Multi-scale Opinion Dynamics

William Oakley[1], Yacoub Kureh[1], P. Jeffrey Brantingham[2], David Kempe[3], and Mason A. Porter[1]

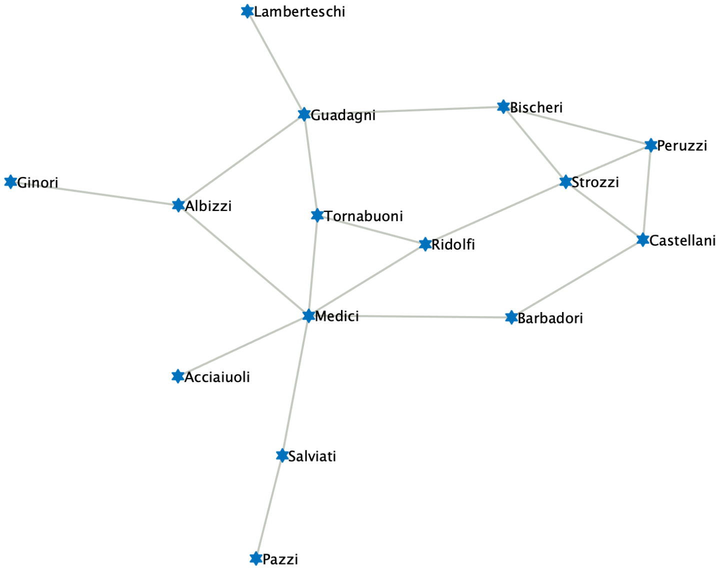
July 1, 2019

[1]UCLA - Department of Mathematics

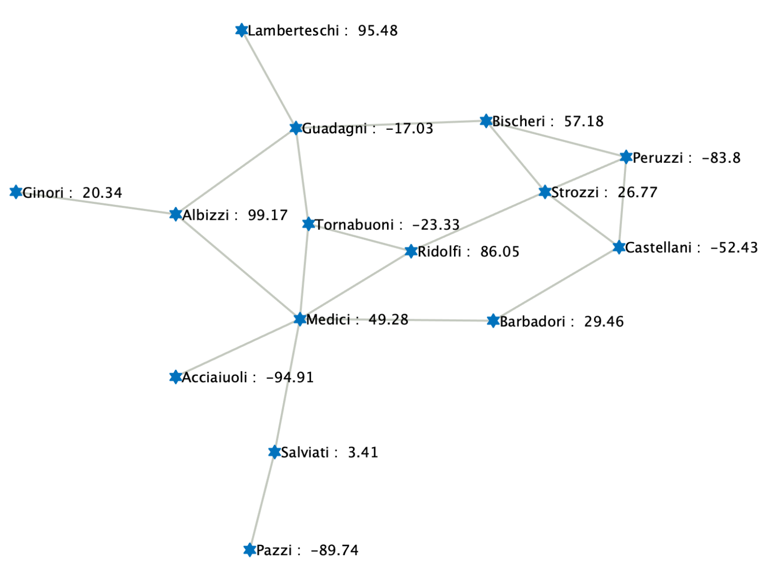
[2]UCLA - Department of Anthropology

[3]USC - Department of Computer Science

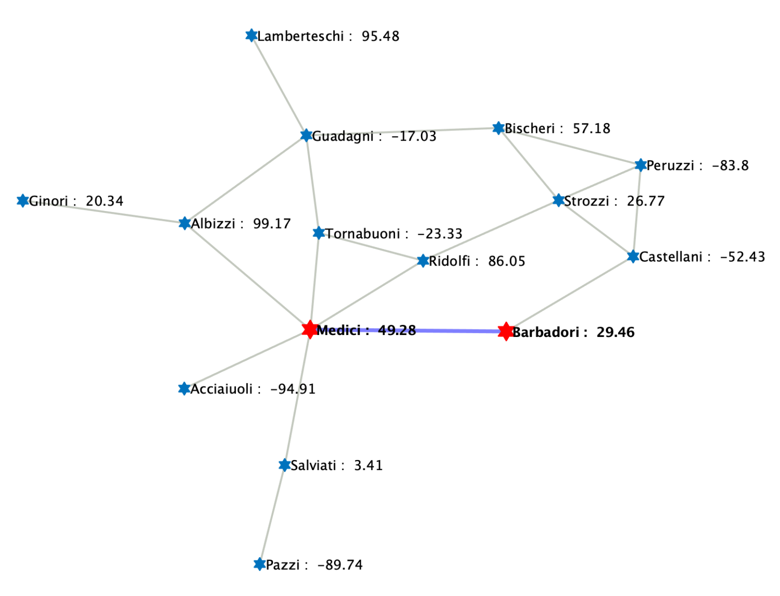
Florentine Families Dataset



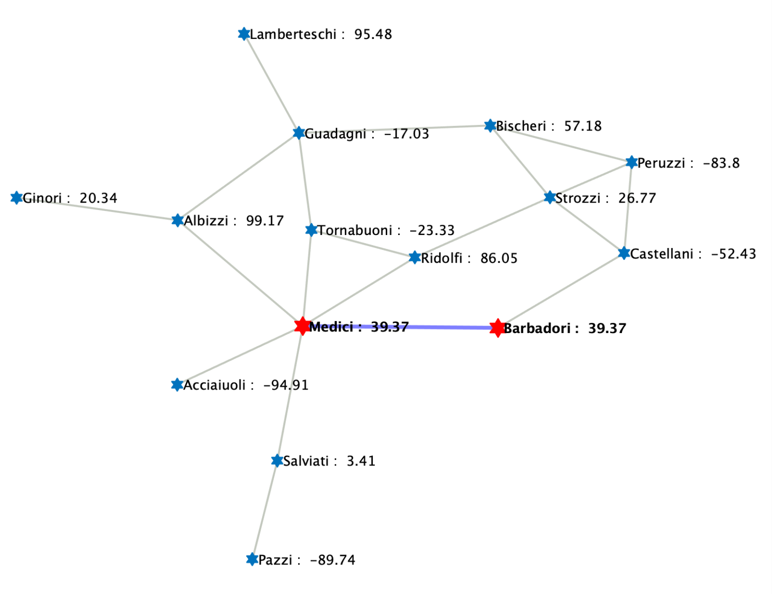
Florentine Families with Opinions



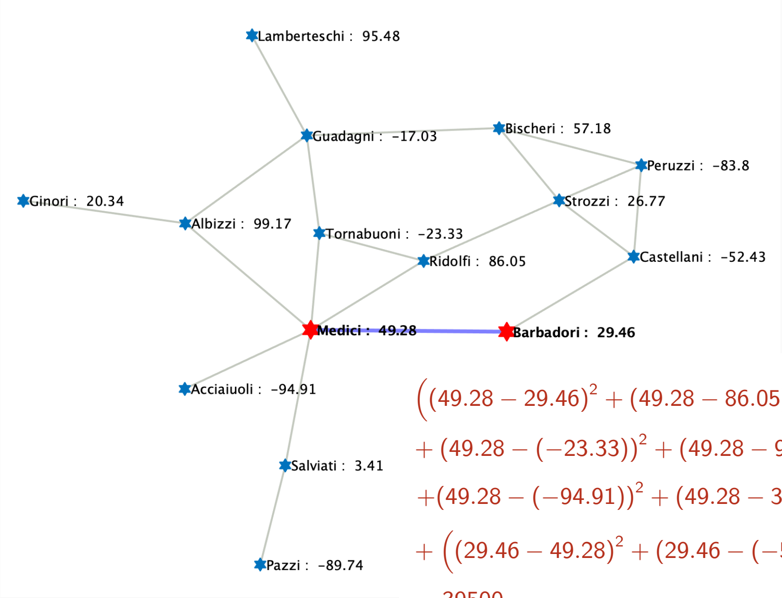
Pairwise Updates



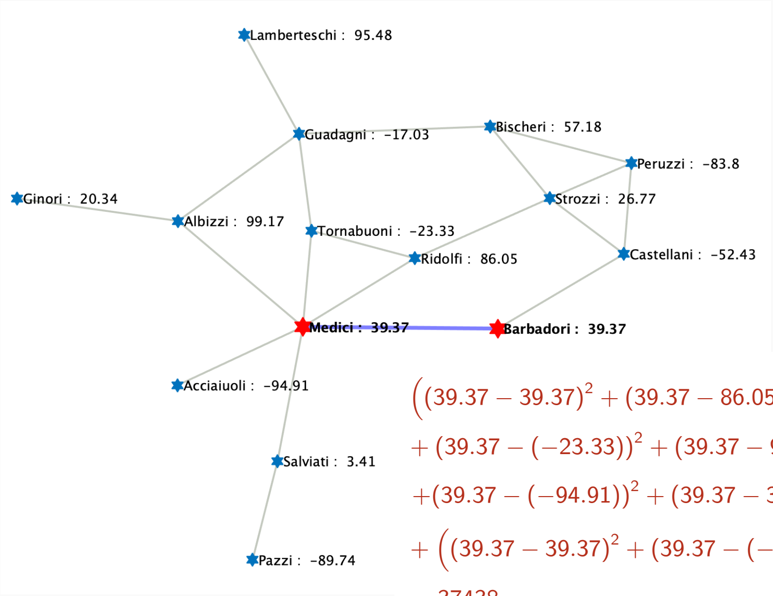
Pairwise Updates



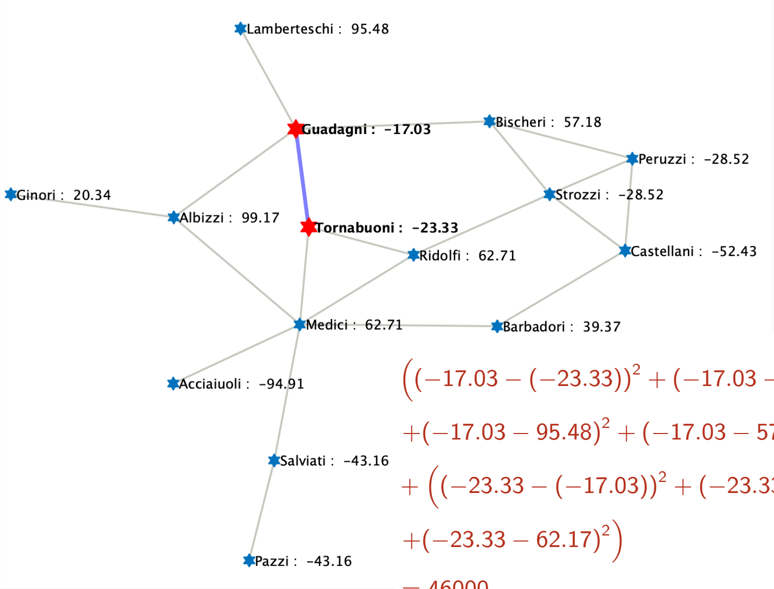
Biased Pairwise Updates



$$\Big((49.28 - 29.46)^2 + (49.28 - 86.05)^2$$
$$+ (49.28 - (-23.33))^2 + (49.28 - 99.17)^2$$
$$+(49.28 - (-94.91))^2 + (49.28 - 3.41)^2\Big)$$
$$+ \Big((29.46 - 49.28)^2 + (29.46 - (-52.43))^2\Big)$$
$$= 39500.$$

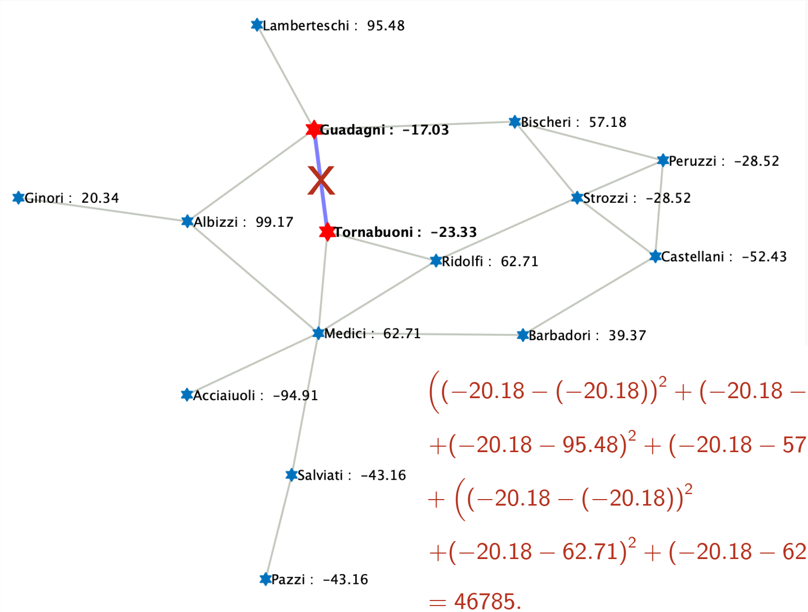


$$\Big((39.37-39.37)^2+(39.37-86.05)^2 +(39.37-(-23.33))^2+(39.37-99.17)^2 +(39.37-(-94.91))^2+(39.37-3.41)^2\Big) +\Big((39.37-39.37)^2+(39.37-(-52.43))^2\Big) =37438.$$



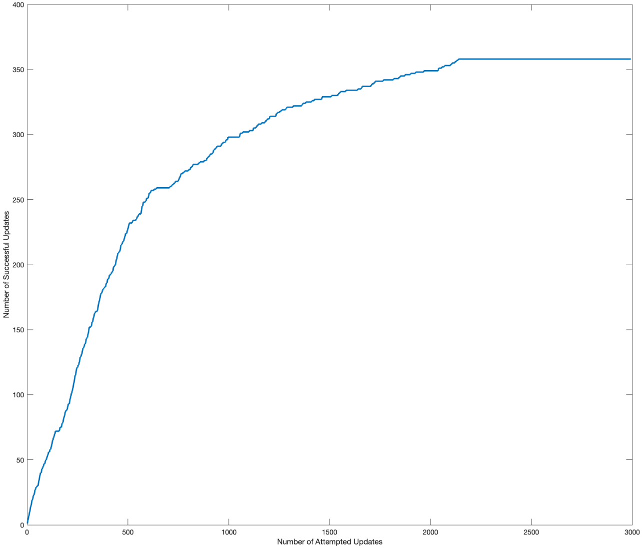
$$
\begin{aligned}
&\Big((-17.03-(-23.33))^2+(-17.03-97.17)^2\\
&+(-17.03-95.48)^2+(-17.03-57.18)^2\Big)\\
&+\Big((-23.33-(-17.03))^2+(-23.33-62.71)^2\\
&+(-23.33-62.17)^2\Big)\\
&=46000.
\end{aligned}
$$

Biased Pairwise Updates

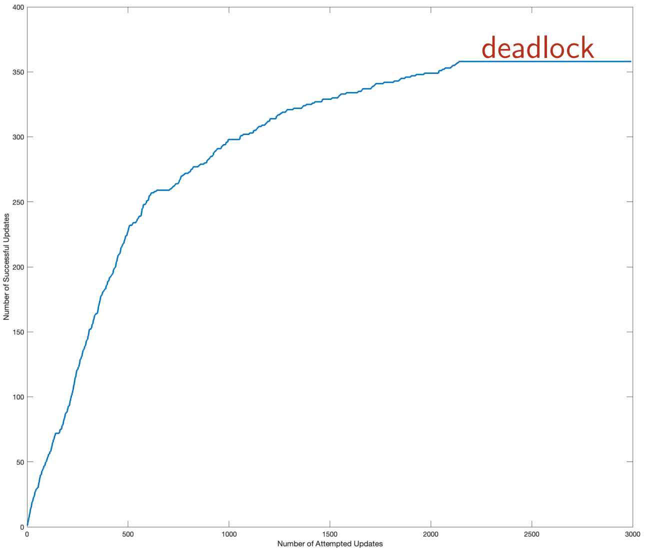


$$\Big((-20.18-(-20.18))^2+(-20.18-97.17)^2$$
$$+(-20.18-95.48)^2+(-20.18-57.18)^2\Big)$$
$$+\Big((-20.18-(-20.18))^2$$
$$+(-20.18-62.71)^2+(-20.18-62.17)^2\Big)$$
$$=46785.$$

Successful Biased Pairwise Updates

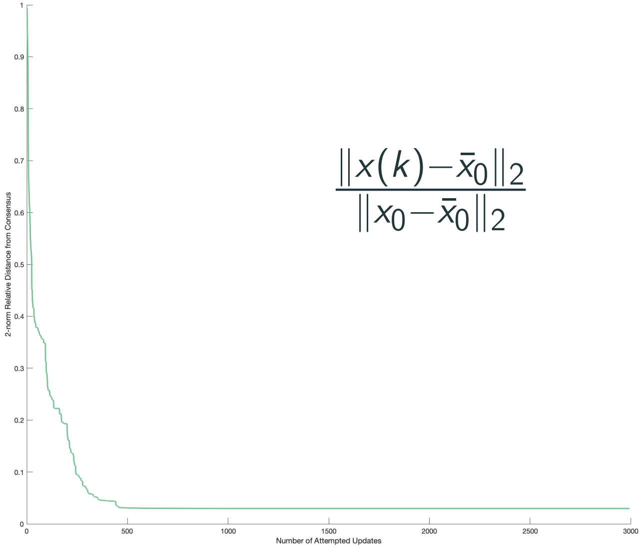


Successful Biased Pairwise Updates



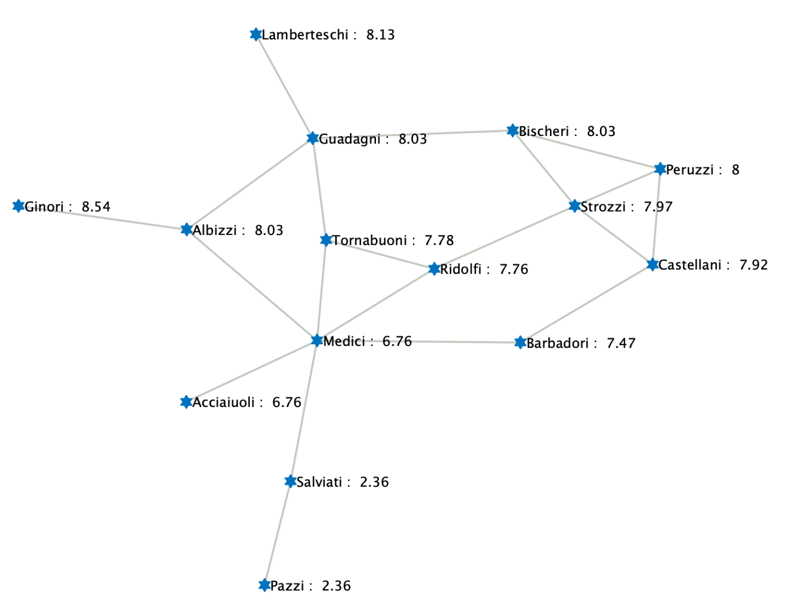
Consensus Error - Biased Pairwise Updates

UCLA USC

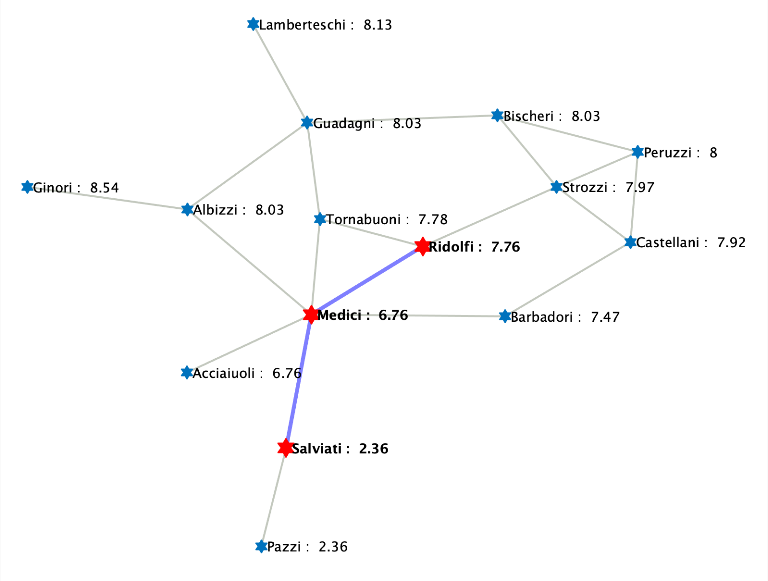


$$\frac{\|x(k)-\bar{x}_0\|_2}{\|x_0-\bar{x}_0\|_2}$$

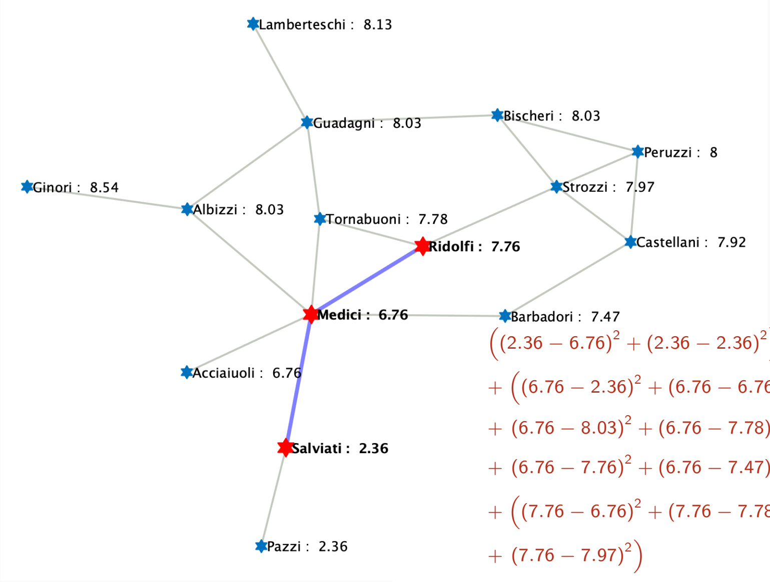
Biased Pairwise Updates - Steady state



Biased Triple-wise Updates

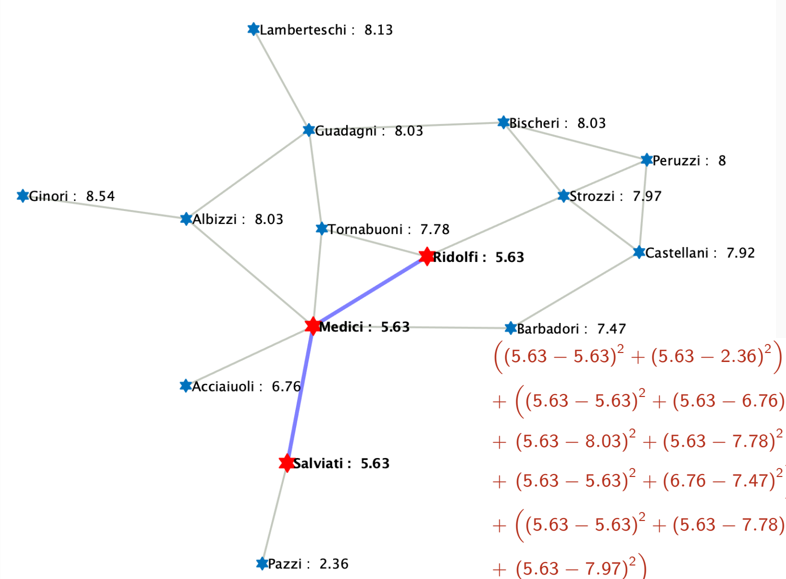


Biased Triple-wise Updates



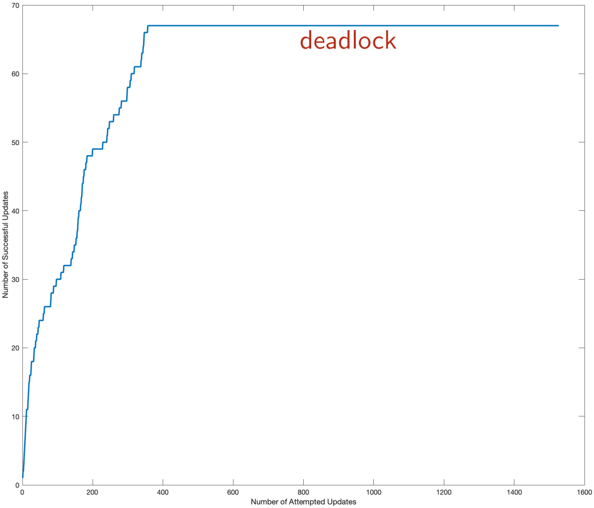
$$\begin{aligned}
&\left((2.36-6.76)^2+(2.36-2.36)^2\right)\\
&+\Big((6.76-2.36)^2+(6.76-6.76)^2\\
&+(6.76-8.03)^2+(6.76-7.78)^2\\
&+(6.76-7.76)^2+(6.76-7.47)^2\Big)\\
&+\Big((7.76-6.76)^2+(7.76-7.78)^2\\
&+(7.76-7.97)^2\Big)\\
&=43.92
\end{aligned}$$

Biased Triple-wise Updates

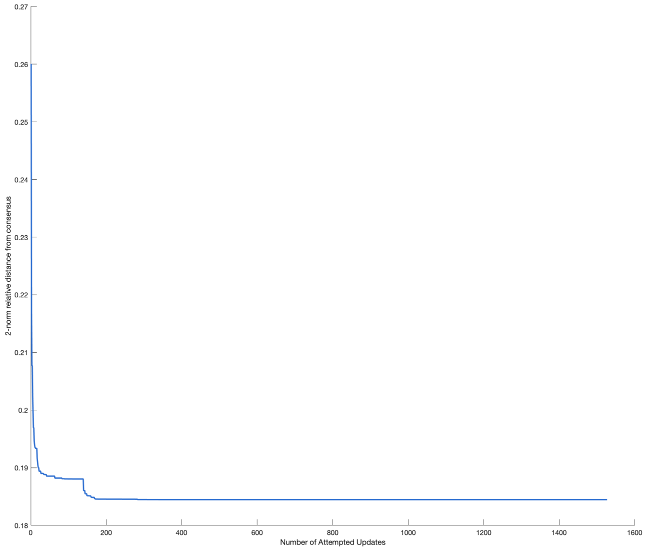


$$
\begin{aligned}
&\left((5.63-5.63)^2+(5.63-2.36)^2\right)\\
&+\Big((5.63-5.63)^2+(5.63-6.76)^2\\
&+(5.63-8.03)^2+(5.63-7.78)^2\\
&+(5.63-5.63)^2+(6.76-7.47)^2\Big)\\
&+\Big((5.63-5.63)^2+(5.63-7.78)^2\\
&+(5.63-7.97)^2\Big)\\
&=32.95
\end{aligned}
$$

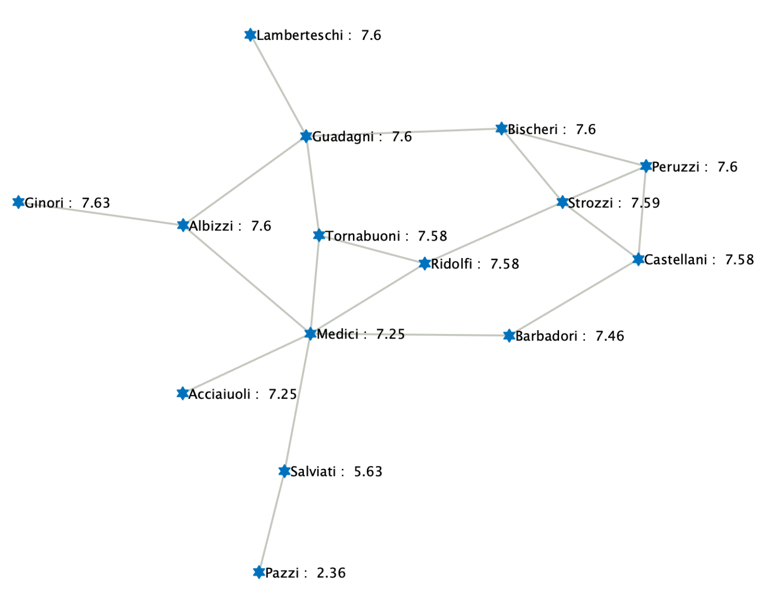
Successful Biased Triple-wise Updates



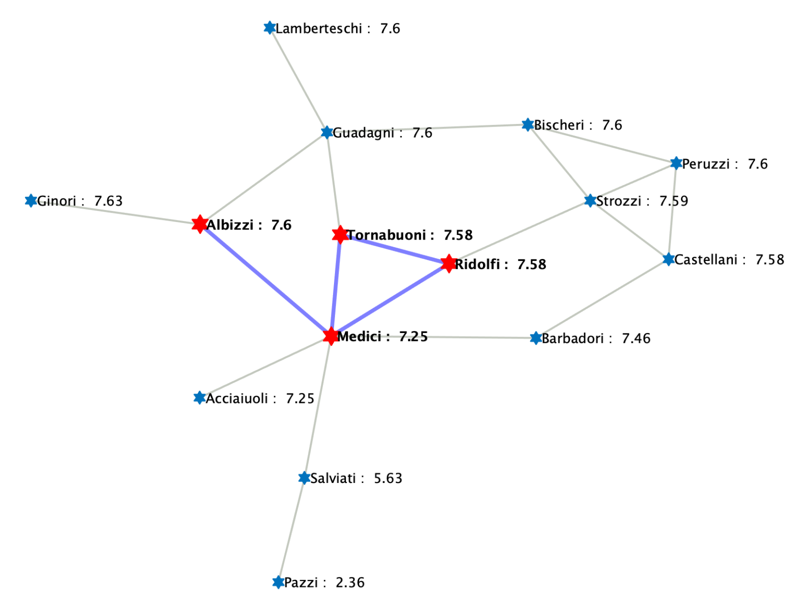
Consensus Error - Biased Triple-wise Updates



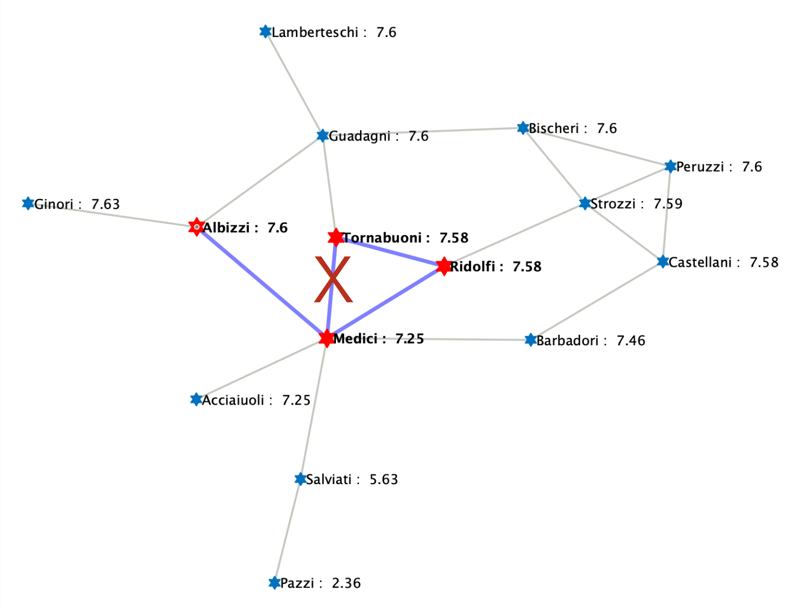
Biased Triple-wise Updates - Steady state



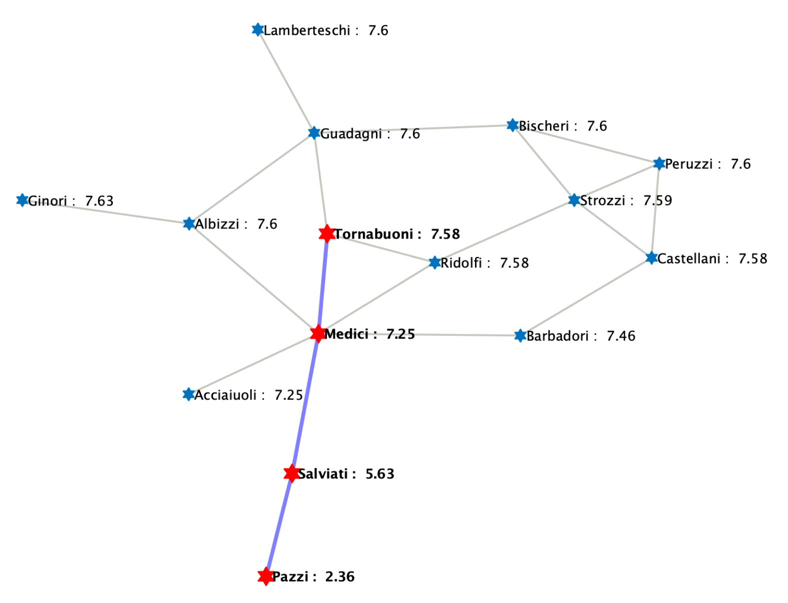
Biased Quadruple-wise Updates



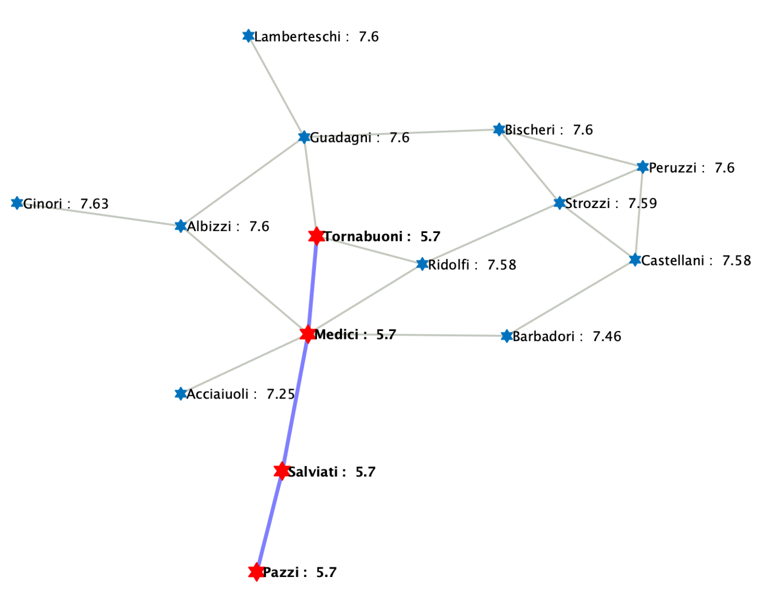
Biased Quadruple-wise Updates



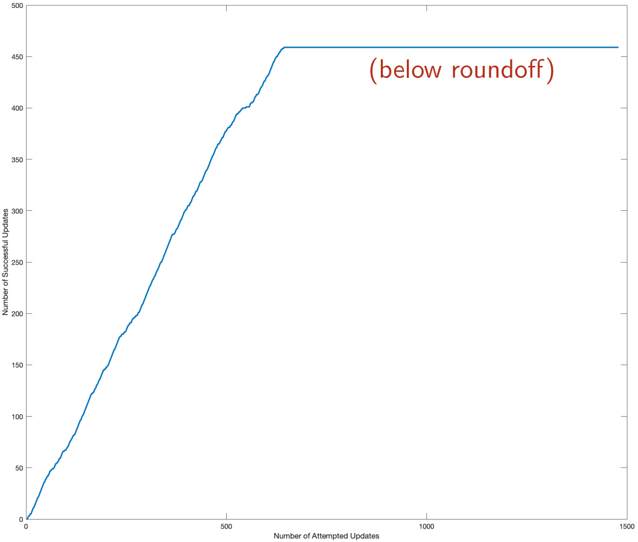
Biased Quadruple-wise Updates



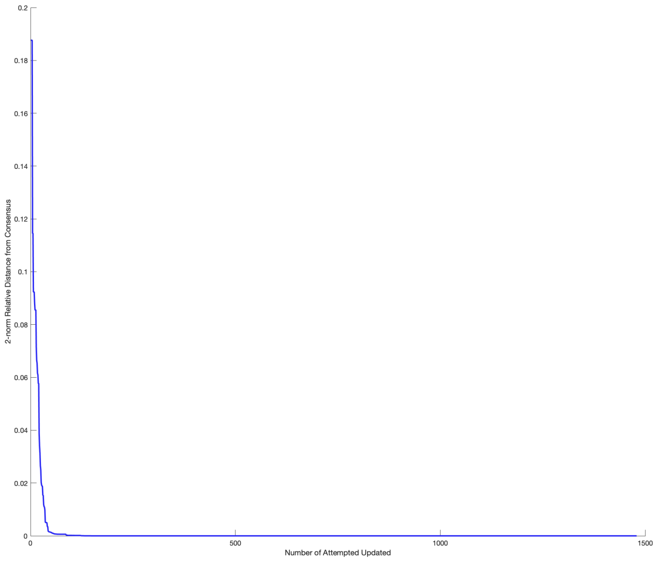
Biased Quadruple-wise Updates



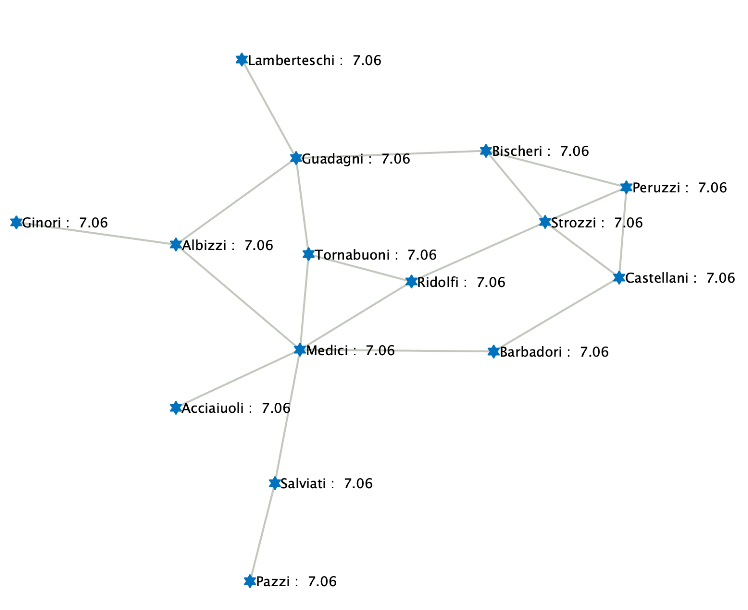
Successful Quadruple-wise Updates



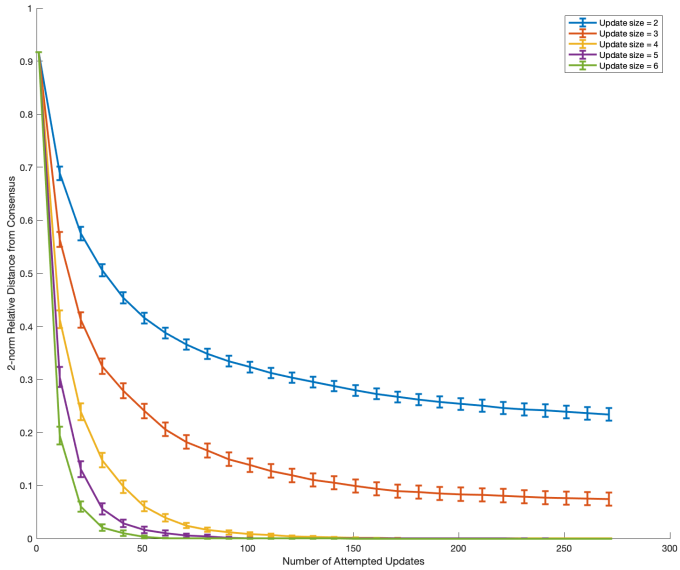
Consensus Error - Biased Quadruple-wise Updates



Biased Quadruple-wise Updates - Steady state



Consensus Error - Biased k-group Updates



Takeaways from Example

Observations:

- Conformity bias affects convergence and may stop it.
- Multi-scale updates may make convergence possible and accelerate convergence.
- Shared neighbors help updates; opinions of unshared neighbors will determine feasibility of update.

Takeaways from Example

Observations:

- Conformity bias affects convergence and may stop it.
- Multi-scale updates may make convergence possible and accelerate convergence.
- Shared neighbors help updates; opinions of unshared neighbors will determine feasibility of update.

$$\Downarrow$$

Takeaways from Example

Observations:

- Conformity bias affects convergence and may stop it.
- Multi-scale updates may make convergence possible and accelerate convergence.
- Shared neighbors help updates; opinions of unshared neighbors will determine feasibility of update.

$$\Downarrow$$

Factors to consider:

1. Network structure
2. Multi-scale communication
3. Biases

1. Rate of Convergence for Unbiased Multi-scale Consensus
2. Rate of Convergence of Biased Partial Consensus
3. Deadlock Formation under Biased Full Consensus

Rate of Convergence for Unbiased Multi-scale Consensus

- Multi-scale updates: Communication occurs between pairs of individuals and/or among individuals within groups of various sizes.

Unbiased Multi-scale Update Rule

- Multi-scale updates: Communication occurs between pairs of individuals and/or among individuals within groups of various sizes.
- Parital consensus: when a connected group of individuals S of size M communicates at timestep k, they update their opinions by a fraction $0 < \beta \leq 1$ to the group average:

$$x_i(k+1) = (1-\beta)x_i(k) + \beta \frac{\sum_{j \in S} x_j(k)}{M}, \; i \in S. \tag{1}$$

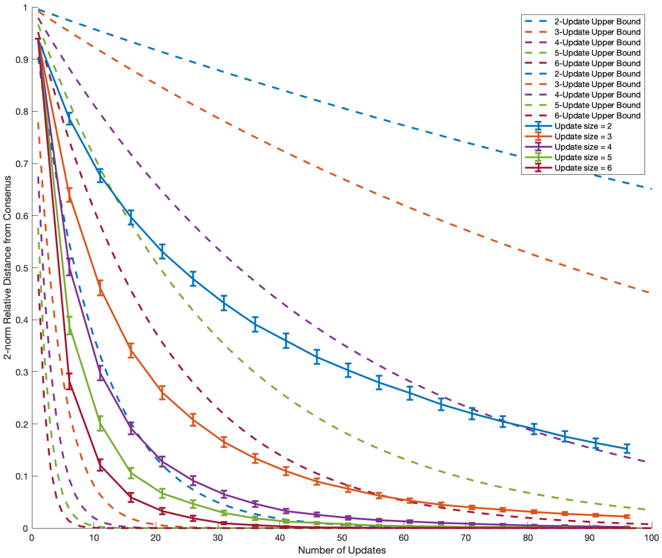
Unbiased Multi-scale Update Rule

- Multi-scale updates: Communication occurs between pairs of individuals and/or among individuals within groups of various sizes.
- Parital consensus: when a connected group of individuals S of size M communicates at timestep k, they update their opinions by a fraction $0 < \beta \leq 1$ to the group average:

$$x_i(k+1) = (1-\beta)x_i(k) + \beta\frac{\sum_{j\in S} x_j(k)}{M}, \; i \in S. \tag{1}$$

- We have derived lower bounds and upper bounds on the *expected* convergence rates to consensus: they depend on 1. group sizes, 2. network topology, 3. partial consensus size β, and 4. communication rates of individuals/groups.

Convergence Results



Rate of Convergence of Biased Partial Consensus

- Upon communication, a pair of individuals updates their opinions as

$$x_i(k+1) = (1-\beta)x_i(k) + \beta\frac{x_i(k)+x_j(k)}{2} \tag{2}$$

Conformity Bias

- Upon communication, a pair of individuals updates their opinions as

$$x_i(k+1) = (1-\beta)x_i(k) + \beta\frac{x_i(k) + x_j(k)}{2} \tag{2}$$

- Assume individuals have a *conformity bias*: in a graph G of size n with adjacency matrix A, a group S of size M will update at timestep k only if

$$\sum_{i \in S}\sum_{j=1}^{n} A_{ij}(x_i(k+1) - x_j(k+1))^2 - \sum_{i \in S}\sum_{j=1}^{n} A_{ij}(x_i(k) - x_j(k))^2 < 0 \tag{3}$$

Proposition (partial consensus is possible somewhere)

For any graph G of m edges and any vector $x \in \mathbb{R}^n$ of values indexed by the nodes of G, there exists a pair of adjacent nodes $p, q \in \mathcal{V}(G)$ and a $\beta \geq 1/2m$ such that an update under conformity bias is possible between nodes p and q.

Proposition (partial consensus is possible somewhere)

For any graph G of m edges and any vector $x \in \mathbb{R}^n$ of values indexed by the nodes of G, there exists a pair of adjacent nodes $p, q \in \mathcal{V}(G)$ and a $\beta \geq 1/2m$ such that an update under conformity bias is possible between nodes p and q.

Theorem ($\implies$ convergence to consensus)

Assume that at each timestep k, the pair of nodes $p, q \in \mathcal{V}(G)$ with the largest possible conformity biased update $\beta_k |x_{p_k}(k) - x_{q_k}(k)|$ are updated. Denote L the graph Laplacian of the graph G, $\lambda_2(L)$ the second smallest eigenvalue of L, and x_{ave} the average opinion. Then, for all k,

$$\beta_k |x_{p_k}(k) - x_{q_k}(k)| \geq \frac{\lambda_2(L)^{1/2}}{\sqrt{2} n m^{3/2}} \|x - x_{ave}\|_2.$$

Deadlock Formation under Biased Full Consensus

Biased Full Consensus

- Groups of individuals update to *full* consensus:

$$x_i(k+1) = \frac{\sum_{j \in S} x_j(k)}{M}, \; i \in S. \tag{4}$$

Biased Full Consensus

- Groups of individuals update to *full* consensus:

$$x_i(k+1) = \frac{\sum_{j \in S} x_j(k)}{M}, \; i \in S. \tag{4}$$

- Individuals have conformity bias:

$$\sum_{i \in S} \sum_{j=1}^{n} A_{ij}(x_i(k+1) - x_j(k+1))^2 - \sum_{i \in S} \sum_{j=1}^{n} A_{ij}(x_i(k) - x_j(k))^2 < 0$$

Biased Full Consensus

- Groups of individuals update to *full* consensus:

$$x_i(k+1) = \frac{\sum_{j \in S} x_j(k)}{M}, \ i \in S. \tag{4}$$

- Individuals have conformity bias:

$$\sum_{i \in S} \sum_{j=1}^{n} A_{ij}(x_i(k+1) - x_j(k+1))^2 - \sum_{i \in S} \sum_{j=1}^{n} A_{ij}(x_i(k) - x_j(k))^2 < 0$$

- Opinion deadlocks at various scales form in some graphs.

Biased Full Consensus

- Groups of individuals update to *full* consensus:

$$x_i(k+1) = \frac{\sum_{j \in S} x_j(k)}{M}, \ i \in S. \tag{4}$$

- Individuals have conformity bias:

$$\sum_{i \in S} \sum_{j=1}^{n} A_{ij}(x_i(k+1) - x_j(k+1))^2 - \sum_{i \in S} \sum_{j=1}^{n} A_{ij}(x_i(k) - x_j(k))^2 < 0$$

- Opinion deadlocks at various scales form in some graphs.
- Multi-scale updates break opinion deadlocks in various graphs.

Biased Full Consensus

- Groups of individuals update to *full* consensus:

$$x_i(k+1) = \frac{\sum_{j \in S} x_j(k)}{M}, \ i \in S. \tag{4}$$

- Individuals have conformity bias:

$$\sum_{i \in S} \sum_{j=1}^{n} A_{ij}(x_i(k+1) - x_j(k+1))^2 - \sum_{i \in S} \sum_{j=1}^{n} A_{ij}(x_i(k) - x_j(k))^2 < 0$$

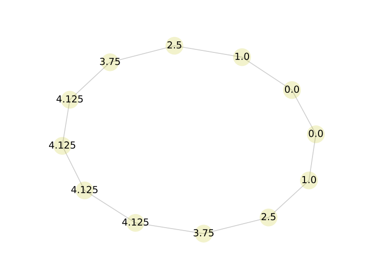
- Opinion deadlocks at various scales form in some graphs.
- Multi-scale updates break opinion deadlocks in various graphs.
- Main thrust: classification

Pairwise Opinion Deadlocked Graphs

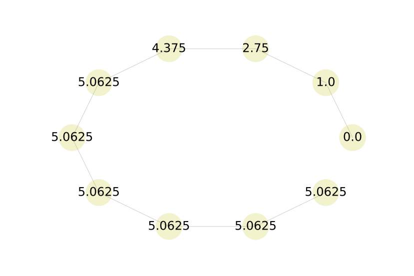
Graphs that can become pairwise deadlocked:

- Cycle Graphs of $n \geq 9$ nodes.
- Star Graphs of $n \geq 5$ leaf nodes.
- Path Graphs of $n \geq 5$ nodes.
- Padgett's Florentine Families Network.

Pairwise Opinion Deadlocked Graphs



(a) Deadlocked cycle graph of 12 nodes.



(b) Deadlocked path graph of 10 nodes.

Figure 1: Examples of pairwise deadlocked graphs.

- Complete Graphs
- Erdos-Renyi Graphs $G(N, p)$ with sufficiently high p seem to have a low probability of becoming deadlocked.

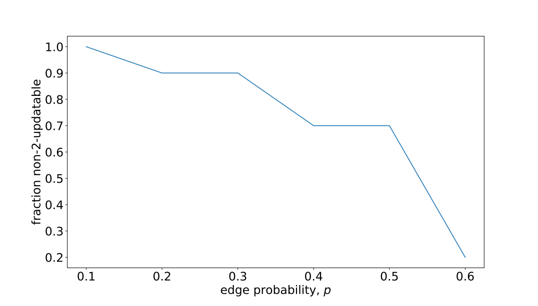


Figure 2: Probability of immunity to deadlock of $G(N, p)$ increases with p.

- Star graphs of $5 \leq n \leq 8$ nodes can become pairwise deadlocked but not triad deadlocked.
- Dumbbell graphs of $n \geq 10$ nodes can become deadlocked for all group sizes less than n.

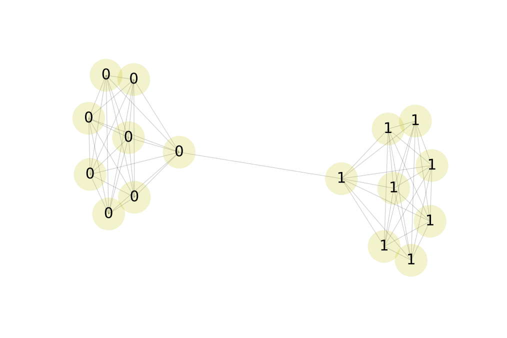


Figure 3: Deadlocked barbell graph of $n = 16$ nodes.

MURI N00014-17-S-F006

Thank you.