Topology QR Solutions – 8 May 2009

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updated September 5, 2010

Morning Session

1. Let $C^0(\mathbb{R}, \mathbb{R})$ denote the set of continuous functions

$$f : \mathbb{R} \to \mathbb{R}$$

For a compact set $K \subset \mathbb{R}$, $\epsilon > 0$ and $f \in C^0(\mathbb{R}, \mathbb{R})$, define

$$B_\epsilon(f, K) = \left\{ g \in C^0(\mathbb{R}, \mathbb{R}) \mid \sup_{x \in K} \{|f(x) - g(x)|\} < \epsilon \right\}.$$

The set of all subsets of $C^0\mathbb{R}, \mathbb{R})$ of the form $B_\epsilon(f, C)$ is a basis for the *topology of compact convergence* on $C^0(\mathbb{R}, \mathbb{R})$. From now on we consider $C^0(\mathbb{R}, \mathbb{R})$ to be endowed with this topology.

(a) Show that $C^0(\mathbb{R}, \mathbb{R})$ is Hausdorff, second countable, connected and simply connected.

(b) Which of the following two sets are compact in the topology of compact convergence?

- $\{f_n(x) = x + \sin nx, n \in N\}$
- The set of all polynomials of degree at most 4 all whose coefficients have absolute value less than 1.

Solution.

2. Let X be the space obtained from o$\mathbb{S}^1 \times \mathbb{R}$ by removing the interior of k disjoint 2-disks.

(a) Compute the fundamental group $\pi_1(X)$.

*with additional input from M. Hochster, G.P. Scott, and others from the U of M Mathematics Department

(b) What would be your answer to part (a) if $\mathbb{S}^1 \times \mathbb{R}$ is replaced by $\mathbb{S}^2 \times \mathbb{R}$ and 2-disks are replaced by 3-balls?

(c) Let Y be the union of two copies of the real projective plane $\mathbb{R}P^2$ having exactly one point y in common. Compute $\pi_1(Y, y)$.

Solution.

3. Let $\mathbb{S}^1 = \{(x, y, 0, 0) \in \mathbb{R}^4 \mid x^2 + y^2 = 1\}$ be the unit circle and consider $M = \mathbb{R}^4 \setminus \mathbb{S}^1$. Compute the fundamental group $\pi_1(M)$ and the homology groups $H_*(M)$ of M.

Solution.

4. (a) Give an example of a local homeomorphism $\mathbb{R}^2 \to \mathbb{S}^2$ which is surjective but not a covering.

   (b) Prove that there is no local homeomorphism $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$.

Solution.

5. (a) For $n \geq 1$, let $\mathbb{R}P^n$ be the real projective space. Compute $\pi_1(\mathbb{R}P^n)$.

   (b) Let $\tau : \mathbb{R}^3 \to \mathbb{R}^3$ be the homeomorphism $\tau(x) = -x$, let $\langle\tau\rangle$ be the cyclic group generated by τ and endow $\mathbb{R}^3/\langle\tau\rangle$ with the quotient topology. Prove that $\mathbb{R}^3/\langle\tau\rangle$ is not a manifold.

Solution.

1. The n-sphere S^n covers $\mathbb{R}P^n$ using the action of $\mathbb{Z}_2$. So $\pi_1(\mathbb{R}P^n) \simeq \mathbb{Z}_2$. $\square$

2. Every point in $\mathbb{R}^3/\langle\tau\rangle$ has two preimages, except for the origin. Suppose a connected neighborhood of the origin is diffeomorphic to an open ball B in $\mathbb{R}^3$. Then $\mathbb{R}^3$ must map to B in a way that every fiber is 2 points, except the origin, which is one point. $\square$

Afternoon Session

1. Endow $\{0, 1\}$ with the discrete topology and $X = \{0, 1\}^{\mathbb{N}}$ with the product topology.

   (a) Prove that X is compact and totally disconnected.

(b) Construct a continuous surjective map $\pi : X \to [0,1]$ such that for every $t \in [0,1]$ the set $\pi^{-1}(t)$ consists of at most 2 points.

Solution.

2. Let S_1 and S_2 be two closed orientable surfaces and $\gamma_1 \subset S_1$ and $\gamma_2 \subset S_2$ be two simple closed curves. Let X be the space obtained by gluing S_1 and S_2 along γ_1 and γ_2. Is X a manifold? If not, is X homotopy equivalent to a manifold?

Solution.

3. Compute $H_*(\mathbb{R}P^2 \sharp \mathbb{R}P^2)$ where $\mathbb{R}P^2$ is the real projective plane and $\sharp$ denotes connected sum.

Solution.

4. Let X be a path-connected space with base point x_0, and consider the function

$$\phi : \pi_1(X, x_0) \to H_1(X, \mathbb{Z})$$

which takes a homotopy class $[\alpha]$ to the singular homology class $\alpha_*([\mathbb{S}^1])$ where $[\mathbb{S}^1] \in H_1(\mathbb{S}^1, \mathbb{Z})$ is a fixed generator. Show that

(a) ϕ is a homomorphism, and

(b) ϕ is surjective.

Solution.

5. Let SO_n be the set of orthogonal n-by-n natrices with real coefficients and determinant 1. Consider SO_n as a subset of $\mathbb{R}^{n^2}$.

(a) Prove that SO_n is a manifold.

(b) Show that SO_n admits a nowhere vanishing vectorfield.

(c) Compute $\chi(SO_n)$ (Hint: use (b)).

Solution.