Portfolio statistics and the CAPM

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A different perspective on the optimal risky portfolio

Suppose an investor currently holds the risky portfolio p, and she is thinking about increasing her holding of investment i.

Remember that she wants to increase the Sharpe ratio of p.

It turns out that adding more of i will do this if

$$SR_i > \rho_{ip} \times SR_p$$

Definition: The “beta” of investment i with respect to portfolio p is

$$\beta_{ip} \equiv \rho_{ip} \times \frac{\sigma_i}{\sigma_p}$$

Then we can say that the investor will want to hold more of i if

$$\mathbb{E}[r_i - r_f] > \beta_{ip} \times \mathbb{E}[r_p - r_f]$$

Investment analysis based on β_{ip} and α_{ip}

- We can think of $\beta_{ip} \times \mathbb{E}[r_p - r_f]$ as the hurdle rate for someone holding portfolio p to decide they want to add more of i.

- This leads us to two important statistics:

$$\beta_{ip} = \rho_{ip} \times \frac{\sigma_i}{\sigma_p}$$

$$\alpha_{ip} = \mathbb{E}[ri - rf] - \beta_{ip} \times \mathbb{E}[r_p - r_f]$$

We call these “the beta and alpha of i with respect to p.”

- β_{ip} determines the hurdle rate someone applies to an investment.

- α_{ip} measures by how much the investment beats the hurdle rate.

Two ways to think about β

1. β is a rescaled version of the correlation ρ: If ρ is low, then the hurdle rate $\beta_{ip} \times \mathbb{E}[r_p - r_f]$ is low, reflecting the value of diversification (as we have seen before).
   But β, unlike ρ, can be greater than 1:

   - $\beta_{ip} > 1$ means that σ_p grows when we add a bit more of i to p.
   - $\beta_{ip} < 1$ means that σ_p shrinks.
   - $\beta_{ip} = 1$ means it does not change.

2. β and α are also the statistics in the following regression model:

$$(r_{it} - r_{ft}) = \alpha_{ip} + \beta_{ip} \times (r_{pt} - r_{ft}) + \varepsilon_{ipt}$$

If you gather data and create a scatter plot of r_{it} against r_{pt}, then β_{ip} is the slope of the best-fit line, and α_{ip} is its intercept.

The information ratio

We can define one more important statistic, the **information ratio**:

$$\mathrm{IR}_{ip} \quad = \quad \frac{\alpha_{ip}}{\sigma(\varepsilon_{ipt})}$$

If an investor currently holds p, and is considering whether to add more of i, the *maximum* Sharpe ratio they can achieve from this is

$$SR_{\max} = \sqrt{SR_p^2 + IR_{ip}^2}$$

(textbook formula 8.26)

$\sigma(\varepsilon_{ipt})$ is the standard deviation of ε_{ipt} from the prior slide. We can also calculate it from this formula:

$$\sigma(\varepsilon_{ipt}) = \sqrt{\sigma_i^2 - \beta_{ip}^2 \sigma_p^2}$$

From portfolio theory to the CAPM

Portfolio theory tells us what a mean-variance investor should do, given their predictions about μ, σ, and ρ.

The Capital Asset Pricing Model (CAPM) asks what would happen, if all investors followed this advice, and made the same forecasts:

- With the same forecasts, they choose the same risky portfolio.
- The only portfolio they can all choose is the market portfolio.
- So, it must in fact have the highest possible Sharpe ratio.
- Then all investments have zero α with respect to it.
- More precisely, the best *prediction* of α is always zero.

  We will always find investments with nonzero α in past data.

  But the CAPM says you could not have picked them in advance.

The Capital Asset Pricing Model (CAPM)

Conclusion: The best prediction about every investment is that

$$\mathbb{E}[r_i - r_f] = \beta_{im} \times \mathbb{E}[r_m - r_f]$$

This appears similar to our earlier formula, but says much more:

- Portfolio theory says what an individual should do, but does not make any predictions about what will actually happen.
- The CAPM does: It predicts that all investments have zero α. If some investment had a positive α, then all investors should try to buy it, and prices should simply adjust until that α disappears.
- In other words, the CAPM describes an equilibrium.

The CAPM uses ideas from portfolio theory

The logic behind the CAPM formula is really just portfolio theory, applied to the case of an investor who holds the market portfolio:

- β_{im} is a (rescaled) correlation with the market portfolio. It is also the slope of the best-fit line from regressing r_i on r_m.
- α_{im} measures whether investment i beats its hurdle rate. It is also the intercept of the same best-fit regression line.
- Suppose we start from the market portfolio and add more of i.
  $\beta_{im} \lesseqgtr 1$ tells us whether portfolio volatility grows or shrinks,
  $\alpha_{im} \lesseqgtr 0$ tells us whether portfolio Sharpe ratio grows or shrinks.

Again, the new thing is the prediction that α_{im} is always zero.

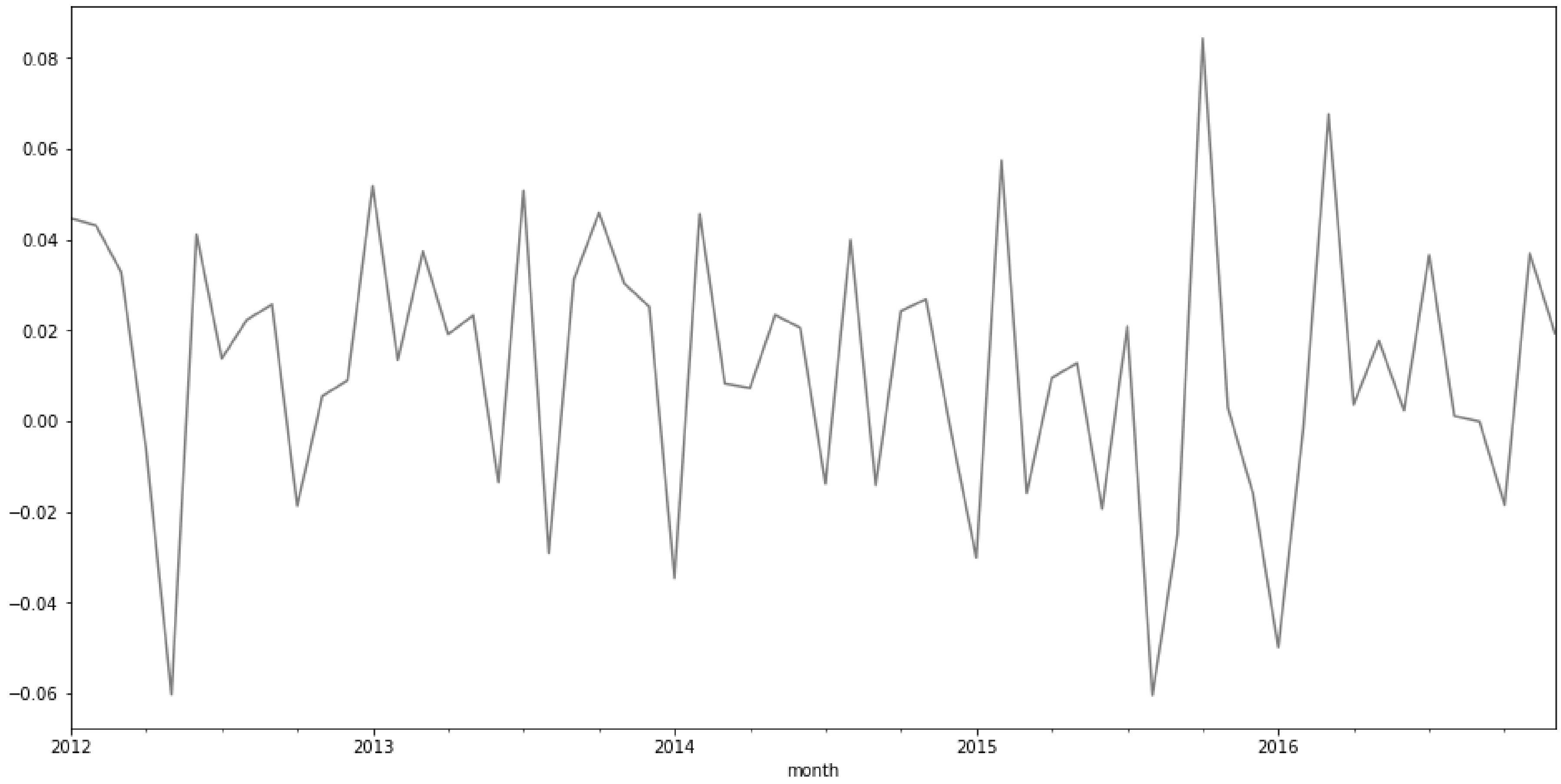
This also means $IR_{im} = 0$, so indexing is the best strategy.

The market portfolio in theory and in practice

- The CAPM gives a special role to the “market portfolio,” which theoretically includes *every* risky investment.
- To use the CAPM, we need to measure the market portfolio’s return, in order to estimate β_{im}. This is clearly a difficult task.
- Most investment in the US is allocated to the stock market, and the average stock investor holds a value-weighted portfolio.
- So, *in practice*, people typically represent the market portfolio with a large, value-weighted portfolio of US stocks.
- Be aware that this is a big simplification of the theory. The “true” market portfolio would include not only stocks, but also corporate bonds, real estate, commodities, etc…

What does market risk look like?

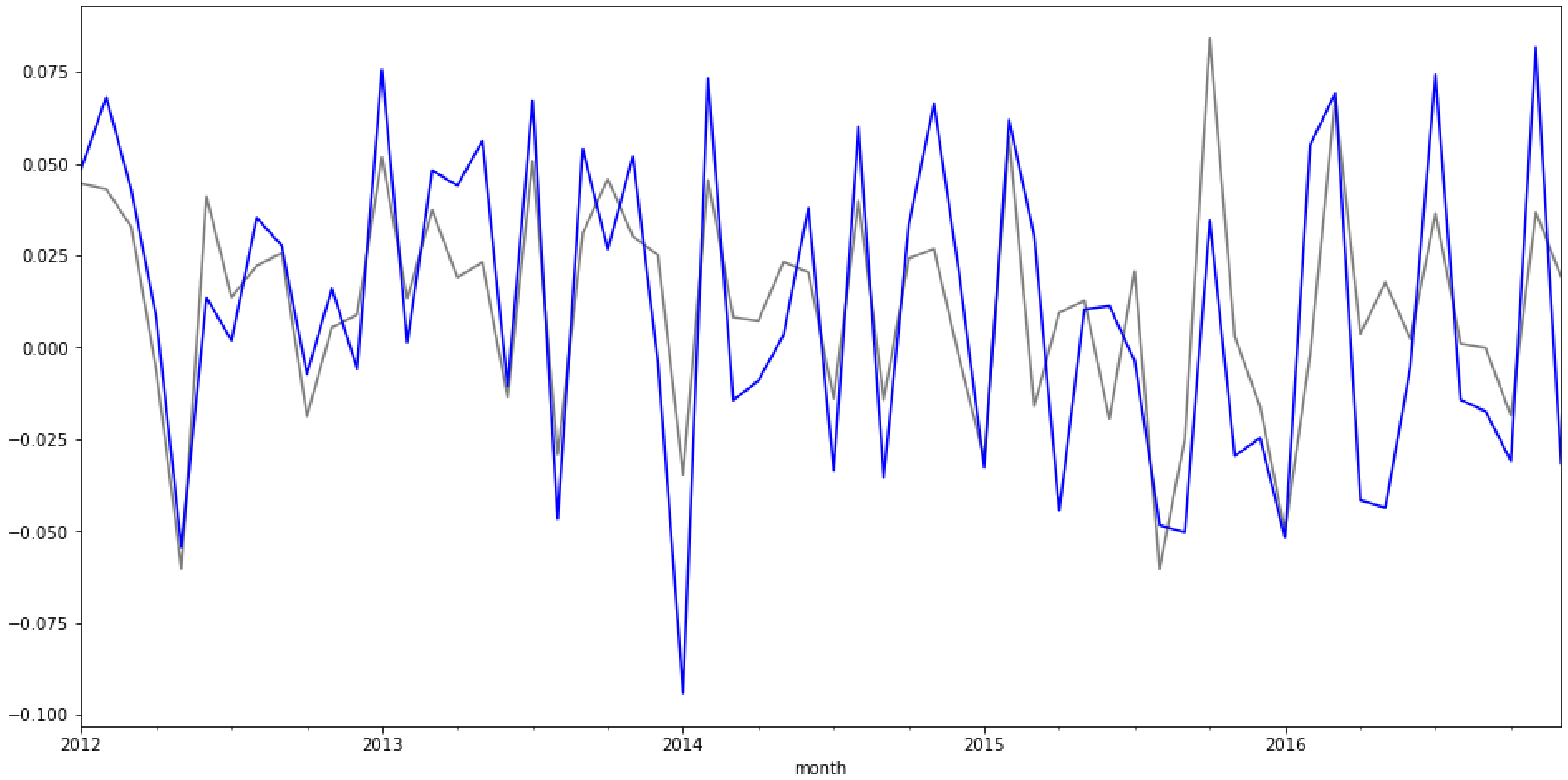
Here's a view of the market portfolio, as it is usually measured: This figure plots monthly returns on the VFINX index fund.



What does market risk look like?

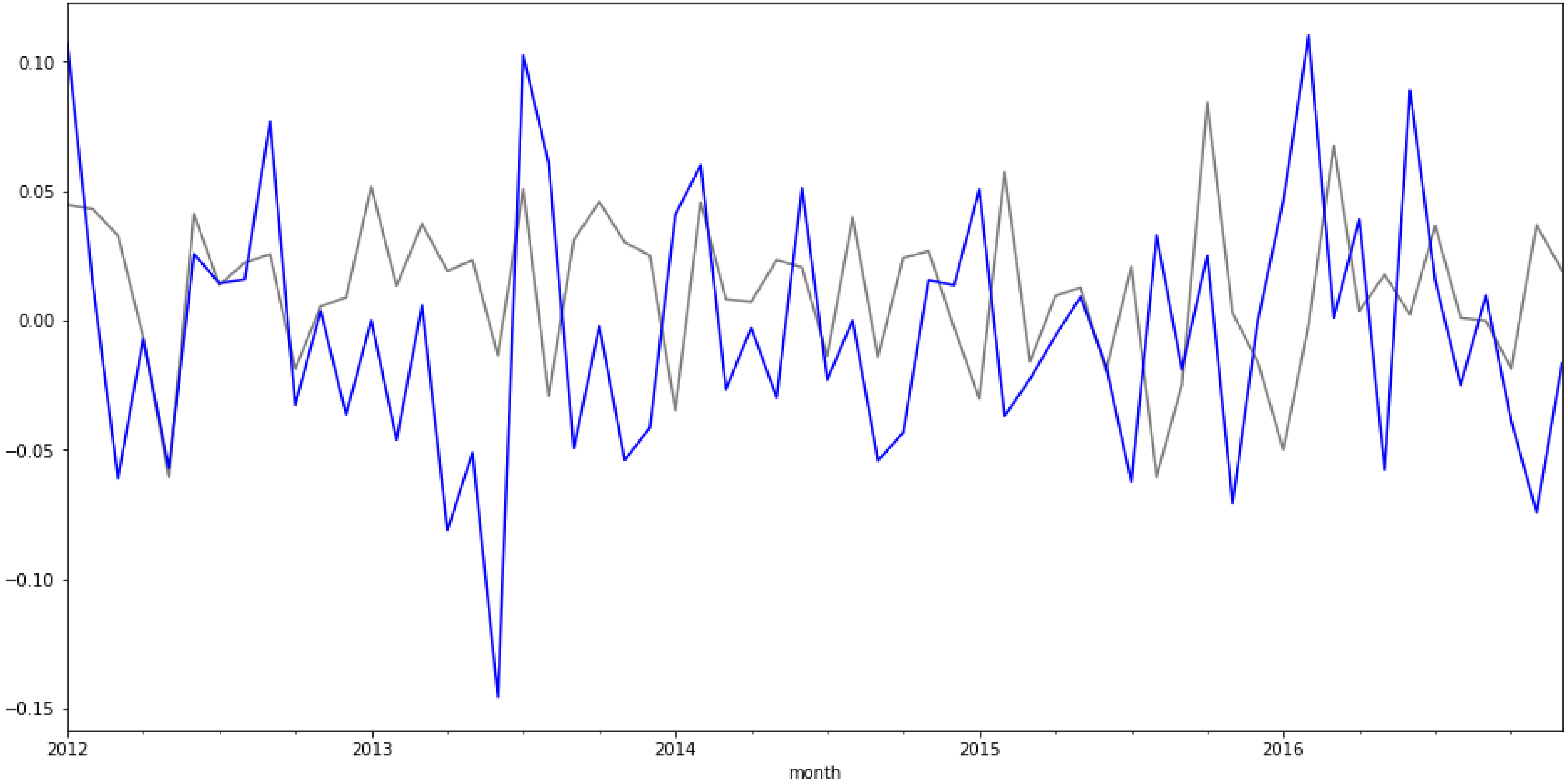
Add in the return of an ETF of retail stocks, with dividends reinvested.

The ETF's monthly return has 3% volatility and a market β of 1.09.



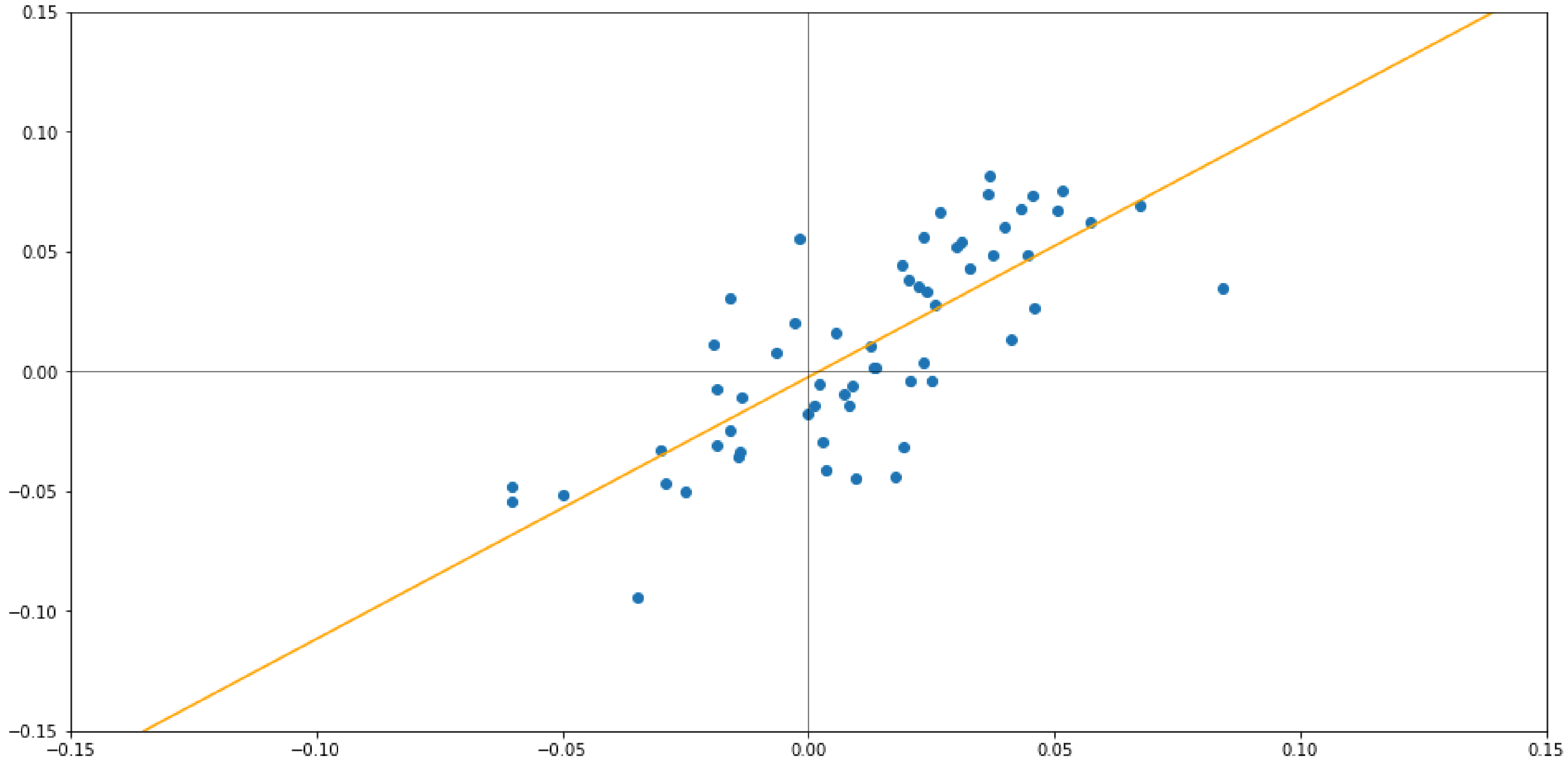
What does market risk look like?

Replace the retail ETF with a gold ETF. This ETF's monthly return has 5% volatility, but a beta of only 0.09.



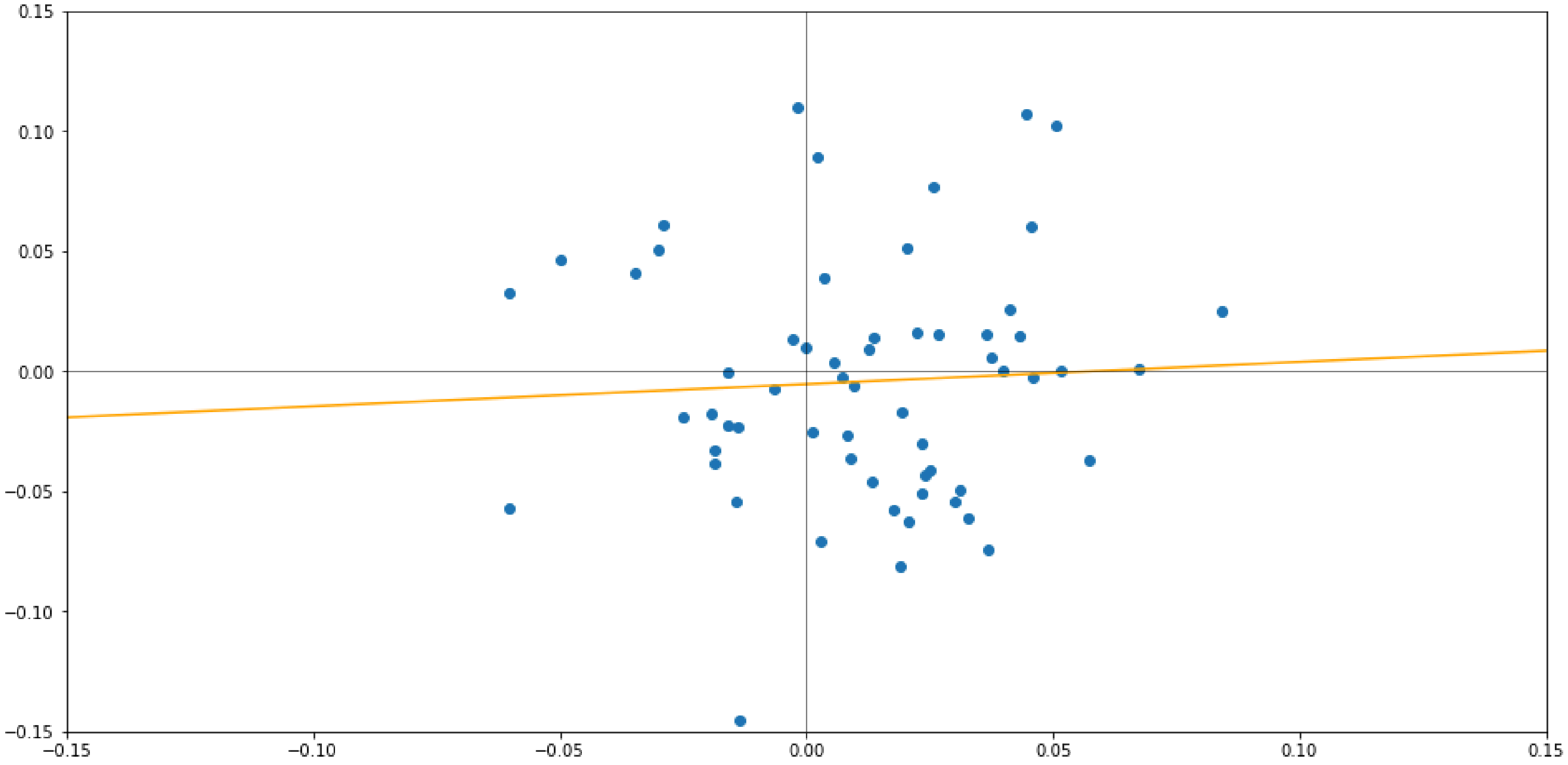
What does market risk look like?

Here's a different view of the same data on the retail ETF return:
β is the slope of the best-fit line in the figure. Here, β = 1.09.



What does market risk look like?

A similar figure using the gold ETF:
The line here is almost flat, with slope (beta) of 0.09



β as a correlation

β is the slope of a regression line like the prior ones.

Again, one way to interpret this number is that it measures a relationship between the investment and the market.

- β = 2: Moves twice as far as the market, on average.
- β = 0.5: Moves half as far as the market, on average.
- The market portfolio itself has a beta of 1.
- A risk-free investment has a beta of zero.
- A risky investment can also have a beta of zero!

  Gold is a risky investment with approximately zero beta.

The CAPM focuses only on β, so it predicts (correctly) that gold earns low average returns despite having high volatility.

β as non-diversifiable risk

A related interpretation is that β measures "non-diversifiable" risk. performance of these stocks is closely tied to the overall market.

- Gold has a low beta because its returns, while very volatile, have very little economic connection to the overall market.
- Adding gold decreases the market portfolio's volatility ($\beta < 1$). We call this idiosyncratic risk or diversifiable risk.
- By contrast, retail stocks have a very high beta, because the
- When you increase holdings of retail stocks in your market portfolio, their risk reinforces the volatility you were already exposed to, and your portfolo volatility slightly increases ($\beta > 1$). We call this market risk, non-diversifiable risk, or systematic risk.

The CAPM says that investors only care about *non*-diversifiable risk.

Diversifiable risk does not matter to them, because it disappears when the investment is added to their portfolio.

Next: What does the evidence say about the CAPM?

The CAPM is testable...with lots of assumptions (such as how to measure the market portfolio, what time frame to use, and so forth).

Next we will start looking at the evidence on how closely the world does, or does not, resemble the main predictions of the CAPM. And we will discuss how to think about the evidence that we see.

To preview: The main prediction of the CAPM is that there is no simple strategy that earns positive α over a long historical time.

We will find that some strategies have done so, at least in the past. But this should not really surprise us. What are the implications?

For purposes like valuation and capital budgeting, the CAPM might be the best framework, even if its predictions don't always hold.

Application: The analysis of Homework 2

We will reproduce the results of that assignment and then see how the formulas from this week connect with that topic.

First let's remind ourselves what we concluded as the optimal portfolio (the one with the highest Sharpe ratio). Using the matrix formula from last week's slides we can calculate:

```
The portfolio with the highest Sharpe ratio is 36.070% gold, 63.930% VFI
NX.
It has a Sharpe ratio of 0.18558.
```

Sharpe ratio formula

Now let's look at the first formula from this document. It say that we want to increase allocation to any investment i as long as $SR_i > \rho_{ip} \times SR_p$, where p is the portfolio we currently hold. Let's apply this formula to the homework, with gold playing the role of investment i and VFINX the portfolio p.

First we calculate the Sharpe ratio of gold:

```
Sharpe ratio of gold: 0.1015
```

Now imagine we start off holding just VFINX, and are considering whether to allocate some to gold. Calculate the Sharpe ratio of VFINX by itself, and the correlation of gold with VFINX.

```
Sharpe ratio of VFINX: 0.1607
Correlation of gold with VFINX: 0.0550
Multiply these two together: 0.0088
```

Because the Sharpe ratio of gold was greater than this amount, we conclude that we can achieve a higher portfolio Sharpe ratio if we allocate *some* amount to gold.

Now suppose we try an allocation of 20% to gold. We can simulate the returns of this allocation, and then redo the calculations just above with this new allocation playing the role of p, our current portfolio.

```
Sharpe ratio of an allocation 20% gold, 80% VFINX: 0.1789
Correlation of gold with this allocation: 0.3048
Multiply these two together: 0.0545
```

The result is still less than the Sharpe ratio of gold by itself that we calculated earlier. So we conclude that we should allocate more than 20% to gold. We can repeat this process (and could even code it up as an optimizer). It will eventually settle down to the same answer as we found in Homework 2.

Finally, we verify that at the correct answer, the two sides of the equation are exactly equal, telling us that this allocation to gold achieves the highest possible Sharpe ratio in our historical data.

```
Sharpe ratio of optimal portfolio from HW2: 0.18570
Optimal rho: 0.5467
Optimal SR times rho: 0.1015
This matches the Sharpe ratio of gold by itself.
```

Formula with alpha

The next major formula at the top of this document says that we want to add more allocation to an investment i as long as it exhibits positive α with respect to our currently portfolio, where α can be calculated in two different ways:

- Slide 4 shows that you can calculate α as the average return on investment i, minus the average return on our current portfolio times the β of investment i with respect to our current portfolio,

$$\alpha_{ip} = \mathbb{E}[r_i - r_f] - \beta_{ip} \times \mathbb{E}[r_p - r_f]$$

- Slide 5 shows that you can calculate α as the intercept from a regression of the returns on investment i against the returns on the current portfolio p,

$$(r_{it} - r_{ft}) = \alpha_{ip} + \beta_{ip} \times (r_{pt} - r_{ft}) + \varepsilon_{ipt}$$

Let's see how these formulas show up in the analysis from Homework 2.

First we check that gold has a positive α with respect to VFINX. In the regression table below, α corresponds to the number labeled "Intercept".

	coef	std err	t	P>|t|	[0.025	0.975]
Intercept	0.0046	0.004	1.225	0.222	-0.003	0.012
VFINX	0.0582	0.079	0.735	0.463	-0.098	0.214

We can interpret α as a monthly return. So, we would say that, in our historical data, gold delivers 0.46% per month more than its appropriate hurdle rate based on its β_{ip} with respect to our current portolio of holding only VFINX.

That value of β_{ip} is labeled "VFINX" in the next row of the table. The value of 0.0582 is very low and is effectively zero, reflecting the fact that gold has almost no correlation with the performance of the overall stock market (represented by VFINX), even though it is very volatile on its own.

As a reminder, the hurdle rate for any investment is $\beta_{ip} \times \mathbb{E}[r_{pt} - r_{ft}]$. If the investment i earns exactly this rate of return, then it will neither increase or decrease our Sharpe ratio when we add it to our existing portfolio p.

So, since we found a positive α for gold in the regression, this tells us that we *can* increase our Sharpe ratio by adding some allocation to gold to our portfolio, and so we should do so. (This is exactly the same as our first step in the previous section using Sharpe ratios, just showing the math differently.)

Notice that the hurdle rate depends directly on β:

- Investments with low β have low correlation to our existing portfolio, which means that they can diversify our existing portfolio risk and lower our overall portfolio volatility. This can increase our Sharpe ratio even if the new investment doesn't earn a very high return on its own. So investments with low β get a low hurdle rate.
- For investments with high β, the logic is exactly the opposite. They have high correlation to our existing portfolio, so they will not do much to diversify our existing risk. In fact if $\beta = 1$ they will not lower our portfolio volatility at all, and if $\beta > 1$ they will actually increase it. Since they don't offer any diversification, these investments are only attractive if they offer very high returns, and so they get a high hurdle rate to reflect that fact.

Let's quickly check that we can match the alpha and beta value from the regression above by calculating directly with the formulas from slide 4:

```
Matching the above regression by direct calculations:
  Beta: 0.0582
  Alpha: 0.0046
```

Reminder: All of the above logic may feel like the CAPM, based on what you have seen in past classes, but it's just plain old portfolio theory! The CAPM is a special case of all this analysis, as we will discuss in class.

Now we again try our allocation of 20% to gold, 80% to VFINX, and evaluate the α of gold with respect to this allocation:

	coef	std err	t	P>|t|	[0.025	0.975]
Intercept	0.0023	0.004	0.649	0.517	-0.005	0.009
Gold20	0.3845	0.090	4.269	0.000	0.207	0.562

In the regression above, there is still some evidence of positive α, suggesting that we could do even better by increasing our allocation to gold some more, which is consistent with what we concluded when we were looking at Sharpe ratios. (However, the standard errors above give us some new information: they suggest that the further improvement might not be statistically significant.)

Finally, we can check that gold has exactly zero alpha with respect to our optimal solution from Homework 2, so we could find that same answer by tinkering with the portfolio until alpha settles down to zero:

	coef	std err	t	P>|t|	[0.025	0.975]
Intercept	-8.674e-19	0.003	-2.73e-16	1.000	-0.006	0.006
Optimal	0.7588	0.087	8.711	0.000	0.587	0.931

Analysis based on information ratios

To calculate the information ratio of investment i with respect to portfolio p, we run the same regression as just above where we calculated α_{ip}, and then:

$$\text{Information ratio}_{ip} = \frac{\alpha_{ip}}{\sigma(\varepsilon_{it})}$$

where $\sigma(\varepsilon_{it})$ is the standard deviation of the residuals from the regression.

The information ratio is just α scaled by a measure of the volatility of returns, so it delivers a similar message. Where α just told is whether it was *possible* to improve Sharpe ratio by allocating to an investment, the information ratio tells us *how much* our performance will improve, and also, how much we ideally want to invest.

The information ratio can help us judge whether it's really worth it to pursue a new invesmtent, given that there is always some fixed cost of adding something new to our portfolio. From the other direction, a fund manager wants to achieve a high information ratio, because this will increase the amount that clients want to allocate to her fund, which is the basis of her fee.

As before, we start out by calculating the information ratio of gold with respect to VFINX:

```
Information ratio of gold with respect to VFINX: 0.0928
```

There are many ways to use this number. One of the most important is a formula in the slides: For an investor who is allocating only between the new investment i (in this case gold), and some other portfolio p (in this case VFINX), the maximum Sharpe ratio they can possibly achieve is given by the following formula (slide 6 and textbook formula 8.26),

$$SR_{max} = \sqrt{SR_p^2 + IR_{ip}^2}$$

Using the numbers we have calculated, we calculate a maximum Sharpe ratio that exactly matches the Sharpe ratio of our solution from Homework 2:

```
Max SR: 0.18558
```

The point is that we can decide how attractive this particular investment is, before sitting down to do an explicit portfolio optimization process. And more importantly, if the manager has a pretty good guess what portfolio their clients hold on average (it is likely to be close to the market portfolio!), then they can attract the most investment into their fund by focusing on maximizing their information ratio with respect to that portfolio.

One last useful connection with regression theory: The information ratio is really just the t-statistic of the regression intercept multiplied by $\sqrt{N}$, where N is the number of observations of data in the regression. (But note, to make this formula line up perfectly requires some further small adjustments that are beyond the scope of the class right now.)