Calculus I (Math 2554)

Summer 2015

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The base for these slides was done by Shannon Dingman, later encoded in LaTeX by Brad Lutes.

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Tips for Success

- Attend class every day. Participate, take notes, and ask questions.
- Don't get behind on MLP homeworks. Stay on top of the book problems.
- Be sure to seek assistance (tutoring, office hours, etc.) if you are struggling.
- Don't rely on success in high school calculus to save you in college calculus.
- Find a study partner(s) to meet with on a regular basis to cover questions and study for quizzes/exams.
- REMEMBER... THE TERM STARTS TODAY! SO DOES THE EVENTUAL EARNING OF YOUR FINAL GRADE!!!

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2.1 The Idea of Limits

Question

How would you define, and then differentiate between, the following pairs of terms?

- instantaneous velocity vs. average velocity?
- tangent line vs. secant line?

(Recall: What is a tangent line and what is a secant line?)

Example

An object is launched into the air. Its position s (in feet) at any time t (in seconds) is given by the equation:

$$s(t) = -4.9t^2 + 30t + 20.$$

(a) Compute the average velocity of the object over the following time intervals: $[1, 3]$, $[1, 2]$, $[1, 1.5]$

(b) As your interval gets shorter, what do you notice about the average velocities? What do you think would happen if we computed the average velocity of the object over the interval $[1, 1.2]$? $[1, 1.1]$? $[1, 1.05]$?

Example, cont.

An object is launched into the air. Its position s (in feet) at any time t (in seconds) is given by the equation:

$$s(t) = -4.9t^2 + 30t + 20.$$

(c) How could you use the average velocities to estimate the instantaneous velocity at $t = 1$?

(d) What do the average velocities you computed in 1. represent on the graph of $s(t)$?

Question

What happens to the relationship between instantaneous velocity and average velocity as the time interval gets shorter?

Answer: The instantaneous velocity at $t = 1$ is the limit of the average velocities as t approaches 1.

Question

What about the relationship between the secant lines and the tangent lines as the time interval gets shorter?

Answer: The slope of the tangent line at $(1, 45.1 = s(1))$ is the limit of the slopes of the secant lines as t approaches 1.

2.1 Book Problems

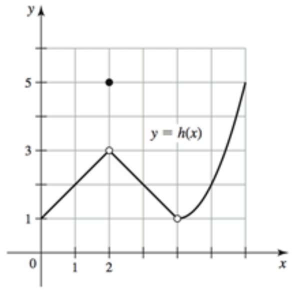
1-3, 7, 9, 11, 13, 17, 21

- Even though book problems aren't turned in, they're a very good way to study for quizzes and tests (wink wink wink).

$\oint$ 2.2 Definition of Limits

Determining Limits from a Graph

Exercise



Determine the following:

(a) $h(1)$

(b) $h(2)$

(c) $h(4)$

(d) $\lim_{x \to 2} h(x)$

(e) $\lim_{x \to 4} h(x)$

(f) $\lim_{x \to 1} h(x)$

Question

Does $\lim_{x \to a} f(x)$ always equal $f(a)$?

(Hint: Look at the example from the previous slide!)

Determining Limits from a Table

Exercise

Suppose $f(x) = \dfrac{x^2 + x - 20}{x - 4}$.

(a) Create a table of values of $f(x)$ when

$$x = 3.9, 3.99, 3.999, \text{ and}$$
$$x = 4.1, 4.01, 4.001$$

(b) What can you conjecture about $\lim_{x \to 4} f(x)$?

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One-Sided Limits

Up to this point we have been working with two-sided limits; however, for some functions it makes sense to examine one-sided limits.

Notice how in the previous example we could approach $f(x)$ from both sides as x approaches a, i.e., when $x > a$ and when $x < a$.

Definition (right-hand limit)

Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

$$\lim_{x \to a^+} f(x) = L$$

and say **the limit of** $f(x)$ **as** x **approaches** a **from the right equals** L.

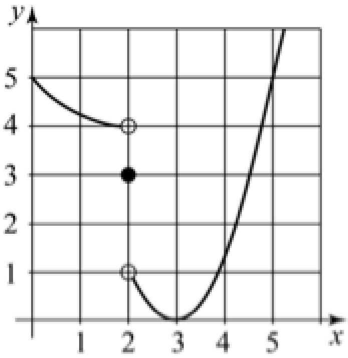
Definition (left-hand limit)

Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

$$\lim_{x \to a^-} f(x) = L$$

and say **the limit of** $f(x)$ **as** x **approaches** a **from the left equals** L.

Exercise



Determine the following:

(a) $g(2)$

(b) $\lim\limits_{x\to 2^+} g(x)$

(c) $\lim\limits_{x\to 2^-} g(x)$

(d) $\lim\limits_{x\to 2} g(x)$

Relationship Between One- and Two-Sided Limits

Theorem

If f is defined for all x near a except possibly at a, then $\lim_{x \to a} f(x) = L$ if and only if both $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$.

In other words, the only way for a two-sided limit to exist is if the one-sided limits equal the same number (L).

2.2 Book Problems

1-4, 10, 12, 16, 18, 20, 25

$\oint$ 2.3 Techniques for Computing Limits

This section provides various laws and techniques for determining limits. These constitute **analytical** methods of finding limits. The following is an example of a very useful limit law:

Limits of Linear Functions: Let a, b, and m be real numbers. For linear functions $f(x) = mx + b$,

$$\lim_{x \to a} f(x) = f(a) = ma + b.$$

This rule says we if $f(x)$ is a linear function, then in taking the limit as $x \to a$, we can just plug in the a for x.

IMPORTANT! Using a table or a graph to compute limits, as in the previous sections, can be helpful. However, "analytical" does not include those techniques.

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Limit Laws

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, c is a real number, and m, n are positive integers.

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1. **Sum:** $\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. **Difference:** $\lim_{x\to a} (f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

In other words, if we are taking a limit of two things added together or subtracted, then we can first compute each of their individual limits one at a time.

Limit Laws, cont.

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, c is a real number, and m, n are positive integers.

---

3. **Constant Multiple:** $\lim_{x\to a} (cf(x)) = c\left(\lim_{x\to a} f(x)\right)$

4. **Product:** $\lim_{x\to a} (f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$

The same is true for products. If one of the factors is a constant, we can just bring it outside the limit. In fact, a constant is its own limit.

Limit Laws, cont.

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, c is a real number, and m, n are positive integers.

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5. Quotient: $\lim_{x\to a}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

(provided $\lim_{x\to a} g(x) \neq 0$)

Question

Why the caveat?

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Limit Laws, cont.

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, c is a real number, and m, n are positive integers.

---

6. **Power:** $\lim_{x\to a} (f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$

7. **Fractional Power:** $\lim_{x\to a} (f(x))^{\frac{n}{m}} = \left(\lim_{x\to a} f(x)\right)^{\frac{n}{m}}$

(provided $f(x) \geq 0$ for x near a if m is even and $\frac{n}{m}$ is in lowest terms)

Question

Explain the caveat in **7.**

Limit Laws, cont.

Laws **1.-6.** hold for one-sided limits as well. But **7.** must be modified:

7. Fractional Power (one-sided limits):

- $$\lim_{x\to a^+} (f(x))^{\frac{n}{m}} = \left(\lim_{x\to a^+} f(x)\right)^{\frac{n}{m}}$$
  (provided $f(x) \geq 0$ for x near a with $x > a$, if m is even and $\frac{n}{m}$ is in lowest terms)

- $$\lim_{x\to a^-} (f(x))^{\frac{n}{m}} = \left(\lim_{x\to a^-} f(x)\right)^{\frac{n}{m}}$$
  (provided $f(x) \geq 0$ for x near a with $x < a$, if m is even and $\frac{n}{m}$ is in lowest terms)

Limits of Polynomials and Rational Functions

Assume that $p(x)$ and $q(x)$ are polynomials and a is a real number.

- **Polynomials:** $\lim_{x\to a} p(x) = p(a)$
- **Rational functions:** $\lim_{x\to a} \dfrac{p(x)}{q(x)} = \dfrac{p(a)}{q(a)}$

  (provided $q(a) \neq 0$)

For polynomials and rational functions we can plug in a to compute the limit, as long as we don't get zero in the denominator. Linear functions count as polynomials. A rational function is a “fraction” made of polynomials.

Exercise

Evaluate the following limits analytically.

1. $\displaystyle\lim_{x\to 1} \frac{4f(x)g(x)}{h(x)}$, given that

$$\lim_{x\to 1} f(x) = 5,\ \lim_{x\to 1} g(x) = -2,\ \text{and}\ \lim_{x\to 1} h(x) = -4.$$

2. $\displaystyle\lim_{x\to 3} \frac{4x^2 + 3x - 6}{2x - 3}$

3. $\displaystyle\lim_{x\to 1^-} g(x)$ and $\displaystyle\lim_{x\to 1^+} g(x)$, given that

$$g(x) = \begin{cases} x^2 & \text{if } x \leq 1; \\ x + 2 & \text{if } x > 1. \end{cases}$$

Additional (Algebra) Techniques

When direct substitution (a.k.a. plugging in a) fails try using algebra:

- Factor and see if the denominator cancels out.

Example

$$\lim_{t \to 2} \frac{3t^2 - 7t + 2}{2 - t}$$

- Look for a common denominator.

Example

$$\lim_{h \to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h}$$

Another Technique: Squeeze Theorem

This method for evaluating limits uses the relationship of functions with each other.

Theorem (Squeeze Theorem)

Assume $f(x) \leq g(x) \leq h(x)$ *for all values of* x *near* a, *except possibly at* a, *and suppose*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.$$

Then since g *is always between* f *and* h *for* x*-values close enough to* a, *we must have*

$$\lim_{x \to a} g(x) = L.$$

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Example

(a) Draw a graph of the inequality

$$-|x| \leq x^2 \ln\left(x^2\right) \leq |x|.$$

(b) Compute $\lim_{x \to 0} x^2 \ln\left(x^2\right)$.

2.3 Book Problems

12-30 (every 3rd problem), 31, 33, 37-47 (odds), 51, 53, 61-65 odds

- In general, review your algebra techniques, since they can save you some headache.

∮2.4 Infinite Limits

In the next two sections, we examine limit scenarios involving infinity. The two situations are:

- **Infinite limits:** as x (i.e., the independent variable) approaches a finite number, y (i.e., the dependent variable) becomes arbitrarily large or small

  $$\text{looks like: } \lim_{x \to \text{number}} f(x) = \pm\infty$$

- **Limits at infinity:** as x approaches an arbitrarily large or small number, y approaches a finite number

  $$\text{looks like: } \lim_{x \to \pm\infty} f(x) = \text{number}$$

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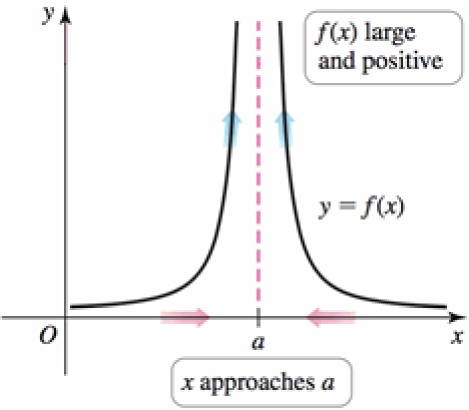
Definition of Infinite Limits

Definition (positively infinite limit)

Suppose f is defined for all x near a. If $f(x)$ grows arbitrarily large for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = \infty$$

and say **the limit of $f(x)$ as x approaches a is infinity**.



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Definition (negatively infinite limit)

Suppose f is defined for all x near a. If $f(x)$ is negative and grows arbitrarily large in magnitude for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = -\infty$$

and say **the limit of $f(x)$ as x approaches a is negative infinity**.

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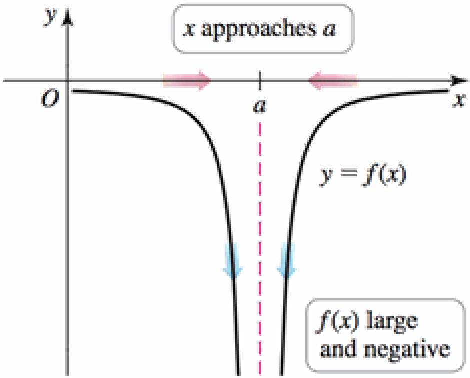
y
x approaches a
O
a
x
y = f(x)
f(x) large
and negative

The definitions work for one-sided limits, too.

Exercise

Using a graph and a table of values, given $f(x) = \dfrac{1}{x^2 - x}$, determine:

(a) $\lim\limits_{x \to 0^+} f(x)$

(b) $\lim\limits_{x \to 0^-} f(x)$

(c) $\lim\limits_{x \to 1^+} f(x)$

(d) $\lim\limits_{x \to 1^-} f(x)$

Definition of Vertical Asymptote

Definition

Suppose a function f satisfies at least one of the following:

- $\lim_{x \to a} f(x) = \pm\infty$,
- $\lim_{x \to a^+} f(x) = \pm\infty$
- $\lim_{x \to a^-} f(x) = \pm\infty$

Then the line $x = a$ is called a **vertical asymptote** of f.

Exercise

Given $f(x) = \dfrac{3x-4}{x+1}$, determine, analytically (meaning without using a table or a graph),

(a) $\displaystyle\lim_{x\to -1^+} f(x)$

(b) $\displaystyle\lim_{x\to -1^-} f(x)$

Remember to check for factoring –

Exercise

(a) What is/are the vertical asymptotes of

$$f(x) = \frac{3x^2 - 48}{x + 4}?$$

(b) What is $\lim_{x \to -4} f(x)$?

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2.4 Book Problems

7-10, 15, 17-26, 36-37

$\oint$ 2.5 Limits at Infinity

Limits at infinity determine what is called the **end behavior** of a function.

Horizontal Asymptotes

Definition

If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write

$$\lim_{x\to\infty} f(x) = L.$$

The line $y = L$ is a **horizontal asymptote** of f.

The limit at negative infinity, $\lim_{x\to-\infty} f(x) = M$, is defined analogously and in this case, the horizontal asymptote is $y = M$.

Infinite Limits at Infinity

Question

Is it possible for a limit to be both an infinite limit and a limit at infinity? (Yes.)

If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write

$$\lim_{x\to\infty} f(x) = \infty.$$

(The limits $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) = -\infty$ are defined similarly.)

Powers and Polynomials: Let n be a positive integer and let $p(x)$ be a polynomial.

- $n =$ even number: $\lim\limits_{x \to \pm\infty} x^n = \infty$
- $n =$ odd number: $\lim\limits_{x \to \infty} x^n = \infty$ and $\lim\limits_{x \to -\infty} x^n = -\infty$

- (again, assuming n is positive)

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = \lim_{x\to\pm\infty} x^{-n} = 0$$

- For a polynomial, only look at the term with the highest exponent:

$$\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} (\text{constant}) \cdot x^n$$

The constant is called the **leading coefficient**, $\mathrm{lc}(p)$. The highest exponent that appears in the polynomial is called the **degree**, $\deg(p)$.

Rational Functions: Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function.

- If $\deg(p) < \deg(q)$, i.e., the numerator has the smaller degree, then
$$\lim_{x\to\pm\infty} f(x) = 0$$
and $y = 0$ is a horizontal asymptote of f.

- If $\deg(p) = \deg(q)$, i.e., numerator and denominator have the same degree, then
$$\lim_{x\to\pm\infty} f(x) = \frac{\mathsf{lc}(p)}{\mathsf{lc}(q)}$$
and $y = \frac{\mathsf{lc}(p)}{\mathsf{lc}(q)}$ is a horizontal asymptote of f.

- If $\deg(p) > \deg(q)$, (numerator has the bigger degree) then

$$\lim_{x \to \pm\infty} f(x) = \infty \quad \text{or} \quad -\infty$$

  and f has no horizontal asymptote.

- Assuming that $f(x)$ is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeroes of q.

  (This is why it is a good idea to check for factoring and cancelling first!)

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Exercise

Determine the end behavior of the following functions (in other words, compute both limits, as $x \to \pm\infty$, for each of the functions):

1. $f(x) = \dfrac{x+1}{2x^2-3}$
2. $g(x) = \dfrac{4x^3-3x}{2x^3+5x^2+x+2}$
3. $h(x) = \dfrac{6x^4-1}{4x^3+3x^2+2x+1}$

Algebraic and Transcendental Functions

Example

Determine the end behavior of the following functions.

1. $f(x) = \dfrac{4x^3}{2x^3 + \sqrt{9x^6 + 15x^4}}$ (radical signs appear)
2. $g(x) = \cos x$ (trig)
3. $h(x) = e^x$ (exponential)

2.5 Book Problems

9-10, 13-35 (odds), 39, 43, 45, 53