There are no invariant skew-symmetric forms on semisimple Lie algebras

I. Let $\mathfrak{g}$ be a real Lie algebra and $\omega : \mathfrak{g} \wedge \mathfrak{g} \to \mathbb{R}$ a skew-symmetric form on $\mathfrak{g}$ which is invariant with respect to $\mathrm{ad}(\mathfrak{g})$, that is,

$$\omega([X, Y], Z) = -\omega(Y, [X, Z])$$

for all $X, Y, Z \in \mathfrak{g}$. Moreover, the skew-symmetry gives us

$$\omega([X, Y], Z) = -\omega(Z, [X, Y]),$$

which combined means

$$\omega(Y, [X, Z]) = \omega(Z, [X, Y]). \tag{$*$}$$

II. The radical of ω,

$$\mathfrak{rad}(\omega) = \{X \in \omega \mid \omega(X, Y) = 0 \text{ for all } Y \in \mathfrak{g}\} \subset \mathfrak{g},$$

is an ideal in $\mathfrak{g}$, due to invariance:

$$\omega([X, Y], Z) = \omega(X, [Y, Z]) = 0,$$

where $X \in \mathfrak{rad}(\omega)$ and $Y, Z \in \mathfrak{g}$.
If we assume $\mathfrak{g}$ to be semisimple, this implies

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{rad}(\omega),$$

where $\mathfrak{g}_0$ is a semisimple ideal in $\mathfrak{g}$ such that $\omega|_{\mathfrak{g}_0 \times \mathfrak{g}_0}$ is non-degenerate.
III. So without loss of generality, we may assume that ω is a *non-degenerate* invariant skew-symmetric form on $\mathfrak{g}$.
But by $(*)$ we have for all $X, Y, Z \in \mathfrak{g}$:

$$\begin{aligned} 0 &= \omega([X, Y], Z) + \omega(Y, [X, Z]), \\ &= -\omega([Y, X], Z) - \omega(X, [Y, Z]). \end{aligned}$$

We add these two equations to obtain:

$$\begin{aligned} 0 &= \omega([X, Y], Z) - \omega([Y, X], Z) + \omega(Y, [X, Z]) - \omega(X, [Y, Z]) \\ &= 2\omega([X, Y], Z) \underbrace{-\omega(Y, [Z, X]) + \omega(X, [Z, Y])}_{=0}, \end{aligned}$$

where the last term is 0 due to $(*)$ (with X and Z exchanged). By non-degeneracy of ω, this implies

$$[X, Y] = 0$$

for all $X, Y \in \mathfrak{g}$.

IV. We have proved the following:

Theorem: *If $\mathfrak{g}$ is a Lie algebra with a non-degenerate invariant skew-symmetric form ω, then $\mathfrak{g}$ is abelian.*

In particular, $\mathfrak{g}$ cannot be semisimple. With II. this means

Corollary: *There exist no invariant skew-symmetric forms on a semisimple Lie algebra other than the 0-form.*

The argument in part III. is a simplifcation of a proof in B-Y. Chu, *Symplectic Homogeneous Spaces*, Trans. Amer. Math. Soc. 197, 1974.