Unit 1, Lesson 3
Continuity and Algebraic Limits

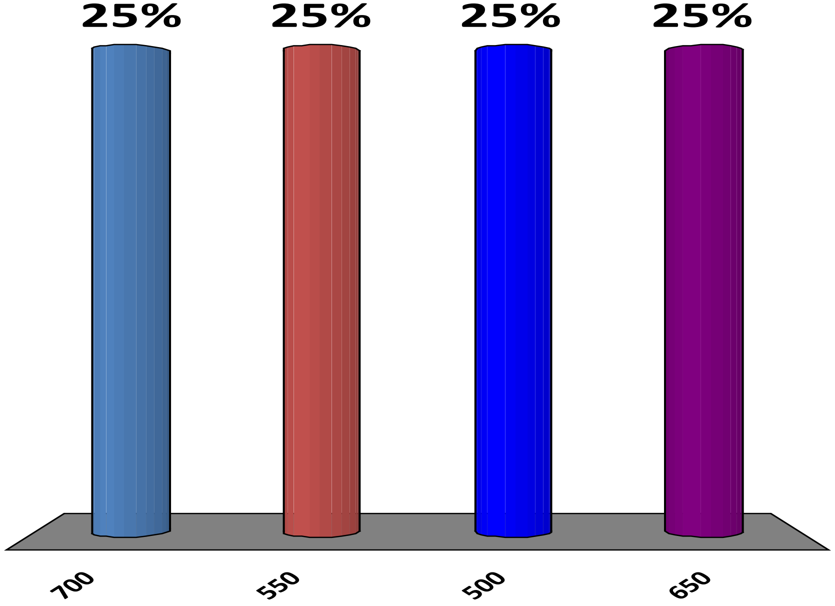
Continuity and Algebraic Limits

OBJECTIVES:

- Use rules of limits
- Evaluate limits algebraically by means of substitution, factoring, and using special limits.
- Use limits to determine whether a function is continuous at a point.

How many lab minutes do you need to earn this semester?

A. 700
B. 550
C. 500
D. 650



Rules for Limits

Let a, A, and B be real numbers, and let f and g be functions such that

$$\lim_{x \to a} f(x) = A \qquad \text{and} \qquad \lim_{x \to a} g(x) = B.$$

1. If k is a constant, then $\lim_{x \to a} k = k$ and $\lim_{x \to a} [k \cdot f(x)] = k \cdot \lim_{x \to a} f(x) = k \cdot A$.

2. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = A \pm B$

(The limit of a sum or difference is the sum or difference of the limits.)

3. $\lim_{x \to a} [f(x) \cdot g(x)] = [\lim_{x \to a} f(x)] \cdot [\lim_{x \to a} g(x)] = A \cdot B$

(The limit of a product is the product of the limits.)

4. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{A}{B}$ if $B \neq 0$

(The limit of a quotient is the quotient of the limits, provided the limit of the denominator is not zero.)

5. If $p(x)$ is a polynomial, then $\lim_{x \to a} p(x) = p(a)$.

6. For any real number k, $\lim_{x \to a} [f(x)]^k = [\lim_{x \to a} f(x)]^k = A^k$, provided this limit exists.*

7. $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ if $f(x) = g(x)$ for all $x \neq a$.

Let $\lim_{x\to 4} f(x) = 9$ and $\lim_{x\to 4} g(x) = 27$. Use the limit rules to find the following:

- $\lim_{x\to 4}[f(x) - g(x)] =$

$$\lim_{x\to 4} f(x) - \lim_{x\to 4} g(x) = 9 - 27 = -18$$

- $\lim_{x\to 4}[5g(x) + 2] =$

$$5 \cdot \lim_{x\to 4} g(x) + 2 = 5(27) + 2 = 137$$

- $\lim_{x\to 4} \sqrt{f(x)} =$

$$\lim_{x\to 4} f(x)^{1/2} = \left[\lim_{x\to 4} f(x)\right]^{1/2} = 9^{1/2} = 3$$

Explain:

Let $p(x)$ and $q(x)$ be polynomials. Explain why the following rules can be used to find $\lim_{x\to\pm\infty}\left(\frac{p(x)}{q(x)}\right)$.

1. If the degree of $p(x)$ is less than the degree of $q(x)$, then the limit is 0.

2. If the degree of $p(x)$ is greater than the degree of $q(x)$, then the limit is ∞ or $-\infty$.

3. If the degree of $p(x)$ is equal to the degree of $q(x)$, then the limit is A/B where A and B are the leading coefficients of $p(x)$ and $q(x)$ respectively.

Let $\lim_{x \to 5} f(x) = 9$ and $\lim_{x \to 5} g(x) = 3$.

Use the limit rules to find

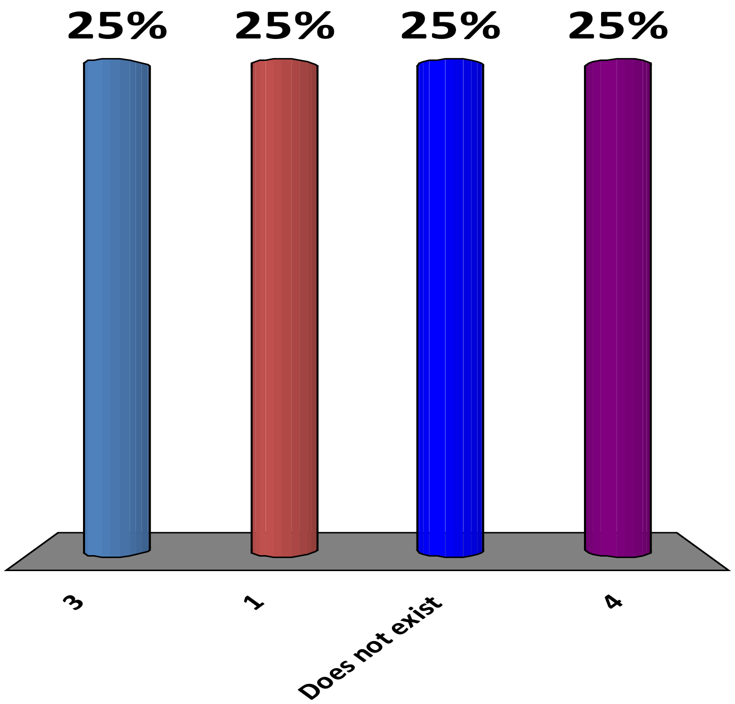
$$\lim_{x \to 5} \frac{f(x) + g(x)}{4g(x)}$$

A. 3

B. 1

C. Does not exist

D. 4



Question:

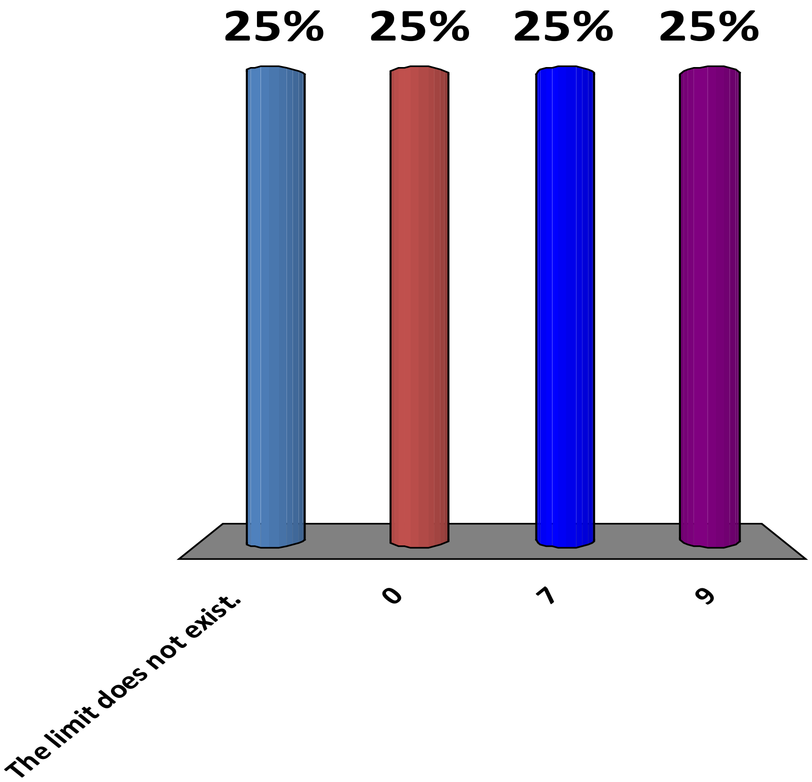
A friend who is curious about limits wonders why you investigate the value of a function closer and closer to a point instead of just finding the value of the function. How would you respond?

Find $\lim_{x \to 2} g(x)$ where $g(x) = \frac{x^3 - 2x^2}{x - 2}$.

Find $\lim_{x \to 5} \frac{x^2 - 3x - 10}{x - 5}$.

$$\lim_{x \to 5} \frac{x^2 - 3x - 10}{x - 5} =$$

A. The limit does not exist.
B. 0
C. 7
D. 9



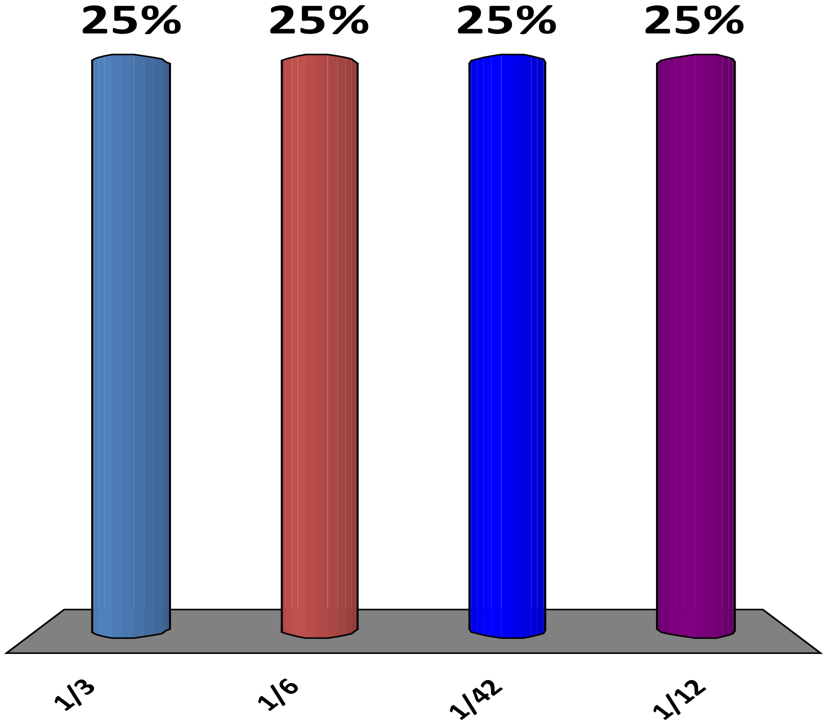
$$\lim_{x \to 36} \frac{\sqrt{x} - 6}{x - 36} =$$

A. 1/3

B. 1/6

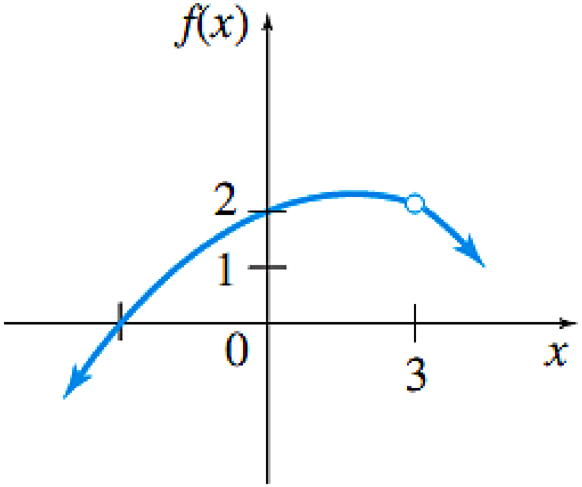
C. 1/42

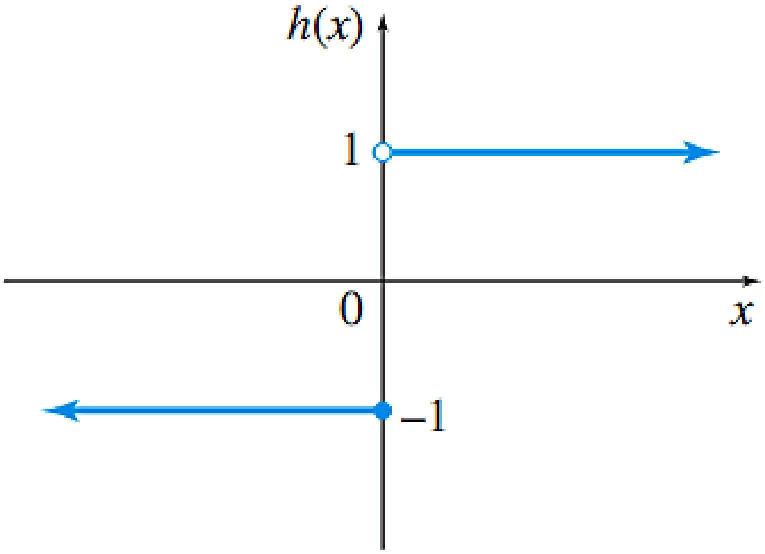
D. 1/12



Question:

How can you tell if a function is continuous?





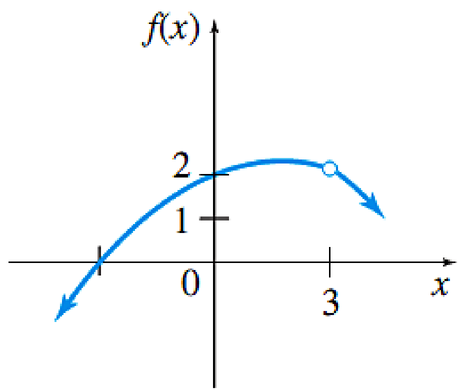
Continuity Checklist:

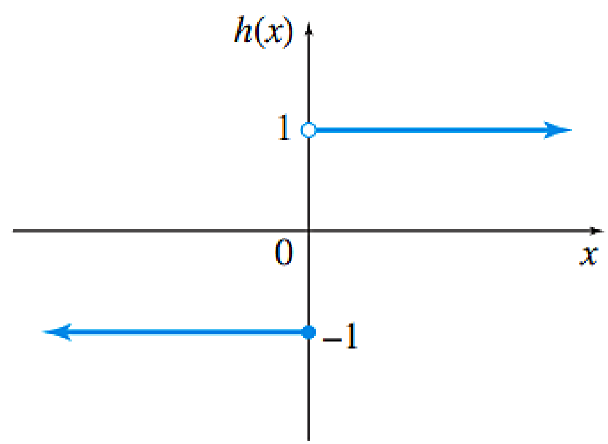
Continuity at $x = c$

A function f is **continuous** at $x = c$ if the following three conditions are satisfied:

1. $f(c)$ is defined,
2. $\lim_{x \to c} f(x)$ exists, and
3. $\lim_{x \to c} f(x) = f(c)$.

If f is not continuous at c, it is **discontinuous** there.





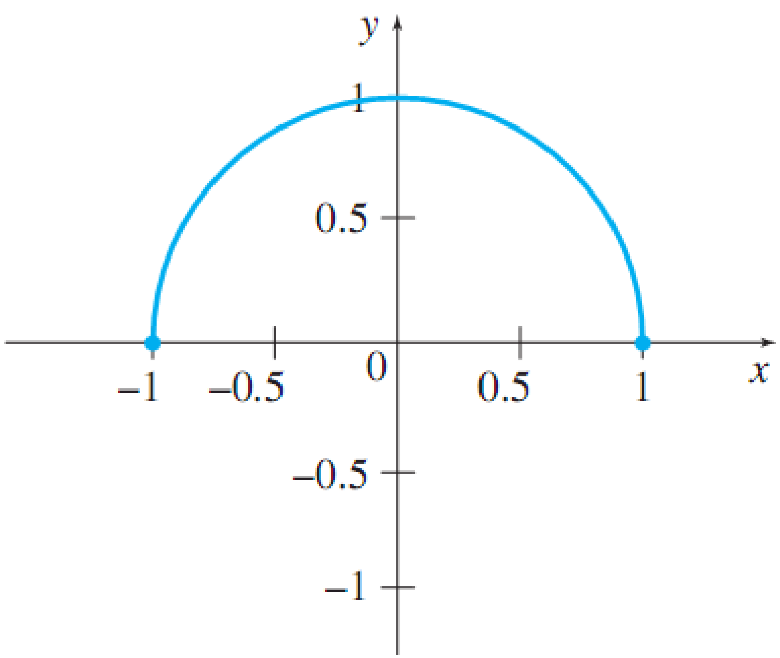
Continuity on a Closed Interval

A function is **continuous on a closed interval** $[a, b]$ if

1. it is continuous on the open interval (a, b),

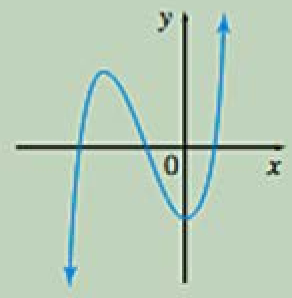
2. it is continuous from the right at $x = a$, and

3. it is continuous from the left at $x = b$.

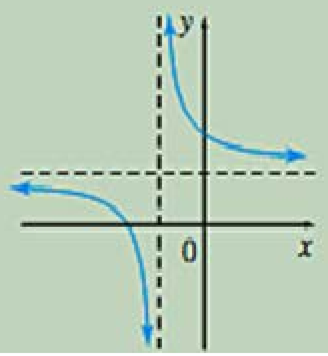


Question:

1. Where is a polynomial function continuous?

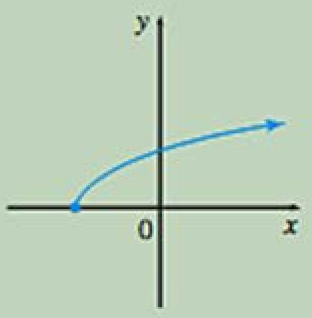
Type of Function	Where It Is Continuous	Graphic Example
Polynomial Function $y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers, not all 0	For all x	

2. Where is a rational function continuous?

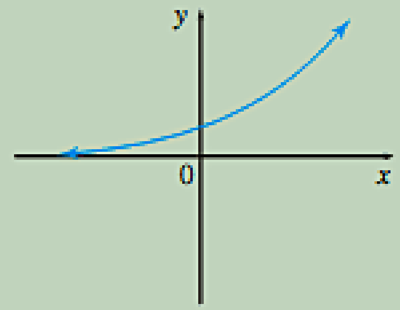
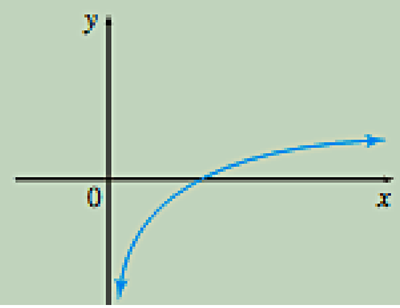
Type of Function	Where It Is Continuous	Graphic Example
Rational Function $y = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, with $q(x) \neq 0$	For all x where $q(x) \neq 0$	

Question:

3. Where is a root function continuous?

Type of Function	Where It Is Continuous	Graphic Example
Root Function $y = \sqrt{ax + b}$, where a and b are real numbers, with $a \neq 0$ and $ax + b \geq 0$	For all x where $ax + b \geq 0$	

Continuous Functions (cont.)

Type of Function	Where It Is Continuous	Graphic Example
Exponential Function $y = a^x$ where $a > 0$	For all x	
Logarithmic Function $y = \log_a x$ where $a > 0$, $a \neq 1$	For all $x > 0$	

True or False:

If $\lim_{x \to c} f(x) = L$ and $f(c) = L$,
then $f(c)$ is continuous at c.

A rational function can have infinitely many x-values at which it is not continuous.

$$f(x) = \sqrt{5x+3}$$

Find all values $x = a$ where the function is discontinuous.

Solution: This root function is discontinuous wherever the radicand is negative.

There is a discontinuity when $5x + 3 < 0$

$$x < -\frac{3}{5}.$$

Find all values of x where the piecewise function is discontinuous.

$$f(x) = \begin{cases} 5x - 4 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 3 \\ x + 6 & \text{if } x > 3 \end{cases}$$

Find the constant a such that the function is continuous on the entire real number line.

$$f(x) = \begin{cases} x^3 & x \le 2 \\ ax^2 & x > 2 \end{cases}$$

A. 2
B. 3
C. 5
D. 1