Calculus I (Math 2554)

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The base for these slides was done by Dr. Shannon Dingman, later encoded in LaTeX by Dr. Brad Lutes.

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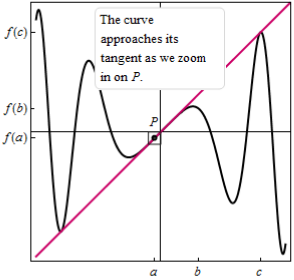
∮ 4.5 Linear Approximation and Differentials

Suppose f is a function such that f' exists at some point P. If you zoom in on the graph, the curve appears more and more like the tangent line to f at P.



Linear Approximation

This idea – that smooth curves (i.e., curves without corners) appear straighter on smaller scales – is the basis of linear approximations.

One of the properties of a function that is differentiable at a point P is that it is locally linear near P (i.e., the curve approaches the tangent line at P.)

Therefore, it makes sense to approximate a function with its tangent line, which matches the value and slope of the function at P.

This is why you've had to do so many "find the equation for the tangent line to the given point" problems!

Definition

Suppose f is differentiable on an interval I containing the point a. The **linear approximation** to f at a is the linear function

$$L(x) = f(a) + f'(a)(x-a) \qquad \text{for } x \text{ in } I.$$

Remarks: Compare this definition to the following: At a given point $P = (a, f(a))$, the slope of the line tangent to the curve at P is $f'(a)$. So the equation of the tangent line is

$$y - f(a) = f'(a)(x-a).$$

(Yes, it is the same thing!)

Exercise

Write the equation of the line that represents the linear approximation to

$$f(x) = \frac{x}{x+1} \qquad \text{at } a = 1.$$

Then *use* the linear approximation to estimate $f(1.1)$.

Solution: First compute

$$f'(x) = \frac{1}{(x+1)^2}, \quad f(a) = \frac{1}{2}, \quad f'(a) = \frac{1}{4}$$

and plug into the equation to get

$$L(x) = \frac{1}{2} + \frac{1}{4}(x-1) = \frac{1}{4}x + \frac{1}{4}.$$

Because $x = 1.1$ is near $a = 1$, we can estimate $f(1.1)$ using $L(1.1)$:

$$f(1.1) \approx L(1.1) = 0.525$$

Note that $f(1.1) = 0.5238$, so the error in this estimation is

$$\frac{0.525 - 0.5238}{0.5238} \times 100 = 0.23\%.$$

Intro to Differentials

Our linear approximation $L(x)$ is used to approximate $f(x)$ when a is fixed and x is a nearby point:

$$f(x) \approx f(a) + f'(a)(x - a)$$

When rewritten,

$$f(x) - f(a) \approx f'(a)(x - a)$$

$$\implies \Delta y \approx f'(a)\Delta x.$$

A change in y can be approximated by the corresponding change in x, magnified or diminished by a factor of $f'(a)$.

This is another way to say that $f'(a)$ is the rate of change of y with respect to x!

$$\Delta y \approx f'(a)\Delta x$$

$$\frac{\Delta y}{\Delta x} \approx f'(a)$$

So if f is differentiable on an interval I containing the point a, then the change in the value of f (the Δy), between two points a and $a + \Delta x$ in I, is approximately $f'(x)\Delta x$.

We now have two different, but related quantities:

- The change in the function $y = f(x)$ as x changes from a to $a + \Delta x$ (which we call Δy).
- The change in the linear approximation $y = L(x)$ as x changes from a to $a + \Delta x$ (called the differential, dy).

$$\Delta y \approx dy$$

When the x-coordinate changes from a to $a + \Delta x$:

- The function change is **exactly** $\Delta y = f(a + \Delta x) - f(a)$.
- The linear approximation change is

$$\begin{aligned}\Delta L &= L(a + \Delta x) - L(a) \\ &= (f(a) + f'(a)(a + \Delta x - a)) - (f(a) + f'(a)(a - a)) \\ &= f'(a)\Delta x\end{aligned}$$

and this is dy.

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We define the differentials dx and dy to distinguish between the change in the function (Δy) and the change in the linear approximation (ΔL):

- dx is simply the change in x, i.e. Δx.
- dy is the change in the linear approximation, which is $\Delta L = f'(a)\Delta x$.

In fact, we can write

$$\Delta L = f'(a)\Delta x$$

$$dy = f'(a)dx$$

$$\frac{dy}{dx} = f'(a) \quad \text{(at } x = a\text{)}$$

Definition

Let f be differentiable on an interval containing x.

- A small change in x is denoted by the **differential** dx.
- The corresponding change in $y = f(x)$ is approximated by the **differential** $dy = f'(x)dx$; that is,

$$\Delta y = f(x + \Delta x) - f(x)$$

$$\approx dy = f'(x)dx.$$

The use of differentials is critical as we approach integration.

Example

Use the notation of differentials $[dy = f'(x)dx]$ to approximate the change in $f(x) = x - x^3$ given a small change dx.

Solution: $f'(x) = 1 - 3x^2$, so $dy = (1 - 3x^2)dx$.
A small change dx in the variable x produces an approximate change of $dy = (1 - 3x^2)dx$ in y.

For example, if x increases from 2 to 2.1, then $dx = 0.1$ and

$$dy = \left(1 - 3(2)^2\right)(0.1) = -1.1.$$

This means as x increases by 0.1, y decreases by 1.1.

4.5 Book Problems

7-9, 11, 12, 29-38

$\oint$ 4.6 Mean Value Theorem

Theorem (Rolle's Theorem)

Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. Then there is at least one point c in (a, b) such that $f'(c) = 0$.

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Theorem (Mean Value Theorem (MVT))

If f is continuous on a closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

See Figure 4.68 in your text for a visual justification of MVT. The slope of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$ is

$$\frac{f(b) - f(a)}{b - a}.$$

MVT says that there is a point c on f where the tangent line at c (whose slope is $f'(c)$) is parallel to this secant line.

Example

Let $f(x) = x^2 - 4x + 3$.

(a) Determine whether the MVT applies to $f(x)$ on the interval $[-2, 3]$.

(b) If so, find the point(s) that are guaranteed to exist by the MVT.

Example

How many points c satisfy the conclusion of the MVT for $f(x) = x^3$ on the interval $[-1, 1]$? Justify your answer.

Consequences of MVT

Theorem (Zero Derivative Implies Constant Function)

If f is differentiable and $f'(x) = 0$ at all points of an interval I, then f is a constant function on I.

Theorem (Functions with Equal Derivatives Differ by a Constant)

If two functions have the property that $f'(x) = g'(x)$ for all x of an interval I, then $f(x) - g(x) = C$ on I, where C is a constant.

Consequences of MVT (cont.)

Theorem (Intervals of Increase and Decrease)

Suppose f is continuous on an interval I and differentiable at all interior points of I.

- *If $f'(x) > 0$ at all interior points of I, then f is increasing on I.*
- *If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.*

4.6 Book Problems

7, 10, 11, 13, 15, 17, 20-22, 24-26, 29

$\oint$ 4.7 L'Hôpital's Rule

In Ch. 2, we examined limits that were computed using analytical techniques. Some of these limits, in particular those that were indeterminate, could not be computed with simple analytical methods. For example,

$$\lim_{x\to 0}\frac{\sin x}{x} \quad \text{and} \quad \lim_{x\to 0}\frac{1-\cos x}{x}$$

are both limits that can't be computed by substitution, because plugging in 0 for x gives $\frac{0}{0}$.

Theorem (L'Hôpital's Rule $\left(\frac{0}{0}\right)$)

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is $\pm\infty$).

(The rule also applies if $x \to a$ is replaced by $x \to \pm\infty$, $x \to a^+$ or $x \to a^-$.)

Example

Evaluate the following limit:

$$\lim_{x \to -1} \frac{x^4 + x^3 + 2x + 2}{x + 1}.$$

Solution: By direct substitution, we obtain $\frac{0}{0}$. So we must apply l'Hôpital's Rule (LR) to evaluate the limit:

$$\begin{aligned} \lim_{x \to -1} \frac{x^4 + x^3 + 2x + 2}{x + 1} &\overset{\text{LR}}{=} \lim_{x \to -1} \frac{\frac{d}{dx}\left(x^4 + x^3 + 2x + 2\right)}{\frac{d}{dx}(x + 1)} \\ &= \lim_{x \to -1} \frac{4x^3 + 3x^2 + 2}{1} \\ &= -4 + 3 + 2 = 1 \end{aligned}$$

Theorem (L'Hôpital's Rule ($\frac{\infty}{\infty}$))

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is $\pm\infty$).

(The rule also applies if $x \to a$ is replaced by $x \to \pm\infty$, $x \to a^+$ or $x \to a^-$.)

Exercise

Evaluate the following limits using l'Hôpital's Rule:

1. $\lim\limits_{x\to\infty} \dfrac{4x^3 - 2x^2 + 6}{\pi x^3 + 4}$
2. $\lim\limits_{x\to 0} \dfrac{\tan 4x}{\tan 7x}$

L'Hôpital's Rule in Disguise

Other indeterminate limits in the form $0 \cdot \infty$ or $\infty - \infty$ cannot be evaluated directly using l'Hôpital's Rule. For $0 \cdot \infty$ cases, we must rewrite the limit in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. A common technique is to divide by the reciprocal:

$$\lim_{x \to \infty} x^2 \sin\left(\frac{1}{5x^2}\right) = \lim_{x \to \infty} \frac{\sin\left(\frac{1}{5x^2}\right)}{\frac{1}{x^2}}$$

Exercise

Compute $\displaystyle\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right)$.

For $\infty - \infty$, we can divide by the reciprocal as well as use a change of variables:

Example

Find $\lim_{x \to \infty} x - \sqrt{x^2 + 2x}$.

Solution:

$$\begin{aligned}\lim_{x\to\infty} x - \sqrt{x^2+2x} &= \lim_{x\to\infty} x - \sqrt{x^2(1+\tfrac{2}{x})}\\ &= \lim_{x\to\infty} x - x\sqrt{1+\tfrac{2}{x}}\\ &= \lim_{x\to\infty} x\left(1-\sqrt{1+\tfrac{2}{x}}\right)\\ &= \lim_{x\to\infty} \frac{1-\sqrt{1+\frac{2}{x}}}{\frac{1}{x}}\end{aligned}$$

This is now in the form $\frac{0}{0}$, so we can apply l'Hôpital's Rule and evaluate the limit. In this case, it may even help to change variables. Let $t = \frac{1}{x}$:

$$\lim_{x\to\infty} \frac{1-\sqrt{1+\frac{2}{x}}}{\frac{1}{x}} = \lim_{t\to 0^+} \frac{1-\sqrt{1+2t}}{t}.$$

Other Indeterminate Forms

Limits in the form 1^∞, 0^0, and ∞^0 are also considered indeterminate forms, and to use l'Hôpital's Rule, we must rewrite them in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Here's how: Assume $\lim\limits_{x\to a} f(x)^{g(x)}$ has the indeterminate form 1^∞, 0^0, or ∞^0.

1. Evaluate $L = \lim\limits_{x\to a} g(x)\ln f(x)$. This limit can often be put in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, which can be handled by l'Hôpital's Rule.
2. Then $\lim\limits_{x\to a} f(x)^{g(x)} = e^L$. Don't forget this step!

Example

Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Solution: This is in the form 1^∞, so we need to examine

$$\begin{aligned} L &= \lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right) \\ &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \\ &\overset{\text{LR}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1. \end{aligned}$$

NOT DONE! Write

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^L = e^1 = e.$$

Examining Growth Rates

We can use l'Hôpital's Rule to examine the rate at which functions grow in comparison to one another.

Definition

Suppose f and g are functions with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Then $\boldsymbol{f}$ **grows faster than** $\boldsymbol{g}$ as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \text{ or } \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty.$$

$g \ll f$ means that f grows faster than g as $x \to \infty$.

Definition

The functions f and g have **comparable growth rates** if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M, \text{ where } 0 < M < \infty.$$

Pitfalls in Using l'Hôpital's Rule

1. L'Hôpital's Rule says that $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$.

   NOT $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \left[\dfrac{f(x)}{g(x)}\right]'$ or $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \left[\dfrac{1}{g(x)}\right]' f'(x)$

   (i.e., don't confuse this rule with the Quotient Rule).

2. Be sure that the limit with which you are working is in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Pitfalls in Using l'Hôpital's Rule (cont.)

3. When using l'Hôpital's Rule more than once, simplify as much as possible before repeating the rule.
4. If you continue to use l'Hôpital's Rule in an unending cycle, another method must be used.

4.7 Book Problems

13-39 (odds), 43-51 (odds), 63-69 (odds)

Exam #3 Review

- $\oint$ 3.10 Related Rates
  - Know the steps to solving related rates problems, and be able to use them to solve problems given variables and rates of change.
  - Be able to solve related rates problems. If, while doing the HW (paper or computer), you were provided a formula in order to solve the problem, then I will do the same. If you were not provided a formula while doing the HW (paper or computer), then I also will not provide the formula.

Exam #3 Review (cont.)

- $\oint$4.1 Maxima and Minima
  - Know the definitions of maxima, minima, and what makes these points local or absolute extrema (both analytically and graphically).
  - Know how to find critical points for a function.
  - Given a function on a given interval, be able to find local and/or absolute extrema.
  - Given specified properties of a function, be able to sketch a graph of that function.

Exam #3 Review (cont.)

- $\oint$4.2 What Derivatives Tell Us
  - Be able to use the first derivative to determine where a function is increasing or decreasing.
  - Be able to use the First Derivative Test to identify local maxima and minima. Be able to explain in words how you arrived at your conclusion.
  - Be able to find critical points, absolute extrema, and inflection points for a function.

Exam #3 Review (cont.)

- Be able to use the second derivative to determine the concavity of a function.
- Be able to use the Second Derivative Test to determine whether a given point is a local max or min. Be able to explain in words how you arrived at your conclusion.
- Know your Derivative Properties!!! (see Figure 4.36)

Exam #3 Review (cont.)

- $\oint$4.3 Graphing Functions
  - Be able to find specific characteristics of a function that are spelled out in the Graphing Guidelines (e.g., know how to find x- and y-intercepts, vertical/horizontal asymptotes, critical points, inflection points, intervals of concavity and increasing/decreasing, etc.).
  - Be able to use these specific characteristics of a function to sketch a graph of the function.

Exam #3 Review (cont.)

- $\oint$4.4 Optimization Problems
  - Be able to solve optimization problems that maximize or minimize a given quantity.
  - Be able to identify and express the constraints and objective function in an optimization problem.
  - Be able to determine your interval of interest in an optimization problem (e.g., what range of x-values are you searching for your extreme points?)
  - **As to formulas, the same comment made above with respect to formulas for related rates problems applies here as well.**

Exam #3 Review (cont.)

- $\oint$ 4.5 Linear Approximation and Differentials
  - Be able to find a linear approximation for a given function.
  - Be able to use a linear approximation to estimate the value of a function at a given point.
  - Be able to use differentials to express how the change in x (dx) impacts the change in y (dy).

Exam #3 Review (cont.)

- $\oint$4.6 Mean Value Theorem
  - Know and be able to state Rolle's Thm and the Mean Value Thm, including knowing the hypotheses and conclusions for both.
  - Be able to apply Rolle's Thm to find a point in a given interval.
  - Be able to apply the MVT to find a point in a given interval.
  - Be able to use the MVT to find equations of secant and tangent lines.

Exam #3 Review (cont.)

- $\oint$ 4.7 L'Hôpital's Rule
  - Know how to use L'Hôpital's Rule, including knowing under what conditions the Rule works.
  - Be able to apply L'Hôpital's Rule to a variety of limits that are in indeterminate forms (e.g., $0/0$, ∞/∞, $0 \cdot \infty$, $\infty - \infty$, 1^∞, 0^0, ∞^0).
  - Be able to use L'Hôpital's Rule to determine the growth rates of two given functions.
  - Be aware of the pitfalls in using L'Hôpital's Rule.
  - PRACTICE THESE. Some of the book problems have non-obvious algebra tricks that simplify an otherwise crazy problem.

Pep Talk

- Read the question!
- Do the book problems.
- Find a buddy who understands concepts a little better than you and work on problems for 2-3 hours. Then find a buddy who is struggling and work with them 2-3 hours. Explaining to someone else tests how deeply you really know the material. This strategy also helps reduce stress because it doesn't require you to devote a full day or night of studying, just 2-3 hours at a time of productive work.

Pep Talk (cont.)

- If you encounter an unfamiliar type of problem on the exam, relax, because it's most likely not a trick. The solutions will always rely on the information from the required reading/assignments. Take your time and do each baby step carefully.
- During the exam, do the problems you are most confident with first! Different people will find different problems easier.
- The exam is not a race. If you finish early take advantage of the time to check your work. You don't want to leave feeling smug about how quickly you finished only to find out next week you lost a letter grade's worth of points from silly mistakes.

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