Mon 6 Oct 2014

- Exam 1 feedback and comments
- Curve posted online
- MIDTERM Wed 15 Oct 2014 6:30-8p covers EVERYTHING up to 3.9

location: POSC A211 (Poultry Science Auditorium)

- Quiz Tues 7 Oct covers 3.5-3.7
- This week: the rest of Chapter 3

Contextual Example

The energy (in joules) released by an earthquake of magnitude *M* is given by the equation

$$E = 25{,}000 \cdot 10^{1.5M}$$

How much energy is released in a magnitude 3.0 earthquake?

What size earthquake releases 8 million joules of energy?

What is $\frac{dE}{dM}$ and what does it tell you?

Derivatives of General Logarithmic Functions

The relationship $y = \log_b x \Leftrightarrow x = b^y$ also applies to logarithms of other bases:

$$y = \ln x \Leftrightarrow x = e^y$$

Using implicit differentiation, we obtain $\dfrac{d}{dx}(\log_b x)$:

$$x = b^y$$

$$1 = b^y \cdot \ln b \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{b^y \ln b} = \frac{1}{x \ln b}$$

So $\dfrac{d}{dx}(\log_b x) = \dfrac{1}{x \ln b}$

Logarithmic Differentiation

Logarithmic Differentiation can be helpful in finding derivatives of functions where using other rules may be tedious.

Example: Compute the derivative of $f(x) = \dfrac{x^2(x-1)^3}{(3+5x)^4}$

We can use logarithmic differentiation by first taking the natural logs of both sides and then using properties of logarithms

Logarithmic Differentiation

$$f(x)=\frac{x^2(x-1)^3}{(3+5x)^4}\Rightarrow \ln(f(x))=\ln\left(\frac{x^2(x-1)^3}{(3+5x)^4}\right)$$

$$\ln(f(x))=\ln x^2+\ln(x-1)^3-\ln(3+5x)^4$$

$$\ln(f(x))=2\ln x+3\ln(x-1)-4\ln(3+5x)$$

Now we take the derivative of both sides:

$$\frac{d}{dx}\big(\ln(f(x))\big)=\frac{d}{dx}\big(2\ln x+3\ln(x-1)-4\ln(3+5x)\big)$$

$$\frac{f'(x)}{f(x)}=2\cdot\frac{1}{x}+3\cdot\frac{1}{x-1}\cdot 1-4\cdot\frac{1}{3+5x}\cdot 5$$

Logarithmic Differentiation

$$\frac{f'(x)}{f(x)} = 2\cdot\frac{1}{x} + 3\cdot\frac{1}{x-1}\cdot 1 - 4\cdot\frac{1}{3+5x}\cdot 5$$

$$\frac{f'(x)}{f(x)} = \frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}$$

Solving for *f'(x)*:

$$f'(x) = f(x)\left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}\right]$$

$$= \frac{x^2(x-1)^3}{(3+5x)^4}\left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}\right]$$

EX: Logarithmic Differentiation

Use logarithmic differentiation to calculate the derivative of

$$f(x)=\frac{(x+1)^{\frac{3}{2}}(x-4)^{\frac{5}{2}}}{(5x+3)^{\frac{2}{3}}}$$

HW from section 3.8

- Do problems 9-27 odd, 31-37 odd, 41-47 odd (pgs. 199-200 in textbook)

Wed 8 Oct 2014

Wed Oct 15: Midterm Exam, 6:30 – 8:00 pm
location: POSC A211 (Poultry Science Auditorium)

Derivatives of Inverse Trigonometric Functions

Recall from work with inverse functions that if

y = f(x), then $f^{-1}(x)$ is the value of *y* so that *x = f(y).*

Example: If *f(x) = 3x+2*, then $f^{-1}(x) = (x-2)/3$

NOTE: $f^{-1}(x)$ is not the same as $\dfrac{1}{f(x)}$!! $\left(f(x)\right)^{-1} = \dfrac{1}{f(x)}$

To avoid this confusion, we use arcsin(x), arccos(x), arctan(x), … to denote inverse trig functions.

Derivative of Inverse Sine

Given our definition, $y = \sin^{-1} x$ is the value of *y* such that *x = sin y*. So the derivative of $y = \sin^{-1} x$ can be found by differentiating both sides of *x = sin y* with respect to x and then finding $\frac{dy}{dx}$:

$$x = \sin y$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$$

$$1 = (\cos y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Derivative of Inverse Sine, cont.

Since $\sin^2 y + \cos^2 y = 1$, then $\cos y = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-x^2}$

Because $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$ (e.g., where $y = \sin^{-1} x$ was originally defined), then $\cos y \ge 0 \Rightarrow \cos y = \sqrt{1-x^2}$

Therefore

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

Examples of derivative of $\sin^{-1}x$

Compute the derivative of the following:

$$\frac{d}{dx}\left[\sin^{-1}(4x^2-3)\right]$$

$$\frac{d}{dx}\left[\cos(\sin^{-1}x)\right]$$

Derivative of Inverse Tangent

Using a similar method as with inverse sine, we can find inverse tangent:

$$x = \tan y$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan y)$$

$$1 = (\sec^2 y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

Since $\sec^2 y = 1 + \tan^2 y$, we have $\sec^2 y = 1 + x^2$

So $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

Derivative of Inverse Secant

Using a similar method, we can find inverse secant:

$$x = \sec y$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sec y)$$

$$1 = \sec y \tan y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

Since $\sec^2 y = 1 + \tan^2 y$, then $\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$

When $x \geq 1 \Rightarrow \tan y > 0$ and when $x \leq 1 \Rightarrow \tan y < 0$

So $$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

All other inverse trig derivatives

Using the facts that $\cos^{-1}x+\sin^{-1}x=\frac{\pi}{2}$, $\cot^{-1}x+\tan^{-1}x=\frac{\pi}{2}$

and $\csc^{-1}x+\sec^{-1}x=\frac{\pi}{2}$, we can differentiate these identities to obtain all trig derivatives:

$$\frac{dy}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$$ (-1 < x < 1) $$\frac{dy}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$ $$\frac{d}{dx}\left(\cot^{-1}x\right)=-\frac{1}{1+x^2}$$

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$$\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{|x|\sqrt{x^2-1}}$$ $$\frac{d}{dx}\left(\csc^{-1}x\right)=-\frac{1}{|x|\sqrt{x^2-1}}$$

Compute the derivative of $f(x) = \tan^{-1}(1/x)$

A. $\dfrac{-1}{x^2+1}$

B. $\dfrac{1}{x^2+1}$

C. $\dfrac{-1}{x^2(x^2+1)}$

D. $\dfrac{1}{x^2(x^2+1)}$

Derivatives of Inverses in General

There is an interesting relationship between the derivative of *f* and the derivative of f^{-1}. With a partner, do the following:

1. Write a linear function in the form of $f(x)=mx+b$ (you provide the *m* and *b*).
2. Compute the derivative of your linear function.
3. Find the inverse of your linear function.
4. Find the derivative of your inverse function.

What do you notice?

Compare your results with someone else to see if you see a pattern.

Example

For *f(x) = 3x+4, f ’(x) = 3*

$$f^{-1}(x) = \frac{1}{3}x - \frac{4}{3}$$

$$\left(f^{-1}(x)\right)' = \frac{1}{3}$$

In General

Let f be differentiable and have an inverse on an interval I. Let x_0 be a point in I at which $f'(x_0) \neq 0$.

Then f^{-1} is differentiable at $y_0 = f(x_0)$, and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}$$

where $y_0 = f(x_0)$.

Homework from Section 3.9

Do problems 7-27 odd, 31-39 odd (p. 209), and read the notes and examples for Section 3.10.

Fri 10 Oct 2014

- MIDTERM: Wed 15 Oct 6:30-8p covers EVERYTHING up to 3.9
- location: POSC A211 (Poultry Science Auditorium)
- Studying:
  - Use the guide posted on MLP
  - Solutions to Exam 1: Give you an idea of what to write when “showing your work”
  - Written HW: Even if it’s a problem you already know how to do, write the answer as if it’s for a friend taking PreCal. Also, the written hw is a good way to get faster with arithmetic/algebra
  - Online HW: Gives more step-by-step guidance on certain patterns of problems.

Related Rates

In this final section of Chapter 3, we use our knowledge of derivatives to examine how variables change with respect to time.

The prime feature of these types of problems is that two or more variables, which are related in a known way, are themselves changing in time.

The goal of these types of problems is to determine the rate of change (think derivative) of one of the variables at a specific moment of time.

Problem

The edges of a cube increase at a constant rate of 2 cm/sec.

How fast is the volume changing when the length of each edge is 50 cm?

Variables:

How Variables are related:

Rates Known:

Rate We Seek:

Problem

The edges of a cube increase at a constant rate of 2 cm/sec.

How fast is the volume changing when the length of each edge is 50 cm?

Variables: V (Volume of cube) and e (length of edge)

How Variables are related: $V = e^3$

Rates Known: $de/dt = 2$ cm/sec

Rate We Seek: dV/dt

Problem

The edges of a cube increase at a constant rate of 2 cm/sec.

How fast is the volume changing when the length of each edge is 50 cm?

Note that both V and e are functions of t (their respective sizes are dependent upon how much time t has passed).

Differentiating **V(t) = [e(t)]^3** provides us with a method to examine their rates of change:

$$V'(t) = 3e(t)^2 \times e'(t)$$

Problem

The edges of a cube increase at a constant rate of 2 cm/sec.

How fast is the volume changing when the length of each edge is 50 cm? $V'(t) = 3e(t)^2 \cdot e'(t)$

Note e(t) is the length of the cube's edges at time t, and e'(t) is the rate at which the edges are changing.

So the volume of the cube at the desired time t (e.g., when edges are 50 cm) is

$$V'(t) = 3 \cdot 50^2 \cdot 2 = 15,000 cm^3 / \sec$$

Steps for Solving Related-Rates Problems

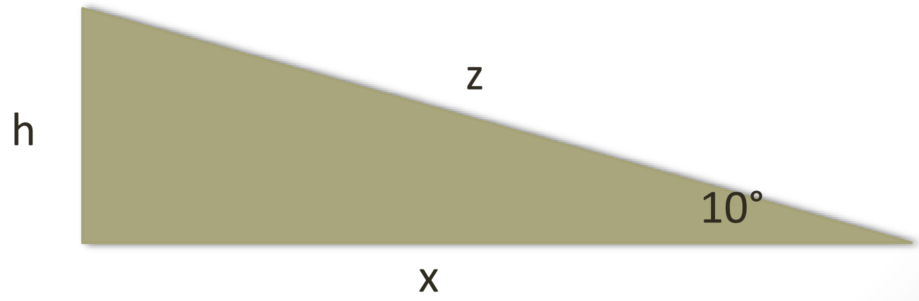
1. Read the problem carefully, making a sketch to organize the given information. Identify the rates that are given and the rate that is to be determined.
2. Write one or more equations that express the basic relationships among the variables.
3. Introduce rates of change by differentiating the appropriate equation(s) with respect to time t.
4. Substitute known values and solve for the desired quantity.
5. Check that the units are consistent and the answer is reasonable.

Problem

A jet ascends at a 10° angle from the horizontal with an airspeed (e.g., the speed along its line of flight) of 550 miles/hr. How fast is the altitude of the jet increasing? If the sun is directly overhead, how fast is the shadow of the jet moving on the ground?

Jet Problem

Step 1: Examining the problem, we can see that we have 3 variables: the distance the shadow has traveled (x), the altitude of the jet (h), and the distance the jet has actually traveled on its line of flight (z). We know that dz/dt = 550 miles/hr, and we need to know dx/dt and dh/dt. We also see the variables are related through a right triangle:



Jet Problem

Step 2: To answer the first question (e.g., how fast is the altitude increasing), we need an equation involving h and z only. Using trigonometry, we see that:

$$\sin(10^\circ) = \frac{h}{z} \Rightarrow h = \sin(10)\cdot z \approx 0.174z$$

Additionally to answer the second question (e.g., how fast is the shadow moving), we need an equation involving x and z only. Again, using trig, we have:

$$\cos(10^\circ) = \frac{x}{z} \Rightarrow x = \cos(10)\cdot z \approx 0.985z$$

Jet Problem

Step 3: We can now differentiate each equation to answer each question:

$$h \approx 0.174z \Rightarrow h'(t) = 0.174z'(t)$$
$$x \approx 0.985z \Rightarrow x'(t) = 0.985z'(t)$$

Step 4: We know that z'(t) is 550 miles/hr. So

$$h'(t) = 0.174 \times 550 = 95.7 mi/hr$$

$$x'(t) = 0.985 \times 550 = 541.64 mi/hr$$

Jet Problem

Step 5: Because both answers are in terms of miles/hr and both answers seem reasonable within the context of the problem, we can assume that the jet is gaining altitude at a rate of 95.7 miles/hr, while the shadow on the ground is moving at about 541.64 miles/hr.

The sides of a cube increase at a rate of R cm/sec. When the sides have a length of 2 cm, the rate of change of the volume is:

A. 10R

B. 8R

C. 12R

D. 16R

Exercise

Two boats leave a dock at the same time. One boat travels south at 30 miles/hr and the other travels east at 40 miles/hr. After half an hour, how fast is the distance between the boats increasing?

Homework from Section 3.10

Do problems 5-12 all, 14-15, 17-18, 30-31 (pgs. 214-216).