Quiz 11 Solutions pg. 1

① Rolle's Thm hypotheses:

① f is continuous on $[a,b]$.

② f is differentiable on (a,b).

③ $f(a)=f(b)$

Mean Value Thm hypotheses:

① f is continuous on $[a,b]$

② f is differentiable on (a,b).

② Since f is continuous on $[-5,3]$ and differentiable on $(-5,3)$ [because f is a polynomial], and because $f(-5)=f(3)=0$, then Rolle's Thm applies.

$$f'(x)=16x^3+24x^2-120x = 8x(2x^2+3x-15)$$

$$x=\frac{-3\pm\sqrt{9-4(2)(-15)}}{4}=\frac{-3\pm\sqrt{129}}{4}=2.089 \;\&\; -3.589. \qquad 8x=0 \Rightarrow x=0.$$

So the points guaranteed to exist by Rolle's Thm are $x=-3.589, 0, +2.089$.

③ Since g is not continuous or differentiable @ $x=2$, then the hypotheses of the Mean Value Thm are not satisfied & thus the MVT cannot be applied.

④ Since h is continuous on $[2,7]$ and differentiable on $(2,7)$ [the only discontinuity of h is @ $x=1$, which is not in the interval $[2,7]$], then the MVT applies to $h(x)$.

$$\frac{h(7)-h(2)}{7-2}=\frac{9.5-7}{5}=\frac{1}{2}.$$

$$h'(x)=1-\frac{3}{(x-1)^2}. \qquad \text{So} \quad 1-\frac{3}{(x-1)^2}=\frac{1}{2}$$

$$\Rightarrow -\frac{3}{(x-1)^2}=-\frac{1}{2}$$

$$\Rightarrow 6=(x-1)^2 \Rightarrow \pm\sqrt{6}=x-1 \Rightarrow x=1\pm\sqrt{6}.$$

Since the point needs to be in the interval $[2,7]$, then

$$x=1+\sqrt{6}.$$

5. $f(x) = Ax^3 - 3x^2 + 5$

$f(3) = 27A - 22$

$f(0) = 5$

$$\frac{f(3)-f(0)}{3-0} = \frac{27A-22-5}{3} = \frac{27A-27}{3} = 9A-9$$

$f'(x) = 3Ax^2 - 6x$

The MVT is guaranteed at $x=2$: $f'(2) = 12A - 12$

So $12A - 12 = 9A - 9 \Rightarrow 3A = 3 \Rightarrow \underline{A=1}$

6. $\displaystyle\lim_{t\to\infty}\left(1+\frac{9}{t^3}\right)^{t^3}$

$$L = \lim_{t\to\infty} \ln\left(1+\frac{9}{t^3}\right)^{t^3} = \lim_{t\to\infty} t^3 \ln\left(1+\frac{9}{t^3}\right) = \lim_{t\to\infty} \frac{\ln\left(1+\frac{9}{t^3}\right)}{\frac{1}{t^3}}$$

Let $u = \frac{1}{t^3}$. So $\displaystyle\lim_{u\to 0^+} \frac{\ln(1+9u)}{u} \overset{LH}{=} \lim_{u\to 0^+} \frac{\frac{1}{1+9u}\cdot 9}{1} = \lim_{u\to 0^+} \frac{9}{1+9u} = 9$

So $\displaystyle\lim_{t\to\infty}\left(1+\frac{9}{t^3}\right)^{t^3} = \boxed{e^9}$.

7. $$\lim_{y\to 0} \ln(1+3y)^{4/y} = \lim_{y\to 0} \frac{4}{y}\ln(1+3y) = \lim_{y\to 0}\frac{4\ln(1+3y)}{y} \overset{LH}{=} \lim_{y\to 0} \frac{4\cdot\frac{1}{1+3y}\cdot 3}{1}$$

$$= \lim_{y\to 0} \frac{12}{1+3y} = \boxed{12}$$

8. $$\lim_{x\to\infty} x - \sqrt{x^2-5x+3} = \lim_{x\to\infty} x - \sqrt{x^2\left(1-\frac{5}{x}+\frac{3}{x^2}\right)} = \lim_{x\to\infty} x - x\sqrt{1-\frac{5}{x}+\frac{3}{x^2}}$$

$$= \lim_{x\to\infty} x\left(1-\sqrt{1-\frac{5}{x}+\frac{3}{x^2}}\right) = \lim_{x\to\infty}\frac{1-\sqrt{1-\frac{5}{x}+\frac{3}{x^2}}}{\frac{1}{x}}$$

Let $t = \frac{1}{x}$

$$= \lim_{t\to 0^+} \frac{1-\sqrt{1-5t+3t^2}}{t} \overset{LH}{=} \lim_{t\to 0^+} \frac{-\frac{1}{2\sqrt{1-5t+3t^2}}\cdot 6t-5}{1} = \lim_{t\to 0^+} \frac{5-6t}{2\sqrt{1-5t+3t^2}}$$

$$= \frac{5}{2\sqrt{1}} = \boxed{\frac{5}{2}}$$

9. $$\lim_{x\to-\infty} (x^2-3x+4)e^x = \lim_{x\to-\infty} \frac{x^2-3x+4}{e^{-x}} \overset{LH}{=} \lim_{x\to-\infty} \frac{2x-3}{-e^{-x}}$$

$$\overset{LH}{=} \lim_{x\to-\infty} \frac{2}{e^{-x}} = \boxed{0}$$

10. $$\lim_{x\to\infty} \frac{h(x)}{f(x)} = \lim_{x\to\infty} \frac{\ln\sqrt{x+1}}{(x-4)^3} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x+1}}\cdot\frac{1}{2\sqrt{x+1}}}{3(x-4)^2} = \lim_{x\to\infty} \frac{1}{2(x+1)}\cdot\frac{1}{3(x-4)^2} = 0,$$

So $h \ll f$

$$\lim_{x\to\infty} \frac{f(x)}{j(x)} = \lim_{x\to\infty} \frac{(x-4)^3}{2^{2x}} = \lim_{x\to\infty} \frac{(x-4)^3}{4^x} \overset{LH}{=} \lim_{x\to\infty} \frac{3(x-4)^2}{4^x \ln 4}$$

$$\overset{LH}{=} \lim_{x\to\infty} \frac{6(x-4)}{4^x(\ln 4)^2} \overset{LH}{=} \lim_{x\to\infty} \frac{6}{4^x(\ln 4)^3} = 0.$$

So $f \ll j$.

$$\lim_{x\to\infty} \frac{j(x)}{g(x)} = \lim_{x\to\infty} \frac{2^{2x}}{e^{x^2-5x}} = \lim_{x\to\infty} \frac{4^x}{e^{(x-5)x}} = \lim_{x\to\infty} \left(\frac{4}{e^{x-5}}\right)^x = 0.$$

So $j \ll g$.

So $h(x) \ll f(x) \ll j(x) \ll g(x)$.