Midterm 11a.m.

STAT 302
Winter Session 2020/2021 - Term 1

QUESTIONS:

Question 1

In a corporate office in Vancouver, employee ID numbers are formed by 5 letters from the alphabet followed by3 digits from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Note that a digit or a letter can only appear once in an employee ID.
If an employee ID number is generated at random, what is the probability (rounded to 3 decimal places) that it contains no even numbers and no vowels?
Note that there are 26 letters in the alphabet, including 5 vowels.

Question 2

Let A and B be two independent events in the sample space Ω. Which of the following two statements are true? Select all that apply.
Note: Assume that $0 < P(A) < 1$ and $0 < P(B) < 1$.

1. The events A and B^c are independent.
2. The events $A \cap B^c$ and B are independent.

Question 3

Suppose that, in a certain part of the world, in any 50-year period the probability of a major plague is 0.39, the probability of a major famine is 0.52, and the probability of both a plague and a famine is 0.15. What is the probability that there was a famine in the last fifty years given that there was a plague during this period?

Question 4

Consider a sequence of independent tosses of a biased coin. On each toss, the probability of tossing a head is 2/3. A game consists of 10 coin tosses, and \$10 are paid out each time that at least 8 tosses are heads. It costs\$4 to play the game.
What is the variance of the winnings of a player in this game?

Question 5

Suppose that $P(A) = 0.20$, $P(B|A) = 0.50$ and $P(B|A^c) = 0.25$.
What is $P(A|B)$?

Question 6

Suppose U is a discrete uniform random variable, taking equal probability 1/5 for the integers 1, 2, 3, 4 and 5. Let $X = e^U$.
(a) Find the probability density function $f_X(x)$ and the cumulative distribution function $FX(x)$.
(b) Calculate the mean and variance of X.

Question 7

The number of calls coming per minute into a hotel reservation center is a Poisson random variable with mean 3. Consider the number of calls in a 2-minute period.
(a) Find the probability that at least four calls will come given that the number of calls in this 2-minute period is known to be at least one.
(b) The cost for each phone call at the hotel reservation center is at \$0.15. What is the variance of the cost for a 2-minute period if it is known that the number of calls in this 2-minute period is at most four.

SOLUTIONS:

Question 1

$\frac{P_{21}^5 P_5^3}{P_{26}^5 P_9^3} = 0.037$

Question 2

$P(A \cap B^c \cap B) = P(\phi) = 0$ but $P(A \cap B^c)P(B)$ does not necessarily have to be 0. $P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c)$. So A and B^c are independent.

Question 3

Define $A = \{\text{Plague in any 50-year period}\}$, $B = \{\text{Famine in any 50-year period}\}$. We have $P(A) = 0.39$, $P(B) = 0.52$ and $P(A \cap B) = 0.15$. So $P(B|A) = \frac{P(A \cap B)}{P(B)} = \frac{0.15}{0.39} = \frac{5}{13}$.

Question 4

Let X be the number of heads and W be winning. $P(win) = P(X \geq 8) = 0.299$. So $E(W) = 6(0.299) - 4(1 - 0.299) = -1$. $E(W^2) = 22$. So $Var(W) = E(W^2) - (E(W))^2 = 21$.

Question 5

$P(A \cap B) = P(B|A)P(A) = 0.1$. $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = 0.1 + 0.25(0.8) = 0.3$. So $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{3}$.

Question 6

(a) The range of X is $\{e, e^2, e^3, e^4, e^5\}$. $f_X(e) = f_x(e^2) = f_x(e^3) = f_x(e^4) = f_x(e^5) = \frac{1}{5}$.

$$F_X(x) = \begin{cases} 0 & \text{if } x < e \\ \frac{1}{5} & \text{if } e \leq x < e^2 \\ \frac{2}{5} & \text{if } e^2 \leq x < e^3 \\ \frac{3}{5} & \text{if } e^3 \leq x < e^4 \\ \frac{4}{5} & \text{if } e^4 \leq x < e^5 \\ 1 & \text{if } x \geq e^5 \end{cases}$$

(b)

$$\begin{aligned} E(X) &= \sum_{k=1}^{5} \frac{1}{5} e^k \\ &= \frac{1}{5}(e + e^2 + e^3 + e^4 + e^5) \end{aligned}$$

$$E(X^2) = \frac{1}{5}(e^2 + e^4 + e^6 + e^8 + e^{10})$$

$$Var(X) = E(X^2) - (E(X))^2$$

Question 7

(a) Let X be the number of calls in a 2-minute period. $\lambda = 3$, $t = 2$, $X \sim Poi(\lambda t) = Poi(6)$.

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - e^{-6} \\ &= 0.9975 \end{aligned}$$

$$\begin{aligned} P(X \geq 4) &= 1 - P(X = 0) - P(X = 1) - P(X = 2) - P(X = 3) \\ &= 1 - e^{-6} - 6e^{-6} - \frac{6^2 e^{-6}}{2} - \frac{6^3 e^{-6}}{3!} \\ &= 0.8488 \end{aligned}$$

$$\begin{aligned} P(X \geq 4 | X \geq 1) &= \frac{P(\{X \geq 4\} \cap \{X \geq 1\})}{P(X \geq 1)} \\ &= \frac{P(X \geq 4)}{P(X \geq 1)} \\ &= 0.8509 \end{aligned}$$

(b) Let Y be the cost for a 2-minute period. The range of Y is $\{0, 0.15, 0.3, 0.45, 0.6\}$.

$$P(X \leq 4) = 1 - P(X \geq 4) + P(X = 4) = 0.2851$$

$$P(Y = 0) = P(X = 0 | X \leq 4) = \frac{P(X = 0)}{P(X \leq 4)} = \frac{e^{-6}}{0.2851} = 0.0087$$

$$P(Y = 0.15) = P(X = 1 | X \leq 4) = \frac{P(X = 1)}{P(X \leq 4)} = \frac{6e^{-6}}{0.2851} = 0.0522$$

$$P(Y = 0.3) = P(X = 2 | X \leq 4) = ... = 0.1565$$

$$P(Y = 0.45) = P(X = 3 | X \leq 4) = ... = 0.3130$$

$$P(Y = 0.6) = P(X = 4 | X \leq 4) = ... = 0.4695$$

$$\begin{aligned} E(Y) &= \sum_{k=0}^{4} 0.15k P(Y = 0.15k) \\ &= 0.4773 \end{aligned}$$

$$\begin{aligned} E(Y^2) &= \sum_{k=0}^{4} (0.15k)^2 P(Y = 0.15k) \\ &= 0.2477 \end{aligned}$$

$$Var(Y) = E(Y^2) - (E(Y))^2 = 0.0198$$