Mon 10 Nov 2014

- This week: We postpone section 4.8 until after 5.3
- Tues 18 Nov: Quiz on 4.6-4.7
- Tues 25 Nov: Quiz on 5.1-5.2
- Tues 2 Dec: Quiz on 5.3, 4.8
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- FINAL is on Mon 15 Dec

Mean Value Theorem

If *f* is continuous on the closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point *c* in (a, b) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$

Figure 4.68 on pg. 276 provides a nice picture of the Mean Value Thm in action between the points (a, f(a)) and (b, f(b)). The slope of the secant line connecting these points is given by $\frac{f(b)-f(a)}{b-a}$. What the MVT states is that we can find a point *c* on *f* where the tangent line (whose slope is *f'(c)*) is parallel to the secant line (e.g., they have the same slope).

Example of MVT

Suppose you leave Fayetteville for a location in Fort Smith that is 60 miles away.

If it takes you 1 hour to get there, what can we say about your speed?

If it takes you 45 minutes to get there, what can we say about your speed?

Problem

Let $f(x) = x^2 - 4x + 3$.

1. Determine whether the MVT applies to *f(x)* on the interval [-2, 3]. How do you know?
2. If so, find the point(s) that are guaranteed to exist by the MVT.

How many points c satisfy the conclusion of the Mean Value Theorem for the function $f(x) = x^3$ on the interval from [-1, 1]?

A. 0

B. 1

C. 2

D. 3

Theorems Resulting from the MVT

The implications of the Mean Value Theorem provide several critical results and theorems:

Theorem: If *f* is differentiable and *f'(x)=0* at all points of an interval *I*, then *f* is a constant function on *I*.

This result is the converse of the idea generated back in chapter 3 that a constant function has a derivative of 0. Note the connection to the MVT:

$$\frac{f(b)-f(a)}{b-a} = f'(c) = 0 \Rightarrow f(b)-f(a) = 0 \Rightarrow f(a) = f(b)$$

Theorems Resulting from the MVT

Theorem: If two functions have the property that

$f'(x) = g'(x)$ for all *x* of an interval *I*, then $f(x) - g(x) = C$ on *I*, where *C* is a constant.

Note in this example, C could be 0, implying $f(x)=g(x)$ for all x in *I*.

Theorems Resulting from the MVT

Theorem: Suppose *f* is continuous on an interval *I* and differentiable at all interior points of *I*.

If *f'(x) > 0* at all interior points of *I*, then *f* is increasing on *I.*

If *f'(x) < 0* at all interior points of *I*, then *f* is decreasing on *I*.

The MVT provides a proof to this result stated in section 4.2. Note that

$$\frac{f(b)-f(a)}{b-a}=f'(c) \Rightarrow f(b)-f(a)=f'(c)(b-a)$$

Homework from Section 4.6

Do problems 7, 10, 11, 13, 15, 17, 20-22, 24-26, 29 (pgs. 279-280).

L' Hopital's Rule

In Chapter 2, we examined limits that were computed using analytical techniques. Some of these limits, in particular those that were indeterminate, could not be computed with simple analytical methods (e.g, substitution).

For example, $\lim_{x \to 0} \frac{\sin x}{x}$ and $\lim_{x \to 0} \frac{1 - \cos x}{x}$ are both limits that can't be computed by substitution, as placing 0 in for x yields 0/0.

L' Hopital's Rule (0/0)

Suppose *f* and *g* are differentiable on an open interval *I* containing *a* with *g'(x) ≠ 0* on *I* when x ≠ a. If

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is ±∞). The rule also applies if $x \to a$ **is replaced by** $x \to \pm\infty$ **,**

$x \to a^+$ **or** $x \to a^-$

Exercise

Evaluate the following limit: $\lim_{x \to -1} \frac{x^4 + x^3 + 2x + 2}{x + 1}$

Exercise

Evaluate the following limit: $\lim_{x \to -1} \frac{x^4 + x^3 + 2x + 2}{x + 1}$

By direct substitution, we obtain 0/0. So we must apply L'Hopital's rule to evaluate the limit:

$$\lim_{x \to -1} \frac{x^4 + x^3 + 2x + 2}{x + 1} = \lim_{x \to -1} \frac{4x^3 + 3x^2 + 2}{1} = -4 + 3 + 2 = 1$$

So $\lim_{x \to -1} \frac{x^4 + x^3 + 2x + 2}{x + 1} = 1$

Wed 12 Nov 2014

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Pop Quiz!

Find the dimensions of a closed box with a square base that has the following characteristics:

1. The minimum surface area, given that the volume is 12 cubic meters.
2. The maximum volume given that the surface area is 20 square meters.

L' Hopital's Rule (∞/∞)

Suppose *f* and *g* are differentiable on an open interval *I* containing *a* with *g'(x)* ≠ *0* on *I* when x ≠ a. If

$$\lim_{x \to a} f(x) = \pm\infty \quad \textbf{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is ±∞). The rule also applies if $x \to a$ **is replaced by** $x \to \pm\infty$ **,**

$x \to a^+$ **or** $x \to a^-$

Problems

Evaluate the following limits using L'Hopital's Rule:

$$\lim_{x \to \infty} \frac{4x^3 - 2x^2 + 6}{\pi x^3 + 4}$$

$$\lim_{x \to 0} \frac{\tan 4x}{\tan 7x}$$

Other Indeterminate Forms

Other indeterminate limits in the form *0 × ∞* or ∞ – ∞ cannot be evaluated directly using L'Hopital's Rule.

For *0 × ∞* cases, we must rewrite the limit in the form of 0/0 or ∞/∞. A common technique is to divide by the reciprocal:

Example: $$\lim_{x \to \infty} x^2\left(1 - \cos\left(\frac{1}{x^2}\right)\right) = \lim_{x \to \infty} \frac{1 - \cos\left(\frac{1}{x^2}\right)}{\frac{1}{x^2}}$$

Other Indeterminate Forms

For ∞ – ∞ , we can divide by the reciprocal as well as use a change of variables:

Example: $$\lim_{x \to \infty} x - \sqrt{x^2+1} = \lim_{x \to \infty} x - \sqrt{x^2(1+1/x^2)}$$

$$= \lim_{x \to \infty} x\left(1 - \sqrt{(1+1/x^2)}\right) = \lim_{x \to \infty} \frac{1 - \sqrt{(1+1/x^2)}}{\frac{1}{x}}$$

This is now in the form 0/0, so we can apply L'Hopital's Rule and evaluate the limit. However, it may help to change variables. Let t = 1/x. So

$$\lim_{x \to \infty} \frac{1 - \sqrt{(1+1/x^2)}}{\frac{1}{x}} = \lim_{t \to 0^+} \frac{1 - \sqrt{1+t^2}}{t}$$

Which of the following limits is equivalent to $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$?

A. $\lim_{y \to 0^+} y \sin y$

B. $\lim_{y \to 0^+} \frac{\sin y}{y}$

C. $\lim_{y \to \infty} \frac{\sin y}{y}$

D. $\lim_{y \to 0^+} y \sin\left(\frac{1}{y}\right)$

Other Indeterminate Forms

Limits in the form 1^{∞}, 0^{0}, and ∞^{0} are also considered indeterminate forms, and to use L'Hopital's Rule, we must rewrite in the form 0/0 or ∞/∞. Here's how:

Assume $\lim_{x \to a} f(x)^{g(x)}$ has the indeterminate form 1^{∞}, 0^{0}, or ∞^{0}.

1. Evaluate $L = \lim_{x \to a} g(x) \ln f(x)$. This limit can often be put in the form 0/0 or ∞/∞, which can be handled by L'Hopital's Rule.
2. Then $\lim_{x \to a} f(x)^{g(x)} = e^{L}$

Example

$\lim_{x \to 0^+} x^x$. This is in the form 0^0, so we need to examine

$$L = \lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0^+} (-x) = 0$$

So $\lim_{x \to 0^+} x^x = e^L = e^0 = 1$

Examining Growth Rates

We can use L'Hopital's Rule to examine the rate at which functions grow in comparison to one another.

Suppose *f* and *g* are functions with $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty$.

Then ***f* grows faster than *g*** as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = 0 \quad \text{or} \quad \lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$$

The functions *f* and *g* have **comparable growth rates** if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = M$$, where $0 < M < \infty$.

Note: f << g means that g grows faster than f as $x \to \infty$

Warnings of Using L'Hopital's Rule

1. L'Hopital's Rule says $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, not

   $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \left[\frac{f(x)}{g(x)} \right]'$ or $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \left[\frac{1}{g(x)} \right]' f'(x)$

   (i.e., don't confuse this rule with the Quotient Rule).
2. Be sure the limit with which you are working is in the form 0/0 or ∞/∞.
3. When using L'Hopital's Rule more than once, simplify as much as possible before repeating rule.
4. If you continue to use L'Hopital's Rule in an unending cycle, another method must be used.

Homework from Section 4.7

Do problems 13-39 odd, 43-51 odd, 63-69 odd (pgs. 290-291).

Fri 14 Nov 2014

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Approximating Areas Under Curves

In the previous 2 chapters, we have come to see the derivative of a function associated with the rate of change of a function as well as the slope of the tangent line to the curve.

In the first section of Chapter 5, we now examine the meaning of the integral.

Question: If we know the velocity function of a particular object, what does that tell us about its position function?

Approximating Areas Under Curves

Example: Suppose you ride your bike at a constant velocity of 8 miles per hour for 1.5 hours.

What is the velocity function that models this scenario?

What does the graph of the velocity function look like?

What is the position function for this scenario?

Where is the displacement (e.g, the distance you've traveled) represented when looking at the graph of the velocity function?

Approximating Areas Under Curves

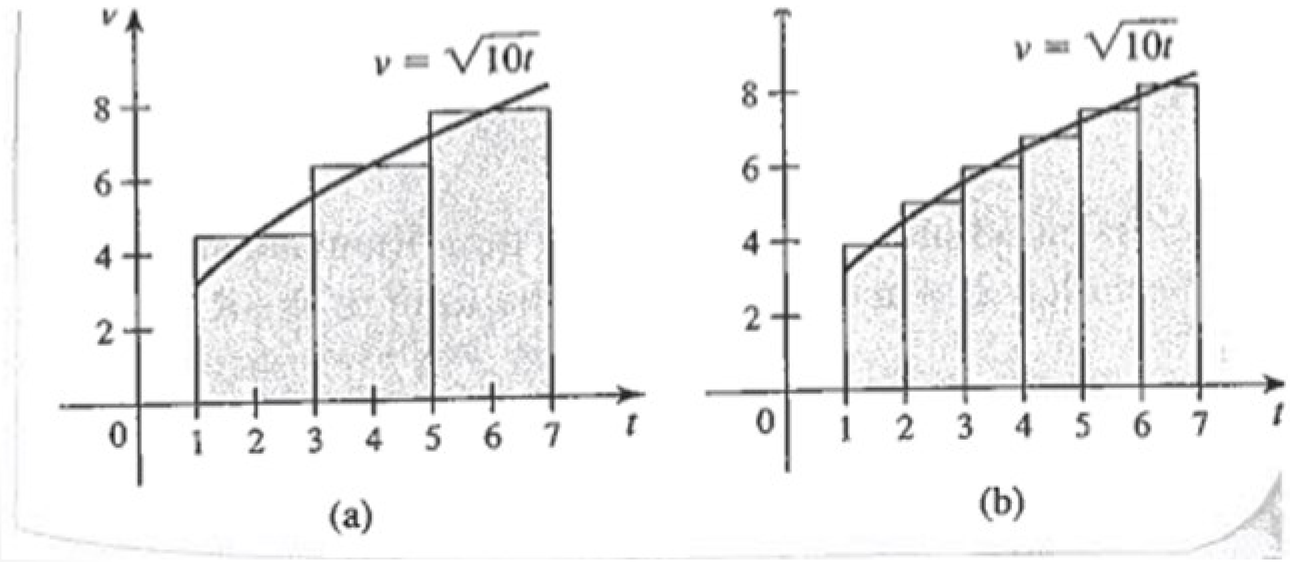
In the previous example, the velocity was constant. In most cases, this is not accurate (or possible). How could we find displacement when the velocity is changing over an interval?

One strategy is to divide the time interval into a particular number of subintervals and approximate the velocity on each subinterval with a constant velocity. Then for each subinterval, the displacement can be calculated and summed.

Note: This provides us with only an approximation, but with a larger number of subintervals, the approximation becomes more accurate.

Approximating Areas Under Curves

Example: Suppose the velocity of an object moving along a line is given by $v(t) = \sqrt{10t}$ on the interval $1 \le t \le 7$. Divide the time interval into $n = 3$ subintervals, assuming the object moves at a constant velocity equal to the value of v evaluated at the midpoint of the subinterval. Estimate the displacement of object on [1,7]. Repeat for $n = 6$ subintervals.



(a)
(b)

Definition of Riemann sum

The more subintervals you divide your time interval into, the more accurate your approximation of displacement will be (see Example 1 on pg. 307-308).

We now examine a method for approximating areas under curves.

Suppose you are studying a function *f* over the interval [a, b]. Suppose we divide [a, b] into *n* subintervals of equal length:

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

with $x_0 = a$ and $x_n = b$. The length of each subinterval is denoted

$$\Delta x = \frac{b-a}{n}$$

Definition of Riemann sum

In each subinterval $[x_{k-1}, x_k]$, we can choose any point $\overline{x_k}$ and create a rectangle with a height of $f(\overline{x_k})$. The area of the rectangle then is height x base, or $f(\overline{x_k})\Delta x$.

Doing this for each subinterval, and then summing each rectangle's area, produces an approximation of the overall area. This approximation is called a **Riemann sum**

$$R = f(\overline{x_1})\Delta x + f(\overline{x_2})\Delta x + \ldots + f(\overline{x_n})\Delta x$$

Definition of Riemann sum

Note: $\overline{x_k}$ is typically chosen to be consistent across all subintervals. The most common are the left Riemann sum, right Riemann sum, and the midpoint Riemann sum.*

For the Riemann sum $R = f(\overline{x_1})\Delta x + f(\overline{x_2})\Delta x + ... + f(\overline{x_n})\Delta x$:

1. R is a left Riemann sum if $\overline{x_k}$ is the left endpoint of each subinterval $[x_{k-1}, x_k]$.
2. R is a right Riemann sum if $\overline{x_k}$ is the right endpoint of each subinterval $[x_{k-1}, x_k]$.
3. R is a midpoint Riemann sum if $\overline{x_k}$ is the midpoint of each subinterval $[x_{k-1}, x_k]$.

*See Figures on pgs. 310-312 for picture of these sums.

Sigma Notation

Because Riemann sums can have a large number of intervals over which to sum areas, it is common to use sigma notation to assist in computing the sums.

Recall how sigma notation works: $\sum_{n=1}^{5} n^2$ means to sum all integer values from the lowest limit (n = 1) to the highest limit (n = 5) in the summand n^2. So

$$\sum_{n=1}^{5} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

Exercise: Evaluate $\sum_{k=0}^{3} 2k-1$

Sigma Notation

Common sums of positive integers. Let *n* be a positive integer.

Sum of a constant c: $\sum_{n=1}^{n} c = cn$

Sum of the first *n* integers: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Sum of squares of the first *n* integers: $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Sum of cubes of the first *n* integers: $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

Riemann Sums Using Sigma Notation

Now using sigma notation, we can write the Riemann sum in a much more compact form:

$$R = f(\overline{x_1})\Delta x + f(\overline{x_2})\Delta x + \ldots + f(\overline{x_n})\Delta x = \sum_{k=1}^{n} f(\overline{x_k})\Delta x$$

To write the left, right, and midpoint Riemann sums in sigma notation, we need to know the point $\overline{x_k}$.

Left, Right and Midpoint Riemann Sums in Sigma Notation

Suppose *f* is defined on a closed interval [a, b], which is divided into *n* subintervals of equal length Δx. If $\overline{x_k}$ is a point in the *k*th subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then the **Riemann sum** of *f* on [a, b] is $\sum_{k=1}^{n} f(\overline{x_k})\Delta x$.

1. $\sum_{k=1}^{n} f(\overline{x_k})\Delta x$ is a left Riemann sum if $\overline{x_k} = a + (k-1)\Delta x$.
2. $\sum_{k=1}^{n} f(\overline{x_k})\Delta x$ is a right Riemann sum if $\overline{x_k} = a + k\Delta x$.
3. $\sum_{k=1}^{n} f(\overline{x_k})\Delta x$ is a midpoint Riemann sum if $\overline{x_k} = a + \left(k - \frac{1}{2}\right)\Delta x$

Exercise

Use sigma notation to write the left, right and midpoint Riemann sums for the function $f(x) = x^2$ on the interval [1, 5] given that $n = 4$.

Based on these approximations, estimate the area bounded by the graph of *f(x)* over [1, 5].

Homework from Section 5.1

Do problems 9, 11, 15-23 odd, 31, 33, 53-57 all (pgs. 315-320).