Principal Minor Ideals with Matroid Theory

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Ideals Generated by Principal Minors

Thank-you for the invitation to speak!

r, s arbitrary positive integers

$K[X]$ = polynomial ring over K with variables $x_{11}, \ldots, x_{rs}$

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1s} \\ \vdots & \ddots & \ddots & \vdots \\ x_{r1} & \cdots & \cdots & x_{rs} \end{pmatrix}$$

The **principal** minors of an $n \times n$ matrix are those whose defining row and column indices are the same.

$\mathfrak{B}_t$ = ideal in $K[X]$ generated by the size t principal minors of the generic square matrix X

Size $t = 2$ Principal Minors

Theorem (–)

For all n, $K[X]/\mathfrak{B}_2$ is a complete intersection, is isomorphic to a K-algebra generated by monomials, and is normal. In particular, it is strongly F-regular (characteristic $p > 0$ case) and Gorenstein.

The proof exploits the fact that $\mathfrak{B}_2$ is toric. For $t > 2$ it becomes more convenient to study components of $\mathcal{V}(\mathfrak{B}_t)$ by fixing their matrix rank.

$$\mathcal{Y}_{n,r,t} = \mathcal{V}(\mathfrak{B}_t) \bigcap \{n \times n \text{ matrices of rank } r\}$$

Size $t = n - 1$ Principal Minors

Lemma (–)

In the localized ring $K[X]_{\det X}$, the K-algebra automorphism $X \to X^{-1}$ induces an isomorphism of the schemes defined, respectively, by $\mathfrak{B}_t \cdot K[X]_{\det X}$ and $\mathfrak{B}_{n-t} \cdot K[X]_{\det X}$.

Theorem (–)

For $n \geq 4$, the minimal primes for $\mathfrak{B}_{n-1}$ are the determinantal ideal I_{n-1} and the contraction of $\ker \phi$ to $K[X]$, which we denote by $\mathfrak{Q}_{n-1}$, where ϕ is the ring homomorphism

$$\phi : K[X]_{\det X} \to \left(\frac{K[X]}{\mathfrak{B}_1}\right)_{\det X}$$
$$X \mapsto (\det X) \cdot X^{-1}$$

A quick corollary: $\operatorname{ht}(\mathfrak{B}_t) \leq \binom{n+1}{2} - \binom{t+2}{2} + 4$ for $n \neq 3$.

An even more immediate corollary: $\operatorname{ht}(\mathfrak{Q}_{n-1}) = n$. Consequently, principal minor ideals are generally not Cohen-Macaulay. By Hochster+Roberts, it follows that, in particular, their quotients cannot be rings of invariants.

Note, the two components of $\mathcal{V}(\mathfrak{B}_{n-1})$ are:

(1) $$\mathcal{V}(\mathrm{I}_{n-1}) = \bigcup_{r'<n-1} \mathcal{Y}_{n,r',n-1}$$

(2) $$\mathcal{V}(\mathfrak{Q}_{n-1}) = \overline{\mathcal{Y}}_{n,n,n-1} \supset \mathcal{Y}_{n,n-1,n-1}$$

Size $t = n - 2$ Principal Minors: Rank $r = n - 2$ Case

When $t \neq 1, 2, n-1, n$ identifying the components of $\mathcal{V}(\mathfrak{B}_t)$ becomes harder.

Theorem (–)

$$\dim \mathcal{Y}_{n,n-2,n-2} = n^2 - 4 - n$$

Note, a matrix of rank r can be decomposed as a product of two matrices, so we can identify $\mathcal{Y}_{n,r,r}$ with a product of two Grassmann varieties.

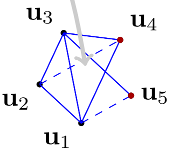
$$\begin{tikzcd} A \in \mathcal{Y}_{n,r,r} \arrow[d, mapsto] \arrow[dr] & \\ (\mathrm{col}\, A, \mathrm{row}\, A) & \in \mathrm{Grass}_K(r,n) \times \mathrm{Grass}_K(r,n) \end{tikzcd}$$

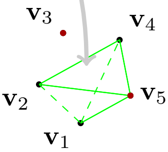
Let $\mathcal{G} = \mathrm{Grass}(n-2, n)$.

Given $\mathbf{g} \in \mathcal{G}$, construct $\operatorname{Graph}(\mathbf{g})$ as follows: a vertex represents an index; an edge joining two vertices indicates the Plücker coordinate with complementary indices vanishes.

Example

$$\mathbf{g}_U \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ b_{41} & b_{42} & b_{43} \\ b_{51} & b_{52} & b_{53} \end{pmatrix} \quad \mathbf{g}_V \sim \begin{pmatrix} 1 & 0 & c_{13} & 0 & c_{15} \\ 0 & 1 & c_{23} & 0 & c_{25} \\ 0 & 0 & c_{33} & 1 & c_{35} \end{pmatrix}$$





Proposition

$\mathrm{Graph}\,(\mathbf{g})$ *is well-defined.*

Given a graph G, if there exists $\mathbf{g} \in \mathcal{G}$ such that $\mathrm{Graph}\,(\mathbf{g}) = G$ then G is called permissible. A subvariety $\mathcal{S} \subseteq \mathcal{G}$ that is the set of all points with the same permissible graph is denoted $\mathrm{Graph}\,(\mathcal{S})$.

Theorem (–)

A product $\mathcal{S} \times \mathcal{T}$ of permissible subvarieties corresponds to a component of $\mathcal{Y}_{n,n-2,n-2}$. Furthermore (modulo transposition of $\mathcal{S}$ and $\mathcal{T}$),

(a) $\mathrm{Graph}\,(\mathcal{S})$ *is the union of a complete graph of order $a > 1$ and $n - a$ isolated vertices;*

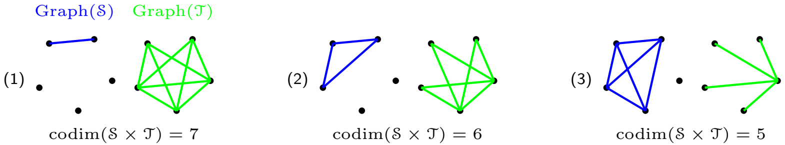
(b) $\mathrm{Graph}\,(\mathcal{T})$ *is the complement of* $\mathrm{Graph}\,(\mathcal{S})$*.*

Theorem (–)

Suppose $\mathcal{S} \times \mathcal{T}$ is a permissible pair and $\mathrm{Graph}(\mathcal{S})$ has a maximal complete subgraph of order a. Then

- *(a)* $\operatorname{codim} \mathcal{S} = a - 1$,
- *(b)* $\operatorname{codim} \mathcal{T} = 2(n - a)$, *and*
- *(c) (corollary)* $2 \leq a \leq n - 1$. *It follows that the minimal codimension of such* $\mathcal{S} \times \mathcal{T}$ *is* n.

Example (Permissible Pairs for $n = 5$)



Connection to Matroid Theory

Matroids are a type of combinatorial data used to describe many seemingly unrelated objects in mathematics, including graphs, transversals, vector spaces, and networks. *Matroid* has many equivalent definitons.

Definition (Independence Axioms)

Let E denote a finite set and 2^E its power set. Suppose $\mathcal{I} \subseteq 2^E$. Then the system $\mathcal{M} = (E, \mathcal{I})$ is a **matroid** if and only if

(1) $\emptyset \in \mathcal{I}$,
(2) if $S \in \mathcal{I}$ and $T \subseteq S$ then $T \in \mathcal{I}$, and
(3) (**Independence Augmentation Axiom**) if $S, T \in \mathcal{I}$ and $|S| > |T|$, then there exists $e \in S \setminus T$ such that $T \cup \{e\} \in \mathcal{I}$.

K-Representable Matroids

A matroid defined by a K-vector space is called **K-representable**. Let A denote an $r \times n$ matrix and put

$$\begin{aligned} E &= \{\text{columns of } A\} \\ \mathcal{I} &= \{\text{collections of linearly independent columns}\} \\ \mathcal{D} &= 2^E \setminus \mathcal{I} \\ \mathcal{B} &= \{\text{sets in } \mathcal{I} \text{ with maximal cardinality}\} \end{aligned}$$

Question

The Independence Augmentation Axiom implies all maximal sets in $\mathcal{I}$ have the same cardinality. What is it? To what do $\mathcal{D}, \mathcal{B}$ correspond?

Using correspondingly prescribed axioms, the matroid $\mathcal{M} = (E, \mathcal{I})$ can also be defined using $\mathcal{D}$ and $\mathcal{B}$. Another equivalent definition:

Definition (Rank Axioms)

A function $r : 2^E \to \mathbb{Z}_+$ is the **rank function** of a matroid $\mathcal{M} = (E, r)$ if and only if for all $S, T \subseteq E$

(a) $0 \leq r(S) \leq |S|$,
(b) if $T \subseteq S$ then $r(T) \leq r(S)$, and
(c) **(Submodularity)** $r(S) + r(T) \geq r(S \cup T) + r(S \cap T)$.

Matroid Subvarieties of a Grassmannian

Fix $r < n$. We get a matroid structure on the finite set of columns of a generic $r \times n$ matrix when we prescribe a subset of Plücker coordinates to vanish; let $\mathcal{D}$ denote the set of indices for the vanishing Plücker coordinates.

Given such a matroid $\mathcal{M}$, the **open matroid variety** is the subset of points in $\mathcal{G} = \mathrm{Grass}\,(r, n)$ whose matroid is $\mathcal{M}$. Its Zariski closure is called a **matroid variety**, which we shall denote by $\mathcal{V}\,(\mathcal{M})$.

Example

Schubert and Richardson varieties are matroid varieties.

The following example shows we cannot, in general, simply use the indices from $\mathcal{D}$ on the Plücker variables to generate the defining ideal for $\mathcal{V}\,(\mathcal{M})$. For any Plücker coordinate with index $\underline{\mathrm{i}}$, let $x_{\underline{\mathrm{i}}}$ denote the correspondingly indexed variable in the homogeneous coordinate ring for $\mathcal{G}$.

Example (Ford)

Put $r = 3, n = 7$, and $\mathcal{D} = \{\{1,2,7\},\{3,4,7\},\{5,6,7\}\}$, the set of indices for Plücker coordinates we require to vanish. One hopes the defining ideal for $\mathcal{V}(E, \mathcal{I})$ is

$$I = (x_{\{1,4,7\}}, x_{\{3,4,7\}}, x_{\{5,6,7\}}).$$

However, the defining ideal is actually

$$J = I + (x_{\{1,2,4\}}x_{\{3,5,6\}} - x_{\{1,2,3\}}x_{\{4,5,6\}}).$$

Positroid Varieties

A particular class of matroid varieties exists, however, where the geometry is better behaved. A **positroid** is a matroid determined by a rank condition on cyclic intervals in $E = \{1, \ldots, n\}$, where a cyclic interval is an ordinary interval or its complement.

Positroid varieties are the matroid varieties we get from positroids.

Theorem (Knutson+Lam+Speyer)

Positroid varieties are normal, Cohen-Macaulay, have rational singularities, and their defining ideals are given by Plücker variables.

Theorem (–)

If an irreducible algebraic set is defined by Plücker variables for $\mathrm{Grass}\,(n-2, n)$ *then it is a positroid variety.*

Question (Current Work)

What about for $\mathrm{Grass}\,(r,n)$ for general r? If irreducible algebraic subsets defined by Plücker variables are positroidal, it will follow that the components of $\mathcal{Y}_{n,r,r} \subset \mathcal{V}\,(\mathfrak{B}_r)$ are normal, Cohen-Macaulay, and have rational singularities.

Proposition

Let $R = K[\wedge^r X] \subset K[X]$ and suppose $P \subset R$ is a prime ideal. Then the $r \times r$ minors in P give a rank r representable matroid.

Question

What are the conditions for two prime ideals in $K[\wedge^r X]$ to minimally cover the homogeneous maximal ideal? When is it possible, if ever, to partition the entries of $\wedge^r X$ so that the respective ideals they generate are prime?

Idea: Use the **circuit** definition of a matroid.

Definition (Circuit Axioms)

A collection $\mathcal{C} \subset 2^E$ is the set of **circuits** of a matroid if and only if

(1) $\emptyset \notin \mathcal{C}$,

(2) if $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$ then $C_1 = C_2$, and

(3) (**Circuit Elimination Axiom**) if $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$, and $e \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.