Calculus I (Math 2554)

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Dr. Ashley K. Wheeler

University of Arkansas

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The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

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∮ 3.7 Implicit Differentiation

Up to now, we have calculated derivatives of functions of the form $y = f(x)$, where y is defined **explicitly** in terms of x. In this section, we examine relationships between variables that are **implicit** in nature, meaning that y either is not defined explicitly in terms of x or cannot be easily manipulated to solve for y in terms of x.

The goal of **implicit differentiation** is to find a single expression for the derivative directly from an equation of the form $F(x, y) = 0$ without first solving for y.

Example

Calculate $\frac{dy}{dx}$ directly from the equation for the circle

$$x^2 + y^2 = 9.$$

Solution: To remind ourselves that x is our independent variable and that we are differentiating with respect to x, we can replace y with $y(x)$:

$$x^2 + (y(x))^2 = 9.$$

Now differentiate each term with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}((y(x))^2) = \frac{d}{dx}(9).$$

By the Chain Rule, $\frac{d}{dx}((y(x))^2) = 2y(x)y'(x)$ (Version 2), or $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$ (Version 1). So

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ \implies \quad \frac{dy}{dx} &= \frac{-2x}{2y} \\ &= -\frac{x}{y}. \end{aligned}$$

The derivative is a function of x and y, meaning we can write it in the form

$$F(x, y) = -\frac{x}{y}.$$

To find slopes of tangent lines at various points along the circle we just plug in the coordinates. For example, the slope of the tangent line at (0,3) is

$$\left.\frac{dy}{dx}\right|_{(x,y)=(0,3)} = -\frac{0}{3} = 0.$$

The slope of the tangent line at $(1, 2\sqrt{2})$ is

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,2\sqrt{2})} = -\frac{1}{2\sqrt{2}}.$$

Question

The following functions are implicitly defined:

- $x^2 + y^2 = 9$
- $x + y^3 - xy = 4$
- $\cos(x - y) + \sin y = \sqrt{2}$

For each of these functions, how would you find $\frac{dy}{dx}$?

Exercise

Find $\frac{dy}{dx}$ for $xy + y^3 = 1$.

Exercise

Find an equation of the line tangent to the curve $x^4 - x^2y + y^4 = 1$ at the point $(-1, 1)$.

Higher Order Derivatives

Example

Find $\dfrac{d^2y}{dx^2}$ if $xy + y^3 = 1.$

Power Rule for Rational Exponents

Implicit differentiation also allows us to extend the power rule to rational exponents: Assume p and q are integers with $q \neq 0$. Then

$$\frac{d}{dx}(x^{\frac{p}{q}}) = \frac{p}{q}x^{\frac{p}{q}-1}$$

(provided $x \geq 0$ when q is even and $\frac{p}{q}$ is in lowest terms).

Exercise

Prove it.

3.7 Book Problems

5-21 (odds), 27-45 (odds)

$\oint$ 3.8 Derivatives of Logarithmic and Exponential Functions

The natural exponential function $f(x) = e^x$ has an inverse function, namely $f^{-1}(x) = \ln x$. This relationship has the following properties:

1. $e^{\ln x} = x$ for $x > 0$ and $\ln(e^x) = x$ for all x.
2. $y = \ln x \quad \Longleftrightarrow \quad x = e^y$
3. For real numbers x and $b > 0$,

$$b^x = e^{\ln(b^x)} = e^{x \ln b}.$$

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Derivative of $y = \ln x$

Using 2. from the last slide, plus implicit differentiation, we can find $\frac{d}{dx}(\ln x)$. Write $y = \ln x$. We wish to find $\frac{dy}{dx}$. From 2.,

$$\frac{d}{dx}(x = e^y) \Rightarrow \frac{d}{dx}x = \frac{d}{dx}(e^y)$$

$$1 = e^y\left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

So $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Derivative of $y = \ln|x|$

Recall, we can only take "$\ln$" of a positive number. However:

- For $x > 0$, $\ln|x| = \ln x$, so

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}.$$

- For $x < 0$, $\ln|x| = \ln(-x)$, so

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln(-x)) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}.$$

In other words, the absolute values do not change the derivative of natural log.

Exercise

Find the derivative of each of the following functions:

- $f(x) = \ln(15x)$
- $g(x) = x \ln x$
- $h(x) = \ln(\sin x)$

Derivative of $y = b^x$

What about other logs? Say $b > 0$. Since $b^x = e^{\ln b^x} = e^{x \ln b}$ (by 3. on the earlier slide),

$$\begin{aligned}\frac{d}{dx}(b^x) &= \frac{d}{dx}(e^{x \ln b}) \\ &= e^{x \ln b} \cdot \ln b \\ &= b^x \ln b.\end{aligned}$$

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Exercise

Find the derivative of each of the following functions:

- $f(x) = 14^x$
- $g(x) = 45(3^{2x})$

Exercise

Determine the slope of the tangent line to the graph $f(x) = 4^x$ at $x = 0$.

Story Problem Example

Example

The energy (in Joules) released by an earthquake of magnitude M is given by the equation

$$E = 25000 \cdot 10^{1.5M}.$$

(a) How much energy is released in a magnitude 3.0 earthquake?

(b) What size earthquake releases 8 million Joules of energy?

(c) What is $\frac{dE}{dM}$ and what does it tell you?

Derivatives of General Logarithmic Functions

The relationship $y = \ln x \iff x = e^y$ applies to logarithms of other bases:

$$y = \log_b x \quad \iff \quad x = b^y.$$

Now taking $\frac{d}{dx}\left(x = b^y\right)$ we obtain

$$1 = b^y \ln b \left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{1}{b^y \ln b}$$

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

Neat Trick: Logarithmic Differentiation

Example

Compute the derivative of $f(x) = \dfrac{x^2(x-1)^3}{(3+5x)^4}$.

Solution: We can use logarithmic differentiation – first take the natural log of both sides and then use properties of logarithms.

$$\begin{aligned}\ln(f(x)) &= \ln\left(\frac{x^2(x-1)^3}{(3+5x)^4}\right)\\ &= \ln x^2 + \ln (x-1)^3 - \ln (3+5x)^4\\ &= 2\ln x + 3\ln(x-1) - 4\ln(3+5x)\end{aligned}$$

Now we take $\frac{d}{dx}$ on both sides:

$$\frac{1}{f(x)}\left(\frac{df}{dx}\right) = 2\left(\frac{1}{x}\right) + 3\left(\frac{1}{x-1}\right) - 4\left(\frac{1}{3+5x}\right)(5)$$

$$\frac{f'(x)}{f(x)} = \frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}$$

Finally, solve for $f'(x)$:

$$f'(x) = f(x)\left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}\right]$$

$$= \frac{x^2(x-1)^3}{(3+5x)^4}\left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}\right]$$

3.8 Book Problems

9-27 (odds), 31-37 (odds), 41-47 (odds)

∮ 3.9 Derivatives of Inverse Trigonometric Functions

Recall: If $y = f(x)$, then $f^{-1}(x)$ is the value of y such that $x = f(y)$.

Example

If $f(x) = 3x + 2$, then what is $f^{-1}(x)$?

NOTE: $f^{-1}(x) \neq f(x)^{-1} \left(= \dfrac{1}{f(x)} \right)$

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Derivative of Inverse Sine

Trig functions are functions, too. Just like with "f", there has to be something to "plug in". It makes no sense to just say $\sin$, without having $\sin($*something*$)$.

$$y = \sin^{-1} x \Longleftrightarrow x = \sin y$$

The derivative of $y = \sin^{-1} x$ can be found using implicit differentiation:

$$\begin{aligned} x &= \sin y \\ \frac{d}{dx}(x) &= \frac{d}{dx}(\sin y) \\ 1 &= (\cos y)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\cos y} \end{aligned}$$

We still need to replace $\cos y$ with an expression in terms of x. We use the trig identity $\sin^2 y + \cos^2 y = 1$ (careful with notation: in this case we mean $(\sin y)^2 + (\cos y)^2 = 1$). Then

$$\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - x^2}.$$

The range of $y = \sin^{-1} x$ is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. In this range, cosine is never negative, so we can just take the positive portion of the square root. Therefore,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}} \implies \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.$$

Exercise

Compute the following:

1. $\dfrac{d}{dx}\left(\sin^{-1}(4x^2-3)\right)$
2. $\dfrac{d}{dx}\left(\cos(\sin^{-1}x)\right)$

Derivative of Inverse Tangent

Similarly to inverse sine, we can let $y = \tan^{-1} x$ and use implicit differentiation:

$$\begin{aligned} x &= \tan y \\ \frac{d}{dx}(x) &= \frac{d}{dx}(\tan y) \\ 1 &= (\sec^2 y)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} \end{aligned}$$

Use the trig identity $\sec^2 y - \tan^2 y = 1$ to replace $\sec^2 y$ with $1 + x^2$:

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$$

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Derivative of Inverse Secant

$$\begin{aligned} y &= \sec^{-1} x \\ x &= \sec y \\ \frac{d}{dx}(x) &= \frac{d}{dx}(\sec y) \\ 1 &= \sec y \tan y \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\sec y \tan y} \end{aligned}$$

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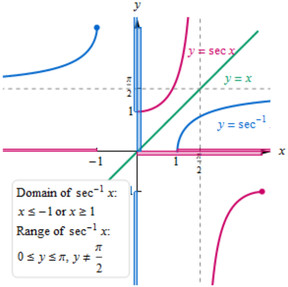
Use the trig identity $\sec^2 y - \tan^2 y = 1$ again to get

$$\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}.$$

This time, the $\pm$ matters:



- If $x \geq 1$, then $0 \leq y < \frac{\pi}{2}$ and so $\tan y > 0$.
- If $x \leq -1$, then $\frac{\pi}{2} < y \leq \pi$ and so $\tan y < 0$.

Therefore,

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

Using other trig identities (which you do not need to prove)

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2} \quad \cot^{-1} x + \tan^{-1} x = \frac{\pi}{2} \quad \csc^{-1} x + \sec^{-1} x = \frac{\pi}{2}$$

we can get the rest of the inverse trig derivatives.

All Other Inverse Trig Derivatives

To summarize:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$
$$(-1 < x < 1)$$
$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$
$$(-\infty < x < \infty)$$
$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$
$$(|x| > 1)$$

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Example

Compute the derivatives of $f(x) = \tan^{-1}\left(\frac{1}{x}\right)$ and $g(x) = \sin\left(\sec^{-1}(2x)\right)$.

Derivatives of Inverse Functions in General

Let f be differentiable and have an inverse on an interval I. Let x_0 be a point in I at which $f'(x_0) \neq 0$. Then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}$$

where $y_0 = f(x_0)$.

Example

Let $f(x) = \frac{1}{4}x^3 + x - 1$. Find $\left(f^{-1}\right)'(3)$.

3.9 Book Problems

7-27 (odds), 31-39 (odds)

Midterm (Exam #2) Review

- $\oint$2.1-2.2
  - **Material may not be explicitly tested, but the topics here are foundational to later sections.**

Midterm (Exam #2) Review (cont.)

- $\oint$2.3 Techniques for Computing Limits
  - **Be able to do questions similar to 1-48.**
  - Know and be able to compute limits using analytical methods (e.g., limit laws, additional techniques).
  - Be able to evaluate one-sided and two-sided limits of functions.
  - Know the Squeeze Theorem and be able to use this theorem to determine limits.

Midterm (Exam #2) Review (cont.)

Exercise (problems from past midterm)

Evaluate the following limits:

- $\displaystyle\lim_{x\to 3}\frac{x^2-x-6}{x^2-9}$
- $\displaystyle\lim_{\theta\to 0}\frac{\sec\theta\tan\theta}{\theta}$

Midterm (Exam #2) Review (cont.)

- $\oint$ 2.4 Infinite Limits
  - **Be able to do questions similar to 17-30.**
  - Be able to use a graph, a table, or analytical methods to determine infinite limits.
  - Be able to use analytical methods to evaluate one-sided limits.
  - Know the definition of a vertical asymptote and be able to determine whether a function has vertical asymptotes.

Midterm (Exam #2) Review (cont.)

- $\oint$2.5 Limits at Infinity
  - **Be able to do questions similar to 9-30 and 38-46.**
  - Be able to find limits at infinity and horizontal asymptotes.
  - Know how to compute the limits at infinity of rational functions and algebraic functions.
  - Be able to list horizontal and/or vertical asymptotes of a function.

Midterm (Exam #2) Review (cont.)

- $\oint$2.6 Continuity
  - **Be able to do questions similar to 9-44.**
  - Know the definition of continuity and be able to apply the continuity checklist.
  - Be able to determine the continuity of a function (including those with roots) on an interval.
  - Be able to apply the Intermediate Value Theorem to a function.

Midterm (Exam #2) Review (cont.)

Exercise (problem from past midterm)

Determine the value of k so the function is continuous on $0 \leq x \leq 2$.

$$f(x) = \begin{cases} x^2 + k & 0 \leq x \leq 1 \\ -2kx + 4 & 1 < x \leq 2 \end{cases}$$

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Midterm (Exam #2) Review (cont.)

- $\oint$3.1 Introducing the Derivative
  - **Be able to do questions similar to 11-32.**
  - Know the definition of a derivative and be able to use this definition to calculate the derivative of a given function.
  - Be able to determine the equation of a line tangent to the graph of a function at a given point.
  - Know the 3 conditions for when a function is not differentiable at a point, and why these three conditions make a function not differentiable at the given point.

Midterm (Exam #2) Review (cont.)

- $\oint$3.2 Rules for Differentiation
  - **Be able to do questions similar to 7-41.**
  - Be able to use the various rules for differentiation (e.g., constant rule, power rule, constant multiple rule, sum and difference rule) to calculate the derivative of a function.
  - Know the derivative of e^x.
  - Be able to find slopes and/or equations of tangent lines.
  - Be able to calculate higher-order derivatives of functions.

Midterm (Exam #2) Review (cont.)

Exercise

Given that $y = 3x + 2$ is tangent to $f(x)$ at $x = 1$ and that $y = -5x + 6$ is tangent to $g(x)$ at $x = 1$, write the equation of the tangent line to $h(x) = f(x)g(x)$ at $x = 1$.

Midterm (Exam #2) Review (cont.)

- $\oint$ 3.3 The Product and Quotient Rules
  - **Be able to do questions similar to 7-42 and 47-52.**
  - Be able to use the product and/or quotient rules to calculate the derivative of a given function.
  - Be able to use the product and/or quotient rules to find tangent lines and/or slopes at a given point.
  - Know the derivative of e^{kx}.
  - Be able to combine derivative rules to calculate the derivative of a function.

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Midterm (Exam #2) Review (cont.)

Note: Functions are not always given by a formula. When faced with a problem where you don't know where to start, go through the rules first.

Exercise

Suppose you have the following information about the functions f and g:

$$f(1) = 6 \quad f'(1) = 2 \quad g(1) = 2 \quad g'(1) = 3$$

- Let $F = 2f + 3g$. What is $F(1)$? What is $F'(1)$?
- Let $G = fg$. What is $G(1)$? What is $G'(1)$?

Midterm (Exam #2) Review (cont.)

- $\oint$3.4 Derivatives of Trigonometric Functions
  - **Be able to do questions similar to 1-55.**
  - Know the two special trigonometric limits

    $$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \text{and} \qquad \lim_{x\to 0}\frac{\cos x-1}{x}=0$$

    and be able to use them to solve other similar limits.
  - Know the derivatives of $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$, and be able to use the quotient rule to derive the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$.
  - Be able to calculate derivatives (including higher order) involving trig functions using the rules for differentiation.

Midterm (Exam #2) Review (cont.)

Exercise

Calculate the derivative of the following functions:

- $f(x) = (1 + \sec x) \sin^3 x$
- $g(x) = \dfrac{\sin x + \cot x}{\cos x}$

Exercise

Evaluate $\lim\limits_{x \to -3} \dfrac{\sin(x+3)}{x^2 + 8x + 15}$.

Midterm (Exam #2) Review (cont.)

- $\oint$3.5 Derivatives as Rates of Change
  - **Be able to do questions similar to 11-18.**
  - Be able to use the derivative to answer questions about rates of change involving:
    - Position and velocity
    - Speed and acceleration
    - Growth rates
    - Business applications

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Midterm (Exam #2) Review (cont.)

- Be able to use a position function to answer questions involving velocity, speed, acceleration, height/distance at a particular time t, maximum height, and time at which a given height/distance is achieved.
- Be able to use growth models to answer questions involving growth rate and average growth rate, and cost functions to answer questions involving average and marginal costs.

Midterm (Exam #2) Review (cont.)

- $\oint$3.6 The Chain Rule
  - **Be able to do questions similar to 7-43.**
  - Be able to use both versions of the Chain Rule to find the derivative of a composition function.
  - Be able to use the Chain Rule more than once in a calculation involving more than two composed functions.
  - Know and be able to use the Chain Rule for Powers:

$$\frac{d}{dx}\left(f(x)\right)^n = n\left(f(x)\right)^{n-1} f'(x)$$

Midterm (Exam #2) Review (cont.)

Exercise

Suppose $f(9) = 10$ and $g(x) = f(x^2)$. What is $g'(3)$?

Midterm (Exam #2) Review (cont.)

- $\oint$ 3.7 Implicit Differentiation
  - **Be able to do questions similar to 5-26 and 33-46.**
  - Be able to use implicit differentiation to calculate $\frac{dy}{dx}$.
  - Be able to use the derivative found from implicit differentiation to find the slope at a given point and/or a line tangent to the curve at the given point.
  - Be able to calculate higher-order derivatives of implicitly defined functions.
  - Be able to calculate $\frac{dy}{dx}$ when working with functions containing rational functions.

Midterm (Exam #2) Review (cont.)

Exercise

Use implicit differentiation to calculate $\frac{dz}{dw}$ for

$$e^{2w} = \sin(wz)$$

Exercise

If $\sin x = \sin y$, then

- $\frac{dy}{dx} = ?$
- $\frac{d^2y}{dx^2} = ?$

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Midterm (Exam #2) Review (cont.)

- $\oint$ 3.8 Derivatives of Logarithmic and Exponential Functions
  - Be able to compute derivatives involving $\ln x$ and $\log_b x$
  - Be able to compute derivatives of exponential functions of the form b^x
  - Be able to use logarithmic differentiation to determine $f'(x)$
- $\oint$ 3.9 Derivatives of Inverse Trig Functions
  - Know the derivatives of the six inverse trig functions.

Also: You are responsible for every derivative rule and every derivative formula we have covered this semester.

Running Out of Time on the Exam

- Do practice problems completely, from beginning to end (as if it were a quiz). You might think you understand something but when it's time to write down the details things are not so clear.
- Find a buddy who understands concepts a little better than you and work on problems for 2-3 hours. Then find a buddy who is struggling and work with them 2-3 hours. Explaining to someone else tests how deeply you really know the material. This strategy also helps reduce stress because it doesn't require you to devote a full day or night of studying, just 2-3 hours at a time of productive work.

Running Out of Time on the Exam (cont.)

- Don't count on cookie cutter problems. If you are doing a practice problem where you've memorized all the steps, make sure you understand why each step is needed. The exam problems may have a small variation from homeworks and quizzes. If you're not prepared, it'll come as a "twist" on the exam...
- If you encounter an unfamiliar type of problem on the exam, relax, because it's most likely not a trick. The solutions will always rely on the information from the required reading/assignments. Take your time and do each baby step carefully.
- During the exam, do the problems you are most confident with first! Different people will find different problems easier.

Running Out of Time on the Exam (cont.)

- During the exam, budget your time. Count the problems and divide by 80 minutes. The easier questions will take less time so doing them first leaves extra time for the harder ones. When studying, aim for 10 problems per hour (i.e., 6 minutes per problem).
- The exam is not a race. If you finish early take advantage of the time to check your work. You don't want to leave feeling smug about how quickly you finished only to find out next week you lost a letter grade's worth of points from silly mistakes.

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Other Study Tips

- Brush up on algebra, especially radicals, logs, common denominators, etc. Many times knowing the right algebra will simplify the problem!
- When in doubt, show steps.
- You will be punished for wrong notation. The slides for $\oint$3.1 show different notations for the derivative. Make sure whichever one you use in your work, that you are using it correctly.
- Read the question!
- Do the book problems.
- Look at the pictures in the book and the interactive applets on MLP.