INTRODUCTION TO COMPLEX NUMBERS

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1. Basics of complex numbers

In this note we denote by $\mathbf{R}$ the set of real numbers.

Definition 1.1. A complex number is an expression of the form $a + bi$, where $a, b \in \mathbf{R}$ and i is the so called *imaginary unit*. We set $i^2 = -1$. The set of all complex numbers will be denoted by $\mathbf{C}$. When $a = 0$, we will simply write bi and when $b = 0$ we will simply write a for the complex number $a + bi$.

All of the following numbers are complex numbers: $2+3i$, $\frac{1}{2}-7i$, $\sqrt{2}i = 0+\sqrt{2}i$, $5 = 5+0i$. Here $\frac{1}{2} - 7i$ stands to $\frac{1}{2} + (-7)i$. This is a standard convention which we will adopt.

Definition 1.2. If $z = a + bi$ is a complex number with $a, b \in \mathbf{R}$, we say that a is the *real part* of z and write $\mathrm{Re}(z) = a$ and b is the *imaginary part* of z and write $\mathrm{Im}(z) = b$.

Note that both the real and the imaginary part of z are real numbers.

Any complex number can be represented as a point on the plane, namely $z \in \mathbf{C}$ corresponds to the point $(\mathrm{Re}(z), \mathrm{Im}(z))$. Addition (respectively subtraction) of complex numbers is defined in the same way as addition (resp. subtraction) of vectors on the plane, i.e., if $a, b, c, d \in \mathbf{R}$, then

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i.$$

Multiplication is defined in a slightly more complicated way:

$$(a + bi)(c + di) = ac + bdi^2 + (ad + bc)i = ac - bd + (ad + bc)i.$$

One can check (you can try this!) that both the addition and the multiplication operations are associative and commutative, as well as the distributive property holds, i.e., that

$$z(v + w) = zv + zw \quad \text{for any complex numbers } z, v, w.$$

Definition 1.3. Let $z \in \mathbf{C}$. The *complex conjugate* of z denoted by $\overline{z}$ is defined to be

$$\overline{z} = \mathrm{Re}(z) - \mathrm{Im}(z)i.$$

Theorem 1.4. *Let $z = a + bi$ be a complex number with $a, b \in \mathbf{R}$. Suppose that either a or b is not zero. Then there exists a complex number v such that $zv = 1$, i.e., z has a multiplicative inverse. More specifically,*

$$v = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.$$

From now on we denote v by z^{-1} or by $\frac{1}{z}$.

Proof. Just multiply z by v and check that you indeed get 1. $\square$

This allows us to define division of complex numbers, namely

$$\frac{z}{w} = z \cdot w^{-1}.$$

Note that w^{-1} was just defined and multiplication was defined earlier, so everything on the right hand side of the above equation was defined before. In practice to divide two complex numbers we multiply the numerator and denominator by the complex conjugate of the denominator. If we do so, and the denominator was $a + bi$, then now the denominator is $a^2 + b^2$ which is a positive real number.

Solve the following problems:

(1) Mark the following complex numbers z on the plane: $-1 - i$, 6, $-2i$, $\frac{1}{3} + 2i$ and for each of them compute $-z$ and $\overline{z}$ and mark them also.

(2) Find the following sums, products and ratios:
   - $(3 - 2i) - (4 + 6i)$
   - $(\sqrt{3} + \sqrt{3}i) + (1 - \sqrt{3}i)$
   - $(3 - 2i)(5 + i)$
   - $(4 + 2i)(-7i + 2)$
   - $i(9 + 2i)$
   - $\frac{1}{2+i}$
   - $\frac{2+6i}{2+5i}$
   - $\frac{i}{1+i}$

(3) For the following numbers z find $\overline{z}$ and $z\overline{z}$: $2 + 3i$, $1 - i$. $3i$, 2, $\sqrt{2} + \sqrt{2}i$.

(4) Show that for any complex numbers z, w one has

$$\overline{z + w} = \overline{z} + \overline{w},$$

$$\overline{z - w} = \overline{z} - \overline{w},$$

$$\overline{z \cdot w} = \overline{z} \cdot \overline{w}.$$

(5) Show that for any complex number z, the real number $z\overline{z}$ is the square of the distance of z from the origin on the plane.

2. Solving quadratic equations

What does it mean when we write $\sqrt{a}$? Can we use any a? Does it make sense to write $\sqrt{5}$? What about $\sqrt{-3}$ or $\sqrt{2+3i}$? And what about $\sqrt[3]{5}$? Not all of these make sense. When we write $\sqrt[3]{5}$, we are usually asking for a number whose third power is 5. In other words we are looking for a solution to the polynomial $x^3 - 5 = 0$. But how do we know it exists and how do we know there is only one? (If there are more than one then this notation is not good - it is ambiguous.) In fact, when we write $\sqrt{4}$ we probably are looking for a solution to the equation $x^2 - 4 = 0$. But you know that there are *two* such solutions: 2 and -2. So, the notation $\sqrt{4}$ is not good, because it is not a priori clear which of the two solutions we have in mind. We are going to put some order into this mess and establish once and for all when we can write $\sqrt[n]{a}$ and when it is a good notation (i.e., represents only one number). The answer to these questions is provided by Theorem 2.2 below.

Remark 2.1. Can you write down the polynomials whose solutions we are looking for when we write the other radical expressions above? Can you in each case find more than one solution to these polynomials?

Theorem 2.2. *Let a be a positive real number and let n be a positive integer. Then there exists a unique positive real number α such that $\alpha^n = a$. We will denote this number α by $\sqrt[n]{a}$.*

This theorem is not easy to prove and we will just accept it. So, whenever we write something like $\sqrt[7]{19}$ we mean the *unique* positive real number whose 7th power is 19. The theorem tells us that such a number exists and that there indeed is only one. In particular in these notes I will never write something like $\sqrt[3]{-2}$ because we reserve the notation $\sqrt[n]{a}$ only for *positive* real numbers a.

Note that i is a solution (in other words a *root*) to a single polynomial equation $x^2 + 1 = 0$. But with this single addition to real numbers we are now able to solve a lot of other polynomials of degree 2 with real coefficients. For example it is easy to see that $x^2 + 9 = 0$ has solutions $3i$ and $-3i$. Similarly, $x^2 + 2 = 0$ has solutions $\sqrt{2}i$ and $-\sqrt{2}i$ (recall that according to our convention $\sqrt{2}$ represents the unique positive real number that is the solution to the polynomial $x^2 - 2 = 0$).

So, what about a general quadratic equation:

$$ax^2 + bx + c = 0, \quad a, b, c \in \mathbf{R}, a \neq 0?$$

Let us try to find its solutions. We first divide by a to get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

and then add to both sides the expression $\frac{b^2}{4a^2}$. We then get

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}. \tag{2.1}$$

The reason for doing so is that

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2 \quad \text{Don't believe me! Check this!}$$

So, we can rewrite (2.1) as

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} = \frac{\Delta}{4a^2}.$$

If $\Delta > 0$, then we have a question: what number squared gives us $\frac{\Delta}{4a^2}$. Since the number $\frac{\Delta}{4a^2}$ is greater than zero we know by our theorem that there is a unique positive real solution, namely $\sqrt{\frac{\Delta}{4a^2}} = \frac{\sqrt{\Delta}}{2a}$. We also then get a negative solution $-\frac{\sqrt{\Delta}}{2a}$. So, we get

$$x + \frac{b}{2a} = \pm\frac{\sqrt{\Delta}}{2a}, \quad \text{so} \quad x = \frac{-b \pm \sqrt{\Delta}}{2a},$$

which is the familiar quadratic formula. But what if $\Delta < 0$? Well, then we can write $\Delta = -(-\Delta)$ and $-\Delta > 0$. So, we get

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{-\Delta}{4a^2}.$$

As $\frac{-\Delta}{4a^2} > 0$, this is like solving an equation of the form $y^2 = -3$. We already know that the solutions to this would be $\sqrt{3}i$ and $-\sqrt{3}i$. So, similarly here we get

$$x + \frac{b}{2a} = \pm\frac{\sqrt{-\Delta}i}{2a}$$

(remember that $-\Delta$ is a positive real number, so $\sqrt{-\Delta}$ is the unique positive real number whose square is $-\Delta$). So, finally

$$x = \frac{-b \pm \sqrt{-\Delta}i}{2a}.$$

So, introducing this one imaginary number i, in reality allows us to solve **all** quadratic polynomials with real coefficients.

Example 2.3. Consider the equation

$$x^2 + 3x + 5 = 0.$$

Here $\Delta = 3^2 - 4 \cdot 1 \cdot 5 = -11 < 0$. So, there are no real roots. However, then $-\Delta = 11 > 0$ and the complex roots of $x^2 + 3x + 5$ are

$$\frac{-3 \pm \sqrt{11}i}{2}.$$

Check that these roots really work by computing

$$\left(\frac{-3 \pm \sqrt{11}i}{2}\right)^2 + 3 \cdot \left(\frac{-3 \pm \sqrt{11}i}{2}\right) + 5.$$

You should get 0.

Solve the following problems:

(1) Find two solutions to each of the following equations:

$$x^2 = -49$$

$$x^2 + 16 = 0$$

$$x^2 + 15 = 0$$

(2) Find two solutions to each of the following equations and check that they work by plugging them into the original equation:

$$x^2 + 2x + 4 = 0$$

$$x^2 - x + 1 = 0$$

$$x^2 - 4x + 13 = 0$$

$$2x^2 + x + 5 = 0$$

3. Trigonomentric form of complex numbers and multiplication

Any point $P = (x, y)$ on the plane can be described by giving its distance from the origin $r = \sqrt{x^2 + y^2}$ and the angle θ that the line OP forms with the positive x-axis (the angle is always measured counter-clockwise). This pair of numbers: the non-negative real number r and the real number θ are sometimes called the *polar coordinates* of the point P. If z is a complex number, then we already know that it can be represented by a point on the plane (x, y), where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. So, such a point can also be written using the polar coordinates. As $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$, we get

$$x = r\cos\theta, \quad y = r\sin\theta.$$

In other words any complex number z can be written as

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

for some non-negative real number r and some real number θ. This is the *trigonometric form* (or *polar*) of the complex number z. What is important is that here θ is in radians and not in degrees.

Remark 3.1. The distance r of the number z from 0 is sometimes called the *modulus* of z and is denoted by $|z|$. From the first homework you know that

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

Remark 3.2. Note that many different values of θ give the same z if they differ by integer multiples of 2π. For example $2(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}) = 2(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6}))$.

Example 3.3. For the complex number $z = \sqrt{2}(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ we have $r = |z| = \sqrt{2}$ and $\theta = \frac{\pi}{3}$, so z can be rewritten in the usual (Cartesian) form as $z = \sqrt{2}\cdot\frac{1}{2} + i\sqrt{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{6}}{2}$.

It is also possible to convert a complex number written in the Cartesian form $z = x + iy$ with $x, y \in \mathbf{R}$ into a polar form. For that we see that

$$r = \sqrt{x^2 + y^2} = |z|, \quad \tan\theta = \frac{y}{x}.$$

Note that if z is in the first or fourth quadrant, then $\theta = \arctan\frac{y}{x}$, but if z is in the second or third quadrant then $\theta = \pi + \arctan\frac{y}{x}$.

Example 3.4. The complex number $z = -\frac{5}{\sqrt{2}} + i\frac{5}{\sqrt{2}}$ can be converted into a polar form by computing

$$r = \sqrt{\left(-\frac{5}{\sqrt{2}}\right)^2 + \left(\frac{5}{\sqrt{2}}\right)^2} = 5, \quad \tan\theta = \frac{\frac{5}{\sqrt{2}}}{-\frac{5}{\sqrt{2}}} = -1.$$

By marking z on the plane we immediately see that the angle is $\frac{3\pi}{4}$. Alternatively, we can also compute it using the formula. We know that $\arctan(-1) = -\frac{\pi}{4}$, but since the point is in the second quadrant, we get

$$\theta = \pi + \arctan(-1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

So, z written in polar form looks like this:

$$z = 5\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right).$$

Example 3.5. We should not be under the impression that the numbers always come out so nice. Indeed, this very simple complex number $z = 2 + i$ has $r = \sqrt{2^2 + 1^2} = \sqrt{5}$, but its angle is not so nice. The number is in the first quadrant and so

$$\theta = \arctan\left(\frac{1}{2}\right) = 0.392699082\dots$$

Here I used google to find this number. So, we have

$$z = 2 + i = \sqrt{5}(\cos 0.392699082 \cdots + +i \sin 0.392699082 \dots).$$

The polar form is very useful for multiplying complex numbers. Suppose we have

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), \quad z_2 = r_1(\cos\theta_2 + i\sin\theta_2).$$

Here $r_1 = |z_1|$ and $r_2 = |z_2|$ with $\theta_1, \theta_2 \in \mathbf{R}$, and we try to compute $z_1 \cdot z_2$. Then we get

$$\begin{aligned} z_1 \cdot z_2 =& r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ =& r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)) \\ =& r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)). \end{aligned} \tag{3.1}$$

Here we used the formulas

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha, \quad \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Formula (3.1) is very useful for multiplying complex numbers in polar form, because it says that to multiply two complex numbers in polar form, you 'multiply the moduli and add the angles'.

Example 3.6. Find the products zv and z^2 for $z = \sqrt{3}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ and $v = 2(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$. We get

$$z \cdot v = 2\sqrt{3}\left(\cos\left(\frac{\pi}{6} + \frac{3\pi}{4}\right) + i\sin\left(\frac{\pi}{6} + \frac{3\pi}{4}\right)\right) = 2\sqrt{3}\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right).$$

$$z^2 = (\sqrt{3})^2\left(\cos\left(\frac{\pi}{6} + \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6} + \frac{\pi}{6}\right)\right) = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).$$

One can see from (3.1) that if $z = r(\cos\theta + i\sin\theta)$ then for any postive integer n one has

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

(To prove this properly one has to use what mathematicians call "induction"). This allows us to very quickly raise complex numbers to very high powers.

Example 3.7. If $z = \sqrt[3]{4}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$, then

$$\begin{aligned} z^{999} =& 4^{333}\left(\cos\frac{999\pi}{6} + i\sin\frac{999\pi}{6}\right) = 4^{333}\left(\cos\frac{333\pi}{2} + i\sin\frac{333\pi}{2}\right) \\ =& 4^{333}\left(\cos\left(83 \cdot 2\pi + \frac{\pi}{2}\right) + i\sin\left(83 \cdot 2\pi + \frac{\pi}{2}\right)\right) \\ =& 4^{333}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 4^{333}(0 + i \cdot 1) = 4^{333}i. \end{aligned} \tag{3.2}$$

The most interesting case is when $r = 1$. The set of complex numbers with modulus 1 forms a circle centered at $z = 0$ and of radius 1. We call it the *unit circle*. So, any complex number on the unit circle can be written as

$$z = \cos\theta + i\sin\theta$$

or some $\theta \in \mathbf{R}$. If we have two complex numbers $z_1 = \cos\theta_1 + i\sin\theta_1$ and $z_2 = \cos\theta_2 + i\sin\theta_2$ on the unit circle (with $\theta_1, \theta_2 \in \mathbf{R}$) then

$$z_1 \cdot z_2 = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2),$$

so we only add the angles and we don't have to worry about multiplying the moduli. If $z = \cos\theta + i\sin\theta$ for some $\theta \in \mathbf{R}$, then for a positive integer n one has

$$z^n = cos(n\theta) + i\sin(n\theta).$$

Example 3.8. Consider the complex number

$$z = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}.$$

Then $z^3 = \cos(2\pi) + i\sin(2\pi) = 1$. So, we just discovered a non-real solution to the equation $x^3 = 1$.

Solve the following problems:

(1) Convert the following complex numbers into polar form (remember to use radians and sometimes you may need to use a calculator):

$$3 - 3i, \frac{1}{2} - \frac{\sqrt{3}}{2}i, -i, -2\sqrt{3} - 2i, 1 + 5i$$

(2) Find z^2, z^{10} and z^{12} for $z = \sqrt{3}(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5})$ (simplify the angle, so that it is between 0 and 2π).

(3) For each of the complex numbers in (1) find their squares, cubes and 2023rd powers (simplify the angle, so that it is between 0 and 2π).

(4) Show that $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ is also a solution to $x^3 = 1$. Now re-write this number in Cartesian form and cube it again to confirm that you get 1.

(5) Prove that $\cos(\frac{2n\pi}{5}) + i\sin(\frac{2n\pi}{5})$ is a solution to the equation $x^5 = 1$ for every integer n. How many *different* solutions do you get? Mark them all on the complex plane (you should get a regular pentagon).