MATH 2554 (Calculus I)

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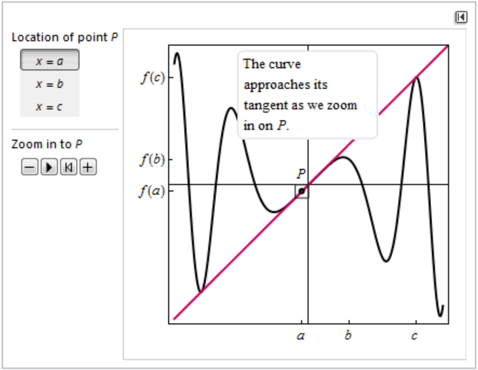
Monday 30 March (Week 11)

- Computer HW this week: $\oint 4.4 - 4.5$
- Quiz #9 due Tues 31 Mar
- Quiz #10 in class Tues 31 Mar on $\oint 4.4$. You can work in groups, get help from your drill instructor, etc. but everyone must turn in their own paper.
- Exam #3 Friday 3 April – up to $\oint 4.5$
- Quiz #11 Take Home given Thurs 9 Apr covers $\oint 4.6 - 4.7$

$\oint$ 4.5 Linear Approximation and Differentials

Suppose f is a function such that f' exists at some point P. If you zoom in on the graph, the curve appears more and more like the tangent line to f at P.



Linear Approximation

This idea – that smooth curves (i.e., curves without corners) appear straighter on smaller scales – is the basis of linear approximations.

One of the properties of a function that is differentiable at a point P is that it is locally linear near P (i.e., the curve approaches the tangent line at P.)

Therefore, it makes sense to approximate a function with its tangent line, which matches the value and slope of the function at P.

This is why you've had to do so many "find the equation for the tangent line to the given point" problems!

Definition

Suppose f is differentiable on an interval I containing the point a. The **linear approximation** to f at a is the linear function

$$L(x) = f(a) + f'(a)(x - a) \qquad \text{for } x \text{ in } I.$$

Remarks: Compare this definition to the following: At a given point $P = (a, f(a))$, the slope of the line tangent to the curve at P is $f'(a)$. So the equation of the tangent line is

$$y - f(a) = f'(a)(x - a).$$

(Yes, it is the same thing!)

Exercise

Write the equation of the line that represents the linear approximation to

$$f(x) = \frac{x}{x+1} \qquad \text{at } a = 1.$$

Then *use* the linear approximation to estimate $f(1.1)$.

Solution: First compute

$$f'(x) = \frac{1}{(x+1)^2}, \quad f(a) = \frac{1}{2}, \quad f'(a) = \frac{1}{4}$$

$$L(x) = \frac{1}{2} + \frac{1}{4}(x-1) = \frac{1}{4}x + \frac{1}{4}.$$

Solution (continued):

Because $x = 1.1$ is near $a = 1$, we can estimate $f(1.1)$ using $L(1.1)$:

$$f(1.1) \approx L(1.1) = 0.525$$

Note that $f(1.1) = 0.5238$, so the error in this estimation is

$$\frac{0.525 - 0.5238}{0.5238} \times 100 = 0.23\%.$$

Intro to Differentials

Our linear approximation $L(x)$ is used to approximate $f(x)$ when a is fixed and x is a nearby point:

$$f(x) \approx f(a) + f'(a)(x-a)$$

When rewritten,

$$f(x) - f(a) \approx f'(a)(x-a)$$

$$\implies \Delta y \approx f'(a)\Delta x.$$

A change in y can be approximated by the corresponding change in x, magnified or diminished by a factor of $f'(a)$.

This is another way to say that $f'(a)$ is the rate of change of y with respect to x!

$$\Delta y \approx f'(a)\Delta x$$

$$\frac{\Delta y}{\Delta x} \approx f'(a)$$

So if f is differentiable on an interval I containing the point a, then the change in the value of f (the Δy), between two points a and $a + \Delta x$ in I, is approximately $f'(x)\Delta x$.

We now have two different, but related quantities:

- The change in the function $y = f(x)$ as x changes from a to $a + \Delta x$ (which we call Δy).
- The change in the linear approximation $y = L(x)$ as x changes from a to $a + \Delta x$ (called the differential, dy).

$$\Delta y \approx dy$$

When the x-coordinate changes from a to $a + \Delta x$:

- The function change is **exactly** $\Delta y = f(a + \Delta x) - f(a)$.
- The linear approximation change is

$$\begin{aligned}\Delta L &= L(a + \Delta x) - L(a) \\ &= (f(a) + f'(a)(a + \Delta x - a)) - (f(a) + f'(a)(a - a)) \\ &= f'(a)\Delta x\end{aligned}$$

and this is dy.

We define the differentials dx and dy to distinguish between the change in the function (Δy) and the change in the linear approximation (ΔL):

- dx is simply the change in x, i.e. Δx.
- dy is the change in the linear approximation, which is $\Delta L = f'(a)\Delta x$.

SO:

$$\Delta L = f'(a)\Delta x$$

$$dy = f'(a)dx$$

$$\frac{dy}{dx} = f'(a) \qquad \text{(at } x = a\text{)}$$

Definition

Let f be differentiable on an interval containing x.

- A small change in x is denoted by the **differential** dx.
- The corresponding change in $y = f(x)$ is approximated by the **differential** $dy = f'(x)dx$; that is,

$$\Delta y = f(x + \Delta x) - f(x)$$
$$\approx dy = f'(x)dx.$$

The use of differentials is critical as we approach integration.

Example

Use the notation of differentials $[dy = f'(x)dx]$ to approximate the change in $f(x) = x - x^3$ given a small change dx.

Solution:

$f'(x) = 1 - 3x^2$, so $dy = (1 - 3x^2)dx$.
A small change dx in the variable x produces an approximate change of $dy = (1 - 3x^2)dx$ in y.

For example, if x increases from 2 to 2.1, then $dx = 0.1$ and

$$dy = \left(1 - 3(2)^2\right)(0.1) = -1.1.$$

This means as x increases by 0.1, y decreases by 1.1.

HW from Section 4.5

Do problems 7–9 all, 11, 12, 29–38 all (p. 273 in textbook)

Wednesday 1 April (Week 11)

- Computer HW this week: $\oint 4.4 - 4.5$
- Exam #3 Friday 3 April – up to $\oint 4.5$
- Quiz #11 Take Home given Thurs 9 Apr covers $\oint 4.6 - 4.7$
- today: Review for Exam 3

3.8 Derivatives of Logarithmic and Exponential Functions

- Be able to compute derivatives involving $\ln x$ and $\log_b x$
- Be able to compute derivatives of exponential functions of the form b^x
- Be able to use logarithmic differentiation to determine $f'(x)$

3.9 Derivatives of Inverse Trig Functions

- Know the derivatives of the six inverse trig functions.
- Also: You are responsible for every derivative rule and every derivative formula we have covered this semester.

3.10 Related Rates

- Know the steps to solving related rates problems, and be able to use them to solve problems given variables and rates of change.
- Be able to solve related rates problems. If, while doing the HW (paper or computer), you were provided a formula in order to solve the problem, then I will do the same. If you were not provided a formula while doing the HW (paper or computer), then I also will not provide the formula.

4.1 Maxima and Minima

- Know the definitions of maxima, minima, and what makes these points local or absolute extrema (both analytically and graphically).
- Know how to find critical points for a function.
- Given a function on a given interval, be able to find local and/or absolute extrema.
- Given specified properties of a function, be able to sketch a graph of that function.

4.2 What Derivatives Tell Us

- Be able to use the first derivative to determine where a function is increasing or decreasing.
- Be able to use the First Derivative Test to identify local maxima and minima. Be able to explain in words how you arrived at your conclusion.
- Be able to find critical points, absolute extrema, and inflection points for a function.
- Be able to use the second derivative to determine the concavity of a function.
- Be able to use the Second Derivative Test to determine whether a given point is a local max or min. Be able to explain in words how you arrived at your conclusion.
- Know your Derivative Properties!!! (see Figure 4.36 on p. 242)

4.3 Graphing Functions

- Be able to find specific characteristics of a function that are spelled out in the Graphing Guidelines on p. 248 (e.g., know how to find x- and y-intercepts, vertical/horizontal asymptotes, critical points, inflection points, intervals of concavity and increasing/decreasing, etc.).
- Be able to use these specific characteristics of a function to sketch a graph of the function.
- TIP: If you are looking for intervals for increasing/decreasing or concave up/down, you should *treat* the naughty points as critical points and/or possible points of inflection.

4.4 Optimization Problems

- Be able to solve optimization problems that maximize or minimize a given quantity.
- Be able to identify and express the constraints and objective function in an optimization problem.
- Be able to determine your interval of interest in an optimization problem (e.g., what range of x-values are you searching for your extreme points?)
- **As to formulas, the same comment made above with respect to formulas for related rates problems applies here as well.**

4.5 Linear Approximation and Differentials

- Be able to find a linear approximation for a given function.
- Be able to use a linear approximation to estimate the value of a function at a given point.
- Be able to use differentials to express how the change in x (dx) impacts the change in y (dy).

Pep Talk

- one 3×5 inch notecard, one-sided only
- Read the question!
- Do the book problems.
- Find a buddy who understands concepts a little better than you and work on problems for 2-3 hours. Then find a buddy who is struggling and work with them 2-3 hours. Explaining to someone else tests how deeply you really know the material. This strategy also helps reduce stress because it doesn't require you to devote a full day or night of studying, just 2-3 hours at a time of productive work.

Running out of Time on the Exam, cont.

- If you encounter an unfamiliar type of problem on the exam, relax, because it's most likely not a trick. The solutions will always rely on the information from the required reading/assignments. Take your time and do each baby step carefully.
- During the exam, do the problems you are most confident with first! Different people will find different problems easier.
- The exam is not a race. If you finish early take advantage of the time to check your work. You don't want to leave feeling smug about how quickly you finished only to find out next week you lost a letter grade's worth of points from silly mistakes.
- Algebra: Look at solutions to Quiz 10 (and others).