Mon 17 Nov 2014

- This week we will cover Section 4.8, then 5.3. It should not change the order or material of the quizzes (see last week's lecture slides for a schedule of the remaining quizzes this term).

Definition of Riemann sum

Note: $\overline{x_k}$ is typically chosen to be consistent across all subintervals. The most common are the left Riemann sum, right Riemann sum, and the midpoint Riemann sum.*

For the Riemann sum $R = f(\overline{x_1})\Delta x + f(\overline{x_2})\Delta x + \ldots + f(\overline{x_n})\Delta x$:

1. R is a left Riemann sum if $\overline{x_k}$ is the left endpoint of each subinterval $[x_{k-1}, x_k]$.
2. R is a right Riemann sum if $\overline{x_k}$ is the right endpoint of each subinterval $[x_{k-1}, x_k]$.
3. R is a midpoint Riemann sum if $\overline{x_k}$ is the midpoint of each subinterval $[x_{k-1}, x_k]$.

*See Figures on pgs. 310-312 for picture of these sums.

Riemann Sums Using Sigma Notation

Using sigma notation, we can write the Riemann sum in a much more compact form:

$$R = f(\overline{x_1})\Delta x + f(\overline{x_2})\Delta x + \ldots + f(\overline{x_n})\Delta x = \sum_{k=1}^{n} f(\overline{x_k})\Delta x$$

To write the left, right, and midpoint Riemann sums in sigma notation, we need to know the point $\overline{x_k}$.

Left, Right and Midpoint Riemann Sums in Sigma Notation

Suppose *f* is defined on a closed interval [a, b], which is divided into *n* subintervals of equal length Δx. If $\overline{x_k}$ is a point in the *k*th subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then the **Riemann sum** of *f* on [a, b] is $\sum_{k=1}^{n} f(\overline{x_k})\Delta x$.

1. $\sum_{k=1}^{n} f(\overline{x_k})\Delta x$ is a left Riemann sum if $\overline{x_k} = a + (k-1)\Delta x$.
2. $\sum_{k=1}^{n} f(\overline{x_k})\Delta x$ is a right Riemann sum if $\overline{x_k} = a + k\Delta x$.
3. $\sum_{k=1}^{n} f(\overline{x_k})\Delta x$ is a midpoint Riemann sum if $\overline{x_k} = a + \left(k - \frac{1}{2}\right)\Delta x$

Sigma Notation

Common sums of positive integers. Let *n* be a positive integer.

Sum of a constant c: $\sum_{n=1}^{n} c = cn$

Sum of the first *n* integers: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Sum of squares of the first *n* integers: $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Sum of cubes of the first *n* integers: $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

Exercise

Use sigma notation to write the left, right and midpoint Riemann sums for the function $f(x) = x^2$ on the interval [1, 5] given that $n = 4$.

Based on these approximations, estimate the area bounded by the graph of *f(x)* over [1, 5].

The sum $(2^2 + 3^2 + 4^2)\square 1$ is a:

A. right Riemann sum for *f* on [1, 4] with *n* = 3.

B. left Riemann sum for *f* on [1, 4] with *n* = 3.

C. midpoint Riemann sum for *f* on [1, 4] with *n* = 3.

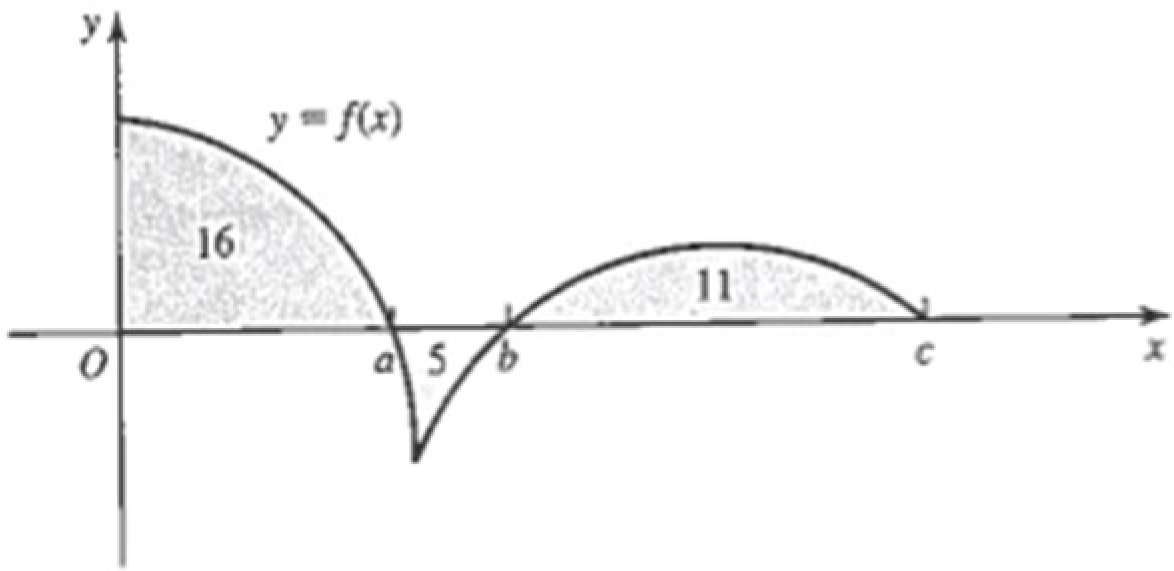
D. right Riemann sum for *f* on [2, 5] with *n* = 3.

Evaluating Definite Integrals

Without formally examining methods to evaluate definite integrals, we need to use geometry and graphs to determine integrals.

Using the picture below, evaluate the following definite integrals:

1. $\int_0^a f(x)\,dx$
2. $\int_0^b f(x)\,dx$
3. $\int_0^c f(x)\,dx$
4. $\int_a^c f(x)\,dx$



Evaluating Definite Integrals

Exercise: Using geometry, evaluate $\int_1^2 (4x+3)\,dx$.

(Hint: The area of a trapezoid is $A = \frac{h(a+b)}{2}$, where *h* is the height of the trapezoid and *a* and *b* are the lengths of the two parallel bases).

Wed 19 Nov 2014

- Exam 2 and the Pop Quiz!
- This week we will cover Section 4.8, then 5.3. It should not change the order or material of the quizzes (see last week's lecture slides for a schedule of the remaining quizzes this term).

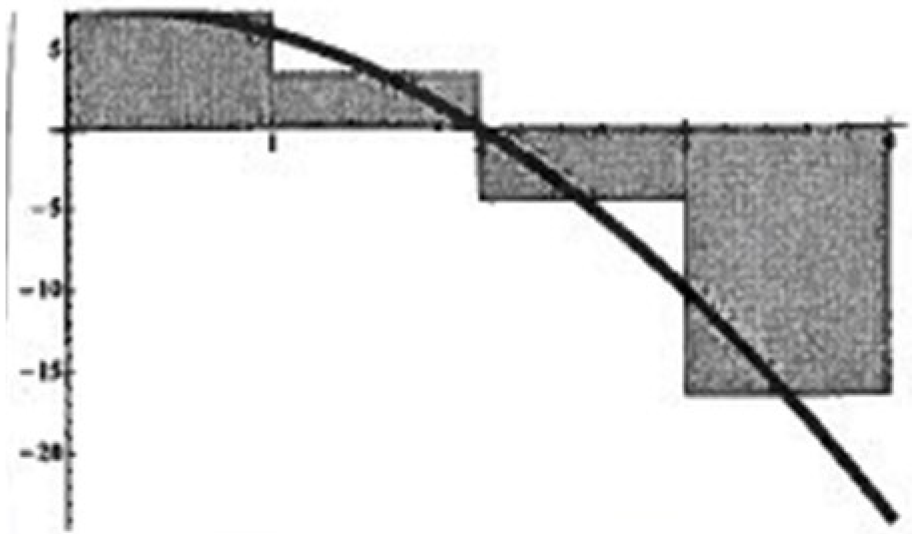
Definite Integral

In Section 5.1, we saw how we can use Riemann sums to approximate the area under a curve. However, the curves we worked with in this section were all non-negative.

What happens when the curve is negative?

Example

Let $f(x) = 8 - 2x^2$ over the interval [0, 4]. Use a left, right, and midpoint Riemann sum with n = 4 to approximate the area under the curve.



Net Area

In the previous example, we saw that the areas where f was positive provided positive contributions to the area, while areas where f was negative provided negative contributions.

The difference between the positive and negative contributions is called the **net area.**

Definition of Net Area

Consider the region *R* bounded by the graph of a continuous function *f* and the *x*-axis between $x = a$ and $x = b$. The **net area** of *R* is the sum of the areas of the parts of *R* that lie above the *x*-axis minus the sum of the areas of the parts of *R* that lie below the *x*-axis on [a, b].

Definite Integral

The Riemann sums give approximations for the area under the curve. To make these approximations more and more accurate, we divide the region into more and more subintervals.

To make these approximations exact, we allow the number of subintervals $n \to \infty$, thereby allowing the length of the subintervals $\Delta x \to 0$. In terms of limits:

$$\text{net area} = \lim_{n \to \infty} \sum_{k=1}^{n} f(\overline{x_k}) \Delta x$$

Definition of Definite Integral

Note that as $n \to \infty$, all of the $\Delta x \to 0$, even the largest of these. So for $\sum_{k=1}^{n} f(\overline{x_k})\Delta x_k$ as $n \to \infty$, then the limit as $\Delta x \to 0$ has the same value over all general partitions.

Definition: A function *f* defined on [a, b] is integrable if $\lim_{\Delta x_k \to 0} \sum_{k=1}^{n} f(\overline{x_k})\Delta x_k$ exists (over all partitions of [a, b] and all choices of $\overline{x_k}$ on a partition). This limit is the **definite integral of *f* from *a* to *b***, which we write

$$\int_a^b f(x)dx = \lim_{\Delta x_k \to 0} \sum_{k=1}^{n} f(\overline{x_k})\Delta x_k$$

Evaluating Definite Integrals

There are specific functions that are integrable.

THM: If *f* is continuous on [a, b] or bounded on [a, b] with a finite number of discontinuities, then *f* is integrable on [a, b]. (See Figure 5.23, pg. 325, for example of noncontinuous function that is integrable).

Knowing the limit of a Riemann sum, we can now translate that to a definite integral.

EX: $\lim_{\Delta \to 0} \sum_{k=1}^{n} (4x_k - 3)\Delta x_k$ on [-1, 4] $\Rightarrow \int_{-1}^{4} (4x-3)\,dx$

Properties of Integrals

The following properties are useful when evaluating definite integrals:

1. (Reversing Limits) $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$
2. (Identical Limits) $$\int_a^a f(x)\,dx = 0$$
3. (Integral of a Sum)

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Properties of Integrals

The following properties are useful when evaluating definite integrals:

4. (Constants in Integrals) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

5. (Integrals over Subintervals) If *c* lies between *a* and *b*, then $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

5. (Integrals of Absolute Values) The function |*f*| is integrable on [a, b] and $\int_a^b |f(x)|\,dx$ is the sum of the areas of regions bounded by the graph of *f* and the *x*-axis on [a, b]. (See Figure 5.31 on pg. 329)

Homework from Section 5.2

Do problems 11-29 odd, 35-44 all, 65 (pgs. 331-333).

Antiderivatives

With differentiation, the goal of problems was to find the function *f'* (e.g, the derivative) given a function f.

With antidifferentiation, the goal is the opposite. Here, given a function *f*, we wish to find a function *F* such that the derivative of *F* is the given function *f*. (e.g., *F' = f*).

Definition of Antiderivative

A function F is called an **antiderivative** of a function f on an interval I provided $F'(x) = f(x)$ for all x in I.

Example: Given $f(x) = 4$, an antiderivative of $f(x)$ is $F(x) = 4x$. Note that $F'(x) = 4 = f(x)$.

Question: Are there other antiderivatives of $f(x)$?

Fri 21 Nov 2014

- Exam 2 and the Pop Quiz! Majority says: E.C. on the next Exam.
- Monday: 5.3 with Dr. Dingman

Definition of Antiderivative

Example: Given $f(x) = 4$, an antiderivative of $f(x)$ is $F(x) = 4x$. Note that $F'(x) = 4 = f(x)$.

Other antiderivatives of $f(x) = 4$ include $F(x) = 4x + 1$, $F(x) = 4x - 5$, $F(x) = 4x + 1{,}000{,}000$. In general, the antiderivatives of $f(x) = 4$ take the form $F(x) = 4x + C$

Theorem: Let F be any antiderivative of f. Then ***all*** the antiderivatives of f have the form $F + C$, where C is an arbitrary constant.

An antiderivative of $5x^4 + 2x - 1$ is:

A. $20x^3 + 2$

B. $x^5 + x^2 - x + 501$

C. $x^5 + x^2$

D. $x^5 + 2x^2 - x$

Problems

Find the antiderivatives of the following functions:

$$f(x) = -6x^{-7}$$

$$g(x) = -4\cos 4x$$

$$h(x) = \csc^2 x$$

Indefinite Integrals

Just as the notation $\frac{d}{dx}(f(x))$ means to find the derivative of *f(x)*, the notation to determine the antiderivative of *f* is the indefinite integral $\int f(x)dx$.

So the indefinite integral sign $\int$ signifies that we find the antiderivative of the function *f(x)* (called the integrand), followed by the differential *dx* (denoting that x is the variable of integration).

Example: $\int 4x^3\,dx = x^4 + C$, where *C* is the constant of integration.

Power Rule for Indefinite Integrals

Just as differentiation had a power rule, constant multiple rule, and sum rule, so too does antidifferentiation.

The Power Rule: $$\int x^p \, dx = \frac{x^{p+1}}{p+1} + C$$

where $p \neq -1$ is a real number and C is an arbitrary constant.

Constant Multiple and Sum Rules for Indefinite Integrals

The Constant Multiple Rule: $\int cf(x)\,dx = c\int f(x)\,dx$

The Sum Rule:

$$\int \big(f(x)+g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Example

Determine the following indefinite integrals:

$$\int \left(3x^{-2} - 4x^{2} + 1\right)dx$$

$$\int 6\sqrt[3]{x}\,dx$$

$$\int 2\cos\left(2x\right)dx$$

Indefinite Integrals of Trig Functions

Table 4.5 (p. 296) provides us with rules for finding indefinite integrals of trig functions.

1. $\dfrac{d}{dx}(\sin ax) = a\cos ax \rightarrow \int \cos ax\,dx = \dfrac{1}{a}\sin ax + C$
2. $\dfrac{d}{dx}(\cos ax) = -a\sin ax \rightarrow \int \sin ax\,dx = -\dfrac{1}{a}\cos ax + C$
3. $\dfrac{d}{dx}(\tan ax) = a\sec^2 ax \rightarrow \int \sec^2 ax\,dx = \dfrac{1}{a}\tan ax + C$
4. $\dfrac{d}{dx}(\cot ax) = -a\csc^2 ax \rightarrow \int \csc^2 ax\,dx = -\dfrac{1}{a}\cot ax + C$
5. $\dfrac{d}{dx}(\sec ax) = a\sec ax\tan ax \rightarrow \int \sec ax\tan ax\,dx = \dfrac{1}{a}\sec ax + C$
6. $\dfrac{d}{dx}(\csc ax) = -a\csc ax\cot ax \rightarrow \int \csc ax\cot ax\,dx = -\dfrac{1}{a}\csc ax + C$

Example

Evaluate the following indefinite integral:

$$\int 2\sec^2 2x \, dx$$

Example

Evaluate the following indefinite integral:

$$\int 2\sec^2 2x\,dx$$

Using rule 3, with a = 2, we have

$$\int 2\sec^2 2x\,dx = 2\int \sec^2 2x\,dx = 2\left(\frac{1}{2}\tan 2x\right) + C = \tan 2x + C$$

So $$\int 2\sec^2 2x\,dx = \tan 2x + C$$

Other Indefinite Integrals

Table 4.6 (p. 297) provides us with rules for finding other indefinite integrals.

7. $\dfrac{d}{dx}(e^{ax}) = ae^{ax} \rightarrow \displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

8. $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x} \rightarrow \displaystyle\int \frac{dx}{x} = \ln|x| + C$

9. $\dfrac{d}{dx}\left(\sin^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{1}{\sqrt{a^2 - x^2}} \rightarrow \displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

10. $\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{a}{a^2 + x^2} \rightarrow \displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

11. $\dfrac{d}{dx}\left(\sec^{-1}\left|\dfrac{x}{a}\right|\right) = \dfrac{a}{x\sqrt{x^2 - a^2}} \rightarrow \displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

Initial Value Problems

In some instances, you have enough information to determine the *C* value in the antiderivative. These are often called Initial Value Problems.

Example: If $f'(x) = 7x^6 - 4x^3 + 12$ and f(1) = 24, find f(x)*.

Solution: $\int 7x^6 - 4x^3 + 12dx = x^7 - x^4 + 12x + C$

Plugging in f(1)=24 (e.g., x=1 and y=24), we find that C = 12. So $f(x) = x^7 - x^4 + 12x + 12$.

*Note: Finding f(x) is often called finding the solution.

Homework from Section 4.8

Do problems 11-45 odd, 55-59 odd, 63, 65 (pgs. 301-302).

Advice: To solve 55-59 odd, 63, and 65, read pgs. 299-300, focusing in on Example 7.

Fundamental Thm of Calculus

Using Riemann sums to evaluate definite integrals is not very efficient or practical. In this sense, we need more effective methods to evaluate integrals.

In this section, we develop methods to find the area under a curve.

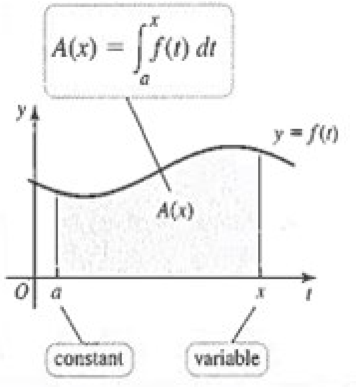
We also tie together the concepts of integration and differentiation through possibly the most important theorem in calculus.

Area Functions

To connect the concepts of differentiation and integration, we first must define the concept of an area function. We start with a continuous function

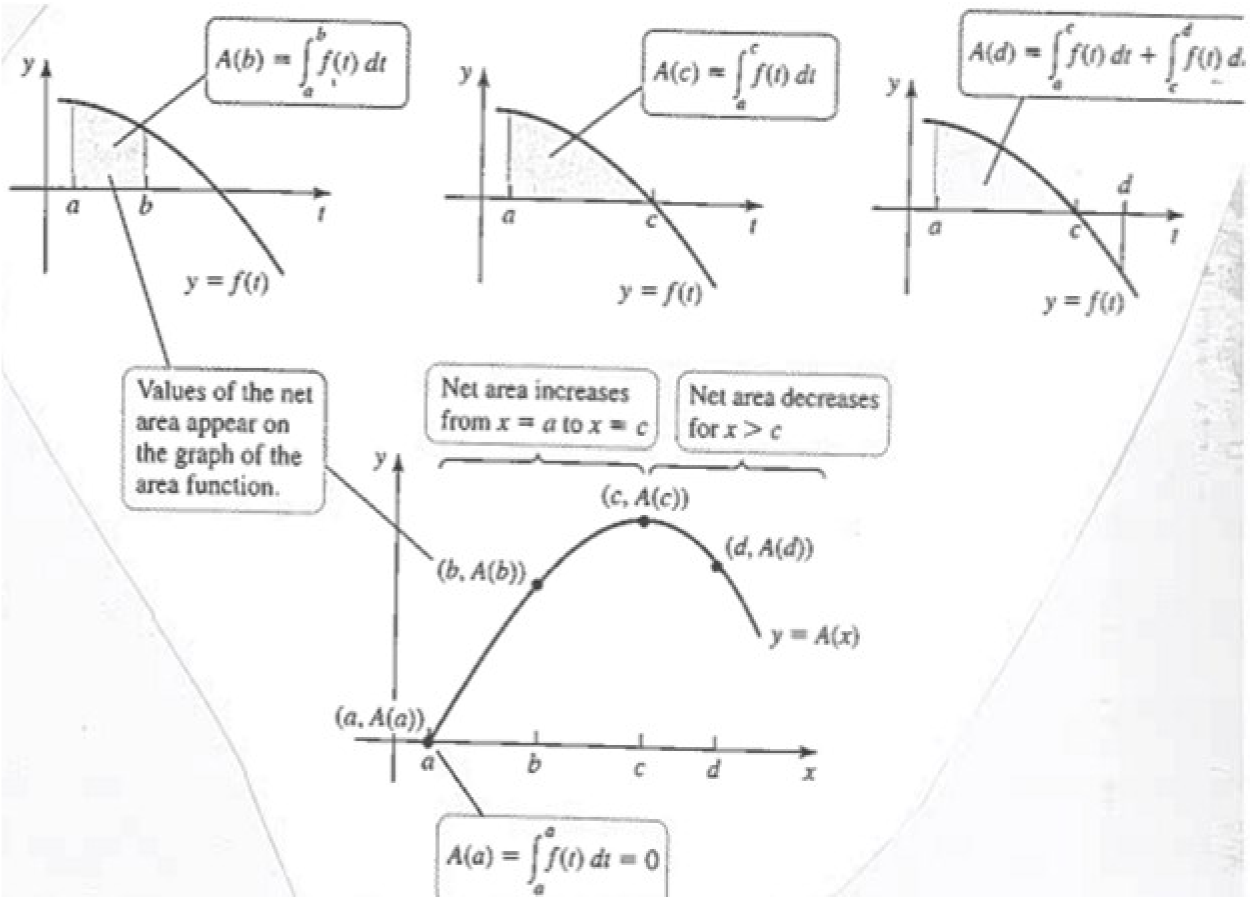
y = f(t) which will be defined for all points $t \geq a$.

The area function for *f* with a left endpoint at *a* is given by $A(x) = \int_a^x f(t)\,dt$. This gives the net area of the region between the graph of *f* and the *t*-axis between the points $t = a$ and $t = x$.



Area Functions

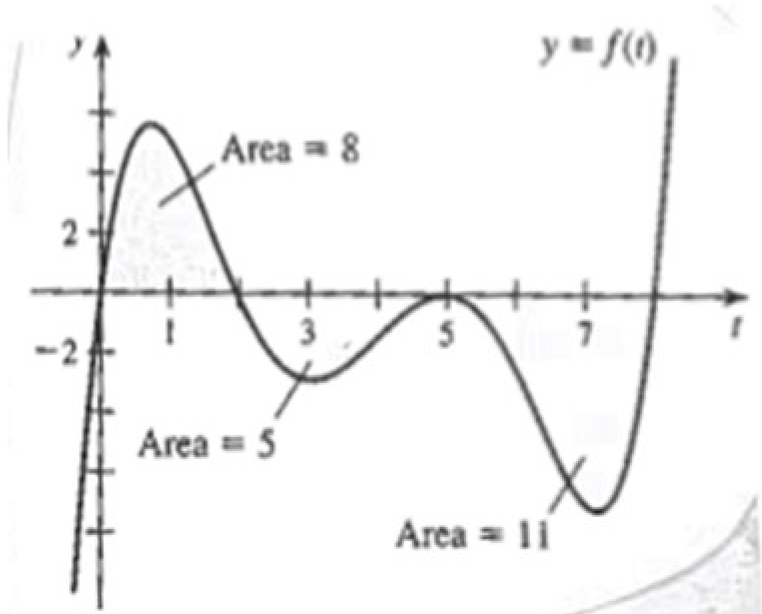
Figure 5.33 (p. 335) illustrates how the area function works.



Exercise

The graph of *f* is shown below. Let $A(x)=\int_0^x f(t)\,dt$ and $F(x)=\int_2^x f(t)\,dt$ be two area functions for *f*.

What is A(2)? F(5)? A(5)? F(8)?



Building the Case

Linear functions help to build the rationale behind the Fundamental Theorem of Calculus

EX: Let *f(t) = 4t + 3* and define $A(x) = \int_1^x f(t)\,dt$.

What is A(2)? A(4)? A(x)? A'(x)?

The Fundamental Theorem of Calculus (Part 1)

In general, the property illustrated with the previous linear function works for all continuous functions and is one part of the FTC.

THEOREM: If *f* is continuous on [a, b], then the area function $A(x) = \int_a^x f(t)dt$ for $a \le x \le b$ is continuous on [a, b] and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$; or equivalently,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

which means that the area function of *f* is an antiderivative of *f*.

The Fundamental Theorem of Calculus (Part 2)

Given that A is an antiderivative of *f*, this provides us with an effective method for evaluating definite integrals and finding areas under curves.

THEOREM: If *f* is continuous on [a, b] and *F* is any antiderivative of *f*, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Overview of FTC

In essence, to evaluate an integral, we

- Find any antiderivative of *f*, and call it *F*;
- Compute *F(b) – F(a)*, the difference in the values of *F* between the upper and lower limits of integration.

The two parts of the FTC illustrate the inverse relationship between differentiation and integration; that is, the integral "undoes" the derivative.

Exercise

The various parts of the FTC can be used to simplify and evaluate integral expressions.

- Use Part 1 of the FTC to simplify $\dfrac{d}{dx}\displaystyle\int_x^{10} \dfrac{dz}{z^2+1}$
- Use Part 2 of the FTC to evaluate $\displaystyle\int_0^{\pi} (1-\sin x)\,dx$

Homework from Section 5.3

Do problems 11-17 all, 19-39 odd, 45-57 odd (pgs. 346-347).