MURI Discordance Games Notes

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1 Definitions

Definition 1. Let G be an undirected graph with adjacency matrix A and let $x \in \mathbb{R}$ be a vector indexed by the nodes $i \in \mathcal{V}(G)$. Denote and define the *discordance* of a node $i \in \mathcal{V}(G)$ as the following function of x:

$$d_i(x|G) = \sum_{k=1}^{n} A_{ik}(x_i - x_k)^2.$$

Naturally, we may also define the discordance of a subset $V \subset \mathcal{V}(G)$ of nodes as follows
Definition 2. Denote and define the discordance of a subset $S \subset \mathcal{V}(G)$ of nodes as follows:

$$d_S(x|G) = \sum_{i \in S} d_i(x|G).$$

If we interpret x as a vector of *opinions* about a topic, then $d_i(x|G)$ is one way to measure how much node i disagrees with its neighbors in the graph G. Under the interpretation of x as a vector of opinions, it is of interest to define a model for how the values x_i change as nodes in G communicate. Accordingly, we make the following definition:
Definition 3. Consider two nodes $p, q \in \mathcal{V}(G)$ with values $x_p, x_q \in \mathbb{R}$. We say that the nodes p and q come to an *α-equal agreement* if the nodes p and q change their values as follows:

$$x_p \mapsto x_p + \operatorname{sgn}(x_q - x_p)\alpha \qquad\qquad x_q \mapsto x_q + \operatorname{sgn}(x_p - x_q)\alpha,$$

where

$$0 < \alpha \leq \frac{|x_p - x_q|}{2}.$$

and $\operatorname{sgn}(x) = x/|x|$ for $x \neq 0$.

We wish to investigate when agreements between two nodes $p, q \in \mathcal{V}(G)$ can occur under the following assumption:
Definition 4. Let $S \subset \mathcal{V}(G)$ be a subset of nodes of the graph G and let $x \in \mathbb{R}^n$ be a vector of values assigned to the nodes in $\mathcal{V}(G)$. Consider an update $x \mapsto x' \in \mathbb{R}^n$ to the values x such that $x'_i = x_i$ for $i \notin S$. We say that the update is *discordance decreasing* if

$$d_S(x') - d_S(x) < 0$$

2 Notes from May 6 Meeting

Question 1. For every choice of $x \in \mathbb{R}^n$ and for every graph G does there exist a pair of adjacent nodes $p, q \in \mathcal{V}(G)$ and an $\alpha > 0$ sufficiently small such that an discordance decreasing α-equal agreement can occur between p and q.

To address the question above, we state the following lemma.
Lemma 1. Let $p, q \in \mathcal{V}(G)$ be two adjacent nodes in a graph G. Let $x \in \mathbb{R}^n$ be a vector of values corresponding to the nodes of G. Assume without loss of generality that $x_p > x_q$. Then there exists an $\alpha > 0$ sufficiently small such that a discordance decreasing update can occur between nodes p and q if the following inequality holds:

$$\operatorname{sgn}(x_p - x_q) \sum_{\substack{k=1 \\ k \neq q}}^{n} A_{ik}(x_k - x_p) + \operatorname{sgn}(x_p - x_q) \sum_{\substack{k=1 \\ k \neq p}}^{n} A_{jk}(x_k - x_q) < 4|x_p - x_q| \tag{1}$$

Proof. Without loss of generality, assume $x_p > x_q$ so that the α-equal agreement between nodes p and q becomes

$$x_p \mapsto x_p - \alpha \qquad\qquad x_q \mapsto x_q + \alpha. \tag{2}$$

Denote x' the new vector of values after the agreement (2), denote $d_{ij}(x) = d_{\{i,j\}}(x)$, and denote k_i the degree of node i. The change in discordance for the set $\{i, j\}$ is given by

$$\begin{aligned}
d_{pq}(x') - d_{pq}(x) &= \sum_{\substack{k=1\\k\neq q}}^{n} A_{pk}((x_p - x_k - \alpha)^2 - (x_p - x_k)^2) \\
&\quad + \sum_{\substack{k=1\\k\neq p}}^{n} A_{qk}((x_q - x_k + \alpha)^2 - (x_q - x_k)^2) + 2(x_p - x_q - 2\alpha)^2 - 2(x_p - x_q)^2 \\
&= -2\alpha \sum_{\substack{k=1\\k\neq q}}^{n} A_{pk}(x_p - x_k) + (k_p - 1)\alpha^2 + 2\alpha \sum_{\substack{k=1\\k\neq p}}^{n} A_{qk}(x_q - x_k) + (k_q - 1)\alpha^2 - 8\alpha(x_p - x_q) + 8\alpha^2 \\
&= 2\alpha \left(\sum_{\substack{k=1\\k\neq q}}^{n} A_{pk}(x_k - x_p) + \sum_{\substack{k=1\\k\neq p}}^{n} A_{qk}(x_q - x_k) - 4(x_p - x_q) \right) + (k_p + k_q + 6)\,\alpha^2.
\end{aligned} \tag{3}$$

Thus, if inequality (1) holds then (3) implies that for α sufficiently small, $d_{pq}(x') - d_{pq}(x) < 0$. □

[[Determine the α in lemma 1 to determine how large of an agreement is possible. When is a consensus to the middle ground possible?]]

The following proposition answers question 1 in the affirmative. The idea of the proof is to build a path (walk of distinct nodes) such that the node values along the path are strictly increasing. If inequality (1) fails for all pairs of adjacent nodes in the graph, then the process by which the path is constructed does not terminate. Since there are a finite number of nodes in the graph, this leads to a contradiction.

Proposition 1. *For any graph G and any vector $x \in \mathbb{R}^n$ of values indexed by the nodes of G, there exists a pair of adjacent nodes $p, q \in \mathcal{V}(G)$ and an $\alpha > 0$ sufficiently small such that a discordance decreasing α-equal agreement is possible between nodes p and q.*

Proof. Let i_0 be the index of a node such that $x_{i_0} \leq x_k$ for all neighbors k of i_0 (such as the minimal value in x) and $x_{i_0} < x_{i_1}$ for some neighbor i_1. We will call a node i satisfying $x_i \leq x_k$ for all neighbors k of i *locally minimal* and a node satisfying the same condition with $\geq$ locally maximal. Inequality (1) with i_0 and i_1 reads

$$\sum_{\substack{k=1\\k\neq i_0}}^{n} A_{i_1k}(x_k - x_{i_1}) + \sum_{\substack{k=1\\k\neq i_1}}^{n} A_{i_0k}(x_{i_0} - x_k) < 4(x_{i_1} - x_{i_0}) \tag{4}$$

If equation (4) holds, we can update nodes i_0 and i_1. If not, then (4) being false and x_{i_0} being minimal among its neighbors implies that

$$\sum_{\substack{k=1\\k\neq i_0}}^{n} A_{i_1k}(x_k - x_{i_1}) \geq 4(x_{i_1} - x_{i_0}) + \sum_{\substack{k=1\\k\neq i_1}}^{n} A_{i_0k}(x_k - x_{i_0}) > 0. \tag{5}$$

Equation (5) implies that there exists a neighbor i_2 of i_1 such that $x_{i_2} > x_{i_1}$. Consider inequality (1) for nodes i_1 and i_2:

$$\sum_{\substack{k=1\\k\neq i_1}}^{n} A_{i_2k}(x_k - x_{i_2}) + \sum_{\substack{k=1\\k\neq i_2}}^{n} A_{i_1k}(x_{i_1} - x_k) < 4(x_{i_2} - x_{i_1}) \tag{6}$$

If inequality (6) holds, then we're done. Otherwise, we write the negation of (6) as follows:

$$\sum_{\substack{k=1\\k\neq i_1}}^{n} A_{i_2k}(x_k - x_{i_2}) + \sum_{\substack{k=1\\k\neq i_0}}^{n} A_{i_1k}(x_{i_1} - x_k) + (x_{i_1} - x_{i_0}) \geq 3(x_{i_2} - x_{i_1}) \tag{7}$$

Reorganizing (7) we obtain

$$\sum_{\substack{k=1\\k\neq i_1}}^{n} A_{i_2k}(x_k - x_{i_2}) \geq 3(x_{i_2} - x_{i_1}) - \sum_{\substack{k=1\\k\neq i_0}}^{n} A_{i_1k}(x_{i_1} - x_k) - (x_{i_1} - x_{i_0}) \tag{8}$$

and by using inequality (5) in (8) we obtain

$$\sum_{\substack{k=1\\k\neq i_1}}^{n} A_{i_2k}(x_k - x_{i_2}) \geq 3(x_{i_2} - x_{i_0}) + \sum_{\substack{k=1\\k\neq i_1}}^{n} A_{i_0k}(x_k - x_{i_0}) > 0. \tag{9}$$

Inequality (9) shows that i_2 has a neighbor i_3 such that $x_{i_3} > x_{i_2}$. Continue in this fashion. Suppose that the inequality (1) does not hold for consecutive values in the sequence $i_0, i_1, \ldots, i_m$ of consecutively adjacent vertices and such that $x_{i_0} < x_{i_1} < \ldots < x_{i_m}$. Suppose inequality (4) fails when applied to nodes i_m and i_{m-1}. Then,

$$\begin{aligned}\sum_{\substack{k=1\\k\neq i_{m-1}}}^{n} A_{i_mk}(x_k - x_{i_m}) &\geq 4(x_{i_m} - x_{i_{m-1}}) + \sum_{\substack{k=1\\k\neq i_m}}^{n} A_{i_{m-1}k}(x_k - x_{i_{m-1}}) \\ &= 3(x_{i_m} - x_{i_{m-1}}) + \sum_{\substack{k=1\\k\neq i_{m-2}}}^{n} A_{i_{m-1}k}(x_k - x_{i_{m-1}}) + (x_{i_{m-2}} - x_{i_{m-1}}) \end{aligned} \tag{10}$$

By the induction hypothesis, we have that a similar inequality holds for i_{m-1} and i_{m-2}:

$$\begin{aligned}\sum_{\substack{k=1\\k\neq i_{m-2}}}^{n} A_{i_{m-1}k}(x_k - x_{i_{m-1}}) &\geq 4(x_{i_{m-1}} - x_{i_{m-2}}) + \sum_{\substack{k=1\\k\neq i_{m-1}}}^{n} A_{i_{m-2}k}(x_k - x_{i_{m-2}}) \\ &= 3(x_{i_{m-1}} - x_{i_{m-2}}) + \sum_{\substack{k=1\\k\neq i_{m-3}}}^{n} A_{i_{m-2}k}(x_k - x_{i_{m-2}}) + (x_{i_{m-3}} - x_{i_{m-2}}) \end{aligned} \tag{11}$$

Plugging in inequality (11) into (13) we obtain

$$\sum_{\substack{k=1\\k\neq i_{m-1}}}^{n} A_{i_mk}(x_k - x_{i_m}) \geq 3(x_{i_m} - x_{i_{m-1}}) + 2(x_{i_{m-1}} - x_{i_{m-2}}) + \sum_{\substack{k=1\\k\neq i_{m-3}}}^{n} A_{i_{m-2}k}(x_k - x_{i_{m-2}}) + (x_{i_{m-3}} - x_{i_{m-2}}). \tag{12}$$

Continuing in this fashion, we obtain the following

$$\sum_{\substack{k=1\\k\neq i_m}}^{n} A_{i_mk}(x_k - x_{i_m}) \geq 3(x_{i_m} - x_{i_{m-1}}) + 2(x_{i_{m-1}} - x_{i_1}) + 3(x_{i_1} - x_{i_0}) + \sum_{\substack{k=1\\k\neq i_1}}^{n} A_{i_0k}(x_k - x_{i_0}) > 0. \tag{13}$$

Which shows that i_m has a neighbor i_{m+1} satisfying $x_{i_{m+1}} > x_{i_m}$.

Since the graph G is finite, the process to obtain the distinct vertices $i_0, i_1, \ldots, i_m$ must terminate. [[one can strengthen the statement of the proposition]] $\square$

3 Notes from May 10 Meeting

3.1 Previous Results and Definitions

We consider the following three aggregate quantities over the graph G for measuring the total difference in opinion between the individuals in the graph:

Definition 5. Denote and define the *discordance* of the nodes $i \in \mathcal{V}(G)$ with values $X_i(t)$ by

$$d(X, t|G) = \sum_{i,j=1}^{n} A_{ij}(X_i(t) - X_j(t))^2 = \|MX(t)\|_2^2.$$

Definition 6. Denote and define the *total discordance* of the nodes $i \in \mathcal{V}(G)$ with values $X_i(t)$ by

$$d(X,t) = \sum_{i,j=1}^{n} (X_i(t) - X_j(t))^2.$$

Definition 7. Denote and define the *consensus error* of the nodes $i \in \mathcal{V}(G)$ with values $X_i(t)$ by

$$e(X,t) = \|X(t) - X_{\text{ave}}\mathbf{1}\|_2^2.$$

For all three definitions 5-7 we will omit the t argument when we are interested in a fixed X. Additionally, when concerned with a sequence $t_0 < t_1 < \cdots$ of times, we will specify just the index rather than the time: $e(X,k) = e(X,t_k)$ for example.

Proposition 2. *The total discordance in definition 6 is related to the consensus error 7 by*

$$d(X,t) = 2ne(X,t).$$

Proposition 3. *Assume that nodes $p,q \in \mathcal{V}(G)$ and $(p,q) \in \mathcal{E}(G)$ are updated at time-step k via the update rule* (33). *Then the change in total discordance at time-step k is given by*

$$d(X,k+1) - d(X,k) = -4\alpha_k(1-\alpha_k)n(X_p(k) - X_q(k))^2 \tag{14}$$

Equivalently, the consensus error changes as

$$e(X,k+1) - e(X,k) = -2\alpha_k(1-\alpha_k)(X_p(k) - X_q(k))^2. \tag{15}$$

Cases:

$$\sum_{\substack{k=1\\k\neq q}}^{n} A_{pk}(x_k - x_p) + \sum_{\substack{k=1\\k\neq p}}^{n} A_{qk}(x_q - x_k) = \sum_{p,q=1}^{n} A_{pq}1\{x_p > x_q\}(k_p^+ + k_q^- - (k_p^- + k_q^+))(x_p - x_q)$$

If a node i is locally minimal, then

3.2 Convergence of the α-equal Agreement Update

Thus, we find that

$$\beta(x_p - x_q) < \frac{4(x_p - x_q) - \left(\sum_{\substack{k=1\\k\neq q}}^{n} A_{pk}(x_k - x_p) + \sum_{\substack{k=1\\k\neq p}}^{n} A_{qk}(x_q - x_k)\right)}{(k_p + k_q + 6)/2} \tag{16}$$

Further, we know that $\beta < 1/2$.

Suppose that β^* and $p^*, q^* \in \mathcal{V}(G)$ are chosen such that

$$\begin{aligned}&(\beta^*, p^*, q^*)\\ &= \max_{0<\beta\leq 1,(p,q)\in\mathcal{E}(G)} \left\{ \beta(x_p - x_q) < \min\left(\frac{x_p - x_q}{2}, \frac{4(x_p - x_q) - \left(\sum_{\substack{k=1\\k\neq q}}^{n} A_{pk}(x_k - x_p) + \sum_{\substack{k=1\\k\neq p}}^{n} (x_q - x_k)\right)}{(k_p + k_q + 6)/2} \right) \right\}\end{aligned} \tag{17}$$

We first prove the following lemma. This lemma implies Proposition 1 and is much simpler.

Lemma 2. Denote k_p^+ the number of neighbors k of node p such that $x_k > x_p$ and denote k_p^- the number of neighbors k of node p such that $x_k < x_p$. There exists at least one adjacent pair of nodes $p,q \in \mathcal{V}(G)$, where $x_p > x_q$ such that

$$k_p^+ + k_q^- - (k_p^- + k_p^+) < 0.$$

Proof. Note the following equalities

$$\begin{aligned} \sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\} k_p^+ &= \sum_{p=1}^{n} k_p^+ k_p^- & \sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\} k_q^- &= \sum_{q=1}^{n} k_q^+ k_q^- \\ \sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\} k_p^- &= \sum_{p=1}^{n} (k_p^-)^2 & \sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\} k_q^+ &= \sum_{q=1}^{n} (k_q^+)^2. \end{aligned} \tag{18}$$

The proof follows from the following, where we use (18):

$$\sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\}((k_p^+ + k_q^-) - (k_p^- + k_q^+)) = -\sum_{p=1}^{n} \left((k_p^+)^2 + (k_p^-)^2 - 2k_p^+ k_p^-\right) = -\sum_{p=1}^{n} (k_p^+ - k_p^-)^2 \tag{19}$$

□

We have the following proposition as a result of Lemma 2,

Proposition 4. *For any graph G and any vector $X \in \mathbb{R}^n$ of values indexed by the nodes of G, there exists a pair of adjacent nodes $p, q \in \mathcal{V}(G)$, where we assume $x_p > x_q$, and a $\beta > 0$ sufficiently small such that a discordance decreasing α-equal agreement, with $\alpha = \beta(x_p - x_q)$ is possible between nodes p and q.*

Proof. Consider equation (3). Substituting $\alpha \mapsto \beta(x_p - x_q)$, we find that for an update at edge (p, q) to be viable,

$$2\left(\sum_{\substack{k=1\\k\neq q}}^{n} A_{pk}(x_k - x_p) + \sum_{\substack{k=1\\k\neq p}}^{n} A_{qk}(x_q - x_k) - 4(x_p - x_q)\right) + (k_p + k_q + 6)\beta(x_p - x_q) < 0. \tag{20}$$

or equivalently

$$2\left(\sum_{k=1}^{n} A_{pk}(x_k - x_p) + \sum_{k=1}^{n} A_{qk}(x_q - x_k) - 2(x_p - x_q)\right) + (k_p + k_q + 6)\beta(x_p - x_q) < 0. \tag{21}$$

We consider summing (20) over all pairs $p, q \in \mathcal{V}(G)$ of adjacent nodes with $x_p > x_q$; i.e. we evaluate

$$\sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\} \left(\left(\sum_{k=1}^{n} A_{pk}(x_k - x_p) + \sum_{k=1}^{n} A_{qk}(x_q - x_k) - 2(x_p - x_q)\right) + \frac{k_p + k_q + 6}{2}\beta(x_p - x_q)\right) \tag{22}$$

We find that

$$\sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\} \left(\sum_{k=1}^{n} A_{pk}(x_k - x_p) + \sum_{k=1}^{n} A_{qk}(x_q - x_k)\right) = \sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\} \left(k_p^+ + k_q^- - (k_p^- + k_q^+)\right)(x_p - x_q) \tag{23}$$

Thus, (22) can be written as

$$\begin{aligned} &\sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\} \left(\left(\sum_{k=1}^{n} A_{pk}(x_k - x_p) + \sum_{k=1}^{n} A_{qk}(x_q - x_k) - 2(x_p - x_q)\right) + \frac{k_p + k_q + 6}{2}\beta(x_p - x_q)\right) \\ &\qquad = \sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\} \left(k_p^+ + k_q^- - (k_p^- + k_q^+) - 2 + \beta\frac{k_p + k_q + 6}{2}\right)(x_p - x_q). \end{aligned} \tag{24}$$

By Lemma 2, we know there exists at least one Th □

$$\begin{aligned} \sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\} \left(\sum_{k=1}^{n} A_{pk}(x_k - x_p) + \sum_{k=1}^{n} A_{qk}(x_q - x_k)\right) &= \sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\} \left(k_p^+ + k_q^- - (k_p^- + k_q^+)\right)(x_p - x_q) \\ &= 2\sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\}(k_p^+ + k_q^-)(x_p - x_q) - \sum_{p,q=1}^{n} A_{pq} 1\{x_p > x_q\}(k_p + k_q)(x_p - x_q) \end{aligned} \tag{25}$$

We prove the following lemma, which gives an alternative proof of Proposition **??** and gives a lower bound for β^*.

Lemma 3. Suppose the nodes $p, q \in \mathcal{V}(G)$ are adjacent and updateable, meaning that equation (??) holds. Denote k_i the degree of node $i \in \mathcal{V}(G)$ and denote . Then

$$\beta^*(p,q) \geq \frac{6}{} \tag{26}$$

In particular, let K be defined by

$$K = \max_{\substack{i,j \in \mathcal{V}(G) \\ (i,j) \in \mathcal{E}(G)}} (k_i + k_j), \tag{27}$$

Lemma 4. There exist constants $c_*(G), c^*(G) \in \mathbb{R}_{>0}$, independent of X, such that

$$c_*(G) e(X)^{1/2} \leq \beta^* |X_{p^*} - X_{q^*}| \leq c^*(G) e(X)^{1/2}, \tag{28}$$

where

$$c_*(G) = \qquad c^*(G) = \sqrt{\frac{n}{2}} \tag{29}$$

Proof. 1. Upper Bound. We know that $\beta \leq 1/2$ and $(x_p - x_q)^2 \leq d(X|G)$, so

$$(\beta^*(x_{p^*} - x_{q^*}))^2 \leq \frac{1}{4} d(X|G) \tag{30}$$

Thus by proposition 2,

$$(\beta^*(x_{p^*} - x_{q^*}))^2 \leq \frac{n}{2} e(X). \tag{31}$$

This bound is tightest bound independent of X, as can be seen by letting the values associated with two adjacent nodes i and j be distinct and letting every other node k have value $(x_i + x_j)/2$. This works for any connected graph of $n \geq 2$ nodes.

2. Lower Bound. We prove the existence of a lower bound by contradiction. Later work will address determining the tightest bound. Let $\epsilon_0 > 0$ be fixed and suppose for all $\epsilon > 0$ there exists an $X \in \mathbb{R}^n$ such that $e(X) > \epsilon_0 > 0$ and $\beta^* |X_{p^*} - X_{q^*}| < \epsilon$. Denote $n(X|G)$ the number of directed non-zero edge differences $X_i - X_j$ for $(i,j) \in \mathcal{V}(G)$. By proposition 2, $e(X) > \epsilon_0$ implies

$$(X_{p^*} - X_{q^*})^2 \geq \frac{d(X)}{2m} > 2n\epsilon_0 \tag{32}$$

□

3.3 Existence of an Update for the "meet-in-the-middle" Update

We consider updates of the following form:

$$\begin{aligned} X_p(k+1) &= X_p(k) - \alpha_k \operatorname{sgn}(X_p(k) - X_q(k)) \frac{X_p(k) - X_q(k)}{2} \\ X_q(k+1) &= X_q(k) + \alpha_k \operatorname{sgn}(X_p(k) - X_q(k)) \frac{X_p(k) - X_q(k)}{2}. \end{aligned} \tag{33}$$

This is the same update as the previous section except we now explicitly consider $\alpha_k \in (0, 1]$ to be a coefficient giving the fraction of the gap the nodes p and q change their opinion upon an α-equal agreement. We make note of the following proposition giving the amount by which the error changes upon an α_k-equal agreement.

Proposition 5. *Assume that nodes $p, q \in \mathcal{V}(G)$ and $(p,q) \in \mathcal{E}(G)$ are updated at time-step k via the update rule* (33). *Then the change in total discordance at time-step k is given by*

$$d(X, k+1) - d(X, k) = -4\alpha_k(1 - \alpha_k) n (X_p(k) - X_q(k))^2 \tag{34}$$

Equivalently, the consensus error changes as

$$e(X, k+1) - e(X, k) = -2\alpha_k(1 - \alpha_k)(X_p(k) - X_q(k))^2. \tag{35}$$

The following corollary directly follows from proposition 5:

Corollary 1. The consensus error satisfies the recurrence relation

$$e(X, k+1) = (1 - \alpha(k|p,q)) e(X, k) \tag{36}$$

where

$$\alpha(k|p,q) = \frac{4\alpha_k(1 - \alpha_k) n (X_p(k) - X_q(k))^2}{d(X,k)} = \frac{2\alpha_k(1 - \alpha_k)(X_p(k) - X_q(k))^2}{e(X,k)}. \tag{37}$$