Quotient groups

Cosets

Definition 1

Let $(G, \star)$ denote a group with subgroup $H < G$ and suppose $g \in G$. The set

$$g \star H := \{g \star h \mid h \in H\}$$

is called a **(left) coset** *of H in G.*

When using multiplicative notation we may write $gH = g \star H$; likewise with additive notation we write $g + H = g \star H$.

Question

What is the condition for a coset to be a subgroup? (In general, a coset is NOT a subgroup!)

Example 1

In $\mathbb{Z}_{12}$, the cosets of $H := \{0, 4, 8\} = 4\mathbb{Z}_{12}$ are:

$$\begin{aligned} H = 0 \oplus_{12} H = 4 \oplus_{12} H = 8 \oplus_{12} H &= \{0, 4, 8\}, \\ 1 \oplus_{12} H = 5 \oplus_{12} H = 9 \oplus_{12} H &= \{1, 5, 9\}, \\ 2 \oplus_{12} H = 6 \oplus_{12} H = 10 \oplus_{12} H &= \{2, 6, 10\}, \\ 3 \oplus_{12} H = 7 \oplus_{12} H = 11 \oplus_{12} H &= \{3, 7, 11\} \end{aligned}$$

Question

What are some observations you can make?

We use G/H to denote the collection of distinct cosets of H in G, called G **modulo** H. The cosets of a subgroup **partition** the group:

Proposition 1

Let G denote a group with subgroup $H < G$.

(a) *The union of all (left, respectively, right) cosets of H in G is the entire group G.*

(b) *For any two cosets $g \star H, h \star H \in G/H$, either*

  (i) $g \star H = h \star H$ *or*

  (ii) $g \star H \cap h \star H = \emptyset$.

$\square$

Exercise 1 (cf. Problem 66)

Let $G = \mathbb{Z}_{30}$ and put $H = 5G$. Using Proposition 1, list the elements of G/H.

Exercise 2 (cf. Problem 67)

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ and let $H = (1,1)G < G$. List the elements of H, then list the cosets of H.

The following proposition gives a way to prove two cosets are equal.

Proposition 2

Suppose $(G, \star)$ is a group with subgroup $H < G$ and suppose $g, h \in G$. Then:

(a) $g \star H = H$ *if and only if* $g \in H$

(b) $g \star H = h \star H$ *if and only if* $g^{-1}h \in H$.

(c) $g \star H = h \star H$ *if and only if* $h \in g \star H$.

Exercise 3 (cf. Problem 68)

Prove Proposition 2. *Hint: Prove (a) first, then use it to prove (b), then use (b) to prove (c).*

Quotient groups

Let $G = (G, \star)$ denote an abelian group with subgroup $H < G$. The notation used thusfar suggest a group structure on G/H with a binary operation $\star_{/H}$ well-defined "up to", or *modulo* elements in H. The natural choice is to define

$$\begin{aligned} \star_{/H} : G/H \times G/H &\to G/H \\ (g \star H, h \star H) &\mapsto (g \star h) \star H. \end{aligned} \tag{1.1}$$

Given $g_1, g_2, h_1, h_2 \in G$, we must verify

$$\begin{aligned} & (g_1 \star H, h_1 \star H) = (g_2 \star H, h_2 \star H) \\ \implies & (g_1 \star H) \star_{/H} (h_1 \star H) = (g_2 \star H) \star_{/H} (h_2 \star H) \\ \implies & (g_1 \star h_1) \star H = (g_2 \star h_2) \star H. \end{aligned}$$

Component-wise, we have, by hypothesis,

$$g_1 \star H = g_2 \star H \quad \text{and} \quad h_1 \star H = h_2 \star H.$$

Along with associativity,

$$\begin{aligned}
(g_1 \star h_1) \star H = g_1 \star (h_1 \star H) \\
= g_1 \star (h_2 \star H) &= g_1 \star (H \star h_2) \\
&= (g_1 \star H) \star h_2 \\
&= (g_2 \star H) \star h_2 \\
&= g_2 \star (H \star h_2) = g_2 \star (h_2 \star H) \\
&\qquad\qquad = (g_2 \star h_2) \star H.
\end{aligned}$$

Theorem 1

Let $G = (G, \star)$ denote an abelian group with subgroup $H < G$. The set G/H is a group, called the **quotient group** *of G by H, equipped with the operation $\star_{/H}$ defined in Equation* (1.1). $\square$

Exercise 4 (cf. Problems 69-70)

Write down the addition table for G/H in

(a) Exercise 1.

(b) Exercise 2.

Non-obvious isomorphisms

Example 2

Let $G = \mathbb{Z} \times \mathbb{Z}$ and define

$$\begin{aligned} H = (3,0)G + (0,2)G &= \{m(3,0) + n(0,2) \mid m, n \in \mathbb{Z}\} \\ &= \{(3m, 2n) \mid m, n \in \mathbb{Z}\}. \end{aligned}$$

Think of the elements in H as movements on a grid indexed by $\mathbb{Z} \times \mathbb{Z}$. The generator $(3,0)$ is right by 3; the generator $(0,2)$ is up by 2.

The cosets of H in G are:

⚅ $:= (0,0) + H = H$	⚂ $:= (0,1) + H$
⚀ $:= (1,1) + H$	⚃ $:= (1,0) + H$
⚁ $:= (2,0) + H$	⚄ $:= (2,1) + H$

Compare the addition table for G/H to the one for $\mathbb{Z}/6\mathbb{Z}$:

G/H	⚅	⚀	⚁	⚂	⚃	⚄
⚅	⚅	⚀	⚁	⚂	⚃	⚄
⚀	⚀	⚁	⚂	⚃	⚄	⚅
⚁	⚁	⚂	⚃	⚄	⚅	⚀
⚂	⚂	⚃	⚄	⚅	⚀	⚁
⚃	⚃	⚄	⚅	⚀	⚁	⚂
⚄	⚄	⚅	⚀	⚁	⚂	⚃

$\mathbb{Z}/6\mathbb{Z}$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Conclusion: $G/H \cong \mathbb{Z}/6\mathbb{Z}$ via the correspondence in the addition tables.

Question

What else is G/H isomorphic to?

Exercise 5 (cf. Problem 72)

"Simplify" the following **group presentations** (a term we define in Section **??**) by exhibting an isomorphism in each case.

1. $\mathbb{Z} \times \mathbb{Z} / \langle (1,1) \rangle$
2. $\mathbb{Z} \times \mathbb{Z} / \langle (2,-1), (-1,2) \rangle$