

## LOCAL COHOMOLOGY AND $D$ -MODULES

(talk given at the University of Michigan Student Commutative Algebra Seminar)

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This talk follows the first half of Mel Hochster's notes *D-modules and Lyubeznik's Finiteness Theorems for Local Cohomology*, which you can find on his website <http://www.math.lsa.umich.edu/~hochster>. The material is hard, so much so that I do not cover any of Lyubeznik's results. In fact, my best judgment found the seemingly excessive number of exercises and footnotes which follow the most effective way to present such a breadth of material, without too many tangents.

### Basic Local Cohomology

In this section the symbol  $I$  will denote an ideal of a Noetherian<sup>1</sup> ring  $R$  and  $M$  will denote an  $R$ -module, which may or may not be Noetherian<sup>2</sup>. Recall<sup>3</sup> one way to define local cohomology is to take a direct limit<sup>4</sup> of Ext modules<sup>5</sup>:

$$H_I^i(M) = \varinjlim_t \operatorname{Ext}_R^i(R/I^t, M).$$

The 0th local cohomology module has the identification

$$\begin{aligned} H_I^0 M &= \varinjlim_t \operatorname{Hom}_R(R/I^t, M) \\ &\cong \bigcup_t \operatorname{Ann}_M(I^t) \end{aligned}$$

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<sup>1</sup>A ring  $R$  is **Noetherian** means all of its ideals are finitely generated. Equivalently, the ideals in  $R$  satisfy the **ascending chain condition** (ACC), that every strictly ascending chain of ideals must terminate, and this is equivalent to the condition that every nonempty collection of ideals in  $R$  has a maximal element. For a proof of the equivalence of these conditions, see Chapter 1 of [Eis], who also shows these statements are equivalent to Hilbert's original definition. A polynomial ring over a field is an example of a Noetherian ring.

<sup>2</sup>An  $R$ -module  $M$  is **Noetherian** means all of its submodules are finitely generated. For reasons analogous to the case of a ring, this is equivalent to  $M$  satisfying ACC on its submodules, which is equivalent to every nonempty collection of submodules having a maximal element.

**Proposition.** *If  $R$  is a Noetherian ring and  $M$  is a finitely generated  $R$ -module, then  $M$  is Noetherian.*

*Proof.* See [Eis] p. 28.  $\square$

<sup>3</sup>Luis Nuñez-Betancourt and Linquan Ma each defined local cohomology for  $M$  with support in  $I$  in seminar talks prior to this one. A good source is [Iyen]; local cohomology is finally defined in Chapter 7. However, the chapters leading up to it provide an excellent explanation of the difficult concepts needed to even define it.

<sup>4</sup>For a construction of direct limits and inverse limits, see [Hoch] p. 150-153.

<sup>5</sup>Ext is called the **extension functor**. Appendix 3 of [Eis] gives the construction of this homological algebra tool, as well as the construction of the Tor functor. For a much more general introduction to  $\delta$ -functors see Chapter 2 of [Weib].

where  $\text{Ann}_M I^t$  is the annihilator of  $I^t$  in  $M$ . So another way we can define local cohomology is to use the right derived functors of  $\Gamma_I(?) = \bigcup_t \text{Ann}_? I^t$ . Explicitly, create an exact sequence of  $R$ -modules

$$0 \rightarrow M \rightarrow N_1 \rightarrow N_2 \rightarrow \cdots$$

where the  $N_i$  satisfy the following property: for every ideal  $J \subset R$ , every homomorphism  $J \rightarrow N_i$  is actually the restriction of a homomorphism defined on all of  $R$ .<sup>6</sup> Omit  $M$  and apply the functor  $\Gamma_I(?)$  to each term of the sequence. The cohomology of the resulting sequence will be the same as the local cohomology defined above. Both of these definitions involve directed systems with ideals  $I^t$ , and this observation makes it clear why local cohomology is the same up to radicals of  $I$ .

Suppose  $f_1, \dots, f_n \in R$  and  $\text{rad } I = \text{rad}((f_1, \dots, f_n)R)$ . A third way to compute local cohomology is to use the direct limit of the Koszul complexes

$$\mathcal{K}^i(f_1^t, \dots, f_n^t; M).$$

Section 7 of [HW11] describes the construction explicitly while Chapter 17 of [Eis] defines more general Koszul complexes and their properties beyond the scope of our context<sup>7</sup>. We first need the notion of the tensor product of complexes. For  $K^\bullet$  and  $L^\bullet$  complexes of  $R$ -modules with respective differentials  $d_K$  and  $d_L$ , let  $M^\bullet = K^\bullet \otimes_R L^\bullet$  denote the complex

- (1)  $M^h = \bigoplus_{i+j=h} K^i \otimes_R L^j$  with
- (2) differential

$$d^h(a \otimes b) = d_K a \otimes b + (-1)^i a \otimes d_L b,$$

where  $a \in K^i$ ,  $b \in L^j$ , and  $i + j = h$ .

**Exercise.**  $M^\bullet$  is a complex of  $R$ -modules.

For  $N$  complexes  $K_{(1)}^\bullet, \dots, K_{(N)}^\bullet$  define the tensor product recursively as

$$K_{(1)}^\bullet \otimes_R \cdots \otimes_R K_{(N)}^\bullet = \left( K_{(1)}^\bullet \otimes_R \cdots \otimes_R K_{(N-1)}^\bullet \right) \otimes_R K_{(N)}^\bullet.$$

Now for  $f \in R$ , define the **Koszul complex**

$$\mathcal{K}^\bullet(f; R) = (0 \rightarrow R \rightarrow R \rightarrow 0),$$

where the middle map is multiplication by  $f$ . Then for  $f_1, \dots, f_n \in R$ , define the Koszul complex

$$\mathcal{K}^\bullet(f_1, \dots, f_n; M) = \left( \bigotimes_{i=1}^n \mathcal{K}^\bullet(f_i; R) \right) \otimes_R M,$$

meaning, tensor  $M$  with each term of the complex  $\mathcal{K}^\bullet(f_1, \dots, f_n; R) = \bigotimes_{i=1}^n \mathcal{K}^\bullet(f_i; R)$ .

<sup>6</sup>Such a module is called **injective**. The exact sequence  $0 \rightarrow M \rightarrow N_1 \rightarrow N_2 \rightarrow \cdots$  is called an **injective resolution of  $M$** . See Appendix 3.4 of [Eis] for a more detailed explanation of injective resolutions.

<sup>7</sup>The Koszul complex we care about will be a Čech complex.

*Notation.* When the context is clear, writing  $\underline{f}$  in place of  $f_1, \dots, f_n$  is easier and quicker, as in  $\mathcal{K}^\bullet(\underline{f}; M)$ . Let

$$\begin{aligned}\mathcal{K}^\bullet(\underline{f}^\infty; M) &= \varinjlim_t \mathcal{K}^\bullet(\underline{f}^t; M) \\ &= \varinjlim_t \mathcal{K}^\bullet(f_1^t, \dots, f_n^t; M).\end{aligned}$$

Let  $H^\bullet(\underline{f}^\infty; M)$  denote its cohomology.

**Theorem.** Suppose  $\text{rad } I = \text{rad } ((f_1, \dots, f_n)R)$ . Then

$$H_I^j M \cong H^\bullet(\underline{f}^\infty; M)$$

canonically as functors of  $M$ .

*Proof.* See [HW11].  $\square$

An immediate consequence is that local cohomology  $H_I^i M$  vanishes when  $i$  exceeds the number of generators for any ideal with the same radical as  $I$ . There are other, slightly less obvious consequences which follow from the various definitions of local cohomology:

**Exercise.** A short exact sequence of  $R$ -modules

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

induces a long exact sequence

$$\begin{aligned}0 \rightarrow H_I^0 M' \rightarrow H_I^0 M \rightarrow H_I^0 M'' \rightarrow H_I^1 M' \rightarrow \dots \\ \rightarrow H_I^i M' \rightarrow H_I^i M \rightarrow H_I^i M'' \rightarrow H_I^{i+1} M' \rightarrow \dots\end{aligned}$$

**Exercise.** If  $R \rightarrow S$  is a ring homomorphism,  $IS$  denotes the ideal generated by the image of  $I$  in  $S$ , and  $M$  is an  $S$ -module (hence an  $R$ -module), then  $H_I^i M = H_{IS}^i M$ .

**Exercise.** Localization at a maximal ideal does not change local cohomology.

## The Ring of Differential Operators

Hochster's treatment of  $D$ -modules in [HD] is a light survey of the results in the theory and omits many technical details. His main source is Jan-Erik Björk's book *Rings of Differential Operators*. The intent of this talk is to keep the material approachable to a Michigan graduate student who has passed the Qualifying Review (i.e., a second or third year student).

The ring we care about will be  $D = D(R, K)$ , constructed as follows. Let  $K$  be a field of characteristic 0 and let  $R$  denote the formal power series ring  $K[[x_1, \dots, x_n]]$  in  $n$  variables over  $K$ . The ring of differential operators  $D(R, K)$  consists of the  $K$ -vector space endomorphisms of  $R$  generated by multiplication of elements in  $R$  and the usual

differential operators  $\delta_1, \dots, \delta_n$  defined formally<sup>8</sup>. For the rest of this talk,  $D$ ,  $R$ , and  $K$  are defined as above.<sup>9</sup>

**Exercise.** The  $\delta_i$  commute with each other.

**Exercise.** Thinking of the  $x_j$  as operators on  $R$ , meaning for  $f \in R$ ,  $x_j(f)$  is just the product  $x_j \cdot f$ , if  $i \neq j$  then  $\delta_i x_j - x_j \delta_i = 0$ .

**Exercise.** Again, thinking of the  $x_i$  as operators on  $R$ ,  $\delta_i x_i - x_i \delta_i = 1$ .

**Exercise.** More generally, for any  $f \in R$ , thought of as an operator on  $R$ ,

$$\delta_i f - f \delta_i = \frac{\partial f}{\partial x_i}.$$

**Exercise.**  $D$  is  $R$ -free on the monomials in the  $\delta_i$ , both as a left and as a right  $R$ -module.

### Holonomic $D$ -Modules

Say  $A$  is an associative (not necessarily commutative) ring with 1, and has an **ascending filtration**

$$\Sigma = (\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \dots),$$

meaning the following conditions hold:

- (1) each  $\Sigma_i$  is an additive subgroup,
- (2)  $1 \in \Sigma_0$ ,
- (3)  $\bigcup_i \Sigma_i = A$ , and
- (4)  $\Sigma_i \Sigma_j \subseteq \Sigma_{i+j}$  for all  $i, j$ .

Furthermore, we insist the filtration is such that the **associated graded ring**

$$\text{gr}(A) = \Sigma_0 \oplus \Sigma_1/\Sigma_0 \oplus \dots \oplus \Sigma_i/\Sigma_{i-1} \oplus \dots$$

is commutative and Noetherian. Such a ring  $A$  is called a **filtered ring**.

**Exercise.** The filtration on  $A$  actually does give  $\text{gr } A$  a ring structure.

$D$  as defined above has a filtration  $\Sigma$  in which  $\Sigma_i$  consists of all  $R$ -linear combinations of monomials in degree at most  $i$  in the  $\delta_j$ .

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<sup>8</sup>When differential operators  $\delta_i$  are defined formally it means for  $f \in R$ ,  $\delta_i f = \frac{\partial f}{\partial x_i}$  where the partials  $\frac{\partial}{\partial x_i}$  are not really limits as in the usual definition of a derivative. Rather, they are simply defined by the power rule for differentiation. This is why we need to be in characteristic 0. A ring of differential operators does exist when  $K$  is characteristic  $p > 0$ , but its construction is more complicated.

<sup>9</sup> $D$  is sometimes called the **Weyl algebra**, though the usual definition uses a polynomial ring  $R$  instead of a formal power series ring.

**Exercise.** The associated graded ring  $\text{gr } D$  is isomorphic to a polynomial ring  $R[\zeta_1, \dots, \zeta_n]$  in  $n$  variables over  $R$ , where  $\zeta_i$  is the image of  $\delta_i$  in  $\Sigma_1/\Sigma_0$ . In particular,  $\text{gr } D$  is commutative, Noetherian, and regular<sup>10</sup>. Its Krull dimension is  $2n$ .

A finitely generated (left)  $A$ -module  $M$  has a **good** filtration

$$\Gamma = (\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \dots)$$

means it satisfies the following:

- (1)  $\{\Gamma_i\}$  are abelian subgroups,
- (2)  $\bigcup_i \Gamma_i = M$ ,
- (3)  $\Sigma_i \Gamma_j \subseteq \Gamma_{i+j}$  for all  $i, j \in \mathbb{N}$

(so far these are just the conditions that make it a filtration), and the **associated graded module**

$$\text{gr}_\Gamma(M) = \bigoplus_{i=1}^{\infty} \Gamma_i / \Gamma_{i-1}$$

is finitely generated over  $\text{gr } A$  (this last condition is the condition for the filtration to be good).

Björk proved the following:

**Proposition.** *A as described above is both left and right Noetherian<sup>11</sup>. Moreover, an  $A$ -module  $M$  has a good filtration if and only if it is finitely generated as an  $A$ -module and for  $\{\Gamma_i\}$  and  $\{\Gamma'_i\}$  filtrations on  $M$  there exists an integer  $c$  with*

$$\Gamma_i \subseteq \Gamma'_{i+c} \text{ and } \Gamma'_i \subseteq \Gamma_{i+c}$$

for all  $i$ .  $\square$

**Corollary.** *By the above proposition, the Krull dimension of  $\text{gr}_\Gamma M$  is independent of the choice of good filtration  $\Gamma$ .*

*Proof.* See p. 5 of [HD], where Hochster cites Björk.  $\square$

From the above discussion,  $A = D = D(R, K)$  is a filtered ring. The finitely generated  $D$ -modules whose associated graded modules have Krull dimension  $n$ , together with the 0 module, constitute the **(left) Bernstein class**<sup>12</sup>. There is a similar notion for the **right Bernstein class** and in fact, a duality between these two categories holds.

<sup>10</sup>A local ring with maximal ideal  $m$  is **regular** means  $m$  can be generated by exactly  $d$  elements, where  $d$  is the Krull dimension of the ring. (The **Krull dimension** of a ring  $R$  is the supremum of the lengths of chains of prime ideals in  $R$ ; the **Krull dimension** of an  $R$ -module  $M$  is the Krull dimension of the ring  $R/\text{Ann } M$ .) More generally, a ring  $R$  is **regular** means all of its localizations at maximal ideals are regular.

<sup>11</sup>When  $A$  is not necessarily commutative the Noetherian condition may hold for left or right ideals.

<sup>12</sup>To define the more general notion for a filtered ring  $A$  requires understanding of weak global dimension.

**Theorem**<sup>13</sup>. *The modules in the Bernstein class have finite length as  $D$ -modules, i.e., each nonzero module in the Bernstein class has a finite filtration by simple  $D$ -modules (each of which is again in the Bernstein class).*

*Proof.* See p. 7 of [HD].  $\square$

*Notation.* The  $D$ -modules in the Bernstein class are called **holonomic**.

**Proposition.**

- (1) *Submodules and quotients of holonomic  $D$ -modules are also holonomic.*
- (2) *For a short exact sequence of  $D$ -modules*

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

*$M_2$  is a holonomic  $D$ -module if and only if both  $M_1$  and  $M_3$  are.*

*Proof.* See p. 6 of [HD].  $\square$

**Exercise.**  $R$  is a holonomic  $D$ -module.

**Exercise.** If  $W$  is any multiplicative system in  $D$  and  $M$  is a  $D$ -module, then  $W^{-1}M$  has the structure of a  $D$ -module in such a way that the map  $M \rightarrow W^{-1}M$  is a homomorphism of  $D$ -modules.

### Applications to Local Cohomology

**Theorem (Björk).** *With  $D = D(R, K)$  as defined above, if  $M$  is a holonomic  $D$ -module and  $f \in R$ , then the localization  $M_f$  is a holonomic  $D$ -module.*  $\square$

This difficult result leads to many applications to local cohomology theory.

**Corollary.** *The local cohomology modules  $H_I^i M$  all have the structure of  $D$ -modules in such a way that if  $M$  is holonomic, then so are  $H_I^i M$ .*

*Proof.* Write  $I = (f_1, \dots, f_s)R$ . We can use the Koszul complex  $\mathcal{K}^\bullet(\underline{f}^\infty; M)$  to compute  $H_I^i M$ . This is

$$0 \rightarrow M \rightarrow \bigoplus_i M_{f_i} \rightarrow \bigoplus_{i < j} M_{f_i f_j} \rightarrow \cdots \rightarrow M_{f_1 \cdots f_s} \rightarrow 0$$

where the maps are alternating sums of the natural localization maps. Thus by an above exercise the Koszul complex is a complex of  $D$ -modules. In particular, the local cohomology modules are  $D$ -modules.

If  $M$  is holonomic then the theorem says so are the  $M_{f_i}$ . To see direct sums of holonomic  $D$ -modules are holonomic, consider the split exact sequence

$$0 \rightarrow M_1 \rightarrow M_1 \oplus M_2 \rightarrow M_2 \rightarrow 0$$

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<sup>13</sup>A more general statement is true for any filtered ring  $A$ .

where  $M_1$  and  $M_2$  are holonomic. Another exercise from above shows the middle term  $M_1 \oplus M_2$  is holonomic as well. Then for more than two summands use induction to show holonomicity. Finally, one of the above exercises shows the kernels, which are submodules, and the cohomology modules, which are quotients, must be holonomic as well. In particular,  $H_I^i M$  must be holonomic if  $M$  is.  $\square$

For any ring  $S$  and  $S$ -module  $N$ , a prime ideal  $P \subset S$  is **associated** to  $N$  means  $P$  annihilates (kills) an element of  $N$ . The set of all primes associated to  $N$  is denoted  $\text{Ass}_S(N)$  and is sometimes called the **assassinator** of  $N$ .

**Exercise.** If  $S$  is Noetherian then  $\text{Ass}_S N$  is nonempty.

**Corollary.** If  $M$  is a simple<sup>14</sup>  $D$ -module then the assassinator of  $M$  as an  $R$ -module contains a unique element  $P$  (note by construction any  $D$ -module is also an  $R$ -module). Hence, if  $M$  is a holonomic  $D$ -module then  $\text{Ass}_R M$  is finite.

*Proof.* Suppose  $M$  is simple. First of all,  $R$  is Noetherian implies  $M$  has an associated prime,  $P$ . Then

$$H_P^0 M \cong \bigcup_t \text{Ann}_M P^t$$

is a nonzero submodule of  $M$ , hence is equal to  $M$ . If  $Q$  is another associated prime then

$$H_Q^0 M = M = H_P^0 M.$$

Local cohomology modules with support in different ideals are the same as long as those ideals have the same radical. But  $P$  and  $Q$  are prime and so they are already radical. Therefore,  $P$  and  $Q$  are the same ideal.

When  $M$  is holonomic it has a finite filtration by simple modules. Therefore if  $P$  annihilates an element of  $M$ , it annihilates an element in one of those simple modules and  $P$  is the unique associated prime for that simple module. So  $M$  can only have finitely many associated primes.  $\square$

By induction, if  $I_1, \dots, I_s$  is a sequence of ideals of  $R$ ,  $i_1, \dots, i_s$  is a sequence of integers, and  $M$  is a holonomic  $D$ -module, then the iterated local cohomology module

$$H_{I_s}^{i_s} \left( H_{I_{s-1}}^{i_{s-1}} \left( \dots H_{I_1}^{i_1} M \dots \right) \right)$$

is a holonomic  $D$ -module. In particular,  $H_{I_s}^{i_s} \left( H_{I_{s-1}}^{i_{s-1}} \left( \dots H_{I_1}^{i_1} R \dots \right) \right)$  is holonomic and so has a finite assassinator.

**Proposition.** Let  $m = (x_1, \dots, x_n)R$  denote the homogeneous maximal ideal<sup>15</sup> in  $R$ . Then  $D/Dm \cong H_m^n R$  as a  $D$ -module.<sup>16</sup>

<sup>14</sup>A module is **simple** means it has no nontrivial submodule.

<sup>15</sup>**Homogeneous** means every element in  $m$  has the same degree in the indeterminates  $x_i$ .

<sup>16</sup> $H_m^n R$  is also the unique smallest injective  $R$ -module containing  $R/m$ , called the **injective hull** of  $R/m$  in  $R$ .

*Proof.* Let  $S$  denote the polynomial ring  $K[x_1, \dots, x_n]$  and let  $Q = (x_1, \dots, x_n)S$ . Exercises from earlier give

$$\begin{aligned} H_m^n(R) &\cong H_{Q S_Q}^n(S_Q) \\ &\cong H_Q^n(S). \end{aligned}$$

So we can instead show  $D/Dm \cong H_Q^n S$ .

The Koszul complex  $\mathcal{K}^\bullet(\underline{x}^\infty; S)$  gives

$$H_Q^n(S) \cong \text{Coker} \left( \bigoplus_{i=1}^n S_{x_1 \cdots \hat{x}_i \cdots x_n} \rightarrow S_{x_1 \cdots x_n} \right)$$

where the hat means that indeterminate is omitted. It turns out  $H_Q^n S$  is the  $K$ -span of the monomials  $x_1^{j_1} \cdots x_n^{j_n}$  where  $j_i$  are strictly negative. From an earlier exercise  $D$  is  $R$ -free as a right  $R$ -module on the monomials in  $\delta_1, \dots, \delta_n$ ; therefore  $D/Dm$  is  $K$ -free on the span of the images of the monomials in the  $\delta_i$ .

There is a  $D$ -linear map  $D \rightarrow H_Q^n S$  that sends  $1 \mapsto x_1^{-1} \cdots x_n^{-1}$ . This map must kill (annihilate)  $m$  and hence also kills the left ideal  $Dm$ , inducing a  $D$ -linear map  $D/Dm \rightarrow H_Q^n S$ . For this map the image in  $H_Q^n S$  of the element represented by  $\delta_1^{i_1} \cdots \delta_n^{i_n}$  is represented by

$$\frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}} (x_1^{-1} \cdots x_n^{-1}) = (-1)^{i_1 + \cdots + i_n} \prod_{j=1}^n i_j! x_j^{-(i_j+1)}.$$

This means the induced  $D$ -linear map carries a  $K$ -vector space basis to a  $K$ -vector space basis, so is a  $D$ -isomorphism.  $\square$

The final results first require some structure theory for injective modules<sup>17</sup>. Let  $S$  be a commutative, associative ring with 1.

A homomorphism of  $S$ -modules  $h : N \rightarrow M$  is an **essential extension** means it is injective and the following equivalent conditions hold:

- (i) Every nonzero submodule of  $M$  has a nonzero intersection with  $h(N)$ .
- (ii) Every nonzero element of  $M$  has a nonzero multiple in  $h(N)$ .
- (iii) If  $\varphi : M \rightarrow Q$  is a homomorphism such that  $\varphi \circ h$  is injective then  $\varphi$  is injective.

Suppose  $S$  is also Noetherian and local with maximal ideal  $\mathfrak{m}$ , and suppose  $M$  is an  $S$ -module such that every element of  $M$  is killed by a power of  $\mathfrak{m}$ . The **socle** in  $M$  is defined as

$$\text{Soc}(M) = \text{Ann}_M \mathfrak{m}.$$

In this special case where every element of  $M$  is killed by a power of  $\mathfrak{m}$ ,  $\text{Soc } M$  is the largest submodule of  $M$  which may be viewed as a vector space over  $S/\mathfrak{m}$ ; any larger submodule would contain an element not killed by  $\mathfrak{m}$ , so could not be a vector space over  $S/\mathfrak{m}$ . Furthermore, the inclusion  $\text{Soc } M \subseteq M$  is an essential extension. To see this, choose  $x \in M$  nonzero and let  $t$  denote the largest integer such that  $\mathfrak{m}^t x \neq (0)$ . Then we can choose  $y \in \mathfrak{m}^t$  such that  $yx \neq 0$ . The inclusion

$$\mathfrak{m}y \subseteq \mathfrak{m}\mathfrak{m}^t x = 0$$

<sup>17</sup>See [HW11].



implies  $y \in \text{Soc } M$  and hence the nonzero multiple  $yx \in \text{Soc } M$ .

**Exercise.** In general, if  $N \subseteq M$  is an essential extension then  $\text{Soc } M \subseteq N$ .

Here is the result on  $D$ -modules:

**Proposition.** *With  $m$  as before, the homogeneous maximal ideal of  $R = K[[x_1, \dots, x_n]]$ , if  $M$  is any  $D$ -module (no finiteness conditions on  $M$  this time) such that every element is killed by a power of  $m$ , then  $M$  is isomorphic with a direct sum of copies of  $D/Dm$ . When  $M$  is holonomic the direct sum is finite.*

*Proof.* By the discussion above  $M$  is an essential extension of a  $K$ -vector space  $V \subseteq M$ , and we may choose a  $K$ -vector space basis  $\{v_\lambda\}_{\lambda \in \Lambda}$  for  $V$ . (The use of  $\Lambda$  to denote the indexing set indicates the basis may be infinite.) Consider a free  $D$ -module  $G$  with free generators  $\{u_\lambda\}$ , also indexed by  $\Lambda$ . Each copy of  $D$  in  $G$  is both a left and right module over  $D$ , so  $G$  is both a left and right module over  $D$ . Therefore  $G$  is both a left and a right module over  $R$ . This will matter later in the proof when we are considering  $G/Gm$  as an  $R$ -module.

Define the  $D$ -linear map  $G \rightarrow M$  by  $u_\lambda \mapsto v_\lambda$  for each  $\lambda \in \Lambda$ . This induces a map  $G/Gm \rightarrow M$  which sends the images of the  $u_\lambda$  to the corresponding elements in the basis  $v_\lambda$  for  $V$ . The goal is to show this induced map is the isomorphism we want.

Because  $G$  was chosen as a direct sum of copies of  $D$  indexed by  $\Lambda$ ,  $G/Gm$  may be identified with a direct sum of copies of  $D/Dm$  indexed by  $\Lambda$ . Thus,  $G/Gm$  is an essential extension of a  $K$ -vector space,  $\text{Soc}(G/Gm)$ . The basis on  $\text{Soc}(G/Gm)$  is induced by the images of the  $u_\lambda$  in  $G/Gm$ , so is indexed by  $\Lambda$ . Each of these basis elements maps to its correspondingly indexed basis element  $v_\lambda$  in  $V$ . That bijection between vector space bases shows  $\text{Soc}(G/Gm)$  is mapped isomorphically onto  $V$ .

As an isomorphism, in particular, the map  $G/Gm \rightarrow M$  is injective on  $\text{Soc}(G/Gm)$ . Suppose a nonzero element in  $G/Gm$  maps to zero. Then because  $\text{Soc}(G/Gm) \subseteq G/Gm$  is an essential extension, a nonzero multiple of that element in  $\text{Soc}(G/Gm)$  is mapped to zero, contradicting injectivity. So  $G/Gm \rightarrow M$  must also be injective. In fact, the map splits as a map of  $R$ -modules: If  $I$  is an ideal in  $R$  and  $\varphi : I \rightarrow G/Gm$  an  $R$ -linear map, then  $\phi$  must kill  $m$ . So we can actually think of  $\varphi$  as a map  $I/Rm \rightarrow G/Gm$ . Since  $m$  is a maximal ideal,  $I/Rm$  must be either zero or the unit ideal. In either case though,  $\phi$  will extend to a map from  $R$ . From this we can conclude  $G/Gm$  is an injective  $R$ -module, therefore its injection into  $M$  splits.<sup>18</sup>

Write  $M \cong G/Gm \oplus M_0$  as the  $R$ -module splitting. As a submodule of  $M$ , every element of  $M_0$  is killed by a power of  $m$  (that was a hypothesis on  $M$ ). Therefore if  $M_0 \neq 0$ , its socle is nonzero. The socle of  $M_0$  is contained in the socle of  $M$ , which is contained in  $V$ . But  $V$  is contained in the image of  $G/Gm$ , which only meets  $M_0$  at zero, by construction. Therefore  $M_0 = 0$  and we have

$$\begin{aligned} M &\cong G/Gm \oplus 0 \\ &\cong G/Gm, \end{aligned}$$

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<sup>18</sup>The splitting of an injective map is a general property of injective modules.

as required.  $\square$

*Notation.* To save writing space, let the symbol  $T$  denote an iteration of local cohomology functors.

**Corollary.** *Let  $T$  be a composition of local cohomology functors as described just above. If  $T(R)$  has the property that every element is killed by a power of  $m$ , then  $T(R)$  is isomorphic as a  $D$ -module to a finite direct sum of copies of  $D/Dm$ , and so is injective.*

*Proof.*  $T(R)$  is a holonomic  $D$ -module by a result above. Then the desired result follows from the last two propositions about  $D$ -modules.  $\square$

For more results, again, see Mel Hochster's notes.

## APPENDIX: SOLUTIONS TO EXERCISES

**Exercise.**  $M^\bullet$  is a complex of  $R$ -modules.

*Solution.* We constructed  $M^\bullet$  so that its terms are  $R$ -modules and its differentials are  $R$ -linear. We just need to check that the compositions  $M^{h-1} \rightarrow M^h \rightarrow M^{h+1}$  are all zero. Say  $a_i \in K^i$  and  $b_j \in L^j$ , for all  $i, j$ . Apply  $d^{h-1}$  to a typical element in  $M^{h-1}$ :

$$\begin{aligned} d^{h-1} \left( \bigoplus_{i+j=h-1} a_i \otimes b_j \right) &= \bigoplus_{i+j=h-1} d^{h-1}(a_i \otimes b_j) \\ &= \bigoplus_{i+j=h-1} (d_K a_i \otimes b_j + (-1)^i a_i \otimes d_L b_j) \end{aligned}$$

The resulting terms are in  $M^h$ . Now apply  $d^h$ :

$$\begin{aligned} &d^h \left( \bigoplus_{i+j=h-1} (d_K a_i \otimes b_j + (-1)^i a_i \otimes d_L b_j) \right) \\ &= \bigoplus_{i+j=h-1} d^h (d_K a_i \otimes b_j + (-1)^i a_i \otimes d_L b_j) \\ &= \bigoplus_{i+j=h-1} (d^h (d_K a_i \otimes b_j) + d^h ((-1)^i a_i \otimes d_L b_j)) \\ &= \bigoplus_{i+j=h-1} ((d_K \circ d_K a_i \otimes b_j + (-1)^{i+1} d_K a_i \otimes d_L b_j) + \\ &\quad (d_K (-1)^i a_i \otimes d_L b_j + (-1)^i a_i \otimes d_L \circ d_L b_j)) \\ &= \bigoplus_{i+j=h-1} (0 + (-1)^{i+1} d_K a_i \otimes d_L b_j + (-1)^i d_K a_i \otimes d_L b_j + 0) \\ &= 0. \end{aligned}$$

We applied the composition to an arbitrary element and got zero, so the composition map itself must be zero.  $\square$

**Exercise. A short exact sequence of  $R$ -modules**

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

**induces a long exact sequence**

$$\begin{aligned} 0 \rightarrow H_I^0 M' \rightarrow H_I^0 M \rightarrow H_I^0 M'' \rightarrow H_I^1 M' \rightarrow \cdots \\ \rightarrow H_I^i M' \rightarrow H_I^i M \rightarrow H_I^i M'' \rightarrow H_I^{i+1} M' \rightarrow \cdots . \end{aligned}$$

*Solution.* Ext is a right derived functor, hence a universal cohomological  $\delta$ -functor. Thus, by definition<sup>19</sup>, induces the long exact sequence

$$\begin{aligned} 0 \rightarrow \operatorname{Hom}_R(R/I^t, M') \rightarrow \operatorname{Hom}_R(R/I^t, M) \rightarrow \operatorname{Hom}_R(R/I^t, M'') \rightarrow \\ \operatorname{Ext}_R^1(R/I^t, M') \rightarrow \operatorname{Ext}_R^1(R/I^t, M) \rightarrow \operatorname{Ext}_R^1(R/I^t, M'') \rightarrow \cdots \rightarrow \\ \operatorname{Ext}_R^i(R/I^t, M') \rightarrow \operatorname{Ext}_R^i(R/I^t, M) \rightarrow \operatorname{Ext}_R^i(R/I^t, M'') \rightarrow \cdots . \end{aligned}$$

To actually get local cohomology though, we have to take the direct limit. But the direct limit is a filtered colimit<sup>20</sup>, so is exact. In other words, applying  $\varinjlim_t$  to each term of the above long exact sequence produces precisely the desired long exact sequence on local cohomology.  $\square$

**Exercise. If  $R \rightarrow S$  is a ring homomorphism,  $IS$  denotes the ideal generated by the image of  $I$  in  $S$ , and  $M$  is an  $S$ -module (hence an  $R$ -module), then  $H_I^i M = H_{IS}^i M$ .**

*Solution.*  $M$  is an  $R$ -module by restriction of scalars<sup>21</sup>, i.e., multiplication of  $M$  by  $f \in R$  is identical to multiplication by the image of  $f$  in  $S$ . Let  $f_1, \dots, f_n$  denote the generators of  $I$  and  $g_1, \dots, g_n$  the respective generators of  $IS$ . Both Koszul complexes  $\mathcal{K}^\bullet(\underline{f}^\infty; M)$  and  $\mathcal{K}^\bullet(\underline{g}^\infty; M)$  consist of localization maps and the remark above gives an identification of  $M_{f_i}$  with  $M_{g_i}$ . Therefore the Koszul complexes and hence, the cohomology, are the same.  $\square$

**Exercise. Localization at a maximal ideal does not change local cohomology.**

*Solution.* Localization at a maximal ideal is a ring homomorphism, so this is a special case of the previous exercise.  $\square$

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<sup>19</sup>Though very formal, [Weib] has the best justification for these statements.

<sup>20</sup>See [Weib]. This exercise also appears as a “Discussion” in [HW11].

<sup>21</sup>See Chapter 2 of [AtyM].

**Exercise. The  $\delta_i$  commute with each other.**

*Solution.* Choose an arbitrary element  $f \in R$  and do the computation:

$$\begin{aligned}
 (\delta_i \delta_j - \delta_j \delta_i)(f) &= \delta_i \delta_j(f) - \delta_j \delta_i(f) \\
 &= \delta_i \left( \frac{\partial}{\partial x_j} f \right) - \delta_j \left( \frac{\partial}{\partial x_i} f \right) \\
 &= \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f - \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} f \\
 &= \frac{\partial^2}{\partial x_i \partial x_j} f - \frac{\partial^2}{\partial x_j \partial x_i} f \\
 &= 0,
 \end{aligned}$$

as desired.  $\square$

**Exercise. Thinking of the  $x_j$  as operators on  $R$ , meaning for  $f \in R$ ,  $x_j(f)$  is just the product  $x_j \cdot f$ , if  $i \neq j$  then  $\delta_i x_j - x_j \delta_i = 0$ .**

*Solution.* As in the previous exercise, choose an arbitrary element  $f \in R$  and do the computation:

$$\begin{aligned}
 (\delta_i x_j - x_j \delta_i)(f) &= \delta_i x_j(f) - x_j \delta_i(f) \\
 &= \frac{\partial}{\partial x_i} (x_j f) - x_j \left( \frac{\partial}{\partial x_i} f \right) \\
 &= \left( 0 \cdot f + x_j \left( \frac{\partial}{\partial x_i} f \right) \right) - x_j \left( \frac{\partial}{\partial x_i} f \right) \\
 &= 0,
 \end{aligned}$$

as desired.  $\square$

**Exercise. Again, thinking of the  $x_i$  as operators on  $R$ ,  $\delta_i x_i - x_i \delta_i = 1$ .**

*Solution.* Applying the left hand side to arbitrary  $f \in R$ , we want the result to be  $f$ .

$$\begin{aligned}
 (\delta_i x_i - x_i \delta_i)(f) &= \delta_i x_i(f) - x_i \delta_i(f) \\
 &= \frac{\partial}{\partial x_i} (x_i f) - x_i \left( \frac{\partial}{\partial x_i} f \right) \\
 &= \left( 1 \cdot f + x_i \left( \frac{\partial}{\partial x_i} f \right) \right) - x_i \left( \frac{\partial}{\partial x_i} f \right) \\
 &= f,
 \end{aligned}$$

so  $\delta_i x_i - x_i \delta_i$  is in fact the identity operator.  $\square$

**Exercise. More generally, for any  $f \in R$ , thought of as an operator on  $R$ ,**

$$\delta_i f - f \delta_i = \frac{\partial f}{\partial x_i}.$$

*Solution.* This time  $f$  is thought of as an operator, so take  $g \in R$ :

$$\begin{aligned}
 (\delta_i f - f \delta_i)(g) &= \delta_i f(g) - f \delta_i(g) \\
 &= \frac{\partial}{\partial x_i}(fg) - f \left( \frac{\partial}{\partial x_i} g \right) \\
 &= \left( \left( \frac{\partial}{\partial x_i} f \right) g + f \left( \frac{\partial}{\partial x_i} g \right) \right) - f \left( \frac{\partial}{\partial x_i} g \right) \\
 &= \left( \frac{\partial}{\partial x_i} f \right) g \\
 &= \left( \frac{\partial f}{\partial x_i} \right) (g),
 \end{aligned}$$

the desired result.  $\square$

**Exercise.**  $D$  is  $R$ -free on the monomials in the  $\delta_i$ , both as a left and as a right  $R$ -module.

*Solution.* The previous few exercises give a way to write an element of  $D$  (that has an expression that is not a sum of two or more elements in  $D$ ) as an element of  $R$  multiplied on the right by a monomial in the  $\delta_i$ : as shown above, the  $\delta_i$  commute with each other;  $R$  is already commutative so we only need a way to write  $\delta_j f$ , for  $f \in R$ , as an expression consisting of an element of  $R$  multiplied on the right by a monomial in the  $\delta_i$ . The last exercise gives

$$\delta_i f = f \delta_i + \frac{\partial f}{\partial x_i}.$$

The commutativity of elements in  $R$  with each other and of the  $\delta_i$  with each other implies such an expression is unique. So  $D$  is  $R$ -free on the monomials in the  $\delta_i$ , as a left  $R$ -module.

Likewise, for  $f \in R$  we have  $f \delta_i = \delta_i f - \frac{\partial f}{\partial x_i}$  by the previous exercise. This fact, along with the commutativity of elements in  $R$  with each other and of the  $\delta_i$  with each other gives every element of  $D$  (which has an expression not a sum of two or more elements in  $D$ ) a unique expression as an element of  $R$  multiplied on the left by a monomial in the  $\delta_i$ . Thus  $D$  is also  $R$ -free on the monomials in the  $\delta_i$ , as a right  $R$ -module.  $\square$

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**Exercise.** The filtration on  $A$  actually does give  $\text{gr } A$  a ring structure.

*Solution.* We just have to check the ring axioms for  $\text{gr } A$ . Define addition and multiplication in  $\text{gr } A$  as the respective induced operations from  $A$ . Any two elements in  $A$  lie in  $\Sigma_i$ , for some  $i$ , and so upon taking quotients,  $\text{gr } A$  is still a commutative group under addition. Similarly, the property  $\Sigma_i \Sigma_j \subseteq \Sigma_{i+j}$  ensures any two elements can be multiplied within one of the subgroups in the filtration  $\Sigma$ , and so associativity is also preserved upon taking quotients. Since  $1 \in \Sigma_0$ , it remains the unit element in  $\text{gr } A$ . Finally, since both ring

operations can be done in one common subgroup  $\Sigma_i$ , the distributive laws remain intact upon taking quotients.  $\square$

**Exercise.** The associated graded ring  $\text{gr } D$  is isomorphic to a polynomial ring  $R[\zeta_1, \dots, \zeta_n]$  in  $n$  variables over  $R$ , where  $\zeta_i$  is the image of  $\delta_i$  in  $\Sigma_1/\Sigma_0$ . In particular,  $\text{gr } D$  is commutative, Noetherian, and regular. Its Krull dimension is  $2n$ .

*Solution.* For each  $i \geq 1$ , elements of  $\Sigma_i/\Sigma_{i-1}$  are in bijection with the  $R$ -linear combinations of monomials in degree exactly  $i$  in the  $\delta_j$ . In particular,  $\Sigma_1/\Sigma_0$  is generated over  $R$  by the images of the  $\delta_i$ . The suggested map

$$\delta_i + \Sigma_0 \mapsto \zeta_i$$

is automatically  $R$ -linear, so addition and multiplication agree with the respective operations in  $R$ .

Commutativity between elements of  $R$  and the variables will follow by induction. The base case is straightforward: for  $f \in R$ ,  $\delta_i f - f \delta_i = \frac{\partial f}{\partial x_i} \in \Sigma_0$ , so vanishes in the quotient. To simplify notation in the inductive case, for any  $d \in D$ , let  $\bar{d}^{(j)}$  denote the image of  $d$  in  $\Sigma_j/\Sigma_{j-1}$ . So in the base case, for example,  $\bar{\delta}_i^{(1)} f - f \bar{\delta}_i^{(1)} = \overline{\left(\frac{\partial f}{\partial x_i}\right)}^{(1)} = 0$ . Now say  $d \in D$  has the expression

$$d = f \prod_{i=1}^n \delta_i^{j_i}$$

where  $f \in R$  and  $j_1 + \dots + j_n = j$ . Assume commutativity holds for elements in  $\Sigma_j/\Sigma_{j-1}$ . Then

$$\begin{aligned} \bar{\delta}_l^{(1)} \bar{d}^{(j)} - \bar{d}^{(j)} \bar{\delta}_l^{(1)} &= \bar{\delta}_l^{(1)} \overline{\left(f \prod_{i=1}^n \delta_i^{j_i}\right)}^{(j)} - \overline{\left(f \prod_{i=1}^n \delta_i^{j_i}\right)}^{(j)} \bar{\delta}_l^{(1)} \\ &= \bar{\delta}_l^{(1)} \bar{f}^{(0)} \overline{\left(\prod_{i=1}^n \delta_i^{j_i}\right)}^{(j)} - \bar{f}^{(0)} \overline{\left(\prod_{i=1}^n \delta_i^{j_i}\right)}^{(j)} \bar{\delta}_l^{(1)} \\ &= \bar{f}^{(0)} \bar{\delta}_l^{(1)} \overline{\left(\prod_{i=1}^n \delta_i^{j_i}\right)}^{(j)} - \bar{f}^{(0)} \overline{\left(\prod_{i=1}^n \delta_i^{j_i}\right)}^{(j)} \bar{\delta}_l^{(1)} \\ &= \bar{f}^{(0)} \overline{\left(\prod_{i=1}^n \delta_i^{j_i}\right)}^{(j+1)} \delta_l - \bar{f}^{(0)} \overline{\left(\prod_{i=1}^n \delta_i^{j_i}\right)}^{(j+1)} \delta_l \\ &= 0. \end{aligned}$$

Verifying the commutativity of the ring  $\text{gr } D$  ensures the map to  $R[\zeta_1, \dots, \zeta_n]$  preserves ring operations. Furthermore, the bijection between  $\bar{\delta}_i^{(1)}$  and  $\zeta_i$  ensures the map is a ring isomorphism.

The additional statements (commutative, Noetherian, regular, Krull dimension  $2n$ ) are already true for  $R[\zeta_1, \dots, \zeta_n]$  and do not change under ring isomorphism, hence are true for  $\text{gr } D$ .  $\square$

—

**Exercise.**  $R$  is a holonomic  $D$ -module.

*Solution.* First of all,  $R$  is finitely generated as a  $D$ -module: define the  $D$ -map

$$\begin{aligned} D &\rightarrow R \\ d &\mapsto d(1) \end{aligned}$$

for each  $d \in D$ . Then  $R \cong D/\text{Ann}_D(1)$ , so only needs one generator.

We have to check the Krull dimension of  $\text{gr}_\Gamma R$ , assuming a good filtration  $\Gamma$  exists. Define  $\Gamma = \{\Gamma_i\}$  by  $\Gamma_i = \Sigma_i R$ , where  $\Sigma_i R$  is again the subgroup of all  $R$ -linear combinations of monomials in degree at most  $i$  in the  $\delta_i$ . Then  $\Sigma_i R = R$  for all  $i$  and hence  $\Gamma$  satisfies all the conditions to be a filtration. The associated graded module is then

$$\begin{aligned} \text{gr}_\Gamma R &= \bigoplus_i \Gamma_i / \Gamma_{i-1} \\ &= R/0 \oplus R/R \oplus \dots \oplus R/R \oplus \dots \\ &= R \end{aligned}$$

where by convention  $\Gamma_i = 0$  for all  $i < 0$ . Recall the associated graded ring for  $D$  is a polynomial ring in  $n$  variables over  $R$ , so  $R$  is again generated by one element over  $D$ . The Krull dimension of  $R$  is  $n$  because  $R = K[[x_1, \dots, x_n]]$ . Therefore  $R$  is a holonomic  $D$ -module.  $\square$

**Exercise.** If  $W$  is any multiplicative system in  $R$  and  $M$  is a  $D$ -module, then  $W^{-1}M$  has the structure of a  $D$ -module in such a way that the map  $M \rightarrow W^{-1}M$  is a homomorphism of  $D$ -modules.

*Proof.* First extend the action of  $D$  to an element  $\frac{m}{w} \in W^{-1}M$ , where  $m \in M$  and  $w \in W$ . Elements in  $R$  already act as usual, by multiplication, since  $w \in R$ . For  $i = 1, \dots, n$ ,

$$\delta_i \left( \frac{m}{w} \right) = \frac{w \cdot \delta_i m - m \cdot \frac{\partial}{\partial x_i} w}{w^2} \in W^{-1}M.$$

The  $D$ -linearity is carried over from  $D$ -linearity over  $M$  and the linearity of multiplication and partial derivatives. The natural map  $M \rightarrow W^{-1}M$  given by  $m \mapsto m/1$  is already  $R$ -linear. To get  $D$ -linearity, take  $i = 1, \dots, n$  and

$$\begin{aligned} \delta_i m \mapsto \frac{\delta_i m}{1} &= \delta_i \left( \frac{m}{1} \right) \\ &= \frac{1 \cdot \delta_i m - m \cdot \frac{\partial}{\partial x_i} 1}{1^2} \\ &= \frac{\delta_i m}{1}, \end{aligned}$$

as it should.  $\square$

---

**Exercise. If  $S$  is Noetherian then  $\text{Ass}_S N$  is nonempty.**

*Proof.* Choose a nonzero element  $u \in N$ . The following set

$$\mathcal{A} = \{\text{Ann } ru \mid r \in S, ru \neq 0\}$$

is a family of ideals in  $S$ .  $\mathcal{A}$  is nonempty because it contains the zero ideal. All the ideals in  $\mathcal{A}$  are proper, because if  $S$  kills  $u$  then so must 1, meaning  $u = 0$ , a contradiction to the choice of  $u$ .

$S$  is Noetherian, so  $\mathcal{A}$  must have a maximal element,  $I = \text{Ann } ru$ , for some  $r \in S$ . By construction  $ru \neq 0$  so we can just rechoose  $u$  so that maximal ideal in  $\mathcal{A}$  will be  $I = \text{Ann } u$ . Showing  $I$  is prime will show  $\text{Ass}_S N$  is nonempty.

Suppose  $a, b \notin I$  and  $ab \in I$ . Since  $b$  is not in  $I$ ,  $bu \neq 0$ . On the other hand,  $I \cdot bu = 0$ , because  $a(bu) = 0$ . Therefore  $I + Sa$  kills  $u$ , so is also in  $\mathcal{A}$ . But  $I \subseteq I + Sa \neq S$  was chosen to be maximal, so  $a$  must also be in  $I$ .  $\square$

**Exercise. In general, if  $N \subseteq M$  is an essential extension then  $\text{Soc } M \subseteq N$ .**

*Proof.* It is enough to show any nonzero element in  $\text{Soc } M$  lies in  $N$ , because zero already does. So choose nonzero  $u \in \text{Soc } M$ . By definition, the maximal ideal  $\mathfrak{m} \in S$  kills  $u$ , but  $S$  does not by the selection of  $u$ . So

$$\begin{aligned} Su &\cong S/\text{Ann } u \\ &\cong S/\mathfrak{m}, \end{aligned}$$

a field. On the other hand,  $N \subseteq M$  is an essential extension, so in particular,  $u$  lies in a nonzero submodule contained in  $N$ . But  $u$  generates  $S/\mathfrak{m}$ , the smallest nonzero  $S$ -module, and so  $S/\mathfrak{m} \subseteq N$ . In particular,  $u \in N$ .  $\square$

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