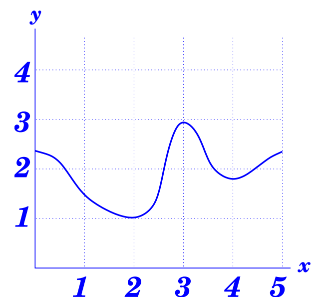
Section 4.2 – Optimization

1. Sketch a continuous, differentiable graph with the following properties:

   - local minima at 2 and 4
   - global minimum at 2
   - local and global maximum at 3
   - no other extrema



2. A warehouse orders and stores boxes. The cost of storing boxes is proportional to q, the quantity ordered. The cost of ordering boxes is proportional to $1/q$, because the warehouse gets a price cut for larger orders. The total cost of operating the warehouse is the sum of ordering costs and storage costs. What value of q gives the minimum cost?

   Let $C = f(q)$ represent the total cost of operating the warehouse as a function of the price, q. Then, from the given information, we have

$$C = f(q) = k_1 q + k_2 \left(\frac{1}{q}\right) = k_1 q + \frac{k_2}{q},$$

   where k_1 and k_2 are positive constants. Rewriting, we see that $f(q) = k_1 q + k_2 q^{-1}$, so $f'(q) = k_1 - k_2 q^{-2}$. To find the critical points of C, we set $f'(q)$ equal to zero and solve for q.

$$\begin{aligned} k_1 - k_2 q^{-2} &= 0 \\ k_1 &= k_2 q^{-2} \\ q &= \pm\sqrt{\frac{k_2}{k_1}} \end{aligned}$$

   Therefore, $q = \sqrt{k_2/k_1}$ is the only relevant critical point. Since $f'(q) < 0$ for all $0 < q < \sqrt{k_2/k_1}$ and $f'(q) > 0$ for all $q > \sqrt{k_2/k_1}$, we see that C is decreasing for $0 < q \le \sqrt{k_2/k_1}$ and increasing for $q > \sqrt{k_2/k_1}$. It follows that $q = \sqrt{k_2/k_1}$ gives the minimum total cost of operating the warehouse.

3. Find the best possible bounds for $f(t) = t + \sin t$ for t between 0 and 2π.

   We begin by noting that $f'(t) = 1 + \cos t$, and we can find the critical points of f by setting $f'(t)$ equal to zero and solving for t.

$$\begin{aligned} 1 + \cos t &= 0 \\ \cos t &= -1 \\ t &= \pi \end{aligned}$$

   Therefore, $t = \pi$ is the only critical point between 0 and 2π. To find the best possible bounds, we compare the values of f at our two endpoints and the critical points (see table to the right). Since 0 is the smallest output value and 2π is the largest output value, we conclude that

t	$f(t)$
0	0
π	π
2π	2π

$$0 \le t + \sin t \le 2\pi$$

   for all t between 0 and 2π, and that these are the best possible bounds.