Mon 29 Sep 2014

- Thurs 2 Oct: Quiz in drill covers 3.2-3.4
- Tues 7 Oct: Quiz in drill covers 3.5-3.7
- Wheeler section Exam 2 will be first week of November
- Wed 15 Oct: MIDTERM see syllabus for time. Location TBA

Derivatives as Rates of Change

Up to now in Chapter 3 we have seen a number of methods and rules to compute derivatives. We now look at the question "When would we ever need this?"

Today we look at four areas where the derivative assists us with determining the rate of change in various contexts.

Position and Velocity

Recall from section 2.1 the definitions of average velocity and instantaneous velocity. In section 2.1, we often were given position functions *s = f(t)*, where the position *s* of an object was measured at any time *t*.

In 2.1, we measured the distance away from a given point in terms of *a* and *a+h*. Now we look at the displacement of the object between $t = a$ and $t = a + \Delta t$ is

$$\Delta s = f(a + \Delta t) - f(a)$$

Here Δt represents how much time has elapsed.

Speed and Acceleration

Continuing to use the position function $s = f(t)$, we can find the speed and acceleration of the object as well.

Question: How are speed and velocity related?

A rock is dropped off a bridge and its distance *s* (in feet) from the bridge after *t* seconds is $s(t) = 16t^2 + 4t$. The velocity of the rock at *t = 2* is ______ and the acceleration at *t = 2* is ________.

A. 64 ft/s; 16 ft/s2

B. 68 ft/s; 32 ft/s2

C. 64 ft/s; 32 ft/s2

D. 68 ft/s; 16 ft/s2

Growth Models

Suppose *p = f(t)* is a function of the growth of some quantity of interest (e.g., population, prices, etc.). The average growth rate of *p* between times *t = a* and a later time *t = a + Δt* is the change in *p* divided by the elapsed time *Δt*. So:

$$\frac{\Delta p}{\Delta t} = \frac{f(a+\Delta t) - f(a)}{\Delta t}$$

Growth Models

As Δt approaches 0, the average growth rate approaches the derivative $\frac{dp}{dt}$, which is the instantaneous growth rate (or just simply the growth rate). Therefore:

$$\frac{dp}{dt} = \lim_{\Delta t \to 0} \frac{f(a+\Delta t) - f(a)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t}$$

Exercise

The population of the state of Georgia (in thousands) from 1995 ($t = 0$) to 2005 ($t = 10$) is modeled by the polynomial

$$p(t) = -0.27t^2 + 101t + 7055$$

- What was the average growth rate from 1995 to 2005?
- What was the growth rate for Georgia in 1997?
- What can you tell about the population growth rate in Georgia between 1995 and 2005?

Average and Marginal Cost

The final example stems from the world of business.

Suppose a company produces a large amount of a particular quantity. Associated with manufacturing the quantity is a **cost function** $C(x)$ that gives the cost of manufacturing x items. This cost may include a **fixed cost** to get started as well as a **unit cost** (or **variable cost**) in producing one item.

Average and Marginal Cost

If a company produces *x* items at a cost of *C(x)*, then the average cost is *C(x) / x.*

This average cost indicates the cost of items already produced. To determine the cost of producing additional items, we would take $C(x + \Delta x) - C(x)$. So the average cost of producing those extra Δx items is

$$\frac{\Delta C}{\Delta x} = \frac{C(x + \Delta x) - C(x)}{\Delta x}$$

Exercise

If the cost of producing *x* items is given by

$$C(x) = -0.04x^2 + 100x + 800$$

for $0 \le x \le 1000$, find the average cost and marginal cost functions.

What is the average and marginal cost when *x = 500?* What do these values mean?

HW from section 3.5

- Do problems 9-12, 17-18, 22-23, 27-37 odd (pgs. 171-175 in textbook)

Chain Rule

The rules up to now have not allowed us to differentiate composition functions $f(g(x))$.

For example, if $f(x) = x^7$ and $g(x) = 2x - 3$, then we have $f(g(x)) = (2x - 3)^7$.

We could multiply the polynomial out, but in general we would need a much more efficient strategy to employ to composition functions.

Version 1 of Chain Rule

Suppose that Yvonne (y) can run twice as fast as Uma (u). Therefore $\frac{dy}{du} = 2$. Suppose that Uma can run four times as fast as Xavier (x). So $\frac{du}{dx} = 4$.

How much faster can Yvonne run than Xavier?

In this case, we would take both our rates and multiply them together:

$$\frac{dy}{du} \cdot \frac{du}{dx} = 2 \cdot 4 = 8$$

Wed 1 Oct 2014

- Exam 2 is not until after the midterm
- Midterm Wed 15 Oct – see syllabus. Location TBA

Version 1 of the Chain Rule

In general, if *g* is differentiable at *x*, and *y = f(u)* is differentiable at *u = g(x)*, then the composite function *y = f(g(x))* is differentiable at *x*, and its derivative can be expressed as:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Guidelines for Using the Chain Rule

Assume the differentiable function $y = f(g(x))$ is given.

1. Identify the outer function f, the inner function g, and let $u = g(x)$.
2. Replace $g(x)$ by u to express y in terms of u:

$$y = f(g(x)) \Rightarrow y = f(u)$$

3. Calculate the product $\dfrac{dy}{du} \cdot \dfrac{du}{dx}$
4. Replace u by $g(x)$ in $\dfrac{dy}{du}$ to obtain $\dfrac{dy}{dx}$.

Example

Use Version 1 of the Chain Rule to calculate $\frac{dy}{dx}$ for $y = (5x^2 + 11x)^{20}$

Solution: The inner function for $y = (5x^2 + 11x)^{20}$ is $u = 5x^2 + 11x$ and the outer function is $y = u^{20}$. So $y = f(g(x)) = (5x^2 + 11x)^{20}$

So $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 20u^{19} \cdot (10x + 11)$

and replacing *u* with *g(x):* $\frac{dy}{dx} = 20(5x^2 + 11x)^{19} \cdot (10x + 11)$

Use the first version of the Chain Rule to calculate $\frac{dy}{dx}$ for

$$y = \cos(5x+1)$$

A. $y' = -\cos(5x+1) \cdot \sin(5x+1)$

B. $y' = -5\sin(5x+1)$

C. $y' = 5\cos(5x+1) - \sin(5x+1)$

D. $y' = -\sin(5x+1)$

Second Version of Chain Rule

Notice if *y = f(u)* and *u = g(x)*, we have *y = f(u) = f(g(x)).*

Then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot g'(x) = f'(g(x)) \cdot g'(x)$

Example

Use Version 2 of the Chain Rule to calculate $\frac{dy}{dx}$ for $y = (7x^4 + 2x + 5)^9$

Solution: The inner function for $y = (7x^4 + 2x + 5)^9$ is $g(x) = 7x^4 + 2x + 5$ and the outer function is $f(u) = u^9$.

So $f'(u) = 9u^8 \Rightarrow f'(g(x)) = 9(7x^4 + 2x + 5)^8$ and $g'(x) = 28x^3 + 2$

So
$$f'(g(x)) \cdot g'(x) = 9(7x^4 + 2x + 5)^8 \cdot (28x^3 + 2)$$
$$= 18(14x^3 + 1) \cdot (7x^4 + 2x + 5)^8$$

Chain Rule for Powers

As we've seen with the previous examples, the power rule we have previously used with functions of the form f(x)=xn also applies with the chain rule:

$$\frac{d}{dx}\left[(g(x))^n\right] = n(g(x))^{n-1} \cdot g'(x)$$

Example: $\frac{d}{dx}\left[(1-e^x)^4\right] = 4(1-e^x)^3 \cdot (-e^x) = -4e^x(1-e^x)^3$

Composition of 3 or more functions

If we have more than two functions, we can continue to use the Chain Rule, only we would need to break the problem down and use the rule repeatedly.

Example:

$$\frac{d}{dx}\left[\sqrt{(3x-4)^2+3x}\right]$$

HW from section 3.6

- Do problems 7-29 odd, 30, 33-43 odd, 49 (pgs. 180-181 in textbook)

Implicit Differentiation

Up to now, we have calculated derivatives of functions of the form $y = f(x)$, where y is defined **explicitly** in terms of x.

In this section, we examine relationships between variables that are **implicit** in nature, meaning that y either is not defined explicitly in terms of x or cannot be easily manipulated to solve for y in terms of x.

Examples of functions implicitly defined

$$x^2 + y^2 = 9$$

$$x + y^3 - xy = 4$$

$$\cos(x - y) + \sin y = \sqrt{2}$$

Fri 3 Oct 2014

- Exams back next week, plus solutions

- Tues 7 Oct: Quiz on 3.5-3.7

- Wed 15 Oct: Midterm Exam, 6:30-8p

Location: POSC A211 (Poultry Science Auditorium)

Conflicts with Midterm

If you have a legitimate, university-related conflict with the midterm date and time, please email me and give to me the following information:

- Name
- ID
- Calculus 1
- Your conflict

Higher Order Derivatives

To find $\frac{d^2y}{dx^2}$, we first have to find $\frac{dy}{dx}$, which we did on a previous slide.

Then we differentiate $\frac{dy}{dx}$ again, substituting our previous answer for $\frac{dy}{dx}$ in our calculations.

$$\frac{d^2y}{dx^2} = ??$$

If $xy + y^3 = 1$, then

A. $\frac{dy}{dx} + 3y^2 = 0$

B. $y + x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$

C. $y + x\frac{dy}{dx} + 3y^2 = 0$

D. $x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$

Higher-Order Derivatives

Finding higher-order derivatives of implicit functions is much like finding higher-order derivatives of explicit functions, with one exception: we must use previous derivatives to substitute in our work to finish the computations.

EX: Find $\frac{d^2y}{dx^2}$ if $xy + y^3 = 1$

Finding tangent lines

Exercise: Find an equation of the line tangent to the curve $x^4 - x^2 y + y^4 = 1$ at the point (-1, 1).

Power Rule for Rational Exponents

Implicit differentiation also allows us to extend the power rule to rational exponents:

Assume *p* and *q* are integers with $q \neq 0$. Then

$$\frac{d}{dx}\left(x^{\frac{p}{q}}\right) = \frac{p}{q} x^{\frac{p}{q}-1}$$

provided that $x \geq 0$ when *q* is even.

HW from section 3.7

- Do problems 5-21 odd, 27-45 odd (pgs. 188-189 in textbook)