Section 4.3 – Families of Functions

1. (Taken from *Hughes Hallett, et. al.*) The number, N, of people who have heard a rumor spread by mass media at time, t, is given by $N(t) = a(1 - e^{-kt})$. There are 200,000 people in the population who hear the rumor eventually. If 10% of them heard it the first day, find a and k, assuming that t is measured in days.

   To begin, we are given the following information

   $$\begin{aligned} N(1) &= 0.1 \cdot 200{,}000 = 20{,}000 \\ \lim_{t\to\infty} N(t) &= 200{,}000 \end{aligned}$$

   Because $\lim_{t\to\infty} N(t) = 200{,}000$, we have

   $$\begin{aligned} \lim_{t\to\infty} a(1 - e^{-kt}) &= 200{,}000 \\ a(1 - 0) &= 200{,}000 \end{aligned}$$

   Therefore, $a = 200{,}000$, which means that $N(t) = 200{,}000(1 - e^{-kt})$. But we also know that $N(1) = 20{,}000$, so we have

   $$\begin{aligned} 200{,}000(1 - e^{-k(1)}) &= 20{,}000 \\ 1 - e^{-k} &= 0.1 \\ e^{-k} &= 0.9 \\ k &= -\ln(0.9) \end{aligned}$$

   Therefore, our answers are $a = 200{,}000$ and $k = -\ln(0.9)$.

2. Let $f(x) = x^4 - ax^2$.

   (a) Find all possible critical points of f in terms of a.

   We have $f'(x) = 4x^3 - 2ax = 2x(2x^2 - a)$, so we can see that $f'(x) = 0$ when $x = 0$ or when $x = \pm\sqrt{a/2}$. Therefore, our critical points are $x = 0$, $x = \sqrt{a/2}$, and $x = -\sqrt{a/2}$.

   (b) If $a < 0$, how many critical points does f have?

   If $a < 0$, then $a/2 < 0$, which means that $\sqrt{a/2}$ and $-\sqrt{a/2}$ are not real numbers. Therefore, using our answer to part (a), we see that $x = 0$ is the only critical point of f, i.e., f has exactly one critical point.

   (c) If $a > 0$, find the x and y coordinates of all critical points of f.

   If $a > 0$, then, by part (a), $x = 0$, $x = \sqrt{a/2}$, and $x = -\sqrt{a/2}$ are all critical points of f. We have

   $$\begin{aligned} f(0) &= 0^4 - a(0)^2 = 0 \\ f\left(\pm\sqrt{\frac{a}{2}}\right) &= \left(\sqrt{\frac{a}{2}}\right)^4 - a\left(\sqrt{\frac{a}{2}}\right)^2 = \frac{a^2}{4} - \frac{a^2}{2} = -\frac{a^2}{4}, \end{aligned}$$

Interval	Sign of $f'(x)$
$x < -\sqrt{a/2}$	$-$
$-\sqrt{a/2} < x < 0$	$+$
$0 < x < \sqrt{a/2}$	$-$
$x > \sqrt{a/2}$	$+$

   so the critical points are

   $$(0, 0), \qquad \left(\sqrt{\frac{a}{2}}, -\frac{a^2}{4}\right), \qquad \text{and} \qquad \left(-\sqrt{\frac{a}{2}}, -\frac{a^2}{4}\right),$$

   which can be classified as local maxima or minima by referring to the above sign chart.

(d) Find a value of a such that the two local minima of f occur at $x = \pm 2$.

The sign chart from part (c) reveals that $(\pm\sqrt{a/2},\ -a^2/4)$ are the coordinates of the two local minima of f. We have

$$\begin{aligned} \sqrt{a/2} &= 2 \\ a/2 &= 4 \\ a &= 8 \end{aligned}$$

Therefore, the local minima of f occur at $x = \pm 2$ when $a = 8$.

3. Let $f(x) = axe^{-bx}$. ASSUME THAT a AND b ARE BOTH POSITIVE.

(a) Find all inflection points of f in terms of a and b.

Interval	Sign of $f''(x)$
$x < 2/b$	$-$
$x > 2/b$	$+$

Since $f'(x) = axe^{-bx} \cdot (-b) + ae^{-bx} = ae^{-bx}(1 - bx)$, we see that

$f''(x) = ae^{-bx} \cdot (-b) + ae^{-bx} \cdot (-b) \cdot (1 - bx) = abe^{-bx}(bx - 2)$.

Therefore, $f''(x) = 0$ if and only if $x = 2/b$, and the sign chart above confirms that an inflection point does indeed occur at this point. Since $f(2/b) = (2a/b)e^{-2}$, we see that the one and only inflection point of f is

$$\left(\frac{2}{b},\ \frac{2a}{b}e^{-2}\right).$$

(b) Find a and b so that the inflection point of f occurs at $(1, 2)$.

In order for the inflection point that we found in part (a) above to occur at $(1, 2)$, we must have $2/b = 1$ and $(2a/b)e^{-2} = 2$. Since $2/b = 1$, we have $b = 2$, which leads to

$$\begin{aligned} \frac{2a}{2}e^{-2} &= 2 \\ a &= 2e^2 \end{aligned}$$

Therefore, we conclude that $a = 2e^2$ and $b = 2$.