Tips for Success

- Attend class every day. Participate, take notes, and ask questions.
- Don't get behind on MLP homeworks. Stay on top of the book problems.
- Be sure to seek assistance (tutoring, office hours, etc.) if you are struggling.
- Don't rely on success in high school calculus to save you in college calculus.
- Find a study partner(s) to meet with on a regular basis to cover questions and study for quizzes/exams.
- REMEMBER... THE TERM STARTS TODAY! SO DOES THE EVENTUAL EARNING OF YOUR FINAL GRADE!!!

§2.1 The Idea of Limits

Question

How would you define, and then differentiate between, the following pairs of terms?

- instantaneous velocity vs. average velocity?
- tangent line vs. secant line?

(Recall: What is a tangent line and what is a secant line?)

Example

An object is launched into the air. Its position s (in feet) at any time t (in seconds) is given by the equation:

$$s(t) = -4.9t^2 + 30t + 20.$$

(a) Compute the average velocity of the object over the following time intervals: $[1, 3]$, $[1, 2]$, $[1, 1.5]$

(b) As your interval gets shorter, what do you notice about the average velocities? What do you think would happen if we computed the average velocity of the object over the interval $[1, 1.2]$? $[1, 1.1]$? $[1, 1.05]$?

Example, cont.

An object is launched into the air. Its position s (in feet) at any time t (in seconds) is given by the equation:

$$s(t) = -4.9t^2 + 30t + 20.$$

(c) How could you use the average velocities to estimate the instantaneous velocity at $t = 1$?

(d) What do the average velocities you computed in 1. represent on the graph of $s(t)$?

Question

What happens to the relationship between instantaneous velocity and average velocity as the time interval gets shorter?

Answer: The instantaneous velocity at $t = 1$ is the limit of the average velocities as t approaches 1.

Question

What about the relationship between the secant lines and the tangent lines as the time interval gets shorter?

Answer: The slope of the tangent line at $(1, 45.1 = s(1))$ is the limit of the slopes of the secant lines as t approaches 1.

2.1 Book Problems

1-3, 7-13, 15, 21, 25, 27, 29

Even though book problems aren't turned in, they're a very good way to study for quizzes and tests (wink wink wink).

§2.2 Definition of Limits

Question

- Based on your everyday experiences, how would you define a "limit"?
- Based on your mathematical experiences, how would you define a "limit"?
- How do your definitions above compare or differ?

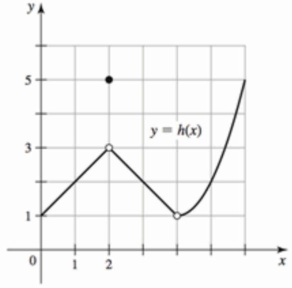
Definition (limit)

Suppose the function f is defined for all x near a, except possibly at a. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = L$$

and say **the limit of** $f(x)$ **as** x **approaches** a **equals** L.

Exercise



Determine the following:

(a) $h(1)$

(b) $h(2)$

(c) $h(4)$

(d) $\lim_{x \to 2} h(x)$

(e) $\lim_{x \to 4} h(x)$

(f) $\lim_{x \to 1} h(x)$

Question

Does $\lim_{x \to a} f(x)$ always equal $f(a)$?

(Hint: Look at the example from the previous slide!)

Exercise

Suppose $f(x) = \dfrac{x^2 + x - 20}{x - 4}$.

(a) Create a table of values of $f(x)$ when

$$x = 3.9, 3.99, 3.999, \text{ and}$$
$$x = 4.1, 4.01, 4.001$$

(b) What can you conjecture about $\lim_{x \to 4} f(x)$?

Up to this point we have been working with two-sided limits; however, for some functions it makes sense to examine one-sided limits.

Notice how in the previous example we could approach $f(x)$ from both sides as x approaches a, i.e., when $x > a$ and when $x < a$.

Definition (right-hand limit)

Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

$$\lim_{x \to a^+} f(x) = L$$

and say **the limit of** $f(x)$ **as** x **approaches** a **from the right equals** L.

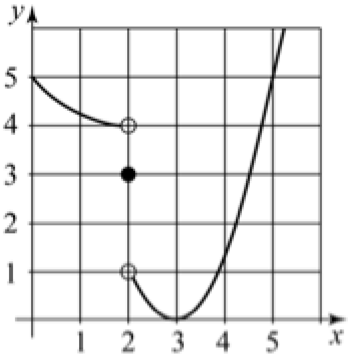
Definition (left-hand limit)

Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

$$\lim_{x \to a^-} f(x) = L$$

and say **the limit of** $f(x)$ **as** x **approaches** a **from the left equals** L.

Exercise



Determine the following:

(a) $g(2)$

(b) $\lim\limits_{x \to 2^+} g(x)$

(c) $\lim\limits_{x \to 2^-} g(x)$

(d) $\lim\limits_{x \to 2} g(x)$

Relationship Between One- and Two-Sided Limits

Theorem

If f is defined for all x near a except possibly at a, then $\lim_{x\to a} f(x) = L$ if and only if both $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.

In other words, the only way for a two-sided limit to exist is if the one-sided limits equal the same number (L).

2.2 Book Problems

1-4, 7, 9, 11, 13, 19, 23, 29, 31

§2.3 Techniques for Computing Limits

Exercise

Given the function $f(x) = 4x + 7$, find $\lim_{x \to -2} f(x)$

(a) graphically;

(b) numerically (i.e., using a table of values near -2)

(c) via a direct computation method of your choosing.

Compare and contrast the methods in this exercise – which is the most favorable?

This section provides various laws and techniques for determining limits. These constitute **analytical** methods of finding limits. The following is an example of a very useful limit law:

Limits of Linear Functions: Let a, b, and m be real numbers. For linear functions $f(x) = mx + b$,

$$\lim_{x \to a} f(x) = f(a) = ma + b.$$

This rule says we if $f(x)$ is a linear function, then in taking the limit as $x \to a$, we can just plug in the a for x.

IMPORTANT! Using a table or a graph to compute limits, as in the previous sections, can be helpful. However, "analytical" does not include those techniques.

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, c is a real number, and m, n are positive integers.

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1. **Sum:** $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. **Difference:** $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

In other words, if we are taking a limit of two things added together or subtracted, then we can first compute each of their individual limits one at a time.

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, c is a real number, and m, n are positive integers.

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3. **Constant Multiple:** $\lim_{x\to a} (cf(x)) = c\left(\lim_{x\to a} f(x)\right)$

4. **Product:** $\lim_{x\to a} (f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$

The same is true for products. If one of the factors is a constant, we can just bring it outside the limit. In fact, a constant is its own limit.

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, c is a real number, and m, n are positive integers.

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5. **Quotient:** $\displaystyle \lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

(provided $\lim_{x \to a} g(x) \neq 0$)

Question

Why the caveat?

Limit Laws, cont.

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, c is a real number, and m, n are positive integers.

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6. **Power:** $\lim_{x\to a} (f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$

7. **Fractional Power:** $\lim_{x\to a} (f(x))^{\frac{n}{m}} = \left(\lim_{x\to a} f(x)\right)^{\frac{n}{m}}$

   (provided $f(x) \geq 0$ for x near a if m is even and $\frac{n}{m}$ is in lowest terms)

Question

Explain the caveat in **7.**

Laws **1.-6.** hold for one-sided limits as well. But **7.** must be modified:

7. **Fractional Power (one-sided limits):**
   - $\displaystyle\lim_{x\to a^+} (f(x))^{\frac{n}{m}} = \left(\lim_{x\to a^+} f(x)\right)^{\frac{n}{m}}$
     (provided $f(x) \geq 0$ for x near a with $x > a$, if m is even and $\frac{n}{m}$ is in lowest terms)
   - $\displaystyle\lim_{x\to a^-} (f(x))^{\frac{n}{m}} = \left(\lim_{x\to a^-} f(x)\right)^{\frac{n}{m}}$
     (provided $f(x) \geq 0$ for x near a with $x < a$, if m is even and $\frac{n}{m}$ is in lowest terms)

Limits of Polynomials and Rational Functions

Assume that $p(x)$ and $q(x)$ are polynomials and a is a real number.

- **Polynomials:** $\lim_{x \to a} p(x) = p(a)$
- **Rational functions:** $\lim_{x \to a} \dfrac{p(x)}{q(x)} = \dfrac{p(a)}{q(a)}$

  (provided $q(a) \neq 0$)

For polynomials and rational functions we can plug in a to compute the limit, as long as we don't get zero in the denominator. Linear functions count as polynomials. A rational function is a "fraction" made of polynomials.

Exercise

Evaluate the following limits analytically.

1. $\displaystyle\lim_{x\to 1} \frac{4f(x)g(x)}{h(x)}$, given that

$$\lim_{x\to 1} f(x) = 5,\ \lim_{x\to 1} g(x) = -2, \text{ and } \lim_{x\to 1} h(x) = -4.$$

2. $\displaystyle\lim_{x\to 3} \frac{4x^2 + 3x - 6}{2x - 3}$

3. $\displaystyle\lim_{x\to 1^-} g(x)$ and $\displaystyle\lim_{x\to 1^+} g(x)$, given that

$$g(x) = \begin{cases} x^2 & \text{if } x \leq 1; \\ x + 2 & \text{if } x > 1. \end{cases}$$

Additional (Algebra) Techniques

When direct substitution (a.k.a. plugging in a) fails try using algebra:

- Factor and see if the denominator cancels out.

Example

$$\lim_{t\to 2} \frac{3t^2 - 7t + 2}{2 - t}$$

- Look for a common denominator.

Example

$$\lim_{h\to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h}$$

Exercise

Evaluate $\lim_{s\to 3} \dfrac{\sqrt{3s+16}-5}{s-3}$.

This method for evaluating limits uses the relationship of functions with each other.

Theorem (Squeeze Theorem)

Assume $f(x) \leq g(x) \leq h(x)$ *for all values of* x *near* a*, except possibly at* a*, and suppose*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.$$

Then since g *is always between* f *and* h *for* x*-values close enough to* a*, we must have*

$$\lim_{x \to a} g(x) = L.$$

Example

(a) Draw a graph of the inequality

$$-|x| \leq x^2 \ln\left(x^2\right) \leq |x|.$$

(b) Compute $\lim_{x \to 0} x^2 \ln\left(x^2\right)$.

2.3 Book Problems

12-30 (every 3rd problem), 33, 39-51 (odds), 55, 57, 61-67 (odds)

In general, review your algebra techniques, since they can save you some headache.