UNIT 3, LESSON 3

Concavity

In this lesson, the second derivative is used to describe the graphical behavior of functions. Students will be able to:

- Find intervals where a function is concave up or concave down.
- Find inflection points.
- Use the second derivative test to find local extrema.

As we begin U3L3 we must have a clear idea about the meaning of a function and its derivative.

$f(x)$:

y –coordinates on the graph of f

$f'(x)$:

Slope of the tangent line to the graph of f

Instantaneous rate of change of f

If f is Increasing/Decreasing

The derivative f' is the “speedometer” for f. It tells us how fast f is changing and whether it is rising or falling.

Suppose you are given the function $f(x) = 3x^2 + 1$ and asked to find the instantaneous rate of change at $x = 1$. Which would give you the correct answer?

A. $f(1)$

B. $f(2)$

C. $f'(1)$

D. $f'(2)$

E. $f\left(f'(1)\right)$

F. $f\left(f'(2)\right)$

Suppose $f(x) = 3x^2 + 1$. If you wanted to find the y –coordinate on the graph of f when $x = 2$ what would you compute?

A. $f(1)$
B. $f(2)$
C. $f'(1)$
D. $f'(2)$

f' is a function. Suppose you were asked to find the instantaneous rate of change of f' at $x = 2$. What would you do?

A. Find the derivative of f' and plug in $x = 2$
B. Plug $x = 2$ into f'
C. None of the above.

Notation for Higher Derivatives

The second derivative of $y = f(x)$ can be written using any of the following notations:

$$f''(x), \quad \frac{d^2y}{dx^2}, \quad \text{or} \quad D_x^2[f(x)].$$

The third derivative can be written in a similar way. For $n \geq 4$, the nth derivative is written $f^{(n)}(x)$.

Find $f''(1)$ if $f(x) = 5x^4 - 4x^3 + 3x$.

Find the second derivative for $f(x) = (x^3 + 1)^2$

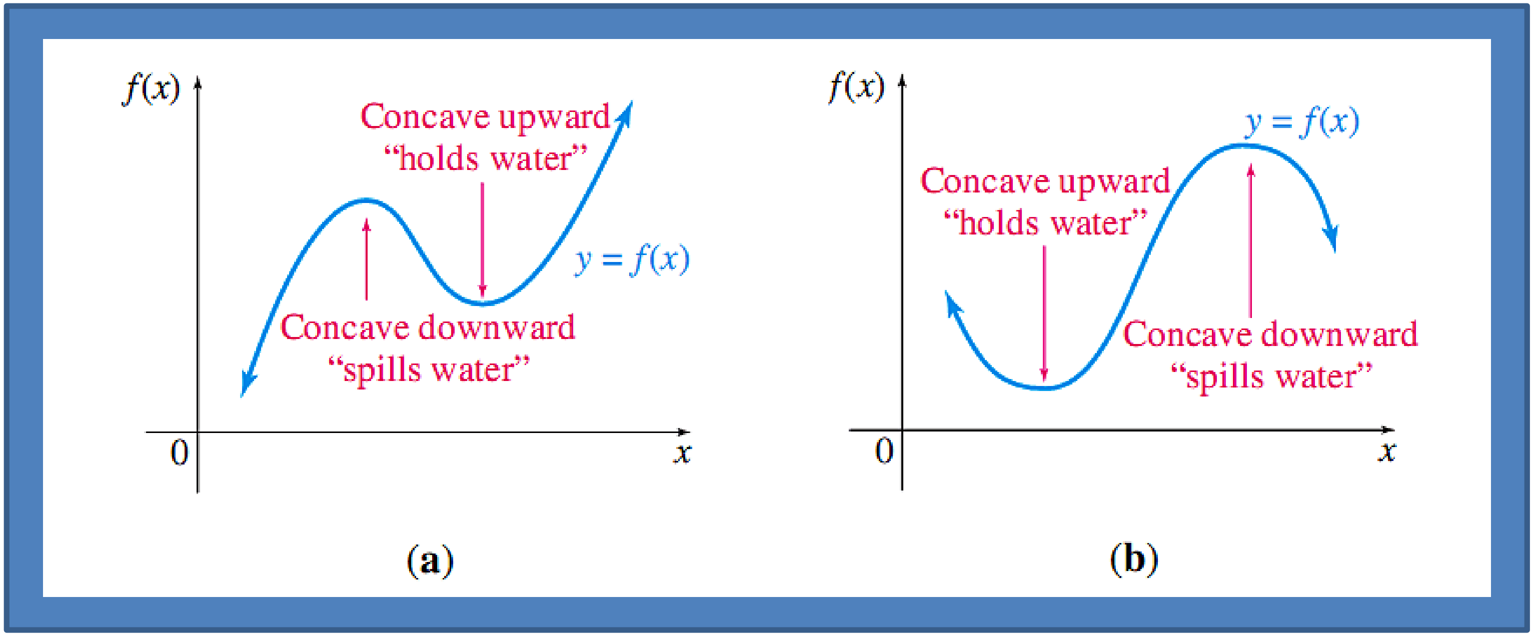
A. $f''(x) = 6x(x^3 + 1)^3$

B. $f''(x) = 30x^4 + 12x$

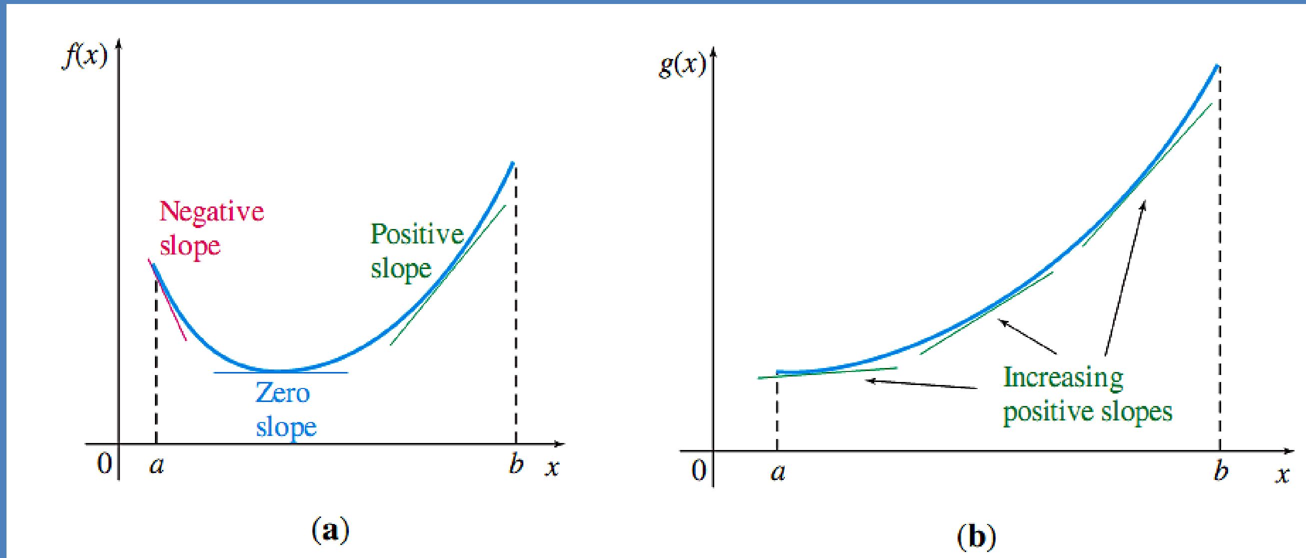
C. $f''(x) = 6x^4 + 12x$

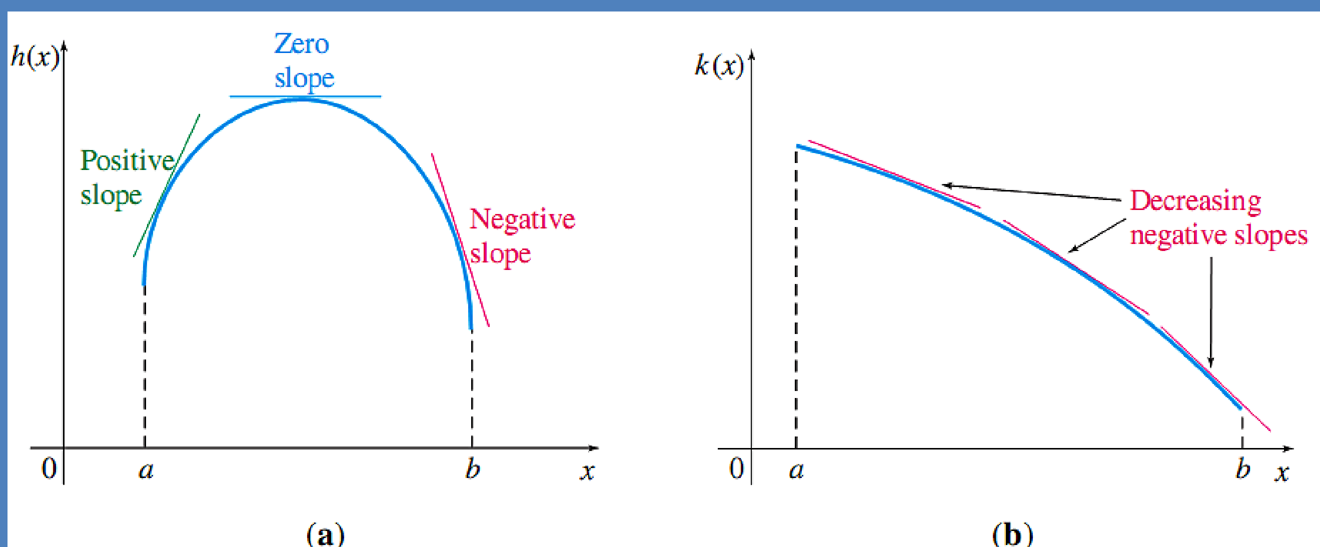
D. $f''(x) = 30x^2 - 12x$

Concavity:



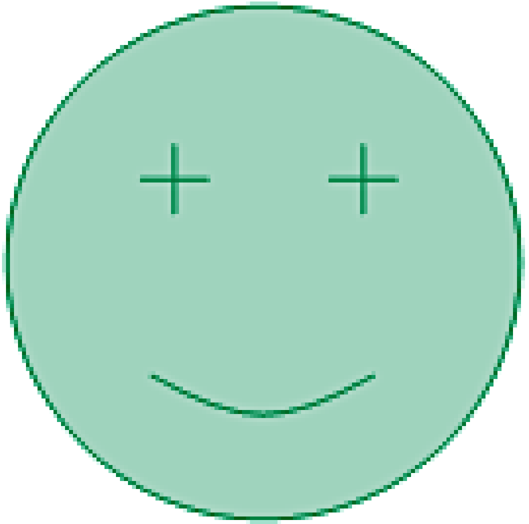
$f(x)$
Concave upward
"holds water"
$y = f(x)$
Concave downward
"spills water"
0
x
(a)
$f(x)$
$y = f(x)$
Concave upward
"holds water"
Concave downward
"spills water"
0
x
(b)

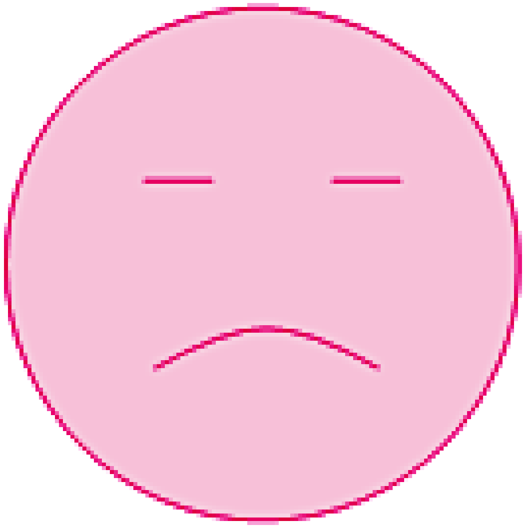

f(x)
Negative
slope
Positive
slope
Zero
slope
0
a
b
x
(a)
g(x)
Increasing
positive slopes
0
a
b
x
(b)


h(x)
Zero
slope
Positive
slope
Negative
slope
0
a
b
x
(a)
k(x)
Decreasing
negative slopes
0
a
b
x
(b)

Test for Concavity

Let f be a function with derivatives f' and f'' existing at all points in an interval (a, b). Then f is concave upward on (a, b) if $f''(x) > 0$ for all x in (a, b) and concave downward on (a, b) if $f''(x) < 0$ for all x in (a, b).





Example: Find all intervals where $f(x) = x^4 - 8x^3 + 18x^2$ is concave upward or downward, and find all inflection points.

Example: Find the open intervals where the function $f(x) = \frac{4}{x-2}$ is concave upward or concave downward. Find any inflection points.

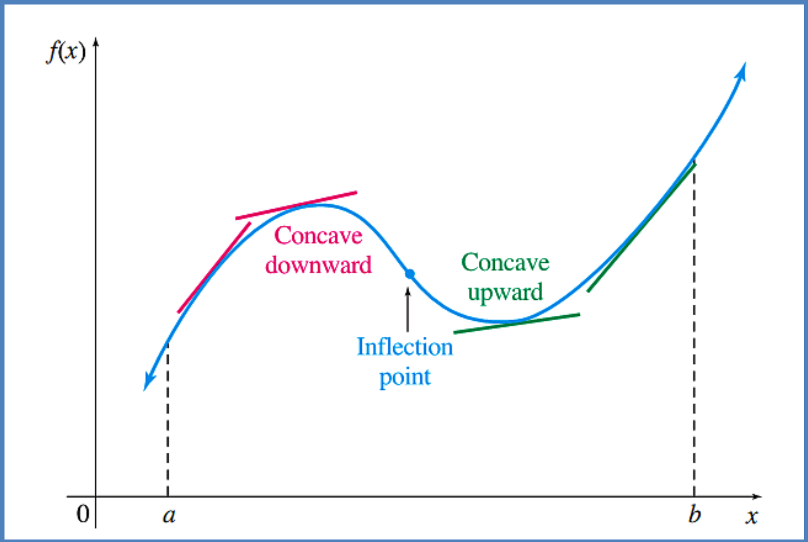
Find any open intervals where the function $f(x) = \ln(x^2 + 100)$ is concave upward.

A. $(-10,10)$

B. $(-\infty, -10) \cup (10, \infty)$

C. $(-\infty, -10)$

D. $(10, \infty)$



Second Derivative Test

Let f'' exist on some open interval containing c, (except possibly at c itself) and let $f'(c) = 0$.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.
3. If $f''(c) = 0$ or $f''(c)$ does not exist, then the test gives no information about extrema, so use the first derivative test.

Example: Use the second derivative test to find where all relative extrema for

$$f(x) = 4x^3 + 7x^2 - 10x + 8$$

Use the second derivative test to find where the relative extrema occur for $f(x) = 2x^3 - 3x^2 - 72x + 15$.

A. Relative max at $x = -3$. Relative min at $x = 4$.

B. Relative min at $x = -3$. Relative max at $x = 4$.

C. Relative max at $x = 3$. Relative min at $x = -4$.

Things that each function tells you

$f(x)$:

y −coordinates on the graph of f

$f'(x)$:

Slope of the tangent line to the graph of f

Instantaneous rate of change of f

Whether f is increasing or decreasing

$f''(x)$:

Concavity of the graph of f