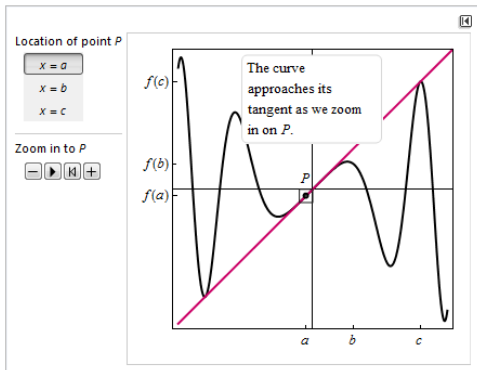


Fri 1 Apr

- Exam 3: next Friday. Covers §3.10-4.6

§4.5 Linear Approximation and Differentials

Suppose f is a function such that f' exists at some point P . If you zoom in on the graph, the curve appears more and more like the tangent line to f at P .



Linear Approximation

This idea – that **smooth** curves (i.e., curves without corners) appear straighter on smaller scales – is the basis of linear approximations.

One of the properties of a function that is **differentiable** at a point P is that it is **locally linear** near P (i.e., the curve approaches the tangent line at P .)

Therefore, it makes sense to approximate a function with its tangent line, which matches the value and slope of the function at P .

This is why you've had to do so many “find the equation for the tangent line to the given point” problems!

Definition

Suppose f is differentiable on an interval I containing the point a . The **linear approximation** to f at a is the linear function

$$L(x) = f(a) + f'(a)(x - a) \quad \text{for } x \text{ in } I.$$

Remarks: Compare this definition to the following: At a given point $P = (a, f(a))$, the slope of the line tangent to the curve at P is $f'(a)$. So the equation of the tangent line is

$$y - f(a) = f'(a)(x - a).$$

(Yes, it is the same thing!)

Exercise

Write the equation of the line that represents the linear approximation to

$$f(x) = \frac{x}{x+1} \quad \text{at } a = 1.$$

Then *use* the linear approximation to estimate $f(1.1)$.

Solution: First compute

$$f'(x) = \frac{1}{(x+1)^2}, \quad f(a) = \frac{1}{2}, \quad f'(a) = \frac{1}{4}$$

$$L(x) = \frac{1}{2} + \frac{1}{4}(x-1) = \frac{1}{4}x + \frac{1}{4}.$$

Solution (continued):

Because $x = 1.1$ is near $a = 1$, we can estimate $f(1.1)$ using $L(1.1)$:

$$f(1.1) \approx L(1.1) = 0.525$$

Note that $f(1.1) = 0.5238$, so the error in this estimation is

$$\frac{0.525 - 0.5238}{0.5238} \times 100 = 0.23\%.$$

Exercise

- (a) The linear approximation to $f(x) = \sqrt{1+x}$ at the point $x = 0$ is (choose one):
- A. $L(x) = 1$
 - B. $L(x) = 1 + \frac{x}{2}$
 - C. $L(x) = x$
 - D. $L(x) = 1 - \frac{x}{2}$
- (b) What is an approximation for $f(0.1)$?

Our linear approximation $L(x)$ is used to approximate $f(x)$ when a is fixed and x is a nearby point:

$$f(x) \approx f(a) + f'(a)(x - a)$$

When rewritten,

$$\begin{aligned} f(x) - f(a) &\approx f'(a)(x - a) \\ \implies \Delta y &\approx f'(a)\Delta x. \end{aligned}$$

A change in y can be approximated by the corresponding change in x , magnified or diminished by a factor of $f'(a)$.

This is another way to say that $f'(a)$ is the rate of change of y with respect to x !

$$\Delta y \approx f'(a)\Delta x$$

$$\frac{\Delta y}{\Delta x} \approx f'(a)$$

So if f is differentiable on an interval I containing the point a , then the change in the value of f (the Δy), between two points a and $a + \Delta x$ in I , is **approximately** $f'(x)\Delta x$.

We now have two different, but related quantities:

- The change in the function $y = f(x)$ as x changes from a to $a + \Delta x$ (which we call Δy).
- The change in the linear approximation $y = L(x)$ as x changes from a to $a + \Delta x$ (called the **differential**, dy).

$$\Delta y \approx dy$$

When the x -coordinate changes from a to $a + \Delta x$:

- The function change is exactly $\Delta y = f(a + \Delta x) - f(a)$.
- The linear approximation change is

$$\begin{aligned}\Delta L &= L(a + \Delta x) - L(a) \\ &= (f(a) + f'(a)(a + \Delta x - a)) - (f(a) + f'(a)(a - a)) \\ &= f'(a)\Delta x\end{aligned}$$

and this is dy .

We define the differentials dx and dy to distinguish between the change in the function (Δy) and the change in the linear approximation (ΔL):

- dx is simply the change in x , i.e. Δx .
- dy is the change in the linear approximation, which is $\Delta L = f'(a)\Delta x$.

SO:

$$\Delta L = f'(a)\Delta x$$

$$dy = f'(a)dx$$

$$\frac{dy}{dx} = f'(a) \quad (\text{at } x = a)$$

Definition

Let f be differentiable on an interval containing x .

- A small change in x is denoted by the **differential** dx .
- The corresponding change in $y = f(x)$ is approximated by the **differential** $dy = f'(x)dx$; that is,

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x) \\ &\approx dy = f'(x)dx.\end{aligned}$$

The use of differentials is critical as we approach integration.

Example

Use the notation of differentials [$dy = f'(x)dx$] to approximate the change in $f(x) = x - x^3$ given a small change dx .

Solution: $f'(x) = 1 - 3x^2$, so $dy = (1 - 3x^2)dx$.

A small change dx in the variable x produces an approximate change of $dy = (1 - 3x^2)dx$ in y .

For example, if x increases from 2 to 2.1, then $dx = 0.1$ and

$$dy = (1 - 3(2)^2)(0.1) = -1.1.$$

This means as x increases by 0.1, y decreases by 1.1.

4.5 Book Problems

13-20, 35-50