Topology Solutions - 4 Sep 2010

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Morning Session

Problem 1. Say that a metric space X has property (A) if the image of every continuous function $f : X \to \mathbb{R}$ is an interval, which may be open, closed, or half-open. Prove that X has property (A) if and only if it is connected.

Solution. Assert rays are also intervals, since they are homeomorphic to them. Suppose X is not connected, so has a nontrivial separation $X = A \amalg B$. Let $[a, b]$ be a closed interval in $\mathbb{R}$ such that $a \neq b$. Urysohn's lemma gives a continuous map $X \to [a, b]$ such that $f(x) = a$ for all $x \in A$ and $f(x) = b$ for all $x \in B$. Then $f(X) = \{a, b\}$ and so X cannot have property (A).

Conversely, suppose X is connected and assume $f(X)$ is not an interval, for some continuous $f : X \to \mathbb{R}$. In particular, $f(X)$ is not connected, so has a nontrivial (relatively) clopen set Z. Then $f^{-1}(Z)$ is clopen in X by continuity of f, and is nonempty by choice of Z. Furthermore, $f^{-1}(Z)$ is strictly contained in X or else $f(X) = Z$, contradicting the choice of Z. Conclude $f^{-1}(Z)$ is a nontrivial clopen set in X. But then that contradicts X connected. So property (A) must hold. □

Problem 2. Consider $\mathrm{SL}_n\mathbb{R}$ as a group and as a topological space with the topology induced from $\mathbb{R}^{n^2}$. Show that if $H \subset \mathrm{SL}_n\mathbb{R}$ is an abelian subgroup, then the closure $\overline{H}$ of $\mathrm{SL}_n\mathbb{R}$ is also an abelian subgroup.

Solution. Matrix multiplication and addition are continuous functions from $\mathbb{R}^{n^2} \times \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$. Suppose A is a limit point for H. So any neighborhood of A contains a point in H. In particular, for any ϵ there exists a δ-neighborhood of A containing $A' \in H$ such that for $B \in H$,

$$\begin{aligned} \|AB - A'B\| &< \frac{\epsilon}{2} \\ \|BA' - BA\| &< \frac{\epsilon}{2}. \end{aligned}$$

By the Triangle Inequality,

$$\begin{aligned} \|AB - BA\| &\leq \|AB - A'B\| + \|BA' - BA\| \\ &< \epsilon. \end{aligned}$$

If B is a limit point of H as well, then the same argument applies since A and B each commute with everything in H. □

Problem 3. Recall that the complex projective space $\mathbb{C}P^d$ is the quotient space of $\mathbb{C}^{d+1} \smallsetminus \{0\}$ under the equivalence relation $x \sim y$ if and only if there is $\lambda \in \mathbb{C}$ with $x = \lambda y$. Prove that $\mathbb{C}P^d$ is a compact, connected, orientable manifold of dimension $2d$.

Solution. $\mathbb{C}^{d+1} \setminus \{0\}$ is homotopy equivalent to $S^{2d+1} \subset \mathbb{R}^{2(d+1)}$, so $\mathbb{C}P^d$ is equivalently constructed as a quotient of S^{2d+1}. S^{2d+1} is compact and connected then implies $\mathbb{C}P^d$ is compact and connected.

In general, $\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a real $2n$-cell via the quotient map $S^{2n-1} \to \mathbb{C}P^{n-1}$. This means $\mathbb{C}P^d$ is a manifold with CW-structure consisting of exactly one cell in each even real dimension, up to $2d$. In particular, its first homology group is trivial. As the abelianization of the fundamental group, this implies $\pi_1(\mathbb{C}P^d)$ is trivial. This implies $\mathbb{C}P^d$ is its own universal cover, so must be orientable, and it is simply connected. □

Problem 4. Consider the 2-dimensional torus $\mathbb{T}^2$ and the topological space

$$X = \mathbb{T}^2 \times [-1,1]/\sim$$

where $(x,t) \sim (x',t')$ if either $(x,t) = (x',t')$, or $t = t' \in \{-1,1\}$. Compute $H_*(X;\mathbb{Z})$.

Solution. Write $X = (\mathbb{T}^2 \times [-1,0]/\sim) \cup (\mathbb{T}^2 \times [0,1]/\sim)$ and let U and V denoted the respective components. Each of U and V is homotopy equivalent to a point because of the equivalence relation in defining X. Their intersection is a torus, T. Use a Mayer-Vietoris sequence to compute the homology:

$$\begin{aligned}\cdots \to 0 \to H_3(X) \to H_2(T) \to H_2(U) \oplus H_2(V) \to H_2(X) \to H_1(T) \to \\ \to H_1(U) \oplus H_1(V) \to H_1(X) \to H_0(T) \to H_0(U) \oplus H_0(V) \to H_0(X) \to 0.\end{aligned}$$

Then the homology groups are

$$\begin{aligned} H_3(X) &\cong H_2(T) \\ &\cong \mathbb{Z}, \\ H_2(X) &\cong H_1(T) \\ &\cong \mathbb{Z} \oplus \mathbb{Z}, \\ H_1(X) &\cong 0, \text{ and} \\ H_0(X) &\cong \mathbb{Z}, \end{aligned}$$

because X is connected. The higher homology groups vanish. □

Problem 5. Let $\mathbb{S}^4$ be the 4-dimensional sphere and suppose that $\pi : \mathbb{S}^4 \to M$ is a local homeomorphism onto an orientable manifold. Prove that π is a homeomorphism and give an example showing that the orientability condition is necessary.

Solution. First, S^4 is compact Hausdorff, so π must be a cover. Since S^4 is simply connected, S^4 must then be the universal cover of M. S^4 is path connected and locally path connected, and the universal cover is regular, so $M \cong S^4/G$ where G acts freely and properly discontinuously on S^4. The only nontrivial free and properly discontinuous group action on S^4 is $\mathbb{Z}_2$ because 4 is even. But $S^4/\mathbb{Z}_2 \cong \mathbb{R}P^4$ is nonorientable. Since M is orientable, it must be the quotient of the trivial action on S^4, hence $M \cong S^4$. $\mathbb{R}P^4$ gives the desired counterexample when M is not orientable. □

Afternoon Session

Problem 1. Let G be a cyclic group, $G \curvearrowright \mathbb{S}^1$ an effective action by rotations and endow the quotient $\mathbb{S}^1/G$ with the quotient topology. Prove that $\mathbb{S}^1/G$ is T_0 if and only if G is finite.

Solution. Suppose G is infinite. Rotation must be by $2\pi q$, where q is irrational, because G is cyclic and must act effecively on $\mathbb{S}^1$. Assume $\mathbb{S}^1/G$ is T_0. Let $x \neq y \in \mathbb{S}^1/G$ and $U \subset \mathbb{S}^1/G$ open with $x \in U$, $y \notin U$. The preimage of U covers $\mathbb{S}^1$, hence intersects the preimage of y. But then that implies $y \in U$, a contradiction. So $\mathbb{S}^1/G$ cannot be T_0.

Conversely, let $n < \infty$ denote the order of G. Choose $x, y \in \mathbb{S}^1/G$. The preimage K of $\{x, y\}$ consists of $2n$ points. $\mathbb{S}^1$ is a metric space with the induced metric d from $\mathbb{R}^2$. Put $\delta = \min_{x' \neq y' \in K} d(x', y')$ and $\epsilon = \frac{\delta}{3}$. The ϵ-balls centered at each point of K with $\mathbb{S}^1$ are disjoint, as are their images in $\mathbb{S}^1/G$. This means $\mathbb{S}^1/G$ is in fact Hausdorff, hence T_0. $\square$

Problem 2. Consider the standard sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3, \|x\| = 1\}$ and

$$T^1\mathbb{S}^2 = \{(x, y) \in \mathbb{S}^2 \times \mathbb{S}^2, x \perp y\}$$

with the induced topology. Prove that $T^1\mathbb{S}^2$ is homeomorphic to SO_3.

Solution. Let e_1, e_2 and e_3 denote the standard unit vectors in $\mathbb{R}^3$. Suppose $(x, y) \in T^1\mathbb{S}^2$. Regarding x and y as vectors, there is a unique rotation $T_{x,y} \in \mathrm{SO}_3$ such that $T_{x,y}(e_1) = x, T_{x,y}(e_2) = y$. Define $h : T^1\mathbb{S}^2 \to \ SO_3$ by $h : (x, y) \mapsto T_{x,y}$. Rotation is continuous, so a neighborhood of $T \in \mathrm{SO}_3$ maps e_1, e_2 to a neighborhood of x, y. Hence h is continuous. Define the inverse $k : \mathrm{SO}_3 \to T^1\mathbb{S}^2$ by $k : T \mapsto (T(e_1), T(e_2))$. By the same argument k is continuous, and by construction h and k are inverses to eachother. Therefore $T^1\mathbb{S}^2 \cong \mathrm{SO}_3$. $\square$

Problem 3. Suppose that $M^d \subset \mathbb{R}^n$ is a d-dimensional smooth submanifold of n-dimensional Euclidean space. Prove that $\mathbb{R}^n \smallsetminus M^d$ is simply connected if $n - d \geq 3$.

Solution. A loop $\gamma \subset \mathbb{R}^n$ is homotopy equivalent to $S^1 \subset P$, where P is a plane. If γ is not contractible in $\mathbb{R}^n \smallsetminus M^d$ then P must intersect M^d in $\mathbb{R}^n$. If $n - d \geq 3$ then $d + 2 < n$. Therefore, if $P \cap M^d \neq \emptyset$ then at no point can the intersection be transversal. A perturbation of P makes the intersection empty. Then γ is contractible on P, so $\mathbb{R}^n \smallsetminus M^d$ is simply connected. $\square$

Problem 4. Let K be the image of an embedding of $\mathbb{S}^1 \times \mathbb{D}^2$ into $\mathbb{S}^3$. Compute $H_1(\mathbb{S}^3 \smallsetminus K; \mathbb{Z})$.

Solution. Write $\mathbb{S}^3 = K \cup \mathbb{S}^3 \smallsetminus K$; since K is the solid torus, the intersection is the boundary ∂K, which is a torus, T. Use a Mayer-Vietoris equence to find $H_1(\mathbb{S}^3 \smallsetminus K; \mathbb{Z})$. Note K is homotopy equivalent to $\mathbb{S}^1$, hence has the same homology groups.

$$\cdots \to 0 \to H_2(\mathbb{S}^3) \to H_1(T) \to H_1(K) \oplus H_1(\mathbb{S}^3 \smallsetminus K) \to H_1(\mathbb{S}^3) \to \cdots$$

becomes

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus H_1(\mathbb{S}^3 \setminus K) \to 0$$

and so $H_1(\mathbb{S}^3 \setminus K) \cong \mathbb{Z}$. $\square$

Problem 5. Suppose X is a connected compact CW-complex. Prove that $H_1(X; \mathbb{Z})$ is finite if and only if every map $X \to \mathbb{S}^1$ is homotopic to a constant map.

Solution. Any map $X \to \mathbb{S}^1$ induces a map on homology, $H_i(X) \to H_i(\mathbb{S}^1)$. Consider the double complex

$$\begin{tikzcd}
 & 0 \arrow[r] & H_2(\mathbb{S}^2) \arrow[r] & H_1(\mathbb{S}^1) \arrow[r] & H_0(\mathbb{S}^1) \arrow[r] & 0 \\
\cdots \arrow[r] & H_3(X) \arrow[r] \arrow[u] & H_2(X) \arrow[r] \arrow[u] & H_1(X) \arrow[r] \arrow[u] & H_0(X) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where the rows are exact and the squares commute (since H_i is a functor). If $H_1(X)$ is finite then it cannot map non-trivially onto $\mathbb{Z}$, so $H_1(X) \to H_1(\mathbb{S}^1)$ is the zero map. Since $H_0(\mathbb{S}^1) \cong H_0(X) \cong \mathbb{Z}$, by commutativity $H_1(X) \to H_0(X)$ must be the zero map. Then every 1-ball in X has a boundary; since X is connected, it is thus contractible. So $X \to \mathbb{S}^1$ factors through a point, which maps to a constant.

Conversely, if $f : X \to \mathbb{S}^1$ is homotopic to a constant map, then the induced map on fundamental groups, and hence first homology groups, is trivial. Since this works for *any* such f, $H_1(X)$ must be either finite or trivial. $\square$