DIVISORS

by Ashley K. Wheeler
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These notes are a work in progress. Stated informally, a (Weil) divisor on a variety $\mathcal{X}$ is an element of the free abelian group with the irreducible codimension 1 subvarieties as generators. The divisor class group is an important invariant of an algebraic variety, and more generally, of a scheme. Unless explicitly stated otherwise, all schemes/varieties are over an algebraically closed field, K. All rings are Noetherian.

We start with the following motivation idea: Consider a monic polynomial $f \in K[x]$, in one variable over an algebraically closed field K. By the Fundamental Theorem of Algebra, f factors completely. The solutions to $f(x) = 0$ have multiplicities. Suppose f has m distinct roots, $C_1, \ldots, C_m$ with respective multiplicities μ_i for $i = 1, \ldots, m$. Such data is enough to define f uniquely up to a constant, which is why we asserted f was monic. To define f, specify points in $\mathbb{A}^1_K \cong K$ where f must vanish, along with their respective multiplicities. Define

$$\operatorname{div} f = \sum_{\substack{C \text{ is a} \\ \text{root of } f}} \mu_C \cdot C,$$

an element of the free abelian group generated by the points in $\mathbb{A}^1_K$. Then $\operatorname{div} f$ uniquely determines f (again, up to a constant).

In fact, if f is a rational function, i.e.,

$$f = \frac{p}{q} \in K(x),$$

then we can regard the poles of f as multiplicities of roots of q, and give them a negative sign. We can define

$$\operatorname{div}_0 f = \operatorname{div} p \qquad \text{and} \qquad \operatorname{div}_\infty f = \operatorname{div} q.$$

Then $\operatorname{div} f = \operatorname{div}_0 f - \operatorname{div}_\infty f$. From this definition it should be clear that $\operatorname{div} f$ doesn't depend on the choices $p, q \in K[x]$.

Divisors generalize this notion to arbitrary varieties, and ultimately, to arbitrary schemes. Let $\mathcal{X}$ be an irreducible variety. A **(Weil) divisor** D on $\mathcal{X}$ is a collection $C_1, \ldots, C_m$ of irreducible codimension 1 subvarieties for some m, with respective "multiplicities" μ_i assigned to them. We write

$$D = \sum_{i=1}^{m} \mu_i \cdot C_i,$$

an element of $\operatorname{Div} \mathcal{X}$, the free abelian group generated by all the irreducible codimension 1 subvarieties of $\mathcal{X}$.

$$\operatorname{Div} \mathcal{X} = \bigoplus_{\substack{C \subset \mathcal{X} \mid C \text{ is an irreducible} \\ \text{subvariety of codimension 1}}} \mathbb{Z}$$

Exercise. How do you add Weil divisors?

The generators for $\operatorname{Div}\mathcal{X}$, the Cs, are called **prime divisors**. The **support** of D, denoted $\operatorname{Supp} D$, is the set of C_is where $\mu_i \neq 0$. A divisor D is **effective** means $\mu_i \geq 0$ for all i and not all μ_is are zero[1].

Divisor of a Function

For now, suppose $\mathcal{X}$ is an irreducible variety, non-singular in codimension 1. Given $f \in K(\mathcal{X})$, i.e., a rational function on $\mathcal{X}$, we associate to it a divisor $\operatorname{div} f$. Divisors that arise in this form are called **principal divisors**.

$$\{\text{rational functions on } \mathcal{X}\} \to \operatorname{Div}\mathcal{X}$$

Suppose C is a prime divisor on $\mathcal{X}$ and intersect it with an open set $\mathcal{U}$ satisfying:
(1) $\mathcal{U}$ is open affine
(2) $\mathcal{U}$ is non-singular
(3) $\mathcal{U} \cap C = \mathcal{V}(\varphi)$ for some $\varphi \in K[\mathcal{U}]$, the ring of regular functions on $\mathcal{U}$. Given $f \in K[\mathcal{U}]$, define

$$v_C(f) = r,$$

such that $f \in (\varphi^r)$ but $f \notin (\varphi^{r+1})$.

Exercise. Such r exists and is unique. Furthermore, it doesn't depend on the set $\mathcal{U}$.

We say "f has a **zero** along C" to mean $v_C(f) > 0$ and we say "f has a **pole** along C" to mean $v_C(f) < 0$. The divisor associated to f is

$$\operatorname{div} f = \bigoplus_{\substack{C \subset \mathcal{X} \mid C \text{ is an irreducible} \\ \text{subvariety of codimension 1}}} v_C(f) \cdot C$$

Exercise. There are only finitely many non-zero Cs in $\operatorname{div} f$. In other words, $\operatorname{div} f$ is well-defined.

We also define

$$\operatorname{div}_0 f = \sum_{v_C(f) > 0} v_C(f) \cdot C \qquad \text{and} \qquad \operatorname{div}_\infty f = \sum_{v_C(f) < 0} |v_C(f) \cdot C|.$$

Exercise. Suppose $f \in K[\mathcal{X}]$, where $\mathcal{X}$ is irreducible and non-singular in codimension 1.
(1) $\operatorname{div} f \geq 0$.
(2) $\operatorname{div} f = 0$ for $f \in K$.

[1] Many contemporary algebraic geometers allow the term effective to include the case where all the μ_is are zero.

Proposition. *Suppose $\mathcal{X}$ is a non-singular irreducible variety. A rational function f is regular on $\mathcal{X}$ if and only if* $\operatorname{div} f \geq 0$.

Proof. From the preceding exercise, $f \in K[\mathcal{X}]$ implies $\operatorname{div} f \geq 0$. It remains to prove the converse. We suppose $\operatorname{div} f \geq 0$.

Assume there exists a point $a \in \mathcal{X}$ where $f = \frac{g}{h}$ is not regular, so $g, h \in \mathcal{O}_a$ but $f \notin \mathcal{O}_a$. Recall, $\mathcal{O}_a$ is the ring of rational functions that have no poles at a. Since $\mathcal{O}_a$ is a UFD we can assert g, h have no common factor. We choose $\pi \in \mathcal{O}_a$ such that $\pi \mid h$ but π does not divide g. The algebraic set $\mathcal{V}(\pi) \subset \mathcal{U}$ is irreducible, for some open neighborhood $\mathcal{U}$ of a, and it has codimension 1. Let C denote its closure in $\mathcal{X}$. Then $v_C(f) < 0$, a contradiction. $\square$

The Divisor Class Group

Given $\mathcal{X}$, an irreducible variety, non-singular in codimension 1, its principal divisors form a subgroup of $\operatorname{Div} \mathcal{X}$.

Exercise. Prove that.

The quotient, $\operatorname{Cl} \mathcal{X}$, is called the **divisor class group** for $\mathcal{X}$.

$$\frac{\operatorname{Div} \mathcal{X}}{\{\text{principal divisors on } \mathcal{X}\}} = \operatorname{Cl} \mathcal{X}$$

Divisors D_1, D_2 in the same coset are called **linearly equivalent** and we write $D_1 \sim D_2$.

Example. $\mathcal{X} = \mathbb{A}^1_K$ $(= \operatorname{Spec} K[x])$

This is the case we considered in the beginning of the talk. An irreducible subvariety of codimension 1 is a point. Suppose

$$D = \sum_{i=1}^{m} \mu_i \cdot C_i$$

is a divisor on $\mathcal{X}$. If we define

$$f = \prod_{i=1}^{m} (x - C_i)^{\mu_i} \in K[x]$$

then it is clear $\operatorname{div} f = D$. Thus every Weil divisor on $\mathcal{X}$ is principal. It follows that $\operatorname{Cl}(\mathbb{A}^1_K) = 0$.

Example. $\mathcal{X} = \mathbb{A}^n$ $(= \operatorname{Spec}(K[x_1, \ldots, x_n]))$.

Any irreducible codimension 1 subvariety C is defined by one equation, $f \in K[\mathcal{X}]$. Then, as in the $n = 1$ case, we have $C = \operatorname{div} f$. Therefore every prime divisor, and hence every divisor, is principal. We conclude $\operatorname{Cl}(\mathbb{A}^n) = 0$.

Example. $\mathcal{X} = \mathbb{P}^n$ $(= \operatorname{Proj}(K[x_0, \ldots, x_n]))$.

Again, any irreducible codimension 1 subvariety C is defined by one equation f, only f is homogeneous, say of degree d. The **degree** of a divisor $D = \sum_i \mu_i C_i$ is

$$\deg D = \sum_i \mu_i \cdot \deg C_i,$$

where $\deg C_i$ is the degree of the hypersurface C_i. The degree of a divisor gives a surjection

$$\deg : \operatorname{Div} \mathcal{X} \twoheadrightarrow \mathbb{Z}.$$

We claim the degree 0 divisors on $\mathbb{P}^n$ are exactly the principal divisors.

A rational function f can be written $f = \frac{g}{h}$, where $g, h \in K[x_0, \ldots, x_n]$ are homogeneous and of the same degree. We factor into irreducibles

$$g = \prod_i g_i^{\mu_i} \qquad \text{and} \qquad h = \prod_j h_j^{\nu_j}.$$

We have

$$\begin{aligned} \operatorname{div} f &= \operatorname{div} g - \operatorname{div} h \\ &= \sum_i \mu_i \cdot \mathcal{V}(g_i) - \sum_j \nu_j \cdot \mathcal{V}(h_j) \end{aligned}$$

which implies $\deg f = \deg g - \deg h = 0$. Conversely, suppose D is a divisor of degree 0. Let

$$f = \prod_i f_i^{\mu_i}$$

where f_i defines the hypersurface C_i, for each i. It follows that $D = \operatorname{div} f$, so is principal.

Since the degree 0 divisors on $\mathcal{X}$ are exactly the principal divisors we get an exact sequence

$$0 \to \{\text{principal divisors}\} \to \operatorname{Div} \mathcal{X} \to \mathbb{Z} \to 0.$$

It follows that $\operatorname{Cl}(\mathbb{P}^n) = \mathbb{Z}$. In particular, any Weil divisor D on $\mathbb{P}^n$ with degree d is linearly equivalent to $d \cdot H$ where H is any hyperplane.

The next two propositions give a couple of shortcuts for computing divisor class groups.

Proposition. *Suppose $\mathcal{X}$ is an affine normal variety. Then $\operatorname{Cl} \mathcal{X} = 0$ if and only if $K[\mathcal{X}]$ is a UFD.*

Proposition. *Suppose $\mathcal{X}$ is an irreducible variety, and regular (i.e., non-singular) in codimension one, with a subvariety $\mathcal{Z} \subsetneq \mathcal{X}$. Let $\mathcal{U}$ denote its complement, $\mathcal{X} \setminus \mathcal{Z}$. If $\mathcal{Z}$ is irreducible and of codimension 1 then there is an exact sequence*

$$\mathbb{Z} \to \operatorname{Cl} \mathcal{X} \to \operatorname{Cl} \mathcal{U} \to 0,$$

where the first map is defined by $1 \mapsto 1 \cdot \mathcal{Z}$.

Example. Let $R = K[x,y,z]/(xy - z^2)$ and let $\mathcal{X} = \mathcal{V}(xy - z^2) \subset \mathbb{A}^3$.

Since $\mathcal{X}$ is an affine quadric cone in $\mathbb{A}^3$ it has dimension 2. Let C denote the line given by $(y,z) \in R$. Note how $R/(y,z) \cong K[x]$ so is a domain of dimension 1. In other words, C really is an irreducible subvariety of codimension 1. On the other hand, C has two generators so is not principal. However,

$$(y,z)^2 = (y^2, yz, z^2) = (y^2, yz, y) = (y)$$

so $2C$ is principal. Consider $\mathcal{X} \setminus C = \operatorname{Spec} R[\frac{1}{y}]$. In the localized ring we can use $xy - z^2 = 0$ to solve for x:

$$R\left[\frac{1}{y}\right] = \frac{K[x, y, \frac{1}{y}, z]}{(xy - z^2)} \cong K\left[y, \frac{1}{y}, z\right],$$

which is a UFD. So $\mathrm{Cl}(\mathcal{X} \setminus C) = 0$. It follows that given an arbitrary divisor on $\mathcal{X}$, if its support is anything other than C then it must be 0. In other words, any Weil divisor on $\mathcal{X}$ is linearly equivalent to a multiple of C. But we saw $2C \sim 0$, so we must have

$$\mathrm{Cl}\left(\operatorname{Spec} \frac{K[x,y,z]}{(xy - y^2)}\right) \cong \mathbb{Z}/2\mathbb{Z}.$$

Example. Let $\mathcal{Y}$ denote a degree d irreducible curve in $\mathbb{P}^2$. Then $\mathrm{Cl}(\mathbb{P}^2 \setminus \mathcal{Y}) \cong \mathbb{Z}/d\mathbb{Z}$.

Example. **Analogue to Number Theory.** Let R denote a Dedekind domain with field of fractions $\mathcal{K}$. An R-submodule M of $\mathcal{K}$ is called a **fractional ideal** means there exists $x \in R \setminus \{0\}$ such that $xM \subseteq R$. The fractional ideals generated by non-zero elements in $\mathcal{K}$ are called **principal**. The fractional ideals form a group with a subgroup given by the principal fractional ideals. The quotient is called the **ideal class group**. When R is the ring of algebraic integers, the ideal class group coincides with $\mathrm{Cl}(\operatorname{Spec} R)$.

Cartier Divisors

In this section we define the notion of a divisor on a scheme. We blackbox as much of the necessary sheaf theory as possible. Most of the time when working with sheaves it helps to pretend they are just rings, in developing intuition. In reality, a sheaf $\mathcal{F}$ (of rings) on the topological space $\mathcal{X}$ assigns to every Zariski-open subset $\mathcal{U} \subseteq \mathcal{X}$ a ring[2], denoted $\Gamma(\mathcal{U}, \mathcal{F})$, in such a way that the rings are consistent on overlapping open sets. Elements in $\Gamma(\mathcal{X}, \mathcal{F})$ are called **global sections**. A scheme is a pair $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, with $\mathcal{X}$ a Zariski topological space and $\mathcal{O}_{\mathcal{X}}$ its **structure sheaf**, which assigns to each open subset $\mathcal{U} \subseteq \mathcal{X}$ its ring of regular functions $\mathcal{O}_{\mathcal{X}}(\mathcal{U}) = \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{X}})$. We often denote a scheme $\mathcal{X}$, with the structure sheaf implied. When context is clear, we may also write $\mathcal{O} = \mathcal{O}_{\mathcal{X}}$.

[2] There are also sheaves of groups, sheaves of modules, etc.

Let $\mathcal{X}$ denote a scheme. We shall only consider schemes that are noetherian, integral, separated, and regular in codimension 1 (see II.6 of [Hart] for the relevant definitions). For each open $\mathcal{U}$, let $S(\mathcal{U})$ denote the multiplicative system of non-zerodivisors in $\mathcal{O}_{\mathcal{X}}(\mathcal{U})$. The rings $S(\mathcal{U})^{-1}\mathcal{O}(\mathcal{U})$ form a presheaf[3] whose associated sheaf $\mathcal{K}$ is called the **sheaf of total quotient rings** of $\mathcal{O}_{\mathcal{X}}$. Think of $\mathcal{K}$ as the sheaf-analogue to the ring of rational functions $K(\mathcal{X})$ when $\mathcal{X}$ is a variety. Similarly, think of $\mathcal{K}^*$ and $\mathcal{O}^*$ as analogues to multiplicative groups.

We get a Cartier divisor on $\mathcal{X}$ by taking an open cover $\{\mathcal{U}_i\}$ of $\mathcal{X}$, and to each i an element $f_i \in \Gamma(\mathcal{U}_i, \mathcal{K}^*)$ such that for any i, j we have

$$\frac{f_i}{f_j} \in \Gamma(\mathcal{U}_i \cap \mathcal{U}_j, \mathcal{O}^*).$$

A **Cartier divisor**, also called a **locally principal (Weil) divisor**, is a global section of the quotient sheaf $\mathcal{K}^*/\mathcal{O}^*$, and is independent of the cover chosen for $\mathcal{X}$. We write

$$D = \sum_C k_C \cdot C,$$

where $k_C = v_C(f_i)$ if $\mathcal{U}_i \cap C \neq \emptyset$ and zero otherwise. Again, the definition is consistent, as it does not depend on the cover of $\mathcal{X}$.

A **principal Cartier divisor** is an element in the image of the natural map

$$\Gamma(\mathcal{X}, \mathcal{K}^*) \to \Gamma(\mathcal{X}, \mathcal{K}^*/\mathcal{O}^*).$$

As with Weil divisors, the principal Cartier divisors give a subgroup of the group of all Cartier divisors, and the quotient is called the **Picard group**.

Connection to Invertible Sheaves

Given a sheaf $\mathcal{F}$ of rings we can form sheaves of modules on those rings; such sheaves are called $\mathcal{F}$-modules. An **invertible sheaf** $\mathcal{L}$ on a scheme $\mathcal{X}$ is a locally free $\mathcal{O}_{\mathcal{X}}$-module of rank 1, that is, its modules $\mathcal{L}(\mathcal{U})$ are free $\mathcal{O}(\mathcal{U})$-modules of rank 1. The reason we call such a sheaf invertible is because there always exists another sheaf, $\mathcal{L}^{-1}$, also invertible, such that

$$\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_{\mathcal{X}}.$$

Tensoring[4] invertible sheaves with each other produces more invertible sheaves and in fact, the isomorphism classes of invertible sheaves on $\mathcal{X}$ form a group under the tensor operation. It turns out this group is isomorphic to the Picard group of $\mathcal{X}$, which is denoted $\operatorname{Pic}\mathcal{X}$.

[3]a technicality; see Chapter II of [Hart].

[4]The tensor product of two sheaves, $\mathcal{F} \otimes \mathcal{G}$, is defined as the sheafification of the presheaf $\mathcal{U} \mapsto \mathcal{F}(\mathcal{U}) \otimes_{\mathcal{O}(\mathcal{U})} \mathcal{G}(\mathcal{U})$.

Now, suppose D is a Cartier divisor given by the cover $\{\mathcal{U}_i\}$ and respective functions f_i. We define a sheaf $\mathcal{L}(D)$ by taking

$$\Gamma(\mathcal{U}_i, \mathcal{L}(D)) = (f_i^{-1}) \cdot \mathcal{K}(\mathcal{U}_i).$$

This works (is well-defined) precisely because of the requirement f_i/f_j is invertible on $\mathcal{U}_i \cap \mathcal{U}_j$. The sheaf $\mathcal{L}(D)$ is called the **sheaf associated to** D and is a sub-$\mathcal{O}_{\mathcal{X}}$-module of $\mathcal{K}$.

It is through this correspondence that the group of invertible sheaves on $\mathcal{X}$, $\operatorname{Pic}\mathcal{X}$, is isomorphic to group of Cartier divisors modulo the principal Cartier divisors.

Linear Systems

Consider $\mathbb{P}^1$. The condition deg $f \leq n$ is the same as saying $\operatorname{div} f + nx_\infty$ is effective. Now say D is any divisor on $\mathcal{X}$, a non-singular variety. Then the set of all non-zero functions $f \in K(\mathcal{X})$ such that

$$D_f = \operatorname{div} f + D \geq 0,$$

along with 0, form a vector space called the **associated vector space** or **Riemann-Roch space** of D and is denoted by $\mathcal{L}(D)$. The **dimension** of D, denoted $l(D)$, is given by the vector space dimension of $\mathcal{L}(D)$.

Theorem. *Linearly equivalent divisors have the same dimension.*

Therefore we may as well speak of the dimension of a divisor class in $\operatorname{Cl}\mathcal{X}$. Let D represent a class in $\operatorname{Cl}\mathcal{X}$. If $f \in \mathcal{L}(D)$ then D_f is effective. Conversely, if D' is effective and linearly equivalent to D, then there exists f such that $D' = D_f$. Since in projective space a function f is uniquely determined up to a constant, we may parametrize the effective divisors which are linearly equivalent to D via $\operatorname{Proj}(\mathcal{L}(D)) \cong \mathbb{P}^{l(D)-1}$.

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