# 1 Boundary Conditions

The `pdepe` function requires that boundary conditions be specified at both the left and right ends of the region. In discussing boundary conditions, it is convenient to denote variables at the left end with the subscript $L$ and at the right end with the subscript $R$. Constraints for `pdepe` are described with the following equation

$$p(x,t,u) + q(x,t)f\left(x,t,u,\frac{\partial u}{\partial x}\right) = 0 \tag{1}$$

As part of the problem definition, a `MATLAB` function must be supplied that returns $p_L, q_L, p_R, q_R$. The value of the flux function, $f$, will be calculated by `pdepe` at the two ends. The specific definition of the flux depends on the particular PDE being solved. Its value at any point $x$ and time, $t$, is defined in the `pdefun` function and returned as the second value from that function. With these values, equation (1) can be rewritten explicitly at the two ends

$$p_L + f_L q_L = 0 \tag{2}$$

$$p_R + f_R q_R = 0 \tag{3}$$

The `pdepe` algorithm will insure that these constraint equations are satisfied.

## 1.1 Specifying the Values of $p$ and $q$

In partial differential equation theory, the different types of boundary conditions are often designated as Dirichlet, Neumann, and Robin. Basically, a Dirichlet condition specifies the value of the dependent variable $u$ at a boundary, a Neumann condition specifies $\partial u/\partial x$, and a Robin condition specifies a linear combination of the two.

We will consider these different cases in a specific example, the classical heat equation

$$\rho C_p \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) \tag{4}$$

where $u$ is the temperature, $\rho$ is the density, $C_p$ is the specific heat, and $k$ is the thermal conductivity.

### 1.1.1 Dirichlet Conditions at Both Ends

We consider an example where we want to specify $u$ at the left end to be equal zero and $u$ at the right end equal to ten. That is, we want $u_L = 0$ and $u_R = 10$. A simple comparison with equations (2) and (3) reveals that the flux variables don't enter into these particular boundary conditions so $q_L$ and $q_R$ will be set to zero. Setting $p_L = u_L$ and $p_R = u_R - 10$ puts the two boundary conditions in the form required by (2) and (3).

Writing the required MATLAB bcfun is straightforward as shown below.

```
function [p_L,q_L,p_R,q_R] = heatEqnBCFunction(x_L,u_L,x_R,u_R,t)
p_L = u_L;
q_L = 0;
p_R =u_R-10;
q_R = 0;
end
```

### 1.1.2 Dirichlet Conditions At Right End, Neumann Condition At Left

We consider an example where we want to specify $u$ at the left end to be equal zero and $\partial u/\partial x$ at the right end equal to three. Obviously the condition at the left end is identical to the previous example. At first glance, the required condition at the right end doesn't appear to match the required form in equation (3). But consider the flux function for the PDE, equation (4)

$$f\left(x, t, u, \frac{\partial u}{\partial x}\right) = k\frac{\partial u}{\partial x} \tag{5}$$

The Neumann boundary condition can be written as

$$3 - \frac{\partial u}{\partial x} = 0 \tag{6}$$

Substituting $\partial u/\partial x$ from equation (5) yields

$$3k - (1)f_R = 0; \tag{7}$$

where, again, $p_R$ and $q_R$ can easily be identified.

The following lines replace the comparable two in the function heatEqnBCFunction

```
p_R = 3*k;
q_R = -1;
```

It might seem that the developers of pdepe simply made it unnecessarily difficult for users to specify Neumann boundary conditions with the particular form they chose for the boundary condition function. However, it is worthwhile to consider the origin of many PDE in mathematical physics. Many PDE originate from a so-called "conversation law" where a particular quantity in a region changes based on how much flows in and how much flows out of the region. The flow of the particular quantify is its flux. In these cases it is much more typical to specify boundary conditions on the flux rather than simply the derivative of the dependent variables. For example, it is common to specify the heat flux entering a boundary or that at an insulated boundary the heat flux is zero. Thus, writing the boundary condition equation in terms of flux is often more natural.

### 1.1.3 Dirichlet Conditions At Right End, Robin Condition At Left

In this example, we will again specify $u$ at the left end to be equal zero but now we will specify $4u + 3\partial u/\partial x = 7t^2$ at the right end. This is a Robin boundary condition that includes terms that are functions of the dependent variable, $u$, and time, $t$. Following the same procedure as in the last section, we write the Robin boundary condition in the form of equation (1) and then substitute from the definition of flux for this PDE, equation (5)

$$4u - 7t^2 + \frac{3}{3}f = 0 \tag{8}$$

Then function `heatEqnBCFunction` is modified with the following two lines

```
p_R = 4*u_R - 7*t^2;
q_R = 3/k;
```

### 1.1.4 PDE Systems

The examples above have all been for a single PDE. However `pdepe` supports an arbitrary number of differential equations, i.e. a system of coupled partial differential equations. In this case, boundary conditions must be specified at both ends for each PDE in the system; the $p$ and $q$ variables are vectors with lengths equal to the number of PDE in the system. The boundary conditions for each equation are treated just like the single-PDE cases described above.