Wed 17 June

- Midterm this Friday. Stay tuned for more info.
  - up to $\oint$3.9
  - 12-13 questions
  - 80 minutes
  - syllabus-approved calculator
- MLP due dates are Friday and Sunday

1 Week 4: 15-19 June
- Wednesday 17 June

∮3.8 Derivatives of Logarithmic and Exponential Functions
- Derivative of $y = b^x$
- Story Problem Example
- Derivatives of General Logarithmic Functions
- Neat Trick: Logarithmic Differentiation
- Book Problems

∮3.9 Derivatives of Inverse Trigonometric Functions
- Derivative of Inverse Sine
- Derivative of Inverse Tangent
- Derivative of Inverse Secant
- All Other Inverse Trig Derivatives
- Derivatives of Inverse Functions in General
- Book Problems

What about other logs? Say $b > 0$. Since $b^x = e^{\ln b^x} = e^{x \ln b}$ (by 3. on the earlier slide),

$$\begin{aligned} \frac{d}{dx}(b^x) &= \frac{d}{dx}(e^{x \ln b}) \\ &= e^{x \ln b} \cdot \ln b \\ &= b^x \ln b. \end{aligned}$$

Exercise

Find the derivative of each of the following functions:

- $f(x) = 14^x$
- $g(x) = 45(3^{2x})$

Exercise

Determine the slope of the tangent line to the graph $f(x) = 4^x$ at $x = 0$.

Story Problem Example

Example

The energy (in Joules) released by an earthquake of magnitude M is given by the equation

$$E = 25000 \cdot 10^{1.5M}.$$

(a) How much energy is released in a magnitude 3.0 earthquake?

(b) What size earthquake releases 8 million Joules of energy?

(c) What is $\frac{dE}{dM}$ and what does it tell you?

The relationship $y = \ln x \iff x = e^y$ applies to logarithms of other bases:

$$y = \log_b x \quad \iff \quad x = b^y.$$

Now taking $\frac{d}{dx}\left(x = b^y\right)$ we obtain

$$1 = b^y \ln b \left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{1}{b^y \ln b}$$

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

Example

Compute the derivative of $f(x) = \dfrac{x^2(x-1)^3}{(3+5x)^4}$.

Solution: We can use logarithmic differentiation – first take the natural log of both sides and then use properties of logarithms.

$$\begin{aligned}\ln(f(x)) &= \ln\left(\frac{x^2(x-1)^3}{(3+5x)^4}\right)\\ &= \ln x^2 + \ln (x-1)^3 - \ln (3+5x)^4\\ &= 2\ln x + 3\ln(x-1) - 4\ln(3+5x)\end{aligned}$$

Now we take $\frac{d}{dx}$ on both sides:

$$\frac{1}{f(x)}\left(\frac{df}{dx}\right) = 2\left(\frac{1}{x}\right) + 3\left(\frac{1}{x-1}\right) - 4\left(\frac{1}{3+5x}\right)(5)$$

$$\frac{f'(x)}{f(x)} = \frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}$$

Finally, solve for $f'(x)$:

$$f'(x) = f(x)\left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}\right]$$

$$= \frac{x^2(x-1)^3}{(3+5x)^4}\left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}\right]$$

3.8 Book Problems

9-27 (odds), 31-37 (odds), 41-47 (odds)

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$\oint$ 3.9 Derivatives of Inverse Trigonometric Functions

Recall: If $y = f(x)$, then $f^{-1}(x)$ is the value of y such that $x = f(y)$.

Example

If $f(x) = 3x + 2$, then what is $f^{-1}(x)$?

NOTE: $f^{-1}(x) \neq f(x)^{-1} \left(= \dfrac{1}{f(x)}\right)$

Trig functions are functions, too. Just like with "f", there has to be something to "plug in". It makes no sense to just say $\sin$, without having $\sin($*something*$)$.

$$y = \sin^{-1} x \Longleftrightarrow x = \sin y$$

The derivative of $y = \sin^{-1} x$ can be found using implicit differentiation:

$$\begin{aligned} x &= \sin y \\ \frac{d}{dx}(x) &= \frac{d}{dx}(\sin y) \\ 1 &= (\cos y)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\cos y} \end{aligned}$$

We still need to replace $\cos y$ with an expression in terms of x. We use the trig identity $\sin^2 y + \cos^2 y = 1$ (careful with notation: in this case we mean $(\sin y)^2 + (\cos y)^2 = 1$). Then

$$\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - x^2}.$$

The range of $y = \sin^{-1} x$ is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. In this range, cosine is never negative, so we can just take the positive portion of the square root. Therefore,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}} \implies \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.$$

Exercise

Compute the following:

1. $\dfrac{d}{dx}\left(\sin^{-1}(4x^2-3)\right)$
2. $\dfrac{d}{dx}\left(\cos(\sin^{-1}x)\right)$

Derivative of Inverse Tangent

Similarly to inverse sine, we can let $y = \tan^{-1} x$ and use implicit differentiation:

$$\begin{aligned}
x &= \tan y \\
\frac{d}{dx}(x) &= \frac{d}{dx}(\tan y) \\
1 &= (\sec^2 y)\frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{1}{\sec^2 y}
\end{aligned}$$

Use the trig identity $\sec^2 y - \tan^2 y = 1$ to replace $\sec^2 y$ with $1 + x^2$:

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

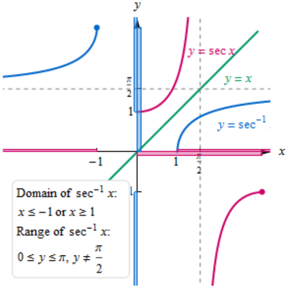
Derivative of Inverse Secant

$$
\begin{aligned}
y &= \sec^{-1} x \\
x &= \sec y \\
\frac{d}{dx}(x) &= \frac{d}{dx}(\sec y) \\
1 &= \sec y \tan y \frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{1}{\sec y \tan y}
\end{aligned}
$$

Use the trig identity $\sec^2 y - \tan^2 y = 1$ again to get

$$\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}.$$

This time, the $\pm$ matters:



- If $x \geq 1$, then $0 \leq y < \frac{\pi}{2}$ and so $\tan y > 0$.
- If $x \leq -1$, then $\frac{\pi}{2} < y \leq \pi$ and so $\tan y < 0$.

Therefore,

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

Using other trig identities (which you do not need to prove)

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2} \quad \cot^{-1} x + \tan^{-1} x = \frac{\pi}{2} \quad \csc^{-1} x + \sec^{-1} x = \frac{\pi}{2}$$

we can get the rest of the inverse trig derivatives.

All Other Inverse Trig Derivatives

To summarize:

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

$(-1 < x < 1)$

$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

$(-\infty < x < \infty)$

$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$

$(|x| > 1)$

Example

Compute the derivatives of $f(x) = \tan^{-1}\left(\frac{1}{x}\right)$ and $g(x) = \sin\left(\sec^{-1}(2x)\right)$.

Derivatives of Inverse Functions in General

Let f be differentiable and have an inverse on an interval I. Let x_0 be a point in I at which $f'(x_0) \neq 0$. Then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}$$

where $y_0 = f(x_0)$.

Example

Let $f(x) = \frac{1}{4}x^3 + x - 1$. Find $\left(f^{-1}\right)'(3)$.

3.9 Book Problems

7-27 (odds), 31-39 (odds)