Mon 22 June

- Midterm Median = 125/175 (before the curve). See webpage for the curve.
- Quiz Medians
    - Quiz 3: 18/20
    - Quiz 4: 11/15
    - Quiz 5: 20/23
- Grades... The syllabus contains the information to compute your grade. For the exams, the curve information is on the course webpage; scale your letter grade range to 10 points to get an exact percentage.
    - Attendance: There is a correlation between good attendance and good scores.

Mon 22 June (cont.)

- MLP: This is the most direct way you can improve your grade. Theoretically, everyone should have an A in this area. If the problems are hard, come to office hours or otherwise get help.
- Don't skip quizzes. Right now there aren't enough to drop any so everyone's lowest scores are hurting them.
- You are always welcome to email me or come to office hours to get information about your grade.

- 2 quizzes this week...
- next Monday Quiz: Optimization ($\oint$4.4) with Related Rates sprinkled in. Closed book. Not collaborative.

$\oint$ 3.10 Related Rates

In this section, we use our knowledge of derivatives to examine how variables change with respect to time. The prime feature of these problems is that two or more variables, which are related in a known way, are themselves changing in time. The goal of these types of problems is to determine the rate of change (i.e., the derivative) of one or more variables at a specific moment in time.

Question

The edges of a cube increase at a rate of 2 cm/sec. How fast is the volume changing when the length of each edge is 50 cm?

Variables:

$$V = \text{volume (cm}^3\text{)}$$

$$x = \text{length of an edge (cm)}$$

Relations:

$$V = x^3 \text{ cm}^3$$

Rates Known:

$$\frac{dx}{dt} = 2 \text{ cm/sec}$$

Want to Find:

$$\left.\frac{dV}{dt}\right|_{x=50 \text{ cm}}$$

Both V and x are functions of t (their respective sizes are dependent upon how much time has passed), where t is in seconds.

We can write $V(t) = x(t)^3$ and then differentiate with respect to t:

$$V'(t) = 3x(t)^2 x'(t) \quad \text{or,}$$
$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

Note that $x = x(t)$ is the length of the cube's edges at time t, and $\frac{dx}{dt} = x'(t)$ is the rate at which the edges are changing at time t.

So the rate of change of the volume when $x = 50$ cm is

$$\left.\frac{dV}{dt}\right|_{x=50} = 3 \cdot 50^2 \cdot 2 = \boxed{15\,000 \text{ cm}^3/\text{sec.}}$$

Steps for Solving Related Rates Problems

1. Read the problem carefully, making a sketch to organize the given information. Identify the rates that are given and the rate that is to be determined.
2. Write one or more equations that express the basic relationships among the variables.
3. Introduce rates of change by differentiating the appropriate equation(s) with respect to time t.
4. Substitute known values and solve for the desired quantity.
5. Check that the units are consistent and the answer is reasonable.

Question

A jet ascends at a 10° angle from the horizontal with an airspeed of 550 mph (i.e., its speed along its line of flight is 550 mph).

(a) How fast is the altitude of the jet increasing?

(b) If the sun is directly overhead, how fast is the shadow of the jet moving on the ground?

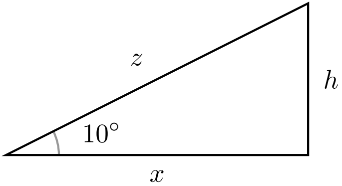
Step 1: There are three variables:

$x =$ distance the shadow has traveled (miles)

$h =$ altitude of the jet (miles)

$z =$ distance the jet has traveled on its line of flight (miles)

These variables are related through a right triangle:



The rate we know is

$$\frac{dz}{dt} = 550 \text{ mph}$$

and the rates we want to find are

$$\frac{dx}{dt} \quad \text{and} \quad \frac{dh}{dt}.$$

Step 2: To answer part (a), how fast the altitude is increasing, we need an equation involving only h and z. Using trigonometry,

$$\sin(10°) = \frac{h}{z} \implies h = \sin(10°) \cdot z.$$

To answer part (b), how fast the shadow is moving, we need an equation involving only x and z. Using trigonometry,

$$\cos(10°) = \frac{x}{z} \implies x = \cos(10°) \cdot z.$$

Step 3: We can now differentiate each equation to answer each question:

$$h = \sin(10^\circ) \cdot z \implies \frac{dh}{dt} = \sin(10^\circ)\frac{dz}{dt}$$
$$x = \cos(10^\circ) \cdot z \implies \frac{dx}{dt} = \cos(10^\circ)\frac{dz}{dt}$$

Step 4: We know that $\frac{dz}{dt} = 550$ mph. So

$$\frac{dh}{dt} = \sin(10^\circ) \cdot 550 \approx 95.5 \text{ mph}$$
$$\frac{dx}{dt} = \cos(10^\circ) \cdot 550 \approx 541.6 \text{ mph}$$

Step 5: Because both answers are in terms of miles per hour and both answers seem reasonable within the context of the problem, we conclude that the jet is gaining altitude at a rate of 95.5 mph, while the shadow on the ground is moving at about 541.6 mph.

Exercise

The sides of a cube increase at a rate of R cm/sec. When the sides have a length of 2 cm, what is the rate of change of the volume?

Exercise

Two boats leave a dock at the same time. One boat travels south at 30 mph and the other travels east at 40 mph. After half an hour, how fast is the distance between the boats increasing?

3.10 Book Problems

5-12, 14-15, 17-18, 30-31

$\oint$ 4.1 Maxima and Minima

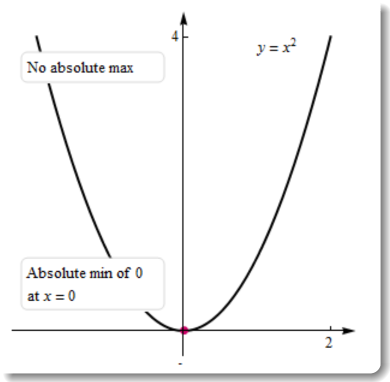
Definition

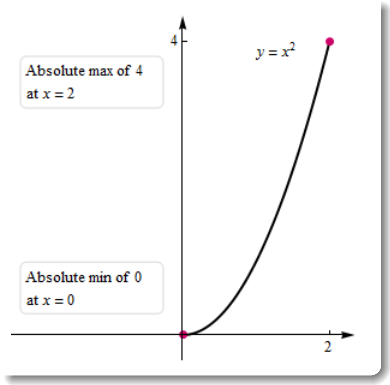
Let f be defined on an interval I containing c.

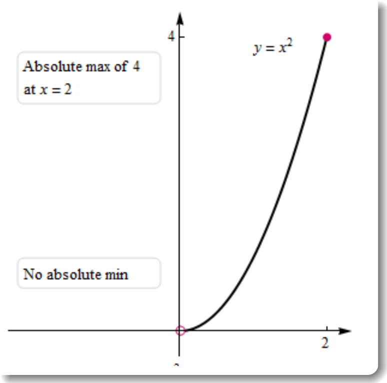
(a) f has an **absolute maximum** value on I at c means $f(c) \geq f(x)$ for every x in I.

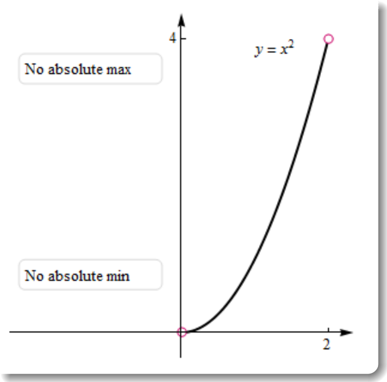
(b) f has an **absolute minimum** value on I at c means $f(c) \leq f(x)$ for every x in I.

The existence and location of absolute extreme values depend on the function and the interval of interest:






Absolute max of 4 at $x = 2$
No absolute min
$y = x^2$
4
2


No absolute max
No absolute min
$y = x^2$
4
2

Theorem (Extreme Value Theorem)

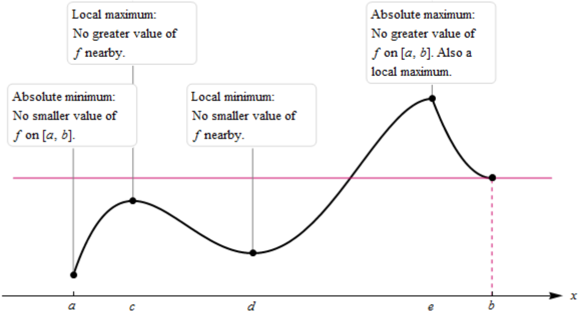
A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.

The EVT provides the criteria to ensure the existence of absolute extrema:

- the function must be continuous on the interval of interest;
- the interval of interest must be closed and bounded.

Local Maxima and Minima

Beyond absolute extrema, a graph may have a number of peaks and dips throughout its interval of interest:



Definition (local extrema)

Suppose I is an interval on which f is defined and c is in I.

(a) $f(c)$ is a **local maximum** value of f means there is some open interval within I and containing c, where $f(c) \geq f(x)$ for all x.

(b) $f(c)$ is a **local minimum** value of f means there is some open interval within I and containing c, where $f(c) \leq f(x)$ for all x.