Math 115
Extra Problems for 5.5

1. The sum of two positive numbers is 48. What is the smallest possible value of the sum of their squares?

   Solution. Let x and y denote the two numbers, so that $x + y = 48$. Write

   $$\begin{aligned} S &= x^2 + y^2 \\ &= x^2 + (48 - x)^2 \\ &= 2x^2 - 96x + 48^2 \end{aligned}$$

   to denote the sum of the squares. Since S is a parabola, its critical point will be a minimum. Now find the critical point:

   $$\begin{aligned} 0 = \frac{dS}{dx} &= 4x - 96 \\ x &= 24. \end{aligned}$$

   Then the smallest possible value of the sum of the squares is $(24)^2 + (48 - (24))^2 = 1152$.

2. The manager of a computer shop has to decide how many computers to order from the manufacturer at one time. If she orders a large number, she has to pay extra in storage costs. If she orders only a small number, she will have to reorder more often, which will involve additional handling costs. She has found that if she orders computers in lots of size x, the storage and handling costs for a year will be C dollars, where

   $$C = 15x + \frac{24,000}{x} + 6000.$$

   How many should she order at one time to minimize her costs?

   Solution. Minimize costs by setting

   $$\begin{aligned} 0 = \frac{dC}{dx} &= 15 - \frac{24000}{x^2} \\ x^2 &= 1600. \end{aligned}$$

   Only the positive square root makes sense in this problem. Then since $\frac{dC}{dx}$ is a parabola with zeros at $x = \pm 40$, $\frac{dC}{dx}$ must go from negative to positive at $x = 40$, which implies C goes from increasing to decreasing at $x = 40$. Conclude $x = 40$ computers per lot will minimize costs.

3. Among all rectangles having perimeter 100 meters, find the dimensions of the one with the largest area. What if the perimeter were L meters?

   Solution. Let x, y denote the dimensions of the rectangle. The perimeter is $2x + 2y = 100$ m, and the area is

   $$\begin{aligned} A &= xy \\ &= x(\frac{100 - 2x}{2}) \\ &= x(50 - x) \\ &= 50x - x^2. \end{aligned}$$

The area function is an upside down parabola, so its critial point will be a maximum. Find the critical point,

$$\begin{aligned} 0 = \frac{dA}{dx} &= 50 - 2x \\ x &= 25, \end{aligned}$$

so $y = 50 - (25) = 25$. The dimensions of the rectangle should be 25×25 m.

On the other hand, if the perimeter is $2x + 2y = L$ m, then, following the same steps above, the area is $A = x(\frac{L-2x}{2})$ and the critical point is at $x = \frac{L}{4}$. So the dimensions should be $\frac{L}{4} \times \frac{L}{4}$ m.

4. The local council is planning to construcct a new sports ground in the shape of a rectangle with semicircular ends. A running track 400 meters long is to go around the perimeter. What choice of dimensions will make the rectangular area in the center as large as possible? What should the dimensions be if the total area enclosed by the running track is to be as large as possible?

Solution. Let x denote the length of the rectangle and let r denote the radius of the semicircles. Then the running track is $2x + 2\pi r = 400$ m long. The rectangular area in the center is given by

$$\begin{aligned} A &= x \cdot 2r \\ &= \left(\frac{400 - 2\pi r}{2}\right) \cdot 2r \\ &= (200 - \pi r) \cdot 2r \\ &= 400r - 2\pi r^2. \end{aligned}$$

This function with respect to r is an upside down parabola, so its critical point will be a maximum:

$$\begin{aligned} 0 = \frac{dA}{dr} &= 400 - 4\pi r \\ r &= \frac{100}{\pi} \approx 31.831 \end{aligned}$$

Then $x = 200 - \pi \left(\frac{100}{\pi}\right) = 100$. So the rectangle should be 100 m long and the semicircles should have radius 31.831 m, in order to maximize the area of the rectangle.

On the other hand, the total area enclosed is given by

$$\begin{aligned} B &= 400r - 2\pi r^2 + \pi r^2 \\ &= 400r - \pi r^2 \end{aligned}$$

which is again an upside down parabola, so its critical point is a maximum. Following the same steps as above, the critical point is $r = \frac{200}{\pi}$ so $x = 200 - \pi \left(\frac{200}{\pi}\right) = 0$. Therefore the entire space should be a circle of radius $r = \frac{200}{\pi} \approx 63.662$ m.

5. A wire two meters long is cut into two pieces. One piece is bent into a square for a stained glass frame while the other piece is bent into a circle for a TV antennna. To reduce storage

space, where should the wire be cut to minimize the total area of both figures? Where should the wire be cut to maximize the total area?

Solution. Let x denote the length of one side of the square and let r denote the radius of the circle. The perimeters should add to 2 m, so $4x + 2\pi r = 2$. The areas add to

$$
\begin{aligned}
A &= x^2 + \pi r^2 \\
&= x^2 + 2\pi \left(\frac{2 - 4x}{2\pi} \right)^2 \\
&= x^2 + 2\pi \left(\frac{1 - 2x}{\pi} \right)^2 \\
&= \frac{\pi^2 x^2 + 2\pi(1 - 4x + 4x^2)}{\pi^2} \\
&= \frac{\pi x^2 + 2 - 8x + 8x^2}{\pi} \\
&= \frac{1}{\pi} \left((8 + \pi)x^2 - 8x + 2 \right)
\end{aligned}
$$

which is a parabola, so its critical point will be a minimum. Find the critical point:

$$
\begin{aligned}
0 = \frac{dA}{dx} &= \frac{1}{\pi} (2(8 + \pi)x - 8) \\
&= 2(8 + \pi)x - 8 \\
x &= \frac{8}{2(8 + \pi)} \\
&= \frac{4}{8 + \pi} \\
&\approx 0.359
\end{aligned}
$$

Then the wire should be cut at $4x \approx 1.436$ m to minimize the total area.

To maximize the total area, the wire should form either the square or the circle, but not both. This is because the only critical point was a minimum. The global maximum will occur on the endpoints of the domain $x \in [0, \frac{1}{4}]$, since the wire is 2 m long. When $x = 0, A = \frac{2}{\pi} \approx 0.627$ m^2. When $x = \frac{1}{4}, A = \frac{1}{2\pi} + \frac{1}{16} \approx 0.222$ m^2. So the wire should be cut at $4x = 0$ m; the entire wire should be used to make the circle.

6. Some airlines have restrictions on the size of items of luggage that passengers are allowed to take with them. One has a rule that the sum of the length, width, and height of any piece of luggage must be less than 158 cm. A passenger wants to take a box of the maximum allowable volume. If the length and width are to be equal, what should the dimensions be and what will the volume be? If the length is to be twice the width, what should the dimensions be?

Solution. If the length equals the width, let $2x + y < 158$ cm. Then the volume is

$$
\begin{aligned}
V &= x^2 y \\
&< x^2 (158 - 2x) \\
&= 158x^2 - 2x^3.
\end{aligned}
$$

Find the critical points of the left-hand side of the above inequality:

$$
\begin{aligned}
0 = \frac{d}{dx}(158x^2 - 2x^3) &= 316x - 6x^2 \\
&= (316 - 6x)x.
\end{aligned}
$$

In the context of the problem, the domain is $x \in (0, \sqrt{158})$, so take the positive critical point. If $x < \frac{316}{6} \approx 52.667$ then $6x - 316 > 0$; if $x > \frac{316}{6}$ then $6x - 316 < 0$. The derivative goes from positive to negative so the critical point is a maximum. Then $y = 158 - 2\left(\frac{316}{6}\right) \approx 52.667$ and the luggage is a cube with dimensions less than $52.667 \times 52.667 \times 52.667$ cm^3.

If the length is twice the width, then $3x + y < 158$ cm. The volume is $V = 2x^2y = 316x^2 - 6x^3$ and the maximum occurs when $x = \frac{632}{18} \approx 35.111$, so $y = 158 - 3\left(\frac{632}{18}\right) \approx 52.667$. So the dimensions must be less than $70.222 \times 35.111 \times 52.667$ cm^3.

7. A wire six meters long is cut into 12 pieces. These pieces are welded together at right angles to form the frame of a box with square base. Where should the cuts be made to maximize the total surface area of the box?

Solution. Let the box have dimensions $x \times x \times y$ m^3. Then the pieces making the frame are $8x + 4y = 12$ m total. The surface area is

$$
\begin{aligned}
A &= 2x^2 + 4xy \\
&= 2x^2 + 4x\left(\frac{12 - 8x}{4}\right) \\
&= 2x^2 + 4x(3 - 2x) \\
&= 2x^2 + 12x - 8x^2 \\
&= 12x - 6x^2
\end{aligned}
$$

and is given by an upside down parabola. Therefore its critical point will be a maximum:

$$
\begin{aligned}
0 = \frac{dA}{dx} &= 12 - 12x \\
x &= 1
\end{aligned}
$$

and so $y = 1$. Thus the wire should be cut into 12 equal pieces each of length 1 m.

8. Suppose that you want to make a fish tank with a volume of two cubic meters whose base is a rectangle twice as long as it is wide. The base and sides are to be made of glass. What shape of tank will use the least amount of glass (and so cost least)?

Solution. Let $2x \times x \times y$ m^3 denote the dimensions of the fish tank. The volume is $V = 2x^2y = 2$ m^3. The total area of the base and sides is

$$
\begin{aligned}
A &= 2x^2 + 6xy \\
&= 2x^2 + 6x\left(\frac{2}{2x^2}\right) \\
&= 2x^2 + \frac{6}{x}.
\end{aligned}
$$

To maximize A,

$$
\begin{aligned}
0 = \frac{dA}{dx} &= 4x - \frac{6}{x^2} \\
&= 4x^3 - 6 \\
x^3 &= \frac{3}{2} \\
x &= \left(\frac{3}{2}\right)^{\frac{1}{3}} \\
&\approx 1.145
\end{aligned}
$$

which implies $y = \frac{2}{2\left(\frac{3}{2}\right)^{\frac{2}{3}}} \approx 0.763$. The second derivative $\frac{d^2 A}{dx^2} = 4x + \frac{12}{x^3}$ is always positive when $x > 0$, so A is concave up when $x > 0$. So the critical point must be a minimum. Then the dimensions of the fish tank must be $2.289 \times 1.145 \times 0.763$ m^3.

9. A factory makes cylindrical cans of volume 500 cubic centimeters. What should they make the diameter and height of the can to use the least amount of metal? If the metal for the top and the bottom of the can costs twice as much as the metal for the sides, what should the dimensions of the can be to minimize the cost?

Solution. Let r denote the radius of the can and h denote the height. The volume is $500 = \pi r^2 h$. The area is

$$
\begin{aligned}
A &= 2\pi r h + 2\pi r^2 \\
&= 2\pi r \left(\frac{500}{\pi r^2}\right) + 2\pi r^2 \\
&= \frac{1000}{r} + 2\pi r^2.
\end{aligned}
$$

The to minimize the area,

$$
\begin{aligned}
0 = \frac{dA}{dx} &= \frac{-1000}{r^2} + 4\pi r \\
&= -1000 + 4\pi r^3 \\
\left(\frac{1000}{4\pi}\right)^{\frac{1}{3}} &= r.
\end{aligned}
$$

When $r < \left(\frac{1000}{4\pi}\right)^{\frac{1}{3}}, \frac{dA}{dx} < 0$ and when $r > \left(\frac{1000}{4\pi}\right)^{\frac{1}{3}}, \frac{dA}{dx} > 0$. Therefore $r = \left(\frac{1000}{4\pi}\right)^{\frac{1}{3}} \approx 4.301$ gives a minimum; $h = \frac{500}{\pi\left(\frac{1000}{4\pi}\right)^{\frac{2}{3}}} \approx 8.603$. So the diameter should be 8.603 cm and the height should be 8.603 cm.

If the top and bottom cost twice as much as the side, then the cost is

$$
\begin{aligned}
C &= 2\pi r h + 4\pi r^2 \\
&= 2\pi r \left(\frac{500}{\pi r^2}\right) + 4\pi r^2 \\
&= \frac{1000}{r} + 4\pi r^2.
\end{aligned}
$$

To minimize,

$$
\begin{aligned}
0 = \frac{dA}{dx} &= \frac{-1000}{r^2} + 8\pi r \\
&= -1000 + 8\pi r^3 \\
\left(\frac{1000}{8\pi}\right)^{\frac{1}{3}} &= r.
\end{aligned}
$$

As before, the critical point is a minimum, so the diameter should be 6.834 cm and the height should be 13.656 cm.

10. Find the coordinates of the point on the curve $y = \sqrt{x}$ closest to the point (1,0).

Solution. Minimize the square of the distance between the curve and (1,0), because it's easier. The square of the distance is

$$
\begin{aligned}
R &= (y-1)^2 + (x-0)^2 \\
&= (\sqrt{x}-1)^2 + x^2.
\end{aligned}
$$

Using a calculator,

$$
0 = \frac{dR}{dx} = 1 - \frac{1}{\sqrt{x}} + 2x
$$

has solution $x \approx 0.348$. The derivative is strictly increasing, so it must go from negative to positive at the critical point, making the critical point a minimum. Then the coordinates of the closest point are $(0.348, \sqrt{0.348}) = (0.348, 0.590)$.

11. A farmer is planning to plant a small orchard and is gathering information about the amount of fruit he can expect to harvest each year once the trees are mature. He is advised that if he plants up to 60 trees of a particular type on his plot of land, the average harvest from each tree will be about 120 kg, but for each additional tree planted the expected yield will go down by an average of 2 kg per tree as a result of overcrowding. He wants to plan for the maximum yield of fruit. How many trees should he plant?

Solution. Let t denote the number of trees. When $0 \le t \le 60$, the harvest is an increasing linear function in t so the maximum harvest is at $t = 60$. When $t > 60$ the harvest is given by

$$
\begin{aligned}
h &= (120 - 2(t-60))t \\
&= -2t^2 + 240t
\end{aligned}
$$

which is an upside down parabola, so its critical point is a maximum:

$$
\begin{aligned}
0 = \frac{dh}{dt} &= -4t + 240 \\
t &= 60
\end{aligned}
$$

which is not in the domain. Therefore $t = 60$ trees will have the maximum yield of fruit.

12. The cost of running a small van at a speed of v km/hr is $25 + 0.01v^2$ dollars per hour. How long would it take for a trip of 100 km at a constant speed of v km/hr, and how much would it cost? How fast should the driver travel in order to minimize the cost?

Solution. Time is the product of distance and speed, so a trip of 100 km at a constant speed of v km/hr takes $\frac{100}{v}$ hrs. The cost of the trip will be the cost per hour, times the time it takes:

$$
\begin{aligned}
C &= (25 + 0.001v^2)\frac{100}{v} \\
&= \frac{2500}{v} + v
\end{aligned}
$$

Minimizing,

$$
\begin{aligned}
0 = \frac{dC}{dv} &= \frac{-2500}{v^2} + 1 \\
&= -2500 + v^2 \\
2500 &= v^2 \\
50 &= v.
\end{aligned}
$$

Take the positive root, for the sake of the context of the problem. When $v < 50$, $\frac{dC}{dv} < 0$ and when $v > 50$, $\frac{dC}{dv} > 0$, so v is a minimum. Therefore the driver should travel 50 km/hr.