BACKGROUND MATH

WILLIAM GOLLINGER

1. Polynomials

The fundamental object in this project is the **polynomial**. A polynomial

$$p(v_1, \ldots, v_n) = \sum_{(i_1,\ldots,i_n)} a_{(i_1,\ldots,i_n)} v_1^{i_1} v_2^{i_2} \ldots v_n^{i_n}$$

is an algebraic expression in some number of variables, where the coefficients come from a specified ring R. The **multidegree** $\iota = (i_1, \ldots, i_n)$ ranges over $\mathbb{N}^n$; a multidegree's **total degree** is the sum of its entries $i_1 + \cdots + i_n$. For each $i \in \mathbb{N}$, let p_i denote the **homogenous part of degree** i, that is all of the terms of p whose total degree is i. The collection of all polynomials $p(v_1, \ldots, v_n)$ with coefficients in R is denoted

$$R[v_1, \ldots, v_n]$$

In this project, polynomials are represented by a hash table

$$\text{Map}\langle\text{multidegree, coefficient}\rangle \text{ terms;}$$

(more specifically $\text{Map}\langle\text{Integer, Map}\langle\text{multidegree, coefficient}\rangle\rangle$ because they have a built-in grading at the moment). A multidegree $(i_1, \ldots, i_n)$ represents the product $v_1^{i_1} \ldots v_n^{i_n}$, so in particular the multidegree $(0, \ldots, 0)$ is the constant term.

We consider polynomial rings where the variables are all truncated, specifically polynomial rings of the form

$$P := R[v_1, \ldots, v_n]/\langle v_1^{t_1+1}, \ldots, v_n^{t_n+1}\rangle$$

where the ring R is either $\mathbb{Z}$ or $\mathbb{Z}/m$ for $m \in \mathbb{N}$. The multidegree $\tau_P := (t_1, \ldots, t_n)$ is called the **truncation** can could have infinite entries. If every entry is finite, then

$$\mu_P := v_1^{t_1} v_2^{t_2} \ldots v_n^{t_n}$$

is the unique product of variables in P with the highest degree $d = t_1 + \cdots + t_n$. Given $p \in P$ and a partition $\pi = (j_1, \ldots, j_l)$ of d, the **characteristic number** of p at π is defined as

$$p(\pi) = p_{j_1} p_{j_2} \ldots p_{j_l}[\tau]$$

Note that since each p_{j_i} has degree j_i their product has homogeneous of degree d, and therefore the product is exactly equal to $p(\pi)\tau$.

Given a polynomial $p(v_1, \ldots, v_n)$ and a multidegree $(i_1, \ldots, i_n)$ let

$$p[i_1, \ldots, i_n] := \text{coefficient of } v_1^{i_1} \ldots v_n^{i_n}$$

in the code, this quantity is given by terms $.get((i_1, \ldots, i_n))$.

Given two sets of variables $\{v_1, \ldots, v_n\}$ and $\{x_1, \ldots, x_n\}$ we can combine polynomial rings together with the tensor product, namely

$$\begin{aligned}&\big(R_1[v_1, \ldots, v_n]/\langle v_1^{t_1+1}, \ldots, v_n^{t_n+1}\rangle\big) \otimes \big(R_2[x_1, \ldots, x_n]/\langle x_1^{s_1+1}, \ldots, x_n^{s_n+1}\big) \\ &\cong (R_1 \otimes R_2)[v_1, \ldots, v_n, x_1, \ldots, x_n]/\langle v_1^{t_1+1}, \ldots, v_n^{t_n+1}, x_1^{s_1+1}, \ldots, x_n^{s_n+1}\rangle\end{aligned}$$

2. Blackboxes: cohomology and characteristic classes

The following can be swallowed as fact, but the interested reader is encouraged read the classic "Characteristic Classes" by Milnor and Stasheff [].

Cohomology is a functor from the category of topological spaces to the category of graded rings. Given an abelian group G, cohomology produces from a space X a sequence of groups denoted $H^i(X; G)$ for $i \in \mathbb{N}$. There is a cup-product operation

$$H^i(X; G) \times H^j(X; G) \to H^{i+j}(X; G)$$

turning this sequence of groups into a graded-commutative ring denoted $H^*(X; G)$. For a commutative ring R, an R-module structure on G induces an R-module structure on $H^*(X; G)$.

A connected, closed, orientable manifold M is a particularly nice topological space, one of whose special properties is that there is a natural number n (the **real dimension** of M) such that $H^n(M; G) \cong G$ and $H^i(M; G) = 0$ for all $i > n$. In the case that $G = \mathbb{Z}$, $H^n(M; \mathbb{Z})$ has two possible generators, a choice of which we will call a **fundamental class** and denoted μ_M. Like the above notion of characteristic numbers of polynomials, if we have a cohomology class $k = k_0 + k_1 + \cdots + k_n$ where $k_i \in H^i(M; \mathbb{Z})$ and a partition $\pi = (j_1, \ldots, j_l)$, we can form

$$k[\pi] := k_{j_1} \ldots k_{j_l} = a\mu_M \in H^n(M; \mathbb{Z}) \text{ for some } a \in \mathbb{Z}$$

We "confuse notation" and let $k[\pi]$ also denote **characteristic number** $a = k[\pi][\mu_M]$. The characteristic number depends on the choice of fundamental class of the orientable manifold M as well as the order of the product, but only up to sign.

If M is a smooth manifold, its tangent bundle induces additional cohomological information called **characteristic classes**. In this project we are only concerned with three types:

1. the **Stiefel-Whitney** classes $w_i(M) \in H^i(M; \mathbb{Z}/2)$ are defined for all manifolds
2. the **Chern** classes $c_i(M) \in H^{2i}(M; \mathbb{Z})$ are defined for complex manifolds
3. the **Pontryagin** classes $p_i(M) \in H^{4i}(M; \mathbb{Z})$ are defined for all manifolds.

In particular if M has real dimension n then it has Stiefel-Whitney classes $w_0, \ldots, w_n$ and Pontryagin classes $p_0, \ldots, p_{\lfloor \frac{n}{4} \rfloor}$; if M is complex then n must be even and M also has Chern classes $c_0, \ldots, c_{\frac{n}{2}}$.

There are various relations between these characteristic classes. The ones we use in this project concern the "mod-2 reduction" map

$$\rho_2 \colon H^i(M;\mathbb{Z}) \to H^i(M;\mathbb{Z}/2)$$

and they state that (1) $\rho_2(c_i(M)) = w_{2i}(M)$, and that (2) $\rho_2(p_i(M)) = w_{2i}(M)^2$.

For the central example in this project, the **complex projective space** $\mathbb{CP}^n$ for $n \in \mathbb{N}$, cohomology and characteristic classes have very simple expressions. The reader can take this information as given, but there are many excellent sources out there in which to find this information.

The integral and mod-2 cohomology of $\mathbb{CP}^n$ are given by

$$H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[u]/\langle u^n + 1\rangle \text{ where } u \text{ has degree 2.}$$

$$H^*(\mathbb{CP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[\tilde{u}]/\langle \tilde{u}^n + 1\rangle \text{ where } \tilde{u} = \rho_2(u).$$

As a graded abelian group, $H^i(\mathbb{CP}^n; A) = A$ if i is even and $0 \leq i \leq 2n$ and 0 otherwise for any abelian group A.

The Chern class is of $\mathbb{CP}^n$ is

$$c(\mathbb{CP}^n) = (1+u)^{n+1}$$

so the homogenous terms are given by $c_i(\mathbb{CP}^n) = \binom{n+1}{i} u^i \in H^{2i}(\mathbb{CP}^n;\mathbb{Z})$.

Since $w = \rho_2(c)$ the Stiefel-Whitney class is given by

$$w(\mathbb{CP}^n) = (1+\tilde{u})^{n+1}$$

and $w_i(\mathbb{CP}^n) = \binom{n+1}{i}\tilde{u}^i \in H^i(\mathbb{CP}^n;\mathbb{Z}/2)$. Lastly, the Pontryagin class is

$$p(\mathbb{CP}^n) = (1+u^2)^{n+1}$$

and so $p_i(\mathbb{CP}^n) = \binom{n+1}{i} u^{2i} \in H^{4i}(\mathbb{CP}^n;\mathbb{Z})$.

Cohomology and characteristic classes both have product rules when computing their values for $M \times N$.

This project combines examples using products, and to formulate the product rule for chomology we need to mention some **Homological Algebra**. For R a ring, A a right R-module, and B a left R-module, we can form their **tensor product** $A \otimes_R B$ which is very much like the tensor product from linear algebra and encompasses the tensor product of polynomial rings, and Homological Algebra provides a construction for the somewhat more obtuse **Tor groups** $\mathrm{Tor}_n^R(A,B)$. If A and B are **graded** R modules then the tensor product and Tor groups also inherit gradings. For this project we won't need to know the intricacies of how these objects work, just their values in some very special cases.

If R is a commutative ring and A is any R-module, then for all spaces X the cohomology groups $H^*(X;A)$ form a graded R-module. Given two spaces X and Y we can thus form the tensor product and Tor groups of their cohomology rings. There is a **cross product** map

$$\times \colon H^*(X;A) \otimes_R H^*(Y;A) \to H^*(X \times Y;A)$$

which is not always an isomorphism, but under some finite-type assumptions on X and Y we have the **Künneth Exact Sequence**:

$$0 \to H^*(X;A) \otimes_R H^*(Y;A) \to H^*(X \times Y;A) \to \Sigma\,\mathrm{Tor}_1^R(H^*(X;A), H^*(Y;A)) \to 0$$

where Σ means that the grading on the Tor group is shifted up by one. This sequence is easiest to apply when the Tor term vanishes outright, such as the case where the cohomology of X or Y is a free R-module. In particular this is satisfied for products of complex projective spaces since $H^*(\mathbb{CP}^n;R)$ is a free R module for any commutative ring R, so

$$H^*(\mathbb{CP}^n \times \mathbb{CP}^m;R) \cong H^*(\mathbb{CP}^n;R) \otimes_R H^*(\mathbb{CP}^m;R) \cong R[u,v]/\langle u^{n+1}, v^{m+1}\rangle$$

The product rule for our characteristic classes, the so-called **Whitney Product Formula**, is now easy to state. For the Stiefel-Whitney and Chern classes (if they are defined) we have simply

$$w(M \times N) = w(M) \times w(N) \in H^*(M \times N;\mathbb{Z}/2)$$
$$c(M \times N) = c(M) \times c(N) \in H^*(M \times N;\mathbb{Z})$$

For Pontryagin classes, the same is true if there is no 2-torsion present:

$$2(p(M \times N) - p(M) \times p(N)) = 0 \in H^*(M \times N;\mathbb{Z})$$

In terms of homogeneous classes, this is expressed this as

$$w_i(M \times N) = \sum_{k=0}^{i} w_k(M) \times w_{i-k}(N)$$

and similarly for the other classes, but we don't use this expression in this project.

Now we move to the key example, a product of complex projective spaces $\mathbb{CP}^n \times \mathbb{CP}^m$.

Since $H^*(\mathbb{CP}^n;\mathbb{Z})$ is a free $\mathbb{Z}$ module the Tor term vanishes in the Künneth Sequence, so if v denotes the variable in the cohomology ring of $\mathbb{CP}^m$ we get

$$H^*(\mathbb{CP}^n \times \mathbb{CP}^m;\mathbb{Z}) \cong \mathbb{Z}[u,v]/\langle u^{n+1}, v^{m+1}\rangle$$

and similarly

$$H^*(\mathbb{CP}^n \times \mathbb{CP}^m;\mathbb{Z}/2) \cong \mathbb{Z}/2[\tilde{u},\tilde{v}]/\langle \tilde{u}^{n+1}, \tilde{v}^{m+1}\rangle$$

We then have that the Chern class is $c(\mathbb{CP}^n \times \mathbb{CP}^m) = (1+u)^{n+1}(1+v)^{m+1}$, the Stiefel-Whitney class is $w(\mathbb{CP}^n \times \mathbb{CP}^m) = (1+\tilde{u})^{n+1}(1+\tilde{v})^{m+1}$, and the Pontryagin class is $p(\mathbb{CP}^n \times \mathbb{CP}^m) = (1+u^2)^{n+1}(1+v^2)^{m+1}$.

Let's take the concrete example of $\mathbb{CP}^2 \times \mathbb{CP}^3$, with $H^*(\mathbb{CP}^2 \times \mathbb{CP}^3;\mathbb{Z}) \cong \mathbb{Z}[u,v]/\langle u^3, u^4\rangle$. It is a complex manifold with complex dimension 5, hence with real dimension 10. Therefore it potentially has Stiefel-Whitney classes $w_0, \ldots, w_{10}$, Chern classes $c_0, \ldots, c_5$, and Pontryagin classes p_0, p_1, and p_2. Then the Whitney product formula tells us $c(\mathbb{CP}^2 \times \mathbb{CP}^3) = (1+u)^3(1+v)^4$, which if expended and collected into homogeneous terms gives

$$c_0 = 1$$

$$c_1 = 3u + 4v$$
$$c_2 = 3u^2 + 12uv + 6v^2$$
$$c_3 = 12u^2v + 18uv^2 + 4v^3$$
$$c_4 = 18u^2v^2 + 12uv^3$$
$$c_5 = 12u^2v^3$$

Reducing mod 2, we find $w(\mathbb{CP}^2 \times \mathbb{CP}^3) = 1 + \tilde{u} + \tilde{u}^2$, i.e.

$$w_0 = 1$$
$$w_2 = \tilde{u}$$
$$w_4 = \tilde{u}^2$$

and all other Stiefel-Whitney classes vanish.

Lastly, $p(\mathbb{CP}^2 \times \mathbb{CP}^3) = (1 + u^2)^3(1 + v^2)^4$ so

$$p_0 = 1$$
$$p_1 = 3u^2 + 4v^2$$
$$p_2 = 12u^2v^2$$

3. Characteristic Numbers

If we apply our characteristic numbers recipe to these characteristic classes, we get the **Stiefel-Whitney**, **Chern**, and **Pontryagin numbers** respectfully.

Given a partition π of 5 we get a Chern number $c[\pi]$ of $\mathbb{CP}^2 \times \mathbb{CP}^3$. For example, if $\pi = (2, 3)$ compute $c_2c_3 = (3u^2 + 12uv + 6v^2)(12u^2v + 18uv^2 + 4v^3) = 218u^2v^3$, therefore $c[\pi] = 218$. Each partition of 10 gives us a Stiefel-Whitney number, but they all vanish. Since the real dimension of $\mathbb{CP}^2 \times \mathbb{CP}^3$ isn't divisible by 4 there are no Pontryagin numbers.

Exercise: compute the Stiefel-Whitney, Chern, and Pontryagin numbers of the 4-manifold $\mathbb{CP}^2 \times \mathbb{CP}^2$. Check your Pontryagin classes by verifying Hirzebruch's Signature Formula in this case:

$$\frac{7p[2] - p[1, 1]}{45} = 1$$

According to this project, a **manifold** is either $\mathbb{CP}^n$ for some $n \geq 0$ or a **product** of manifolds. The cohomology and characteristic classes of $\mathbb{CP}^n$ are determined using the standard formulae, and for products we can just use the tensor product of cohomology rings and cross product of characteristic classes since all of the integral cohomologies involved are free $\mathbb{Z}$-modules.