Tues 7 July

- MLP homework 4.6-4.7 is due tonight
- Exam 3 feedback
  - median $= 69/125$ ($\approx 55.2\%$)
  - You may redo problems for up to 90% of points back. With this setup, everyone has the potential to still get an A.
  - You may collaborate and use resources, including office hours.
  - The deadline to submit is 4pm Friday. NO EXCEPTIONS.
  - If you don't redo your problems, the grade sticks.
- Expect 2 quizzes this week and one next week.

Indefinite Integrals of Trig Functions

Table 4.5 (in the text) provides us with rules for finding indefinite integrals of trig functions.

1. $\dfrac{d}{dx}(\sin ax) = a\cos ax \quad \longrightarrow \quad \displaystyle\int \cos ax\ dx = \frac{1}{a}\sin ax + C$

2. $\dfrac{d}{dx}(\cos ax) = -a\sin ax \quad \longrightarrow \quad \displaystyle\int \sin ax\ dx = -\frac{1}{a}\cos ax + C$

3. $\dfrac{d}{dx}(\tan ax) = a\sec^2 ax \quad \longrightarrow \quad \displaystyle\int \sec^2 ax\ dx = \frac{1}{a}\tan ax + C$

Indefinite Integrals of Trig Functions (cont.)

$$4.\ \frac{d}{dx}(\cot ax) = -a\csc^2 ax \quad \longrightarrow \quad \int \csc^2 ax\ dx = -\frac{1}{a}\cot ax + C$$

$$5.\ \frac{d}{dx}(\sec ax) = a\sec ax \tan ax \quad \longrightarrow \quad \int \sec ax \tan ax\ dx = \frac{1}{a}\sec ax + C$$

$$6.\ \frac{d}{dx}(\csc ax) = -a\csc ax \cot ax \quad \longrightarrow \quad \int \csc ax \cot ax\ dx = -\frac{1}{a}\csc ax + C$$

Table 4.6 provides us with rules for finding other indefinite integrals.

7. $\dfrac{d}{dx}(e^{ax}) = ae^{ax} \longrightarrow \displaystyle\int e^{ax}\ dx = \frac{1}{a}e^{ax} + C$

8. $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x} \longrightarrow \displaystyle\int \frac{dx}{x} = \ln|x| + C$

9. $\dfrac{d}{dx}\left(\sin^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{1}{\sqrt{a^2 - x^2}} \longrightarrow \displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

Other Indefinite Integrals (cont.)

10. $\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{a}{a^2+x^2} \quad \longrightarrow \int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$

11. $\dfrac{d}{dx}\left(\sec^{-1}\left|\dfrac{x}{a}\right|\right) = \dfrac{a}{x\sqrt{x^2-a^2}} \longrightarrow \int \dfrac{dx}{x\sqrt{x^2-a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{x}{a}\right| + C$

Example

Evaluate the following indefinite integral: $\int 2\sec^2 2x \; dx$.

Solution: Using Rule 3., with $a = 2$, we have

$$\int 2\sec^2 2x \; dx = 2\int \sec^2 2x \; dx = 2\left[\frac{1}{2}\tan 2x\right] + C = \tan 2x + C.$$

Exercise

Evaluate $\int 2\cos(2x) \; dx$.

In some instances, we have enough information to determine the value of C in the antiderivative. These are often called **initial value problems**.

Example

If $f'(x) = 7x^6 - 4x^3 + 12$ and $f(1) = 24$, find $f(x)$.

Solution: $f(x) = \displaystyle\int (7x^6 - 4x^3 + 12)\, dx = x^7 - x^4 + 12x + C$. Now find out which C gives $f(1) = 24$:

$$24 = f(1) = 1 - 1 + 12 + C,$$

so $C = 12$. Hence, $f(x) = x^7 - x^4 + 12x + 12$.

Exercise

Find the function f that satisfies $f''(t) = 6t$ with $f'(0) = 1$ and $f(0) = 2$.

4.8 Book Problems

11-45 (odds), 55-59 (odds), 63, 65

- To solve 55-59 (odds), 63, and 65, look through the section, focusing in on Example 7.

5.1 Approximating Area Under Curves

Example

Suppose you ride your bike at a constant velocity of 8 miles per hour for 1.5 hours.

(a) What is the velocity function that models this scenario?

(b) What does the graph of the velocity function look like?

(c) What is the position function for this scenario?

(d) Where is the displacement (the distance you've traveled) represented when looking at the graph of the velocity function?

Question

In the previous example, the velocity was constant. In most cases, this is not accurate (or possible). How could we find displacement when the velocity is changing over an interval?

One strategy is to divide the time interval into a particular number of subintervals and approximate the velocity on each subinterval with a constant velocity. Then for each subinterval, the displacement can be evaluated and summed.

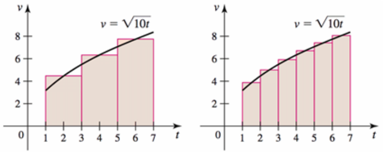
Note: This provides us with only an approximation, but with a larger number of subintervals, the approximation becomes more accurate.

Example

Suppose the velocity of an object moving along a line is given by $v(t) = \sqrt{10t}$ on the interval $1 \leq t \leq 7$.

(a) Divide the time interval into $n = 3$ subintervals, assuming the object moves at a constant velocity equal to the value of v evaluated at the midpoint of the subinterval. Estimate the displacement of the object on $[1, 7]$.

(b) Repeat for $n = 6$ subintervals.



We can see that our approximation gets more accurate when we use more subintervals of the time interval (see Example 1 in the text). Using this idea, we now examine a method for approximating areas under curves.

Consider a function f over the interval $[a, b]$. Divide $[a, b]$ into n subintervals of equal length:

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

with $x_0 = a$ and $x_n = b$. The length of each subinterval is denoted

$$\Delta x = \frac{b-a}{n}.$$

In each subinterval $[x_{k-1}, x_k]$ (where k ranges from 1 to n), we can choose any point, call it $\overline{x}_k$, and create a rectangle of height $f(\overline{x}_k)$. The base of the rectangle has length

$$x_k - x_{k-1} = \Delta x,$$

so the area of that rectangle is $f(\overline{x}_k)\Delta x$.

Doing this for each subinterval, and then summing the rectangles' areas, produces an approximation of the overall area. This approximation is called a **Riemann sum**:

$$R = f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x.$$

Note: We let k vary from 1 to n, and we always have $x_{k-1} \leq \overline{x}_k \leq x_k$.

We usually choose $\overline{x}_k$ so that it is consistent across all the subintervals. The most common ways to do this are with **left Riemann sums**, **right Riemann sums**, and **midpoint Riemann sums**.

Definition

Let $R = f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x$.

1. R is a **left Riemann sum** when we choose $\overline{x}_k = x_{k-1}$ for each k.
2. R is a **right Riemann sum** when we choose $\overline{x}_k = x_k$ for each k.
3. R is a **midpoint Riemann sum** when we take $\overline{x}_k$ to be the midpoint between x_{k-1} and x_k, for each k.

(See Figures 5.9–5.11 for pictures of these sums.)

Sigma Notation

Although Riemann sums become more accurate when we make n (the number of rectangles) bigger, writing it all down becomes a pain. Sigma notation gives a shorthand.

Example

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

We sum all integer values from the lowest limit ($k = 1$) to the highest limit ($k = 5$) in the summand k^2.

Exercise

Evaluate $\sum_{k=0}^{3}(2k-1)$.

Σ-Shortcuts

(n is always a positive integer)

$$\sum_{k=1}^{n} c = cn \text{ (where } c \text{ is a constant)}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Riemann Sums Using Sigma Notation

We can use sigma notation to write the Riemann sum in a much more compact form:

$$
\begin{aligned}
R &= f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x \\
&= \sum_{k=1}^{n} f(\overline{x}_k)\Delta x.
\end{aligned}
$$

To write the left, right, and midpoint Riemann sums in sigma notation, we need to know the point $\overline{x}_k$.

Left, Right, and Midpoint Riemann Sums in Sigma Notation

Suppose f is defined on a closed interval $[a, b]$ which is divided into n subintervals of equal length Δx.

1. $\displaystyle\sum_{k=1}^{n} f(a + (k-1)\Delta x)\Delta x$ gives a left Riemann sum.
2. $\displaystyle\sum_{k=1}^{n} f(a + k\Delta x)\Delta x$ gives a right Riemann sum.
3. $\displaystyle\sum_{k=1}^{n} f\left(a + \left(k - \frac{1}{2}\right)\Delta x\right)\Delta x$ gives a midpoint Riemann sum.

Exercise

(a) Use sigma notation to write the left, right, and midpoint Riemann sums for the function $f(x) = x^2$ on the interval $[1, 5]$ given that $n = 4$.

(b) Based on these approximations, estimate the area bounded by the graph of $f(x)$ over $[1, 5]$.

5.1 Book Problems

9, 11, 15-23 (odds), 31, 33, 53-57