The fundamental group of a connected Lie group is abelian

Let G be a connected Lie group with universal covering group $\tilde{G}$ and fundamental group Γ. The Lie group $\tilde{G}$ is also connected.

I. It is a standard fact that the canonical projection $\pi : \tilde{G} \to G$ is a Lie group homomorphism with kernel Γ. In particular, Γ is a normal subgroup in $\tilde{G}$.

II. Γ is also a discrete subgroup of $\tilde{G}$, as it is the fundamental group of G.

III. So Γ is a discrete normal subgroup in a connected Lie group $\tilde{G}$. Now let $g(t)$ be a continuous path in $\tilde{G}$ with $g(0) = 1$. For every element $\gamma \in \Gamma$,

$$\gamma(t) = g(t) \cdot \gamma \cdot g(t)^{-1}$$

is again a continuous path, and it is a path in Γ because Γ is normal. But because Γ is discrete, the continuous path is constant,

$$\gamma(t) = \gamma(0) = \gamma.$$

As every element $g \in \tilde{G}$ can be connected to 1 by a continuous path, this means every $\gamma \in \Gamma$ commutes with every g.

In other words, *Γ is contained in the centre of $\tilde{G}$, so it is abelian.*