Symplectic Lie Groups

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History

Symplectic homogeneous spaces appeared in 1970 in the work of

- B. Kostant, in an attempt to extend Kirillov's orbit method to solvable Lie groups.

  An element of a coadjoint orbit defines a symplectic form, so that the orbit becomes a symplectic homogeneous space.

- J.-M. Souriau, who studied dynamical systems.

  "Elementary" systems correspond to transitive actions of Lie groups on symplectic manifolds.

See Sternberg (1975) for a review.

Ph.D. thesis by Ph. Zwart in 1965 on compact symplectic homogeneous spaces, published only in 1980.

I
Definition and Basic Properties

Definition

A symplectic Lie group G is a Lie group with a left-invariant symplectic form ω on G.

That is,

- $\omega_g : T_g G \times T_g G \to T_g G$ is a non-degenerate bilinear form
- ω_g is skew-symmetric: $\omega_g(x, y) = -\omega_g(y, x)$
- ω is closed: $\mathrm{d}\omega = 0$
- $\omega_g(X_g, Y_g) = \omega_{1_G}(X_{1_G}, Y_{1_G})$
  for left-invariant vector fields $X, Y \in \mathfrak{g}$

That ω is closed can be expressed on $\mathfrak{g}$ as

$$\omega(X,[Y,Z]) + \omega(Z,[X,Y]) + \omega(Y,[Z,X]) = 0.$$

So ω is a (non-degenerate) 2-cocycle on $\mathfrak{g}$.

What *is not* a symplectic Lie group?

Symplectic Lie Group $\neq \mathbf{Sp}_{2n}(\mathbb{R})$

Some unfortunate terminology...

The classical symplectic group

$$\mathbf{Sp}_{2n}(\mathbb{R}) = \{g \in \mathbf{GL}_{2n}(\mathbb{R}) \mid \omega_0(gx, gy) = \omega_0(x, y) \text{ for all } x, y \in \mathbb{R}^{2n}\}$$

is not a symplectic Lie group!

No semisimple symplectic Lie groups

Theorem (Chu, 1974)
A semisimple Lie group does not admit a left-invariant symplectic form.

Proof:

- If G is a symplectic Lie group, then $\mathfrak{g}$ admits a multiplication $\circ$ such that $X \circ Y - Y \circ X = [X, Y]$, and $L_X(Y) = X \circ Y$ is a representation; $\circ$ is defined by $\omega(X \circ Y, \cdot) = \mathscr{L}_X(\omega(Y, \cdot))$.
- For semisimple G, such a product cannot exist (consequence of $H^1(\mathfrak{g}, L, \mathfrak{g}) = \{0\}$). $\square$

Dimension 4

Corollary

A symplectic Lie group of dimension 4 is solvable.

Proof:

- Assume $\mathfrak{g}$ is not solvable. Also, $\mathfrak{g}$ is not semisimple.
- Then the Levi decomposition is $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{a}$ with $\dim \mathfrak{s} = 3$ and $\dim \mathfrak{a} = 1$, $\mathfrak{s}$ semisimple and $\mathfrak{a}$ abelian.
- Then $[\mathfrak{s}, \mathfrak{a}] = \{0\}$, and as ω is closed, this implies $\omega(\mathfrak{s}, \mathfrak{a}) = \{0\}$.
- As $\dim \mathfrak{a} = 1$, $\omega(\mathfrak{a}, \mathfrak{a}) = \{0\}$. So ω is degenerate, a contradiction. $\square$

Compact symplectic Lie groups

Corollary

A compact connected symplectic Lie group is a torus group.

Proof:

- If G is compact, then $\mathfrak{g}$ is reductive:
  $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{a}$ with $\mathfrak{s}$ semisimple and $\mathfrak{a}$ the centre of $\mathfrak{g}$.
- Then $[\mathfrak{s}, \mathfrak{a}] = \{0\}$, and as ω is closed, this implies $\omega(\mathfrak{s}, \mathfrak{a}) = \{0\}$.
- Then $\mathfrak{s} = \{0\}$, otherwise it would be a semisimple symplectic Lie algebra.
- Now $\mathfrak{g} = \mathfrak{a}$ follows. $\square$

Bi-invariant symplectic forms

Theorem (Chu, 1974)

If a connected Lie group G admits a bi-invariant symplectic form ω, then G is abelian.

Proof:

- $\mathrm{Ad}(g)^*\omega = \omega$ for all $g \in G$ means $\mathscr{L}_X\omega = 0$ for all $X \in \mathfrak{g}$.
- The product $\circ$ defined by $\omega(X \circ Y, \cdot) = \mathscr{L}_X(\omega(Y, \cdot))$ vanishes.
- Now $[X, Y] = X \circ Y - Y \circ X = 0$, so $\mathfrak{g}$ is abelian. $\square$

Example: Non-solvable symplectic Lie group

Agaoka (2001): On the 6-dimensional affine algebra

$$\mathfrak{aff}(\mathbb{R}^2) = \left\{ \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \;\middle|\; A \in \mathfrak{gl}_2(\mathbb{R}), v \in \mathbb{R}^2 \right\},$$

a non-degenerate 2-cocycle ω is given by

$$\omega(X, Y) = \alpha([X, Y]),$$

where

$$\alpha = E_{1,2}^* + E_{2,3}^* \in \mathfrak{aff}(\mathbb{R}^2)^*$$

for the canonical basis $E_{i,j}$ ($i = 1, 2$, $j = 1, 2, 3$) of $\mathfrak{aff}(\mathbb{R}^2)$.

II
Cotangent Groups

Solvable symplectic Lie groups seem to form an important class.

Recall

A Lie group G is solvable if the derived subgroups

$$\mathscr{D}^1 G = [G, G], \quad \mathscr{D}^k G = [\mathscr{D}^{k-1} G, \mathscr{D}^{k-1} G]$$

yield a normal series

$$G \supset \mathscr{D}^1 G \supset \mathscr{D}^2 G \supset \ldots \supset \mathscr{D}^m G = \{1_G\}.$$

G is completely solvable if for all k

$$\dim(\mathscr{D}^k G / \mathscr{D}^{k+1} G) = 1.$$

Cotangent construction

A well-studied class of solvable symplectic Lie groups are cotangent groups:

- Let H be a solvable Lie group.
- Then $G = T^*H \cong H \ltimes \mathfrak{h}^*$ is a Lie group with Lie algebra $\mathfrak{h} \ltimes \mathfrak{h}^*$ given by

$$\big[(X_1, \xi_1), (X_2, \xi_2)\big] = \big([X_1, X_2],\ \mathrm{ad}^*(X_1)\xi_2\big)$$

and a closed 2-form on $\mathfrak{h} \times \mathfrak{h}^*$ by

$$\omega\big((X_1, \xi_1), (X_2, \xi_2)\big) = \xi_2(X_1) - \xi_1(X_2).$$

Theorem (Chu, 1974)
ω is a left-invariant symplectic form on $G = T^*H$ if and only if H is abelian.

Cotangent construction: deformations

Generalise the cotangent construction for solvable Lie groups H

- with a flat affine connection ∇ on H
- and a deformation cocycle $\varphi : \mathfrak{h} \wedge \mathfrak{h} \to \mathfrak{h}^*$
  (that is, $\varphi(X,Y)(Z) + \varphi(Z,X)(Y) + \varphi(Y,Z)(X) = 0$).

∇ and φ induce a Lie product on $\mathfrak{h} \ltimes \mathfrak{h}^*$:

$$\big[(X_1,\xi_1),(X_2,\xi_2)\big] = \big([X_1,X_2],\ \xi_2 \circ \nabla_{X_1} - \xi_1 \circ \nabla_{X_2} + \varphi(X,Y)\big),$$

and on $\mathfrak{h} \times \mathfrak{h}^*$ set

$$\omega\big((X_1,\xi_1),(X_2,\xi_2)\big) = \xi_2(X_1) - \xi_1(X_2).$$

(ω is closed $\Leftrightarrow$ φ is a deformation.)

Characteristion of cotangent groups

Theorem (Boyom, 1993)

Solvable Lie groups H with flat affine connection ∇ and deformation cocycle φ

$$\overset{1:1}{\longleftrightarrow}$$

solvable cotangent groups $G = T^*H$ with left-invariant symplectic form ω.

Proof:

- Given (H, ∇, φ): Check that ω as defined previously is left-invariant.
- Given (T^*H, ω): Lie product on $\mathfrak{h} \times \mathfrak{h}^*$ induces 2-cocycle φ by $[(X_1, 0), (X_2, 0)] = ([X_1, X_2], \varphi(X_1, X_2))$ and connection ∇ by $[(X, 0), (0, \xi)] = (0, -\xi \circ \nabla_X)$.

  Closedness of ω provides flatness of ∇ and deformation property of ω. $\square$

Claim 1 (Boyom, 1993)

Let G be a simply connected (completely?) solvable symplectic Lie group.
Then G is a semidirect product of symplectic cotangent groups.

Claim 2 (Boyom, 1993)

Let G be a completely solvable symplectic Lie group.
Then G is a symplectic cotangent group.

~~Proof:~~ Problems:

- Unclear which assumption Boyom makes in Claim 1.
- Claim 2 is based on the assumption that $\mathfrak{g}$ contains a Lagrange ideal.
  This is not true in general, as counterexamples by Baues & Cortés (2013) show.

III
Lagrange Subgroups

Definition

Let $\mathfrak{g}$ be a Lie algebra with symplectic form ω.

A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called

- isotropic, if $\omega|_{\mathfrak{h}\times\mathfrak{h}} = 0$.
- Lagrange subalgebra, if $\mathfrak{h}^\perp = \mathfrak{h}$
  (equivalent: $\mathfrak{h}$ is isotropic and $\dim \mathfrak{h} = \frac{1}{2} \dim \mathfrak{g}$).

Define the notions for Lie groups accordingly.

Symplectic reduction

If $\mathfrak{g}$ contains an isotropic ideal $\mathfrak{h}$, then

$$\mathfrak{g} \cong \mathfrak{h} \oplus (\mathfrak{h}^{\perp}/\mathfrak{h}) \oplus \mathfrak{h}^*$$

and $\mathfrak{g}_1 = \mathfrak{h}^{\perp}/\mathfrak{h}$ with the symplectic form ω_1 induced by ω is called the symplectic reduction of $\mathfrak{g}$ by $\mathfrak{h}$ (write $(\mathfrak{g}, \omega) \dashrightarrow (\mathfrak{g}_1, \omega_1)$).

$\mathfrak{g}$ is called irreducible if it contains no (non-trivial) isotropic ideals.

Theorem (Baues & Cortés, 2013)

If G is a completely solvable symplectic Lie group, then G is completely reducible in the sense

$$(\mathfrak{g}, \omega) \dashrightarrow (\mathfrak{g}_1, \omega_1) \dashrightarrow \ldots \dashrightarrow (\mathfrak{g}_k, \omega_k) = (0, 0).$$

In particular, $\mathfrak{g}$ has a non-trivial isotropic ideal.

Proof:

- By complete solvability, a 1-dimensional ideal $\mathfrak{h}$ exists in $\mathfrak{g}$.
- Clearly, $\mathfrak{h}$ is isotropic with respect to ω.
- The reduction $\mathfrak{g}_1$ is again completely solvable. $\square$

Remark

Non-solvable completely reducible groups exist.

Example: Solvable without Lagrange ideal

However... there exist completely solvable $\mathfrak{g}$ without a *Lagrange* ideal: Let $\alpha\beta \neq 0$ and set

$$X_1 = \begin{pmatrix} \alpha & & & \\ & -\alpha & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \beta & \\ & & & -\beta \end{pmatrix}$$

act on $\mathbb{R}^4 = \mathrm{span}\{e_1, e_2, e_3, e_4\}$. Set

$$\mathfrak{g} = \mathrm{span}\{X_1, X_2\} \ltimes \mathbb{R}^4$$

and define a symplectic form (in the dual basis) by

$$\omega = e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + X_1^* \wedge X_2^*.$$

Baues & Cortés (2013):
$\mathfrak{g}$ is completely solvable without Lagrange ideal.

Unique reduction base

Theorem (Baues & Cortés, 2013)
Let $\mathfrak{g}$ *be a symplectic Lie algebra and*

$$(\mathfrak{g},\omega) \dashrightarrow (\mathfrak{g}_1,\omega_1) \dashrightarrow \dots \dashrightarrow (\mathfrak{g}_k,\omega_k)$$
$$(\mathfrak{g},\omega) \dashrightarrow (\mathfrak{g}'_1,\omega'_1) \dashrightarrow \dots \dashrightarrow (\mathfrak{g}'_m,\omega'_m)$$

such that $(\mathfrak{g}_k,\omega_k)$ *and* $(\mathfrak{g}'_m,\omega'_m)$ *are irreducible. Then:*

$$(\mathfrak{g}_k,\omega_k) \cong (\mathfrak{g}'_m,\omega'_m).$$

Proof: Essentially induction on the reduction's length. $\square$

Alternative

A sequence of symplectic reductions ends

- either with an irreducible $(\mathfrak{g}_k, \omega_k)$ in the last step,
- or with $(0, 0)$, then $(\mathfrak{g}_{k-1}, \omega_{k-1})$ in the next to last step has a Lagrange ideal.

Irreducible symplectic Lie algebras

Theorem (Baues & Cortés, 2013)
Let $\mathfrak{g}$ be an irreducible symplectic Lie algebra.

Then there exist

- *a real abelian Lie algebra $\mathfrak{a}$,*
- *a complex vector space (V, J) with basis $\{v_1, \dots, v_m\}$,*
- *a set of weights $\lambda_1, \dots, \lambda_m$ spanning $\mathfrak{a}^*$*

such that

$$\mathfrak{g} \cong \mathfrak{a} \ltimes V$$

with Lie product given by

$$[A, v_j] = \lambda_j(A) J v_j \ \in V.$$

Lagrange ideals in nilpotent Lie algebras

Existence results for nilpotent symplectic Lie algebras:

1. If $\mathfrak{g}$ is k-step nilpotent, then $\mathscr{C}^j\mathfrak{g}$ is an isotropic ideal for $j \geq \frac{k}{2}$.
2. If $\mathfrak{g}$ is k-step nilpotent, then there exists an isotropic ideal of dimension $\geq \frac{k}{2}$.
3. If $\mathfrak{g}$ is nilpotent and $\dim \mathfrak{g} \leq 6$, then $\mathfrak{g}$ has a Lagrange ideal.
4. If $\mathfrak{g}$ is 2-step nilpotent, then $\mathfrak{g}$ has a Lagrange ideal.
5. If $\mathfrak{g}$ is 3-step nilpotent and $\dim \mathfrak{g} \leq 10$, then $\mathfrak{g}$ has a Lagrange ideal.

Classification of Lagrange extensions

If $\mathfrak{g}$ has a Lagrange ideal $\mathfrak{h}^*$, then (as a vector space)

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*,$$

so $\mathfrak{g}$ is an Lagrangian extension of $\mathfrak{h}$ by $\mathfrak{h}^*$.

Theorem (Baues & Cortés, 2013)
Lie groups H with flat affine connection ∇ and Lagrangian extension cocycle φ

$$\overset{1:1}{\longleftrightarrow}$$

cotangent groups $G = T^*H$ with left-invariant symplectic form ω.

IV
Outlook: Symplectic Homogeneous Spaces

Definition

A symplectic homogeneous space $M = G/K$ is a symplectic manifold with a Lie transformation group $G \subset \mathbf{Symp}(M)$ acting transitively on M.

1. The class of Lie groups G with symplectic homogeneous spaces G/K is much larger than the class of symplectic Lie groups.
2. For example, there exist semisimple G with symplectic homogeneous spaces G/K.
3. Here, only the existence of a possibly degenerate 2-cocycle ω on G is required.
   Factoring out K induces a non-degenerate form on G/K, where
   $$\mathfrak{k} = \{X \in \mathfrak{g} \mid \omega(X, \cdot) = 0\}.$$

Simply connected G/H

Theorem (Chu, 1974)

Let G be a simply connected Lie group. Then:

Simply connected symplectic homogeneous space $M = G/K$ where $\mathfrak{k} = \{X \in \mathfrak{g} \mid \omega(X, \cdot) = 0\}$

$$\overset{1:1}{\longleftrightarrow}$$

2-cocycles $\omega \in Z^2(\mathfrak{g})$.

Compact symplectic homogeneous manifolds

Zwart & Boothby (1980) studied compact G/K where K acts almost effectively: K contains only discrete normal subgroups of G.

Compact symplectic homogeneous manifolds

Theorem (Zwart & Boothby, 1980)

Let $M = G/K$ be a compact symplectic homogeneous manifold, and let $G = S \cdot R$ be the Levi decomposition of G.

Then S is compact and G/K splits as a product of symplectic spaces:

$$G/K = (S_1/K_1 \times \cdots \times S_m/K_m) \ \times \ R/K_R$$

where the $S_j \subset S$ are simple, $K_j = S_j \cap K$ and $K_R = R \cap K$.

Compact symplectic homogeneous solvmanifolds

Theorem (Zwart & Boothby, 1980)

Let G be a (simply connected) solvable Lie group and $M = G/K$ a compact symplectic homogeneous manifold.

Then:

$$G = A \cdot N,$$

where A and N are abelian and N is the nilradical of G.

Moreover,

$$K = \Gamma \cdot M,$$

where Γ is a lattice in A and $M \subset N$ a uniform subgroup.

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