Fri 10 July

- MLP assignments are ALL reopened. They close the night before the final (Wednesday night).
- FINAL! is next Thursday, in class. Covers $\oint$3.10-5.5.
- one more quiz next week, probably Tuesday

Idea: Suppose we have $F(g(x))$, where F is an antiderivative of f. Then

$$\frac{d}{dx}\Big[F(g(x))\Big] = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$$

and $\displaystyle\int f(g(x)) \cdot g'(x)\ dx = F(g(x)) + C$

Let $u = g(x)$, where g' is continuous on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

u-Substitution is the Chain Rule, backwards.

Example

Evaluate $\displaystyle\int 8x\cos(4x^2+3)\;dx.$

Solution: Look for a function whose derivative also appears.

$$u(x) = 4x^2 + 3$$

$$\text{and } u'(x) = \frac{du}{dx} = 8x$$

$$\implies du = 8x\;dx.$$

Now rewrite the integral and evaluate. Replace u at the end with its expression in terms of x.

$$\begin{aligned}\int 8x\cos(4x^2+3)\ dx &= \int \cos(\underbrace{4x^2+3}_{u})\ \underbrace{8x\ dx}_{du}\\ &= \int \cos u\ du\\ &= \sin u + C\\ &= \sin(4x^2+3)+C\end{aligned}$$

We can check the answer – by the Chain Rule,

$$\frac{d}{dx}\left(\sin\left(4x^2+3\right)+C\right) = 8x\cos\left(4x^2+3\right).$$

Procedure for Substitution Rule (Change of Variables)

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\ dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

Warning: Not all integrals yield to the Substitution Rule.

Exercise

Evaluate the following integrals. Check your work by differentiating each of your answers.

1. $\displaystyle\int \sin^{10} x \cos x \; dx$
2. $\displaystyle -\int \frac{\csc x \cot x}{1+\csc x} \; dx$
3. $\displaystyle\int \frac{1}{(10x-3)^2} \; dx$
4. $\displaystyle\int (3x^2+8x+5)^8(3x+4) \; dx$

There are times when the u-substitution is not obvious or that more work must be done.

Example

Evaluate $\displaystyle\int \frac{x^2}{(x+1)^4}\,dx$.

Solution: Let $u = x + 1$. Then $x = u - 1$ and $du = dx$. Hence,

$$\begin{aligned}\int \frac{x^2}{(x+1)^4}\,dx &= \int \frac{(u-1)^2}{u^4}\,du \\ &= \int \frac{u^2 - 2u + 1}{u^4}\,du\end{aligned}$$

$$
\begin{aligned}
&= \int \left(u^{-2} - 2u^{-3} + u^{-4}\right)\, du \\
&= \frac{-1}{u} + \frac{1}{u^2} + \frac{-1}{3u^3} + C \\
&= \frac{-1}{x+1} + \frac{1}{(x+1)^2} - \frac{1}{3(x+1)^3} + C
\end{aligned}
$$

Exercise

Check it.

This type of strategy works, usually, on problems where u can be written as a linear function of x.

We can use the Substitution Rule for Definite Integrals in two different ways:

1. Use the Substitution Rule to find an antiderivative F, and then use the Fundamental Theorem of Calculus to evaluate $F(b) - F(a)$.
2. Alternatively, once you have changed variables from x to u, you may also change the limits of integration and complete the integration with respect to u. Specifically, if $u = g(x)$, the lower limit $x = a$ is replaced by $u = g(a)$ and the upper limit $x = b$ is replaced by $u = g(b)$.

The second option is typically more efficient and should be used whenever possible.

Example

Evaluate $\int_0^4 \frac{x}{\sqrt{9+x^2}}\, dx$.

Solution: Let $u = 9 + x^2$. Then $du = 2x\ dx$. Because we have changed the variable of integration from x to u, the limits of integration must also be expressed in terms of u. Recall, u is a function of x (the $g(x)$ in the Chain Rule). For this example,

$$\begin{aligned} x = 0 &\implies u(0) = 9 + 0^2 = 9 \\ x = 4 &\implies u(4) = 9 + 4^2 = 25 \end{aligned}$$

We had $u = 9 + x^2$ and $du = 2x\ dx \implies \frac{1}{2}du = x\ dx$. So:

$$\int_0^4 \frac{x}{\sqrt{9+x^2}}\ dx = \frac{1}{2}\int_9^{25} \frac{du}{\sqrt{u}}$$
$$= \frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right)\Bigg|_9^{25}$$
$$= \sqrt{25} - \sqrt{9}$$
$$= 5 - 3 = 2.$$

Exercise

Evaluate $\displaystyle\int_0^2 \frac{2x}{(x^2+1)^2}\,dx.$

5.5 Book Problems

9-39 (odds), 53-63 (odds)