Wed 8 July

- Exam 3 redo
  - You may collaborate and use resources, including office hours.
  - The deadline to submit is 4pm Friday. NO EXCEPTIONS.
  - If you don't redo your problems, the grade sticks.
- FINAL! is next Thursday, in class. Covers 3.10-5.5.
- Quiz 10 tomorrow covers 4.7 and 5.1. Collaborative+open resources.
- Quiz 11 Friday covers 4.8,5.2, some 5.3.

Left, Right, and Midpoint Riemann Sums in Sigma Notation

Suppose f is defined on a closed interval $[a, b]$ which is divided into n subintervals of equal length Δx.

1. $\displaystyle\sum_{k=1}^{n} f(a + (k-1)\Delta x)\Delta x$ gives a left Riemann sum.
2. $\displaystyle\sum_{k=1}^{n} f(a + k\Delta x)\Delta x$ gives a right Riemann sum.
3. $\displaystyle\sum_{k=1}^{n} f\left(a + \left(k - \frac{1}{2}\right)\Delta x\right)\Delta x$ gives a midpoint Riemann sum.

Exercise

(a) Use sigma notation to write the left, right, and midpoint Riemann sums for the function $f(x) = x^2$ on the interval $[1, 5]$ given that $n = 4$.

(b) Based on these approximations, estimate the area bounded by the graph of $f(x)$ over $[1, 5]$.

5.1 Book Problems

9, 11, 15-23 (odds), 31, 33, 53-57

$\oint$ 5.2 Definite Integrals

In $\oint$5.1, we saw how we can use Riemann sums to approximate the area under a curve. However, the curves we worked with were all non-negative.

Question

What happens when the curve is negative?

Example

Let $f(x) = 8 - 2x^2$ over the interval $[0, 4]$. Use a left, right, and midpoint Riemann sum with $n = 4$ to approximate the area under the curve.

In the previous example, the places where f was positive provided positive contributions to the area, while places where f was negative provided negative contributions. The difference between positive and negative contributions is called the net area.

Definition

Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis, minus the sum of the areas of the parts of R that lie below the x-axis, on $[a, b]$.

The Riemann sums give approximations for the area under the curve and to make these approximations more and more accurate, we divide the region into more and more subintervals.

To make these approximations *exact*, we allow the number of subintervals $n \to \infty$, and consequently the length of the subintervals $\Delta x \to 0$.

In terms of limits:

$$\text{Net Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(\overline{x}_k)\Delta x.$$

General Riemann Sums

Suppose

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

so that $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ are subintervals of $[a, b]$.

The subintervals might have different lengths, so given each k, we let Δx_k be the length of the subinterval $[x_{k-1}, x_k]$. Let $\overline{x}_k$ denote any point in $[x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$.

General Riemann Sums (cont.)

If f is defined on $[a,b]$, then the sum

$$\sum_{k=1}^{n} f(\overline{x}_k)\Delta x_{k} = f(\overline{x}_1)\Delta x_{1} + f(\overline{x}_2)\Delta x_{2} + \cdots + f(\overline{x}_n)\Delta x_{n}$$

is called a **general Riemann sum for f on $[a,b]$**.

(In this definition, the lengths of the subintervals do not have to be equal.)

As $n \to \infty$, all of the Δx_ks approach 0, even the largest of these. Let Δ be the largest of the Δx_ks.

Definition

A function f defined on $[a, b]$ is **integrable** means the limit

$$\int_a^b f(x)\ dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(\overline{x}_k)\Delta x_k$$

exists – over all partitions of $[a, b]$ and all choices of $\overline{x}_k$ on a partition. This limit is called the **definite integral of f from a to b**.

Theorem

If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$.

See Figure 5.23 for an example of a noncontinuous function that is integrable.

Knowing the limit of a Riemann sum, we can now translate that to a definite integral.

Example

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} (4\overline{x}_k - 3)\Delta x_k \text{ on } [-1, 4] \iff \int_{-1}^{4} (4x - 3)\, dx.$$

Exercise

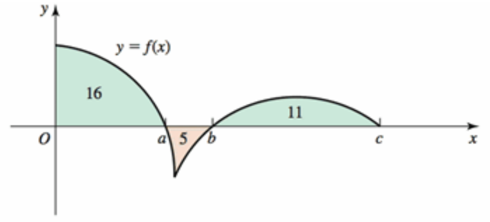
Using geometry, evaluate $\int_1^2 (4x-3)\,dx$.

(*Hint:* The area of a trapezoid is $\frac{h(a+b)}{2}$, where h is the height of the trapezoid and a and b are the lengths of the two parallel bases.)

Exercise

Using the picture below, evaluate the following definite integrals:

1. $\int_0^a f(x)\,dx$
2. $\int_0^b f(x)\,dx$
3. $\int_0^c f(x)\,dx$
4. $\int_a^c f(x)\,dx$



Properties of Integrals

1. (Reversing Limits) $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$
2. (Identical Limits) $\int_a^a f(x)\ dx = 0$
3. (Integral of a Sum)
   $$\int_a^b (f(x) + g(x))\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$
4. (Constants in Integrals) $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$

Properties of Integrals (cont.)

5. (Integrals over Subintervals) If c lies between a and b, then

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

6. (Integrals of Absolute Values) The function $|f|$ is integrable on $[a, b]$ and $\int_a^b |f(x)|\ dx$ is the sum of the areas of regions bounded by the graph of f and the x-axis on $[a, b]$. (See Figure 5.31)

   (This is the total area, no negative signs.)

Exercise

Given

$$\int_2^4 f(x)\ dx = 3 \quad \text{and} \quad \int_4^6 f(x)\ dx = -2,$$

compute

$$\int_2^6 f(x)\ dx.$$

5.2 Book Problems

11-29 (odds), 35-44, 65

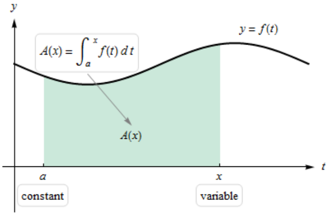
$\oint$ 5.3 Fundamental Theorem of Calculus

Using Riemann sums to evaluate definite integrals is usually neither efficient nor practical. We will develop methods to evaluate integrals and also tie together the concepts of differentiation and integration.

Area Functions

Let $y = f(t)$ be a continuous function which is defined for all $t \geq a$, where a is a fixed number. The area function for f with left endpoint at a is given by

$$A(x) = \int_a^x f(t)\,dt.$$



This gives the net area of the region between the graph of f and the t-axis between the points $t = a$ and $t = x$. (See Figure 5.33 for pictures of the area function in action.)

Exercise

The graph of f is shown below. Let

$$A(x) = \int_0^x f(t)\ dt \quad \text{and} \quad F(x) = \int_2^x f(t)\ dt$$

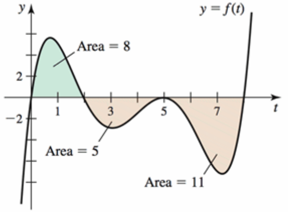
be two area functions for f. Compute:

(a) $A(2)$

(b) $F(5)$

(c) $A(5)$

(d) $F(8)$



Theorem (FTOC I)

If f is continuous on $[a, b]$, then the area function is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$; or equivalently,

$$A'(x) = \frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$

(the area function of f is an antiderivative of f).

Since A is an antiderivative of f, we now have a way to evaluate definite integrals and find areas under curves.

Theorem (FTOC II)

If f is continuous on $[a,b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

We use the notation $F(x)|_a^b = F(b) - F(a)$.

In essence, to evaluate an integral, we

- Find any antiderivative of f, and call if F.
- Compute $F(b) - F(a)$, the difference in the values of F between the upper and lower limits of integration.

The two parts of the FTOC illustrate the inverse relationship between differentiation and integration – the integral "undoes" the derivative.

Exercise

1. Use Part 1 of the FTOC to simplify $\dfrac{d}{dx}\displaystyle\int_x^{10} \frac{dz}{z^2+1}$.
2. Use Part 2 of the FTOC to evaluate $\displaystyle\int_0^{\pi} (1-\sin x)\, dx$.
3. Compute $\displaystyle\int_1^{y} h'(p)\, dp$.

5.3 Book Problems

11-17, 19-39 (odds), 45-57 (odds)