Matroid Varieties

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Matroid varieties are projective subvarieties of the Grassmannian. In studying ideals generated by principal minors of a generic matrix, we look at the locally closed sets given by a rank condition. Let

$$\mathcal{Y}_{n,r,t} = \{n \times n \text{ matrices of rank } r, \text{ whose size } t \text{ princpal minors vansish}\}.$$

When $t = r$ studying these components reduces to studying pairs in $\mathrm{Grass}(t, n) \times \mathrm{Grass}(t, n)$. A particular type of subvariety of the Grassmannian, called a positroid variety, has many nice properties shown by Knutson, Lam, and Speyer in 2013. The author observed, the following year, that components of $\mathcal{Y}_{n,n-2,n-2}$ are products of positroid varieties.

Although positroid varieties, defined in what follows, are a special type of projective variety, a less restrictive notion is that of a matroid variety. We wish to use the axioms defining a matroid in order to study the components of $Y_{n,t,t}$. Primary references are [Ford] and [KLS].

Grassmannians

In this section we review the construction of the Grassmann variety. Recall,

$$\mathrm{Grass}_K(r, n) = \{r\text{-dimensional vector subspaces of } K^n\}$$

where K is some arbitrary algebraically closed field. The Grassmannian embeds into $\mathbb{P}_K^{\binom{n}{r}-1}$ under the **Plücker embedding**. Its homogeneous coordinate ring is $K[\wedge^r X]$, where X is a generic $r \times n$ matrix and $\wedge^r X$ is the $\binom{n}{r} \times \binom{r}{r}$ matrix whose entries are the $r \times r$ minors of X. Notice, the matrix $\wedge^r X$ is a column vector of length $\binom{n}{r}$, so it makes since that the embedding should be into $\mathbb{P}^{\binom{n}{r}-1}$. The entries of $\wedge^r X$ are called **Plücker variables**.

Given an $r \times n$ matrix A with entries in K, the entries of $\wedge^r A$ are called **Plücker coordinates** for the point row $A \in \mathrm{Grass}(r, n)$. An $r \times n$ matrix B has the same row span of A if and only if it is a change of basis away, i.e., there is an invertible $r \times r$ matrix C such that $A = CB$. Such matrices C can be parametrized by their respective determinants:

$$\begin{aligned}\wedge^r A &= \wedge^r C \cdot \wedge^r B \\ &= (\det C) \cdot \wedge^r B\end{aligned}$$

and so we really do get an equivalence relation on Plücker coordinates given by scalar multiples – as we should for a projective variety.

The open affine cover of $\mathrm{Grass}(r,n)$ is given as follows: let U denote a fixed $r \times r$ submatrix of X and invert its determinant. Then X can be normalized to a matrix X', where the image of U is an $r \times r$ identity matrix $I_{r\times r}$. The new matrix X' has the same column space as X, but now coordinates are uniquely determined by the entries in the complementary submatrix $X' \setminus I_{r\times r}$. This open set is isomorphic to $\mathbb{A}^{r(n-r)}$ and its coordinate ring is $K[\wedge^r X][\frac{1}{\det U}]$.

Finally, as an alternate way of thinking of the coordinate ring for $\mathrm{Grass}(r,n)$, define the surjection

$$\varphi : K[Z] \to K[\wedge^r X]$$

where $Z = (z_{ij})$ is a generic $\binom{n}{r} \times \binom{r}{r}$ matrix, and φ takes entries to entries. The generators for $\ker\varphi$ are homogeneous quadratics (in the Plücker variables) called the **Plücker relations**. We can think of the homogeneous coordinate ring for $\mathrm{Grass}(r,n)$ as $K[Z]/(\ker\varphi) \cong K[\wedge^r X]$.

Matroid Varieties

For the moment we hold off on giving a precise definition for a matroid and instead introduce matroid varieties. Given a subset $\mathcal{B}$ of indices of Plücker coordinates from $\mathrm{Grass}(r,n)$, its associated **open matroid variety** $\mathcal{V}^\circ(\mathcal{B})$ is the set of points in $\mathrm{Grass}(r,n)$ whose Plücker coordinates indexed by sets in $\mathcal{B}$ do not vanish. Note, non-vanishing of polynomials is a Zariski-open condition. The **matroid variety** $\mathcal{V}(\mathcal{B})$ is the open matroid variety's closure. Finding a defining ideal for a matroid variety is generally nowhere as easy as it seems; the näive approach is to take the complementary set of Plücker coordinates to vanish, but that does not work, as the following example shows.

Example. ([Ford]) Let $\binom{[7]}{3}$ denote the collection of 3-element subsets of $\{1,\ldots,7\}$. For any $S \in \binom{[7]}{3}$, let x_S denote the Plücker variable given by columns of a generic 3×7 matrix X, indexed by the entries in S. Consider the Plücker coordinates indexed by sets in the collection

$$\mathcal{B} = \binom{[7]}{3} \setminus \{\{1,2,7\},\,\{3,4,7\},\,\{5,6,7\}\}\,.$$

One would hope the defining ideal for $\mathcal{V}(\mathcal{I}) \subset \mathrm{Grass}_K(3,7)$ is

$$I = \left(x_{\{1,2,7\}},\, x_{\{3,4,7\}},\, x_{\{5,6,7\}}\right) \subset K[\wedge^3 X].$$

However, the defining ideal is actually

$$J = I + \left(x_{\{1,2,4\}}x_{\{3,5,6\}} - x_{\{1,2,3\}}x_{\{4,5,6\}}\right).$$

Matroid varieties actually do arise naturally. In the following examples, however, we do not explicitly show the matroid construction.

Example. Schubert varieties are an example. A sequence $0 < a_1 < \cdots < a_r \leq n$ together with a chain of vector spaces $W_1 \subset \cdots \subset W_r$ with $\dim W_i = a_i$ for all $i = 1, \ldots, r$ is called a **flag**. Given a fixed flag, the **Schubert variety** is defined as

$$\Omega(a_1, \ldots, a_r) = \{V \in \text{Grass}(r, n) \mid \dim(V \cap W_i) \geq i \text{ for all } i = 1, \ldots r\}$$

In terms of matrices we can define a Schubert variety as the points $V \in \text{Grass}(r, n)$ such that if V is the rowspan of an $r \times n$ matrix A, then for all $i = 1, \ldots r$, the span of the first a_i columns has dimension at most i.

Example. Richardson varieties are another example. Given two sequences $0 < a_1 < \cdots < a_r \leq n$ and $0 < b_1 < \cdots < b_r \leq n$, we take all $V \in \text{Grass}(r, n)$ such that if V is the rowspan of an $r \times n$ matrix A, then for all $i = 1, \ldots, r$ the span of first a_i columns of A has rank at most i and the span of the last b_i columns of A has rank at most i.

Application: Positroid Varieties

As noted above, the defining ideal for a matroid variety in general is hard to determine, although not so much in the case of Schubert varieties and Richardson varieties. There is a reason for this: both examples fall into the class of **positroid varieties**. Given the finite set $E = \{1, \ldots, n\}$, let $\mathcal{B}$ denote a collection of sets in $\binom{E}{r}$ where the elements of each set are consecutive, in the sense that cyclic permutations count. The corresponding matroid variety is called a positroid variety.

Theorem (Knutson+Lam+Speyer 2013). *Positroid varieties are normal, Cohen–Macaulay, have rational singularities, and their defining ideals are given by Plücker coordinates.*

Theorem (– 2014). *If a subset of Plücker coordinates for* $\text{Grass}(n-2, n)$ *defines an irreducible algebraic set, i.e., is a variety, then it is positroidal. Furthermore, every irreducible component of every matroid scheme in* $\text{Grass}(n-2, n)$ *(one defined by vanishing of a subset of Plücker coordinates) is of this form, and so is positroidal.*

For fixed $t \leq r \leq n$, let $\mathcal{Y}_{n,r,t}$ denote the locally closed set of $n \times n$ matrices of rank r whose size t principal minors vanish.

Corollary (– 2014). *The components for* $\mathcal{Y}_{n,n-2,n-2}$ *are normal, Cohen-Macaulay, have rational singularities, and their defining ideals are given by Plücker coordinates.*

In studying ideals generated by principal minors of a generic matrix, the components for $\mathcal{Y}_{n,n,n-2} \cong \mathcal{Y}_{n,n,2}$ are known. It remains to understand components of $\mathcal{Y}_{n,n-1,n-2}$ in order to get a full understanding of the algebraic set $\mathcal{V}(\mathfrak{P}_{n-2})$.

Defining a Matroid

In this section we give a more precise notion for what a matroid is. In the following section we shall apply matroid theory to the study of the sets $\mathcal{Y}_{n,t,t}$. Hopefully after the discussion of matroid varieties one has an intuitive idea of what a matroid is – they seem to come from prescribed rank conditions on a collection of column vectors. In fact, matroids are a type of combinatorial data used to describe many seemingly unrelated objects in mathematics, including graphs, transversals, vector spaces, and networks. The word **matroid** even has many equivalent definitions, including a characterization using the Greedy Algorithm.

Let E denote a finite set and let 2^E denote its power set. We write $S = \{e_1, \ldots, e_x\} \in 2^E$ if and only if $|S| = s$ (in other words we assert elements in a set are distinct). We call elements of E **columns**.

Definition (Dependence Axioms). *A set $\mathcal{D} \subseteq 2^E$ is the collection of* **dependent** *sets of a matroid $\mathcal{M} = (E_\mathcal{M}, \mathcal{D}_\mathcal{M})$, where $E_\mathcal{M} = E$ and $\mathcal{D}_\mathcal{M} = \mathcal{D}$, if and only if*

(1) $\emptyset \notin \mathcal{D}$,
(2) if $S \in \mathcal{D}$ and $S \subseteq T$, then $T \in \mathcal{D}$, and
(3) if $S, T \in \mathcal{D}$ and $S \cap T \notin \mathcal{D}$, then for every $e \in E$, $(S \cap T) \setminus \{e\} \in \mathcal{D}$.

Definition (Independence Axioms). *A set $\mathcal{I} \subseteq 2^E$ is the collection of* **independent** *sets of a matroid $\mathcal{M} = (E_\mathcal{M}, \mathcal{I}_\mathcal{M})$, with $E_\mathcal{M} = E$ and $\mathcal{I}_\mathcal{M} = \mathcal{I}$, if and only if*

(1) $\emptyset \in \mathcal{I}$,
(2) if $S \in \mathcal{I}$ and $T \subseteq S$, then $T \in \mathcal{I}$, and
(3) (Independence Augmentation Axiom) if $S, T \in \mathcal{I}$ and $|S| > |T|$ then there exists $e \in S \setminus T$ such that $T \cup \{e\} \in \mathcal{I}$.

Example. Let A denote an $r \times n$ matrix of full rank, with $r \leq n$. Suppose two collections of r columns give a basis for the column span of A. If we let $E = \{1, \ldots, n\}$ index the columns of A then we have two r-element subsets $S, T \subseteq E$ corresponding to the above collections of spanning vectors. We get an independence system by letting $\mathcal{I}$ denote all subsets whose elements correspond to indices for linearly independent columns of A. It is not hard to see $\mathcal{I}$ satisfies the independence augmentation axiom; given two collections V, W of independent columns, if $|V| > |W|$ then V contains a column that is linearly independent of all columns in W.

When the context is unambiguous we shall always write $E = \{1, \ldots, n\}$ to denote the finite set $E_\mathcal{M}$ for a given matroid $\mathcal{M}$. We observe for any matroid $\mathcal{M} = (E, \mathcal{I}_\mathcal{M})$, the complement in 2^E of $\mathcal{I}_\mathcal{M}$ satisfies the Dependence Axioms. It follows that we may specify $\mathcal{M}$ using either $\mathcal{I}_\mathcal{M}$ or $\mathcal{D}_\mathcal{M}$.

Proposition. *Let $\mathcal{M}$ be a matroid. If two dependent sets S, T of cardinality s satisfy $|S \cap T| = s - 1$, then either*

(a) $S \cap T$ is dependent or
(b) all s-tuples in $2^{S \cap T}$ are dependent.

Proof. Put $\mathcal{D} = \mathcal{D}_\mathcal{M}$ and $\mathcal{I} = I_\mathcal{M}$. If (a) does not hold then Dependence Axiom (3) says all sets in the family

$$\mathcal{F} = \{(S \cup T) \setminus \{e\} \mid e \in E\}$$

are dependent. The s-tuples n $2^{S \cup T}$ are all in $\mathcal{F}$ so (b) holds.

On the other hand, suppose there exists an s-tuple $R \in 2^{S \cup T}$ that is independent, and assume $S \cap T \in \mathcal{I}$. Then by the Independence Augmentation Axiom, since $|R| = s > |S \cap T| = s-1$, there exists $e \in R \setminus (S \cap T)$ such that $(S \cap T) \cup \{e\} \in \mathcal{I}$. Such e is in either S or T, since $R \setminus (S \cap T) \subset R \subset S \cap T$, so without loss of generality, say $e \in S$. The independent set $(S \cap T) \cup \{e\}$ is contained in S, and $|(S \cup T) \cup \{e\}| = |S|$ implies the sets are equal. Particularly, we get $S \in \mathcal{I}$, contradicting our hypothesis $S \in \mathcal{D}$. Therefore we must have $S \cap T \in \mathcal{D}$. $\square$

The maximal independent sets for a matroid $\mathcal{M}$ are called **bases**, and we denote this collection $\mathcal{B}_{\mathcal{M}} \subseteq \mathcal{I}_{\mathcal{M}}$. Similarly, sets in the collection of minimal dependent sets, $\mathcal{C}_{\mathcal{M}} \subseteq \mathcal{D}_{\mathcal{M}}$, are called **circuits**. To define a matroid it is enough to list either its bases or its circuits.

Definition (Circuit Axioms). *A collection $\mathcal{C} \subseteq 2^E$ is the set of circuits of a matroid $\mathcal{M} = (E, \mathcal{C}_{\mathcal{M}})$, where $\mathcal{C}_{\mathcal{M}} = \mathcal{C}$, if and only if*
(1) $\emptyset \notin \mathcal{C}$,
(2) if $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$, and
(3) (Circuit Elimination Axiom) if $C_1, C_2 \in \mathcal{C}$, with $C_1 \neq C_2$ and $e \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.

Definition (Basis Axioms). *A collection $\mathcal{B} \subseteq 2^E$ is the set of bases of a matroid $\mathcal{M} = (E, \mathcal{B}_{\mathcal{M}})$, with $\mathcal{B}_{\mathcal{M}} = \mathcal{B}$, if and only if*
(1) $\neq \emptyset$ and
(2) (Base Exchange Axiom) if $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$, then there exists $e' \in B_2 \setminus B_1$ such that $(B \setminus \{e\}) \cup \{e'\} \in \mathcal{B}$.

Proposition. *All elements in $\mathcal{B}$ have the same cardinality.*

In the situation where was are dealing with vector spaces, the proposition is obvious. Given a matroid $\mathcal{M}$, we call the cardinality of its basis elements $r = r_{\mathcal{M}}$. In fact, prescribing a set $\mathcal{B}_{\mathcal{M}}$ is equivalent to defining a **rank function**, also denoted $r_{\mathcal{M}}$, by

$$\begin{aligned} r_{\mathcal{M}} : 2^E &\mapsto \mathbb{Z}_{\geq 0} \\ S &\mapsto \max\{|T| \mid T \subseteq S \text{ and } T \in \mathcal{I}_{\mathcal{M}}\} \end{aligned}$$

Definition (Rank Axioms). *A function $r : 2^E \to \mathbb{Z}_{\geq 0}$ is the rank function of a matroid if and only if for all $S, T \subseteq E$,*
(1) $0 \leq r(S) \leq |S|$,
(2) if $T \subseteq S$, then $r(T) \leq r(S)$, and
(3) (Submodularity) $r(S) + r(T) \geq r(S \cup T) + r(S \cap T)$.

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