Topology QR Solutions – 12 Sep 2009

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Morning Session

1. Prove that if X is a (non-empty) countable compact Hausdorff space, then X is not connected. (You may use the fact that an intersection of countably many dense open sets in a compact Hausdorff space is dense.)

   Solution.

2. Let P be a polygon with an even number of sides. Suppose that the sides are identified in pairs in any way whatsoever. Prove that the quotient space is a manifold.

   Solution.

3. Prove that if M is a non-empty compact smooth manifold with boundary, then there is no smooth retraction from M to its boundary ∂M. (You may use Sard's Theorem.)

   Solution. Assume there is a smooth retraction $f : M \to \partial M$. Sard's Theorem gives a regular value $y \in \partial M$ for f and $f \mid_{\partial M} = \mathrm{id}_M$, so $f^{-1}(y)$ is a smooth manifold of dimension $\dim M - \dim \partial M = 1$. Now y is closed implies $f^{-1}(y)$ is closed, hence compact. So since $f^{-1}(y)$ is 1-dimensional, it is a finite disjoint union of circles and line segments. In particular, its boundary is an even number of points. But

$$\begin{aligned} \partial \left(f^{-1}(y)\right) &= f^{-1}(y) \cap \partial M \\ &= f \circ f^{-1}(y) \\ &= y, \end{aligned}$$

*with additional input from M. Hochster, G.P. Scott, and others from the U of M Mathematics Department a contradiction. □

4. Let X be a path-connected topological space. For $n > 1$ an integer, denote by S_n the symmetric group on n-letters. State and prove a bijective correspondence between degree n covering spaces of X and group homomorphisms $\pi_1(X) \to S_n$. (Note that finding an accurate statement is part of the problem.)

   Solution. For a degree n covering space $p : \tilde{X} \to X$, a path γ in X has a unique lift $\tilde{\gamma}$ starting at a given point in $p^{-1}(\gamma(0))$ so this gives a well-defined map $p^{-1} \circ \gamma(0) \to p^{-1} \circ \gamma(1)$. Its inverse L_γ is similarly defined using $\bar{\gamma}$, the reverse path of γ. Thus $L_{\gamma_1\gamma_2} = L_{\gamma_1} L_{\gamma_2}$ for any paths γ_1, γ_2 implies L_γ only depends on the homotopy class of γ. Thus L_γ induces a bijection $\pi_1(X, x_0) \to G$ where $G \subset S_n$ is the group of permutations of $p^{-1}(x_0)$ and x_0 is any base point for X, since X is path connected. □

5. For integers k, n with $\leq k \leq n$, let

$$S^n = \left\{(x_1, \ldots, x_{n+1}) \mid x_1^2 + \cdots + x_{n+1}^2 = 1\right\} \subset \mathbb{R}^{n+1}$$

   and let

$$D_k = \left\{(x_1, \ldots, x_{n+1}) \mid x_1^2 + \cdots + x_k^2 \leq 1, x_{k+1} = \cdots = x_{n+1} = 0\right\}.$$

   Calculate the homology of $X_{k,n} = S^n \cup D_k$.

   Solution. Note that S^n is the union of an n-ball and a point. Include the disc D_k to get a CW-complex for $X_{k,n}$. The groups are $\mathbb{Z}$ in the $n, k, 1$ positions and trivial elsewhere. Hence the homology groups are

$$\begin{aligned} H_n\left(X_{k,n}\right) &= \begin{cases} \mathbb{Z} & \text{if } k \neq n-1, n \neq 1 \\ 0 & \text{otherwise} \end{cases} \\ H_k\left(X_{k,n}\right) &= \begin{cases} \mathbb{Z} & \text{if } k \neq n-1, n \neq 1 \\ 0 & \text{otherwise} \end{cases} \\ H_0\left(X_{k,n}\right) &= \mathbb{Z}. \end{aligned}$$

   All other homology groups are zero. □

Afternoon Session

1. Prove that the one point compactification $X \cup \{\infty\}$ is Hausdorff if and only if X is locally compact and Hausdorff.

   Solution. Suppose X is locally compact Hausdorff, and choose $x \in X$. Choose a compact set C containing a neighborhood U 0f x. Then U

and $Y \diagdown C$ are disjoint neighborhoods of x, ∞, respectively and hence Y is Hausdorrf.

Conversely, suppose $Y = X \cup \{\infty\}$ is Hausdorff. Then X is automatically Hausdorff. For $x \in X$, choose disjoint neighborhoods U, V around x, ∞, respectively. Then $C = Y \diagdown V$ is closed in Y, hence compact; $C \subset X$ implies C is compact in X. And, C contains the neighborhood U of $x \in X$. Hence X is locally compact Hausdorff. $\square$

2. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. The point $(x, y) \in \mathbb{R}^2$ is the stereographic projection of the point $(\xi, \eta, \zeta) \in S^2$ if and only if the three points $(0, 0, 1), (x, y, 0)$, and (ξ, η, ζ) are collinear; this defines a map $\sigma : \mathbb{R}^2 \to S^2$, $\sigma(x, y) = (\xi, \eta, \zeta)$. Show that σ maps $\mathbb{R}^2$ diffeomorphically onto the complement of a point in S^2.

   Solution.

3. By definition, a topological group is a set G with both a toplogy and a group structure, such that the map $G \to G$ sending x to x^{-1} and the map $G \times G \to G$ sending (x, y) to xy are both continuous. Let $a \in G$ denote the identity of this topological group G. Show that $\pi_1(G, 1)$ is abelian.

   Solution.

4. Show that the map $\phi : S^1 \times S^1 \to \mathbb{R}^3$ defined by

$$\phi(u, v) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 + \cos\beta \\ 0 \\ \sin\beta \end{pmatrix}$$

   for $u = (\cos\alpha, \sin\alpha)$ and $v(\cos\beta, \sin\beta)$ is an embedding.

   Solution. Note ϕ also defines a map $[0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$ by

$$(\alpha, \beta) \mapsto \begin{pmatrix} (\cos\alpha)(2 + \cos\beta) \\ (\sin\alpha)(2 + \cos\beta) \\ \sin\beta \end{pmatrix}$$

   which has Jacobian

$$J = \begin{pmatrix} -(\sin\alpha)(2 + \cos\beta & (\cos\alpha)(-\sin\beta) \\ (\cos\alpha)(2 + \cos\beta) & (\sin\alpha)(-\sin\beta) \\ 0 & \cos\beta \end{pmatrix}.$$

If J maps $(\alpha_0, \beta_0) \in [0, 2\pi] \times [0, 2\pi]$ to zero, then $\beta_0 \cos\beta = 0$. If $\cos\beta = 0$ then $\sin\alpha = \cos\alpha = 0$, a contradiction. So $\beta_0 = 0$, and this then implies $\alpha_0 = 0$. Therefore J is injective, i.e., ϕ is an immersion.

Now, $[0, 2\pi] \times [0, 2\pi]$ (and $S^1 \times S^1$) are compact so the image of ϕ is compact, hence closed as well. So a compact set in $\mathbb{R}^3$ intersects the image of ϕ in a closed set, C, which is also closed in the image of ϕ. Then by continuity of ϕ, $\phi^{-1}(C)$ is a closed subset of the compact space $[0, 2\pi] \times [0, 2\pi]$, so is compact. Conclude ϕ is proper. Together with the conlusion ϕ is an immersion, this implies ϕ is an embedding. $\square$

5. Let X be a finite simplicial complex of dimension 1. Prove that either $\pi_1 X \cong \mathbb{Z}$, or every continuous map $f : X \to X$ homotopic to the identity has a fixed point.

Solution. Let $f : X \to X$ be continuous and homotopic to the identity. So the Lefshetz number for f is equal to the Euler characteristic $\chi(X)$. If f does not have a fixed point then $\chi(X) = 0$. Equivalently, X has an equal number of vertices and edges.

Given any finite simplicial complex of dimension 1 with V vertices and E edges, what happens when adding an edge? Then E becomes $E + 1$ and either V remains fixed or becomes $V + 1$. If V remains fixed then the Euler characteristic goes down by 1. If not, then the Euler characteristic remains fixed.

When $E = 1$ the only possible complex is a closed line segment, so $\chi(X) = -1$. Similarly, when $E = 2$, $\chi(X) = -1$. For $E \geq 3$, add each edge one at a time to construct X from the $E = 2$ case. Then V must increase exactly once for $\chi(X) = 0$. If this happens it means a circuit has formed. Then X is homotopy equivalent to a circle, so $\pi_1(X) \simeq \mathbb{Z}$.

Conversely, a small rotation of a circle is homotopic to the identity but fixes no points. $\square$