§3.5 Derivatives of Trigonometric Functions

Trig functions are commonly used to model cyclic or periodic behavior in everyday settings. Therefore it is important to know how these functions change across time.

Fact: Derivative formulas for sine and cosine can be derived using the following limits:

- $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$
- $\displaystyle\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$

(We will prove these limits in Chapter 4.)

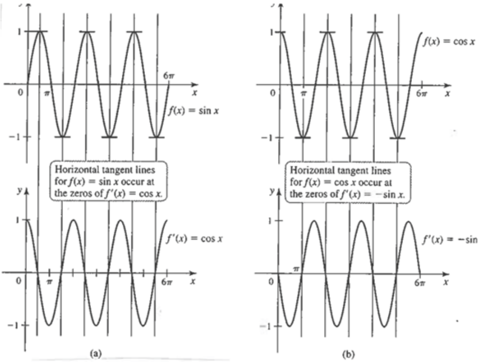
Exercise

Evaluate $\lim_{x\to 0} \frac{\sin 9x}{x}$ and $\lim_{x\to 0} \frac{\sin 9x}{\sin 5x}$.

Using the previous limits and the definition of the derivative, we obtain

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$

Examining the graphs of sine and cosine illustrate the relationship between the functions and their derivatives.



Trig Identities You Should Know

- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = 1 - 2 \sin^2 x$
- $\cos^2 x = \dfrac{1 + \cos 2x}{2}$
- $\sin^2 x = \dfrac{1 - \cos 2x}{2}$
- $\tan x = \dfrac{\sin x}{\cos x}$
- $\cot x = \dfrac{\cos x}{\sin x}$
- $\cot x = \dfrac{1}{\tan x}$
- $\sec x = \dfrac{1}{\cos x}$
- $\csc x = \dfrac{1}{\sin x}$

Derivatives of Other Trig functions

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{\cos x\cos x - (-\sin x)\sin x}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

So $\frac{d}{dx}(\tan x) = \sec^2 x$.

By using trig identities and the Quotient Rule, we obtain

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\csc x \cot x$$
$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \sec x \tan x$$
$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\tan x}\right) = -\csc^2 x$$

Exercise

Compute the derivative of the following functions:

$$f(x) = \frac{\tan x}{1 + \tan x} \qquad g(x) = \sin x \cos x$$

Exercise

Use the difference and product rules to fine the derivative of the function $y = \cos x - x \sin x$.

A. $-\sin x + x\cos x$

B. $x \cos x$

C. $-2\sin x - x\cos x$

D. $x\cos x - 2\sin x$

There is a cyclic relationship between the higher order derivatives of $\sin x$ and $\cos x$:

$$
\begin{aligned}
f(x) &= \sin x \\
f'(x) &= \cos x \\
f''(x) &= -\sin x \\
f^{(3)}(x) &= -\cos x \\
f^{(4)}(x) &= \sin x
\end{aligned}
\qquad\qquad
\begin{aligned}
g(x) &= \cos x \\
g'(x) &= -\sin x \\
g''(x) &= -\cos x \\
g^{(3)}(x) &= \sin x \\
g^{(4)}(x) &= \cos x
\end{aligned}
$$

3.5 Book Problems

7-47 (odds), 57, 59, 61

§3.6 Derivatives as Rates of Change

Question

Why do we need derivatives in real life?

We look at four areas where the derivative assists us with determining the rate of change in various contexts.

Suppose an object moves along a straight line and its location at time t is given by the position function $s = f(t)$. The **displacement** of the object between $t = a$ and $t = a + \Delta t$ is

$$\Delta s = f(a + \Delta t) - f(a).$$

Here Δt represents how much time has elapsed.

We now define average velocity as

$$\frac{\Delta s}{\Delta t} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

Recall that the limit of the average velocities as the time interval approaches 0 was the instantaneous velocity (which we denote here by v). Therefore, the instantaneous velocity at a is

$$v(a) = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = f'(a).$$

In mathematics, speed and velocity are related but not the same – if the velocity of an object at any time t is given by $v(t)$, then the speed of the object at any time t is given by

$$|v(t)| = |f'(t)|.$$

By definition, acceleration (denoted by a) is the instantaneous rate of change of the velocity of an object at time t. Therefore,

$$a(t) = v'(t)$$

and since velocity was the derivative of the position function $s = f(t)$, then

$$a(t) = v'(t) = f''(t).$$

Summary: Given the position function $s = f(t)$, the velocity at time t is the first derivative, the speed at time t is the absolute value of the first derivative, and the acceleration at time t is the second derivative.

Question

Given the position function $s = f(t)$ of an object launched into the air, how would you know:

- The highest point the object reaches?
- How long it takes to hit the ground?
- The speed at which the object hits the ground?

Exercise

A rock is dropped off a bridge and its distance s (in feet) from the bridge after t seconds is $s(t) = 16t^2 + 4t$. At $t = 2$ what are, respectively, the velocity of the rock and the acceleration of the rock?

A. 64 ft/s; 16 ft/s^2

B. 68 ft/s; 32 ft/s^2

C. 64 ft/s; 32 ft/s^2

D. 68 ft/s; 16 ft/s^2

Suppose $p = f(t)$ is a function of the growth of some quantity of interest. The average growth rate of p between times $t = a$ and a later time $t = a + \Delta t$ is the change in p divided by the elapsed time Δt:

$$\frac{\Delta p}{\Delta t} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

As Δt approaches 0, the average growth rate approaches the derivative $\frac{dp}{dt}$, which is the instantaneous growth rate (or just simply the growth rate). Therefore,

$$\frac{dp}{dt} = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t}.$$

Exercise

The population of the state of Georgia (in thousands) from 1995 ($t = 0$) to 2005 ($t = 10$) is modeled by the polynomial

$$p(t) = -0.27t^2 + 101t + 7055.$$

(a) What was the average growth rate from 1995 to 2005?

(b) What was the growth rate for Georgia in 1997?

(c) What can you say about the population growth rate in Georgia between 1995 and 2005?

Suppose a company produces a large amount of a particular quantity. Associated with manufacturing the quantity is a **cost function** $C(x)$ that gives the cost of manufacturing x items. This cost may include a **fixed cost** to get started as well as a **unit cost** (or **variable cost**) in producing one item.

If a company produces x items at a cost of $C(x)$, then the average cost is $\frac{C(x)}{x}$. This average cost indicates the cost of items already produced. Having produced x items, the cost of producing another Δx items is $C(x+\Delta x) - C(x)$. So the average cost of producing these extra Δx items is

$$\frac{\Delta C}{\Delta x} = \frac{C(x+\Delta x) - C(x)}{\Delta x}.$$

If we let Δx approach 0, we have

$$\lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = C'(x)$$

which is called the **marginal cost**. The marginal cost is the approximate cost to produce one additional item after producing x items.

Note: In reality, we can't let Δx approach 0 because Δx represents whole numbers of items.

Exercise

If the cost of producing x items is given by

$$C(x) = -0.04x^2 + 100x + 800$$

for $0 \leq x \leq 1000$, find the average cost and marginal cost functions. Also, determine the average and marginal cost when $x = 500$.

3.6 Book Problems

9-19, 21-24, 30-33 (odds)

§3.7 The Chain Rule

The rules up to now have not allowed us to differentiate composite functions

$$f \circ g(x) = f(g(x)).$$

Example

If $f(x) = x^7$ and $g(x) = 2x - 3$, then $f(g(x)) = (2x - 3)^7$. To differentiate we could mulitply the polynomial out... but in general we should use a much more efficient strategy to emply to composition functions.

Example

Suppose that Yvonne (y) can run twice as fast as Uma (u). Then write $\frac{dy}{du} = 2$.

Suppose that Uma can run four times as fast as Xavier (x). So $\frac{du}{dx} = 4$.

How much faster can Yvonne run than Xavier? In this case, we would take both our rates and multiply them together:

$$\frac{dy}{du} \cdot \frac{du}{dx} = 2 \cdot 4 = 8.$$

Version 1 of the Chain Rule

If g is differentiable at x, and $y = f(u)$ is differentiable at $u = g(x)$, then the composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Guidelines for Using the Chain Rule

Assume the differentiable function $y = f(g(x))$ is given.

1. Identify the outer function f, the inner function g, and let $u = g(x)$.
2. Replace $g(x)$ by u to express y in terms of u:

$$y = f(g(x)) \implies y = f(u)$$

3. Calculate the product $\dfrac{dy}{du} \cdot \dfrac{du}{dx}$
4. Replace u by $g(x)$ in $\dfrac{dy}{du}$ to obtain $\dfrac{dy}{dx}$.

Example

Use Version 1 of the Chain Rule to calculate $\frac{dy}{dx}$ for $y = (5x^2 + 11x)^{20}$.

- inner function: $u = 5x^2 + 11x$
- outer function: $y = u^{20}$

We have $y = f(g(x)) = (5x^2 + 11x)^{20}$. Differentiate:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 20u^{19} \cdot (10x + 11)$$
$$= 20(5x^2 + 11x)^{19} \cdot (10x + 11)$$

Exercise

Use the first version of the Chain Rule to calculate $\frac{dy}{dx}$ for

$$y = \left(\frac{3x}{4x+2}\right)^5.$$

Exercise

Use the first version of the Chain Rule to calculate $\frac{dy}{dx}$ for

$$y = \cos(5x + 1).$$

A. $y' = -\cos(5x + 1) \cdot \sin(5x + 1)$
B. $y' = -5\sin(5x + 1)$
C. $y' = 5\cos(5x + 1) - \sin(5x + 1)$
D. $y' = -\sin(5x + 1)$

Version 2 of the Chain Rule

Notice if $y = f(u)$ and $u = g(x)$, then $y = f(u) = f(g(x))$, so we can also write:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= f'(u) \cdot g'(x) \\ &= f'(g(x)) \cdot g'(x).\end{aligned}$$

Example

Use Version 2 of the Chain Rule to calculate $\frac{dy}{dx}$ for $y = (7x^4 + 2x + 5)^9$.

- inner function: $g(x) = 7x^4 + 2x + 5$
- outer function: $f(u) = u^9$

Then

$$f'(u) = 9u^8 \implies f'(g(x)) = 9(7x^4 + 2x + 5)^8$$
$$g'(x) = 28x^3 + 2.$$

Putting it together,

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x) = 9(7x^4 + 2x + 5)^8 \cdot (28x^3 + 2)$$

Exercise

Use Version 2 of the Chain Rule to calculate $\frac{dy}{dx}$ for

$$y = \tan\left(5x^5 - 7x^3 + 2x\right).$$

If g is differentiable for all x in the domain and n is an integer, then

$$\frac{d}{dx}\left[\,(g(x))^n\,\right] = n(g(x))^{n-1} \cdot g'(x).$$

Example

$$\frac{d}{dx}\left[(1-e^x)^4\right] = ?$$

Answer:

$$\begin{aligned}\frac{d}{dx}\left[(1-e^x)^4\right] &= 4(1-e^x)^3 \cdot (-e^x)\\ &= -4e^x(1-e^x)^3\end{aligned}$$

Composition of 3 or More Functions

Example

Compute $\dfrac{d}{dx}\left[\sqrt{(3x-4)^2+3x}\right]$.

Answer:

$$\begin{aligned}
\frac{d}{dx}\left[\sqrt{(3x-4)^2+3x}\right] &= \frac{1}{2}\left((3x-4)^2+3x\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left[(3x-4)^2+3x\right] \\
&= \frac{1}{2\sqrt{\left((3x-4)^2+3x\right)}} \cdot \left[2(3x-4)\frac{d}{dx}(3x-4)+3\right] \\
&= \frac{1}{2\sqrt{\left((3x-4)^2+3x\right)}} \cdot \left[2(3x-4)\cdot 3+3\right] \\
&= \frac{18x-21}{2\sqrt{\left((3x-4)^2+3x\right)}}
\end{aligned}$$

3.7 Book Problems

7-33 (odds), 38, 45-67 (odds)

§3.8 Implicit Differentiation

Up to now, we have calculated derivatives of functions of the form $y = f(x)$, where y is defined **explicitly** in terms of x. In this section, we examine relationships between variables that are **implicit** in nature, meaning that y either is not defined explicitly in terms of x or cannot be easily manipulated to solve for y in terms of x.

The goal of **implicit differentiation** is to find a single expression for the derivative directly from an equation of the form $F(x, y) = 0$ without first solving for y.

Example

Calculate $\frac{dy}{dx}$ directly from the equation for the circle

$$x^2 + y^2 = 9.$$

Solution: To remind ourselves that x is our independent variable and that we are differentiating with respect to x, we can replace y with $y(x)$:

$$x^2 + (y(x))^2 = 9.$$

Now differentiate each term with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}((y(x))^2) = \frac{d}{dx}(9).$$

By the Chain Rule, $\frac{d}{dx}((y(x))^2) = 2y(x)y'(x)$ (Version 2), or $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$ (Version 1). So

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ \implies \frac{dy}{dx} &= \frac{-2x}{2y} \\ &= -\frac{x}{y}. \end{aligned}$$

The derivative is a function of x and y, meaning we can write it in the form

$$F(x,y) = -\frac{x}{y}.$$

To find slopes of tangent lines at various points along the circle we just plug in the coordinates. For example, the slope of the tangent line at (0,3) is

$$\left.\frac{dy}{dx}\right|_{(x,y)=(0,3)} = -\frac{0}{3} = 0.$$

The slope of the tangent line at $(1, 2\sqrt{2})$ is

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,2\sqrt{2})} = -\frac{1}{2\sqrt{2}}.$$

The point is that, in some cases it is difficult to solve an implicit equation in terms of y and then differentiate with respect to x. In other cases, although it may be easier to solve for y in terms of x, you may need two or more functions to do so, which means two or more derivatives must be calculated (e.g., circles).

The goal of implicit differentiation is to find one single expression for the derivative directly given $F(x, y) = 0$ (i.e., some equation with xs and ys in it), without solving first for y.

Question

The following functions are implicitly defined:

- $x + y^3 - xy = 4$
- $\cos(x - y) + \sin y = \sqrt{2}$

For each of these functions, how would you find $\frac{dy}{dx}$?

Exercise

Find $\frac{dy}{dx}$ for $xy + y^3 = 1$.

Exercise

Find an equation of the line tangent to the curve $x^4 - x^2y + y^4 = 1$ at the point $(-1, 1)$.

Example

Find $\frac{d^2y}{dx^2}$ if $xy + y^3 = 1$.

Exercise

If $\sin x = \sin y$, then $\frac{dy}{dx} = ?$ and $\frac{d^2y}{dx^2} = ?$

A. $\dfrac{\cos y}{\cos x}; \quad \dfrac{\tan y \cos^2 x - \sin x \cos y}{\cos^2 x}$

B. $\dfrac{\cos x}{\cos y}; \quad \dfrac{\tan y \cos^2 x - \sin x \cos y}{\cos^2 y}$

C. $\dfrac{\cos x}{\cos y}; \quad \dfrac{\cos y(\sin x - \sin y)}{\cos^2 y}$

D. $\dfrac{\cos y}{\cos x}; \quad \dfrac{\cos y(\sin x - \sin y)}{\cos^2 x}$

Implicit differentiation also allows us to extend the power rule to rational exponents: Assume p and q are integers with $q \neq 0$. Then

$$\frac{d}{dx}(x^{\frac{p}{q}}) = \frac{p}{q}x^{\frac{p}{q}-1}$$

(provided $x \geq 0$ when q is even and $\frac{p}{q}$ is in lowest terms).

Exercise

Prove it.

3.8 Book Problems

5-25 (odds), 31-49 (odds)