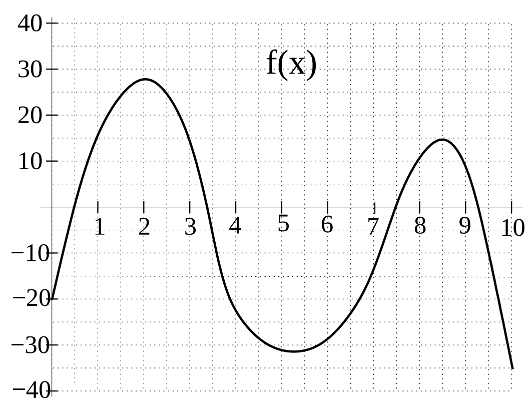
Section 4.1 – Using the First and Second Derivatives

Definitions. Let f be a function.

1. A *critical point* of f is a point p in the domain of f such that either $f'(p) = 0$ or $f'(p)$ is undefined.
2. We say that f has a *local minimum* at p if $f(p)$ is less than or equal to the values of f for points near p.
3. We say that f has a *local maximum* at p if $f(p)$ is greater than or equal to the values of f for points near p.
4. An *inflection point* of f is a point at which the function f changes concavity.

Example. Given to the right is the graph of a function f.



(a) Estimate the critical point(s) of f.

$$x = 2, \quad x = 5.3, \quad x = 8.5$$

(b) Estimate the inflection point(s) of f.

$$x = 3.4, \quad x = 7.2$$

(c) Does f have any local maximum or local minimum values? If so, list them, making it clear which are which.

$$\begin{aligned} f(2) &= 27 \text{ is a local maximum.} \\ f(5.3) &= -31 \text{ is a local minimum.} \\ f(8.5) &= 15 \text{ is a local maximum.} \end{aligned}$$

First Derivative Test. Suppose that p is a critical point of a continuous function f.

1. If f' changes from negative to positive at p, then f has a <u>local minimum</u> at $x = p$.
2. If f' changes from positive to negative at p, then f has a <u>local maximum</u> at $x = p$.

Second Derivative Test.

1. If $f'(p) = 0$ and $f''(p) > 0$, then f has a <u>local minimum</u> at $x = p$.
2. If $f'(p) = 0$ and $f''(p) < 0$, then f has a <u>local maximum</u> at $x = p$.

EXERCISES. Please do the following on a separate sheet of paper.

1. Let $f(x) = x^{2/3}(4-x)^{1/3}$.

   (a) Given that $f'(x) = \dfrac{8-3x}{3x^{1/3}(4-x)^{2/3}}$, find the intervals on which f is increasing/decreasing.

   First, we note that $f'(x) = 0$ when $8 - 3x = 0$, so solving for x indicates that $x = 8/3$ is one critical point. In addition, we can see that $x = 0$ and $x = 4$ are critical points since $f'(x)$ is undefined for these two values of x. We therefore obtain the sign chart shown to the right:

Interval	Sign of $f'(x)$
$x < 0$	$-$
$0 < x < 8/3$	$+$
$8/3 < x < 4$	$-$
$x > 4$	$-$

   We therefore conclude from the sign chart that f is increasing on $0 \le x \le 8/3$ and decreasing for $x \le 0$ and $8/3 \le x \le 4$.

   (b) Given that $f''(x) = \dfrac{-32}{9x^{4/3}(4-x)^{5/3}}$, find the intervals on which f is concave up/concave down.

   We can see from the formula for $f''(x)$ that $f''(x)$ never equals zero; however, $f''(x)$ is undefined for $x = 0$ and $x = 4$. We therefore obtain the sign chart shown to the right:

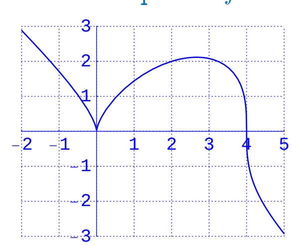
Interval	Sign of $f''(x)$
$x < 0$	$-$
$0 < x < 4$	$-$
$x > 4$	$+$

   We therefore conclude that f is concave up for $x > 4$ and concave down for $x < 0$ and $0 < x < 4$.

   (c) Find all local maxima, local minima, and inflection points of f.

   From the sign chart for f', we can see that f has a local minimum at $x = 0$ and a local maximum at $x = 8/3$, and we can see from the sign chart for f'' that f has an inflection point at $x = 4$. We summarize this information below:

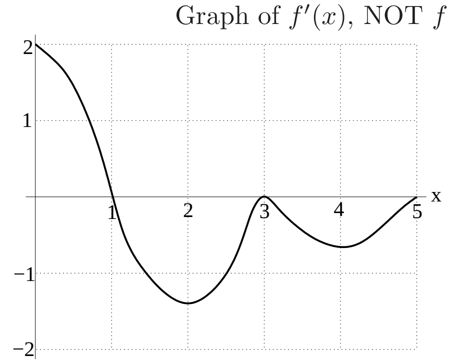
   Graph of f

   $$\begin{aligned} f(0) &= 0^{2/3}(4-0)^{1/3} = 0 \text{ is a local minimum.} \\ f(8/3) &= \left(\frac{8}{3}\right)^{2/3}\left(4 - \frac{8}{3}\right)^{1/3} = \frac{4}{3}\sqrt[3]{4} \text{ is a local maximum.} \\ (4, f(4)) &= (4, 0) \text{ is an inflection point} \end{aligned}$$

2. Given to the right is the graph of the DERIVATIVE of a function. Use this graph to help you answer the following questions about the ORIGINAL FUNCTION f.

   Graph of $f'(x)$, NOT $f(x)$

   (a) What are the critical points of f?

   $x = 1, \; x = 3, \; x = 5$

   (b) Where is f increasing? decreasing?

   increasing on $0 \le x \le 1$
   decreasing on $1 \le x \le 5$

   (c) Does f have any local maxima? If so, where?

   Yes, f has a local maximum at $x = 1$.

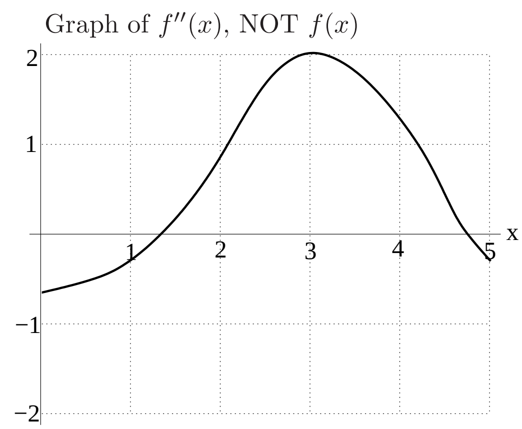
   (d) Does f have any local minima? If so, where?

   No, f has no local minima.

   (e) Where is f concave up? concave down?

   concave up on $2 < x < 3$ and $4 < x < 5$
   concave down on $0 < x < 2$ and $3 < x < 4$

3. Given to the right is the graph of the SECOND DERIVATIVE of a function. Use this graph to help you answer the following questions about the ORIGINAL FUNCTION f.



   (a) Where is f concave up? concave down?

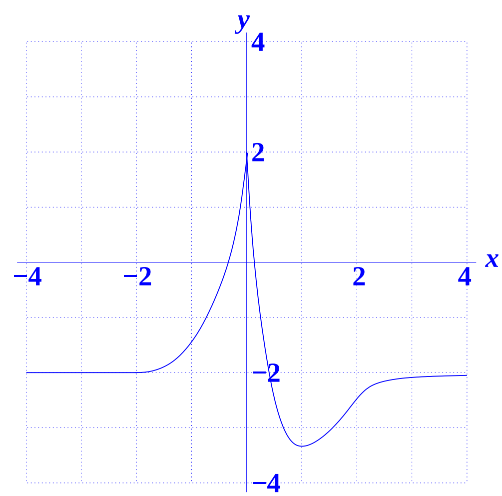
   `concave up on` $1.3 < x < 4.7$
   `concave down on` $0 < x < 1.3$ `and` $4.7 < x < 5$

   (b) Does f have any inflection points? If so, where?

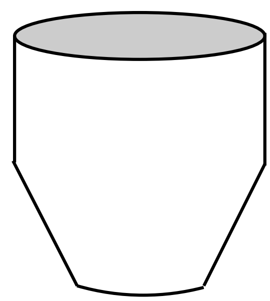
   `Yes, at` $x = 1.3$ `and` $x = 4.7$

4. Sketch the graph of ONE FUNCTION f that has ALL of the following properties.

   - f is continuous everywhere.
   - $f(0) = 2$.
   - $f'(x) = 0$ for $-4 \le x \le -2$.
     $f'(x) < 0$ for $0 < x < 1$.
     $f'(x) > 0$ for $-2 < x < 0$ and for $1 < x < 4$.
   - $f''(x) > 0$ for $-2 < x < 0$ and for $0 < x < 2$.
     $f''(x) < 0$ for $2 < x < 4$
   - $\lim_{x\to\infty} f(x) = -2$.



5. If water is flowing at a constant rate (i.e. constant volume per unit time) into the urn pictured to the right, sketch a graph of the depth of the water in the urn against time. Mark on the graph the time at which the water reaches the corner of the urn.



`Because the urn increases in width from floor level to the corner of the urn, the graph of depth versus time should be increasing at a decreasing rate (and therefore concave down) until` t^*`, the time when the water level reaches the corner. After this time, the width of the urn becomes constant, so the water level should increase at a constant rate, meaning that depth is a linear function of time to the right of` t^* `(see graph to the right).`