Topology QR Solutions – 3 Jan 2009

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Morning Session

1. Let X be a compact metric space, and let Y be a Hausdorff space. Suppose that there is a continuous mapping f of X onto Y. Show that Y has a countable basis.

   Solution. X is a metric space, so has a countable basis of $\frac{1}{n}$-balls. Choose $y \in Y$ and a neighborhood N of y. X is compact so $f^{-1}(y)$ is finite. Choose n such that the $\frac{1}{n}$-balls around points in $f^{-1}(y)$ are all contained in $f^{-1}(N)$. Now since Y is Hausdorff, each of these $\frac{1}{n}$-balls maps homeomorphically to some neighborhood around y; each must be contained in N. Let the images of these $\frac{1}{n}$-balls define a countable basis for y. Conclude Y has a countable basis since the choice of y was arbitrary. $\square$

2. Let X be a compact Hausdorff space with the property that each of its points is a G_δ set, i.e. is the intersection of countably many open subsets. Prove that X is "first countable", i.e. has a countable basis at each of its points.

   Solution. Choose $x \in X$, so $x = \cap_{n=1}^{\infty} U_n$, and let U be a neighborhood of x. Put $V_n := \cap_{i=1}^{n} U_i$. Then

   $$x = \bigcap_{n=1}^{\infty} V_n = \bigcap_{n=1}^{\infty} V_n \cap U.$$

   Assume there does not exist n such that $V_n \subseteq U$. Choose $x_n \in V_n \setminus U$. Since X is compact Hausdorff, the sequence $\{x_n\}_n$ contains a subsequence which converges uniquely to x. But U is a neighborhood of x means it must contain one of the sequence elements, a contradiction.

*with additional input from M. Hochster, G.P. Scott, and others from the U of M Mathematics Department

So since there does exist n such that $V_n \subseteq U$, define a countable basis by such sets. $\square$

3. For all $a \in \mathbb{R}$ let Z_a be the subset of those $(x, y, z) \in \mathbb{R}^3$ with $(x^2+y^2)z = a$.
    - Determine for which a is Z_a a topological manifold.
    - Determine for which a is Z_a connected.
    - Determine for which a is Z_a simply connected.
    - Determine for which a is Z_a contractible.

    Solution. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $(x, y, z) \mapsto (x^2+y^2)z$. Then the Jacobian
    $$df_{(x,y,z)} = \begin{pmatrix} 2xz & 2yz & x^2 + y^2 \end{pmatrix}$$
    has rank 1 for all (x, y, z) except when $x = y = 0$. Thus Z_a is a manifold of dimension 2 when $a \neq 0$. On the other hand, $Z_0 = f^{-1}(0)$ is the union of the xy-plane and the z-axis, so is not a manifold. However, Z_0 is connected, simply connected, and contractible. For $a \neq 0$, Z_a is homeomorphic to a cylinder so is connected, but not simply connected nor contractible. $\square$

4. Identify the 2-dimensional sphere $\mathbb{S}^2$ with the subset of $\mathbb{R}^3$ consisting of vectors of length 1 and for $x, y \in \mathbb{S}^2$ let $d(x, y)$ be their distance in $\mathbb{R}^3$. Prove that if X is any topological space and $f, g : X \to \mathbb{S}^2$ are such that $d(f(x), g(x)) < 2$ for all $x \in X$, then f and g are homotopic to each other.

    Solution. Define $F : X \times [0, 1] \to \mathbb{S}^2$ by
    $$(x, t) \mapsto \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}.$$
    Then F is continuous when $f(x), g(x)$ are not antipodal. But the hypothesis gives $d((f(x), g(x)) < 2$ for all $x \in X$ and antipodal points are distance 2 apart. Therefore F is continuous, and defines a homotopy from f to g. $\square$

5. Let M be a simply connected manifold. Prove that every continuous map $f : M \to \mathbb{S}^1 \times \mathbb{S}^1$ is homotopic to a constant map.

    Solution. Since M is simply connected, it is homotopy equivalent to a point. Now f composed with this homotopy factors through a point which maps constantly to $\mathbb{S}^1 \times \mathbb{S}^1$. $\square$

Afernoon Session

1. Let X be a compact Hausdorff space, and let $\cdots \subset X_n \subset X_{n-1} \subset \cdots \subset X_2 \subset X_1$ be a nested sequence of closed, nonempty connected subsets of X. Prove that $\cap_{i=1}^{\infty} X_1$ is nonempty and connected.

   Solution. Put $Y := \cap_{i=1}^{\infty}$. The sequence is nested and X_i is connected for all i, so Y is connected. The sequence also has the finite intersection property. So, since X is compact Y must be nonempty. $\square$

2. Let N and M be compact manifolds without boundary and of the same dimension. Prove that any immersion $f : N \to M$ is a covering map.

   Solution. If $f : N \to M$ is an immersion then $df_x : T_xN \to T_{f(x)}M$ is injective for all $x \in N$. $T_xN, T_{f(x)}$ are vector spaces of the same dimension so df_x must also be subjective, i.e., a submersion. Then all points of M are regular and N is a local diffeomorphism. For $y \in M$, $f^{-1}(y)$ is finite, by compactness. There exists a neighborhood V around y whose preimage in N is a disjoint union of neighborhoods around each point of $f^{-1}(y)$, each of which maps diffeomorphically to V. Hence f is a covering map. $\square$

3. Let M be a connected manifold with $\pi_1(M)$ finite with odd order. Prove that there is no degree 2 cover $\pi : N \to M$ with N connected. Deduce that N is orientable.

   Solution. Let $\pi : N \to M$ be a cover of degree 2. Then the index of $\pi_*(\pi_1(N)$ in $\pi_1(M)$ is 2, contradicting the odd order for $\pi_1(M)$ when N is connected. Conclude N has 2 components, each of which is homeomorphic to M. Then M orientable implies N orientable. $\square$

4. For $g = 0, 1, 2, \ldots$ let Σ_g be the compact orientable surface of genus g with empty boundary. For which g is there a non-trivial covering map $f : \Sigma_g \to \Sigma_g$?

   Solution. If a covering map has finite degree d, then the Euler characteristic gets divided by d. In this case, Σ_g is compact, so any cover g is finite. If the degree is d then the relationship between Euler characteristics is

$$d\chi(\Sigma_g) = \chi(\Sigma_g).$$

This implies $\chi(\Sigma_g) = 0$. By classification of surfaces Σ_g must be a torus, so $g = 1$. $\square$

5. Let X and Y be two compact manifolds, $A \subset X$ and $B \subset Y$ two closed connected submaifolds, and $f : A \to B$ a homeomorphism. Denote by $Z = X \cup_f Y$ the space obtained by identifying $x \in A$ with $f(x) \in B$. Compute the Euler-characteristic $\chi(Z)$ of Z in terms of $\chi(X), \chi(Y)$ and $\chi(A)$.

   Solution. Representing each space as a CW-complex, the subcomplex A gets identified bijectively with B. So $\chi(X) + \chi(Y)$ counts $\chi(A)$ twice. Therefore
$$\chi(Z) = \chi(X) + \chi(Y) - \chi(A).$$
   $\square$