highest possible Sharpe ratio

- In the figure, a straight line from the risk-free investment to any risky investment is called a **capital allocation line (CAL).**
- Along the line, there is a tradeoff between σ and μ. The steeper is the line, the better is the reward (higher μ) that we get for accepting higher volatility (higher σ).
- The exact formula for the line is

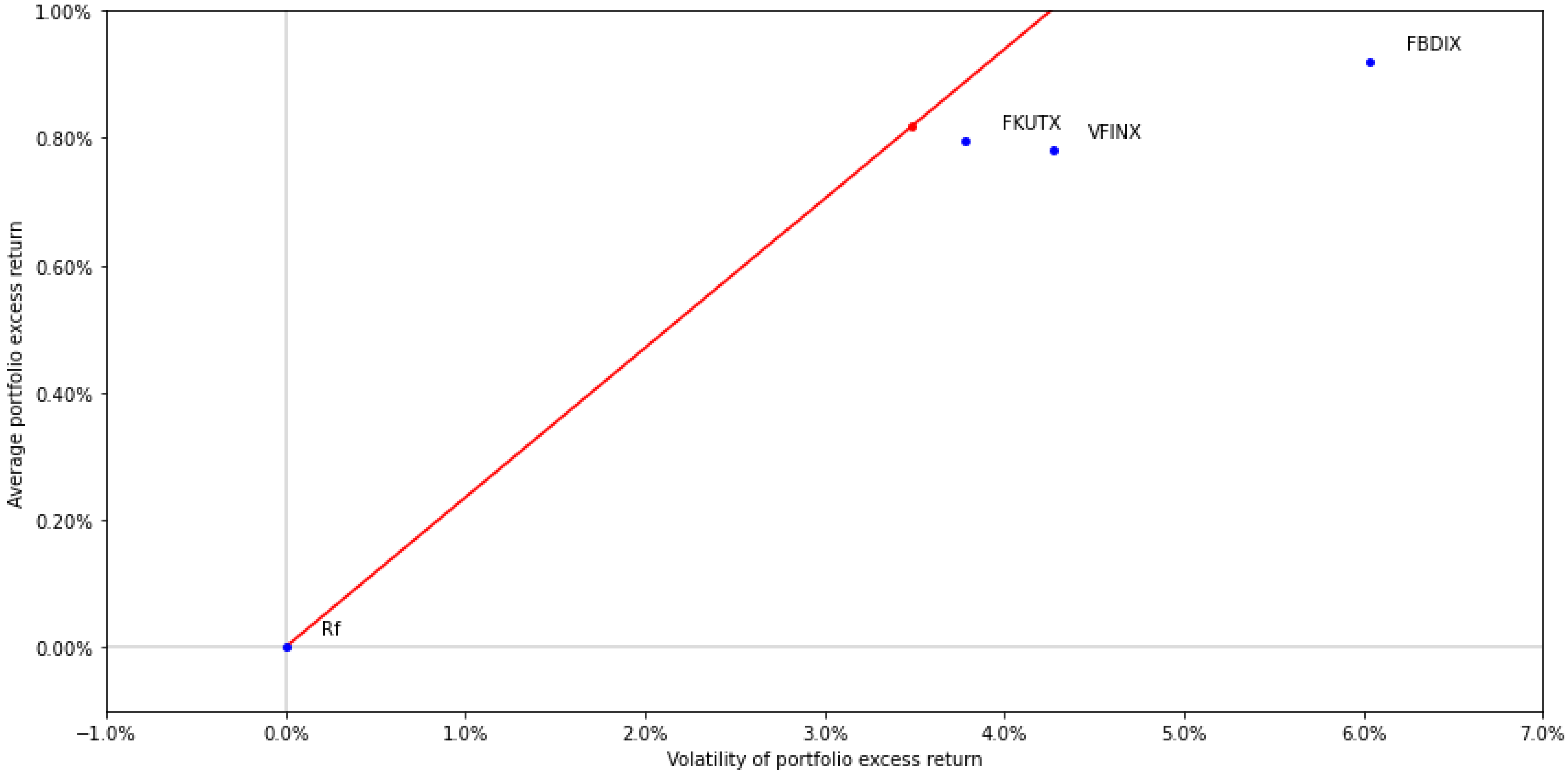
  $$\mu_{\text{portfolio}} = \frac{\mu_{\text{risky}}}{\sigma_{\text{risky}}} \times \sigma_{\text{portfolio}}$$

  This is the average return we can get for any choice of volatility.
- Notice that $\frac{\mu_{\text{risky}}}{\sigma_{\text{risky}}}$ is exactly the Sharpe ratio of the risky investment. So the investor wants to find the risky investment with <u>the highest Sharpe ratio</u> (the steepest CAL).
- This means that if they had to choose <u>one</u> fund, they would choose FKUTX.

But what is the highest possible Sharpe ratio?

- To repeat carefully the above conclusion: FKUTX would be the best of the three funds, if the investor had to allocate their non-risk-free investing entirely to one of them.
- But of course the investor doesn't have to do that. They are free to allocate their money across all three funds.
- And it turns out that this might attain a higher Sharpe ratio than any of the three funds attains on its own.
- For example, suppose the investor allocates their risky investing to a portfolio that is 60% FKUTX, 20% VFINX, 20% FBDIX…

Allocations between risk-free and a 60/20/20 portfolio



How can a portfolio have a higher SR than its ingredients?

Math reasoning: Because volatilities do not average like returns.

- Imagine any portfolio you could form of the three funds.
- Unlike the average return, the volatility of the portfolio is not just a simple average of the individual funds' volatilities. In fact it is always less than that average.
- When σ falls faster than μ, the ratio μ/σ increases.

Economic reasoning: Diversification.

- The funds' returns are all risky, but not perfectly correlated.
- When combined together, the correlations partially offset, giving us average returns at below-average volatility.
- Harry Markowitz called this "the only free lunch in investing."

Illustrating diversification: Perfect positive correlation

Suppose two investments have perfect positive correlation, $\rho_{AB} = 1$.

In this extreme case, portfolio volatility *is* just a weighted average.

To see this, substitute $\rho = 1$ into formula (7.7) and simplify:

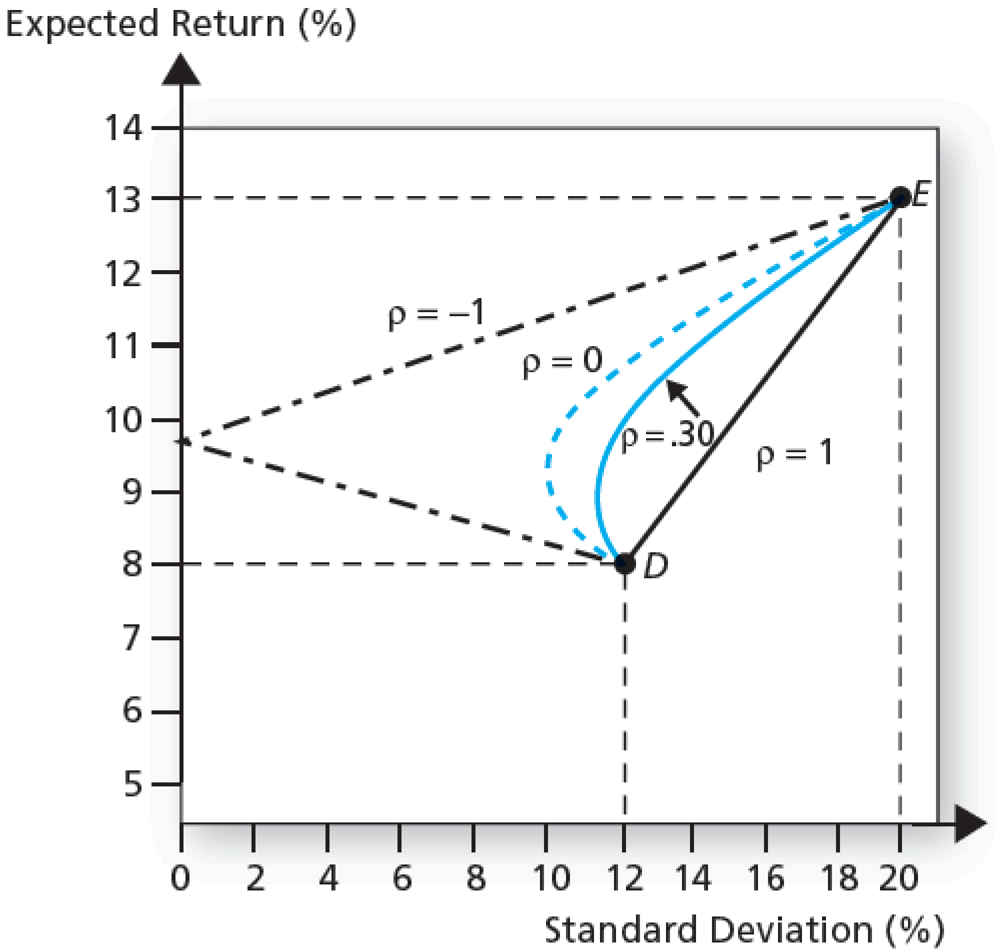
$$\sigma_p^2 = (w_A \sigma_A + w_B \sigma_B)^2$$

When we take the square root of both sides,

$$\sigma_p = w_A \sigma_A + w_B \sigma_B$$

Figure 7.6: Two risky investments in a portfolio

The solid black line traces out possible σ and μ values when $\rho_{AB} = 1$.



Illustrating diversification: Zero correlation

Suppose two investments have zero correlation, $\rho_{AB} = 0$. Then formula (7.7) becomes:

$$\sigma_p^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2$$

When we take the square root of both sides,

$$\sigma_p = \sqrt{w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2}$$

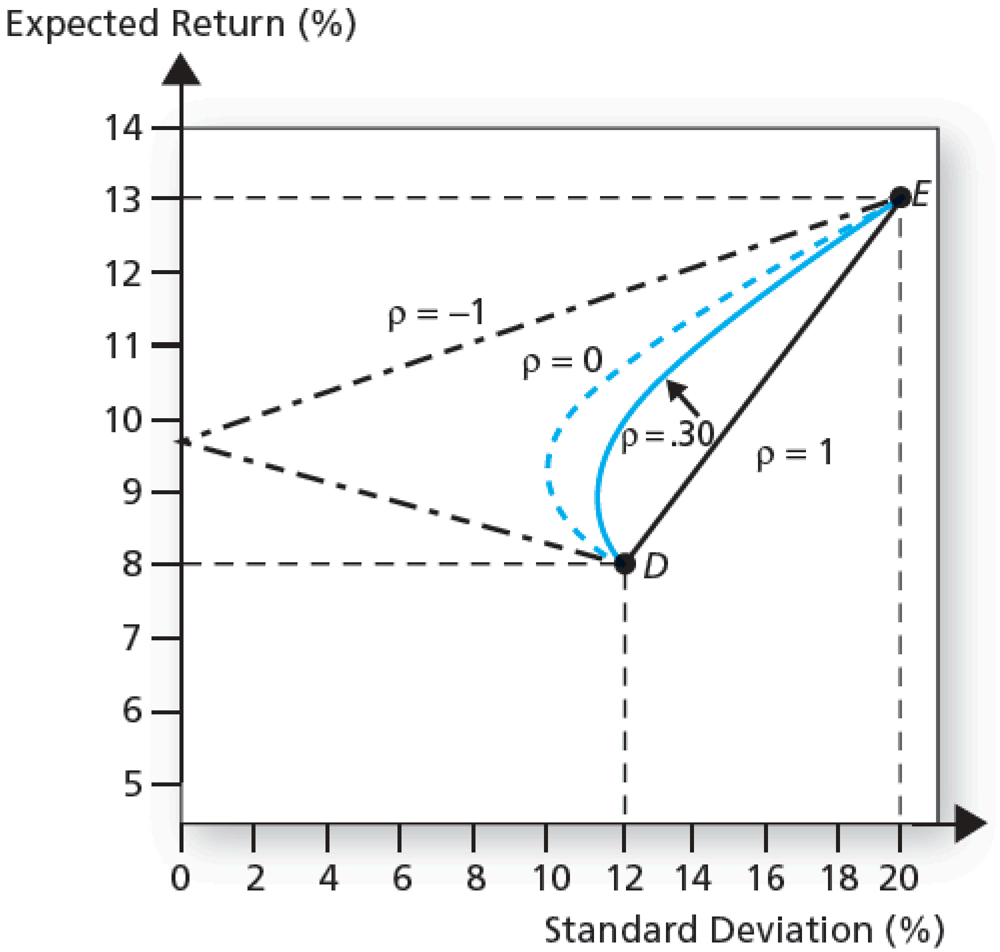
This is *less* than the weighted average $w_A \times \sigma_A + w_B \times \sigma_B$.

In general, volatility is less than a weighted average whenever $\rho < 1$. This effect gets more powerful as ρ gets smaller.

Visually, it causes the curves in the figures to bend leftward.

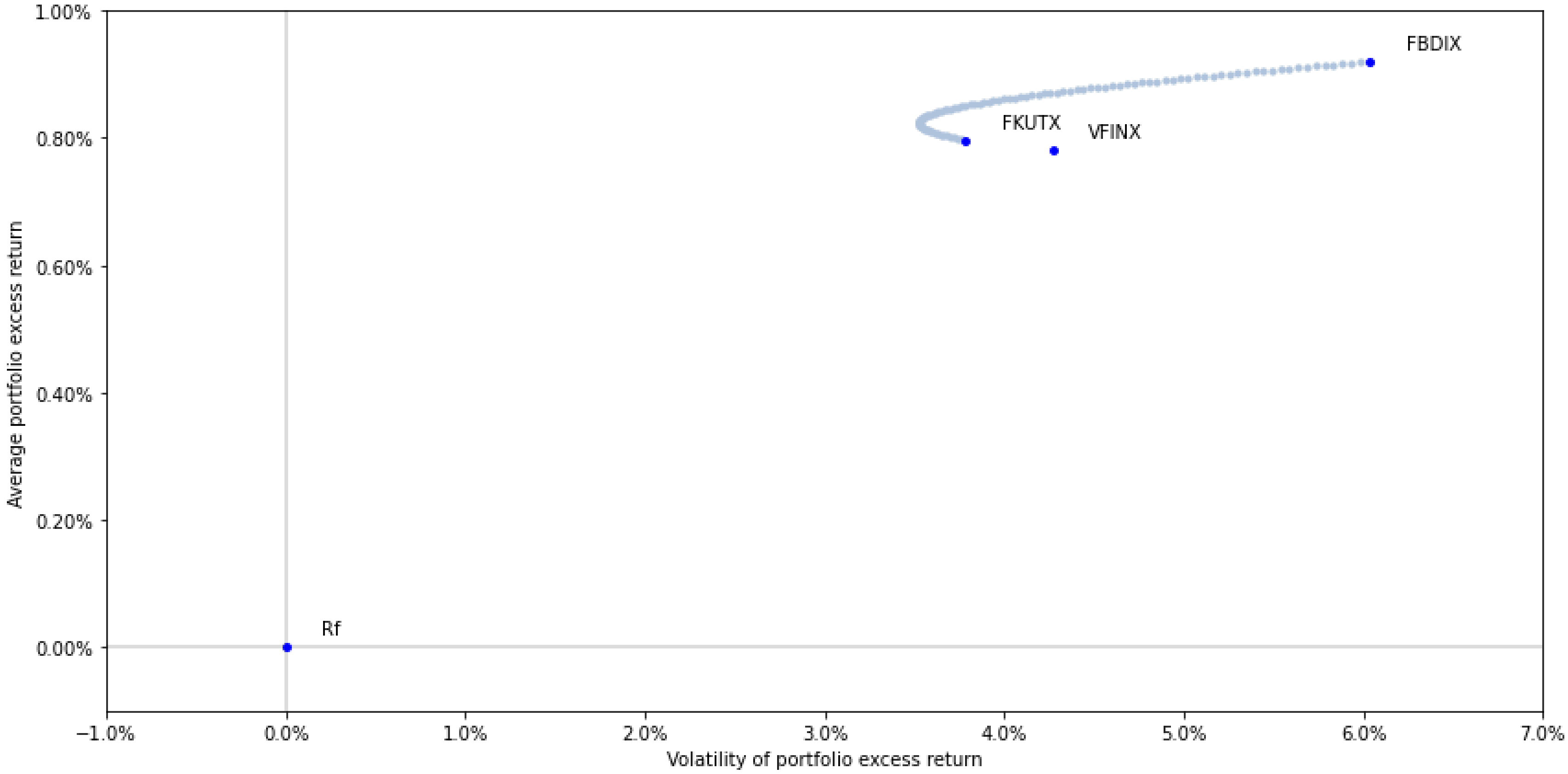
Figure 7.6: Two risky investments in a portfolio

The dashed blue line traces out possible σ and μ values when $\rho_{AB} = 0$.



Possible portfolios using just two of the three funds

In our data, the correlation between FKUTX and FBDIX is about 0.25.



Steepest CAL using just two of the three funds

This is the steepest CAL we can find using just FKUTX and FBDIX:

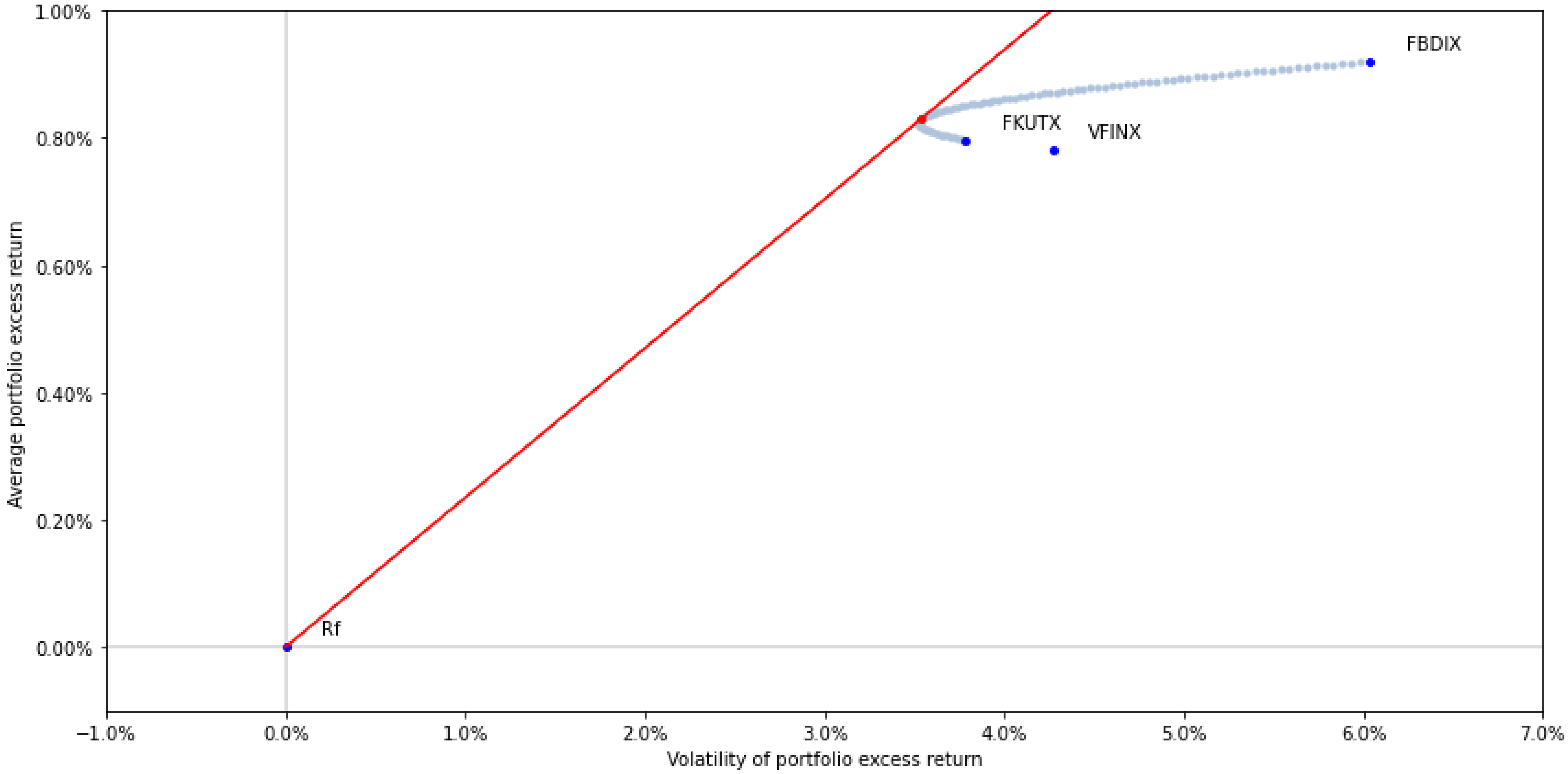
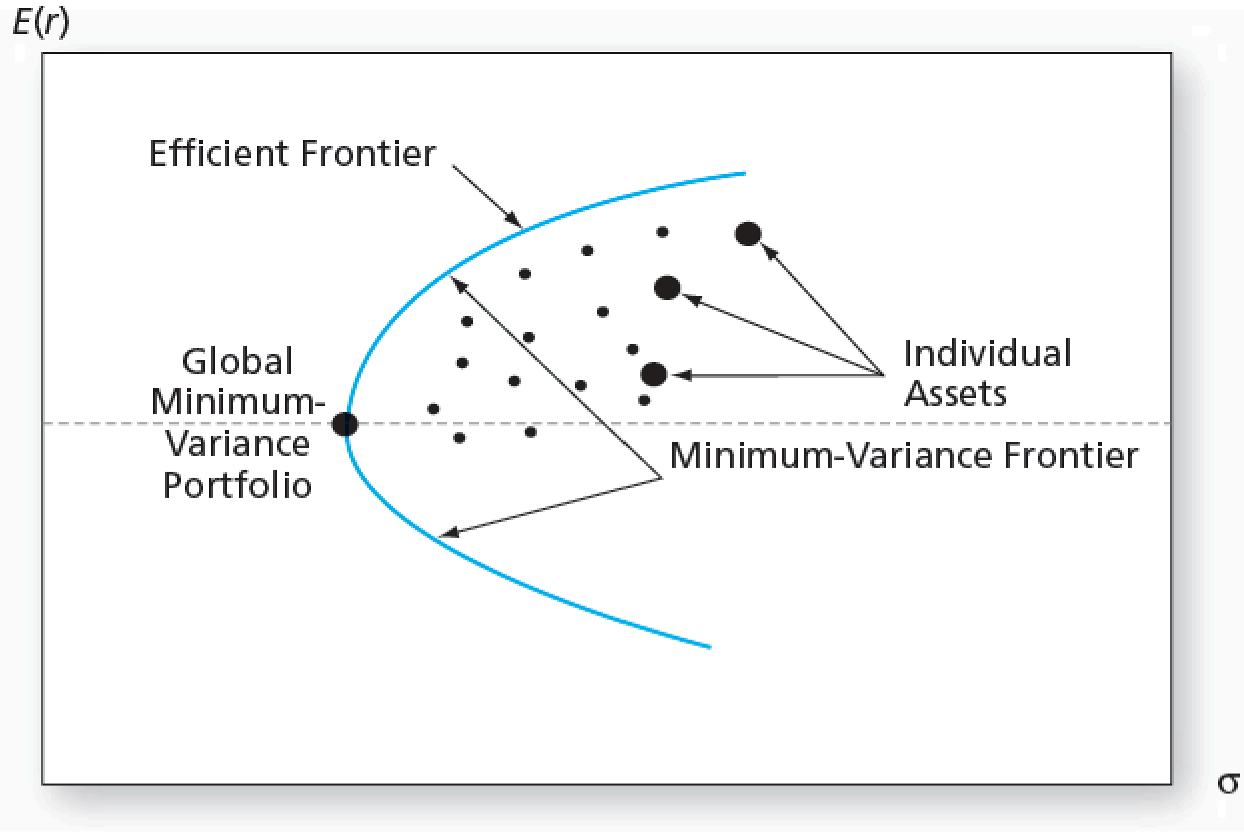
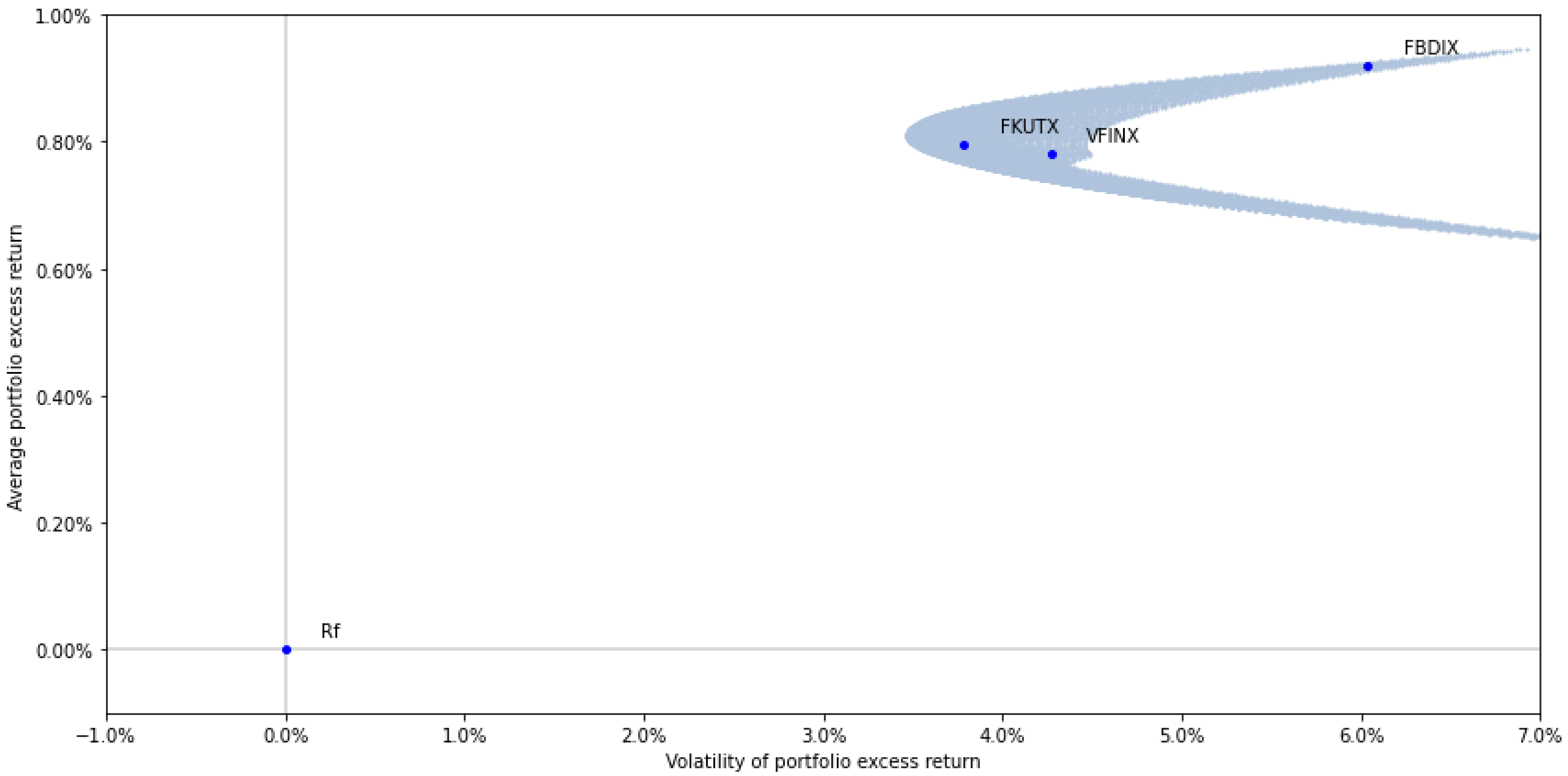


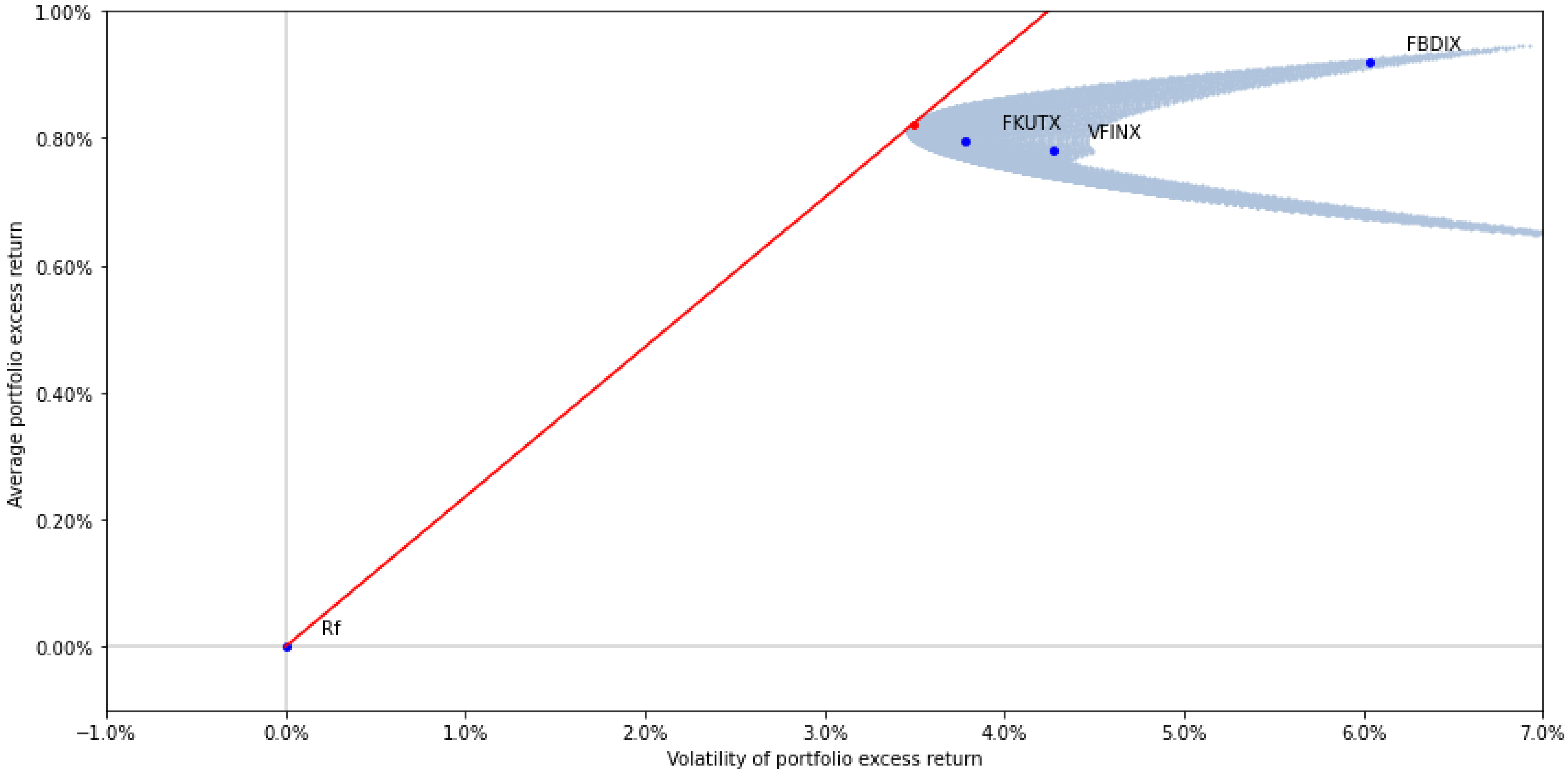
Figure 7.10: Many risky investments in a portfolio



Possible portfolios using all three funds



Portfolio with the highest Sharpe ratio and steepest CAL

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The weights are roughly 10.69% VFINX, 66.74% FKUTX, 22.57% FBDIX.
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