Survey of Calculus Unit 1 — Fall 2016

Practice Problems, Part II (with Solutions)

① Find $\dfrac{3x}{x^2+2x-15} - \dfrac{2x}{x^2+x-12}$.

Solution

$$\frac{3x}{x^2+2x-15} - \frac{2x}{x^2+x-12}$$

$$= \frac{3x}{(x+5)(x-3)} - \frac{2x}{(x+4)(x-3)}$$

$$= \frac{3x(x+4)}{(x+5)(x-3)(x+4)} - \frac{2x(x+5)}{(x+4)(x-3)(x+5)}$$

$$= \frac{3x^2+12x-(2x^2+10x)}{(x+5)(x-3)(x+4)}$$

$$\boxed{= \frac{x^2-2x}{(x+5)(x-3)(x+4)}} \quad \text{OR} \quad \boxed{\frac{x(x-2)}{(x+5)(x-3)(x+4)}}$$

→

(2) Solve for x:

$$0 = -5 + \sqrt{x^2+16}$$

$\boxed{\text{Solution}}$

(<u>Note</u>: You can't simplify to $0 = -5 + x + 4$)

$$0 = -5 + \sqrt{x^2+16}$$

$$5 = \sqrt{x^2+16}$$

$$25 = x^2 + 16$$

$$9 = x^2$$

$$0 = x^2 - 9$$

$$= (x+3)(x-3)$$

$$\Rightarrow \boxed{x = \pm 3}$$

(3) Solve for x:

$$0 = x^3 - 3x^2 - 4x$$

$\boxed{\text{Solution}}$

(<u>Note</u>: You can't cancel out x)

$$0 = x^3 - 3x^2 - 4x$$

$$= x(x^2 - 3x - 4)$$

$$= x(x-4)(x+1)$$

$$\Rightarrow \boxed{x = 0, 4, -1}$$

④ Simplify $\left(\frac{9c^2}{a^7}\right)^{3/2}\left(c^{\frac{1}{5}}\right)^3$.

Solution

$$\left(\frac{9c^2}{a^7}\right)^{3/2}\left(c^{\frac{1}{5}}\right)^3 = \frac{9^{3/2}\left(c^2\right)^{3/2}}{\left(a^7\right)^{3/2}}\left(c^{3/5}\right)$$

$$= \frac{\left(9^{\frac{1}{2}}\right)^3 c^{6/2}}{a^{21/2}}\left(c^{3/5}\right)$$

$$= \frac{3^3 c^{6/2+3/5}}{a^{21/2}}$$

$$= \frac{27c^{3+3/5}}{a^{21/2}} = \boxed{\frac{27c^{18/5}}{a^{21/2}}}$$

$\longrightarrow$

5) In 2000, the population of a country was approximately 5.88 million. The population is projected to grow exponentrally to 10 million in 2050. Determine the exponential function of the form

$$A(t) = A_0 e^{kt}$$

that models the population $A(t)$ of this country (in millions) t years after 2000.

$\boxed{\text{Solution}}$

In 2000, $t=0$, So

$$A(0) = A_0 \cancel{e^{k(0)}} = 5.88 \text{ million}$$

$$\Rightarrow A_0 = 5.88$$

In 2050, $t = 50$. So

$$A(50) = A_0 e^{k(50)}$$

$$= 5.88 e^{50k} = 10 \text{ (million)}$$

$$e^{50k} = \frac{10}{5.88}$$

$$50k = \ln\left(\frac{10}{5.88}\right)$$

$$k = \frac{\ln\left(\frac{10}{5.88}\right)}{50} \approx 0.0106$$

$\longrightarrow$

The formula becomes

$$A(t) = 5.88e^{\frac{\ln\left(\frac{10}{5.88}\right)}{50}t} \approx \boxed{5.88e^{0.0106t}} \text{ (approx.)}$$

$$= 5.88\left(e^{\ln\left(\frac{10}{5.88}\right)}\right)^{t/50}$$

OR $= \boxed{5.88\left(\frac{10}{5.88}\right)^{t/50}}$ (exact)

OR $\approx \boxed{5.88(1.7)^{t/50}}$

6) If $h(x) = x^2 - 2x$, then find $h(c+5)$.

Solution

$$h(c+5) = (c+5)^2 - 2(c+5)$$

$$= (c+5)\big((c+5) - 2\big)$$

$$= (c+5)(c+3)$$

$$= \boxed{c^2 + 8c + 15}$$

⑦ a. Write down the Continuity √List.

Solution

$f(x)$ is continuous at $x=c$ if:

① $f(c)$ is defined,

② $\lim_{x\to c} f(x)$ exists, and

③ $\lim_{x\to c} f(x) = f(c)$

b. At what value(s) is the function $h(x) = \frac{3x^2-9x-12}{x-4}$ discontinuous?

Solution

$x=4$

c. Rewrite $h(x)$ to make it continuous at all values of x.

Solution

Apply the Continuity √List to $x=4$.

In particular, ③ has to be satisfied.

$$\lim_{x\to 4} h(x) = \lim_{x\to 4} \frac{3x^2-9x-12}{x-4}$$

⟶

$$= \lim_{x\to 4} \frac{(3x+3)\cancel{(x-4)}}{\cancel{x-4}}$$

$$= \lim_{x\to 4} (3x+3) = 3(4)+3$$

$$= 15.$$

$$\Rightarrow h(4) = 15.$$

Rewriting,

$$h(x) = \begin{cases} \dfrac{3x^2-9x-12}{x-4} & x \neq 4 \\ 15 & x = 4 \end{cases}$$

⑧ Write (and simplify) the difference quotient for $f(x)=\sqrt{3x+1}$.

$\boxed{\text{Solution}}$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h}\left(\frac{\sqrt{3(x+h)+1}+\sqrt{3x+1}}{\sqrt{3(x+h)+1}+\sqrt{3x+1}}\right)$$

$$= \lim_{h\to 0} \frac{\left[\left(\sqrt{3(x+h)+1}\right)^2 \cancel{-\sqrt{3x+1}\left(\sqrt{3(x+h)+1}\right)} \cancel{+\sqrt{3(x+h)+1}\left(\sqrt{3x+1}\right)} + \left(\sqrt{3x+1}\right)^2\right]}{h\left(\sqrt{3(x+h)+1}+\sqrt{3x+1}\right)}$$

$\longrightarrow$

$$= \lim_{h\to 0} \frac{\left(3(x+h)+1\right)-\left(3x+1\right)}{h\left(\sqrt{3(x+h)+1}+\sqrt{3x+1}\right)}$$

$$= \lim_{h\to 0} \frac{3\cancel{x}+3h+\cancel{1}-3\cancel{x}-\cancel{1}}{h\left(\sqrt{3(x+h)+1}+\sqrt{3x+1}\right)}$$

$$= \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}\left(\sqrt{3(x+h)+1}+\sqrt{3x+1}\right)}$$

$$= \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)+1}+\sqrt{3x+1}}$$

$$= \frac{3}{\sqrt{3(x+0)+1}+\sqrt{3x+1}}$$

$$\boxed{= \frac{3}{2\sqrt{3x+1}}}$$

<u>Notes</u> about this problem:

- Use online resources or ask a friend to find $f'(x)$ (the derivative). You should get the same answer.
- This problem used the Conjugate Trick — see the U1L3 slides' solutions. In general, this is how you get radical signs to "cancel".