Calculus I (Math 2554)

Spring 2016

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University of Arkansas

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The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

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Tips for Success

- Attend class every day. Participate in math discussions. Do the lecture-cises fully, not just on a scratch paper.
- Don't get behind on MLP homeworks. Stay on top of the book problems.
- Find a study partner(s) to meet with on a regular basis. Don't be afraid to seek further assistance (tutoring, office hours, etc.) if you are struggling.
- high school calculus $\neq$ college calculus
- REMEMBER... THE TERM STARTS TODAY! SO DOES THE EVENTUAL EARNING OF YOUR FINAL GRADE!!!

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Wed 20 Jan

Welcome to Cal I!

- comp.uark.edu/~ashleykw/Cal1Spring2016/cal1spr16.html
  Course website. All information is here, including a link to MLP, lecture slides, administrative information, etc. You should have already seen the **syllabus** by now.
- MyLabsPlus (MLP) has the graded homework. Solutions to Quizzes and Drill exercises will be posted there, under "Menu $\rightarrow$ Course Tools $\rightarrow$ Document Sharing".

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Wed 20 Jan (cont.)

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- For old Calculus materials, see the parent page `comp.uark.edu/~ashleykw` and look for links under "Previous Semesters".

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§2.1 The Idea of Limits

Question

How would you define, and then differentiate between, the following pairs of terms?

- instantaneous velocity vs. average velocity?
- tangent line vs. secant line?

(Recall: What is a tangent line and what is a secant line?)

Example

An object is launched into the air. Its position s (in feet) at any time t (in seconds) is given by the equation:

$$s(t) = -4.9t^2 + 30t + 20.$$

(a) Compute the average velocity of the object over the following time intervals: $[1, 3]$, $[1, 2]$, $[1, 1.5]$

(b) As your interval gets shorter, what do you notice about the average velocities? What do you think would happen if we computed the average velocity of the object over the interval $[1, 1.2]$? $[1, 1.1]$? $[1, 1.05]$?

Example, cont.

An object is launched into the air. Its position s (in feet) at any time t (in seconds) is given by the equation:

$$s(t) = -4.9t^2 + 30t + 20.$$

(c) How could you use the average velocities to estimate the instantaneous velocity at $t = 1$?

(d) What do the average velocities you computed in 1. represent on the graph of $s(t)$?

Question

What happens to the relationship between instantaneous velocity and average velocity as the time interval gets shorter?

Answer: The instantaneous velocity at $t = 1$ is the limit of the average velocities as t approaches 1.

Question

What about the relationship between the secant lines and the tangent lines as the time interval gets shorter?

Answer: The slope of the tangent line at $(1, 45.1 = s(1))$ is the limit of the slopes of the secant lines as t approaches 1.

2.1 Book Problems

1-3, 7-13, 15, 21, 25, 27, 29

Even though book problems aren't turned in, they're a very good way to study for quizzes and tests (wink wink wink).

§2.2 Definition of Limits

Question

- Based on your everyday experiences, how would you define a "limit"?
- Based on your mathematical experiences, how would you define a "limit"?
- How do your definitions above compare or differ?

Fri 22 Jan

- comp.uark.edu/~ashleykw/Cal1Spring2016/cal1spr16.html
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Fri 22 Jan (cont.)

- Lecture slides are available on the course website. I'll try to have the week's slides posted in advance but the individual lectures might not be posted until right before class. **Don't try to take notes from the slides.** Instead, print out the slides beforehand or else follow along on your tablet/phone/laptop. You should, however, take notes when we do exercises during lecture.
- For old Calculus materials, see the parent page `comp.uark.edu/~ashleykw` and look for links under "Previous Semesters".
- Next week: Attendance using clickers.

Definition of a Limit of a Function

Definition (limit)

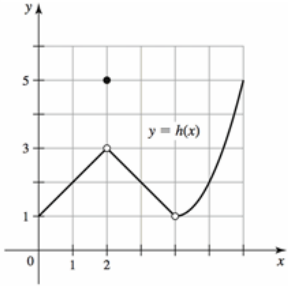
Suppose the function f is defined for all x near a, except possibly at a. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = L$$

and say **the limit of** $f(x)$ **as** x **approaches** a **equals** L.

Determining Limits from a Graph

Exercise



Determine the following:

(a) $h(1)$

(b) $h(2)$

(c) $h(4)$

(d) $\lim_{x\to 2} h(x)$

(e) $\lim_{x\to 4} h(x)$

(f) $\lim_{x\to 1} h(x)$

Question

Does $\lim_{x \to a} f(x)$ always equal $f(a)$?

(Hint: Look at the example from the previous slide!)

Determining Limits from a Table

Exercise

Suppose $f(x) = \dfrac{x^2 + x - 20}{x - 4}$.

(a) Create a table of values of $f(x)$ when

$$x = 3.9, 3.99, 3.999, \text{ and}$$
$$x = 4.1, 4.01, 4.001$$

(b) What can you conjecture about $\lim_{x \to 4} f(x)$?

One-Sided Limits

Up to this point we have been working with two-sided limits; however, for some functions it makes sense to examine one-sided limits.

Notice how in the previous example we could approach $f(x)$ from both sides as x approaches a, i.e., when $x > a$ and when $x < a$.

Definition (right-hand limit)

Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

$$\lim_{x \to a^+} f(x) = L$$

and say **the limit of** $f(x)$ **as** x **approaches** a **from the right equals** L.

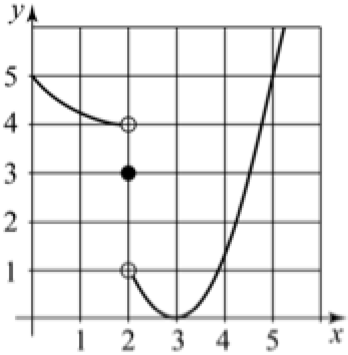
Definition (left-hand limit)

Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

$$\lim_{x \to a^-} f(x) = L$$

and say **the limit of** $f(x)$ **as** x **approaches** a **from the left equals** L.

Exercise



Determine the following:

(a) $g(2)$

(b) $\lim_{x\to 2^+} g(x)$

(c) $\lim_{x\to 2^-} g(x)$

(d) $\lim_{x\to 2} g(x)$

Relationship Between One- and Two-Sided Limits

Theorem

If f is defined for all x near a except possibly at a, then $\lim_{x\to a} f(x) = L$ if and only if both $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.

In other words, the only way for a two-sided limit to exist is if the one-sided limits equal the same number (L).

2.2 Book Problems

1-4, 7, 9, 11, 13, 19, 23, 29, 31

§2.3 Techniques for Computing Limits

Exercise

Given the function $f(x) = 4x + 7$, find $\lim_{x \to -2} f(x)$

(a) graphically;

(b) numerically (i.e., using a table of values near -2)

(c) via a direct computation method of your choosing.

Compare and contrast the methods in this exercise – which is the most favorable?

This section provides various laws and techniques for determining limits. These constitute **analytical** methods of finding limits. The following is an example of a very useful limit law:

Limits of Linear Functions: Let a, b, and m be real numbers. For linear functions $f(x) = mx + b$,

$$\lim_{x \to a} f(x) = f(a) = ma + b.$$

This rule says we if $f(x)$ is a linear function, then in taking the limit as $x \to a$, we can just plug in the a for x.

IMPORTANT! Using a table or a graph to compute limits, as in the previous sections, can be helpful. However, "analytical" does not include those techniques.

Limit Laws

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, c is a real number, and m, n are positive integers.

---

1. **Sum:** $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. **Difference:** $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

In other words, if we are taking a limit of two things added together or subtracted, then we can first compute each of their individual limits one at a time.

Limit Laws, cont.

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, c is a real number, and m, n are positive integers.

---

3. **Constant Multiple:** $\lim_{x\to a}(cf(x)) = c\left(\lim_{x\to a} f(x)\right)$

4. **Product:** $\lim_{x\to a}(f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$

The same is true for products. If one of the factors is a constant, we can just bring it outside the limit. In fact, a constant is its own limit.

Limit Laws, cont.

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, c is a real number, and m, n are positive integers.

---

5. Quotient: $$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

(provided $\lim_{x \to a} g(x) \neq 0$)

Question

Why the caveat?

Limit Laws, cont.

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, c is a real number, and m, n are positive integers.

---

6. **Power:** $\lim_{x\to a} (f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$

7. **Fractional Power:** $\lim_{x\to a} (f(x))^{\frac{n}{m}} = \left(\lim_{x\to a} f(x)\right)^{\frac{n}{m}}$

   (provided $f(x) \geq 0$ for x near a if m is even and $\frac{n}{m}$ is in lowest terms)

Question

Explain the caveat in **7.**

Limit Laws, cont.

Laws **1.-6.** hold for one-sided limits as well. But **7.** must be modified:

7. **Fractional Power (one-sided limits):**
   - $$\lim_{x \to a^+} (f(x))^{\frac{n}{m}} = \left(\lim_{x \to a^+} f(x) \right)^{\frac{n}{m}}$$
     (provided $f(x) \geq 0$ for x near a with $x > a$, if m is even and $\frac{n}{m}$ is in lowest terms)
   - $$\lim_{x \to a^-} (f(x))^{\frac{n}{m}} = \left(\lim_{x \to a^-} f(x) \right)^{\frac{n}{m}}$$
     (provided $f(x) \geq 0$ for x near a with $x < a$, if m is even and $\frac{n}{m}$ is in lowest terms)

Limits of Polynomials and Rational Functions

Assume that $p(x)$ and $q(x)$ are polynomials and a is a real number.

- **Polynomials:** $\lim_{x \to a} p(x) = p(a)$
- **Rational functions:** $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$

  (provided $q(a) \neq 0$)

For polynomials and rational functions we can plug in a to compute the limit, as long as we don't get zero in the denominator. Linear functions count as polynomials. A rational function is a "fraction" made of polynomials.

Exercise

Evaluate the following limits analytically.

1. $\displaystyle \lim_{x \to 1} \frac{4f(x)g(x)}{h(x)}$, given that

$$\lim_{x \to 1} f(x) = 5, \ \lim_{x \to 1} g(x) = -2, \text{ and } \lim_{x \to 1} h(x) = -4.$$

2. $\displaystyle \lim_{x \to 3} \frac{4x^2 + 3x - 6}{2x - 3}$

3. $\displaystyle \lim_{x \to 1^-} g(x)$ and $\displaystyle \lim_{x \to 1^+} g(x)$, given that

$$g(x) = \begin{cases} x^2 & \text{if } x \le 1; \\ x + 2 & \text{if } x > 1. \end{cases}$$

Mon 25 Jan

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Mon 25 Jan (cont.)

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- GET YOUR CLICKER
- Note: There is no Blackboard for this course.
- Stay on top of the MLP! First deadline is coming up. Don't wait till the last minute.
- MLP issues...
- Quiz 1 is due in drill tomorrow. See MLP for a copy.

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Additional (Algebra) Techniques

When direct substitution (a.k.a. plugging in a) fails try using algebra:

- Factor and see if the denominator cancels out.

Example

$$\lim_{t\to 2} \frac{3t^2 - 7t + 2}{2 - t}$$

- Look for a common denominator.

Example

$$\lim_{h\to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h}$$

Exercise

Evaluate $\lim_{s\to 3} \dfrac{\sqrt{3s+16}-5}{s-3}$.

Another Technique: Squeeze Theorem

This method for evaluating limits uses the relationship of functions with each other.

Theorem (Squeeze Theorem)

Assume $f(x) \leq g(x) \leq h(x)$ *for all values of* x *near* a*, except possibly at* a*, and suppose*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.$$

Then since g *is always between* f *and* h *for* x*-values close enough to* a*, we must have*

$$\lim_{x \to a} g(x) = L.$$

Example

(a) Draw a graph of the inequality

$$-|x| \leq x^2 \ln\left(x^2\right) \leq |x|.$$

(b) Compute $\lim_{x\to 0} x^2 \ln\left(x^2\right)$.

2.3 Book Problems

12-30 (every 3rd problem), 33, 39-51 (odds), 55, 57, 61-67 (odds)

In general, review your algebra techniques, since they can save you some headache.

§2.4 Infinite Limits

We have examined a number of laws and methods to evaluate limits.

Question

Consider the following limit:

$$\lim_{x \to 0} \frac{1}{x}$$

How would you evaluate this limit?

In the next two sections, we examine limit scenarios involving infinity. The two situations are:

- **Infinite limits:** as x (i.e., the independent variable) approaches a finite number, y (i.e., the dependent variable) becomes arbitrarily large or small

$$\text{looks like: } \lim_{x \to \text{number}} f(x) = \pm\infty$$

- **Limits at infinity:** as x approaches an arbitrarily large or small number, y approaches a finite number

$$\text{looks like: } \lim_{x \to \pm\infty} f(x) = \text{number}$$

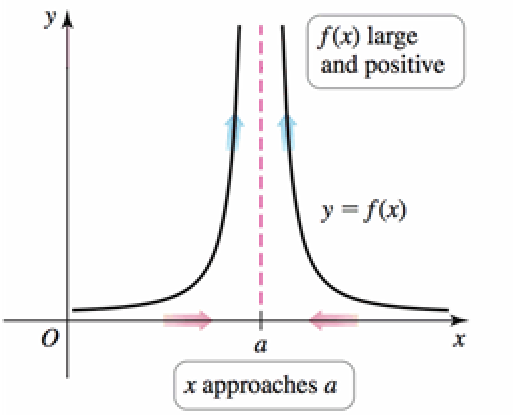
Definition of Infinite Limits

Definition (positively infinite limit)

Suppose f is defined for all x near a. If $f(x)$ grows arbitrarily large for all x sufficiently close (but not equal) to a, we write

$$\lim_{x\to a} f(x) = \infty$$

and say **the limit of $f(x)$ as x approaches a is infinity**.



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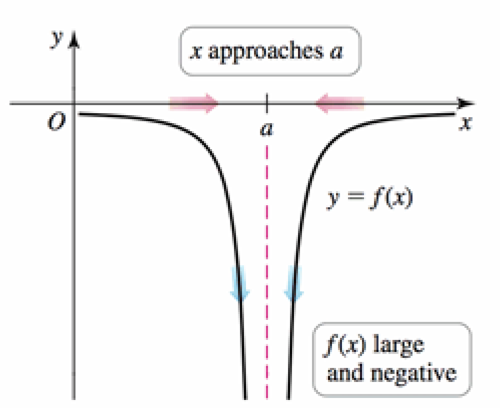
Definition (negatively infinite limit)

Suppose f is defined for all x near a. If $f(x)$ is negative and grows arbitrarily large in magnitude for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = -\infty$$

and say **the limit of $f(x)$ as x approaches a is negative infinity**.



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The definitions work for one-sided limits, too.

Exercise

Using a graph and a table of values, given $f(x) = \dfrac{1}{x^2 - x}$, determine:

(a) $\displaystyle\lim_{x\to 0^+} f(x)$

(b) $\displaystyle\lim_{x\to 0^-} f(x)$

(c) $\displaystyle\lim_{x\to 1^+} f(x)$

(d) $\displaystyle\lim_{x\to 1^-} f(x)$

Wed 27 Jan

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- Stay on top of the MLP! First deadline is coming up. Don't wait till the last minute.

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Definition of Vertical Asymptote

Definition

Suppose a function f satisfies at least one of the following:

- $\lim_{x \to a} f(x) = \pm\infty$,
- $\lim_{x \to a^+} f(x) = \pm\infty$
- $\lim_{x \to a^-} f(x) = \pm\infty$

Then the line $x = a$ is called a **vertical asymptote** of f.

Exercise

Given $f(x) = \dfrac{3x-4}{x+1}$, determine, analytically (meaning using "number sense" and without a table or a graph),

(a) $\lim\limits_{x \to -1^+} f(x)$

(b) $\lim\limits_{x \to -1^-} f(x)$

Summary Statements

Here is a common way you can summarize your solutions involving limits:

"Since the numerator approaches (#) and the denominator approaches 0, and is (positive/negative), and since (analyze signs here), (insert limit problem)=$(+\infty\ /-\infty)$."

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Remember to check for factoring –

Exercise

(a) What is/are the vertical asymptotes of

$$f(x) = \frac{3x^2 - 48}{x + 4}?$$

(b) What is $\lim_{x \to -4} f(x)$? Does that correspond to your earlier answer?

2.4 Book Problems

7-10, 15, 17-23, 31-34, 44-45

§2.5 Limits at Infinity

Limits at infinity determine what is called the **end behavior** of a function.

Exercise

(a) Evaluate the following functions at the points $x = \pm 100, \pm 1000, \pm 10000$;

$$f(x) = \frac{4x^2 + 3x - 2}{x^2 + 2} \qquad g(x) = -2 + \frac{\cos x}{\sqrt[3]{x}}$$

(b) What is your conjecture about $\lim_{x \to \infty} f(x)$? $\lim_{x \to -\infty} f(x)$? $\lim_{x \to -\infty} g(x)$? $\lim_{x \to \infty} g(x)$?

Horizontal Asymptotes

Definition

If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write

$$\lim_{x\to\infty} f(x) = L.$$

The line $y = L$ is a **horizontal asymptote** of f.

The limit at negative infinity, $\lim_{x\to-\infty} f(x) = M$, is defined analogously and in this case, the horizontal asymptote is $y = M$.

Infinite Limits at Infinity

Question

Is it possible for a limit to be both an infinite limit and a limit at infinity? (Yes.)

If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write

$$\lim_{x\to\infty} f(x) = \infty.$$

(The limits $\lim\limits_{x\to\infty} f(x) = -\infty$, $\lim\limits_{x\to-\infty} f(x) = \infty$, and $\lim\limits_{x\to-\infty} f(x) = -\infty$ are defined similarly.)

Powers and Polynomials: Let n be a positive integer and let $p(x)$ be a polynomial.

- $n =$ even number: $\lim\limits_{x \to \pm\infty} x^n = \infty$
- $n =$ odd number: $\lim\limits_{x \to \infty} x^n = \infty$ and $\lim\limits_{x \to -\infty} x^n = -\infty$

- (again, assuming n is positive)

$$\lim_{x \to \pm\infty} \frac{1}{x^n} = \lim_{x \to \pm\infty} x^{-n} = 0$$

- For a polynomial, only look at the term with the highest exponent:

$$\lim_{x \to \pm\infty} p(x) = \lim_{x \to \pm\infty} (\text{constant}) \cdot x^n$$

The constant is called the **leading coefficient**, $\mathrm{lc}(p)$. The highest exponent that appears in the polynomial is called the **degree**, $\deg(p)$.

Rational Functions: Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function.

- If $\deg(p) < \deg(q)$, i.e., the numerator has the smaller degree, then
$$\lim_{x \to \pm\infty} f(x) = 0$$
and $y = 0$ is a horizontal asymptote of f.

- If $\deg(p) = \deg(q)$, i.e., numerator and denominator have the same degree, then
$$\lim_{x \to \pm\infty} f(x) = \frac{\mathrm{lc}(p)}{\mathrm{lc}(q)}$$
and $y = \frac{\mathrm{lc}(p)}{\mathrm{lc}(q)}$ is a horizontal asymptote of f.

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- If $\deg(p) > \deg(q)$, (numerator has the bigger degree) then

$$\lim_{x\to\pm\infty} f(x) = \infty \quad \text{or} \quad -\infty$$

  and f has no horizontal asymptote.

- Assuming that $f(x)$ is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeroes of q.

  (This is why it is a good idea to check for factoring and cancelling first!)

When evaluating limits at infinity for rational functions, it is not enough to use the previous rule to show the limit analytically.

To evaluate these limits, we divide both numerator and denominator by x^n, where n is the degree of the polynomial in the denominator.

Fri 29 Jan

- Today: You MUST sign in if your name is highlighted. Everyone must click in, if possible.
- GET YOUR CLICKER NOW. Starting next week, no attendance sheet, clickers only.
- There is no Blackboard for this course.
- Stay on top of the MLP! First deadline is SUNDAY. Don't wait till the last minute.
- EXAM 1 is in two weeks, covers up to §3.1 (see the semester schedule of material on the course webpage). You must attend your own lecture on exam day.

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Exercise

Determine the end behavior of the following functions (in other words, compute both limits, as $x \to \pm\infty$, for each of the functions):

1. $f(x) = \dfrac{x+1}{2x^2-3}$
2. $g(x) = \dfrac{4x^3-3x}{2x^3+5x^2+x+2}$
3. $h(x) = \dfrac{6x^4-1}{4x^3+3x^2+2x+1}$

Algebraic and Transcendental Functions

Example

Determine the end behavior of the following functions.

1. $f(x) = \dfrac{4x^3}{2x^3 + \sqrt{9x^6 + 15x^4}}$ (radical signs appear)
2. $g(x) = \cos x$ (trig)
3. $h(x) = e^x$ (exponential)

Mon 1 Feb

- GET YOUR CLICKER NOW.
- EXAM 1 is one week from Friday. Covers up to §3.1 (see the semester schedule of material on the course webpage). You must attend your own lecture on exam day.

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Exercise

What are the vertical and horizontal asymptotes of $f(x) = \dfrac{x^2}{2x+1}$?

2.5 Book Problems

9-14, 15-33 (odds), 41-49 (odds), 53-59 (odds), 67

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§2.6 Continuity

Informally, a function f is "continuous at $x = a$" means for x-values anywhere close enough to a the graph can be drawn without lifting a pencil. In other words, no holes, breaks, asymptotes, etc.

Definition

A function f is **continuous** at a means

$$\lim_{x \to a} f(x) = f(a).$$

If f is not continuous at a, then a is a **point of discontinuity**.

Continuity Checklist

In order to claim something is continuous, you must verify all three:

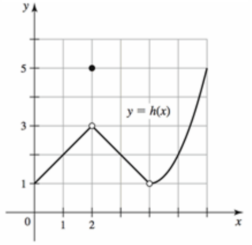
1. $f(a)$ is defined (i.e., a is in the domain of f – no holes, asymptotes).
2. $\lim_{x \to a} f(x)$ exists. You must check both sides and make sure they equal the same number.
3. $\lim_{x \to a} f(x) = f(a)$ (i.e., the value of f equals the limit of f at a).

Question

What is an example of a function that satisfies this condition?

Example

- Where are the points of discontinuity of the function below?
- Which aspects of the checklist fail?



recall (Continuity Checklist):

1. function is defined
2. the two-sided limit exists
3. 2. = 1.

Continuity Rules

If f and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$, provided $g(a) \neq 0$
6. $[f(x)]^n$

From the rules above, we can deduce:

1. Polynomials are continuous for all $x = a$.
2. Rational functions are continuous at all $x = a$ except for the points where the denominator is zero.
3. If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

Continuity on an Interval

Consider the cases where f is not defined past a certain point.

Definition

A function f is **continuous from the left** (or **left-continuous**) at a means

$$\lim_{x \to a^-} f(x) = f(a);$$

a function f is **continuous from the right** (or **right-continuous**) at a means

$$\lim_{x \to a^+} f(x) = f(a).$$

Definition

A function f is **continuous on an interval** I means it is continuous at all points of I.

Notation: Intervals are usually written

$$[a, b]\,,\ (a, b]\,,\ [a, b)\,,\ \text{or}\ (a, b)\,.$$

When I contains its endpoints, "continuity on I" means continuous from the right or left at the endpoints.

Wed 3 Feb

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- Look at old Wheeler exams to study.

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Example

Let $f(x) = \begin{cases} x^3 + 4x + 1 & \text{if } x \leq 0 \\ 2x^3 & \text{if } x > 0. \end{cases}$

1. Use the continuity checklist to show that f is not continuous at 0.
2. Is f continuous from the left or right at 0?
3. State the interval(s) of continuity.

Continuity of Functions with Roots

(assuming m and n are positive integers and $\frac{n}{m}$ is in lowest terms)

- If m is odd, then $[f(x)]^{\frac{n}{m}}$ is continuous at all points at which f is continuous.
- If m is even, then $[f(x)]^{\frac{n}{m}}$ is continuous at all points a at which f is continuous and $f(a) \geq 0$.

Question

Where is $f(x) = \sqrt[4]{4 - x^2}$ continuous?

Continuity of Transcendental Functions

Trig Functions: The basic trig functions are all continuous at all points IN THEIR DOMAIN. Note there are points of discontinuity where the functions are not defined – for example, $\tan x$ has asymptotes everywhere that $\cos x = 0$.

Exponential Functions: The exponential functions b^x and e^x are continuous on all points of their domains.

Inverse Functions: If a continuous function f has an inverse on an interval I (meaning if $x \in I$ then $f^{-1}(y)$ passes the vertical line test), then its inverse f^{-1} is continuous on the interval J, which is defined as all the numbers $f(x)$, given x is in I.

Intermediate Value Theorem (IVT)

Theorem (Intermediate Value Theorem)

Suppose f is continuous on the interval $[a, b]$ and L is a number satisfying

$$f(a) < L < f(b) \quad \text{or} \quad f(b) < L < f(a).$$

Then there is at least one number $c \in (a, b)$, i.e., $a < c < b$, satisfying

$$f(c) = L.$$

Example

Let $f(x) = -x^5 - 4x^2 + 2\sqrt{x} + 5$. Use IVT to show that $f(x) = 0$ has a solution in the interval $(0, 3)$.

Fri 5 Feb

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- Look at old Wheeler exams to study. `comp.uark.edu/~ashleykw`

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

Exercise

Which of the following functions is continuous for all real values of x?

(A) $f(x) = \dfrac{x^2}{2x+1}$

(B) $g(x) = \sqrt{3x^2 - 2}$

(C) $h(x) = \dfrac{5x}{|x^8 - 1|}$

(D) $j(x) = \dfrac{5x}{x^8 + 1}$

2.6 Book Problems

9-25 (odds), 35-45 (odds), 59, 61, 63, 83, 85

The base for these slides was done by Dr. Shannon Dingman, later encoded in LATEX by Dr. Brad Lutes.

§2.7 Precise Definitions of Limits

So far in our dealings with limits, we have used informal terms such as "sufficiently close" and "arbitrarily large". Now we will formalize what these terms mean mathematically.

Recall: $|f(x) - L|$ and $|x - a|$ refer to the distances between $f(x)$ and L and between x and a.

Also, recall that when we worked informally with limits, we wanted x to approach a, but not necessarily equal a. Likewise, we wanted f to get arbitrarily close to L, but not necessarily equal L.

Definition

Assume that $f(x)$ exists for all x in some open interval (open means: neither of the endpoints not included) containing a, except possibly at a. **"The limit of $f(x)$ as x approaches a is L"**, i.e.,

$$\lim_{x \to a} f(x) = L,$$

means **for any $\epsilon > 0$ there exists $\delta > 0$ such that**

$$|f(x) - L| < \epsilon \quad \textbf{whenever} \quad 0 < |x - a| < \delta.$$

Question

Why $0 < |x - a|$ but not for $|f(x) - L|$?

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ϵ and δ

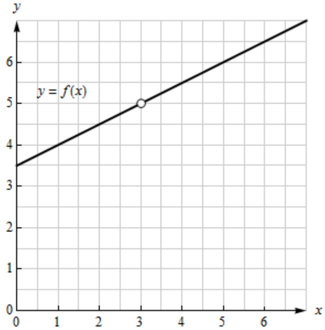
When we worked informally with limits, we saw $f(x)$ get closer and closer to L as x got closer and closer to a.

Question

If we want the distance between $f(x)$ and L to be less than 1, how close does x have to be to a? What if we want $|f(x) - L| < 0.5$? 0.5? 0.1? 0.01?

Seeing ϵs and δs on a Graph

Example



Using the graph, for each $\epsilon > 0$, determine a value of $\delta > 0$ to satisfy the statement

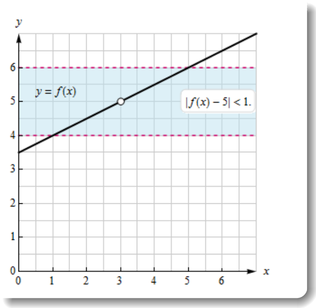
$$|f(x) - 5| < \epsilon \quad \text{whenever} \quad 0 < |x - 3| < \delta.$$

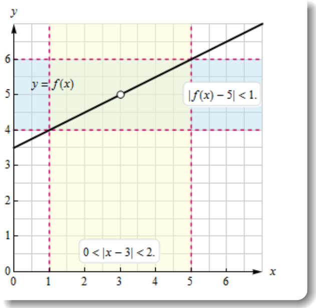
(a) $\epsilon = 1$

(b) $\epsilon = 0.5$.

Seeing ϵs and δs on a Graph, cont.

When $\epsilon = 1$:

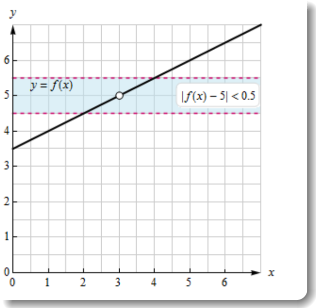


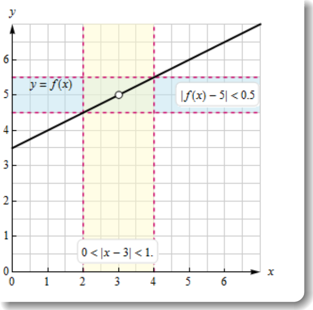


$\ldots\ \delta = 2$

Seeing ϵs and δs on a Graph, cont.

When $\epsilon = 0.5$:



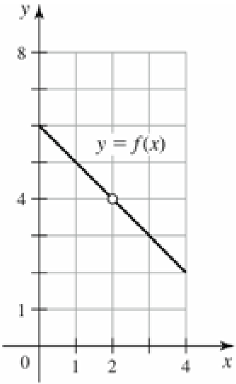


$\ldots\ \delta = 1$

The ϵs and δs give a way to visualize computing the limit, and prove it exists. As the ϵs get smaller and smaller, we want there to always be a δ. In this example,

$$\lim_{x \to 3} f(x) = 5.$$

Exercise



Using the graph, for each $\epsilon > 0$, determine a value of $\delta > 0$ to satisfy the statement

$$|f(x) - 4| < \epsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

(a) $\epsilon = 1$

(b) $\epsilon = 0.5$.

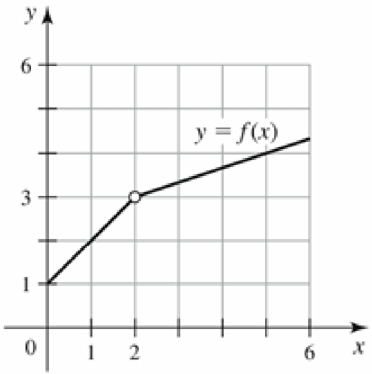
Finding a Symmetric Interval

Question

When finding an interval $(a-\delta, a+\delta)$ around the point a, what happens if you compute two different δs?

Answer: To obtain a symmetric interval around a, use the smaller of the two δs as your distance around a.

Exercise



The graph of $f(x)$ shows

$$\lim_{x \to 2} f(x) = 3.$$

For $\epsilon = 1$, find the corresponding value of $\delta > 0$ so that

$$|f(x) - 3| < \epsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

Mon 8 Feb

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Mon 8 Feb (cont.)

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Mon 8 Feb (cont.)

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Exercise

Let $f(x) = x^2 - 4$. For $\epsilon = 1$, find a value for $\delta > 0$ so that

$$|f(x) - 12| < \epsilon \quad \text{whenever} \quad 0 < |x - 4| < \delta.$$

In this example, $\lim\limits_{x \to 4} f(x) = 12$.

2.7 Book Problems

1-7, 9-18

§3.1 Introducing the Derivative

Recall from Ch 2: We said that the slope of the tangent line at a point is the limit of the slopes of the secant lines as the points get closer and closer.

- slope of secant line: $\dfrac{f(x) - f(a)}{x - a}$ (average rate of change)
- slope of tangent line: $\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ (instantaneous rate of change)

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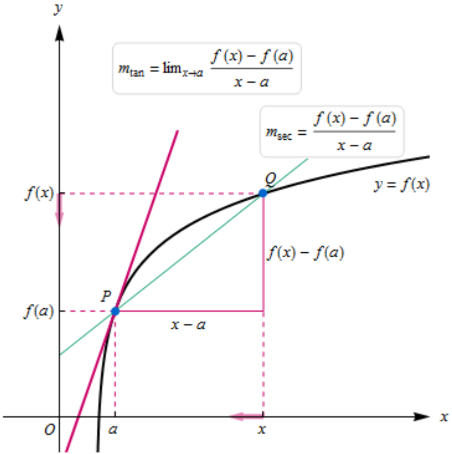

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Exercise

Use the relationship between secant lines and tangent lines, specifically the slope of the tangent line, to find the equation of a line tangent to the curve $f(x) = x^2 + 2x + 2$ at the point $P = (1, 5)$.

In the preceding exercise, we considered two points

$$P = (a, f(a)) \quad \text{and} \quad Q = (x, f(x))$$

that were getting closer and closer together.

Instead of looking at the points approaching one another, we can also view this as the distance h between the points approaching 0. For

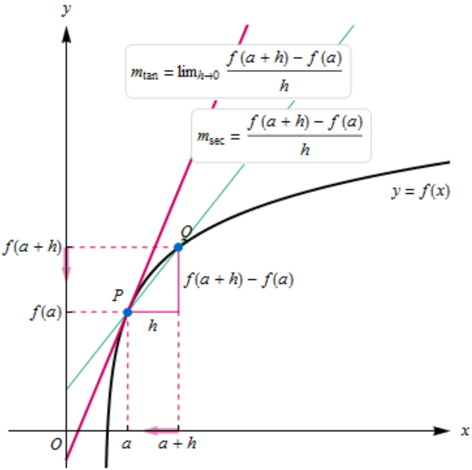
$$P = (a, f(a)) \quad \text{and} \quad Q = (a+h, f(a+h)),$$

- slope of secant line:

$$\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

- slope of tangent line:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$


y
$m_{\tan} = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$
$m_{\sec} = \dfrac{f(a+h) - f(a)}{h}$
$y = f(x)$
Q
$f(a+h)$
$f(a+h) - f(a)$
P
$f(a)$
h
x
O
a
$a+h$

Exercise

Find the equation of a line tangent to the curve $f(x) = x^2 + 2x + 2$ at the point $P = (2, 10)$.

Derivative Defined as a Function

The slope of the tangent line for the function f is itself a function of x (in other words, there is an expression where we can plug in any value $x = a$ and get the derivative at that point), called the derivative of f.

Definition

The **derivative** of f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. If $f'(x)$ exists, we say f is **differentiable** at x. If f is differentiable at every point of an open interval I, we say that f is differentiable on I.

Exercise

Use the definition of the derivative to find the derivative of the function $f(x) = x^2 + 2x + 2$.

Leibniz Notation

A standard notation for change involves the Greek letter Δ.

$$\frac{f(x+h)-f(x)}{h} = \frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}.$$

Apply the limit:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

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Other Notation

The following are alternative ways of writing $f'(x)$ (i.e., the derivative as a function of x):

$$\frac{dy}{dx} \qquad \frac{df}{dx} \qquad \frac{d}{dx}(f(x)) \qquad D_x(f(x)) \qquad y'(x)$$

The following are ways to notate the derivative of f evaluated at $x = a$:

$$f'(a) \qquad y'(a) \qquad \left.\frac{df}{dx}\right|_{x=a} \qquad \left.\frac{dy}{dx}\right|_{x=a}$$

Wed 10 Feb

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Wed 10 Feb (cont.)

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Question

Do the words "derive" and "differentiate" mean the same thing?

Graphing the Derivative

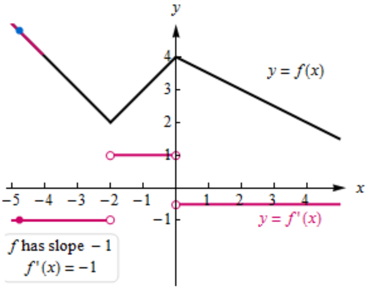
The graph of the derivative is the graph of the collection of slopes of tangent lines of a graph. If you just have a graph (without an equation for the graph), the best you can do is approximate the graph of the derivative.

The base for these slides was done by Dr. Shannon Dingman, later encoded in LATEX by Dr. Brad Lutes.

Example

Simple checklist:

1. Note where $f'(x) = 0$.
2. Note where $f'(x) > 0$.
   (What does this look like?)
3. Note where $f'(x) < 0$.
   (What does this look like?)



Differentiability vs. Continuity

Key points about the relationship between differentiability and continuity:

- If f is differentiable at a, then f is continuous at a.
- If f is not continuous at a, then f is not differentiable at a.
- f can be continuous at a, but not differentiable at a.

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A function f is not differentiable at a if at least one of the following conditions holds:

1. f is not continuous at a.
2. f has a corner at a.

Question

Why does this make f not differentiable?

3. f has a vertical tangent at a.

Question

Why does this make f not differentiable?

3.1 Book Problems

9-45 (odds), 49-53 (odds)

- **NOTE:** You do not know any rules for differentiation yet (e.g., Power Rule, Chain Rule, etc.) In this section, you are strictly using the definition of the derivative and the definition of slope of tangent lines we have derived.

Exam #1 Review

- §2.1 The Idea of Limits
  - Understand the relationship between average velocity & instantaneous velocity, and secant and tangent lines
  - Be able to compute average velocities and use the idea of a limit to approximate instantaneous velocities
  - Be able to compute slopes of secant lines and use the idea of a limit to approximate the slope of the tangent line
- §2.2 Definitions of Limits
  - Know the definition of a limit
  - Be able to use a graph of a table to determine a limit
  - Know the relationship between one- and two-sided limits

Exam #1 Review (cont.)

- §2.3 Techniques for Computing Limits
  - Know and be able to compute limits using analytical methods (e.g., limit laws, additional techniques)
  - Know the Squeeze Theorem and be able to use it to determine limits

Example

Evaluate $\lim_{x \to 0} x \sin \frac{1}{x}$.

Exam #1 Review (cont.)

- §2.4 Infinite Limits
  - Be able to use a graph, a table, or analytical methods to determine infinite limits
  - Know the definition of a vertical asymptote and be able to determine whether a function has vertical asymptotes
- §2.5 Limits at Infinity
  - Be able to find limits at infinity and horizontal asymptotes
  - Know how to compute the limits at infinity of rational functions

Exam #1 Review (cont.)

Example

Determine the end behavior of $f(x)$. If there is a horizontal asymptote, then say so. Next, identify any vertical asymptotes. If $x = a$ is a vertical asymptote, then evaluate $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$.

$$f(x) = \frac{2x^3 + 10x^2 + 12x}{x^3 + 2x^2}$$

Exam #1 Review (cont.)

- §2.6 Continuity
    - Know the definition of continuity and be able to apply the continuity checklist
    - Be able to determine the continuity of a function (including those with roots) on an interval
    - Be able to apply the Intermediate Value Theorem to a function

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Exam #1 Review (cont.)

Example

Determine the value for a that will make $f(x)$ continuous.

$$f(x) = \begin{cases} \dfrac{x^2 + 3x + 2}{x + 1} & x \neq -1 \\ a & x = -1 \end{cases}$$

Example

Show that $f(x) = 2$ has a solution on the interval $(-1, 1)$, with

$$f(x) = 2x^3 + x.$$

Exam #1 Review (cont.)

Exercise

What value of k makes

$$f(x) = \begin{cases} \dfrac{\sqrt{2x-5}-\sqrt{x+7}}{x-2} & x \neq 2 \\ k & x = 2 \end{cases}$$

continuous everywhere?

- $\oint$2.7 Precise Definition of Limits
  - Understand the δ, ϵ relationship for limits

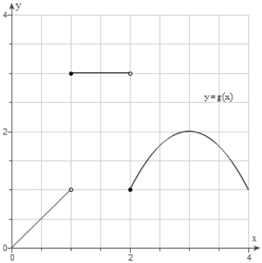
The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

Exam #1 Review (cont.)

- Be able to use a graph or analytical methods to find a value for $\delta > 0$ given an $\epsilon > 0$ (including finding symmetric intervals)

Exam #1 Review (cont.)

Example



Use the graph to find the appropriate δ.

(a) $|g(x) - 2| < \frac{1}{2}$ whenever
$$0 < |x - 3| < \delta$$

(b) $|g(x) - 1| < \frac{3}{2}$ whenever
$$0 < |x - 2| < \delta$$

In this example, the two-sided limits at $x = 1$ and $x = 2$ do not exist.

Exam #1 Review (cont.)

- §3.1 Introducing the Derivative
  - Know the definition of a derivative and be able to use this definition to calculate the derivative of a given function
  - Be able to determine the equation of a line tangent to the graph of a function at a given point
  - Know the 3 conditions for when a function is not differentiable at a point, and why these three conditions make a function not differentiable at the given point

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Exam #1 Review (cont.)

Example

(a) Use the limit definition of the derivative to find an equation for the line tangent to $f(x)$ at a, where

$$f(x) = \frac{1}{x}; \quad a = -5.$$

(b) Using the same $f(x)$ from part (a), find a formula for $f'(x)$ (using the limit definition).

(c) Plug -5 into your answer for (b) and make sure it matches your answer for (a).

Other Study Tips

- Brush up on algebra, especially radicals.
- When in doubt, show steps. Defer to class notes and old exams to get an idea of what's expected.
- You will be punished for wrong notation; e.g., the limit symbol.
- Read the question! Several students always lose points because they didn't answer the question or they didn't follow directions.
- Do the book problems.
- Budget your time. You don't have to do the problems in order. Do the easier ones first.

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Mon 15 Feb

- Expect Exam back on Thursday.
- Quizzes:
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Mon 15 Feb (cont.)

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§3.2 Graphing the Derivative

Recall: The graph of the derivative is essentially the graph of the collection of slopes of the tangent lines of a graph. If you just have a graph (without an equation for the graph), the best you can do is approximate the graph of the derivative.

Simple Checklist:

1. Note where $f'(x) = 0$.
2. Note where $f'(x) > 0$.

Question

What does this look like?

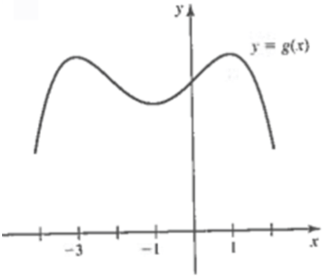
3. Note where $f'(x) < 0$.

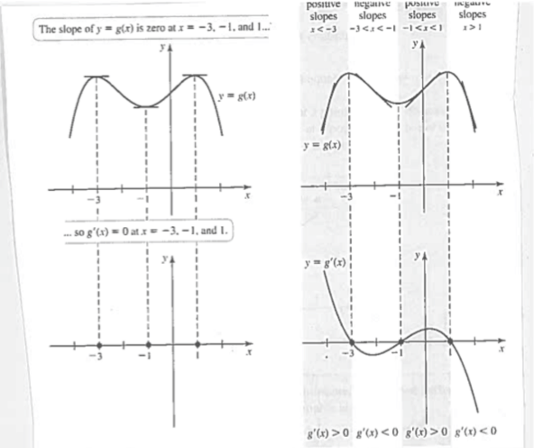
Question

What does this look like?

Example

Given the graph of $g(x)$, sketch the graph of $g'(x)$.

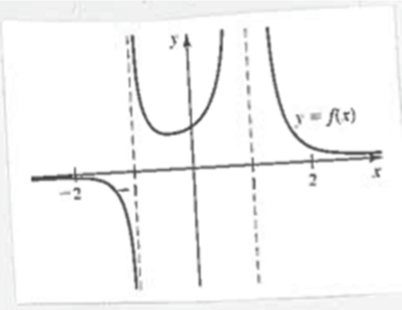


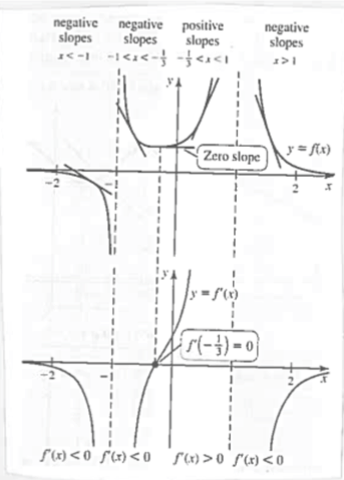


The base for these slides was done by Dr. Shannon Dingman, later encoded in LATEX by Dr. Brad Lutes.

Example (With Asymptotes)

Given the graph of $f(x)$, sketch the graph of $f'(x)$.




negative slopes
$x < -1$
negative slopes
$-1 < x < -\frac{1}{3}$
positive slopes
$-\frac{1}{3} < x < 1$
negative slopes
$x > 1$
Zero slope
$y = f(x)$
-2
-1
2
x
y
$y = f'(x)$
$f'\left(-\frac{1}{3}\right) = 0$
$f'(x) < 0$
$f'(x) < 0$
$f'(x) > 0$
$f'(x) < 0$

Recall the relationship between differentiability and continuity.

Exercise

If a function g is not continuous at $x = a$, then g

A. must be undefined at $x = a$.

B. is not differentiable at $x = a$.

C. has an asymptote at $x = a$.

D. all of the above.

E. A. and B. only.

3.2 Book Problems

5-14

§3.3 Rules of Differentiation

Recall the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(as a function of x, i.e., a formula).
And, for any particular point a, we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Constant Functions

The constant function $f(x) = c$ is a horizontal line with a slope of 0 at every point. This is consistent with the definition of the derivative:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{c - c}{h} \\ &= \lim_{h \to 0} 0 = 0. \end{aligned}$$

Therefore, for constant functions, $\frac{d}{dx} c = 0$.

The base for these slides was done by Dr. Shannon Dingman, later encoded in LATEX by Dr. Brad Lutes.

Power Rule

Fact: For any positive integer n, we can factor

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

For example, when $n = 2$, we get

$$x^2 - a^2 = (x-a)(x+a),$$

which is the difference of squares formula.

Power Rule, cont.

Suppose $f(x) = x^n$ where n is a positive integer. Then at a point a,

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a} \\ &= \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{x - a} \\ &= (a^{n-1} + a^{n-2} \cdot a + \cdots + a \cdot a^{n-2} + a^{n-1}) = na^{n-1}. \end{aligned}$$

Using the formula for the derivative as a function of x, one can show $\frac{d}{dx}(x^n) = nx^{n-1}$.

Constant Multiple Rule

Consider a function of the form $cf(x)$, where c is a constant. Just like with limits, we can factor out the constant:

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} \\ &= \lim_{h\to 0}\frac{c[f(x+h)-f(x)]}{h} = c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= cf'(x)\end{aligned}$$

Therefore, $\frac{d}{dx}[cf(x)] = cf'(x)$.

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

Sum Rule

Sums of functions also behave under the same limit laws when we differentiate:

$$\begin{aligned}
\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0}\left[\frac{[f(x+h)-f(x)]}{h}+\frac{[g(x+h)-g(x)]}{h}\right] \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= f'(x)+g'(x)
\end{aligned}$$

The base for these slides was done by Dr. Shannon Dingman, later encoded in LATEX by Dr. Brad Lutes.

So if f and g are differentiable at x,

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x).$$

The Sum Rule can be generalized for more than two functions to include n functions.

Note: Using the Sum Rule and the Constant Multiple Rule produces the Difference Rule:

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x).$$

Exercise

Using the differentiation rules we have discussed, calculate the derivatives of the following functions. Note which rule(s) you are using.

1. $y = x^5$
2. $y = 4x^3 - 2x^2$
3. $y = -1500$
4. $y = 3x^3 - 2x + 4$

Exponential Functions

Let $f(x) = b^x$, where $b > 0$, $b \neq 1$. To differentiate at 0, we write

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{b^x - b^0}{x} = \lim_{x \to 0} \frac{b^x - 1}{x}.$$

It is not obvious what this limit should be. However, consider the cases $b = 2$ and $b = 3$. By constructing a table of values, we can see that

$$\lim_{x \to 0} \frac{2^x - 1}{x} \approx 0.693 \quad \text{and} \quad \lim_{x \to 0} \frac{3^x - 1}{x} \approx 1.099.$$

So, $f'(0) < 1$ when $b = 2$ and $f'(0) > 1$ when $b = 3$. As it turns out, there is a particular number b, with $2 < b < 3$, whose graph has a tangent line with slope 1 at $x = 0$. In other words, such a number b has the property that

$$\lim_{x \to 0} \frac{b^x - 1}{x} = 1.$$

Question

What number is it?

Answer: This number is $e = 2.718281828459\ldots$ (known as the Euler number). The function $f(x) = e^x$ is called the **natural exponential function**.

Now, using $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$, we can find the formula for $\frac{d}{dx}(e^x)$:

$$\begin{aligned}
\frac{d}{dx}(e^x) &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} \\
&= e^x \left(\lim_{h\to 0} \frac{e^h - 1}{h} \right) \\
&= e^x \cdot 1 = e^x
\end{aligned}$$

Wed 17 Feb

- Expect Exam back on Thursday. Feedback on Friday. Scores $\rightarrow$ MLP?
- Instructions for when you get your exam back:
  - Look over your test, but don't write on it.
  - If you find discrepancies on points or grading, write your grievances on a separate sheet of paper.
  - Return that paper with your exam to your drill instructor by the end of drill.
  - Once you leave the room with your exam you lose this opportunity.
  - This is the only way you can get points back on the exam.

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

Wed 17 Feb (cont.)

- MIDTERM in less than three weeks.
  - Tuesday 8 March 6-7:30p
  - If you have legitimate conflict, i.e., anything that is also scheduled in ISIS, I need to know now. If you are not sure if it conflicts with a course, please have that instructor contact me ASAP.
  - Morning Section: Walker rm 124
    Afternoon Section: Walker rm 218
- Later this month: Sub on Friday 26 Feb and Monday 29 Feb.

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

Exercise

(a) Find the slope of the line tangent to the curve $f(x) = x^3 - 4x - 4$ at the point $(2, -4)$.

(b) Where does this curve have a horizontal tangent?

Higher-Order Derivatives

If we can write the derivative of f as a function of x, then we can take *its* derivative, too. The derivative of the derivative is called the **second derivative** of f, and is denoted f''.

In general, we can differentiate f as often as needed. If we do it n times, the nth derivative of f is

$$f^{(n)}(x) = \frac{d^n f}{dx^n} = \frac{d}{dx}[f^{(n-1)}(x)].$$

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

3.3 Book Problems

9-48 (every 3rd problem), 51-53, 58-60

- For these problems, use only the rules we have derived so far.

§3.4 The Product and Quotient Rules

Issue: Derivatives of products and quotients do NOT behave like they do for limits.

As an example, consider $f(x) = x^2$ and $g(x) = x^3$. We can try to differentiate their product in two ways:

- $$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}\left(x^5\right) = 5x^4$$
- $$f'(x)g'(x) = (2x)(3x^2) = 6x^3$$

Question

Which answer is the correct one?

Product Rule

If f and g are any two functions that are differentiable at x, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x).$$

In the example from the previous slide, we have

$$\begin{aligned}\frac{d}{dx}[x^2 \cdot x^3] &= \frac{d}{dx}(x^2) \cdot (x^3) + x^2 \cdot \frac{d}{dx}(x^3) \\ &= (2x) \cdot (x^3) + x^2 \cdot (3x^2) \\ &= 2x^4 + 3x^4 \\ &= 5x^4\end{aligned}$$

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

Derivation of the Product Rule

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)g(x+h) + [-f(x)g(x+h) + f(x)g(x+h)] - f(x)g(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h}\right) + \left(\lim_{h\to 0}\frac{f(x)g(x+h) - f(x)g(x)}{h}\right)$$

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

Derivation of the Product Rule (cont.)

$$= \lim_{h\to 0}\left(g(x+h)\frac{f(x+h)-f(x)}{h}\right) + \left(\lim_{h\to 0} f(x)\frac{g(x+h)-g(x)}{h}\right)$$

$$= g(x)f'(x) + f(x)g'(x)$$

Exercise

Use the product rule to find the derivative of the function $(x^2 + 3x)(2x - 1)$.

A. $2(2x + 3)$

B. $6x^2 + 10x - 3$

C. $2x^3 + 5x^2 - 3x$

D. $2x(x + 3) + x(2x - 1)$

Derivation of Quotient Rule

Question

Let $q(x) = \dfrac{f(x)}{g(x)}$. What is $\dfrac{d}{dx}q(x)$?

Answer: We can write $f(x) = q(x)g(x)$ and then use the Product Rule:

$$f'(x) = q'(x)g(x) + g'(x)q(x)$$

and now solve for $q'(x)$:

$$q'(x) = \frac{f'(x) - q(x)g'(x)}{g(x)}.$$

Then, to get rid of $q(x)$, plug in $\dfrac{f(x)}{g(x)}$:

$$q'(x) = \frac{f'(x) - g'(x)\frac{f(x)}{g(x)}}{g(x)}$$

$$= \frac{g(x)\left(f'(x) - g'(x)\frac{f(x)}{g(x)}\right)}{g(x) \cdot g(x)}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

"LO-D-HI minus HI-D-LO over LO squared"

Quotient Rule

Just as with the product rule, the derivative of a quotient is not a quotient of derivatives, i.e.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \neq \frac{f'(x)}{g'(x)}.$$

Here is the correct rule, the Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}.$$

Exercise

Use the Quotient Rule to find the derivative of

$$\frac{4x^3 + 2x - 3}{x + 1}.$$

Exercise

Find the slope of the tangent line to the curve

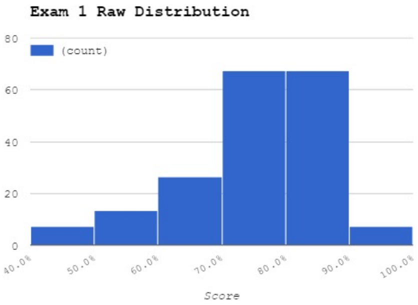
$$f(x) = \frac{2x - 3}{x + 1} \text{ at the point } (4, 1).$$

Mon 22 Feb

- Exam 1 Feedback

		Problem														
	Total	1	2	3(a)	(b)	(c)	(d)	4	5(a)	(b)	(c)	6(a)	(b)	(c)	7	8
out of	75	10	10	3	3	3	3	10	3	3	3	5	5	3	5	5
Median ->	48.0	8	7	2	0	1	2	8	2	3	1	5	1	1	4	3

Mon 22 Feb (cont.)



The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

Mon 22 Feb (cont.)

- MIDTERM
  - Tuesday 8 March 6-7:30p
  - If you have legitimate conflict, i.e., anything that is also scheduled in ISIS, I need to know now. If you are not sure if it conflicts with a course, please have that instructor contact me ASAP.
  - Cumulative. Covers up to §3.9
  - Morning Section: Walker rm 124
    Afternoon Section: Walker rm 218
- Sub on Friday 26 Feb and Monday 29 Feb.
- Exam 2: Friday 4 March. Covers up to §3.8.

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

The Quotient Rule also allows us to extend the Power Rule to negative numbers – if n is any integer, then

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}.$$

Question

How?

Exercise

If $f(x) = \dfrac{x(3-x)}{2x^2}$, find $f'(x)$.

Derivative of e^{kx}

For any real number k,

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}.$$

Exercise

What is the derivative of $x^2 e^{3x}$?

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

Rates of Change

The derivative provides information about the instantaneous rate of change of the function being differentiated (compare to the limit of the slopes of the secant lines from §2.1).

For example, suppose that the population of a culture can be modeled by the function $p(t)$. We can find the instantaneous growth rate of the population at any time $t \geq 0$ by computing $p'(t)$ as well as the **steady-state population** (also called the **carrying capacity** of the population). The steady-state population equals

$$\lim_{t \to \infty} p(t).$$

3.4 Book Problems

9-49 (every 3rd problem), 57, 59, 63, 75-79 (odds)

Wed 24 Feb

- Exam 1: see the course webpage for the curve
- MIDTERM
    - Tuesday 8 March 6-7:30p
    - If you have legitimate conflict, i.e., anything that is also scheduled in ISIS, I need to know now. If you are not sure if it conflicts with a course, please have that instructor contact me ASAP.
    - Cumulative. Covers up to §3.9
    - Morning Section: Walker rm 124
      Afternoon Section: Walker rm 218
- Sub on Friday 26 Feb and Monday 29 Feb.

Wed 24 Feb (cont.)

- Possible sub on Wednesday 2 Mar.
- Exam 2: Friday 4 March. Covers up to §3.8.
- Quizzes: Only some of the quiz problems are graded now.

§3.5 Derivatives of Trigonometric Functions

Trig functions are commonly used to model cyclic or periodic behavior in everyday settings. Therefore it is important to know how these functions change across time.

Fact: Derivative formulas for sine and cosine can be derived using the following limits:

- $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}=1$
- $\displaystyle\lim_{x\to 0}\frac{\cos x-1}{x}=0$

(We will prove these limits in Chapter 4.)

Exercise

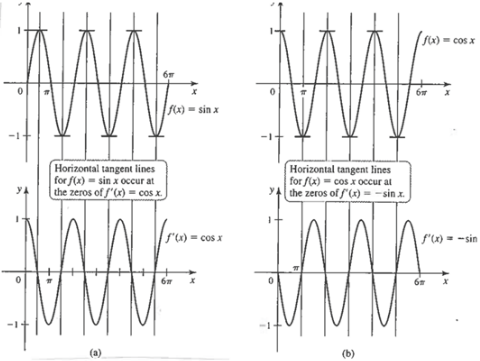
Evaluate $\lim_{x \to 0} \dfrac{\sin 9x}{x}$ and $\lim_{x \to 0} \dfrac{\sin 9x}{\sin 5x}$.

Derivatives of Sine and Cosine Functions

Using the previous limits and the definition of the derivative, we obtain

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$

Examining the graphs of sine and cosine illustrate the relationship between the functions and their derivatives.



The base for these slides was done by Dr. Shannon Dingman, later encoded in LATEX by Dr. Brad Lutes.

Trig Identities You Should Know

- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = 1 - 2\sin^2 x$
- $\cos^2 x = \dfrac{1 + \cos 2x}{2}$
- $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

- $\tan x = \dfrac{\sin x}{\cos x}$
- $\cot x = \dfrac{\cos x}{\sin x}$
- $\cot x = \dfrac{1}{\tan x}$
- $\sec x = \dfrac{1}{\cos x}$
- $\csc x = \dfrac{1}{\sin x}$

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

Derivatives of Other Trig functions

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{\cos x \cos x - (-\sin x)\sin x}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

So $\frac{d}{dx}(\tan x) = \sec^2 x$.

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

By using trig identities and the Quotient Rule, we obtain

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\csc x \cot x$$

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\tan x}\right) = -\csc^2 x$$

Exercise

Compute the derivative of the following functions:

$$f(x) = \frac{\tan x}{1 + \tan x} \qquad g(x) = \sin x \cos x$$

Exercise

Use the difference and product rules to find the derivative of the function $y = \cos x - x \sin x$.

A. $-\sin x + x \cos x$

B. $x \cos x$

C. $-2 \sin x - x \cos x$

D. $x \cos x - 2 \sin x$

Higher-Order Trig Derivatives

There is a cyclic relationship between the higher order derivatives of $\sin x$ and $\cos x$:

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f^{(3)}(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \end{aligned} \qquad \begin{aligned} g(x) &= \cos x \\ g'(x) &= -\sin x \\ g''(x) &= -\cos x \\ g^{(3)}(x) &= \sin x \\ g^{(4)}(x) &= \cos x \end{aligned}$$

3.5 Book Problems

7-47 (odds), 57, 59, 61

§3.6 Derivatives as Rates of Change

Question

Why do we need derivatives in real life?

We look at four areas where the derivative assists us with determining the rate of change in various contexts.

Position and Velocity

Suppose an object moves along a straight line and its location at time t is given by the position function $s = f(t)$. The **displacement** of the object between $t = a$ and $t = a + \Delta t$ is

$$\Delta s = f(a + \Delta t) - f(a).$$

Here Δt represents how much time has elapsed.

We now define average velocity as

$$\frac{\Delta s}{\Delta t} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

Recall that the limit of the average velocities as the time interval approaches 0 was the instantaneous velocity (which we denote here by v). Therefore, the instantaneous velocity at a is

$$v(a) = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = f'(a).$$

Speed and Acceleration

In mathematics, speed and velocity are related but not the same – if the velocity of an object at any time t is given by $v(t)$, then the speed of the object at any time t is given by

$$|v(t)| = |f'(t)|.$$

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

By definition, acceleration (denoted by a) is the instantaneous rate of change of the velocity of an object at time t. Therefore,

$$a(t) = v'(t)$$

and since velocity was the derivative of the position function $s = f(t)$, then

$$a(t) = v'(t) = f''(t).$$

Summary: Given the position function $s = f(t)$, the velocity at time t is the first derivative, the speed at time t is the absolute value of the first derivative, and the acceleration at time t is the second derivative.

Question

Given the position function $s = f(t)$ of an object launched into the air, how would you know:

- The highest point the object reaches?
- How long it takes to hit the ground?
- The speed at which the object hits the ground?

Exercise

A rock is dropped off a bridge and its distance s (in feet) from the bridge after t seconds is $s(t) = 16t^2 + 4t$. At $t = 2$ what are, respectively, the velocity of the rock and the acceleration of the rock?

A. 64 ft/s; 16 ft/s^2

B. 68 ft/s; 32 ft/s^2

C. 64 ft/s; 32 ft/s^2

D. 68 ft/s; 16 ft/s^2

Growth Models

Suppose $p = f(t)$ is a function of the growth of some quantity of interest. The average growth rate of p between times $t = a$ and a later time $t = a + \Delta t$ is the change in p divided by the elapsed time Δt:

$$\frac{\Delta p}{\Delta t} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

As Δt approaches 0, the average growth rate approaches the derivative $\frac{dp}{dt}$, which is the instantaneous growth rate (or just simply the growth rate). Therefore,

$$\frac{dp}{dt} = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t}.$$

Exercise

The population of the state of Georgia (in thousands) from 1995 ($t = 0$) to 2005 ($t = 10$) is modeled by the polynomial

$$p(t) = -0.27t^2 + 101t + 7055.$$

(a) What was the average growth rate from 1995 to 2005?

(b) What was the growth rate for Georgia in 1997?

(c) What can you say about the population growth rate in Georgia between 1995 and 2005?

Average and Marginal Cost

Suppose a company produces a large amount of a particular quantity. Associated with manufacturing the quantity is a **cost function** $C(x)$ that gives the cost of manufacturing x items. This cost may include a **fixed cost** to get started as well as a **unit cost** (or **variable cost**) in producing one item.

If a company produces x items at a cost of $C(x)$, then the average cost is $\frac{C(x)}{x}$. This average cost indicates the cost of items already produced. Having produced x items, the cost of producing another Δx items is $C(x + \Delta x) - C(x)$. So the average cost of producing these extra Δx items is

$$\frac{\Delta C}{\Delta x} = \frac{C(x + \Delta x) - C(x)}{\Delta x}.$$

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

If we let Δx approach 0, we have

$$\lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = C'(x)$$

which is called the **marginal cost**. The marginal cost is the approximate cost to produce one additional item after producing x items.

Note: In reality, we can't let Δx approach 0 because Δx represents whole numbers of items.

Exercise

If the cost of producing x items is given by

$$C(x) = -0.04x^2 + 100x + 800$$

for $0 \leq x \leq 1000$, find the average cost and marginal cost functions. Also, determine the average and marginal cost when $x = 500$.

3.6 Book Problems

9-19, 21-24, 30-33 (odds)

Wed 2 Mar

- Exam 2:
  - Friday 4 Mar. Covers up to §3.8.
  - Spring 2015 Practice Exam. Also look for quizzes on the old webpages for more problems.
  - For more problems study the evens in each of the sections covered.
  - Basic scientific calculator is allowed.

Wed 2 Mar (cont.)

- Midterm:
  - Tuesday 8 March. Covers everything up to §3.9.
  - Morning Section: Walker room 124
    Afternoon Section: Walker room 218
    You must take the test with your officially scheduled section.
  - Stay tuned for conflict resolutions. If you haven't emailed me already regarding a conflict, do it NOW.
  - Stay tuned for a study guide.
  - Basic scientific calculator is allowed....? Stay tuned.

The base for these slides was done by Dr. Shannon Dingman, later encoded in LATEX by Dr. Brad Lutes.

Wed 2 Mar (cont.)

- Quiz 6 next Thurs. Only some of the quiz problems are graded now. You are always welcome to my office for feeback on your work.

§3.7 The Chain Rule

The rules up to now have not allowed us to differentiate composite functions

$$f \circ g(x) = f(g(x)).$$

Example

If $f(x) = x^7$ and $g(x) = 2x - 3$, then $f(g(x)) = (2x - 3)^7$. To differentiate we could mulitply the polynomial out... but in general we should use a much more efficient strategy to emply to composition functions.

Example

Suppose that Yvonne (y) can run twice as fast as Uma (u). Then write $\frac{dy}{du} = 2$.

Suppose that Uma can run four times as fast as Xavier (x). So $\frac{du}{dx} = 4$.

How much faster can Yvonne run than Xavier? In this case, we would take both our rates and multiply them together:

$$\frac{dy}{du} \cdot \frac{du}{dx} = 2 \cdot 4 = 8.$$

Version 1 of the Chain Rule

If g is differentiable at x, and $y = f(u)$ is differentiable at $u = g(x)$, then the composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Guidelines for Using the Chain Rule

Assume the differentiable function $y = f(g(x))$ is given.

1. Identify the outer function f, the inner function g, and let $u = g(x)$.
2. Replace $g(x)$ by u to express y in terms of u:

$$y = f(g(x)) \implies y = f(u)$$

3. Calculate the product $\dfrac{dy}{du} \cdot \dfrac{du}{dx}$
4. Replace u by $g(x)$ in $\dfrac{dy}{du}$ to obtain $\dfrac{dy}{dx}$.

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

Example

Use Version 1 of the Chain Rule to calculate $\frac{dy}{dx}$ for $y = (5x^2 + 11x)^{20}$.

- inner function: $u = 5x^2 + 11x$
- outer function: $y = u^{20}$

We have $y = f(g(x)) = (5x^2 + 11x)^{20}$. Differentiate:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 20u^{19} \cdot (10x + 11)$$
$$= 20(5x^2 + 11x)^{19} \cdot (10x + 11)$$

Exercise

Use the first version of the Chain Rule to calculate $\frac{dy}{dx}$ for

$$y = \left(\frac{3x}{4x+2}\right)^5.$$

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Exercise

Use the first version of the Chain Rule to calculate $\frac{dy}{dx}$ for

$$y = \cos(5x + 1).$$

A. $y' = -\cos(5x + 1) \cdot \sin(5x + 1)$
B. $y' = -5\sin(5x + 1)$
C. $y' = 5\cos(5x + 1) - \sin(5x + 1)$
D. $y' = -\sin(5x + 1)$

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Version 2 of the Chain Rule

Notice if $y = f(u)$ and $u = g(x)$, then $y = f(u) = f(g(x))$, so we can also write:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= f'(u) \cdot g'(x) \\ &= f'(g(x)) \cdot g'(x). \end{aligned}$$

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Example

Use Version 2 of the Chain Rule to calculate $\frac{dy}{dx}$ for $y = (7x^4 + 2x + 5)^9$.

- inner function: $g(x) = 7x^4 + 2x + 5$
- outer function: $f(u) = u^9$

Then

$$f'(u) = 9u^8 \implies f'(g(x)) = 9(7x^4 + 2x + 5)^8$$
$$g'(x) = 28x^3 + 2.$$

Putting it together,

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x) = 9(7x^4 + 2x + 5)^8 \cdot (28x^3 + 2)$$

Exercise

Use Version 2 of the Chain Rule to calculate $\frac{dy}{dx}$ for

$$y = \tan\left(5x^5 - 7x^3 + 2x\right).$$

Chain Rule for Powers

If g is differentiable for all x in the domain and n is an integer, then

$$\frac{d}{dx}\left[(g(x))^n\right] = n(g(x))^{n-1} \cdot g'(x).$$

Chain Rule for Powers (cont.)

Example

$$\frac{d}{dx}\left[(1-e^x)^4\right] = ?$$

Answer:

$$\begin{aligned}\frac{d}{dx}\left[(1-e^x)^4\right] &= 4(1-e^x)^3\cdot(-e^x)\\ &= -4e^x(1-e^x)^3\end{aligned}$$

Composition of 3 or More Functions

Example

Compute $\dfrac{d}{dx}\left[\sqrt{(3x-4)^2+3x}\right]$.

Composition of 3 or More Functions (cont.)

Answer:

$$\begin{aligned}
\frac{d}{dx}\left[\sqrt{(3x-4)^2+3x}\right] &= \frac{1}{2}\left((3x-4)^2+3x\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left[(3x-4)^2+3x\right] \\
&= \frac{1}{2\sqrt{\left((3x-4)^2+3x\right)}} \cdot \left[2(3x-4)\frac{d}{dx}(3x-4)+3\right] \\
&= \frac{1}{2\sqrt{\left((3x-4)^2+3x\right)}} \cdot \left[2(3x-4)\cdot 3+3\right] \\
&= \frac{18x-21}{2\sqrt{\left((3x-4)^2+3x\right)}}
\end{aligned}$$

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3.7 Book Problems

7-33 (odds), 38, 45-67 (odds)

§3.8 Implicit Differentiation

Up to now, we have calculated derivatives of functions of the form $y = f(x)$, where y is defined **explicitly** in terms of x. In this section, we examine relationships between variables that are **implicit** in nature, meaning that y either is not defined explicitly in terms of x or cannot be easily manipulated to solve for y in terms of x.

The goal of **implicit differentiation** is to find a single expression for the derivative directly from an equation of the form $F(x, y) = 0$ without first solving for y.

Example

Calculate $\frac{dy}{dx}$ directly from the equation for the circle

$$x^2 + y^2 = 9.$$

Solution: To remind ourselves that x is our independent variable and that we are differentiating with respect to x, we can replace y with $y(x)$:

$$x^2 + (y(x))^2 = 9.$$

Now differentiate each term with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}((y(x))^2) = \frac{d}{dx}(9).$$

By the Chain Rule, $\frac{d}{dx}((y(x))^2) = 2y(x)y'(x)$ (Version 2), or $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$ (Version 1). So

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ \implies \frac{dy}{dx} &= \frac{-2x}{2y} \\ &= -\frac{x}{y}. \end{aligned}$$

The derivative is a function of x and y, meaning we can write it in the form

$$F(x,y) = -\frac{x}{y}.$$

To find slopes of tangent lines at various points along the circle we just plug in the coordinates. For example, the slope of the tangent line at (0,3) is

$$\left.\frac{dy}{dx}\right|_{(x,y)=(0,3)} = -\frac{0}{3} = 0.$$

The slope of the tangent line at $(1, 2\sqrt{2})$ is

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,2\sqrt{2})} = -\frac{1}{2\sqrt{2}}.$$

The point is that, in some cases it is difficult to solve an implicit equation in terms of y and then differentiate with respect to x. In other cases, although it may be easier to solve for y in terms of x, you may need two or more functions to do so, which means two or more derivatives must be calculated (e.g., circles).

The goal of implicit differentiation is to find one single expression for the derivative directly given $F(x, y) = 0$ (i.e., some equation with xs and ys in it), without solving first for y.

Question

The following functions are implicitly defined:

- $x + y^3 - xy = 4$
- $\cos(x - y) + \sin y = \sqrt{2}$

For each of these functions, how would you find $\frac{dy}{dx}$?

Exercise

Find $\frac{dy}{dx}$ for $xy + y^3 = 1$.

Exercise

Find an equation of the line tangent to the curve $x^4 - x^2y + y^4 = 1$ at the point $(-1, 1)$.

Higher Order Derivatives

Example

Find $\frac{d^2y}{dx^2}$ if $xy + y^3 = 1$.

Exercise

If $\sin x = \sin y$, then $\frac{dy}{dx} = ?$ and $\frac{d^2y}{dx^2} = ?$

A. $\dfrac{\cos y}{\cos x}; \quad \dfrac{\tan y \cos^2 x - \sin x \cos y}{\cos^2 x}$

B. $\dfrac{\cos x}{\cos y}; \quad \dfrac{\tan y \cos^2 x - \sin x \cos y}{\cos^2 y}$

C. $\dfrac{\cos x}{\cos y}; \quad \dfrac{\cos y(\sin x - \sin y)}{\cos^2 y}$

D. $\dfrac{\cos y}{\cos x}; \quad \dfrac{\cos y(\sin x - \sin y)}{\cos^2 x}$

Power Rule for Rational Exponents

Implicit differentiation also allows us to extend the power rule to rational exponents: Assume p and q are integers with $q \neq 0$. Then

$$\frac{d}{dx}(x^{\frac{p}{q}}) = \frac{p}{q}x^{\frac{p}{q}-1}$$

(provided $x \geq 0$ when q is even and $\frac{p}{q}$ is in lowest terms).

Exercise

Prove it.

3.8 Book Problems

5-25 (odds), 31-49 (odds)

Mon 7 Mar

- Exam 2: Solutions posted as soon as make-ups are in. But we will go through them today. You'll get your test back in drill tomorrow and the curve will be posted.
- Midterm:
  - Covers everything up to §3.9. All the slides are up. We will work fast through them today, but solutions to the exercises in the slides will be posted.
  - Morning Section: Walker room 124
    Afternoon Section: Walker room 218
    You must take the test with your officially scheduled section.

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

Mon 7 Mar (cont.)

- If you have questions about your exam conflicts, contact me NOW.
- Study guide is in MLP.
- Basic scientific calculator is allowed...? Yes.
- Sit in every other seat.
- 15 Questions, 10 points each.
- Don't expect a curve. :(

§3.9 Derivatives of Logarithmic and Exponential Functions

The natural exponential function $f(x) = e^x$ has an inverse function, namely $f^{-1}(x) = \ln x$. This relationship has the following properties:

1. $e^{\ln x} = x$ for $x > 0$ and $\ln(e^x) = x$ for all x.
2. $y = \ln x \quad \Longleftrightarrow \quad x = e^y$
3. For real numbers x and $b > 0$,

$$b^x = e^{\ln(b^x)} = e^{x \ln b}.$$

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Derivative of $y = \ln x$

Using 2. from the last slide, plus implicit differentiation, we can find $\frac{d}{dx}(\ln x)$. Write $y = \ln x$. We wish to find $\frac{dy}{dx}$. From 2.,

$$\frac{d}{dx}\left(x = e^y\right) \Rightarrow \frac{d}{dx}x = \frac{d}{dx}(e^y)$$

$$1 = e^y\left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

So $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

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Derivative of $y = \ln|x|$

Recall, we can only take "ln" of a positive number. However:

- For $x > 0$, $\ln|x| = \ln x$, so

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}.$$

- For $x < 0$, $\ln|x| = \ln(-x)$, so

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln(-x)) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}.$$

In other words, the absolute values do not change the derivative of natural log.

Exercise

Find the derivative of each of the following functions:

- $f(x) = \ln(15x)$
- $g(x) = x \ln x$
- $h(x) = \ln(\sin x)$

Derivative of $y = b^x$

What about other logs? Say $b > 0$. Since $b^x = e^{\ln b^x} = e^{x \ln b}$ (by 3. on the earlier slide),

$$\frac{d}{dx}(b^x) = \frac{d}{dx}(e^{x \ln b})$$

$$= e^{x \ln b} \cdot \ln b$$

$$= b^x \ln b.$$

Exercise

Find the derivative of each of the following functions:

- $f(x) = 14^x$
- $g(x) = 45(3^{2x})$

Exercise

Determine the slope of the tangent line to the graph $f(x) = 4^x$ at $x = 0$.

Story Problem Example

Example

The energy (in Joules) released by an earthquake of magnitude M is given by the equation

$$E = 25000 \cdot 10^{1.5M}.$$

(a) How much energy is released in a magnitude 3.0 earthquake?

(b) What size earthquake releases 8 million Joules of energy?

(c) What is $\frac{dE}{dM}$ and what does it tell you?

Derivatives of General Logarithmic Functions

The relationship $y = \ln x \iff x = e^y$ applies to logarithms of other bases:

$$y = \log_b x \quad \iff \quad x = b^y.$$

Now taking $\frac{d}{dx}\left(x = b^y\right)$ we obtain

$$1 = b^y \ln b \left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{1}{b^y \ln b}$$

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

Example

The derivative of $f(x) = \log_2(10x)$ is

A. $\frac{1}{10x}$

B. $\frac{1}{x\ln 2}$

C. $\frac{1}{x}$

D. $\frac{10}{x\ln 2}$

Neat Trick: Logarithmic Differentiation

Example

Compute the derivative of $f(x) = \dfrac{x^2(x-1)^3}{(3+5x)^4}$.

Solution: We can use logarithmic differentiation – first take the natural log of both sides and then use properties of logarithms.

$$\begin{aligned}\ln(f(x)) &= \ln\left(\frac{x^2(x-1)^3}{(3+5x)^4}\right)\\ &= \ln x^2 + \ln (x-1)^3 - \ln (3+5x)^4\\ &= 2\ln x + 3\ln(x-1) - 4\ln(3+5x)\end{aligned}$$

Now we take $\frac{d}{dx}$ on both sides:

$$\frac{1}{f(x)}\left(\frac{df}{dx}\right) = 2\left(\frac{1}{x}\right) + 3\left(\frac{1}{x-1}\right) - 4\left(\frac{1}{3+5x}\right)(5)$$

$$\frac{f'(x)}{f(x)} = \frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}$$

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Finally, solve for $f'(x)$:

$$f'(x) = f(x)\left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}\right]$$

$$= \frac{x^2(x-1)^3}{(3+5x)^4}\left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}\right]$$

Exercise

Use logarithmic differentiation to calculate the derivative of

$$f(x) = \frac{(x+1)^{\frac{3}{2}}(x-4)^{\frac{5}{2}}}{(5x+3)^{\frac{2}{3}}}.$$

3.9 Book Problems

9-29 (odds), 55-67 (odds)

Midterm Review

- §2.1-2.2
  - **Material may not be explicitly tested, but the topics here are foundational to later sections.**
- §2.3 Techniques for Computing Limits
  - **Be able to do questions similar to 1-48.**
  - Know and be able to compute limits using analytical methods (e.g., limit laws, additional techniques).
  - Be able to evaluate one-sided and two-sided limits of functions.
  - Know the Squeeze Theorem and be able to use this theorem to determine limits.

Midterm Review (cont.)

Exercise (problems from past midterm)

Evaluate the following limits:

- $$\lim_{x \to 3} \frac{x^2 - x - 6}{x^2 - 9}$$
- $$\lim_{\theta \to 0} \frac{\sec\theta \tan\theta}{\theta}$$

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Midterm Review (cont.)

- §2.4 Infinite Limits
  - **Be able to do questions similar to 17-30.**
  - Be able to use a graph, a table, or analytical methods to determine infinite limits.
  - Be able to use analytical methods to evaluate one-sided limits.
  - Know the definition of a vertical asymptote and be able to determine whether a function has vertical asymptotes.

Midterm Review (cont.)

- §2.5 Limits at Infinity
  - **Be able to do questions similar to 9-30 and 38-46.**
  - Be able to find limits at infinity and horizontal asymptotes.
  - Know how to compute the limits at infinity of rational functions and algebraic functions.
  - Be able to list horizontal and/or vertical asymptotes of a function.

Midterm Review (cont.)

Exercise

Determine the horizontal asymptote(s) for the function

$$f(x) = \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$$

A. $y = 2$

B. $y = 0$

C. $y = -2$

D. $y = \pm 2$

Running Out of Time on the Exam Plus other Study Tips

- Do practice problems completely, from beginning to end (as if it were a quiz). You might think you understand something but when it's time to write down the details things are not so clear.
- Find a buddy who understands concepts a little better than you and work on problems for 2-3 hours. Then find a buddy who is struggling and work with them 2-3 hours.
- Don't count on cookie cutter problems. If you are doing a practice problem where you've memorized all the steps, make sure you understand why each step is needed. The exam problems may have a small variation from homeworks and quizzes. If you're not prepared, it'll come as a "twist" on the exam...

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Running Out of Time on the Exam Plus other Study Tips (cont.)

- If you encounter an unfamiliar type of problem on the exam, relax, because it's most likely not a trick! The solutions will always rely on the information from the required reading/assignments. Take your time and do each baby step carefully.
- During the exam, do the problems you are most confident with first!
- During the exam, budget your time. Count the problems and divide by 50 minutes. The easier questions will take less time so doing them first leaves extra time for the harder ones. When studying, aim for 10 problems per hour (i.e., 6 minutes per problem).

Running Out of Time on the Exam Plus other Study Tips (cont.)

- Always make sure you answer the question. This is also a good strategy if you're not sure how to start a problem, figure out what the question wants first.
- The exam is not a race. If you finish early take advantage of the time to check your work. You don't want to leave feeling smug about how quickly you finished only to find out next week you lost a letter grade's worth of points from silly mistakes.

Other Study Tips

- Brush up on algebra, especially radicals, logs, common denominators, etc. Many times knowing the right algebra will simplify the problem!
- When in doubt, show steps.
- You will be punished for wrong notation. The slides for §3.1 show different notations for the derivative. Make sure whichever one you use in your work, that you are using it correctly.
- Read the question!
- Do the book problems.
- Look at the pictures in the book and the interactive applets on MLP.

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