BACKGROUND MATH

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1. Polynomials

The fundamental object in this project is the **polynomial**. A polynomial

$$p(v_1, \dots, v_n) = \sum_{(i_1,\dots,i_n)} a_{(i_1,\dots,i_n)} v_1^{i_1} v_2^{i_2} \dots v_n^{i_n}$$

is an algebraic expression in some number of variables, where the coefficients come from a specified ring R. The **multidegree** $\iota = (i_1, \dots, i_n)$ ranges over $\mathbb{N}^n$; a multidegree's **total degree** is the sum of its entries $i_1 + \dots + i_n$. For each $i \in \mathbb{N}$, let p_i denote the **homogenous part of degree** i, that is all of the terms of p whose total degree is i. The collection of all polynomials $p(v_1, \dots, v_n)$ with coefficients in R is denoted

$$R[v_1, \dots, v_n]$$

In this project, polynomials are represented by a hash table

$$\text{Map}\langle\text{multidegree, coefficient}\rangle \ \text{terms;}$$

A multidegree $(i_1, \dots, i_n)$ represents the product $v_1^{i_1} \dots v_n^{i_n}$, so in particular the multidegree $(0, \dots, 0)$ maps to the constant term.

We consider polynomial rings where the variables are all truncated, specifically polynomial rings of the form

$$P := R[v_1, \dots, v_n]/\langle v_1^{t_1+1}, \dots, v_n^{t_n+1}\rangle$$

where the ring R is either $\mathbb{Z}$ or $\mathbb{Z}/m$ for $m \in \mathbb{N}$. The multidegree $\tau_P := (t_1, \dots, t_n)$ is called the **truncation** and could have infinite entries. If every entry is finite, then

$$\mu_P := v_1^{t_1} v_2^{t_2} \dots v_n^{t_n}$$

is the unique product of variables in P with the highest degree $d = t_1 + \dots + t_n$. Given $p \in P$ and a partition $\pi = (j_1, \dots, j_l)$ of d, the **characteristic number** of p at π is defined as

$$p(\pi) = [\mu_P] p_{j_1} p_{j_2} \dots p_{j_l}$$

or in English, it is the coefficient of μ_P in the product $p_{j_1} \dots p_{j_l}$. Note that since each p_{j_i} has degree j_i their product is homogeneous of degree d, and therefore the product is exactly equal to $p(\pi)\mu_P$.

Given a polynomial $p(v_1, \dots, v_n)$ and a multidegree $(i_1, \dots, i_n)$ let

$$p[i_1, \dots, i_n] := \text{coefficient of } v_1^{i_1} \dots v_n^{i_n}$$

in the code, this quantity is given by terms . get$((i_1, \ldots, i_n))$.

Given two sets of variables $\{v_1, \ldots, v_n\}$ and $\{x_1, \ldots, x_m\}$ we can combine polynomial rings together with the tensor product, namely

$$\begin{aligned}&\big(R_1[v_1, \ldots, v_n]/\langle v_1^{t_1+1}, \ldots, v_n^{t_n+1}\rangle\big) \otimes \big(R_2[x_1, \ldots, x_m]/\langle x_1^{s_1+1}, \ldots, x_n^{s_m+1}\rangle\big)\\ &\cong (R_1 \otimes R_2)[v_1, \ldots, v_n, x_1, \ldots, x_m]/\langle v_1^{t_1+1}, \ldots, v_n^{t_n+1}, x_1^{s_1+1}, \ldots, x_n^{s_m+1}\rangle\end{aligned}$$

2. Blackboxes: cohomology and characteristic classes

The following can be swallowed as fact, but the interested reader is encouraged read the classic “Characteristic Classes” by Milnor and Stasheff [1].

Cohomology is a functor from the category of topological spaces to the category of graded rings. Given an abelian group G, cohomology produces from a space X a sequence of groups denoted $H^i(X; G)$ for $i \in \mathbb{N}$, with a natural cup-product operation

$$H^i(X; G) \times H^j(X; G) \to H^{i+j}(X; G)$$

turning this sequence of groups into a graded-commutative ring, denoted $H^*(X; G)$. For a commutative ring R, an R-module structure on G induces a graded R-module structure on $H^*(X; G)$.

A connected, closed, orientable manifold M is a particularly nice topological space, one of whose special properties is that there is a natural number n (the **real dimension** of M) such that $H^n(M; G) \cong G$ and $H^i(M; G) = 0$ for all $i > n$. In the case that $G = \mathbb{Z}$, $H^n(M; \mathbb{Z})$ has two possible generators, a choice of which we will call a **fundamental class** and denoted μ_M. Like the above notion of characteristic numbers of polynomials, if we have a cohomology class $k = k_0 + k_1 + \cdots + k_n$ where $k_i \in H^i(M; \mathbb{Z})$ and a partition $\pi = (j_1, \ldots, j_l)$ of n, we can form

$$k[\pi] := k_{j_1} \ldots k_{j_l} = a\mu_M \in H^n(M; \mathbb{Z}) \text{ for some } a \in \mathbb{Z}$$

We “confuse notation” and let $k[\pi]$ also denote **characteristic number** $a = k[\pi][\mu_M]$. The characteristic number depends on the choice of fundamental class of the orientable manifold M as well as the order of the product, but only up to sign.

If M is a smooth manifold, its tangent bundle induces additional cohomological information called **characteristic classes**. In this project we are only concerned with three types:

(1) the **Stiefel-Whitney** classes $w_i(M) \in H^i(M; \mathbb{Z}/2)$ are defined for all manifolds
(2) the **Chern** classes $c_i(M) \in H^{2i}(M; \mathbb{Z})$ are defined for complex manifolds
(3) the **Pontryagin** classes $p_i(M) \in H^{4i}(M; \mathbb{Z})$ are defined for all manifolds.

In particular if M has real dimension n then it has Stiefel-Whitney classes $w_0, \ldots, w_n$ and Pontryagin classes $p_0, \ldots, p_{\lfloor \frac{n}{4} \rfloor}$; if M is complex then n must be even and M also has Chern classes $c_0, \ldots, c_{\frac{n}{2}}$.

There are various relations between these characteristic classes. The ones we use in this project concern the "mod-2 reduction" map

$$\rho_2\colon H^i(M;\mathbb{Z}) \to H^i(M;\mathbb{Z}/2)$$

and they state that (1) $\rho_2(c_i(M)) = w_{2i}(M)$, and that (2) $\rho_2(p_i(M)) = w_{2i}(M)^2$.

This project currently models two basic manifolds: the **complex projective space** $\mathbb{CP}^n$, which is complex and has real dimension $2n$; and the **quaternionic projective space** $\mathbb{HP}^n$, which is *not* complex and has real dimension $4n$. Both of their cohomology and characteristic classes have very simple expressions. The reader can take this information as given, but there are many excellent sources out there in which to find this information.

The integral and mod-2 cohomology of $\mathbb{CP}^n$ are given by

$$H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[u]/\langle u^n+1\rangle \text{ where } u \text{ has degree 2.}$$

$$H^*(\mathbb{CP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[\tilde{u}]/\langle \tilde{u}^n+1\rangle \text{ where } \tilde{u} = \rho_2(u).$$

As a graded abelian group, $H^i(\mathbb{CP}^n;A) = A$ if i is even and $0 \le i \le 2n$ and 0 otherwise for any abelian group A.

The Chern class is of $\mathbb{CP}^n$ is

$$c(\mathbb{CP}^n) = (1+u)^{n+1}$$

so the homogenous terms are given by $c_i(\mathbb{CP}^n) = \binom{n+1}{i}u^i \in H^{2i}(\mathbb{CP}^n;\mathbb{Z})$.

Since $w = \rho_2(c)$ the Stiefel-Whitney class is given by

$$w(\mathbb{CP}^n) = (1+\tilde{u})^{n+1}$$

and $w_i(\mathbb{CP}^n) = \binom{n+1}{i}\tilde{u}^i \in H^i(\mathbb{CP}^n;\mathbb{Z}/2)$. Lastly, the Pontryagin class is

$$p(\mathbb{CP}^n) = (1+u^2)^{n+1}$$

and so $p_i(\mathbb{CP}^n) = \binom{n+1}{i}u^{2i} \in H^{4i}(\mathbb{CP}^n;\mathbb{Z})$.

The integral and mod-2 cohomology of $\mathbb{HP}^n$ are given by

$$H^*(\mathbb{HP}^n;\mathbb{Z}) \cong \mathbb{Z}[v]/\langle v^n+1\rangle \text{ where } v \text{ has degree 4.}$$

$$H^*(\mathbb{HP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[\tilde{v}]/\langle \tilde{v}^n+1\rangle \text{ where } \tilde{v} = \rho_2(v).$$

As a graded abelian group, $H^i(\mathbb{HP}^n;A) = A$ if i is divisible by 4 and $0 \le i \le 4n$, and 0 otherwise for any abelian group A.

$\mathbb{HP}^n$ is not complex, so it has no Chern class.

The Stiefel-Whitney class is given by

$$w(\mathbb{HP}^n) = (1+\tilde{v})^{n+1}$$

and the Pontryagin class is given by the slightly less simple formula

$$p(\mathbb{HP}^n) = (1+v)^{2n+2}(1+4v)^{-1}$$

Here the right factor of the Pontryagin class is to be interpreted using the power series expression

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

Cohomology and characteristic classes both have product rules when computing their values for $M \times N$.

This project combines examples using products, and to formulate the product rule for chomology we need to mention some **Homological Algebra**. For R a ring, A a right R-module, and B a left R-module, we can form their **tensor product** $A \otimes_R B$ which is very much like the tensor product from linear algebra and encompasses the tensor product of polynomial rings, and Homological Algebra provides a construction for the somewhat more obtuse **Tor groups** $\operatorname{Tor}_n^R(A, B)$. If A and B are **graded** R modules then the tensor product and Tor groups also inherit gradings. For this project we won't need to know the intricacies of how these objects work, just their values in some very special cases.

If R is a commutative ring and A is any R-module, then for all spaces X the cohomology groups $H^*(X; A)$ form a graded R-module. Given two spaces X and Y we can thus form the tensor product and Tor groups of their cohomology rings. There is a **cross product** map

$$\times \colon H^*(X; A) \otimes_R H^*(Y; A) \to H^*(X \times Y; A)$$

which is not always an isomorphism, but under some finite-type assumptions on X and Y we have the **Künneth Exact Sequence**:

$$0 \to H^*(X; A) \otimes_R H^*(Y; A) \to H^*(X \times Y; A) \to \Sigma \operatorname{Tor}_1^R(H^*(X; A), H^*(Y; A)) \to 0$$

where Σ means that the grading on the Tor group is shifted up by one. This sequence is easiest to apply when the Tor term vanishes outright, such as the case where the cohomology of X or Y is a free R-module. In particular this is satisfied for products of complex and quaternionic projective spaces since $H^*(\mathbb{CP}^n; R)$ and $H^*(\mathbb{HP}^m; R)$ are free R modules for any commutative ring R, so

$$H^*(\mathbb{CP}^n \times \mathbb{HP}^m; R) \cong H^*(\mathbb{CP}^n; R) \otimes_R H^*(\mathbb{HP}^m; R) \cong R[u, v]/\langle u^{n+1}, v^{m+1} \rangle$$

The product rule for our characteristic classes, the so-called **Whitney Product Formula**, is now easy to state. For the Stiefel-Whitney and Chern classes (if they are defined) we have simply

$$w(M \times N) = w(M) \times w(N) \in H^*(M \times N; \mathbb{Z}/2)$$
$$c(M \times N) = c(M) \times c(N) \in H^*(M \times N; \mathbb{Z})$$

For Pontryagin classes, the same is true if there is no 2-torsion present, specifically

$$2(p(M \times N) - p(M) \times p(N)) = 0 \in H^*(M \times N; \mathbb{Z})$$

In terms of homogeneous classes, this is expressed this as

$$w_i(M \times N) = \sum_{k=0}^{i} w_k(M) \times w_{i-k}(N)$$

and similarly for the other classes, but we don't use this expression in this project.

Let's take the concrete example of $\mathbb{HP}^2 \times \mathbb{CP}^3$, with

$$H^*(\mathbb{HP}^2 \times \mathbb{CP}^3; \mathbb{Z}) \cong \mathbb{Z}[v, u]/\langle v^3, u^4 \rangle \text{ where } |v| = 4 \text{ and } |u| = 2.$$

It is a manifold without complex structure, and has real dimension 14. Therefore it potentially has Stiefel-Whitney classes $w_0, \ldots, w_{14}$, and Pontryagin classes $p_0, \ldots, p_3$. The Whitney product formula tells us $w(\mathbb{HP}^2 \times \mathbb{CP}^3) = (1 + \tilde{v})^3(1 + \tilde{u})^4 = (1 + \tilde{v})^3$, since $\binom{4}{i}$ is even for all $i > 0$. In homogeneous terms this gives

$$w_0 = 1$$

$$w_4 = \tilde{v}$$

$$w_8 = \tilde{v}^2$$

and all other Stiefel-Whitney classes vanish.

Lastly, $p(\mathbb{HP}^2 \times \mathbb{CP}^3) = (1 + v)^6(1 + 4v)^{-1}(1 + u)^4$ so

$$p_0 = 1$$

$$p_1 = 2v + 4u^2$$

$$p_2 = 7v^2 + 8vu^2$$

$$p_3 = 28v^2u^2$$

3. Bordism and Characteristic Numbers

Bordism is a very beautiful relation between manifolds: put very simply, two manifolds of the same dimension are bordant if there is a manifold of one higher dimension whose boundary is their disjoint union. More structure can be added to the manifolds to give different flavours of bordism groups, such as **oriented bordism** and **complex bordism**. (Complex bordism, in reality, is slightly more subtle than just adding the word "complex", but we will treat it as such here.) For example, every S^n is the oriented boundary of D^{n+1} so S^n is orientedly bordant to the empty manifold for every n; however, most S^n do not admit complex structures, and even though S^2 does it is not null-bordant as a complex manifold. By taking bordism classes, the collection of all manifolds of a given flavour becomes a graded ring: the 0 element is the empty manifold, addition is given by disjoint union, multiplication is given by cartesian product, and the additive inverse is typically given by "reversing" the structure on the manifold in some way, such as an opposite orientation (and in fact, in the absence of extra structure every bordism class is its own inverse, because $2M$ is the boundary of $M \times I$). We adopt the following notation for these rings:

$$\mathcal{N}_* = \text{bordism ring without added structure}$$

$$\Omega_* = \text{oriented bordism ring}$$

$$\Omega^U_* = \text{complex bordism ring}$$

Now, given characteristic classes $w(M)$, $p(M)$, and possibly $c(M)$, we can form their characteristic numbers in the same manor as the recipe for polynomials in a truncated polynomial ring. The remarkable facts are that (1) characteristic numbers are invariant under the appropriate type of bordism, and (2) bordism classes are determined the appropriate characteristic numbers. Namely:

The collection of Stiefel-Whitney numbers is a complete invariant for $\mathcal{N}_*$.

The collection of Stiefel-Whitney and Pontryagin numbers is a complete invariant for Ω_*.

The collection of Chern numbers is a complete invariant for Ω_*^U.

For example, by computing characteristic numbers, the program is able to determine that the class of $\mathbb{HP}^3$ in Ω_* is

$$-8\mathbb{CP}^6 + 24(\mathbb{CP}^2 \times \mathbb{CP}^4) - 16(\mathbb{CP}^2)^3$$

Another important invariant of Ω_* is the **signature**, which is a shadow of a representative manifold's intersection form. It is a well defined ring invariant

$$\sigma \colon \Omega_* \to \mathbb{Z}$$

In particular $\sigma(M) = 0$ if the dimension of M is not divisible by 4, $\sigma(\mathbb{CP}^{2n}) = 1$, and $\sigma(\mathbb{HP}^n)$ is 1 when n is even and 0 when n is odd. Amazingly, the following is true:

Theorem 1 (Hirzebruch's Signature Theorem)**.** *For each n there is a homogeneous polynomial $L_n(x_1, \ldots, x_n)$ of degree $4n$ such that*

$$\forall M \in \Omega_{4n},\ \sigma(M) = L_n(p_1(M), \ldots, p_n(M))[\mu_M]$$

Since L_n is homogenous of degree $4n$, it's actually a linear combination of Pontryagin numbers. This theorem has tremendous theoretical implications, but it is also invaluable for testing wether Pontryagin numbers are correct, since the signature is typically far easier to compute and any minor error will cause everything to go wrong.

Exercise: compute the Stiefel-Whitney, Chern, and Pontryagin numbers of the 4-manifold $\mathbb{CP}^2 \times \mathbb{CP}^2$. Check your Pontryagin classes by verifying Hirzebruch's Signature Formula in this case:

$$\frac{7p[2] - p[1,1]}{45} = 1$$

References

[1] John Milnor and James D Stasheff. *Characteristic Classes.(AM-76)*, volume 76. Princeton university press, 2016.