# Week 14: Monday before Thanksgiving

# Fundamental Thm of Calculus

Using Riemann sums to evaluate definite integrals is not very efficient or practical. In this sense, we need more effective methods to evaluate integrals.

In this section, we develop methods to find the area under a curve.

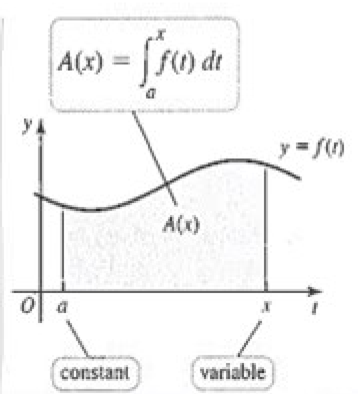
We also tie together the concepts of integration and differentiation through possibly the most important theorem in calculus.

# Area Functions

To connect the concepts of differentiation and integration, we first must define the concept of an area function. We start with a continuous function

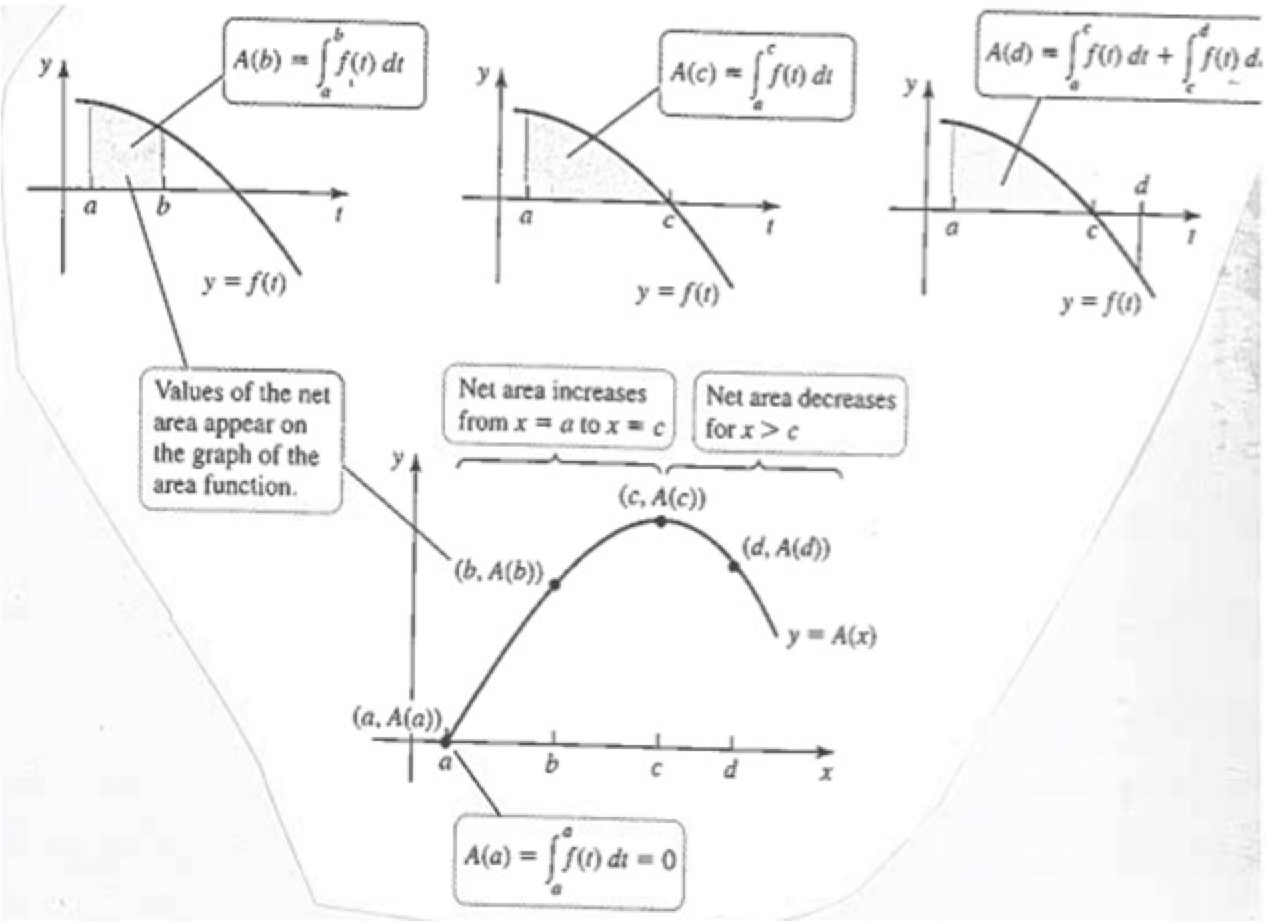
$y = f(t)$ which will be defined for all points $t \geq a$.

The area function for $f$ with a left endpoint at $a$ is given by $A(x) = \int_a^x f(t)\,dt$ . This gives the net area of the region between the graph of $f$ and the $t$-axis between the points $t = a$ and $t = x$.



# Area Functions

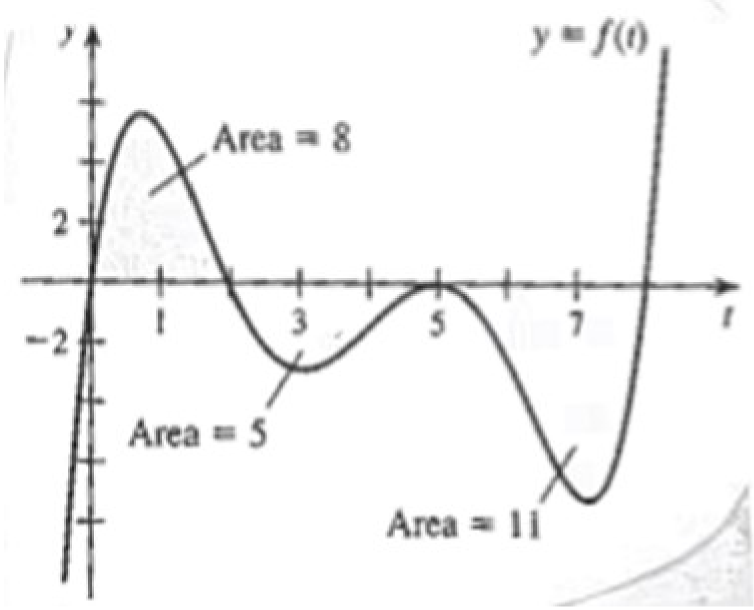
Figure 5.33 (p. 335) illustrates how the area function works.



# Exercise

The graph of *f* is shown below. Let $A(x) = \int_0^x f(t)\,dt$ and $F(x) = \int_2^x f(t)\,dt$ be two area functions for *f*.

What is A(2)? F(5)? A(5)? F(8)?



# Building the Case

Linear functions help to build the rationale behind the Fundamental Theorem of Calculus

EX: Let *f(t) = 4t + 3* and define $A(x) = \int_1^x f(t)\,dt$.

What is A(2)? A(4)? A(x)? A'(x)?

# The Fundamental Theorem of Calculus (Part 1)

In general, the property illustrated with the previous linear function works for all continuous functions and is one part of the FTC.

**THEOREM:** If *f* is continuous on [a, b], then the area function $A(x) = \int_a^x f(t)dt$ for $a \le x \le b$ is continuous on [a, b] and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$; or equivalently,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

which means that the area function of *f* is an antiderivative of *f*.

# The Fundamental Theorem of Calculus (Part 2)

Given that A is an antiderivative of *f*, this provides us with an effective method for evaluating definite integrals and finding areas under curves.

**THEOREM:** If *f* is continuous on [a, b] and *F* is any antiderivative of *f*, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

# Overview of FTC

In essence, to evaluate an integral, we

- Find any antiderivative of $f$, and call it $F$;
- Compute $F(b) – F(a)$, the difference in the values of $F$ between the upper and lower limits of integration.

The two parts of the FTC illustrate the inverse relationship between differentiation and integration; that is, the integral “undoes” the derivative.

# Exercise

The various parts of the FTC can be used to simplify and evaluate integral expressions.

- Use Part 1 of the FTC to simplify $\dfrac{d}{dx}\displaystyle\int_x^{10} \frac{dz}{z^2+1}$
- Use Part 2 of the FTC to evaluate $\displaystyle\int_0^{\pi} (1-\sin x)\,dx$

# Click in your Response

The value of $\int_{1}^{y} h'(p)dp$ is :

A. $h'(y) - h'(1)$.

B. $h(y) - h(1)$.

C. $h(y)$.

D. $h(p) - h(1)$.

# Homework from Section 5.3

Do problems 11-17 all, 19-39 odd, 45-57 odd (pgs. 346-347).

# Week 15: Mon 1 Dec 2014

- All the remaining webwork is now live.
- Tues 2 Dec: Quiz on 4.8-5.3
- Wed 3 Dec: Review
- Fri 5 Dec: Exam 3 covers 4.6-5.4
- Mon 8 Dec: 5.5
- Tues 9 Dec: Quiz on 5.5
- Wed 10 Dec: REVIEW (look at Rolle's Thm)
- FINAL ON MON 15 DEC 6-8p in

  HILLSIDE 206

# Working with Integrals

Now that we have methods to use in integrating functions, we now can examine applications of integration.

These applications include:

- Integration of even and odd functions;
- Finding the average value of a function;
- Developing the Mean Value Theorem for Integrals

# Integrating Even and Odd Functions

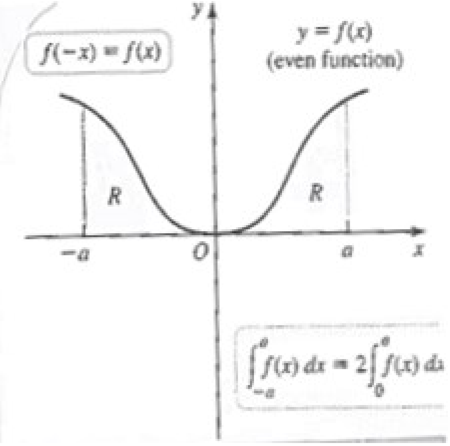
Recall the definition of an even function ( $f(-x) = f(x)$) and of an odd function ( $f(-x) = - f(x)$)

The properties of these functions yield interesting results when integrated over an interval centered at the origin.

# Integrating Even Functions

Recall that even functions are symmetric across the y-axis. So $\int_{-a}^{0} f(x)dx = \int_{0}^{a} f(x)dx$ (e.g., the area under the curve to the left of the y-axis is equal to the area under the curve to the right).

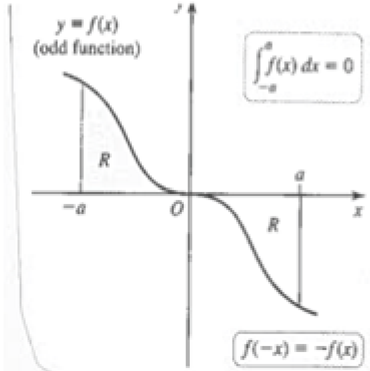
So $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$ for even functions



# Integrating Odd Functions

Recall that even functions have 180° rotation symmetry around the origin (e.g., take the function to the left of the origin, rotate it 180°, and you now have the function to the right of the origin). So $\int_{-a}^{0} f(x)dx = -\int_{0}^{a} f(x)dx$

(e.g., the area under the curve to the left of the origin is the negative of the area under the curve to the right of the origin). So $\int_{-a}^{a} f(x)dx = 0$ for odd functions



# Exercises

Evaluate the following integrals using the properties of even and odd functions:

$$\int_{-4}^{4} (3x^2 - x)\,dx$$

$$\int_{-1}^{1} (1 - |x|)\,dx$$

$$\int_{-\pi}^{\pi} (\sin x)\,dx$$

# Average Value of a Function

The average value of a function is similar to finding the average of a set of numbers.

Suppose we partition [a, b] into *n* equally sized sections and choose a point $\overline{x_k}$ in each section. Then $\dfrac{f(\overline{x_1})+f(\overline{x_2})+\ldots+f(\overline{x_n})}{n}$ is the average of the function values over [a, b]. Since $n\cdot \Delta x = b-a \Rightarrow n = \dfrac{b-a}{\Delta x}$. So

$$\frac{f(\overline{x_1})+f(\overline{x_2})+\ldots+f(\overline{x_n})}{n} = \frac{f(\overline{x_1})+f(\overline{x_2})+\ldots+f(\overline{x_n})}{\frac{b-a}{\Delta x}} = \frac{1}{b-a}(f(\overline{x_1})+f(\overline{x_2})+\ldots+f(\overline{x_n}))\Delta x$$

As $n \to \infty$, this is $\dfrac{1}{b-a}\int_a^b f(x)dx$

# Average Value of a Function

The average value of an integrable function *f* on the interval [a, b] is

$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

# Exercise

Find the average value for the function $f(x) = x(1-x)$ on the interval from [0, 1].

# Homework from Section 5.4

Do problems 7-27 odd, 31-35 odd (pgs. 354-355).

# Wed 3 Dec 2014

- Fri 5 Dec: Exam 3 (4.6 to 5.4)
- Mon 8 Dec: 5.5
- Tues 9 Dec: Quiz on 5.5
- Wed 10 Dec: REVIEW (look at Rolle's Thm)
- FINAL ON MON 15 DEC 6-8p in

  HILLSIDE 206

# Contextual Example (averages)

The elevation of a path is given by $f(x) = x^3 - 5x^2 + 10$, where *x* measures horizontal distances. Draw a graph of the elevation function and find its average value for $0 \le x \le 4$.

# Mean Value Theorem for Integrals

The average value of a function leads to the Mean Value Theorem for Integrals. Similar to the Mean Value Theorem from section 4.6, the MVT for integrals says we can find a point *c* between *a* and *b* so that *f(c)* is the average value of the function.

**THEOREM:** Let *f* be continuous on the interval [a, b]. There exists a point *c* in [a, b] such that

$$f(c) = \overline{f} = \frac{1}{b-a}\int_a^b f(x)dx$$

# Exercise

Find or approximate the point(s) at which the function $f(x) = x^2 - 2x + 1$ equals its average value on the interval [0, 2].

# Review for Exam 3

4.6: Mean Value Thm

- Know and be able to state Rolle's Thm and the Mean Value Thm, including knowing the hypothesis and conclusions for both.
- Be able to apply Rolle's Thm to find a point in a given interval.
- Be able to apply the MVT to find a point in a given interval
- Be able to use the MVT to find equations of secant and tangent lines.

# Review for Exam 3

4.7: L'Hopital's Rule

- Know how to use L'Hopital's Rule, including knowing under what conditions the rule works
- Be able to apply L'Hopital's Rule to a variety of limits that are in indeterminate forms (e.g., 0/0, ∞/∞, 0×∞, ∞−∞, 1^∞, 0^0, ∞^0).
- Be able to use L'Hopital's Rule to determine which of two functions grows at a faster rate.
- Beware of the pitfalls of using L'Hopital's Rule.

# Review for Exam 3

5.1: Approximating Areas under Curves

- Be able to use rectangles to approximate area under the curve for a given function.
- Know how to calculate left Riemann sums, right Riemann sums, and midpoint Riemann sums for a function (these will not have a large *n* value).
- Be able to sum a series of numbers written with sigma notation.
- Be able to identify whether a given Riemann sum written in sigma notation is a left, right or midpoint sum.

# Review for Exam 3

5.2: Definite Integrals

- Be able to compute left, right, or midpoint Riemann sums for curves that have negative components, and understand the concept of net area.
- Be able to evaluate a definite integral using geometry or a given graph.
- Know the properties of definite integrals and be able to use them to evaluate a definite integral.

# Review for Exam 3

4.8: Antiderivatives

- Know the definition of an antiderivative and be able to find one or all antiderivatives of a function.
- Be able to evaluate indefinite integrals, including using known properties of indefinite integrals (e.g., power rule, constant multiple rule, sum rule).
- Know how to find indefinite integrals of trig functions, of e^(ax), of ln x, and of the three given inverse trig functions
- Be able to solve initial value problems to find specific antiderivatives.
- Be able to use antiderivatives to work with motion problems.

# Review for Exam 3

5.3: Fundamental Theorem of Calculus

- Understand the concept of an area function, and be able to evaluate an area function as *x* changes.
- Know the two parts of the Fundamental Theorem of Calculus and its significance (i.e., the inverse relationship between differentiation and integration).
- Use the FTC to evaluate definite integrals or simplify given expressions.

# Review for Exam 3

5.4: Working with Integrals

- Be able to integrate even and odd functions knowing the "shortcuts" provided by these functions characteristics
- Be able to find the average value of a function
- Know the Mean Value Theorem for Integrals and be able to use it to find points associated with the average value of a function