Calculus I (Math 2554)

Summer 2015

Dr. Ashley K. Wheeler

University of Arkansas

last updated: July 3, 2015

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

Table of Contents

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

4.8 Antiderivatives

With differentiation, the goal of problems was to find the function $f'(x)$, given the function $f(x)$.

With antidifferentiation, the goal is the opposite; in this case we wish to find a function F whose derivative is f, i.e.,

$$F'(x) = f(x).$$

Definition

A function F is called an **antiderivative** of a function f on an interval I means

$$F'(x) = f(x) \quad \text{for all } x \text{ in } I.$$

Example

Given $f(x) = 4$, an antiderivative of $f(x)$ is $F(x) = 4x$.

NOTE: Antiderivatives are not unique!

They differ by a constant (C):

Theorem

Let F be any antiderivative of f. Then all the antiderivatives of f have the form $F + C$, where C is an arbitrary constant.

Recall: $\frac{d}{dx} f(x) = f'(x)$ is the derivative of $f(x)$.

Now: $\displaystyle\int f(x)dx = F(x) + C$ is the antiderivative of $f(x)$.

It doesn't matter which F you choose, since writing the C will show you are talking about all the antiderivatives at once. The C is also why we use the term **indefinite integral**.

Indefinite Integrals

Example

$$\int 4x^3 \ dx = x^4 + C.$$

The C is called the **constant of integration**. The dx is called the **differential** (and it is the same dx from $\oint$4.5). Like the $\frac{d}{dx}$, it shows which variable we are talking about. The function written between the $\int$ and the dx is called the **integrand**.

Rules for Indefinite Integrals

Power Rule: $\displaystyle\int x^p dx = \frac{x^{p+1}}{p+1} + C$
(p is any real number except -1)

Constant Multiple Rule: $\displaystyle\int cf(x)dx = c\int f(x)dx$

Sum Rule: $\displaystyle\int (f(x)+g(x))\,dx = \int f(x)dx + \int g(x)dx$

Exercise

Evaluate the following indefinite integrals:

1. $\displaystyle\int \left(3x^{-2} - 4x^2 + 1\right)\, dx$

2. $\displaystyle\int 6\sqrt[3]{x}\; dx$

Indefinite Integrals of Trig Functions

Table 4.5 (in the text) provides us with rules for finding indefinite integrals of trig functions.

1. $\dfrac{d}{dx}(\sin ax) = a\cos ax \quad \longrightarrow \quad \displaystyle\int \cos ax \ dx = \frac{1}{a}\sin ax + C$
2. $\dfrac{d}{dx}(\cos ax) = -a\sin ax \quad \longrightarrow \quad \displaystyle\int \sin ax \ dx = -\frac{1}{a}\cos ax + C$
3. $\dfrac{d}{dx}(\tan ax) = a\sec^2 ax \quad \longrightarrow \quad \displaystyle\int \sec^2 ax \ dx = \frac{1}{a}\tan ax + C$

Indefinite Integrals of Trig Functions (cont.)

4. $\dfrac{d}{dx}(\cot ax) = -a\csc^2 ax \quad \longrightarrow \quad \displaystyle\int \csc^2 ax\ dx = -\frac{1}{a}\cot ax + C$

5. $\dfrac{d}{dx}(\sec ax) = a\sec ax\tan ax \quad \longrightarrow \quad \displaystyle\int \sec ax\tan ax\ dx = \frac{1}{a}\sec ax + C$

6. $\dfrac{d}{dx}(\csc ax) = -a\csc ax\cot ax \quad \longrightarrow \quad \displaystyle\int \csc ax\cot ax\ dx = -\frac{1}{a}\csc ax + C$

Other Indefinite Integrals

Table 4.6 provides us with rules for finding other indefinite integrals.

7. $\dfrac{d}{dx}(e^{ax}) = ae^{ax} \longrightarrow \displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

8. $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x} \longrightarrow \displaystyle\int \frac{dx}{x} = \ln|x| + C$

9. $\dfrac{d}{dx}\left(\sin^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{1}{\sqrt{a^2 - x^2}} \longrightarrow \displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

Other Indefinite Integrals (cont.)

10. $$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{a^2+x^2} \longrightarrow \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

11. $$\frac{d}{dx}\left(\sec^{-1}\left|\frac{x}{a}\right|\right) = \frac{a}{x\sqrt{x^2-a^2}} \longrightarrow \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$

Example

Evaluate the following indefinite integral: $\int 2\sec^2 2x \; dx.$

Solution: Using Rule 3., with $a = 2$, we have

$$\int 2\sec^2 2x \; dx = 2\int \sec^2 2x \; dx = 2\left[\frac{1}{2}\tan 2x\right] + C = \tan 2x + C.$$

Exercise

Evaluate $\int 2\cos(2x) \; dx.$

Initial Value Problems

In some instances, you have enough information to determine the value of C in the antiderivative. These are often called **initial value problems**.

Example

If $f'(x) = 7x^6 - 4x^3 + 12$ and $f(1) = 24$, find $f(x)$.

Solution: $f(x) = \int (7x^6 - 4x^3 + 12)\ dx = x^7 - x^4 + 12x + C$. Now find out which C gives $f(1) = 24$:

$$24 = f(1) = 1 - 1 + 12 + C,$$

so $C = 12$. Hence, $f(x) = x^7 - x^4 + 12x + 12$.

Exercise

Find the function f that satisfies $f''(t) = 6t$ with $f'(0) = 1$ and $f(0) = 2$.

4.8 Book Problems

11-45 (odds), 55-59 (odds), 63, 65

- To solve 55-59 (odds), 63, and 65, look through the section, focusing in on Example 7.

∮5.1 Approximating Area Under Curves

Example

Suppose you ride your bike at a constant velocity of 8 miles per hour for 1.5 hours.

(a) What is the velocity function that models this scenario?

(b) What does the graph of the velocity function look like?

(c) What is the position function for this scenario?

(d) Where is the displacement (the distance you've traveled) represented when looking at the graph of the velocity function?

Question

In the previous example, the velocity was constant. In most cases, this is not accurate (or possible). How could we find displacement when the velocity is changing over an interval?

One strategy is to divide the time interval into a particular number of subintervals and approximate the velocity on each subinterval with a constant velocity. Then for each subinterval, the displacement can be evaluated and summed.

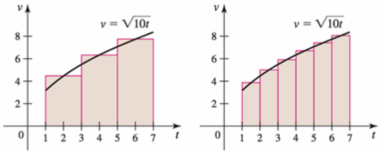
Note: This provides us with only an approximation, but with a larger number of subintervals, the approximation becomes more accurate.

Example

Suppose the velocity of an object moving along a line is given by $v(t) = \sqrt{10t}$ on the interval $1 \le t \le 7$.

(a) Divide the time interval into $n = 3$ subintervals, assuming the object moves at a constant velocity equal to the value of v evaluated at the midpoint of the subinterval. Estimate the displacement of the object on $[1, 7]$.

(b) Repeat for $n = 6$ subintervals.



Riemann Sums

We can see that our approximation gets more accurate when we use more subintervals of the time interval (see Example 1 in the text). Using this idea, we now examine a method for approximating areas under curves.

Consider a function f over the interval $[a, b]$. Divide $[a, b]$ into n subintervals of equal length:

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

with $x_0 = a$ and $x_n = b$. The length of each subinterval is denoted

$$\Delta x = \frac{b-a}{n}.$$

In each subinterval $[x_{k-1}, x_k]$ (where k ranges from 1 to n), we can choose any point, call it $\overline{x}_k$, and create a rectangle of height $f(\overline{x}_k)$. The base of the rectangle has length

$$x_k - x_{k-1} = \Delta x,$$

so the area of that rectangle is $f(\overline{x}_k)\Delta x$.

Doing this for each subinterval, and then summing the rectangles' areas, produces an approximation of the overall area. This approximation is called a **Riemann sum**:

$$R = f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x.$$

Note: We let k vary from 1 to n, and we always have $x_{k-1} \leq \overline{x}_k \leq x_k$.

We usually choose $\overline{x}_k$ so that it is consistent across all the subintervals. The most common ways to do this are with **left Riemann sums**, **right Riemann sums**, and **midpoint Riemann sums**.

Definition

Let $R = f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x$.

1. R is a **left Riemann sum** when we choose $\overline{x}_k = x_{k-1}$ for each k.
2. R is a **right Riemann sum** when we choose $\overline{x}_k = x_k$ for each k.
3. R is a **midpoint Riemann sum** when we take $\overline{x}_k$ to be the midpoint between x_{k-1} and x_k, for each k.

(See Figures 5.9–5.11 for pictures of these sums.)

Sigma Notation

Although Riemann sums become more accurate when we make n (the number of rectangles) bigger, writing it all down becomes a pain. Sigma notation gives a shorthand.

Example

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

We sum all integer values from the lowest limit $(k = 1)$ to the highest limit $(k = 5)$ in the summand k^2.

Exercise

Evaluate $\displaystyle\sum_{k=0}^{3}(2k-1)$.

Σ-Shortcuts

(n is always a positive integer)

$$\sum_{k=1}^{n} c = cn \text{ (where } c \text{ is a constant)}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Riemann Sums Using Sigma Notation

We can use sigma notation to write the Riemann sum in a much more compact form:

$$\begin{aligned} R &= f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x \\ &= \sum_{k=1}^{n} f(\overline{x}_k)\Delta x. \end{aligned}$$

To write the left, right, and midpoint Riemann sums in sigma notation, we need to know the point $\overline{x}_k$.

Left, Right, and Midpoint Riemann Sums in Sigma Notation

Suppose f is defined on a closed interval $[a, b]$ which is divided into n subintervals of equal length Δx.

1. $\displaystyle\sum_{k=1}^{n} f(a + (k-1)\Delta x)\Delta x$ gives a left Riemann sum.
2. $\displaystyle\sum_{k=1}^{n} f(a + k\Delta x)\Delta x$ gives a right Riemann sum.
3. $\displaystyle\sum_{k=1}^{n} f\left(a + \left(k - \frac{1}{2}\right)\Delta x\right)\Delta x$ gives a midpoint Riemann sum.

Exercise

(a) Use sigma notation to write the left, right, and midpoint Riemann sums for the function $f(x) = x^2$ on the interval $[1, 5]$ given that $n = 4$.

(b) Based on these approximations, estimate the area bounded by the graph of $f(x)$ over $[1, 5]$.

5.1 Book Problems

9, 11, 15-23 (odds), 31, 33, 53-57

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

$\oint$ 5.2 Definite Integrals

In $\oint$5.1, we saw how we can use Riemann sums to approximate the area under a curve. However, the curves we worked with were all non-negative.

Question

What happens when the curve is negative?

Example

Let $f(x) = 8 - 2x^2$ over the interval $[0, 4]$. Use a left, right, and midpoint Riemann sum with $n = 4$ to approximate the area under the curve.

In the previous example, the places where f was positive provided positive contributions to the area, while places where f was negative provided negative contributions. The difference between positive and negative contributions is called the net area.

Definition

Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis, minus the sum of the areas of the parts of R that lie below the x-axis, on $[a, b]$.

The Riemann sums give approximations for the area under the curve and to make these approximations more and more accurate, we divide the region into more and more subintervals.

To make these approximations *exact*, we allow the number of subintervals $n \to \infty$, and consequently the length of the subintervals $\Delta x \to 0$.

In terms of limits:

$$\text{Net Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(\overline{x}_k)\Delta x.$$

General Riemann Sums

Suppose

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

so that $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ are subintervals of $[a, b]$.

The subintervals might have different lengths, so given each k, we let Δx_k be the length of the subinterval $[x_{k-1}, x_k]$. Let $\overline{x}_k$ denote any point in $[x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$.

General Riemann Sums (cont.)

If f is defined on $[a, b]$, then the sum

$$\sum_{k=1}^{n} f(\overline{x}_k)\Delta x_k = f(\overline{x}_1)\Delta x_1 + f(\overline{x}_2)\Delta x_2 + \cdots + f(\overline{x}_n)\Delta x_n$$

is called a **general Riemann sum for f on $[a, b]$**.

(In this definition, the lengths of the subintervals do not have to be equal.)

The *Definite* Integral

As $n \to \infty$, all of the Δx_ks approach 0, even the largest of these. Let Δ be the largest of the Δx_ks.

Definition

A function f defined on $[a, b]$ is **integrable** means the limit

$$\int_a^b f(x)\ dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(\overline{x}_k) \Delta x_k$$

exists – over all partitions of $[a, b]$ and all choices of $\overline{x}_k$ on a partition. This limit is called the **definite integral of f from a to b**.

Evaluating Definite Integrals

Theorem

If f is continuous on $[a,b]$ or bounded on $[a,b]$ with a finite number of discontinuities, then f is integrable on $[a,b]$.

See Figure 5.23 for an example of a noncontinuous function that is integrable.

Evaluating Definite Integrals (cont.)

Knowing the limit of a Riemann sum, we can now translate that to a definite integral.

Example

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} (4\overline{x}_k - 3)\Delta x_k \text{ on } [-1, 4] \iff \int_{-1}^{4} (4x - 3)\, dx.$$

Exercise

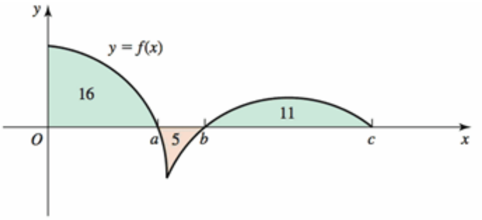
Using geometry, evaluate $\displaystyle\int_1^2 (4x-3)\ dx$.

(*Hint:* The area of a trapezoid is $\frac{h(a+b)}{2}$, where h is the height of the trapezoid and a and b are the lengths of the two parallel bases.)

Exercise

Using the picture below, evaluate the following definite integrals:

1. $\displaystyle\int_0^a f(x)\,dx$ 2. $\displaystyle\int_0^b f(x)\,dx$ 3. $\displaystyle\int_0^c f(x)\,dx$ 4. $\displaystyle\int_a^c f(x)\,dx$



The base for these slides was done by Dr. Shannon Dingman, later encoded in LATEX by Dr. Brad Lutes.

Properties of Integrals

1. (Reversing Limits) $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$
2. (Identical Limits) $\int_a^a f(x)\ dx = 0$
3. (Integral of a Sum)
$$\int_a^b (f(x) + g(x))\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$
4. (Constants in Integrals) $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$

Properties of Integrals (cont.)

5. (Integrals over Subintervals) If c lies between a and b, then

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

6. (Integrals of Absolute Values) The function $|f|$ is integrable on $[a, b]$ and $\int_a^b |f(x)|\ dx$ is the sum of the areas of regions bounded by the graph of f and the x-axis on $[a, b]$. (See Figure 5.31)

   (This is the total area, no negative signs.)

Exercise

Given

$$\int_2^4 f(x)\ dx = 3 \quad \text{and} \quad \int_4^6 f(x)\ dx = -2,$$

compute

$$\int_2^6 f(x)\ dx.$$

5.2 Book Problems

11-29 (odds), 35-44, 65

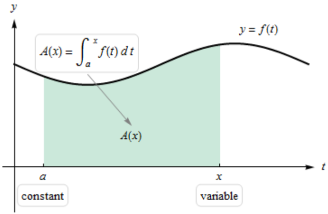
5.3 Fundamental Theorem of Calculus

Using Riemann sums to evaluate definite integrals is usually neither efficient nor practical. We will develop methods to evaluate integrals and also tie together the concepts of differentiation and integration.

Area Functions

Let $y = f(t)$ be a continuous function which is defined for all $t \geq a$, where a is a fixed number. The area function for f with left endpoint at a is given by

$$A(x) = \int_a^x f(t)\,dt.$$



This gives the net area of the region between the graph of f and the t-axis between the points $t = a$ and $t = x$. (See Figure 5.33 for pictures of the area function in action.)

Exercise

The graph of f is shown below. Let

$$A(x) = \int_0^x f(t)\ dt \quad \text{and} \quad F(x) = \int_2^x f(t)\ dt$$

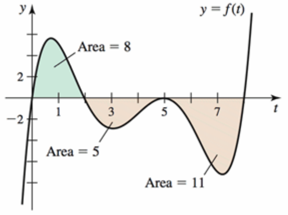
be two area functions for f. Compute:

(a) $A(2)$

(b) $F(5)$

(c) $A(5)$

(d) $F(8)$



The Fundamental Theorem of Calculus (Part 1)

Theorem (FTOC I)

If f is continuous on $[a, b]$, then the area function is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$; or equivalently,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

(the area function of f is an antiderivative of f).

The Fundamental Theorem of Calculus (Part 2)

Since A is an antiderivative of f, we now have a way to evaluate definite integrals and find areas under curves.

Theorem (FTOC II)

If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

We use the notation $F(x)|_a^b = F(b) - F(a)$.

In essence, to evaluate an integral, we

- Find any antiderivative of f, and call if F.
- Compute $F(b) - F(a)$, the difference in the values of F between the upper and lower limits of integration.

The two parts of the FTOC illustrate the inverse relationship between differentiation and integration – the integral "undoes" the derivative.

Exercise

1. Use Part 1 of the FTOC to simplify $\dfrac{d}{dx}\displaystyle\int_x^{10} \frac{dz}{z^2+1}$.
2. Use Part 2 of the FTOC to evaluate $\displaystyle\int_0^{\pi} (1-\sin x)\ dx$.
3. Compute $\displaystyle\int_1^{y} h'(p)\ dp$.

5.3 Book Problems

11-17, 19-39 (odds), 45-57 (odds)

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

$\oint$ 5.4 Working with Integrals

Recall the definition of an even function,

$$f(-x) = f(x),$$

and of an odd function,

$$f(-x) = -f(x).$$

These properties are just examples of ways we can simplify integrals.

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

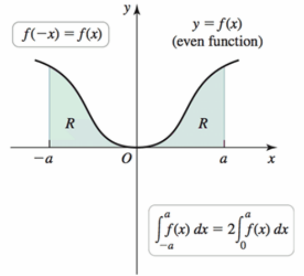
Integrating Even Functions

Even functions are symmetric about the y-axis. So

$$\int_{-a}^{0} f(x)\ dx = \int_{0}^{a} f(x)\ dx$$

i.e., the area under the curve to the left of the y-axis is equal to the area under the curve to the right.

Integrating Even Functions (cont.)



Hence, $\displaystyle\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$ for even functions.

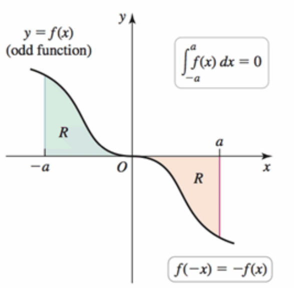
Integrating Odd Functions

On the other hand, odd functions have 180° rotation symmetry about the origin. So

$$\int_{-a}^{0} f(x)\ dx = -\int_{0}^{a} f(x)\ dx$$

i.e., the area under the curve to the left of the origin is the negative of the area under the curve to the right of the origin.

Integrating Odd Functions (cont.)



Hence, $\int_{-a}^{a} f(x)\ dx = 0$ for odd functions.

Exercise

Evaluate the following integrals using the properties of even and odd functions:

1. $\displaystyle\int_{-4}^{4} (3x^2 - x)\ dx$
2. $\displaystyle\int_{-1}^{1} (1 - |x|)\ dx$
3. $\displaystyle\int_{-\pi}^{\pi} \sin x\ dx$

Average Value of a Function

To find the average of $f(x)$ between points a and b, we can estimate by choosing y-values $\overline{x}_k$. If we take n of them, then the average is:

$$\overline{f} \approx \frac{f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)}{n}$$

Average Value of a Function (cont.)

Since $n = \frac{b-a}{\Delta x}$, we have

$$\begin{aligned}\frac{f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)}{n} &= \frac{f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)}{\left(\frac{b-a}{\Delta x}\right)} \\ &= \frac{1}{b-a}\left(f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)\right)\Delta x \\ &= \frac{1}{b-a}\sum_{k=1}^{n} f(\overline{x}_k)\Delta x.\end{aligned}$$

The estimate gets more accurate, the more y-values we take. We let $n \to \infty$.

Average Value of a Function (cont.)

Then the average value of an integrable function f on the interval $[a, b]$ is

$$\overline{f} = \frac{1}{b-a} \int_a^b f(x)\ dx.$$

Exercise

Find the average value of the function $f(x) = x(1-x)$ on the interval $[0, 1]$.

Mean Value Theorem for Integrals

Theorem

If f is continuous on $[a, b]$, then there is at least one point c in $[a, b]$ such that

$$f(c) = \overline{f} = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

In other words, the horizontal line $y = \overline{f} = f(c)$ intersects the graph of f for some point c in $[a, b]$. (See Figure 5.54)

Exercise

Find or approximate the point(s) at which $f(x) = x^2 - 2x + 1$ equals its average value on $[0, 2]$.

5.4 Book Problems

7-27 (odds), 31-35 (odds)