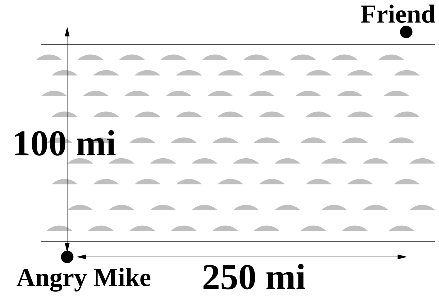
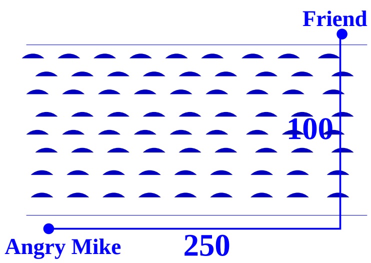
Section 4.4 – Optimization, Geometry, and Modeling with Angry Mike

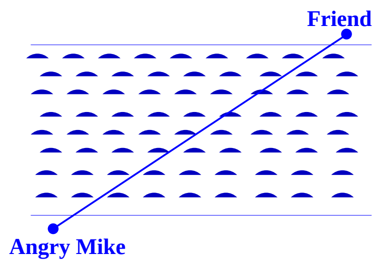
Angry Mike is on the run. He finally couldn't outsmart his brother sheriff Tommy Boy Hopkins any longer and there is a warrant out for his arrest. In his desperation Angry Mike has hijacked a dirt bike and is making a run for the state line. A friend is waiting for him 100 miles north and 250 miles east with a car. Unfortunately, a range of hills stretches all the way to the north and east from Angry Mike's starting position. He knows that he can go 70 mph as long as he stays out of the hills, but that he can go no faster than 40 mph in the hills.



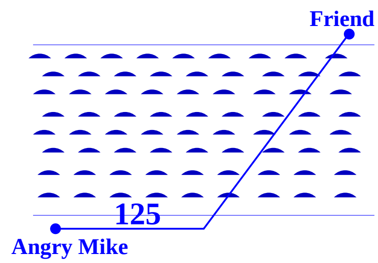
1. Draw three different possible routes (straight lines, or line segments) Angry Mike can take. One of them should maximize the part of the trip outside the hills, one of them should maximize the part of the trip in the hills.



Route 1



Route 2



Route 3

2. Compute the total driving time for those three routes.

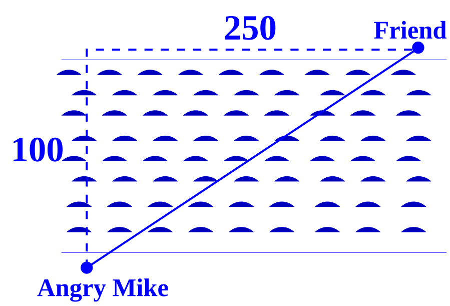
First, we note that for an object moving at a constant speed, the travel time equals the distance traveled divided by the speed. We use this principle for each of the three routes below.

Route 1: Since Angry Mike drives 250 miles at a speed of 70 miles per hour and 100 miles at a speed of 40 miles per hour, his travel time is

$$\frac{250}{70} + \frac{100}{40} \approx 6.07 \text{ hours.}$$

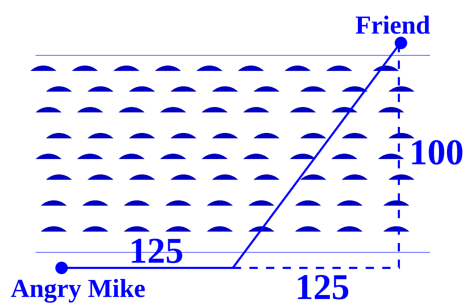
Route 2: Along this route, Angry Mike does all of his traveling through the hills, so his speed is 40 mph for the entire distance he travels. Referring to the diagram to the right, we use Pythagorean's Theorem to conclude that he travels a distance of $\sqrt{100^2 + 250^2} = 50\sqrt{29}$ miles. Therefore, his travel time is

$$\frac{50\sqrt{29}}{40} \approx 6.73 \text{ hours.}$$



Route 3: Referring to the diagram to the right, we see that Angry Mike travels the first 125 miles at a speed of 70 mph. Then, he travels a distance of $\sqrt{125^2 + 100^2} = 25\sqrt{41}$ miles through the hills at a speed of 40 mph. Therefore, his total travel time is

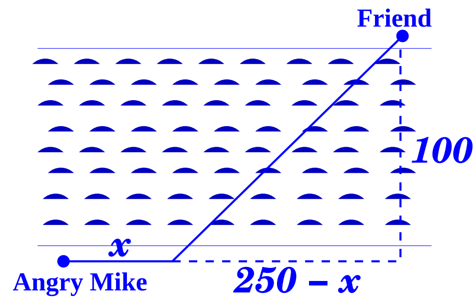
$$\frac{125}{70} + \frac{25\sqrt{41}}{40} \approx 5.79 \text{ hours.}$$



3. There are many more possible routes Mike could take. Out of all of them, which one should Angry Mike take to get to his waiting friend as fast as possible?

In general, suppose that Angry Mike first travels a distance of x miles outside of the hills (for $0 \le x \le 250$), and then cuts across the hills for the remaining distance, which, by Pythagorean's Theorem, would equal

$$\sqrt{100^2 + (250-x)^2} \text{ miles.}$$



Since his speed is 70 mph for the first leg of the trip and 40 mph for the second leg of the trip, his total travel time is given by the function

$$t(x) = \frac{x}{70} + \frac{\sqrt{100^2 + (250-x)^2}}{40} \text{ hours.}$$

To find the fastest travel time, we need to find the global minimum value of t on the interval $0 \le x \le 250$. We begin by solving the equation $t'(x) = 0$ for x to find the critical points.

$$\begin{aligned}
0 &= \frac{1}{70} + \frac{1}{40} \cdot \frac{1}{2}(100^2 + (250-x)^2)^{-1/2} \cdot 2(250-x) \cdot (-1) \\
\frac{250-x}{40\sqrt{100^2 + (250-x)^2}} &= \frac{1}{70} \\
7(250-x) &= 4\sqrt{100^2 + (250-x)^2} \\
49(250-x)^2 &= 16 \cdot 100^2 + 16(250-x)^2 \\
33(250-x)^2 &= 16 \cdot 100^2 \\
x &= 250 \pm \frac{400}{\sqrt{33}}
\end{aligned}$$

Therefore, only one of the critical points above lies in the interval $0 \le x \le 250$, and its approximate value is 180.37. We now compare the value of the function t at this critical point with the approximate values of t at our two interval endpoints (see table to the right). Note that Angry Mike's fastest route is to drive the first 180.37 miles outside of the hills, and then to cut diagonally across the hills straight toward his destination.

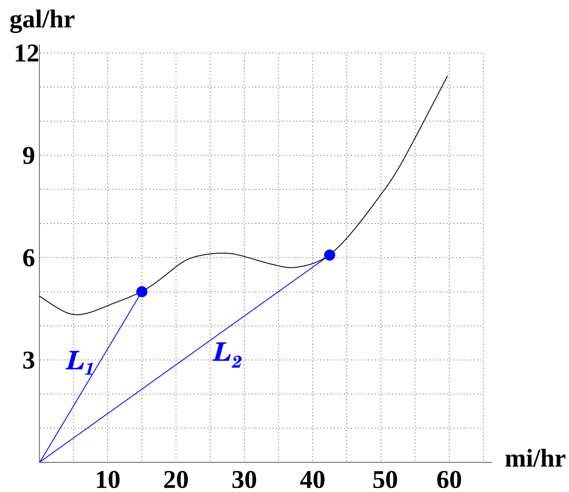
x	$t(x)$
0	6.73
180.37	5.62
250	6.07

4. How long is the part of the trip that Angry Mike has to ride through the hills?

Using the diagram from question (3) above as a guide, we see that Angry Mike's distance traveled through the hills is $\sqrt{100^2 + (250-x)^2}$ miles. Therefore, since $x = 180.37$ for his most time efficient route, we conclude that the length of his journey through the hills is

$$\sqrt{100^2 + (250-180.37)^2} \approx 121.9 \text{ miles.}$$

Meanwhile, Tommy Boy is in hot pursuit and facing an entirely different problem. He is driving a county-issued cross country vehicle. The hills are absolutely no problem, but he can only use one tank of gas plus one reserve tank to follow Angry Mike. The tank of his vehicle holds 25 gallons; the reserve tank holds 16 gallons. Since Tommy Boy knows his brother and his friends, he quickly figured out where Angry Mike is going, and so he is taking the most direct route through the hills which is about 270 miles long. The fuel consumption (in gallons per hour) of Tommy's vehicle as a function of speed (in mph) is given in the graph to the right. Tommy Boy knows that he can maximize the fuel efficiency of his usual patrol car (in miles per gallon) by going 50 mph. He figures that the cross country vehicle is close enough to a normal car to have the same properties.



1. Is he right?

   To figure out how to analyze this question, let us first take a "nice" point on the provided graph, like $(15, 5)$, and try to interpret it in terms of fuel efficiency. The presence of this point on the graph indicates that when this vehicle is driven at a speed of 15 miles per hour, it uses fuel at a rate of 5 gallons per hour. Therefore, the fuel used per mile for this vehicle when driven at a speed of 15 miles per hour is

   $$\frac{5 \text{ gal/hr}}{15\text{mi/hr}} = \frac{5 \text{ gal}}{\text{hr}} \times \frac{1 \text{ hr}}{15 \text{ mi}} = \frac{1}{3} \text{ gal/mi}.$$

   Note that the number calculated above is the slope of the line segment L_1 from the origin to $(5, 15)$ that we drew on the above diagram. Also, note that if we take the reciprocal of the preceding calculation, we get a fuel efficiency figure of 3 miles per gallon at a driving speed of 15 mph.

   From the above demonstration, we deduce that the fuel efficiency, in miles per gallon, can be determined by taking the reciprocal of the slope of a line segment drawn from the origin to the relevant point in the diagram above. To maximize fuel efficiency, we are therefore looking for the point on the curve above for which this line segment has the smallest possible slope. Using a straightedge to help us graphically estimate, we see that the segment labeled L_2 in the diagram above has the smallest possible slope, and that it intersects the graph at the approximate point $(42, 6)$. Therefore, fuel efficiency for this vehicle is maximized at a driving speed of about 42 miles per hour, so Tommy Boy was incorrect in his assumption.

2. Will he actually make it to Angry Mike's destination at a driving speed of 50 mph?

   First, note that Tommy Boy's travel time to the destination at a speed of 50 mph is given by

   $$\frac{270 \text{ mi}}{50 \text{ mi/hr}} = 5.4 \text{ hours}.$$

   The above graph indicates that, at a speed of 50 mph, Tommy Boy's vehicle uses about 7.9 gallons of fuel per hour. Therefore, the total amount of fuel he needs for the trip is $(5.4) \cdot (7.9) = 42.66$ gallons. Since his total gas supply is only 41 gallons, it appears that he will not make it to the destination at this driving speed.

3. How fast should he go to maximize fuel efficiency?

   According to our calculations in problem (1) above, he should drive at a speed of about 42 miles per hour to maximize fuel efficiency.

4. How much gas would he have to spare if he drove with optimal speed?

As in problem (2) above, we first calculate Tommy Boy's traveling time to the destination; in this case, it is

$$\frac{270 \text{ mi}}{42 \text{ mi/hr}} \approx 6.43 \text{ hours.}$$

The graph above indicates that his vehicle will use about 6.1 gallons of fuel per hour at the optimal speed, so the total amount of fuel used is $(6.43) \cdot (6.1) \approx 39.2$ gallons. He will therefore have about 1.8 gallons of gas to spare.