U3L6

Extrema of Multivariable Functions

Relative Maxima and Minima

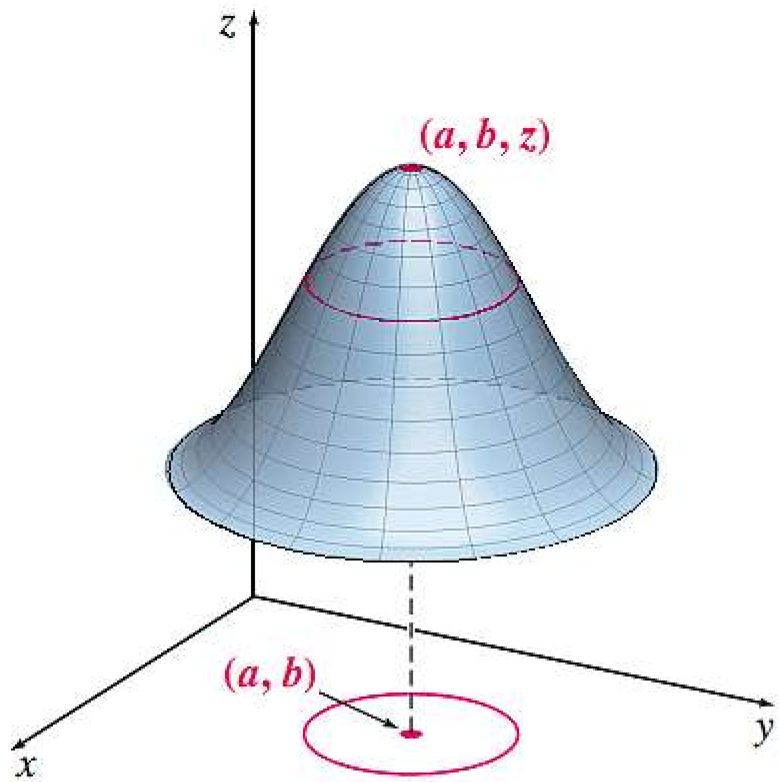
Let (a, b) be the center of a circular region contained in the xy-plane. Then, for a function $z = f(x, y)$ defined for every (x, y) in the region, $f(a, b)$ is a ***relative maximum*** if

$$f(a, b) \geq f(x, y)$$

for all points (x, y) in the circular region, and $f(a, b)$ is a ***relative minimum*** if

$$f(a, b) \leq f(x, y)$$

for all points (x, y) in the circular region.



Relative maximum at (a, b)

Critical Points

For a function $f(x, y)$, the points (a, b) such that

$$f_x(a, b) = 0 \quad and \quad f_y(a, b) = 0$$

are called ***critical points.***

Location of Extrema

Let a function $z = f(x, y)$ have a relative maximum or relative minimum at the point (a, b). Let $f_x(a, b)$ and $f_y(a, b)$ both exist.

Then,

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0$$

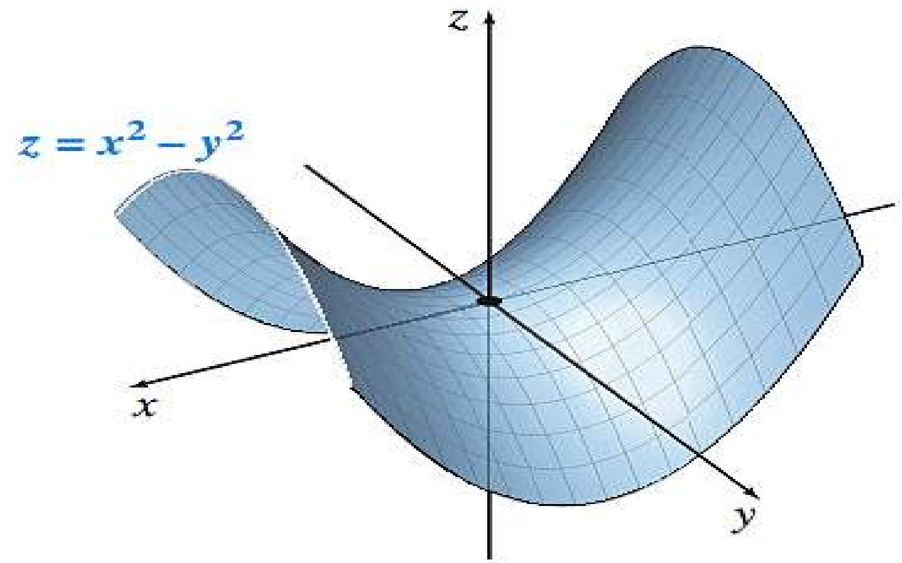
This is to say that relative extrema occur at critical points.

Saddle Points

It is possible to have a situation where

$$f_x(a,b) = 0 \quad and \quad f_y(a,b) = 0,$$

and yet (a, b) does not correspond to a relative maximum or a relative minimum for the function.

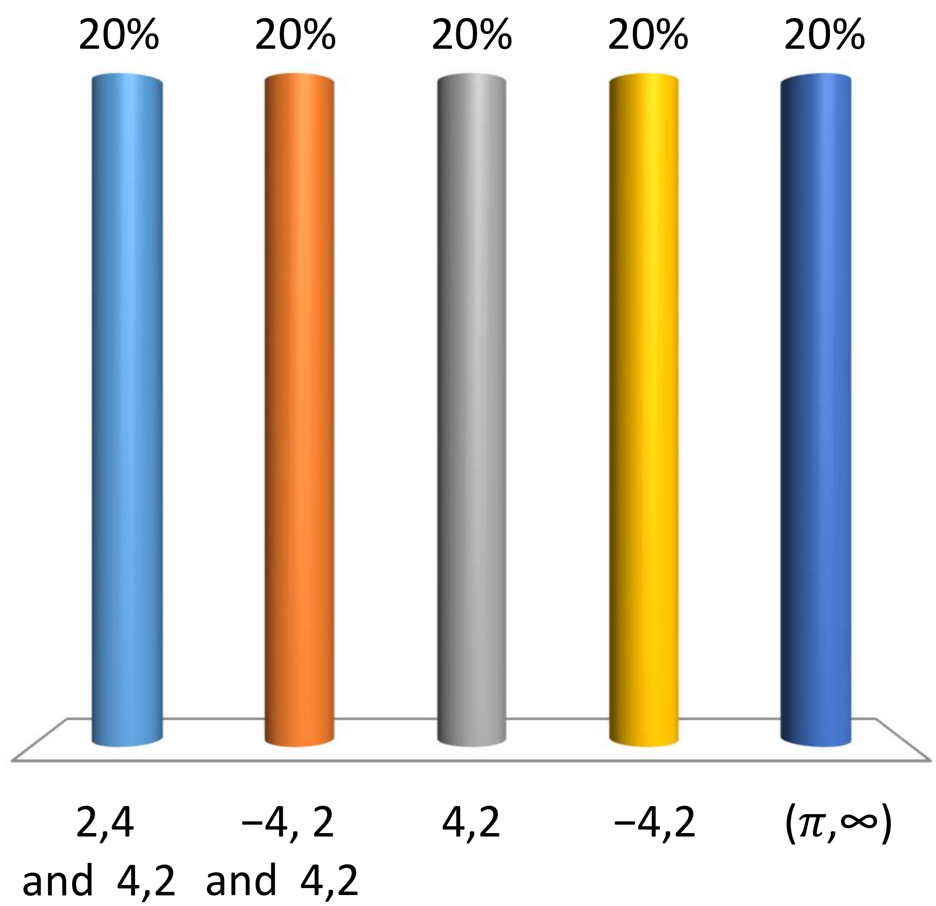


The point (a, b) is a ***saddle point*** if it is a relative maximum from one direction and a relative minimum from the other direction.

Find all critical points for $f(x, y) = 4x^3 + 3xy + 4y^3$.

Find all critical points for
$$f(x,y) = 6x^2 + 6y^2 + 6xy + 36x - 5$$

A. $(2,4)$ and $(4,2)$
B. $(-4,2)$ and $(4,2)$
C. $(4,2)$
D. $(-4,2)$
E. (π,∞)



Test for Relative Extrema

For a function $z = f(x, y)$, let f_{xx}, f_{yy}, and f_{xy} all exist in a circular region contained in the xy-plane with center (a, b). Further, let

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

Define the number D, known as **the discriminant**, by

$$D = f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

Then

a. $f(a, b)$ is a relative maximum if $D > 0$ and $f_{xx}(a, b) < 0$;

b. $f(a, b)$ is a relative minimum if $D > 0$ and $f_{xx}(a, b) > 0$;

c. $f(a, b)$ is a saddle point (neither a maximum nor a minimum) if $D < 0$;

d. if $D = 0$, the test gives no information.

Find all points where the function

$$f(x, y) = 9xy - x^3 - y^3 - 6$$

has any relative maxima or relative minima.

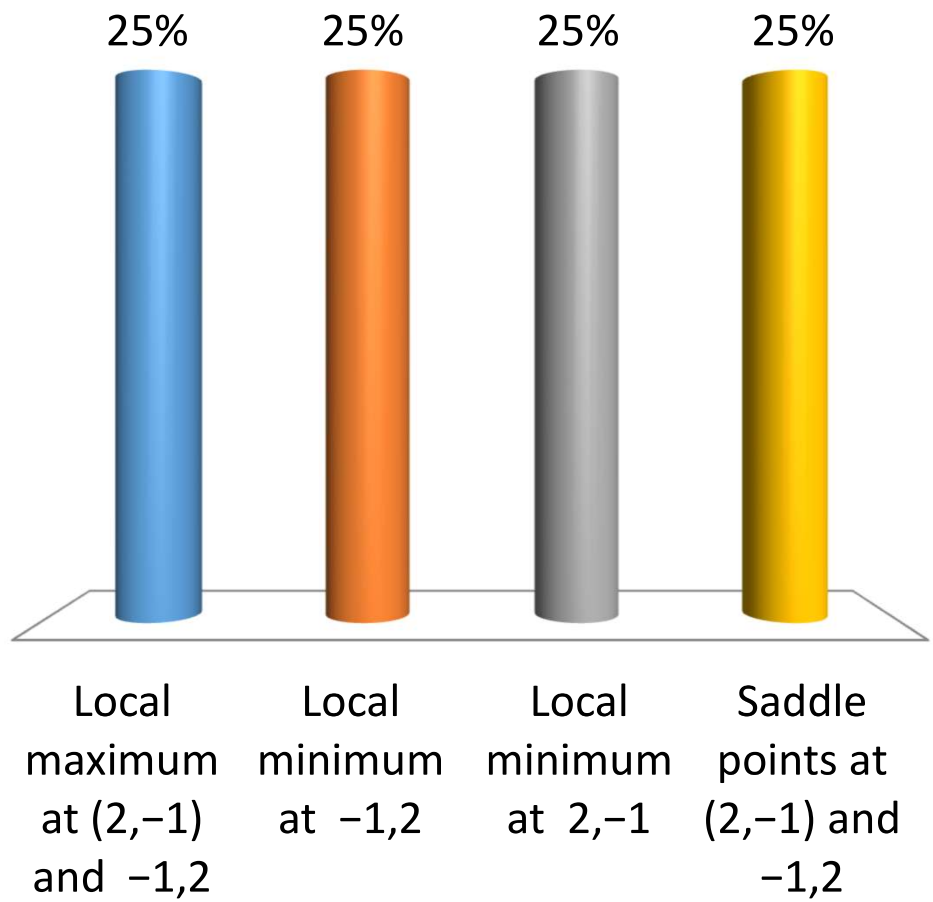
Find all the local maxima, local minima, and saddle points of the given function:

$$f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$$

Find all the local maxima, local minima, and saddle points of the given function:

$$f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$$

A. Local maximum at $(2, -1)$ and $(-1,2)$
B. Local minimum at $(-1,2)$
C. Local minimum at $(2, -1)$
D. Saddle points at $(2, -1)$ and $(-1,2)$



Show that $f(x, y) = 1 - x^4 - y^4$ has a relative maximum, even though D in the theorem is 0.

Suppose the labor cost (in dollars) for manufacturing a camera can be approximated by

$$L(x, y) = \frac{3}{2}x^2 + y^2 - 5x - 6y - 2xy + 120$$

where x is the number of hours required by a skilled craftsperson and y is the number of hours required by a semiskilled person. Find the values of x and y that minimize the labor cost. Find the minimum labor cost.