Quiz 3 Calculus 1
Sections 2.5-2.6 Solutions

1.
$$\lim_{x\to\infty}\frac{4x^5-2x^3}{3x^5-2}=\lim_{x\to\infty}\frac{4x^5-2x^3}{3x^5-2}\cdot\frac{\frac{1}{x^5}}{\frac{1}{x^5}}=\lim_{x\to\infty}\frac{4-\frac{2}{x^2}}{3-\frac{2}{x^5}}=\frac{4}{3}.$$

2.
$$\lim_{y\to-\infty}\frac{y^2-2y+3}{y}=\lim_{y\to-\infty}\frac{y^2-2y+3}{y}\cdot\frac{\frac{1}{y}}{\frac{1}{y}}=\lim_{y\to-\infty}\frac{y-2+\frac{3}{y}}{1}=-\infty$$

3. (Solution 1) Notice that, since $-1\le\sin\theta\le 1$ and $-1\le\cos\theta\le 1$, then
$$-2\le\cos\theta+\sin\theta\le 2.$$
Any other estimate – as long as it is correct and justified – is fine, e.g. $-37\le\cos\theta+\sin\theta\le 42$ or, the sharpest, $-\sqrt{2}\le\cos\theta+\sin\theta\le\sqrt{2}$. We have
$$\lim_{\theta\to\infty}\frac{-2}{\theta^2}=0\quad\text{and}\quad\lim_{\theta\to\infty}\frac{2}{\theta^2}=0,$$
and
$$\frac{-2}{\theta^2}\le\frac{\cos\theta+\sin\theta}{\theta^2}\le\frac{2}{\theta^2}.$$
By the **squeeze theorem**,
$$\lim_{\theta\to\infty}\frac{\cos\theta+\sin\theta}{\theta^2}=0.$$
(Solution 2) We have that $-1\le\sin\theta\le 1$ and $-1\le\cos\theta\le 1$. Since
$$\lim_{\theta\to\infty}\frac{-1}{\theta^2}=0\quad\text{and}\quad\lim_{\theta\to\infty}\frac{1}{\theta^2}=0,$$
and
$$\frac{-1}{\theta^2}\le\frac{\cos\theta}{\theta^2}\le\frac{1}{\theta^2},$$
by the **squeeze theorem**
$$\lim_{\theta\to\infty}\frac{\cos\theta}{\theta^2}=0.$$
Similarly, since
$$\lim_{\theta\to\infty}\frac{-1}{\theta^2}=0\quad\text{and}\quad\lim_{\theta\to\infty}\frac{1}{\theta^2}=0,$$
and
$$\frac{-1}{\theta^2}\le\frac{\sin\theta}{\theta^2}\le\frac{1}{\theta^2},$$
by the **squeeze theorem**
$$\lim_{\theta\to\infty}\frac{\cos\theta}{\theta^2}=0.$$
Since $\lim_{\theta\to\infty}\frac{\cos\theta}{\theta^2}$ and $\lim_{\theta\to\infty}\frac{\sin\theta}{\theta^2}$ **exist**,
$$\lim_{\theta\to\infty}\frac{\cos\theta+\sin\theta}{\theta^2}=\lim_{\theta\to\infty}\frac{\cos\theta}{\theta^2}+\lim_{\theta\to\infty}\frac{\sin\theta}{\theta^2}=0+0=0.$$

4. Since

$$\lim_{z\to\infty} \ln(z) = \infty,$$

then

$$\lim_{z\to\infty} 1 - \ln(z) = -\infty.$$

5. Candidates for **vertical asymptotes**: $2x - 5 = 0$, that is, $x = 5/2$. **Checking that the function is defined around $x = 5/2$**: since

$$(5/2)^2 - 2(5/2) + 3 = 17/4 > 0, \quad \text{(the argument of the square root)}$$

the function is defined around $x = 5/2$.

$$\lim_{x\to\frac{5}{2}^-} \frac{x+1-\sqrt{x^2-2x+3}}{2x-5} = \lim_{x\to\frac{5}{2}^-} \frac{x+1-\sqrt{x^2-2x+3}}{2x-5} \cdot \frac{x+1+\sqrt{x^2-2x+3}}{x+1+\sqrt{x^2-2x+3}} =$$
$$= \lim_{x\to\frac{5}{2}^-} \frac{(x+1)^2-(x^2-2x+3)}{(2x-5)(x+1+\sqrt{x^2-2x+3})} = \lim_{x\to\frac{5}{2}^-} \frac{x^2+2x+1-x^2+2x-3}{(2x-5)(x+1+\sqrt{x^2-2x+3})} =$$
$$= \lim_{x\to\frac{5}{2}^-} \frac{4x-2}{(2x-5)(x+1+\sqrt{x^2-2x+3})}$$

We have:

$$\lim_{x\to\frac{5}{2}^-} (4x-2) = 4\cdot\frac{5}{2} - 2 = 8 > 0, \quad \lim_{x\to\frac{5}{2}^-} (x+1+\sqrt{x^2-2x+3}) = \frac{5}{2} + 1 + \sqrt{\frac{17}{4}} > 0.$$

So,

$$\lim_{x\to\frac{5}{2}^-} \frac{4x-2}{(2x-5)(x+1+\sqrt{x^2-2x+3})} = -\infty.$$

This means that $x = 5/2$ is a vertical asymptote.

Similarly, we could have checked that

$$\lim_{x\to\frac{5}{2}^+} \frac{x+1-\sqrt{x^2-2x+3}}{2x-5} = \infty$$

to see that $x = 5/2$ is a vertical asymptote.

Checking for **horizontal asymptotes**. Since $\lim_{x\to\infty} x^2-2x+x = \infty$, the **function is defined**

as $\boldsymbol{x \to \infty}$, so we can compute the limit.

$$\lim_{x\to\infty} \frac{x+1-\sqrt{x^2-2x+3}}{2x-5} = \ldots = \lim_{x\to\infty} \frac{4x-2}{(2x-5)(x+1+\sqrt{x^2-2x+3})} =$$
$$= \lim_{x\to\infty} \frac{4x-2}{2x^2-3x-5+(2x-5)\sqrt{x^2-2x+3}} =$$
$$= \lim_{x\to\infty} \frac{4x-2}{2x^2-3x-5+(2x-5)\sqrt{x^2(1-\frac{2}{x}+\frac{3}{x^2})}} =$$
$$= \lim_{x\to\infty} \frac{4x-2}{2x^2-3x-5+(2x-5)x\sqrt{1-\frac{2}{x}+\frac{3}{x^2}}} = \quad (\text{since } \sqrt{x^2} = x \text{ when } x > 0)$$
$$= \lim_{x\to\infty} \frac{4x-2}{2x^2-3x-5+(2x-5)x\sqrt{1-\frac{2}{x}+\frac{3}{x^2}}} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} =$$
$$= \lim_{x\to\infty} \frac{\frac{4}{x}-\frac{2}{x^2}}{2-\frac{3}{x}-\frac{5}{x^2}+(2-\frac{5}{x})\sqrt{1-\frac{2}{x}+\frac{3}{x^2}}} = 0,$$

since the numerator goes to 0 while the denominator goes to $4 \neq 0$. So $y = 0$ is the horizontal asymptote as $x \to \infty$.

Since $\lim_{x\to-\infty} x^2 - 2x + x = \infty$, the **function is defined as $\boldsymbol{x \to -\infty}$**, so we can compute the limit.

$$\lim_{x\to-\infty} \frac{x+1-\sqrt{x^2-2x+3}}{2x-5} = \lim_{x\to-\infty} \frac{x+1-\sqrt{x^2(1-\frac{2}{x}+\frac{3}{x^2})}}{2x-5} =$$
$$= \lim_{x\to-\infty} \frac{x+1-(-x)\sqrt{1-\frac{2}{x}+\frac{3}{x^2}}}{2x-5} = \quad (\text{since } \sqrt{x^2} = -x \text{ when } x < 0)$$
$$= \lim_{x\to-\infty} \frac{x+1+x\sqrt{1-\frac{2}{x}+\frac{3}{x^2}}}{2x-5} = \lim_{x\to-\infty} \frac{x+1+x\sqrt{1-\frac{2}{x}+\frac{3}{x^2}}}{2x-5} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} =$$
$$= \lim_{x\to-\infty} \frac{1+\frac{1}{x}+\sqrt{1-\frac{2}{x}+\frac{3}{x^2}}}{2-\frac{5}{x}} = \frac{1+1}{2} = 1.$$

So $y = 1$ is the horizontal asymptote as $x \to -\infty$.

6. - $f(2) = 2^2 - 3$ is defined;
   - $\lim_{x\to 2} x^2 - 3 = 2^2 - 3$ exists since $f(x) = x^2 - 3$ is a polynomial;
   - $f(2) = \lim_{x\to 2} x^2 - 3$.

   So the function $f(x) = x^2 - 3$ continuous at $x = 2$ by the continuity checklist.

7. The function $f(x) = \ln(2-x)$ is not defined at $x = 3$, since $2-3 < 0$, so it cannot be continuous at $x = 3$.

8. (Solution 1) The function
$$f(x) = \frac{x^3 + 2x^2 - 1}{x^2 - 3}$$
is not defined when $x^2 - 3 = 0$, that is, when $x = \pm\sqrt{3}$. On the other hand, if $a \neq \pm\sqrt{3}$,

   - $f(a)$ is defined;
   - $\lim_{x\to a} \frac{x^3+2x^2-1}{x^2-3}$ exists since $\lim_{x\to a} x^2 - 3 \neq 0$ and $f(x)$ is a rational function;
   - since $\lim_{x\to a} x^2 - 3 \neq 0$,
$$\lim_{x\to a} \frac{x^3 + 2x^2 - 1}{x^2 - 3} = \frac{a^3 + 2a^2 - 1}{a^2 - 3} = f(a);$$

   so $f(x)$ is continuous when $x \neq \pm\sqrt{3}$ by the continuity checklist. Thus the intervals of continuity are $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, \sqrt{3})$, $(\sqrt{3}, \infty)$.

   (Solution 2) The function
$$f(x) = \frac{x^3 + 2x^2 - 1}{x^2 - 3}$$
is not defined when $x^2 - 3 = 0$, that is, when $x = \pm\sqrt{3}$. Since $f(x)$ is a rational function, it is continuous on its domain, that is, when $x \neq \pm\sqrt{3}$. Thus the intervals of continuity are $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, \sqrt{3})$, $(\sqrt{3}, \infty)$.

9. Since $2x + a$ is a linear function, it is continuous for all $x < 0$; since $x^2 + 1$ is a polynomial it is continuous for all $0 < x < 2$; since $bx - 2$ is a linear function it is continuous for all $x > 2$. The only possible points of discontinuity are at $x = 0$ and $x = 2$.

   - $f(0)$ is defined;
   - to guarantee that $\lim_{x\to 0} f(x)$ exists, we need $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$ to exists, and $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$.
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2x + a = a,$$
while
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 + 1 = 1.$$
So we need $a = 1$.
   - When $a = 1$, $\lim_{x\to 0} f(x) = 1 = f(0)$.

   - $f(2)$ is defined;
   - to guarantee that $\lim_{x\to 2} f(x)$ exists, we need $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$ to exists, and $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$.
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 + 1 = 5,$$
while
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} bx - 2 = 2b - 2.$$
So we need $5 = 2b - 2$, that is, $b = 3/2$.

- When $b = 3/2$, $\lim_{x\to 2} f(x) = 5 = f(2)$.

10. Since $\frac{x^2-4}{x-2}$ is a rational function, it is continuous at all points where $x - 2 \neq 0$, so the only possible discontinuity is at $x = 2$.

    - $f(2)$ is defined;
    - $$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x + 2 = 4,$$
      so the limit exists;
    - we need $a = f(2) = \lim_{x\to 2} f(x) = 4$, so $a = 4$.