Example: Find two nonnegative numbers x and y for which $2x + y = 30$ and xy^2 is maximized.

Objective Function: $P(x,y) = xy^2$

Constraint(s): $2x + y = 30$ and $x, y \geq 0$

$\Rightarrow y = 30 - 2x$

$\Rightarrow P(x) = x(30-2x)^2$ domain: $x \geq 0$, $30 - 2x \geq 0$ $\Rightarrow 15 \geq x$ $[0, 15]$

$P'(x) = (1)(30-2x)^2 + x(2(30-2x))(-2)$

$= \underbrace{(30-2x)}_{x=15}\underbrace{(30-2x-4x)}_{\substack{30-6x \\ x=5}} = 0$

CP's

$P(0) = 0(30-2(0))^2 = 0$

$P(5) = 5(\underbrace{30-2(5)}_{30-10})^2 = 2000 \leftarrow$ max

$P(15) = 15(\underbrace{30-2(15)}_{0})^2 = 0$

If $x=5$ then $y = 30 - 2(5) = 20$,

So $\boxed{\begin{matrix} x = 5 \\ y = 20 \end{matrix}}$

Find two nonnegative numbers x and y for which $x + 3y = 30$ such that x^2y is maximized.

Objective Function:
$P(x,y) = x^2 y$

Constraint(s): $x + 3y = 30$ and $x, y \geq 0$

$\Rightarrow x = 30 - 3y$

A. $x = 20, y = \frac{10}{3}$

B. $x = 10, y = \frac{20}{3}$

C. $x = \frac{5}{3},\ y = 20$

D. $x = \cancel{10}\ 25, y = \frac{5}{3}$

$\Rightarrow P(y) = (30-3y)^2 y$

$P'(y) = 2(30-3y)(-3)y + (30-3y)^2(1)$

$= (30-3y)(-6y + 30 - 3y) = 0$

$y = 10$; $30 - 9y$; $y = \frac{30}{9} = \frac{5}{3}$

$\Rightarrow$ If $y = 5/3$ then $x = 30 - 3\left(\frac{5}{3}\right) = 25$

domain: $y \geq 0$

$x \geq 0 \Rightarrow 30 - 3y \geq 0 \Rightarrow 10 \geq y$

$P(0) = (30 - 3(0))^2(0) = 0$

$P\left(\frac{5}{3}\right) = \left(30 - 3\left(\frac{5}{3}\right)\right)^2\left(\frac{5}{3}\right) = (25)^2\left(\frac{5}{3}\right) \leftarrow$ max

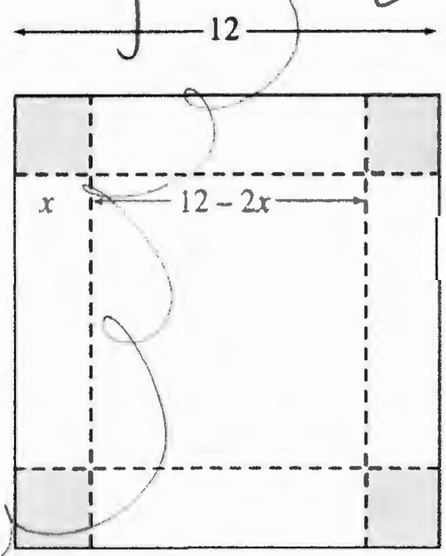
$P(10) = (30 - 3(10))^2(10) = 0$

An open box is to be made by cutting a square from each corner of a $12in$ by $12in$ piece of metal and then folding up the sides. What size square should be cut from each corner to produce a box of maximum volume? Objective: maximize volume

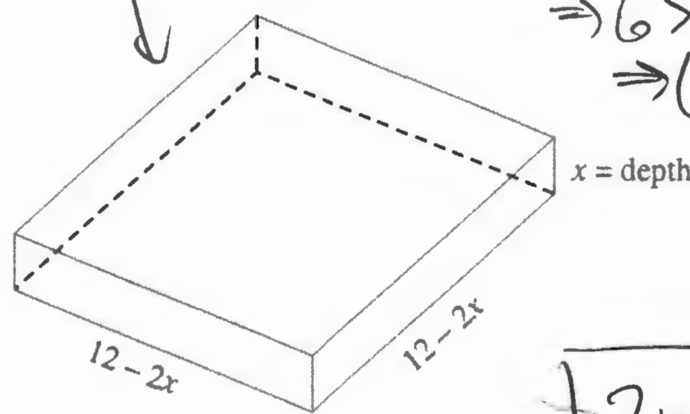
$V(x)=(12-2x)^2 x$

*Note: Even though the domain $(0,6)$ is not closed it is bounded. So we can make it a little bigger by adding endpoints $x=0,6$

Constraint(s): and domain: $x>0$ and $12-2x>0$ $\Rightarrow 6>x$ $\Rightarrow (0,6)$ is the domain



(a)



(b)

$\boxed{2in \times 2in}$

$V'(x)=2(12-2x)(-2)x+(12-2x)^2(1)$

$=(12-2x)(-4x+12-2x)=0$

$x=6 \qquad 12-6x \Rightarrow x=2$

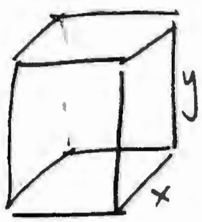
CP's: $x=6,2$

$V(0)=(12-2(0))^2(0)=0$

$V(2)=(12-2(2))^2(2)=128 \leftarrow$ max

$V(6)=(12-2(6))^2(6)=0$

Suppose you are constructing an open-top rectangular box with a square base and a volume of $32in^3$. What dimensions of the box will ~~maximize~~ minimize the surface area?



$x, y > 0$

Objective: maximize surface area

$$A(x,y) = \begin{pmatrix}\text{area of}\\ \text{base}\end{pmatrix} + 4\begin{pmatrix}\text{area of}\\ \text{a wall}\end{pmatrix}$$

$$= x^2 + 4xy$$

Constraint(s): Volume $= x^2 y = 32 \Rightarrow y = \frac{32}{x^2} > 0 \Rightarrow \frac{x^2}{32} > 0$
$\Rightarrow x^2 > 0$

domain: $(0, \infty)$

Surface area:

$$A(x) = x^2 + 4\cancel{x}\left(\frac{32}{x^{\cancel{2}}}\right) = x^2 + \frac{128}{x}$$

$128x^{-1}$

$$A'(x) = 2x - \frac{128}{x^2} = 0$$

$128x^{-2}$

multiply by x^2 $\Rightarrow 2x^3 - 128 = 0$

$$x^3 = 64 \Rightarrow x = 4$$

The domain is not closed so use the 2nd Deriv Test.

$$A''(x) = 2 - \frac{128(-2)}{x^3} = 2 + \frac{256}{x^3}$$

$$A''(4) = 2 + \frac{256}{4^3} > 0 \checkmark \leftarrow \text{min}$$

$$x = 4 \Rightarrow y = \frac{32}{4^2} = 2$$

$\boxed{4in \times 4in \times 2in}$

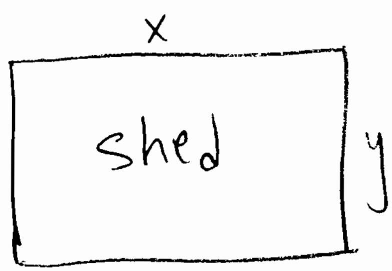
A carpenter is building a rectangular shed with a fixed perimeter of 52 feet. What are the dimensions of the largest shed that can be built?

A. 12ft x 20ft

B. 13ft x 26ft

C. 12ft x 13ft

D. 13ft x 13ft



$x, y \geq 0$

Objective: maximize area

$A(x,y) = xy$

Constraint(s): $2x + 2y = 52$

$\Rightarrow x + y = 26$

$y = 26 - x \geq 0$

domain: $[0, 26]$

$A(x) = x(26 - x)$

$= 26x - x^2$

$A'(x) = 26 - 2x = 0$

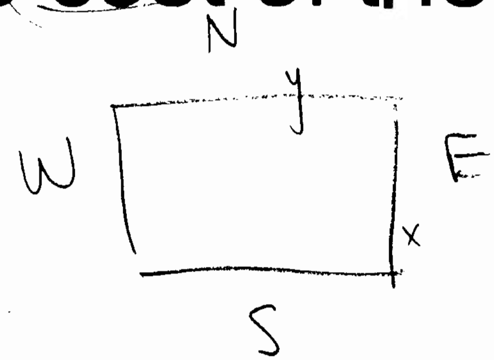
$\Rightarrow x = 13$

$A(0) = 0(26 - 0) = 0$

$A(13) = 13(26 - 13) = 13^2 \leftarrow$ max

$A(26) = 26(26 - 0) = 0$

A fence must be built to enclose an area of 20,000 ft^2. Fencing costs \$1 per foot for the two sides facing North and South and \$2 per foot for the sides facing East and West. Find the cost of the least expensive fence.



Obj: minimize price

$P(x,y) = 2(\text{cost of } y) + 2(\text{cost of } x)$

$= 2(\$1)y + 2(\$2)x$

$= 2y + 4x$

Constraint(s):

$xy = 20000 \, ft^2 \rightarrow y = \frac{20000}{x}$

Check $P(100) = \frac{40000}{100} + 4(100) = \$800$

A. $\$\pi$

B. \$800

C. \$100

D. \$200

$P(x) = 2\left(\frac{20000}{x}\right) + 4x$ on $(0,\infty)$

$P'(x) = -\frac{40000}{x^2} + 4 = 0$

$\Rightarrow -\frac{40000}{x^2} = -4$

$x^2 = 10000$

$\Rightarrow x = \pm 100$ (− not in domain)

$P''(x) = -\frac{2(-40000)}{x^3}$

$P''(100) = \frac{80000}{10000} > 0$ min.

The llama population of a certain area can be modeled using the function $L(t) = 7te^{\frac{-t}{13}}$ where t is the number of years after 2015 and $L(t)$ is hundreds of llamas. In what year will the area be populated by the most llamas?

A. 2015

B. 2013

C. 2030

D. 2028

$L'(t) = 7\left(e^{-t/13}\right) + 7t\left(e^{-t/13}\right)\left(\frac{-1}{13}\right) = 0$

$\Rightarrow \underset{\text{never } 0}{e^{-t/13}}\left(7 - \frac{7}{13}t\right) = 0 \Rightarrow t = \frac{-7}{-7/13} = 13 \leftarrow CP$

$L''(t) = -\frac{1}{13}e^{-t/13}\left(7 - \frac{7}{13}t\right) + e^{-t/13}\left(-\frac{7}{13}\right)$

$= \underset{\text{pos}}{e^{-t/13}}\left(-\frac{1}{13}\left(7 - \frac{7}{13}t\right) - \frac{7}{13}\right)$

$L''(13) = e^{-13/13}\left(-\frac{1}{13}\left(7 - \frac{7}{13}(13)\right) - \frac{7}{13}\right) = -\frac{7}{13}e^{-1} < 0 \leftarrow \text{max}$

$$\begin{array}{r} 2015 \\ +\ 13 \\ \hline 2028 \end{array}$$