Supplementary material

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I. Proof of Model Equivalence

Given hydraulic equations

$$
\begin{aligned}
& H_j - H_k = K_i \dot{m}_i^2 \operatorname{sign}(\dot{m}_i), \\
& \quad \text{where } j,k \text{ are the endpoints of pipe } i,\ i \in \mathbb{E}, && \text{(1a)} \\
& \sum_{i\in\mathbb{E}_j^{\text{in}}} \dot{m}_i - \sum_{i\in\mathbb{E}_j^{\text{out}}} \dot{m}_i - \dot{m}_j^{\text{in}} = 0, \quad j \in \mathbb{V}, && \text{(1b)} \\
& H_i - H_i^{\text{set}} = 0, \quad i \in V_{\text{slack}}, && \text{(1c)}
\end{aligned}
$$

and the equivalent optimization model

$$
\begin{aligned}
\min_{\dot{m}_i,\, \dot{m}_i^{\text{in}}} \quad & \mathbf{z} = \sum_{i\in\mathbb{E}} \frac{K_i |\dot{m}_i|^3}{3} - \sum_{i\in V_{\text{slack}}} H_i^{\text{set}} \dot{m}_i^{\text{in}} \\
\text{s.t.} \quad & \sum_{i\in\mathbb{E}_j^{\text{in}}} \dot{m}_i - \sum_{i\in\mathbb{E}_j^{\text{out}}} \dot{m}_i = \dot{m}_j^{\text{in}}, \quad j \in \mathbb{V}.
\end{aligned} \tag{2}
$$

The following proves that (2) is a convex optimization model and that its global optimal solution satisfies the hydraulic equations (1) under leakage fault scenarios.

1) Convexity

Since the constraints in (2) are linear, according to the definition of convex optimization, we only need to prove that the objective function is convex. We have:

$$
\frac{\partial \mathbf{z}}{\partial \dot{m}_i} = K_i |\dot{m}_i|^2 \cdot \operatorname{sign}(\dot{m}_i)
$$

$$
\frac{\partial^2 \mathbf{z}}{\partial \dot{m}_i^2} = 2K_i |\dot{m}_i| \cdot (\operatorname{sign}(\dot{m}_i))^2 \geq 0,
$$

$$
\frac{\partial^2 \mathbf{z}}{\partial \dot{m}_i \partial \dot{m}_j} = \frac{\partial^2 \mathbf{z}}{\partial \dot{m}_j \partial \dot{m}_i} = 0,
$$

$$
\frac{\partial \mathbf{z}}{\partial \dot{m}_i^{\text{in}}} = H_i^{\text{set}}, \quad \frac{\partial^2 \mathbf{z}}{\partial (\dot{m}_i^{\text{in}})^2} = 0.
$$

Therefore, the Hessian matrix $\nabla^2 \mathbf{z}$ of the objective function is a positive semidefinite matrix with non-negative diagonal elements, making the objective function convex.

2) Equivalence between global optimal solution and power flow solution

Construct the Lagrangian function of model (2):

$$
\begin{aligned}
L(\dot{m}) = & \sum_{i\in\mathbb{E}} \frac{K_i |\dot{m}_i|^3}{3} - \sum_{i\in V_{\text{slack}}} H_i^{\text{set}} \dot{m}_i^{\text{in}} \\
& - \sum_{j\in V} \lambda_j \left(\sum_{i\in\mathbb{E}_j^{\text{in}}} \dot{m}_i - \sum_{i\in\mathbb{E}_j^{\text{out}}} \dot{m}_i - \dot{m}_j^{\text{in}} \right).
\end{aligned} \tag{3}
$$

According to the KKT conditions, the global optimal solution $\dot{m}$ of optimization model (2) and its corresponding Lagrange multipliers λ_i must satisfy:

$$
\begin{aligned}
& \frac{\partial L(\dot{m})}{\partial \dot{m}_i} = K_i \dot{m}_i^2 \operatorname{sign}(\dot{m}_i) - \lambda_j + \lambda_k = 0, \\
& \qquad \text{where } j,k \text{ are the endpoints of pipe } i,\ i \in \mathbb{E},
\end{aligned} \tag{4}
$$

$$
\frac{\partial L(\dot{m})}{\partial \dot{m}_i^{\text{in}}} = -H_i^{\text{set}} + \lambda_i = 0, \quad i \in V_{\text{slack}}, \tag{5}
$$

$$
\frac{\partial L(\dot{m})}{\partial \lambda_j} = \sum_{i\in\mathbb{E}_j^{\text{in}}} \dot{m}_i - \sum_{i\in\mathbb{E}_j^{\text{out}}} \dot{m}_i - \dot{m}_j^{\text{in}}, \quad j \in \mathbb{V}. \tag{6}
$$

Let $H_i \triangleq \lambda_i$. Then (4)-(6) can be rewritten as (1). Therefore, the global optimal solution of the convex optimization model (2) must satisfy the hydraulic equations (1) considering multiple slack nodes.

II. Definition of the Minmod function

$$
\operatorname{minmod}(\chi_1, \chi_2, \chi_3) = \begin{cases} \min(\chi_1, \chi_2, \chi_3) & \text{if } \forall \chi_i > 0, \\ \max(\chi_1, \chi_2, \chi_3) & \text{if } \forall \chi_i < 0, \\ 0 & \text{otherwise.} \end{cases}
$$

$$
\begin{aligned}
\chi_1 &= \frac{T_j(0) - T_{j-1}(0)}{\Delta x}, \\
\chi_2 &= \frac{T_{j+1}(0) - T_{j-1}(0)}{2\Delta x}, \\
\chi_3 &= \frac{T_{j+1}(0) - T_j(0)}{\Delta x}.
\end{aligned} \tag{7}
$$

III. Barry Island Case

The diagram of Barry Island case can be found in Fig.1.

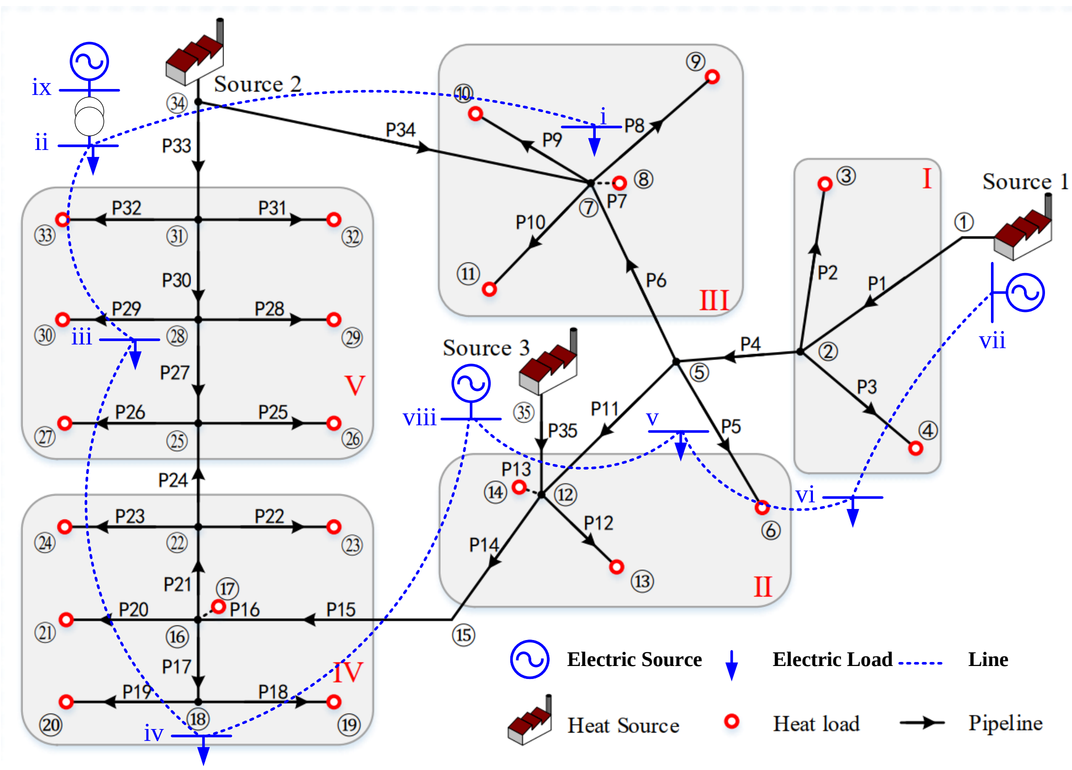

Fig. 1. Diagram of Barry Island.