# Tues 16 June

- Midterm this Friday. Stay tuned for more info.
  - up to $\oint 3.9$
  - 12-13 questions
  - 80 minutes
  - syllabus-approved calculator
- Quiz solutions are up, but please bear with me on getting the grading done.

## Exercise

Find $\frac{dy}{dx}$ for $xy + y^3 = 1$.

## Exercise

Find an equation of the line tangent to the curve $x^4 - x^2y + y^4 = 1$ at the point $(-1, 1)$.

### Example

Find $\dfrac{d^2y}{dx^2}$ if $xy + y^3 = 1$.

## Power Rule for Rational Exponents

Implicit differentiation also allows us to extend the power rule to rational exponents: Assume $p$ and $q$ are integers with $q \neq 0$. Then

$$\frac{d}{dx}(x^{\frac{p}{q}}) = \frac{p}{q}x^{\frac{p}{q}-1}$$

(provided $x \geq 0$ when $q$ is even and $\frac{p}{q}$ is in lowest terms).

**Exercise**

Prove it.

## 3.7 Book Problems

5-21 (odds), 27-45 (odds)

# $\oint$ 3.8 Derivatives of Logarithmic and Exponential Functions

The natural exponential function $f(x) = e^x$ has an inverse function, namely $f^{-1}(x) = \ln x$. This relationship has the following properties:

1. $e^{\ln x} = x$ for $x > 0$ and $\ln(e^x) = x$ for all $x$.
2. $y = \ln x \quad \Longleftrightarrow \quad x = e^y$
3. For real numbers $x$ and $b > 0$,

$$b^x = e^{\ln(b^x)} = e^{x \ln b}.$$

# Derivative of $y = \ln x$

Using 2. from the last slide, plus implicit differentiation, we can find $\frac{d}{dx}(\ln x)$. Write $y = \ln x$. We wish to find $\frac{dy}{dx}$. From 2.,

$$\frac{d}{dx}\left(x = e^y\right) \Rightarrow \frac{d}{dx}x = \frac{d}{dx}(e^y)$$

$$1 = e^y \left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

So $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

## Derivative of $y = \ln|x|$

Recall, we can only take "$\ln$" of a positive number. However:

- For $x > 0$, $\ln|x| = \ln x$, so

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}.$$

- For $x < 0$, $\ln|x| = \ln(-x)$, so

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln(-x)) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}.$$

In other words, the absolute values do not change the derivative of natural log.

## Exercise

Find the derivative of each of the following functions:

- $f(x) = \ln(15x)$
- $g(x) = x \ln x$
- $h(x) = \ln(\sin x)$

# Derivative of $y = b^x$

What about other logs? Say $b > 0$. Since $b^x = e^{\ln b^x} = e^{x \ln b}$ (by 3. on the earlier slide),

$$\begin{aligned} \frac{d}{dx}(b^x) &= \frac{d}{dx}(e^{x \ln b}) \\ &= e^{x \ln b} \cdot \ln b \\ &= b^x \ln b. \end{aligned}$$

### Exercise

Find the derivative of each of the following functions:

- $f(x) = 14^x$
- $g(x) = 45(3^{2x})$

### Exercise

Determine the slope of the tangent line to the graph $f(x) = 4^x$ at $x = 0$.

# Story Problem Example

### Example

The energy (in Joules) released by an earthquake of magnitude $M$ is given by the equation

$$E = 25000 \cdot 10^{1.5M}.$$

(a) How much energy is released in a magnitude 3.0 earthquake?

(b) What size earthquake releases 8 million Joules of energy?

(c) What is $\frac{dE}{dM}$ and what does it tell you?

# Derivatives of General Logarithmic Functions

The relationship $y = \ln x \iff x = e^y$ applies to logarithms of other bases:

$$y = \log_b x \quad \iff \quad x = b^y.$$

Now taking $\frac{d}{dx}(x = b^y)$ we obtain

$$1 = b^y \ln b \left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{1}{b^y \ln b}$$

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

### Example

Compute the derivative of $f(x) = \dfrac{x^2(x-1)^3}{(3+5x)^4}$.

**Solution:** We can use logarithmic differentiation – first take the natural log of both sides and then use properties of logarithms.

$$\ln(f(x)) = \ln\left(\frac{x^2(x-1)^3}{(3+5x)^4}\right)$$
$$= \ln x^2 + \ln (x-1)^3 - \ln (3+5x)^4$$
$$= 2\ln x + 3\ln(x-1) - 4\ln(3+5x)$$

Now we take $\frac{d}{dx}$ on both sides:

$$\frac{1}{f(x)}\left(\frac{df}{dx}\right) = 2\left(\frac{1}{x}\right) + 3\left(\frac{1}{x-1}\right) - 4\left(\frac{1}{3+5x}\right)(5)$$

$$\frac{f'(x)}{f(x)} = \frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}$$

Finally, solve for $f'(x)$:

$$f'(x) = f(x)\left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}\right]$$

$$= \frac{x^2(x-1)^3}{(3+5x)^4}\left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}\right]$$

### 3.8 Book Problems

9-27 (odds), 31-37 (odds), 41-47 (odds)