Calculus I (Math 2554)

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The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

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∮3.10 Related Rates

In this section, we use our knowledge of derivatives to examine how variables change with respect to time. The prime feature of these problems is that two or more variables, which are related in a known way, are themselves changing in time. The goal of these types of problems is to determine the rate of change (i.e., the derivative) of one or more variables at a specific moment in time.

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Question

The edges of a cube increase at a rate of 2 cm/sec. How fast is the volume changing when the length of each edge is 50 cm?

Variables:

$$V = \text{volume (cm}^3\text{)}$$

$$x = \text{length of an edge (cm)}$$

Relations:

$$V = x^3 \text{ cm}^3$$

Rates Known:

$$\frac{dx}{dt} = 2 \text{ cm/sec}$$

Want to Find:

$$\left.\frac{dV}{dt}\right|_{x=50 \text{ cm}}$$

Both V and x are functions of t (their respective sizes are dependent upon how much time has passed), where t is in seconds.

We can write $V(t) = x(t)^3$ and then differentiate with respect to t:

$$V'(t) = 3x(t)^2 x'(t) \quad \text{or,}$$
$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

Note that $x = x(t)$ is the length of the cube's edges at time t, and $\frac{dx}{dt} = x'(t)$ is the rate at which the edges are changing at time t.

So the rate of change of the volume when $x = 50$ cm is

$$\left.\frac{dV}{dt}\right|_{x=50} = 3 \cdot 50^2 \cdot 2 = \boxed{15\,000 \text{ cm}^3/\text{sec.}}$$

Steps for Solving Related Rates Problems

1. Read the problem carefully, making a sketch to organize the given information. Identify the rates that are given and the rate that is to be determined.
2. Write one or more equations that express the basic relationships among the variables.
3. Introduce rates of change by differentiating the appropriate equation(s) with respect to time t.
4. Substitute known values and solve for the desired quantity.
5. Check that the units are consistent and the answer is reasonable.

The Jet Problem

Question

A jet ascends at a 10° angle from the horizontal with an airspeed of 550 mph (i.e., its speed along its line of flight is 550 mph).

(a) How fast is the altitude of the jet increasing?

(b) If the sun is directly overhead, how fast is the shadow of the jet moving on the ground?

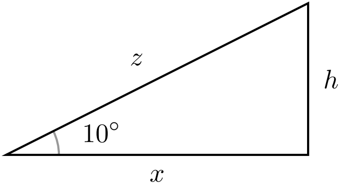
Step 1: There are three variables:

$x =$ distance the shadow has traveled (miles)

$h =$ altitude of the jet (miles)

$z =$ distance the jet has traveled on its line of flight (miles)

These variables are related through a right triangle:



The rate we know is

$$\frac{dz}{dt} = 550 \text{ mph}$$

and the rates we want to find are

$$\frac{dx}{dt} \quad \text{and} \quad \frac{dh}{dt}.$$

Step 2: To answer part (a), how fast the altitude is increasing, we need an equation involving only h and z. Using trigonometry,

$$\sin(10^\circ) = \frac{h}{z} \implies h = \sin(10^\circ) \cdot z.$$

To answer part (b), how fast the shadow is moving, we need an equation involving only x and z. Using trigonometry,

$$\cos(10^\circ) = \frac{x}{z} \implies x = \cos(10^\circ) \cdot z.$$

Step 3: We can now differentiate each equation to answer each question:

$$h = \sin(10^\circ) \cdot z \implies \frac{dh}{dt} = \sin(10^\circ)\frac{dz}{dt}$$
$$x = \cos(10^\circ) \cdot z \implies \frac{dx}{dt} = \cos(10^\circ)\frac{dz}{dt}$$

Step 4: We know that $\dfrac{dz}{dt} = 550$ mph. So

$$\frac{dh}{dt} = \sin(10^\circ) \cdot 550 \approx 95.5 \text{ mph}$$
$$\frac{dx}{dt} = \cos(10^\circ) \cdot 550 \approx 541.6 \text{ mph}$$

Step 5: Because both answers are in terms of miles per hour and both answers seem reasonable within the context of the problem, we conclude that the jet is gaining altitude at a rate of 95.5 mph, while the shadow on the ground is moving at about 541.6 mph.

Exercise

The sides of a cube increase at a rate of R cm/sec. When the sides have a length of 2 cm, what is the rate of change of the volume?

Exercise

Two boats leave a dock at the same time. One boat travels south at 30 mph and the other travels east at 40 mph. After half an hour, how fast is the distance between the boats increasing?

3.10 Book Problems

5-12, 14-15, 17-18, 30-31

$\oint$ 4.1 Maxima and Minima

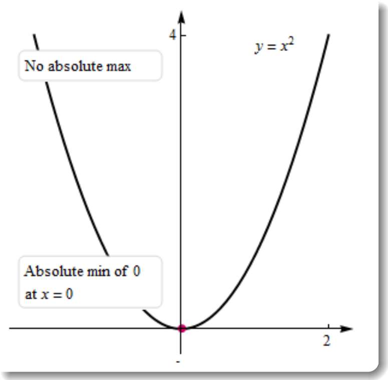
Definition

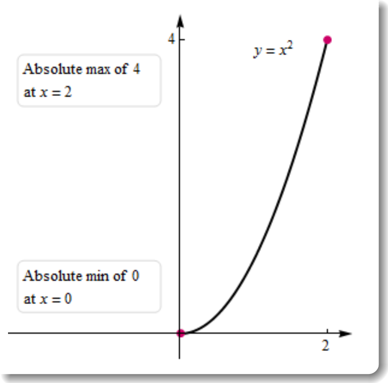
Let f be defined on an interval I containing c.

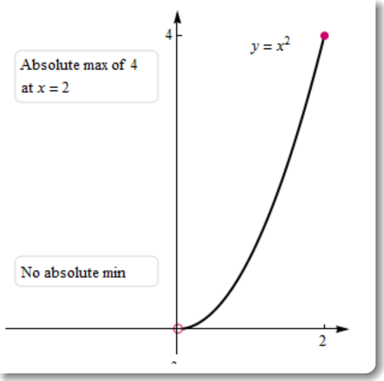
(a) f has an **absolute maximum** value on I at c means $f(c) \geq f(x)$ for every x in I.

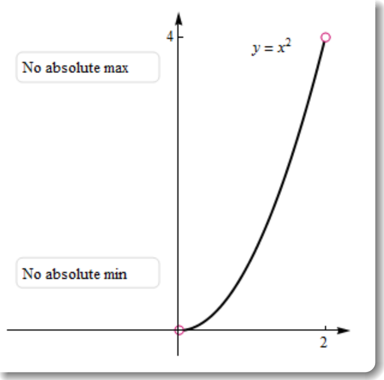
(b) f has an **absolute minimum** value on I at c means $f(c) \leq f(x)$ for every x in I.

The existence and location of absolute extreme values depend on the function and the interval of interest:









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Extreme Value Theorem

Theorem (Extreme Value Theorem)

A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.

The EVT provides the criteria to ensure the existence of absolute extrema:

- the function must be continuous on the interval of interest;
- the interval of interest must be closed and bounded.

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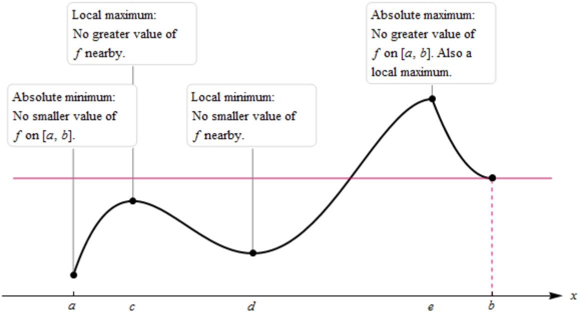
Local Maxima and Minima

Beyond absolute extrema, a graph may have a number of peaks and dips throughout its interval of interest:



Definition (local extrema)

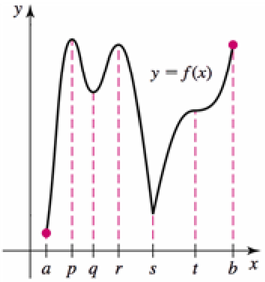
Suppose I is an interval on which f is defined and c is in I.

(a) $f(c)$ is a **local maximum** value of f means there is some open interval within I and containing c, where $f(c) \geq f(x)$ for all x.

(b) $f(c)$ is a **local minimum** value of f means there is some open interval within I and containing c, where $f(c) \leq f(x)$ for all x.

Exercise

Use the graph below to identify the points on the interval $I = [a, b]$ at which local and absolute extreme values occur.



Critical Points

Question

Based on the previous graph, how is the derivative related to where the local extrema occur?

Answer: Local extrema occur where the derivative either does not exist or is equal to 0.

Definition

An interior point c of the domain of f at which $f'(c) = 0$ or $f'(c)$ fails to exist is called a **critical point** of f.

Local Extreme Point Theorem

Theorem (Local Extreme Point Theorem)

If f has a local minimum or maximum value at c and $f'(c)$ exists, then $f'(c) = 0$. **(Converse is not true!)**

It is possible for $f'(c) = 0$ or $f'(c)$ not to exist at a point, yet the point not be a local min or max.

Question

What is an example?

Critical points only provide candidates for local extrema; they do not guarantee that the points are local extrema.

Locating Absolute Min and Max

Two facts help us in the search for absolute extrema:

- Absolute extrema in the interior of an interval are also local extrema, which occur at critical points of f.
- Absolute extrema may occur at the endpoints of f.

Assume that the function f is continuous on $[a, b]$. To find absolute extrema, use the following procedure:

1. Locate the critical points c in (a, b), where $f'(c) = 0$ or $f'(c)$ does not exist. Again, these points are candidates for absolute extrema.
2. Evaluate f at the critical points and at the endpoints of $[a, b]$.
3. Choose the largest and smallest values of f from 2. for the absolute max and min values, respectively.

Exercise

Given $f(x) = (x+1)^{4/3}$ on $[-8, 8]$, determine the critical points and the absolute extreme values of f.

4.1 Book Problems

11-25 (odds), 31-45 (odds)

- **Note:** So far, we only know how to find absolute extrema from an equation. Techniques for locating local extrema, given an equation, come in later sections.

$\oint$ 4.2 What Derivatives Tell Us

Definition

Suppose a function f is defined on an interval I.

(a) f is **increasing** on I means for any two points x_1, x_2 in I, with $x_2 > x_1$, we have $f(x_2) > f(x_1)$.

(b) f is **decreasing** on I means for any two points x_1, x_2 in I, with $x_2 > x_1$, we have $f(x_1) > f(x_2)$.

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We can rephrase the definition from the previous slide in the following way:

Definition

Suppose f is continuous on an interval I and differentiable at every interior point of I.

(a) f is **increasing** on I means $f'(x) > 0$ for all interior points of I.

(b) f is **decreasing** on I means $f'(x) < 0$ for all interior points of I.

Example

Sketch a function that is continuous on $(-\infty, \infty)$ that has the following properties:

- $f'(-1)$ is undefined;
- $f'(x) > 0$ on $(-\infty, -1)$;
- $f'(x) < 0$ on $(-1, \infty)$.

Example

Find the intervals on which

$$f(x) = 3x^3 - 4x + 12$$

is increasing and decreasing.

First Derivative Test

The **First Derivative Test** is used to find local extrema.

Suppose that f is continuous on an interval that contains a critical point c and assume f is differentiable on an interval containing c, except perhaps at c itself.

- If f' changes sign from positive to negative as x increases through c, then f has a **local maximum** at c.
- If f' changes sign from negative to positive as x increases through c, then f has a **local minimum** at c.
- If f' does not change sign at c (from positive to negative or vice versa), then f has no local extreme value at c.

Exercise

If $f(x) = 2x^3 + 3x^2 - 12x + 1$, identify the critical points on the interval $[-3, 4]$, and use the First Derivative Test to locate the local maximum and minimum values. What are the absolute max and min?

Note: Again, the First Derivative Test does NOT test for increasing/decreasing, only local max/min. Use it on critical points.

Absolute Extremes on any Interval

The Extreme Value Theorem (recall, from Section 4.1) stated that we were guaranteed extreme values only on closed intervals.

However: Suppose f is continuous on an interval I that contains only one local extremum, at $(x =)c$.

- If it is a local minimum, then $f(c)$ is the absolute minimum of f on I.
- If it is a local maximum, then $f(c)$ is the absolute maximum of f on I.

What the Derivative of the Derivative Tells Us

Definition (concavity)

Let f be differentiable on an open interval I.

- f is **concave up** on I means f' is increasing on I.
- f is **concave down** on I means f' is decreasing on I.

Definition

Suppose f is continuous at c and f changes concavity at $x = c$ (from up to down, or vice versa). The point c is called an **inflection point**.

Test for Concavity

Suppose that f'' exists on an interval I.

- If $f'' > 0$ on I, then f is concave up on I.
- If $f'' < 0$ on I, then f is concave down on I.
- If c is a point of I at which $f''(c) = 0$ and f'' changes signs at c, then f has an inflection point at c.

In other words, just as the first derivative f' told us whether the function f was increasing or decreasing, the second derivative f'' tells us whether f' is increasing or decreasing.

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Second Derivative Test

Just like with the First Derivative Test, the **Second Derivative Test** locates local extrema.

Suppose that f'' is continuous on an open interval containing c with $f'(c) = 0$.

- If $f''(c) > 0$, then f has a local minimum at c.
- If $f''(c) < 0$, then f has a local maximum at c.
- If $f''(c) = 0$, then the test is inconclusive.

See the Recap of Derivative Properties (Figure 4.36 in the text) for a summary.

Exercise

Let $f(x) = 2x^3 - 6x^2 - 18x$.

(a) Determine the intervals on which f is concave up or down, and identify any inflection points.

(b) Locate the critical points, and use the Second Derivative Test to determine whether they correspond to local minima or maxima, or whether the test is inconclusive.

4.2 Book Problems

11-35 (odds), 47-61 (odds), 67

$\oint$ 4.3 Graphing Functions

Graphing Guidelines:

1. Identify the domain or interval of interest.
2. Exploit symmetry.
3. Find the first and second derivatives.
4. Find critical points and possible inflection points.
5. Find intervals on which the function is increasing or decreasing, and concave up/down.
6. Identify extreme values and inflection points.

4.3 Graphing Functions (cont.)

7. Locate vertical/horizontal asymptotes and determine end behavior.
8. Find the intercepts.
9. Choose an appropriate graphing window and make a graph.

Exercise

According to the graphing guidelines, sketch a graph of

$$f(x) = \frac{x^2}{x^2 - 4}.$$

4.3 Book Problems

7, 9, 13-19 (odds), 23, 29, 43, 45

4.4 Optimization Problems

In many scenarios, it is important to find a maximum or minimum value under given constraints. Given our use of derivatives from the previous sections, optimization problems follow directly from what we have studied.

Example

Given all nonnegative real numbers x and y between 0 and 50 such that their sum is 50 (i.e., $x + y = 50$), which pair has the maximum product?

This is a basic optimization problem. In this problem, we are given a **constraint** $(x + y = 50)$ and asked to maximize an **objective function** $(A = xy)$.

The first step is to express the objective function $A = xy$ in terms of a single variable by using the constraint:

$$\begin{aligned} y &= 50 - x \\ \implies A(x) &= x(50 - x). \end{aligned}$$

Then to maximize A, we find the critical points:

$$A'(x) = 50 - 2x = 0$$

means $x = 25$ is a critical point.

The domain of $A(x)$ is $[0,50]$. To maximize A we evaluate A at the endpoints of the domain and at the critical point:

$$\begin{aligned} A(0) &= 0(50-0) = 0 \\ A(25) &= 25(50-25) = 625 \\ A(50) &= 50(50-0) = 0 \end{aligned}$$

So 625 is the maximum value of A and A is maximized when $x = 25$ (which means $y = 25$).

To answer the question, the pair of nonnegative numbers summing to 50 with the maximum product is 25 and 25.

Essential Feature of Optimization Problems

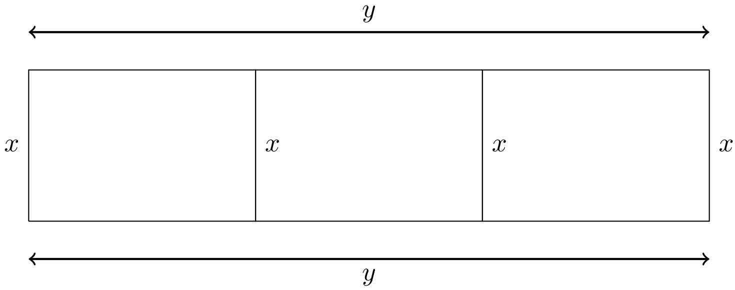
All optimization problems take the following form:

What is the maximum (or minimum) value of an objective function subject to the given constraint(s)?

Most optimization problems have the same basic structure as the previous problem: An objective function (possibly with several variables and/or constraints) with methods of calculus used to find the maximum/minimum values.

Example

Suppose you wish to build a rectangular pen with two interior parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen?



From looking at the picture, we can identify the constraints:

$$2y + 4x = 500 \implies y = -2x + 250.$$

The objective function is what we must maximize. In this case it is the area, $A = xy$. So we write

$$A(x) = x(-2x + 250) = -2x^2 + 250x.$$

Taking the derivative and setting it to zero, we get

$$A'(x) = -4x + 250 = 0$$
$$\implies x = 62.5 \quad \text{is a critical point.}$$

We use the picture to identify the domain. Since we have 500 ft of fencing available we must have $0 \le x \le 125$. Now we find the max:

$$
\begin{aligned}
A(0) &= 0(-2(0) + 250) = 0 \\
A(62.5) &= 62.5(-2(62.5) + 250) = 7812.5 \\
A(125) &= 125(-2(125) + 250) = 0
\end{aligned}
$$

The maximum area is 7812.5 ft^2. The pen's dimensions (answer the question!) are $x = \boxed{62.5 \text{ ft}}$ by $y = -2(62.5) + 250 = \boxed{125 \text{ ft}}$.

Guidelines for Optimization Problems

1. READ THE PROBLEM carefully, identify the variables, and organize the given information with a picture.
2. Identify the objective function (i.e., the function to be optimized). Write it in terms of the variables of the problem.
3. Identify the constraint(s). Write them in terms of the variables of the problem.
4. Use the constraint(s) to eliminate all but one independent variable of the objective function.
5. With the objective function expressed in terms of a single variable, find the interval of interest for that variable.

Guidelines for Optimization Problems (cont.)

6. Use methods of calculus to find the absolute maximum or minimum value of the objective function on the interval of interest. If necessary, check the endpoints.

Exercise

An open rectangular box with square base is to be made from 48 ft^2 of material. What dimensions will result in a box with the largest possible volume?

Exercise

Find the dimensions of the rectangle of largest area which can be inscribed in the closed region bounded by the x-axis, y-axis, and the graph of $y = 8 - x^3$.

4.4 Book Problems

5-13, 18-20, 26