Fundamental Theorem of Finitely Generated Abelian Groups

Presentation of a group

Definition 1

Suppose $\varphi : \mathbb{Z}^n \to \mathbb{Z}^m$ *is a homomorphism.*

(a) *The* **cokernel** *of* φ *is the quotient group* $\operatorname{coker}(\varphi) := \mathbb{Z}^m / \operatorname{image}(\varphi)$.

(b) *Suppose* φ *is given by an* $m \times n$ *matrix* A*. Write* $A\mathbb{Z}^n < \mathbb{Z}^m$ *to denote* $\operatorname{image}(\varphi)$*. Any isomorphism*

$$\psi : \mathbb{Z}^m / A\mathbb{Z}^n \xrightarrow{\cong} G$$

is called a **presentation** *of a finitely generated abelian group* G*, and* A *is called a* **presentation matrix** *for* G*.*

Exercise 1

Let $\varphi : G \to H$ denote a group homomorphism. Prove the following:

(a) φ is injective if and only if $\ker \varphi = \{1_G\}$.

(b) φ is surjective if and only if $\operatorname{coker} \varphi = \{1_H\}$.

Example 1

Suppose $\varphi : \mathbb{Z}^2 \to \mathbb{Z}^2$ is a group homomorphism given by the matrix

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}.$$

We wish to find G such that there is an isomorphism

$$\psi : \mathbb{Z}^2 / A\mathbb{Z}^2 \xrightarrow{\cong} G.$$

Looking at the entire composition

$$\Psi : \mathbb{Z}^2 \overset{\bar{\varphi}}{\twoheadrightarrow} \mathbb{Z}^2 / A\mathbb{Z}^2 \xrightarrow{\psi} G, \tag{1.1}$$

where $\bar{\varphi}$ denotes the composition of φ with the natural map $\mathbb{Z}^2 \to \mathbb{Z}^2 / \ker(\varphi) = \mathbb{Z}^2 / A\mathbb{Z}^2$; should such an isomorphism ψ exist then put $g_1 := \Psi(\mathbf{e}_1)$ and $g_2 := \Psi(\mathbf{e}_2)$.

By construction, we must have $\ker\Psi = A\mathbb{Z}^2$. In other words,

$$A\mathbf{e}_1 = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \mapsto 0 \quad \text{and}$$
$$A\mathbf{e}_2 = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \mapsto 0.$$

To stay consistent with the definition of a group homomorphism, this means

$$\begin{aligned} &\Psi(2\mathbf{e}_1) + \Psi(\mathbf{e}_2) = 0 = \Psi(-1\mathbf{e}_1) + \Psi(2\mathbf{e}_2) \\ &\quad \implies 2g_1 + g_2 = 0 = -g_1 + 2g_2. \end{aligned}$$

We get a system of 2 equations with 2 unknowns; solving it,

$$\begin{cases} \quad 2g_1 + g_2 = 0 & \implies \boxed{g_2 = -2g_1} \\ -g_1 + 2g_2 = 0 & \end{cases} \tag{1.2}$$

$$\begin{aligned} \implies -g_1 + 2g_2 &= -g_1 + 2(-2g_1) = 0 \\ &= -5g_1 = \boxed{0 = 5g_1.} \end{aligned} \tag{1.3}$$

Remember, working over $\mathbb{Z}$ (versus $\mathbb{R}$), we are only allowed to divide by ± 1. The the boxed equations in (1.2), (1.3) cannot be simplified any further.

G is generated by

$$\langle g_1, g_2 \rangle = \langle g_1, -2g_1 \rangle = \langle g_1 \rangle = G,$$

i.e., G is cyclic with generator g_1. The condition $5g_1 = 0$ allows a direct isomorphism

$$\begin{aligned} G &\xrightarrow{\cong} \mathbb{Z}/5\mathbb{Z} \\ g_1 &\mapsto 1. \end{aligned}$$

Example 2

A "better" presentation for $\mathbb{Z}/5Z$ is given by $A = 5$ (a 1×1 integer matrix is just an integer). Left multiplication by A is a homomorphism

$$\begin{aligned} \varphi : \mathbb{Z}^1 &\to \mathbb{Z}^1 \\ n &\mapsto 5n \end{aligned}$$

whose cokernel is $\mathbb{Z}^1/\operatorname{image}(\varphi) = \mathbb{Z}/5\mathbb{Z}$.

Row and column operations

The key step in Example 1 was solving the system of equations

$$\begin{cases} 2g_1 + g_2 = 0 \\ -g_1 + 2g_2 = 0 \end{cases}$$

or, equivalently, performing the following row and column operations

$$\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \xrightarrow{C_1 \mapsto C_1 - 2C_2} \begin{pmatrix} 0 & 1 \\ -5 & 2 \end{pmatrix} \xrightarrow{R_2 \mapsto R_2 - 2R_1} \begin{pmatrix} 0 & 1 \\ -5 & 0 \end{pmatrix} \xrightarrow{R_2 \mapsto -R_2} \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}$$

where C_i (resp. R_i) denotes the ith column (resp. row). Then of course, if we want to, we can interchange columns to standardize the process.

$$\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

The point is, killing the subgroup $A\mathbb{Z}^2 < \mathbb{Z}^2$ is equivalent to killing each of the subgroups $\mathbb{Z} < \mathbb{Z}$ and $5\mathbb{Z} < \mathbb{Z}$, and then taking the direct product:

$$\mathbb{Z}^2/A\mathbb{Z}^2 \cong \mathbb{Z}/\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \cong \mathbb{Z}/5\mathbb{Z}.$$

Proposition 1

Suppose A is an $m \times n$ presentation matrix for a finitely generated abelian group G. Then any of the following operations will result in a presentation matrix for G:

(i) *add an integer multiple of one column (resp. row) to another;*

(ii) *interchange two columns (resp. rows);*

(iii) *multiply a column (resp. row) by ± 1;*

(iv) *delete a column of zeros;*

(v) *delete the ith row and jth column,* **provided** *the jth column is $\mathbf{e}_i$.*

Proof.

Write $\psi : \mathbb{Z}^m / A\mathbb{Z}^n \xrightarrow{\cong} G$ and let the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ denote the respectively indexed columns of A. Since $\ker \psi = A\mathbb{Z}^n$, its generators are $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Operations (i)-(iii) are permissible by Theorem **??**.

We first prove (iv). Attaining a matrix with a column consisting of zeros can be done using only operations (i)-(iii). So **wolog** (without loss of generality), assert the jth column of A consists of zeros. Deleting $\mathbf{a}_j = \mathbf{0}$ does not change the kernel, but it does produce an $m \times (n-1)$ matrix A' defining a homomorphism $\varphi' : \mathbb{Z}^{n-1} \to \mathbb{Z}^m$. The kernel of ψ is unchanged means $A'\mathbb{Z}^{n-1} = A\mathbb{Z}^n$ and hence $\psi : \mathbb{Z}^m / A\mathbb{Z}^n = \mathbb{Z}^m / A'\mathbb{Z}^{n-1} \xrightarrow{\cong} G$.

To prove (v), again, by the operations (i)-(iii) it suffices to suppose the jth column of A is the ith unit vector, i.e., $\mathbf{a}_j = \mathbf{e}_i$. Let $\Psi : \mathbb{Z}^n \to G$ denote the composition as in Equation (1.1) (with 2 replaced by n). The images of the standard basis vectors under Ψ generate G; all vectors $\mathbf{w} \in A\mathbb{Z}^n$ have zeros in the ith entry, which is determined by the ith row of A so we omit it. Likewise, $\Psi(\mathbf{e}_i) = 0$ means we can omit $\mathbf{e}_i \in \mathbb{Z}^n$. $\square$

Example 3

Proposition 1 speeds up the process of finding a group presentation. For example,

$$A = \begin{pmatrix} 3 & 8 & 7 & 9 \\ 2 & 4 & 6 & 6 \\ 1 & 2 & 2 & 1 \end{pmatrix} \xrightarrow[]{\substack{R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 - 2R_3}} \begin{pmatrix} 0 & 2 & 1 & 6 \\ 0 & 0 & 2 & 4 \\ 1 & 2 & 2 & 1 \end{pmatrix} \xrightarrow{\cancel{C_1} \quad \cancel{R_3}} \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 4 \end{pmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 2 & 1 & 6 \\ -4 & 0 & -8 \end{pmatrix} \xrightarrow{\cancel{C_2} \quad \cancel{R_1}} \begin{pmatrix} -4 & -8 \end{pmatrix}$$

$$\xrightarrow{C_2 \to C_2 + 2C_1} \begin{pmatrix} -4 & 0 \end{pmatrix} \xrightarrow{\substack{\cancel{C_2} \\ C_1 \to -C_1}} \begin{pmatrix} 4 \end{pmatrix}.$$

So $(4)\mathbb{Z}$ is a 1×1 matrix defining a homomorphism $\varphi : \mathbb{Z}^1 \to \mathbb{Z}^1$. A represents its cokernel, $\mathbb{Z}/(4)\mathbb{Z} = \mathbb{Z}/4\mathbb{Z}$.

Theorem 1 (Fundamental Theorem of Finitely Generated Abelian Groups)

Let G denote a finitely generated abelian group. Then there exist positive integers $d_1, \ldots, d_k$ and non-negative integer $r \geq 0$, with $d_1 | \cdots | d_k$ such that

$$G \cong \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_k} \times \mathbb{Z}^r.$$

The integers $d_1, \ldots, d_k$ are uniquely determined and called **invariant** *factors. The integer r is also unique and is called the* **free rank of G**.

Example 4

Recall, in Example **??** we reduced the matrix A to its Smith normal form.

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 5 & 1 & -5 \\ -3 & -3 & 29 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 66 \end{pmatrix}.$$

By Proposition 1, we can reduce further to $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 66 \end{smallmatrix}\right)$. Thus A is a presentation matrix for $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/66\mathbb{Z}$.

Exercise 2 (cf. Problem 79)

What direct product of cyclic groups is presented by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$?

(Compare to part **??**. of Exercise **??**).

Exercise 3 (cf. Problem 80)

Use your computations from Problem **??** to find a direct product of cyclic groups presented by the transposed Laplacian of the graph in Figure **??**.

Exercise 4 (cf. Problem 81)

Compute, by hand, a direct product of cyclic groups isomorphic to the abelian groups presented by the following matrices:

(a) $\begin{pmatrix} 5 & 0 & 0 \end{pmatrix}$,

(b) $\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$,

(c) $\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}$.

Exercise 5 (cf. Problem 82)

Compute, using Sage, a direct product of cyclic groups isomorphic to the abelian groups presented by the following matrices

(a) $\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & -1 & 3 \end{pmatrix}$

Question

Compute the determinants for the matrices in Exercises 4 and 5. What is the pattern?

Exercise 6 (cf. Problem 83)

For any positive integer n, consider an $n \times n$ matrix A_n described by Pascal's triangle, exemplified by

$$A_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix}.$$

What finitely generated abelian group G_n is presented by the matrix A_n?