# MURI Discordance Games Notes

## MURI LA

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## 1 Definitions

*Definition* 1. Let $G$ be an undirected graph with adjacency matrix $A$ and let $x \in \mathbb{R}$ be a vector indexed by the nodes $i \in \mathcal{V}(G)$. Denote and define the *discordance* of a node $i \in \mathcal{V}(G)$ as the following function of $x$:

$$d_i(x|G) = \sum_{k=1}^{n} A_{ik}(x_i - x_k)^2.$$

Naturally, we may also define the discordance of a subset $V \subset \mathcal{V}(G)$ of nodes as follows

*Definition* 2. Denote and define the discordance of a subset $S \subset \mathcal{V}(G)$ of nodes as follows:

$$d_S(x|G) = \sum_{i \in S} d_i(x|G).$$

If we interpret $x$ as a vector of *opinions* about a topic, then $d_i(x|G)$ is one way to measure how much node $i$ disagrees with its neighbors in the graph $G$. Under the interpretation of $x$ as a vector of opinions, it is of interest to define a model for how the values $x_i$ change as nodes in $G$ communicate. Accordingly, we make the following definition:

*Definition* 3. Consider two nodes $p, q \in \mathcal{V}(G)$ with values $x_p, x_q \in \mathbb{R}$. We say that the nodes $p$ and $q$ come to an $\alpha$-*equal agreement* if the nodes $p$ and $q$ change their values as follows:

$$x_p \mapsto x_p + \text{sgn}(x_q - x_p)\alpha \qquad x_q \mapsto x_q + \text{sgn}(x_p - x_q)\alpha,$$

where

$$0 < \alpha \leq \frac{|x_p - x_q|}{2}.$$

and $\text{sgn}(x) = x/|x|$ for $x \neq 0$.

We wish to investigate when agreements between two nodes $p, q \in \mathcal{V}(G)$ can occur under the following assumption:

*Definition* 4. Let $S \subset \mathcal{V}(G)$ be a subset of nodes of the graph $G$ and let $x \in \mathbb{R}^n$ be a vector of values assigned to the nodes in $\mathcal{V}(G)$. Consider an update $x \mapsto x' \in \mathbb{R}^n$ to the values $x$ such that $x_i' = x_i$ for $i \notin S$. We say that the update is *discordance decreasing* if

$$d_S(x') - d_S(x) < 0$$

## 2 Notes from May 6 Meeting

*Question* 1. For every choice of $x \in \mathbb{R}^n$ and for every graph $G$ does there exist a pair of adjacent nodes $p, q \in \mathcal{V}(G)$ and an $\alpha > 0$ sufficiently small such that an discordance decreasing $\alpha$-equal agreement can occur between $p$ and $q$.

To address the question above, we state the following lemma.

*Lemma* 1. Let $p, q \in \mathcal{V}(G)$ be two adjacent nodes in a graph $G$. Let $x \in \mathbb{R}^n$ be a vector of values corresponding to the nodes of $G$. Assume without loss of generality that $x_p > x_q$. Then there exists an $\alpha > 0$ sufficiently small such that a discordance decreasing update can occur between nodes $p$ and $q$ if the following inequality holds:

$$\text{sgn}(x_p - x_q) \sum_{\substack{k=1 \\ k \neq q}}^{n} A_{ik}(x_k - x_p) + \text{sgn}(x_p - x_q) \sum_{\substack{k=1 \\ k \neq p}}^{n} A_{jk}(x_k - x_q) < 4|x_p - x_q| \tag{1}$$

*Proof.* Without loss of generality, assume $x_p > x_q$ so that the $\alpha$-equal agreement between nodes $p$ and $q$ becomes

$$x_p \mapsto x_p - \alpha \qquad x_q \mapsto x_q + \alpha. \tag{2}$$

Denote $x'$ the new vector of values after the agreement (2), denote $d_{ij}(x) = d_{\{i,j\}}(x)$, and denote $k_i$ the degree of node $i$. The change in discordance for the set $\{i,j\}$ is given by

$$
\begin{aligned}
d_{pq}(x') - d_{pq}(x) &= \sum_{\substack{k=1 \\ k \neq q}}^{n} A_{pk}((x_p - x_k - \alpha)^2 - (x_p - x_k)^2) \\
&\quad + \sum_{\substack{k=1 \\ k \neq p}}^{n} A_{qk}((x_q - x_k + \alpha)^2 - (x_q - x_k)^2) + 2(x_p - x_q - 2\alpha)^2 - 2(x_p - x_q)^2 \\
&= -2\alpha \sum_{\substack{k=1 \\ k \neq q}}^{n} A_{pk}(x_p - x_k) + (k_p - 1)\alpha^2 + 2\alpha \sum_{\substack{k=1 \\ k \neq p}}^{n} A_{qk}(x_q - x_k) + (k_q - 1)\alpha^2 - 8\alpha(x_p - x_q) + 8\alpha^2 \\
&= 2\alpha \left( \sum_{\substack{k=1 \\ k \neq q}}^{n} A_{pk}(x_k - x_p) + \sum_{\substack{k=1 \\ k \neq p}}^{n} A_{qk}(x_q - x_k) - 4(x_p - x_q) \right) + (k_p + k_q + 6)\alpha^2.
\end{aligned}
\tag{3}
$$

Thus, if inequality (1) holds then (3) implies that for $\alpha$ sufficiently small, $d_{pq}(x') - d_{pq}(x) < 0$. $\square$

[[Determine the $\alpha$ in lemma 1 to determine how large of an agreement is possible. When is a consensus to the middle ground possible?]]

The following proposition answers question 1 in the affirmative. The idea of the proof is to build a path (walk of distinct nodes) such that the node values along the path are strictly increasing. If inequality (1) fails for all pairs of adjacent nodes in the graph, then the process by which the path is constructed does not terminate. Since there are a finite number of nodes in the graph, this leads to a contradiction.

**Proposition 1.** *For any graph $G$ and any vector $x \in \mathbb{R}^n$ of values indexed by the nodes of $G$, there exists a pair of adjacent nodes $p, q \in \mathcal{V}(G)$ and an $\alpha > 0$ sufficiently small such that a discordance decreasing $\alpha$-equal agreement is possible between nodes $p$ and $q$.*

*Proof.* Let $i_0$ be the index of a node such that $x_{i_0} \leq x_k$ for all neighbors $k$ of $i_0$ (such as the minimal value in $x$) and $x_{i_0} < x_{i_1}$ for some neighbor $i_1$. We will call a node $i$ satisfying $x_i \leq x_k$ for all neighbors $k$ of $i$ *locally minimal* and a node satisfying the same condition with $\geq$ locally maximal. Inequality (1) with $i_0$ and $i_1$ reads

$$\sum_{\substack{k=1 \\ k \neq i_0}}^{n} A_{i_1 k}(x_k - x_{i_1}) + \sum_{\substack{k=1 \\ k \neq i_1}}^{n} A_{i_0 k}(x_{i_0} - x_k) < 4(x_{i_1} - x_{i_0}) \tag{4}$$

If equation (4) holds, we can update nodes $i_0$ and $i_1$. If not, then (4) being false and $x_{i_0}$ being minimal among its neighbors implies that

$$\sum_{\substack{k=1 \\ k \neq i_0}}^{n} A_{i_1 k}(x_k - x_{i_1}) \geq 4(x_{i_1} - x_{i_0}) + \sum_{\substack{k=1 \\ k \neq i_1}}^{n} A_{i_0 k}(x_k - x_{i_0}) > 0. \tag{5}$$

Equation (5) implies that there exists a neighbor $i_2$ of $i_1$ such that $x_{i_2} > x_{i_1}$. Consider inequality (1) for nodes $i_1$ and $i_2$:

$$\sum_{\substack{k=1 \\ k \neq i_1}}^{n} A_{i_2 k}(x_k - x_{i_2}) + \sum_{\substack{k=1 \\ k \neq i_2}}^{n} A_{i_1 k}(x_{i_1} - x_k) < 4(x_{i_2} - x_{i_1}) \tag{6}$$

If inequality (6) holds, then we're done. Otherwise, we write the negation of (6) as follows:

$$\sum_{\substack{k=1 \\ k \neq i_1}}^{n} A_{i_2 k}(x_k - x_{i_2}) + \sum_{\substack{k=1 \\ k \neq i_0}}^{n} A_{i_1 k}(x_{i_1} - x_k) + (x_{i_1} - x_{i_0}) \geq 3(x_{i_2} - x_{i_1}) \tag{7}$$

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Reorganizing (7) we obtain

$$\sum_{\substack{k=1\\k\neq i_1}}^{n} A_{i_2 k}(x_k - x_{i_2}) \geq 3(x_{i_2} - x_{i_1}) - \sum_{\substack{k=1\\k\neq i_0}}^{n} A_{i_1 k}(x_{i_1} - x_k) - (x_{i_1} - x_{i_0}) \tag{8}$$

and by using inequality (5) in (8) we obtain

$$\sum_{\substack{k=1\\k\neq i_1}}^{n} A_{i_2 k}(x_k - x_{i_2}) \geq 3(x_{i_2} - x_{i_0}) + \sum_{\substack{k=1\\k\neq i_1}}^{n} A_{i_0 k}(x_k - x_{i_0}) > 0. \tag{9}$$

Inequality (9) shows that $i_2$ has a neighbor $i_3$ such that $x_{i_3} > x_{i_2}$. Continue in this fashion. Suppose that the inequality (1) does not hold for consecutive values in the sequence $i_0, i_1, \ldots, i_m$ of consecutively adjacent vertices and such that $x_{i_0} < x_{i_1} < \ldots < x_{i_m}$. Suppose inequality (4) fails when applied to nodes $i_m$ and $i_{m-1}$. Then,

$$\sum_{\substack{k=1\\k\neq i_{m-1}}}^{n} A_{i_m k}(x_k - x_{i_m}) \geq 4(x_{i_m} - x_{i_{m-1}}) + \sum_{\substack{k=1\\k\neq i_m}}^{n} A_{i_{m-1} k}(x_k - x_{i_{m-1}})$$

$$= 3(x_{i_m} - x_{i_{m-1}}) + \sum_{\substack{k=1\\k\neq i_{m-2}}}^{n} A_{i_{m-1} k}(x_k - x_{i_{m-1}}) + (x_{i_{m-2}} - x_{i_{m-1}}) \tag{10}$$

By the induction hypothesis, we have that a similar inequality holds for $i_{m-1}$ and $i_{m-2}$:

$$\sum_{\substack{k=1\\k\neq i_{m-2}}}^{n} A_{i_{m-1} k}(x_k - x_{i_{m-1}}) \geq 4(x_{i_{m-1}} - x_{i_{m-2}}) + \sum_{\substack{k=1\\k\neq i_{m-1}}}^{n} A_{i_{m-2} k}(x_k - x_{i_{m-2}})$$

$$= 3(x_{i_{m-1}} - x_{i_{m-2}}) + \sum_{\substack{k=1\\k\neq i_{m-3}}}^{n} A_{i_{m-2} k}(x_k - x_{i_{m-2}}) + (x_{i_{m-3}} - x_{i_{m-2}}) \tag{11}$$

Plugging in inequality (11) into (13) we obtain

$$\sum_{\substack{k=1\\k\neq i_{m-1}}}^{n} A_{i_m k}(x_k - x_{i_m}) \geq 3(x_{i_m} - x_{i_{m-1}}) + 2(x_{i_{m-1}} - x_{i_{m-2}}) + \sum_{\substack{k=1\\k\neq i_{m-3}}}^{n} A_{i_{m-2} k}(x_k - x_{i_{m-2}}) + (x_{i_{m-3}} - x_{i_{m-2}}). \tag{12}$$

Continuing in this fashion, we obtain the following

$$\sum_{\substack{k=1\\k\neq i_m}}^{n} A_{i_m k}(x_k - x_{i_m}) \geq 3(x_{i_m} - x_{i_{m-1}}) + 2(x_{i_{m-1}} - x_{i_1}) + 3(x_{i_1} - x_{i_0}) + \sum_{\substack{k=1\\k\neq i_1}}^{n} A_{i_0 k}(x_k - x_{i_0}) > 0. \tag{13}$$

Which shows that $i_m$ has a neighbor $i_{m+1}$ satisfying $x_{i_{m+1}} > x_{i_m}$.

Since the graph $G$ is finite, the process to obtain the distinct vertices $i_0, i_1, \ldots, i_m$ must terminate. [[one can strengthen the statement of the proposition]]  $\square$

# 3   Notes following May 10 Meeting

## 3.1   Previous Results and Definitions

We consider the following three aggregate quantities over the graph $G$ for measuring the total difference in opinion between the individuals in the graph:

*Definition* 5. Denote and define the *discordance* of the nodes $i \in \mathcal{V}(G)$ with values $X_i(t)$ by

$$d(X, t|G) = \sum_{i,j=1}^{n} A_{ij}(X_i(t) - X_j(t))^2 = \|MX(t)\|_2^2.$$

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*Definition* 6. Denote and define the *total discordance* of the nodes $i \in \mathcal{V}(G)$ with values $X_i(t)$ by

$$d(X,t) = \sum_{i,j=1}^{n} (X_i(t) - X_j(t))^2.$$

*Definition* 7. Denote and define the *consensus error* of the nodes $i \in \mathcal{V}(G)$ with values $X_i(t)$ by

$$e(X,t) = \|X(t) - X_{\mathrm{ave}}\mathbf{1}\|_2^2.$$

For all three definitions 5-7 we will omit the $t$ argument when we are interested in a fixed $X$. Additionally, when concerned with a sequence $t_0 < t_1 < \cdots$ of times, we will specify just the index rather than the time: $e(X,k) = e(X,t_k)$ for example.

**Proposition 2.** *The total discordance in definition 6 is related to the consensus error 7 by*

$$d(X,t) = 2ne(X,t).$$

*Proof.* We omit the time $t$ in the following calculation for brevity:

$$d(X) = \sum_{i,j=1}^{n} (X_i - X_j)^2$$

$$= \sum_{i,j=1}^{n} (X_i - X_{\mathrm{ave}})^2 + \sum_{i,j=1}^{n} (X_j - X_{\mathrm{ave}})^2 - 2\sum_{i,j=1}^{n} (X_i - X_{\mathrm{ave}})(X_j - X_{\mathrm{ave}}) \tag{14}$$

$$= 2ne(X) - 2(nX_{\mathrm{ave}} - nX_{\mathrm{ave}})(nX_{\mathrm{ave}} - nX_{\mathrm{ave}}) \tag{15}$$

$$= 2ne(X). \tag{16}$$

where in (14) use $X_i - X_j = X_i - X_{\mathrm{ave}} + X_{\mathrm{ave}} - X_j$ and in (15) we pass the sums inside $(X_i - X_{\mathrm{ave}})(X_j - X_{\mathrm{ave}})$. $\square$

## 3.2 Convergence of the $\alpha$-equal Agreement Update

### 3.2.1 Notation and Setup

We consider updates of the following form:

$$\begin{aligned} X_p(k+1) &= X_p(k) - \beta_k \operatorname{sgn}(X_p(k) - X_q(k))(X_p(k) - X_q(k)) \\ X_q(k+1) &= X_q(k) + \beta_k \operatorname{sgn}(X_p(k) - X_q(k))(X_p(k) - X_q(k)). \end{aligned} \tag{17}$$

This is the same update as the previous section except we now explicitly consider $\beta_k \in (0, 1/2]$ to be a coefficient giving the fraction of the gap the nodes $p$ and $q$ change their opinion upon an $\beta$-equal agreement. We make note of the following proposition giving the amount by which the error changes upon an $\beta_k$-equal agreement.

Consider equation (3). Substituting $\alpha \mapsto \beta(x_p - x_q)$, we find that for an update at edge $(p,q)$ to be viable,

$$2\left( \sum_{\substack{k=1 \\ k \neq q}}^{n} A_{pk}(x_k - x_p) + \sum_{\substack{k=1 \\ k \neq p}}^{n} A_{qk}(x_q - x_k) - 4(x_p - x_q) \right) + (k_p + k_q + 6)\beta(x_p - x_q) < 0, \tag{18}$$

We rewrite (18) as

$$\beta(X_p - X_q) < \frac{4(X_p - X_q) - \left( \sum_{\substack{k=1 \\ k \neq q}}^{n} A_{pk}(X_k - X_p) + \sum_{\substack{k=1 \\ k \neq p}}^{n} A_{qk}(X_q - X_k) \right)}{(k_p + k_q + 6)/2} \tag{19}$$

Define $\beta(p,q)$ as follows

$$\beta(p,q) = \max_{0 < \beta \leq 1/2} \left\{ \beta(X_p - X_q) < \frac{4(X_p - X_q) - \left( \sum_{\substack{k=1 \\ k \neq q}}^{n} A_{pk}(X_k - X_p) + \sum_{\substack{k=1 \\ k \neq p}}^{n} (X_q - X_k) \right)}{(k_p + k_q + 6)/2} \right\}. \tag{20}$$

or $\beta(p,q) = 0$ if the equation (20) does not yield a value for $\beta(p,q)$. Additionally, define $p^*, q^* \in \mathcal{V}(G)$ as an adjacent pair of nodes satisfying

$$(p^*, q^*) \in \underset{\substack{(p,q) \in \mathcal{E}(G) \\ X_p > X_q}}{\operatorname{argmax}} \beta(p,q)(X_p - X_q). \tag{21}$$

We first prove a lemma which relates the graph discordance to the error:

*Lemma 2.* Denote $L$ the graph Laplacian of the graph $G$ and denote $\lambda_k(L)$ the $k^{th}$ smallest eigenvalue of $L$. If $G$ is connected, then $\lambda_n(L), \lambda_2(L) > 0$ and

$$\lambda_2(L)e(X) \leq d(X|G) \leq \lambda_n(L)e(X).$$

*Proof.* Clearly,

$$\left( \inf_{X \in \mathbb{R}^n} \frac{d(X|G)}{e(X)} \right) e(X) \leq d(X|G) \leq \left( \sup_{X \in \mathbb{R}^n} \frac{d(X|G)}{e(X)} \right) e(X) \tag{22}$$

Denote $1_n \in \mathbb{R}^n$ the vector of all ones. Notice that the coefficient

$$\frac{d(X|G)}{e(X)}$$

is unchanged by the transformation $X \mapsto X + c1_n$ for a constant $c$. Accordingly, if we denote $v_1, \ldots, v_n$ the eigenvectors of $L$ corresponding to eigenvalues $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n$, we can write

$$\begin{aligned} \inf_{X \in \mathbb{R}^n} \frac{d(X|G)}{e(X)} &= \inf_{X \perp 1_n} \frac{X^T L X}{X^T X} = \lambda_2(L) \\ \sup_{X \in \mathbb{R}^n} \frac{d(X|G)}{e(X)} &= \frac{v_n^T L v_n}{v_n^T v_n} = \lambda_n(L). \end{aligned} \tag{23}$$

If $G$ is connected, it is known that $\lambda_2(L) > 0$. The lemma follows from (22) and (23). $\qquad \square$

The folllowing proposition relates the "updateability" of the nodes on the graph to the distance of the nodes from consensus. Note: it is not important here to use $\beta(p^*, q^*)(X_{p^*} - X_{q^*})$ in the statement of the proposition. The proposition should be restated to work for any "norm" of the update size. This is important as it will allow us to establish convergence for random choice of updateable node (the expected value of the update size generates a norm, and the usual application of Markov's inequality will prove convergence in probability).

**Proposition 3.** *There exist constants $c_*(G), c^*(G) \in \mathbb{R}_{>0}$, independent of $X$, such that*

$$c_*(G)e(X)^{1/2} \leq \beta(p^*, q^*)|X_{p^*} - X_{q^*}| \leq c^*(G)e(X)^{1/2}, \tag{24}$$

*where $c_*(G) \leq 1/2$ and $c^*(G) = \sqrt{n/2}$.*

*Proof.* 1. Upper Bound. We know that $\beta \leq 1/2$ and $(X_p - X_q)^2 \leq d(X|G)$, so

$$(\beta^*(X_{p^*} - X_{q^*}))^2 \leq \frac{1}{4} d(X|G) \tag{25}$$

Thus by proposition 2,

$$(\beta^*(X_{p^*} - X_{q^*}))^2 \leq \frac{n}{2} e(X). \tag{26}$$

This bound is tightest bound independent of $X$, as can be seen by letting the values associated with two adjacent nodes $i$ and $j$ be distinct and letting every other node $k$ have value $(x_i + x_j)/2$. This works for any connected graph of $n \geq 2$ nodes.

2. Lower Bound. First, note that $c^*(G) \leq 1/2$, which is attained in a connected graph of 2 nodes.

We prove the existence of a lower bound by contradiction. Later work will address determining the tightest bound. Let $\epsilon_0 > 0$ be fixed and suppose for all $\epsilon > 0$ there exists an $X \in \mathbb{R}^n$ such that $e(X) \geq \epsilon_0^2 > 0$ and $\beta^*|X_{p^*} - X_{q^*}| < \epsilon$. By lemma 2, $e(X) \geq \epsilon_0^2$ implies that for some adjacent pair of nodes $i_1, i_0 \in \mathcal{V}(G)$, where we let $X_{i_1} > X_{i_0}$,

$$(X_{i_1} - X_{i_0})^2 \geq \frac{d(X|G)}{2m} \geq \frac{\lambda_2(L)}{2m} \epsilon_0^2. \tag{27}$$

In this proof we will use the assumption that $\beta|X_p - X_q| < \epsilon$ to generate an edge-disjoint path, monotonic with regard to the node values, that passes through the edge $(i_0, i_1)$. Each edge difference, $|X_p - X_q|$, along this path will be of order $\epsilon_0$ rather than of order $\epsilon$. Intuitively, this is is possible since if a node $j$ is only able to update by an amount $\epsilon$ with a node $i$, where we assume $X_j < X_i$, then there must be a neighbor $k$ of $j$ whose value satisfies $X_k < X_j$ (or a neighbor $k$ of $i$ satisfying $X_k > X_i$) and such that $X_j - X_k$ is large enough, in particular of order $\epsilon_0$. Indeed, the function $f(x) = x^2$ is small near zero but grows quickly for $x$ large, so if $X_j \approx X_k$, then node $j$ will not be affected by moving its opinion away from node $k$; however, if $X_k \ll X_j$ then the opinion of node $X_j$ may only be able to move slightly away from $X_k$. The edge-disjoint path generated in this manner, being monotonic in the node values, will stretch from a locally minimal node to a locally maximal node in the graph. However, at this point it is impossible to find a neighbor of the locally maximal node that is larger or a neighbor of the locally minimal node that is smaller. This will be the contradiction. We use big Oh notation below to make the proof easier to read. In this notation, we have that $\epsilon = O(\epsilon_0)$ and $\epsilon_0 = \Omega(\epsilon)$. From the definition 20 of $\beta(p,q)$ and from the assumption that $(X_{i_1} - X_{i_0})^2 \geq \frac{\lambda_2(L)}{2m}\epsilon_0^2$, we have

$$\Theta(\epsilon_0) = 4(X_{i_1} - X_{i_0}) - \frac{k_{i_0} + k_{i_1} + 6}{2}\epsilon < \sum_{\substack{k=1 \\ k \neq i_1}}^{n} A_{i_0 k}(X_{i_0} - X_k) + \sum_{\substack{k=1 \\ k \neq i_0}}^{n} A_{i_1 k}(X_k - X_{i_1}). \tag{28}$$

(28) implies that there must exist some neighbor $i_2 \in \mathcal{V}(G)$ of $i_1$ or some neighbor $i_{-1} \in \mathcal{V}(G)$ of $i_0$ such that $X_{i_2} > X_{i_1}$ or $X_{i_{-1}} < X_0$, respectively, with

$$X_{i_2} - X_{i_1} \geq \frac{4(X_{i_1} - X_{i_0})}{k_{i_0} + k_{i_1} - 2} - \frac{k_{i_0} + k_{i_1} + 6}{2(k_{i_0} + k_{i_1} - 2)}\epsilon \geq \frac{4}{k_{i_0} + k_{i_1} - 2}\sqrt{\frac{\lambda_2(L)\epsilon_0^2}{2m}} - \frac{k_{i_0} + k_{i_1} + 6}{2(k_{i_0} + k_{i_1} - 2)}\epsilon = \Theta(\epsilon_0) \tag{29}$$

or with the same inequality holding for $X_{i_0} - X_{i_{-1}}$. Thus, there are 3 cases: 1.) there exists a neighbor $i_2$ of $i_1$ such that $X_{i_2} > X_{i_1}$ and $X_{i_2} - X_{i_1} = \Theta(\epsilon_0)$, 2.) there exists a neighbor $i_{-1}$ of $i_0$ such that $X_{i_0} > X_{i_{-1}}$ and $X_{i_0} - X_{i_{-1}} = \Theta(\epsilon_0)$, or 3.) both 1.) and 2.) hold.

1. If we obtained $i_2$ in the previous step, we apply the definition 20 of $\beta(p,q)$ to $i_2$ and $i_1$ to obtain the following inequality

$$4(X_{i_2} - X_{i_1}) - \sum_{\substack{k=1 \\ k \neq i_2}}^{n} A_{i_1 k}(X_{i_1} - X_k) - \frac{k_{i_1} + k_{i_2} + 6}{2}\epsilon < \sum_{\substack{k=1 \\ k \neq i_1}}^{n} A_{i_2 k}(X_k - X_{i_2}), \tag{30}$$

which we rearrange to write as

$$3(X_{i_2} - X_{i_1}) - (X_{i_1} - X_{i_0}) + \sum_{\substack{k=1 \\ k \neq i_0}}^{n} A_{i_1 k}(X_k - X_{i_1}) - \frac{k_{i_1} + k_{i_2} + 6}{2}\epsilon < \sum_{\substack{k=1 \\ k \neq i_1}}^{n} A_{i_2 k}(X_k - X_{i_2}). \tag{31}$$

Now (28) can be rewritten as

$$-4(X_{i_1} - X_{i_0}) + \sum_{\substack{k=1 \\ k \neq i_0}}^{n} A_{i_1 k}(X_k - X_{i_1}) > -\frac{k_{i_0} + k_{i_1} + 6}{2}\epsilon - \sum_{\substack{k=1 \\ k \neq i_1}}^{n} A_{i_0 k}(X_{i_0} - X_k) \tag{32}$$

and applied to (31),

$$\Theta(\epsilon_0) = 3(X_{i_2} - X_{i_1}) + 3(X_{i_1} - X_{i_0}) - \frac{k_{i_0} + k_{i_1} + 6}{2}\epsilon - \frac{k_{i_1} + k_{i_2} + 6}{2}\epsilon - \sum_{\substack{k=1 \\ k \neq i_1}}^{n} A_{i_0 k}(X_{i_0} - X_k)$$

$$< \sum_{\substack{k=1 \\ k \neq i_1}}^{n} A_{i_2 k}(X_k - X_{i_2}) \tag{33}$$

2. If we had obtained a node $i_{-1}$ as a result of (28), then the definition 20 of $\beta(p,q)$ implies

$$4(X_{i_0} - X_{i_{-1}}) - \sum_{\substack{k=1 \\ i \neq i_{-1}}}^{n} A_{i_0 k}(X_k - X_{i_0}) - \frac{k_{i_0} + k_{i_{-1}} + 6}{2}\epsilon < \sum_{\substack{k=1 \\ i \neq i_0}}^{n} A_{i_{-1} k}(X_{i_{-1}} - X_k), \tag{34}$$

which we rearrange to write as

$$3(X_{i_0} - X_{i_{-1}}) - (X_{i_1} - X_{i_0}) - \sum_{\substack{k=1 \\ k \neq i_1}}^{n} A_{i_0 k}(X_k - X_{i_0}) - \frac{k_{i_0} + k_{i_{-1}} + 6}{2}\epsilon < \sum_{\substack{k=1 \\ i \neq i_0}}^{n} A_{i_{-1} k}(X_{i_{-1}} - X_k). \quad (35)$$

Now (28) can be rewritten as

$$-4(X_{i_1} - X_{i_0}) + \sum_{\substack{k=1 \\ k \neq i_1}}^{n} A_{i_0 k}(X_{i_0} - X_k) > -\frac{k_{i_0} + k_{i_1} + 6}{2}\epsilon - \sum_{\substack{k=1 \\ k \neq i_0}}^{n} A_{i_1 k}(X_k - X_{i_1})$$

and applied to (34),

$$\Theta(\epsilon_0) = 3(X_{i_0} - X_{i_{-1}}) + 3(X_{i_1} - X_{i_0}) - \frac{k_{i_0} + k_{i_{-1}} + 6}{2}\epsilon - \frac{k_{i_1} + k_{i_0} + 6}{2}\epsilon - \sum_{\substack{k=1 \\ k \neq i_0}}^{n} A_{i_1 k}(X_k - X_{i_1})$$
$$< \sum_{\substack{k=1 \\ k \neq i_0}}^{n} A_{i_{-1} k}(X_{i_{-1}} - X_k) \quad (36)$$

3. In this case, we begin with (33) and apply (35) to write

$$\Theta(\epsilon_0) = 3(X_{i_2} - X_{i_1}) + 2(X_{i_1} - X_{i_0}) + 3(X_{i_0} - X_{i_1}) - \frac{k_{i_{-1}} + k_{i_0} + 6}{2}\epsilon - \frac{k_{i_0} + k_{i_1} + 6}{2}\epsilon - \frac{k_{i_1} + k_{i_2} + 6}{2}\epsilon$$
$$< \sum_{\substack{k=1 \\ k \neq i_1}}^{n} A_{i_2 k}(X_k - X_{i_2}) + \sum_{\substack{k=1 \\ k \neq i_0}}^{n} A_{i_{-1} k}(X_{i_{-1}} - X_k) \quad (37)$$

To finish the proof, suppose we have a sequence $i_{-m}, \ldots, i_0, i_1, \ldots, i_l$ of neighboring nodes that are monotonic in their node values and that

$$3(X_{i_l} - X_{i_{l-1}}) + 2(X_{i_{l-1}} - X_{i_{l-2}}) + \cdots + 2(X_{i_{-m+2}} - X_{i_{-m+1}}) + 3(X_{i_{-m+1}} - X_{i_{-m}}) + O(\epsilon)$$
$$< \sum_{\substack{k=1 \\ k \neq i_{l-1}}}^{n} A_{i_l k}(X_k - X_{i_l}) + \sum_{\substack{k=1 \\ k \neq i_{-m+1}}}^{n} A_{i_{-m} k}(X_{i_{-m}} - X_k) \quad (38)$$

We consider several cases:

1. There exists a neighbor $i_{l+1}$ of $i_l$ such that $X_{i_{l+1}} > X_{i_l}$ and $X_{i_{l+1}} - X_{i_l} = \Theta(\epsilon_0)$. In this case, the definition 20 of $\beta(p, q)$ implies

$$4(X_{i_{l+1}} - X_{i_l}) - \sum_{\substack{k=1 \\ k \neq i_{l+1}}}^{n} A_{i_l k}(X_{i_l} - X_k) - \frac{k_{i_l} + k_{i_{l+1}} + 6}{2}\epsilon < \sum_{\substack{k=1 \\ k \neq i_l}}^{n} A_{i_{l+1} k}(X_k - X_{i_{l+1}}), \quad (39)$$

which we rewrite as

$$3(X_{i_{l+1}} - X_{i_l}) - (X_{i_l} - X_{i_{l-1}}) - \frac{k_{i_l} + k_{i_{l+1}} + 6}{2}\epsilon - \sum_{\substack{k=1 \\ k \neq i_l}}^{n} A_{i_{l+1} k}(X_k - X_{i_{l+1}}) < \sum_{\substack{k=1 \\ k \neq i_{l-1}}}^{n} A_{i_l k}(X_{i_l} - X_k), \quad (40)$$

and which we apply to (38) to obtain

$$\Theta(\epsilon_0) = 3(X_{i_{l+1}} - X_{i_l}) + 2(X_{i_l} - X_{i_{l-1}}) + \cdots + 2(X_{i_{-m+2}} - X_{i_{-m+1}}) + 3(X_{i_{-m+1}} - X_{i_{-m}}) + O(\epsilon)$$
$$< \sum_{\substack{k=1 \\ k \neq i_l}}^{n} A_{i_{l+1} k}(X_k - X_{i_{l+1}}) + \sum_{\substack{k=1 \\ k \neq i_{-m+1}}}^{n} A_{i_{-m} k}(X_{i_{-m}} - X_k) \quad (41)$$

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2. There exists a neighbor $i_{-m-1}$ of $i_{-m}$ such that $X_{i_{l+1}} - X_{i_l}$ and $X_{i_{-m}} - X_{i_{-m-1}} = \Theta(\epsilon_0)$. This case is essentially the same as the previous case. We use the inequality

$$3(X_{i_{-m}} - X_{i_{-m-1}}) - (X_{i_{-m+1}} - X_{i_{-m}}) - \frac{k_{i_{-m-1}} + k_{i_{-m}} + 6}{2}\epsilon - \sum_{\substack{k=1 \\ k \neq i_{-m}}}^{n} A_{i_{-m-1}k}(X_{i_{-m-1}} - X_k)$$

$$< \sum_{\substack{k=1 \\ k \neq i_{-m+1}}}^{n} A_{i_{-m}k}(X_k - X_{i_{-m}}) \quad (42)$$

on (38) to obtain

$$\Theta(\epsilon_0) = 3(X_{i_l} - X_{i_{l-1}}) + 2(X_{i_{l-1}} - X_{i_{l-2}}) + \cdots + 2(X_{i_{-m+1}} - X_{i_{-m}}) + 3(X_{i_{-m}} - X_{i_{-m-1}}) + O(\epsilon)$$

$$< \sum_{\substack{k=1 \\ k \neq i_{l-1}}}^{n} A_{i_l k}(X_k - X_{i_l}) + \sum_{\substack{k=1 \\ k \neq i_{-m}}}^{n} A_{i_{-m-1}k}(X_{i_{-m-1}} - X_k) \quad (43)$$

3. Both cases 1. and 2. hold. Here, we obtain both $i_{l+1}$ and $i_{-m-1}$ as in cases 1. and 2. and we apply (42) to (41) to obtain

$$\Theta(\epsilon_0) = 3(X_{i_{l+1}} - X_{i_l}) + 2(X_{i_l} - X_{i_{l-1}}) + \cdots + 2(X_{i_{-m+1}} - X_{i_{-m}}) + 3(X_{i_{-m}} - X_{i_{-m-1}}) + O(\epsilon)$$

$$< \sum_{\substack{k=1 \\ k \neq i_l}}^{n} A_{i_{l+1}k}(X_k - X_{i_{l+1}}) + \sum_{\substack{k=1 \\ k \neq i_{-m}}}^{n} A_{i_{-m-1}k}(X_{i_{-m-1}} - X_k) \quad (44)$$

The procedure above generates an arbitrarily long finite sequence of adjacent nodes with monotonically increasing node values. This is a contradiction, since the graph is finite.

$\square$

The following proposition shows how the total discordance in defintion 6, or equivalently, the consensus error in definition 7 change when a $\alpha$ update occurs.

**Proposition 4.** *Assume that nodes $p, q \in \mathcal{V}(G)$ and $(p, q) \in \mathcal{E}(G)$ are updated at time-step $k$ via the update rule (17). Then the change in total discordance at time-step $k$ is given by*

$$d(X, k+1) - d(X, k) = -4\beta_k(1 - \beta_k)n(X_p(k) - X_q(k))^2 \quad (45)$$

*Equivalently, the consensus error changes as*

$$e(X, k+1) - e(X, k) = -2\beta_k(1 - \beta_k)(X_p(k) - X_q(k))^2. \quad (46)$$

*Proof.* We need to consider the difference $d(X, k+1) - d(X, k)$. Assume that nodes $p, q \in \mathcal{V}(G)$ are updated at timestep $k + 1$; thus, the random variables in the calculation below are conditioned on $p, q$ being updated at time $k + 1$. What follows is just a calculation of $d(X, k+1) - d(X, k)$:

$$d(X, k+1) - d(X, k) = \sum_{i,j=1}^{n} \left( (X_i(k+1) - X_j(k+1))^2 - (X_i(k) - X_j(k))^2 \right)$$

$$= 2 \sum_{\substack{j=1 \\ j \neq p,q}}^{n} \left( (X_p(k+1) - X_j(k))^2 - (X_p(k) - X_j(k))^2 \right)$$

$$+ 2 \sum_{\substack{j=1 \\ j \neq p,q}}^{n} \left( (X_q(k+1) - X_j(k))^2 - (X_q(k) - X_j(k))^2 \right)$$

$$+ 2 \left( (X_p(k+1) - X_q(k+1))^2 - (X_p(k) - X_q(k))^2 \right)$$

$$
\begin{aligned}
&= 2\sum_{\substack{j=1\\j\neq p,q}}^{n}((\beta_k X_p(k) + (1-\beta_k)X_q(k)) - X_j(k))^2 + 2\sum_{\substack{j=1\\j\neq p,q}}^{n}(((1-\beta_k)X_p(k) + \beta_k X_q(k)) - X_j(k))^2 \\
&\quad - 2\sum_{\substack{j=1\\j\neq p,q}}^{n}\left((X_p(k) - X_j(k))^2 + (X_q(k) - X_j(k))^2\right) \\
&\quad + 2\left((\beta_k X_p(k) + (1-\beta_k)X_q(k) - ((1-\beta_k)X_p(k) + \beta_k X_q(k))^2 - (X_p(k) - X_q(k))^2\right) \quad (47)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\sum_{\substack{j=1\\j\neq p,q}}^{n}\big(\beta_k^2(X_p(k) - X_j(k))^2 + 2\beta_k(1-\beta_k)(X_p(k) - X_j(k))(X_q(k) - X_j(k)) \\
&\quad + (1-\beta_k)^2(X_q(k) - X_j(k))^2\big) + 2\sum_{\substack{j=1\\j\neq p,q}}^{n}\big((1-\beta_k)^2(X_p(k) - X_j(k))^2 \\
&\quad + 2\beta_k(1-\beta_k)(X_p(k) - X_j(k))(X_q(k) - X_j(k)) + \beta_k^2(X_q(k) - X_j(k))^2\big) \\
&\quad - 2\sum_{\substack{j=1\\j\neq p,q}}^{n}\left((X_p(k) - X_j(k))^2 + (X_q(k) - X_j(k))^2\right) \\
&\quad - 2\left(1 - (\beta_k - (1-\beta_k))^2\right)(X_p(k) - X_q(k))^2 \quad (48)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\sum_{\substack{j=1\\j\neq p,q}}^{n}\big(\beta_k^2(X_p(k) - X_j(k))^2 + 2\beta_k(1-\beta_k)(X_p(k) - X_j(k))(X_q(k) - X_j(k)) \\
&\quad + (1-\beta_k)^2(X_q(k) - X_j(k))^2\big) + 2\sum_{\substack{j=1\\j\neq p,q}}^{n}\big((1-\beta_k)^2(X_p(k) - X_j(k))^2 \\
&\quad + 2\beta_k(1-\beta_k)(X_p(k) - X_j(k))(X_q(k) - X_j(k)) + \beta_k^2(X_q(k) - X_j(k))^2\big) \\
&\quad - 2\sum_{\substack{j=1\\j\neq p,q}}^{n}(\beta_k^2 + 2\beta_k(1-\beta_k) + (1-\beta_k)^2)(X_p(k) - X_j(k))^2 \\
&\quad - 2\sum_{\substack{j=1\\j\neq p,q}}^{n}((1-\beta_k)^2 + 2\beta_k(1-\beta_k) + \beta_k^2)(X_q(k) - X_j(k))^2 \\
&\quad - 2\left(1 - (\beta_k - (1-\beta_k))^2\right)(X_p(k) - X_q(k))^2 \quad (49)
\end{aligned}
$$

$$
\begin{aligned}
&= 4\sum_{\substack{j=1\\j\neq p,q}}^{n}\beta_k(1-\beta_k)(X_p(k) - X_j(k))(X_q(k) - X_j(k)) \\
&\quad + 4\sum_{\substack{j=1\\j\neq p,q}}^{n}\beta_k(1-\beta_k)(X_p(k) - X_j(k))(X_q(k) - X_j(k)) - 4\sum_{\substack{j=1\\j\neq p,q}}^{n}\beta_k(1-\beta_k)(X_p(k) - X_j(k))^2 \\
&\quad - 4\sum_{\substack{j=1\\j\neq p,q}}^{n}2\beta_k(1-\beta_k)(X_q(k) - X_j(k))^2 - 2\left(1 - (\beta_k - (1-\beta_k))^2\right)(X_p(k) - X_q(k))^2 \quad (50)
\end{aligned}
$$

$$
= -4\beta_k(1-\beta_k)\sum_{\substack{j=1\\j\neq p,q}}^{n}(X_p(k) - X_q(k))^2 - 2\left(1 - (\beta_k - (1-\beta_k))^2\right)(X_p(k) - X_q(k))^2 \quad (51)
$$

$$
= -(4\beta_k(1-\beta_k)(n-2) + 2(1 - (\beta_k - (1-\beta_k))^2)(X_p(k) - X_q(k))^2 \quad (52)
$$

Lastly, equation (46) is due to proposition 2. $\qquad\square$

Finally, we state the convergence result for the $\alpha$-update. Note: this needs to be generalized to cover update rules other than the greedy update. This shouldn't be hard to do.

**Theorem 1.** *Assume that at each timestep $k$, the pair of nodes $p^*, q^*$ from (21) are updated. Then, for any initial condition $X_0$, $e(X, k) \to 0$.*

*Proof.* First, note that $1 - \beta(p, q) \geq \beta(p, q)$ since $\beta(p, q) \leq 1/2$. By propositions 4 and 3,

$$e(X, k+1) = \left(1 - \frac{2\beta(p_k^*, q_k^*)(1 - \beta(p_k^*, q_k^*))(X_{p_k^*}(k) - X_{q_k^*}(k))^2}{e(X, k)}\right) e(X, k) \tag{53}$$

$$\leq \left(1 - \frac{2\beta(p_k^*, q_k^*)^2 (X_{p_k^*}(k) - X_{q_k^*}(k))^2}{e(X, k)}\right) e(X, k) \tag{54}$$

$$\leq (1 - 2c_*(G)^2) e(X, k) \tag{55}$$

By proposition 3, $0 < c^*(G) \leq 1/2$. Therefore, (53) implies that the greedy $\beta$-update converges. $\qquad\square$

## 3.3 Existence of an Update for the "meet-in-the-middle" Update

The following corollary directly follows from proposition 4:

*Corollary* 1. The consensus error satisfies the recurrence relation

$$e(X, k+1) = (1 - \alpha(k|p, q))e(X, k) \tag{56}$$

where

$$\alpha(k|p, q) = \frac{4\alpha_k(1 - \alpha_k)n(X_p(k) - X_q(k))^2}{d(X, k)} = \frac{2\alpha_k(1 - \alpha_k)(X_p(k) - X_q(k))^2}{e(X, k)}. \tag{57}$$

## 3.4 Miscellaneous

In this section, we include some stuff that may be useful later.

We first prove the following lemma, which says that the total number of nodes "between" pairs of adjacent nodes is greater than the total number of nodes above and below pairs of adjacent nodes.

*Lemma* 3. Denote $k_p^+$ the number of neighbors $k$ of node $p$ such that $x_k > x_p$ and denote $k_p^-$ the number of neighbors $k$ of node $p$ such that $x_k < x_p$. The following holds

$$\sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\}(k_p^+ + k_q^-) - \sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\}(k_p^- + k_q^+) = -\sum_{p=1}^{n}(k_p^+ - k_p^-)^2. \tag{58}$$

*Proof.* Note the following equalities

$$\sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\}k_p^+ = \sum_{p=1}^{n} k_p^+ k_p^- \qquad \sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\}k_q^- = \sum_{q=1}^{n} k_q^+ k_q^-$$

$$\sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\}k_p^- = \sum_{p=1}^{n}(k_p^-)^2 \qquad \sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\}k_q^+ = \sum_{q=1}^{n}(k_q^+)^2. \tag{59}$$

The proof follows from the following, where we use (59):

$$\sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\}((k_p^+ + k_q^-) - (k_p^- + k_q^+)) = -\sum_{p=1}^{n}\left((k_p^+)^2 + (k_p^-)^2 - 2k_p^+ k_p^-\right) = -\sum_{p=1}^{n}(k_p^+ - k_p^-)^2 \tag{60}$$

$\qquad\square$

*Definition* 8. We define a *minimum value decomposition* of a graph $G$ with values $X_i \in \mathbb{R}$ assigned to the nodes $i \in \mathcal{V}(G)$ as a sequence of sets $M_0, \ldots, M_d$ of nodes, where $d$ is referred to as the *depth* of the decomposition, such that

1. For all $i \in M_k$ and for all $j \in \mathcal{V}(G)$ adjacent to $i$ satisfying $X_j < X_i$, we have $j \in \cup_{l=0,k-1} M_l$.

2.

$$\cup_{l=0}^{d} M_l = \mathcal{V}(G) \tag{61}$$

.

In the following proposition, we present an algorithm for obtaining a minimum value decomposition.

**Proposition 5.** *A minimal value decomposition can be constructed via the following algorithm:*

1. *Let $M_0$ be the set of all locally minimal nodes, where a node $i$ is locally minimal if all nodes $j$ adjacent to $i$ satisfy $X_i \leq X_j$.*

2. *Given $M_0, \ldots, M_k$, if $M_0 \cup \cdots \cup M_k \neq \mathcal{V}(G)$, let*

$$M_{k+1} = \{i \in \mathcal{V}(G) - M_0 \cup \cdots \cup M_k : j \in \mathcal{V}(G) \text{ and } X_j \leq X_i \implies j \in M_0 \cup \cdots \cup M_k \text{ or } X_j = X_i\} \tag{62}$$

*Proof.* $M_0$ is nonempty since the graph $G$ is finite. We show that given $M_0, \ldots, M_k$ constructed via the algorithm above, where $M_0 \cup \cdots \cup M_k \neq \mathcal{V}(G)$, $M_{k+1}$ defined by (62) is nonempty. Indeed, this is clearly true since the set $\mathcal{V}(G) - M_0 \cup \cdots \cup M_k$ is finite, so there must exist a set of minimal nodes $i_1, \ldots, i_r$ satisfying $x_{i_l} \leq x_j$ for all neighbors $j$ of $i_l$ and $j \in \mathcal{V}(G) - M_0 \cup \cdots \cup M_k$. Therefore, the algorithm must cease and the depth $d$ of the resulting minimal value decomposition satisfies $d \leq n$. $\qquad\square$

**Proposition 6.** *Given a graph $G$ with values $X_i \in \mathbb{R}$ assigned to the nodes $i \in \mathcal{V}(G)$, the algorithm in Proposition 5 generates a minimal value decomposition of minimal depth.*

*Proof.* Let $M_0, \ldots, M_d$ be the minimal value decomposition generated by the algorithm in Proposition 5 and let $\tilde{M}_0, \ldots, \tilde{M}_{\tilde{d}}$ be any other minimal value decomposition. We wish to show that

$$M_0 \cup \ldots \cup M_k \supseteq \tilde{M}_0 \cup \ldots \cup \tilde{M}_k \implies M_0 \cup \ldots \cup M_{k+1} \supseteq \tilde{M}_0 \cup \ldots \cup \tilde{M}_{k+1}. \tag{63}$$

Suppose $i \in \tilde{M}_1$. Then all $j \in \mathcal{V}(G)$ adjacent to $i$ satisfying $X_j < X_i$ satisfy $j \in \tilde{M}_0 = M_0$. Thus, $M_1$ constructed via (62) by definition contains $i$. Therefore, (63) holds with $k = 0$. Suppose (63) holds. Suppose $i \in \tilde{M}_{k+1}$. Then for all $j \in \mathcal{V}(G)$ adjacent to $i$ such that $X_j < X_i$ we have

$$j \in \tilde{M}_0 \cup \cdots \cup \tilde{M}_k \subseteq M_0 \cup \cdots \cup M_k. \tag{64}$$

Therefore, by the definition of $M_l$ in (62), (64) implies that $i \in M_l$ for some $l \leq k + 1$. Thus,

$$M_0 \cup \cdots \cup M_{k+1} \supseteq \tilde{M}_0 \cup \cdots \cup \tilde{M}_{k+1}. \tag{65}$$

$\qquad\square$

# 4 Notes following May 31 Meeting

## 4.1 Characterization of Meet-in-the-Middle Graphs

Given an assignment $X \in \mathbb{R}^n$ of values to the nodes of a graph $G$, we call a pair of adjacent nodes $p, q \in \mathcal{V}(G)$ *meet-in-the-middle updateable* if the meet-in-the-middle update of $p$ and $q$ is discordance decreasing. We call a graph $G$ *meet-in-the-middle updateable* if for every assignment of values $X \in \mathbb{R}^n$ to the nodes of the graph $G$ there exists an adjacent pair of nodes $i, j \in \mathcal{V}(G)$ that is meet-in-the-middle updatable.

**Theorem 2.** *Denote $k_i$ the degree of node $i \in \mathcal{V}(G)$ and denote $s(i, j)$ the number of shared neighbors of $i$ and $j$, where the number of shared neighbors does not include $i, j$. A graph $G$ is meet-in-the-middle updateable iff there is no cut-set of edges $C \subset \mathcal{E}(G)$ satisfying*

$$\frac{k_i + k_j + 6}{4} \geq 4 + s(i, j). \tag{66}$$

*for all $(i, j) \in C$.*

*Example* 1. Consider a graph $G$ of $n$ nodes with no triangles. In this case, $s(i, j) = 0$ for all $i, j$, so for theorem 2 to imply that the graph $G$ is meet-in-the-middle updateable, we require that every cut-set has an edge $(i, j)$ such that

$$k_i + k_j < 10 \tag{67}$$

In particular, if a tree has a single pair of adjacent nodes satisfying (67) then there exist assignments of values to the nodes of the graph such that there are no meet-in-the-middle updateable pairs of nodes. Even more in particular

1. A star graph of $n$ total nodes. By (67), we must have $n < 10$ for the graph to be meet-in-the-middle updateable.

2. A lattice in $\mathcal{Z}^n$ is meet-in-the-middle updateable iff $n < 3$.

3. etc.?

*Example* 2. Consider the complete graph of $n$ nodes $G$ with $k < (n-1)/2$ edges removed from each vertex. Then, the minimum number of shared neighbors between two nodes $i$ and $j$ is $n - 1 - 2k$. For theorem 2 to imply that $G$ is meet-in-the-middle updateable, every cut set must have a pair of nodes $i, j \in \mathcal{V}(G)$ satisfying

$$2n - 2 = k_i + k_j < 10 + 4s(i,j). \tag{68}$$

Now,

$$s(i,j) \geq n - 1 - 2k. \tag{69}$$

so combining (68) and (69) we find that if

$$k < \frac{n}{4} - 1. \tag{70}$$

then all pairs of nodes satisfy (68) so $G$ is meet-in-them-middle updateable. Thus, for all graphs obtained from the complete graph by removing less than $n/4 - 1$ edges from each node, theorem 2 tells us that a meet-in-the-middle update is always possible.

[[Need more examples of graphs that are meet-in-the-middle updateable or not. Non-trivial to determine whether a cut-set of the form required in theorem 2 exist?]]

*Proof of Theorem 2.*     1. ( $\implies$ ). Assume there is a cut-set $C$ of edges satisfying (66) for all $(i,j) \in C$. We wish to construct an assignment of values $X \in \mathbb{R}^n$ to the nodes of the graph such that no meet-in-the-middle update is possible among the adjacent nodes of the graph. Accordingly, denote by $G_1$ and $G_2$ the connected components of the graph $G$ after removal of $C$. For all $i \in G_1$, set $X_i = a$ and for all $i \in G_2$, set $X_i = b$, where $a, b \in \mathbb{R}$ such that $a \neq b$.

With $X$ constructed as above, no edges $(i,j) \in \mathcal{E}(G_1)$ or $(i,j) \in \mathcal{E}(G_2)$ are meet-in-the-middle updateable, since $X_i = X_j$ for such $(i,j)$. It remains to show that the edges $(i,j) \in C$ are not meet-in-the-middle updateable.

Denote $S(i,j)$ the set of shared neighbors of $i$ and $j$. Assuming $X_i > X_j$, in order for the meet-in-the-middle update between nodes $i$ and $j$ to not be possible, it is required that

$$\frac{k_i + k_j + 6}{2}(X_i - X_j) \geq (4 + s(i,j))(X_i - X_j) - \sum_{\substack{k=1 \\ k \notin S(i,j) \cup \{j\}}}^{n} A_{ik}(X_k - X_i) - \sum_{\substack{k=1 \\ k \notin S(i,j) \cup \{i\}}}^{n} A_{jk}(X_j - X_k). \tag{71}$$

$\square$

## 4.2   Convergence Improved

**Proposition 7.** *There exist constants* $c_*(G), c^*(G) \in \mathbb{R}_{>0}$, *independent of* $X$, *such that*

$$c_*(G)e(X)^{1/2} \leq \beta(p^*, q^*)|X_{p^*} - X_{q^*}| \leq c^*(G)e(X)^{1/2}, \tag{72}$$

*where* $c^*(G) = \sqrt{n/2}$ *and*

$$c_*(G) \geq \frac{1}{m(m+n)}, \tag{73}$$

*where* $m = |\mathcal{E}(G)|$ *and* $n = |\mathcal{V}(G)|$.

*Proof.*     1. Upper Bound. We know that $\beta \leq 1/2$ and $(X_p - X_q)^2 \leq d(X|G)$, so

$$(\beta^*(X_{p^*} - X_{q^*}))^2 \leq \frac{1}{4}d(X|G) \tag{74}$$

Thus by proposition 2,

$$(\beta^*(X_{p^*} - X_{q^*}))^2 \leq \frac{n}{2}e(X). \tag{75}$$

This bound is tightest bound independent of $X$, as can be seen by letting the values associated with two adjacent nodes $i$ and $j$ be distinct and letting every other node $k$ have value $(x_i + x_j)/2$. This works for any connected graph of $n \geq 2$ nodes.

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2. Lower Bound. Given adjacent nodes $p, q \in \mathcal{V}(G)$ with $X_p > X_q$, let $\beta'(p, q)$ satsify the following equality

$$\sum_{k=1}^{n} A_{pk}(X_k - X_p) = \left(2 - \frac{k_p + k_q + 6}{2}\beta'(p, q)\right)(X_p - X_q) + \sum_{k=1}^{n} A_{qk}(X_k - X_q) \tag{76}$$

for a given $X \in \mathbb{R}^n$.

Let $\mathcal{P}(G, X)$ denote the set of all paths that are monotonically increasing in the values $X_l$ and such that if $i_l$ is a node in the path, then $i_{l+1}$ is such that $X_{i_{l+1}} - X_{i_l}$ is maximized among nodes adjacent to $i_l$. Denote $P = (i_{-L}, \ldots, i_M) \in \mathcal{P}(G, X)$ define the value $\beta_*$ as follows:

$$\beta_* = \inf_{X \in \mathbb{R}^n} \inf_{P=(i_{-L},\ldots,i_M)\in\mathcal{P}(\mathcal{G},\mathcal{X})} \left( \frac{(X_{i_M} - X_{i_{-L}}) + \sum_{k=1}^{n} A_{i_M k} 1\{X_k < X_{i_M}\}(X_{i_M} - X_k)}{\frac{1}{2}\sum_{l=-L}^{M-1}(k_{i_{l+1}} + k_{i_l})(X_{i_{l+1}} - X_{i_l})} \right.$$
$$\left. + \frac{\sum_{k=1}^{n} A_{i_{-L} k} 1\{X_i > X_{i_{-L}}\}(X_k - X_{i_{-L}})}{\frac{1}{2}\sum_{l=-L}^{M-1}(k_{i_{l+1}} + k_{i_l})(X_{i_{l+1}} - X_{i_l})} \right) \tag{77}$$

First, note that

$$\beta_* \geq \inf_{X \in \mathbb{R}^n} \inf_{P=(i_{-L},\ldots,i_M)\in\mathcal{P}(\mathcal{G},\mathcal{X})} \frac{1}{\frac{\sum_{l=-L}^{M-1}(k_{i_{l+1}}+k_{i_l})(X_{i_{l+1}}-X_{i_l})}{2(X_{i_M}-X_{i_L})}} \tag{78}$$

$$\geq \inf_{P=(i_{-L},\ldots,i_M)\in\mathcal{P}(\mathcal{G},\mathcal{X})} \frac{1}{\frac{1}{2}\sum_{l=-L}^{M-1}(k_{i_{l+1}}+k_{i_l})} \tag{79}$$

$$\geq \frac{1}{2m} \tag{80}$$

where $m = |\mathcal{E}(G)|$. Therefore, $\beta_*$ is bounded below.

Again, let $P = (i_{-L}, \ldots, i_M) \in \mathcal{P}(G, X)$ be a monotone path of nodes. We prove the following equality

$$\sum_{l=-L+1}^{M} \beta'(i_l, i_{l-1}) \frac{k_{i_l} + k_{i_{l-1}} + 6}{2}(X_{i_l} - X_{i_{l-1}}) = 2(X_{i_M} - X_{i_{-L}}) + \sum_{k=1}^{n} A_{i_M k}(X_{i_M} - X_k) + \sum_{k=1}^{n} A_{i_{-L} k}(X_k - X_{i_{-L}}). \tag{81}$$

Indeed, applying equation (76) to the adjacent pair $(i_{l-1}, i_l) \in \mathcal{V}(G)$, we obtain

$$\frac{k_{i_l} + k_{i_{l-1}} + 6}{2}\beta'(i_l, i_{l-1})(X_{i_l} - X_{i_{l-1}}) = 2(X_{i_l} - X_{i_{l-1}}) + \sum_{k=1}^{n} A_{i_l k}(X_k - X_{i_l}) = \sum_{k=1}^{n} A_{i_{l-1} k}(X_k - X_{i_{l-1}}). \tag{82}$$

Summing (82) for all $i_{l-1}, i_l$ between $l = -L+1, \ldots, M$, we obtain (81). Now, assume that $\beta'(i_l, i_{l-1}) < \beta_*$ for all $l = -L+1, \ldots, M$. Then, (81) and the definition (77) of $\beta_*$ implies

$$2(X_{i_M} - X_{i_{-L}}) + \sum_{k=1}^{n} A_{i_M k}(X_{i_M} - X_k) + \sum_{k=1}^{n} A_{i_{-L} k}(X_k - X_{i_{-L}}) \leq \beta^* \sum_{l=-L+1}^{M} \frac{k_{i_l} + k_{i_{l-1}} + 6}{2}(X_{i_l} - X_{i_{l-1}})$$
$$\leq (X_{i_M} - X_{i_{-L}}) + \sum_{k=1}^{n} A_{i_M k} 1\{X_i < X_{i_M}\}(X_{i_M} - X_k) + \sum_{k=1}^{n} A_{i_{-L} k} 1\{X_k > X_{i_{-L}}\}(X_k - X_{i_{-L}}) \tag{83}$$

and thus

$$(X_{i_M} - X_{i_{-L}}) \leq \sum_{k=1}^{n} A_{i_M k} 1\{X_k > X_{i_M}\}(X_{i_M} - X_k) + \sum_{k=1}^{n} A_{i_{-L} k} 1\{X_k < X_{i_{-L}}\}(X_k - X_{i_{-L}}) \tag{84}$$

If $i_M$ and $i_{-L}$ are locally maximal and minimal nodes respectively, then (84) leads to a contradiction. Therefore, we have shown that any path $P \in \mathcal{P}(G, X)$ between a local maximal and a local minimal node must contain a pair $i_p, i_{p+1}$ with $\beta'(i_p, i_{p_1}) > \beta_*$. We now use (84) to show that there must exist a path of nodes that is "large enough."

It follows immediately by the definition of $d(X|G)$ and Lemma 2 that there exists an adjacent pair of nodes $i_1$ and $i_0$ such that

$$(X_{i_1} - X_{i_0}) \geq \frac{d(X|G)}{2m} \geq \frac{\lambda_2(L)}{2m} e(X)^{1/2}. \tag{85}$$

We now construct a path $P = (i_{q_*}, i_{p_*}, \ldots, i_0, i_1, \ldots, i_{q^*}, i_{p^*}) \in \mathcal{P}(G, X)$ such that $P$ passes through the nodes $i_0, i_1$ of (85) and such that at least one of $i_{q_*}, i_{p_*}$ and $i_{q^*}, i_{p^*}$ satisfy

$$X_{i_p} - X_{i_q} \geq \frac{X_{i_1} - X_{i_0}}{2m - \frac{1}{2}\sum_{l=q_*+1}^{p^*}(k_{i_l} + k_{i_{l+1}})} \geq \frac{X_{i_1} - X_{i_0}}{2m} \tag{86}$$

To begin, consider the pair $i_0, i_1$, which satisfies (83) with $M = 1, -L = 0$. If $i_1$ and $i_0$ are locally minimal nodes, then we have shown already that $\beta'(i_1, i_0) > \beta_*$. Otherwise, assume $\beta'(i_1, i_0) < \beta_*$ so that either $i_1$ and $i_0$ is not locally minimal and (??) holds with $M = 1$ and $-L = 0$. By (??), there must exist a $i_2$ adjacent to $i_1$ with

$$X_{i_2} - X_{i_1} \geq \frac{X_{i_1} - X_{i_0}}{k_{i_1}^+ + k_{i_0}^-} \tag{87}$$

or there must exist an $i_{-1}$ adjacent to $i_0$ with

$$X_{i_0} - X_{i_{-1}} \geq \frac{X_{i_1} - X_{i_0}}{k_{i_1}^+ + k_{i_0}^-}. \tag{88}$$

Assume that $i_2$ and/or $i_{-1}$ are the largest or smallest such nodes satisfying (87) or (94), respsectively.

Set $i_{p^*}, i_{q^*}$ either equal to $i_2, i_1$ or $i_1, i_1$ depending on if (87) holds for some $i_2$ or not, and set $i_{p_*}, i_{q_*}$ either equal to $i_0, i_{-1}$ or $i_0, i_0$ depending on if (94) holds for some $i_{-1}$ or not. If $i_{p^*}$ and $i_{q_*}$ are locally maximal and minimal respectively, then it must be the case that $\beta'(i_{p^*}, i_{q^*}) > \beta_*$ or $\beta'(i_{p_*}, i_{q_*}) > \beta_*$. Otherwise, assume $\beta'(i_{p^*}, i_{q^*}) < \beta_*$ and $\beta'(i_{p_*}, i_{q_*}) < \beta_*$ and by (??) applied to to $i_{p^*}$ and $i_{q_*}$ we obtain that there must exist an $i_{p^*+1}$ adjacent to $i_{p^*}$ with

$$X_{i_{p^*+1}} - X_{i_{p^*}} \geq \frac{X_{i_1} - X_{i_0}}{k_{i_{p^*}}^+ + k_{i_{q_*}}^-} \tag{89}$$

or there must exist an $i_{q_*-1}$ adjacent to $i_{q_*}$ such that

$$X_{i_{q_*-1}} - X_{i_{q_*}} \geq \frac{X_{i_1} - X_{i_0}}{k_{i_{p^*}}^+ + k_{i_{q_*}}^-}. \tag{90}$$

Set $i_{p^*}, i_{q^*}$ equal to $i_{p^*} + 1, i_{p^*}$ or $i_{p^*}, i_{p^*}$ depending on if (95) holds or not and set $i_{p_*}, i_{q_*}$ equal to $i_{q_*}, i_{q_*} - 1$ or $i_{q_*}, i_{q_*}$ depending on if (96) holds or not. Eventually, this process must terminate by finding a pair of nodes $p, q$

Let $P = (i_{-L}, \ldots, i_M) \in \mathcal{P}(G, X)$ from a local minimal to a local maximal node such that $i_0, i_1$ are the nodes of (85). For any path $P$, we showed that there exist nodes $i_p, i_q$ along the path with $\beta'(i_p, i_q) > \beta_*$. In particular, let $i_{p^*}, i_{q^*}$ be such a pair of nodes closest to $i_1, i_0$ with $q^* \geq 0$, i.e. for all $l = 1, \ldots, q^*$, $\beta'(i_l, i_l - 1) < \beta_*$, and $\beta(i_{p^*}, i_{q^*}) > \beta_*$, and let $i_{p_*}, i_{q_*}$ be such a pair of nodes closest to $i_1, i_0$ with $p_* \leq 1$, i.e. for all $l = p_*, \ldots, 0$, $\beta(i_{l+1}, i_l) < \beta_*$, and $\beta'(i_{p_*}, i_{q_*}) > \beta_*$. We use the convention that if $q^* = i_M$ then there is no pair $i_l, i_{l-1}$ satisfying $\beta'(i_l, i_{l-1}) > \beta_*$ for $l \geq 1$ and similarly for $p_* = i_{-L}$. Apply (81) to $q^*$ and $p^*$ to obtain

$$2(X_{i_{q^*}} - X_{i_{p_*}}) + \sum_{k=1}^n A_{i_{q^*}k}(X_{i_{q^*}} - X_k) + \sum_{k=1}^n A_{i_{p_*}k}(X_k - X_{i_{p_*}}) \leq \beta^* \sum_{l=p_*+1}^{q^*} \frac{k_{i_l} + k_{i_{l-1}} + 6}{2}(X_{i_l} - X_{i_{l-1}}), \tag{91}$$

which implies

$$\sum_{k=1}^n A_{i_{q^*}k} 1\{X_k > X_{i_{q^*}}\}(X_k - X_{i_{q^*}}) + \sum_{k=1}^n A_{i_{p_*}k} 1\{X_k < X_{p_*}\}(X_{i_{p_*}} - X_k)$$

$$\geq 2(X_{i_q^*} - X_{i_{p_*}}) - \beta^* \sum_{l=p_*+1}^{q^*} \frac{k_{i_l} + k_{i_{l-1}} + 6}{2}(X_{i_l} - X_{i_{l-1}})$$

$$+ \sum_{k=1}^n A_{i_{q^*}k} 1\{X_k < X_{i_{q^*}}\}(X_{i_{q^*}} - X_k) + \sum_{k=1}^n A_{i_{p_*}k} 1\{X_k > X_{p_*}\}(X_k - X_{i_{p_*}})$$

$$> (X_{i_{q^*}} - X_{i_{p_*}}) \tag{92}$$

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Thus, (92) implies that either

For each path $P$ in the network, (81) requires that the update sizes $\beta'(i_l, i_{l-1})(X_{i_l} - X_{i_{l-1}})$ must not all be too small since the right hand side of (81) is strictly positive. Accordingly, we first find the value, as a fraction of the consensus error $e(X)^{1/2}$, that at least one of the update sizes in the path $P$ must exceed in order for (81) to hold. Assume that

$$\beta'(i_l, i_{l-1})(X_{i_l} - X_{i_{l-1}}) \leq c'(P, X)e(X)^{1/2} \tag{93}$$

for all adjacent nodes $i_l, i_{l-1} \in \mathcal{V}(G)$ in the path $P \in \mathcal{P}(G, X)$, where $c(P, X)$ is a constant depending on the path $P$ and the values $X$. Then (93) applied to (81) implies

$$c'(P, X)\left(\frac{1}{2}\sum_{l=1}^{L}(k_{i_l} + k_{i_{l-1}}) + 3(L-1)\right)e(X)^{1/2} \geq 2(X_{i_L} - X_{i_0}) + \sum_{k=1}^{n} A_{i_L k}(X_{i_L} - X_k) + \sum_{k=1}^{n} A_{i_0 k}(X_k - X_{i_0}), \tag{94}$$

and thus,

$$c'(P, X) \geq \frac{2(X_{i_L} - X_{i_0}) + \sum_{k=1}^{n} A_{i_L k}(X_{i_L} - X_k) + \sum_{k=1}^{n} A_{i_0 k}(X_k - X_{i_0})}{\left(\frac{1}{2}\sum_{l=1}^{L}(k_{i_l} + k_{i_{l-1}}) + 3(L-1)\right)e(X)^{1/2}}. \tag{95}$$

Therefore, at least one pair $i_l, i_{l-1}$ of adjacent nodes in the path $P$ must have an update of size at least $c(P, X)e(X)^{1/2}$, where we set

$$c(P, X) = \frac{2(X_{i_L} - X_{i_0}) + \sum_{k=1}^{n} A_{i_L k}(X_{i_L} - X_k) + \sum_{k=1}^{n} A_{i_0 k}(X_k - X_{i_0})}{\left(\frac{1}{2}\sum_{l=1}^{L}(k_{i_l} + k_{i_{l-1}}) + 3(L-1)\right)e(X)^{1/2}}. \tag{96}$$

$\square$