Calculus I (Math 2554)

Summer 2015

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last updated: May 30, 2015

The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

Table of Contents

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$\oint$ 2.6 Continuity

Informally, a function f is "continuous at $x = a$" means for x-values anywhere close enough to a the graph can be drawn without lifting a pencil. In other words, no holes, breaks, asymptotes, etc.

Definition

A function f is **continuous** at a means

$$\lim_{x \to a} f(x) = f(a).$$

If f is not continuous at a, then a is a **point of discontinuity**.

Continuity Checklist

In order to claim something is continuous, you must verify all three:

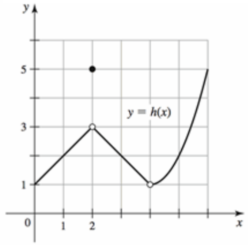
1. $f(a)$ is defined (i.e., a is in the domain of f – no holes, asymptotes).
2. $\lim_{x\to a} f(x)$ exists. You must check both sides and make sure they equal the same number.
3. $\lim_{x\to a} f(x) = f(a)$ (i.e., the value of f equals the limit of f at a).

Question

What is an example of a function that satisfies this condition?

Example

- Where are the points of discontinuity of the function below?
- Which aspects of the checklist fail?



recall (Continuity Checklist):

1. function is defined
2. the two-sided limit exists
3. 2. = 1.

Continuity Rules

If f and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$, provided $g(a) \neq 0$
6. $[f(x)]^n$

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From the rules above, we can deduce:

1. Polynomials are continuous for all $x = a$.
2. Rational functions are continuous at all $x = a$ except for the points where the denominator is zero.
3. If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

Continuity on an Interval

Consider the cases where f is not defined past a certain point.

Definition

A function f is **continuous from the left** (or **left-continuous**) at a means

$$\lim_{x \to a^-} f(x) = f(a);$$

a function f is **continuous from the right** (or **right-continuous**) at a means

$$\lim_{x \to a^+} f(x) = f(a).$$

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Definition

A function f is **continuous on an interval** I means it is continuous at all points of I.

Notation: Intervals are usually written

$$[a, b]\,,\ (a, b]\,,\ [a, b)\,,\ \text{or}\ (a, b)\,.$$

When I contains its endpoints, "continuity on I" means continuous from the right or left at the endpoints.

Example

Let $f(x) = \begin{cases} x^3 + 4x + 1 & \text{if } x \leq 0 \\ 2x^3 & \text{if } x > 0. \end{cases}$

1. Use the continuity checklist to show that f is not continuous at 0.
2. Is f continuous from the left or right at 0?
3. State the interval(s) of continuity.

Continuity of Functions with Roots

(assuming m and n are positive integers and $\frac{n}{m}$ is in lowest terms)

- If m is odd, then $[f(x)]^{\frac{n}{m}}$ is continuous at all points at which f is continuous.
- If m is even, then $[f(x)]^{\frac{n}{m}}$ is continuous at all points a at which f is continuous and $f(a) \geq 0$.

Question

Where is $f(x) = \sqrt[4]{4 - x^2}$ continuous?

Continuity of Transcendental Functions

Trig Functions: The basic trig functions are all continuous at all points IN THEIR DOMAIN. Note there are points of discontinuity where the functions are not defined – for example, $\tan x$ has asymptotes everywhere that $\cos x = 0$.

Exponential Functions: The exponential functions b^x and e^x are continuous on all points of their domains.

Inverse Functions: If a continuous function f has an inverse on an interval I (meaning if $x \in I$ then $f^{-1}(y)$ passes the vertical line test), then its inverse f^{-1} is continuous on the interval J, which is defined as all the numbers $f(x)$, given x is in I.

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Intermediate Value Theorem (IVT)

Theorem (Intermediate Value Theorem)

Suppose f is continuous on the interval $[a, b]$ and L is a number satisfying

$$f(a) < L < f(b) \quad \text{or} \quad f(b) < L < f(a).$$

Then there is at least one number $c \in (a, b)$, i.e., $a < c < b$, satisfying

$$f(c) = L.$$

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Example

Let $f(x) = -x^5 - 4x^2 + 2\sqrt{x} + 5$. Use IVT to show that $f(x) = 0$ has a solution in the interval $(0, 3)$.

2.6 Book Problems

9-23 (odds), 29-37 (odds), 45, 49, 51, 53

∮ 2.7 Precise Definitions of Limits

Assume that $f(x)$ exists for all x in some open interval (open means: neither of the endpoints not included) containing a, except possibly at a. "The limit of $f(x)$ as x approaches a is L", i.e.,

$$\lim_{x\to a} f(x) = L,$$

means for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

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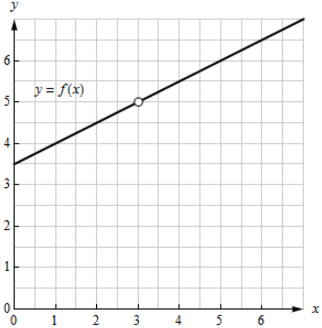
Seeing ϵs and δs on a Graph

Question



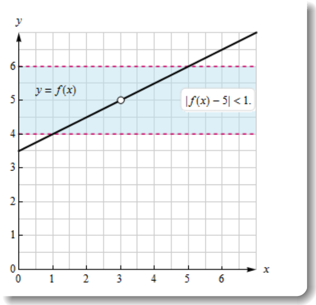
Using the graph, for each $\epsilon > 0$, determine a value of $\delta > 0$ to satisfy the statement

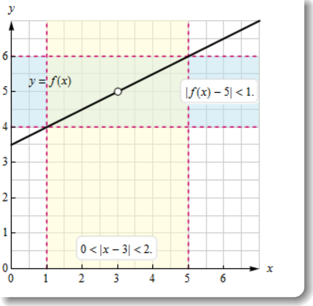
$$|f(x) - 5| < \epsilon \quad \text{whenever} \quad 0 < |x - 3| < \delta.$$

- $\epsilon = 1$
- $\epsilon = 0.5.$

Seeing ϵs and δs on a Graph, cont.

When $\epsilon = 1$:

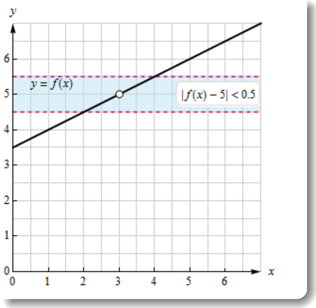


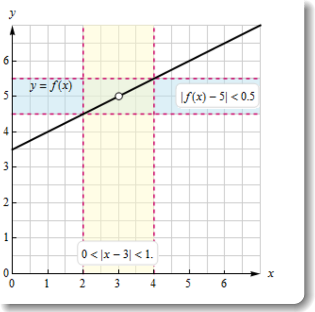


$\ldots\ \delta = 2$

Seeing ϵs and δs on a Graph, cont.

When $\epsilon = 0.5$:





$\ldots\ \delta = 1$

The ϵs and δs give a way to visualize computing the limit, and prove it exists. As the ϵs get smaller and smaller, we want there to always be a δ. In this example,

$$\lim_{x \to 3} f(x) = 5.$$

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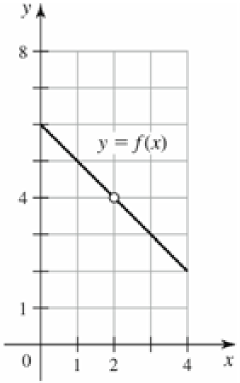
Exercise



Using the graph, for each $\epsilon > 0$, determine a value of $\delta > 0$ to satisfy the statement

$$|f(x) - 4| < \epsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

- $\epsilon = 1$
- $\epsilon = 0.5$.

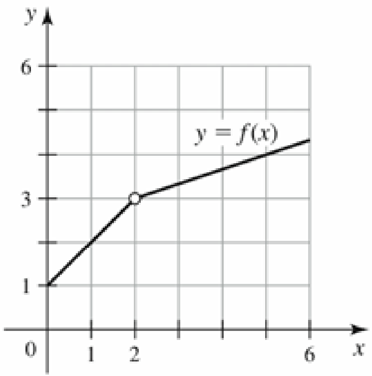
Finding a Symmetric Interval

Question

When finding an interval $(a - \delta, a + \delta)$ around the point a, what happens if you compute two different δs?

Answer: To obtain a symmetric interval around a, use the smaller of the two δs as your distance around a.

Exercise



The graph of $f(x)$ shows

$$\lim_{x \to 2} f(x) = 3.$$

For $\epsilon = 1$, find the corresponding value of $\delta > 0$ so that

$$|f(x) - 3| < \epsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

Exercise

Let $f(x) = x^2 - 4$. For $\epsilon = 1$, find a value for $\delta > 0$ so that

$$|f(x) - 12| < \epsilon \quad \text{whenever} \quad 0 < |x - 4| < \delta.$$

In this example, $\lim_{x \to 4} f(x) = 12$.

2.7 Book Problems

1-7, 9-18

$\oint$ 3.1 Introducing the Derivative

Recall from Ch 2: We said that the slope of the tangent line at a point is the limit of the slopes of the secant lines as the points get closer and closer.

- slope of secant line: $\dfrac{f(x)-f(a)}{x-a}$ (average rate of change)
- slope of tangent line: $\displaystyle\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ (instantaneous rate of change)

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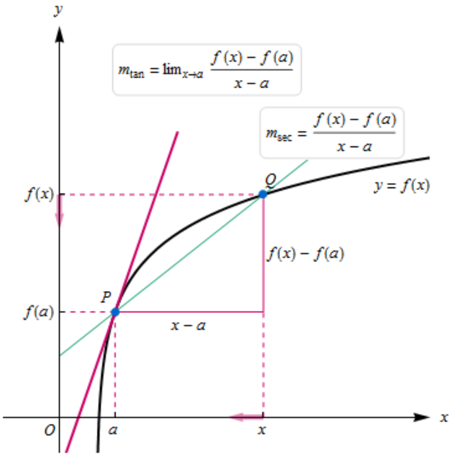

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Example

Use the relationship between secant lines and tangent lines, specifically the slope of the tangent line, to find the equation of a line tangent to the curve $f(x) = x^2 + 2x + 2$ at the point $P = (1, 5)$.

In the preceding example, we considered two points

$$P = (a, f(a)) \quad \text{and} \quad Q = (x, f(x))$$

that were getting closer and closer together.

Instead of looking at the points approaching one another, we can also view this as the distance h between the points approaching 0. For

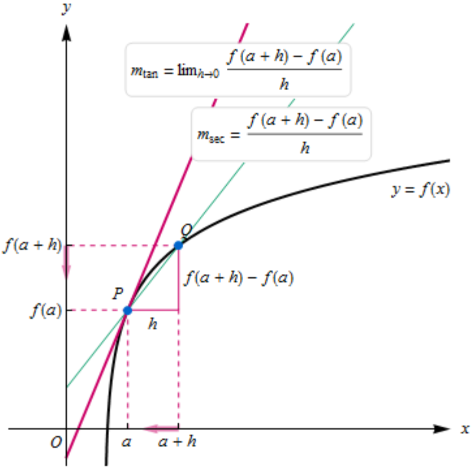
$$P = (a, f(a)) \quad \text{and} \quad Q = (a+h, f(a+h)),$$

- slope of secant line:

$$\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

- slope of tangent line:

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$



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Example

Find the equation of a line tangent to the curve $f(x) = x^2 + 2x + 2$ at the point $P = (2, 10)$.

Derivative Defined as a Function

The slope of the tangent line for the function f is a function of x, called the derivative of f.

Definition

The **derivative** of f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. If $f'(x)$ exists, we say f is **differentiable** at x. If f is differentiable at every point of an open interval I, we say that f is differentiable on I.

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Exercise

Use the definition of the derivative to find the derivative of the function $f(x) = x^2 + 2x + 2$.

Leibniz Notation

A standard notation for change involves the Greek letter Δ.

$$\frac{f(x+h)-f(x)}{h}=\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{\Delta y}{\Delta x}.$$

Apply the limit:

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}$$

Other Notation

The following are alternative ways of writing $f'(x)$ (i.e., the derivative as a function of x):

$$\frac{dy}{dx} \qquad \frac{df}{dx} \qquad \frac{d}{dx}(f(x)) \qquad D_x(f(x)) \qquad y'(x)$$

The following are ways to notate the derivative of f evaluated at $x = a$:

$$f'(a) \qquad y'(a) \qquad \left.\frac{df}{dx}\right|_{x=a} \qquad \left.\frac{dy}{dx}\right|_{x=a}$$

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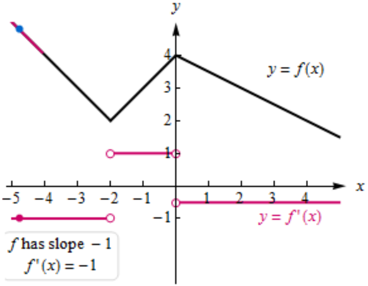
Graphing the Derivative

The graph of the derivative is the graph of the collection of slopes of tangent lines of a graph. If you just have a graph (without an equation for the graph), the best you can do is approximate the graph of the derivative.

Example

Simple checklist:

1. Note where $f'(x) = 0$.
2. Note where $f'(x) > 0$.
   (What does this look like?)
3. Note where $f'(x) < 0$.
   (What does this look like?)



Differentiability vs. Continuity

Key points about the relationship between differentiability and continuity:

- If f is differentiable at a, then f is continuous at a.
- If f is not continuous at a, then f is not differentiable at a.
- f can be continuous at a, but not differentiable at a.

A function f is not differentiable at a if at least one of the following conditions holds:

1. f is not continuous at a.
2. f has a corner at a.

Question

Why does this make f not differentiable?

3. f has a vertical tangent at a.

Question

Why does this make f not differentiable?

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3.1 Book Problems

11-12, 19-20, 23-26, 31-33, 35-36, 39-43, 45, 49-52

- **NOTE:** You do not know any rules for differentiation yet (e.g., Power Rule, Chain Rule, etc.) In this section, you are strictly using the definition of the derivative and the definition of slope of tangent lines we have derived.

Exam #1 Review

- $\oint$2.1 The Idea of Limits
  - Understand the relationship between average velocity & instantaneous velocity, and secant and tangent lines
  - Be able to compute average velocities and use the idea of a limit to approximate instantaneous velocities
  - Be able to compute slopes of secant lines and use the idea of a limit to approximate the slope of the tangent line

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Exam #1 Review (cont.)

- $\oint$ 2.2 Definitions of Limits
  - Know the definition of a limit
  - Be able to use a graph of a table to determine a limit
  - Know the relationship between one- and two-sided limits
- $\oint$ 2.3 Techniques for Computing Limits
  - Know and be able to compute limits using analytical methods (e.g., limit laws, additional techniques)
  - Know the Squeeze Theorem and be able to use it to determine limits

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Exam #1 Review (cont.)

Example

Evaluate $\lim_{x \to 0} x \sin \frac{1}{x}$.

- $\oint$ 2.4 Infinite Limits
  - Be able to use a graph, a table, or analytical methods to determine infinite limits
  - Know the definition of a vertical asymptote and be able to determine whether a function has vertical asymptotes

Exam #1 Review (cont.)

- $\oint$2.5 Limits at Infinity
  - Be able to find limits at infinity and horizontal asymptotes
  - Know how to compute the limits at infinity of rational functions

Exam #1 Review (cont.)

Example

Determine the end behavior of $f(x)$. If there is a horizontal asymptote, then say so. Next, identify any vertical asymptotes. If $x = a$ is a vertical asymptote, then evaluate $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$.

$$f(x) = \frac{2x^3 + 10x^2 + 12x}{x^3 + 2x^2}$$

Exam #1 Review (cont.)

- $\oint$ 2.6 Continuity
  - Know the definition of continuity and be able to apply the continuity checklist
  - Be able to determine the continuity of a function (including those with roots) on an interval
  - Be able to apply the Intermediate Value Theorem to a function

Exam #1 Review (cont.)

Example

Determine the value for a that will make $f(x)$ continuous.

$$f(x) = \begin{cases} \dfrac{x^2+3x+2}{x+1} & x \neq -1 \\ a & x = -1 \end{cases}$$

Example

Show that $f(x) = 2$ has a solution on the interval $(-1, 1)$, with

$$f(x) = 2x^3 + x.$$

Exam #1 Review (cont.)

- $\oint$2.7 Precise Definition of Limits
  - Understand the δ, ϵ relationship for limits
  - Be able to use a graph or analytical methods to find a value for $\delta > 0$ given an $\epsilon > 0$ (including finding symmetric intervals)

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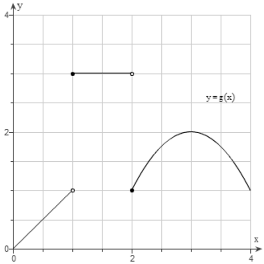
Exam #1 Review (cont.)

Example



Use the graph to find the appropriate δ.

(a) $|g(x) - 2| < \frac{1}{2}$ whenever

$$0 < |x - 3| < \delta$$

(b) $|g(x) - 1| < \frac{3}{2}$ whenever

$$0 < |x - 2| < \delta$$

In this example, the two-sided limits at $x = 1$ and $x = 2$ do not exist.

Exam #1 Review (cont.)

- $\oint$3.1 Introducing the Derivative
  - Know the definition of a derivative and be able to use this definition to calculate the derivative of a given function
  - Be able to determine the equation of a line tangent to the graph of a function at a given point
  - Know the 3 conditions for when a function is not differentiable at a point, and why these three conditions make a function not differentiable at the given point

Exam #1 Review (cont.)

Example

(a) Use the limit definition of the derivative to find an equation for the line tangent to $f(x)$ at a, where

$$f(x) = \frac{1}{x}; \quad a = -5.$$

(b) Using the same $f(x)$ from part (a), find a formula for $f'(x)$ (using the limit definition).

(c) Plug -5 into your answer for (b) and make sure it matches your answer for (a).

Other Study Tips

- Brush up on algebra, especially radicals.
- If your answer is something like $\sqrt{2}$, don't plug that into your calculator, just leave it as is.
- When in doubt, show steps. Defer to class notes and old exams to get an idea of what's expected.
- You will be punished for wrong notation; e.g., the limit symbol.
- Read the question! Several students always lose points because they didn't answer the question or they didn't follow directions.
- Do the book problems.
- Look at the pictures in the book and the interactive applets on MLP.

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