Generalizing Macaulay's Theorem for Hilbert Functions

(talk given at the University of Michigan Student Commutative Algebra Seminar)
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by Ashley K. Wheeler

The point of this talk is to introduce my prospective thesis project, the Eisenbud Green Harris (EGH) conjecture. Anywhere in this paper you should assume $S = k[x_1, \dots, x_n]$, the polynomial ring in n variables over a field k. For the sake of simplifying notation, $\mathbb{A}$ will always refer to a sequence of integers $a_1, \dots, a_n$ satisfying $1 \le a_1 \le \cdots \le a_n$.

Definition. *An ideal $L \subset S$ is* **lex plus powers** *with respect to $\mathbb{A}$, or $\mathbb{A}$-LPP, means L is minimally generated by monomials $u_1, \dots, u_l$, $x_1^{a_1}, \dots, x_n^{a_n}$, and if a monomial $u = x_1^{b_1} \cdots x_n^{b_n}$ satisfies either of the following*

(a) $\deg u > \deg u_i$ for some $i = 1, \dots, l$;

(b) for some $u_i = x_1^{b(i)_1} \cdots x_n^{b(i)_n}$ having the same degree as u, $b_j > b(i)_j$ for the first index where $b_j \neq b(i)_j$;

then $u \in L$.

Definition. *A* **regular sequence** *is a sequence of elements $f_1, \dots, f_n \in S$ such that f_1 is not a zero divisor in S, and f_i $(i = 2, \dots, n)$ is not a zero divisor in $S/(f_1, \dots, f_{i-1})S$.*

Eisenbud Green Harris (EGH) Conjecture. *Let $I \subset S$ be an ideal containing a regular sequence of degrees $a_1, \dots, a_n$. Then there exists an $\mathbb{A}$-LPP ideal with the same Hilbert function as I.*

Intuitively, if $\mathbb{V}(I)$ denotes the geometric object given by the vanishing of all polynomials in I, then the Hilbert function counts the number of conditions needed for hypersurfaces to contain $\mathbb{V}(I)$. Eisenbud, Green, and Harris [EGH1] first considered this problem in the special case $a_1 = \cdots = a_n = 2$, which implies the following generalized Cayley-Bacharach conjecture:

Generalized Cayley-Bacharach Conjecture. *Let $\Omega \subset \mathbb{P}^n$ be a complete intersection of quadrics. Any hypersurface of degree d that contains a subscheme $\Gamma \subset \Omega$ of degree strictly greater than $2^n - 2^{n-d}$ must contain Ω.*

Eisenbud, Green, and Harris prove the generalized Cayley-Bacharach conjecture for $n \le 7$ in [EGH2] and this is the best known progress. Meanwhile, the more general EGH conjecture has more results. This expository follows S.M. Cooper's account [Coop] of the EGH conjecture arising as a generalization of Macaulay's Theorem. Macaulay's Theorem is a 1927 result which characterizes Hilbert functions of homogeneous k-algebras. After the section on Macaulay's Theorem is a brief statement of combinatorial results which Cooper-Roberts use to generalize Macaulay's Theorem. Following that is the statement of a stronger conjecture, the Lex Plus Powers conjecture, and how it relates to the EGH conjecture. Finally is a list of the known results for the EGH conjecture.

Recall:

Definition.

(1) A ring R is **graded** *means R is a direct sum $R = R_0 \oplus \cdots \oplus R_n \oplus \cdots$ of abelian groups, and the summands satisfy $R_i R_j \subset R_{i+j}$ for all i, j.*
(2) An R-module M is **graded** *means M is a direct sum of abelian groups M_i, indexed over the integers $\mathbb{Z}$, and $R_i M_j \subseteq M_{i+j}$ for all i, j.*

Definition. *Let R denote a graded ring.*
(1) An element of R belonging to exactly one of the summands R_i is called **homogeneous***.*
(2) An ideal in R is **homogeneous** *means its generators are homogeneous.*
(3) R is a **homogeneous R_0-algebra** *means R is generated over R_0 by elements in R_1 (elements in R_1 are also called* **1-forms***).*

Definition. *Let M denote a finitely generated S-module. The* **Hilbert function**[1] *for M is given by*

$$\begin{aligned} H_M : \mathbb{N} \to& \mathbb{N} \\ d \mapsto& \dim_k M_d. \end{aligned}$$

Exercise. The Hilbert function is additive, i.e., if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of graded k-algebras, then $H_B(d) = H_A(d) + H_C(d)$ for all $d = 0, 1, \ldots$.

Algebraically, the Hilbert function of a graded ring or module just measures the dimension of each graded piece. Geometrically, if R is the homogeneous coordinate ring of a projective variety X, then the Hilbert function $H_X(d)$ (which is defined as $H_R(d)$) gives the number of conditions imposed by X on hypersurfaces of degree d which contain X.

Example. [EGH2] Let $\Gamma \subseteq \mathbb{P}^2$ denote a finite set of distinct points. Γ imposes l independent conditions on polynomials of degree d means the vector space of degree d polynomials vanishing on Γ has codimension l in the vector space of all degree d polynomials. Thus $H_\Gamma(d) = l$.

Macaulay's Theorem

Let $R = S/I$, where I is a homogeneous ideal. F.S. Macaulay characterizes the sequences which occur as the Hilbert function of such R (sometimes called **O-sequences**). Macaulay's Theorem says for each $d = 0, 1, \ldots$, there is a sharp upper bound for $H_R(d+1)$ in terms of $H_R(d)$. The primary source for this section is [BrH]. Use of the word "monomial" will always mean monic monomial in this talk.

[1]Usually authors reserve the notation $H_M(t)$ to denote the **Hilbert series** of a graded module, which is the formal power series in the variable t whose coefficients correspond to values of the Hilbert function. Hilbert series never come up in this talk, so I use $H_M(d)$ rather than the usual, and more cumbersome, $H(M, d)$ to denote the Hilbert function.

Definition. *A* **monomial ideal** *in S is an ideal generated by monomials.*

Definition. *A nonempty set $\mathcal{M}$ of monomials in S is called an* **order ideal of monomials** *means if $u \in \mathcal{M}$ and a monomial u' divides u, then $u' \in \mathcal{M}$.*

Example. Let $S = k[x, y, z]$. To have an order ideal of monomials $\mathcal{M}$ that contains the monomial x^2yz^4, $\mathcal{M}$ must also contain

$$\begin{aligned}
&x, y, z,\\
&x^2, xy, xz, yz, z^2,\\
&x^2y, x^2z, xyz, xz^2, yz^2, z^3,\\
&x^2yz, xyz^2, yz^3, z^4,\\
&x^2yz^2, xyz^3, yz^4,\\
&x^2yz^3, xyz^4.
\end{aligned}$$

Definition.

(1) A monomial $u = x_1^{a_1} \cdots x_n^{a_n}$ is less than a monomial $u' = x_1^{a'_1} \cdots x_n^{a'_n}$ in the **lexicographic (lex) order** *order means $a_i < a'_i$ for the first index where $a_i \neq a'_i$. The notation is $u <_{\text{lex}} u'$.*

(2) A monomial $u = x_1^{a_1} \cdots x_n^{a_n}$ is less than a monomial $u' = x_1^{a'_1} \cdots x_n^{a'_n}$ in the **homogeneous lexicographic (hlex) order** *order means $\deg u < \deg u'$ or the degrees are the same, and $a_i < a'_i$ for the first index where $a_i \neq a'_i$. The notation is $u <_{\text{hlex}} u'$.*

(3) A monomial $u = x_1^{a_1} \cdots x_n^{a_n}$ is less than a monomial $u' = x_1^{a'_1} \cdots x_n^{a'_n}$ in the **reverse lexicographic (rev or revlex) order** *means $\deg u < \deg u'$ or the degrees are the same and $a_i > a'_i$ for the last index where $a_i \neq a'_i$. The notation is $u <_{\text{rev}} u'$.*

Example. In the polynomial ring $S = k[x, y, z]$,

(1) $xz^2 <_{\text{lex}} xy$.

(2) $xz^2 >_{\text{hlex}} xy$ and $xz >_{\text{hlex}} y^2$.

(3) $xz <_{\text{rev}} y^2$.

The lex, hlex, and revlex orders can also apply to n-tuples over the natural numbers $\mathbb{N}$. The conditions which apply to the exponents of a monomial $x_1^{a_1} \cdots x_n^{a_n}$ correspond to conditions on the components of the n-tuples $(a_1, \ldots, a_n)$.

Theorem (Macaulay). *Let R be a homogeneous k-algebra with generators $X_1, \ldots, X_n$, and define the surjective k-algebra homomorphism*

$$\begin{aligned}
\pi : S = k[x_1, \ldots, x_n] &\twoheadrightarrow R\\
x_i &\mapsto X_i \, (i = 1, \ldots, n).
\end{aligned}$$

Then there exists an order ideal $\mathcal{M}$ of monomials from S such that $\pi(\mathcal{M})$ is a k-basis of R.

Proof. $\mathcal{M}$ will consist of monomials $u_1, u_2, \ldots$ recursively defined as follows: put $u_1 = 1$. For $u_1, \ldots, u_i$ already defined, let u_{i+1} denote the lex smallest monomial such that

$\pi(u_1), \ldots, \pi(u_{i+1})$ are k-linearly independent. This makes sense because the exponents on monomials are indexed by n-tuples in $\mathbb{N}$, so any nonempty set of monomials has a unique lex minimal element. By construction $\mathcal{M}$ is a k-basis for R.

To see $\mathcal{M}$ is an order ideal of monomials assume the contrary. Then there exists $u_i \in \mathcal{M}$ such that $u_i = ux_j$ for some $j = 1, \ldots, n$, and $u \notin \mathcal{M}$. Assert i is the smallest index where this happenes. Now $u \notin \mathcal{M}$ and $u <_{\text{lex}} u_i$, therefore $\pi(u)$ has a unique expression as a linear comination of $\pi(u_1), \ldots, \pi(u_{i-1})$. Furthermore, $u_r <_{\text{lex}} u$ for all $r = 1, \ldots, i-1$, because otherwise there would exist $r < i$ such that $u_r = ux_{j'}$, contradicting the assertion that i is the smallest index where this happens. Multiplying $\pi(u)$ by $\pi(x_j)$ gives $\pi(u_i)$ an expression as a linear combination of the elements $\pi(u_1x_j), \ldots, \pi(u_{i-1}x_j)$, and implies $u_rx_j <_{\text{lex}} u_i$ for all $r = 1, \ldots, i-1$. The construction of $\mathcal{M}$ thus dictates each such $\pi(u_rx_j)$ have a unique expression as a linear combination of $\pi(u_1), \ldots, \pi(u_{i-1})$, and putting them all together gives a unique expression for $\pi(u_i)$ as a linear combination of $\pi(u_1), \ldots, \pi(u_{i-1})$. That directly contradicts $u_i \in \mathcal{M}$, so the assumption $\mathcal{M}$ is not an order ideal of monomials fails. $\square$

Corollary. *With S, R, and $\mathcal{M}$ as in the theorem, let J denote the ideal generated by the monomials not in $\mathcal{M}$. Then R and S/J have the same Hilbert function. In particular, all Hilbert functions of homogeneous rings arise as Hilbert functions of homogeneous rings whose defining ideal is generated by monomials.* $\square$

For a degree d monomial $u \in S$, define the sets

$$\begin{aligned}\mathcal{L}_u =& \{\text{monomials } u' \in S_d : u' >_{\text{rev}} u\} \\ \mathcal{R}_u =& \{\text{monomials } u' \in S_d : u' \leq_{\text{rev}} u\}.\end{aligned}$$

The monomial sets $\mathcal{R}_u$ are called **lexsegments** (of degree d).

Exercise. A lexsegment of degree d spans a lexsegment of degree $d+1$, i.e., $\mathcal{R}_{x_1}\mathcal{R}_u = \mathcal{R}_{x_1u}$.

The sets $\mathcal{L}_u$ admit a natural decomposition: let i be the largest integer such that x_i divides u. Then we can express $\mathcal{L}_u$ as a disjoint union

$$\mathcal{L}_u = [x_1, \ldots, x_{i-1}]_d \cup \mathcal{L}_{x_i^{-1}u}x_i,$$

where $[x_1, \ldots, x_{i-1}]_d$ denotes the set of all degree d monomials in the variables $x_1, \ldots, x_{i-1}$. Thus if we write the degree d monomial

$$u = x_{j(1)}x_{j(2)} \cdots x_{j(d)}$$

with $1 \leq j(1) \leq j(2) \leq \cdots \leq j(d)$, then

$$\begin{aligned}\mathcal{L}_u =& [x_1, \ldots, x_{j(d)-1}]_d \cup \mathcal{L}_{x_{j(d)}^{-1}u}x_{j(d)} \\ =& [x_1, \ldots, x_{j(d)-1}]_d \cup [x_1, \ldots, x_{j(d)-2}]_{d-1}x_{j(d)} \cup \mathcal{L}_{x_{j(d-1)}^{-1}x_{j(d)}^{-1}u}x_{j(d-1)}x_{j(d)} \\ =& \cdots \\ =& \bigcup_{i=1}^{d}[x_1, \ldots, x_{j(i)-1}]_i x_{j(i+1)} \cdots x_{j(d)}\end{aligned}$$

(this is sometimes called the **natural decomposition** of $\mathcal{L}_u$). The cardinality is

$$|\mathcal{L}_u| = \sum_{i=1}^{d} \binom{j(i)+i-2}{i}.$$

Writing $k(i) = j(i)+i-2$, note that $k(d) > k(d-1) > \cdots > k(1) \geq 0$.[2] It turns out any nonnegatigve integer has such a binomial sum expansion.

Lemma. *Let d be a positive integer. Any $a \in \mathbb{N}$ can be written uniquely in the form*

$$a = \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \cdots + \binom{k(1)}{1},$$

where $k(d) > k(d-1) > \cdots > k(1) \geq 0$.

Proof. See [BrH]. □

Definition. *For positive integers a, d, the expression*

$$a^{\langle d \rangle} = \binom{k(d)+1}{d+1} + \binom{k(d-1)+1}{d-1} + \cdots + \binom{k(1)+1}{2}$$

*is called the d***th Macaulay representation of** *a. The integers $k(d), \ldots, k(1)$ are called the d***th Macaulay coefficients of** *a. By convention, we set $0^{\langle d \rangle} = 0$.*

Proposition. *Let u be a monomial of degree d in the polynomial ring S. Then $|\mathcal{L}_{x_1 u}| = |\mathcal{L}_u|^{\langle d \rangle}$.*

Proof. Write $u = x_{j(1)} \cdots x_{j(d)}$ and use $l(1), \ldots, l(d+1)$ to similarly index the factors of $x_1 u$ (so $l(1) = 1$, $l(2) = j(1), \ldots,$ $l(d+1) = j(d)$). Let $k(i) = j(i)+i-2$ and $k'(i) = l(i)+i-2$. Then

$$\binom{k'(1)}{1} = \binom{1+1-2}{1} = 0$$

and for $i = 2, \ldots, d+1$,

$$\begin{aligned} k'(i) &= l(i)+i-2 \\ &= j(i-1)+i-2 \\ &= k(i-1)+1. \end{aligned}$$

Therefore

$$|\mathcal{L}_{x_1 u}| = \sum_{i=1}^{d} \binom{k'(i)}{i} = \sum_{i=2}^{d+1} \binom{k(i-1)+1}{i} = |\mathcal{L}_u|^{\langle d \rangle},$$

as required. □

[2] By convention, $\binom{k}{l} = 0$ for $0 \leq k \leq l$.

Definition. *An ideal in which each degree is k-spanned by a lexsegment is called a* **lexsegment ideal**.

Corollary. *Let $I \subset S$ be a lexsegment ideal, and set $R = S/I$. Then*

$$H_R(d+1) \leq H_R(d)^{\langle d \rangle}$$

for all d. Equalitiy holds for a given d if and only if $I_{d+1} = (x_1, \ldots, x_n)I_d$. $\square$

The final result in this section is the full statement of Macaulay's Thereom:

Theorem (Macaulay). *Let k be a field, and let $h : \mathbb{N} \to \mathbb{N}$ be a numerical function. The following conditions are equivalent:*

(i) there exists a homogeneous k-algebra with Hilbert function identically equal to h;
(ii) there exists a homogeneous k-algebra with monomial relations and with Hilbert function identically equal to h;
(iii) $h(0) = 1$, and $h(d+1) \leq h(d)^{\langle d \rangle}$ for all $d \geq 1$;
(iv) $\mathcal{M}$ is an order ideal of monomials, where $\mathcal{M}$ is defined as follows: let $n = h(1)$, and for each $d \geq 0$ let $\mathcal{M}_d$ be the revlex first $h(d)$ monomials of degree d in the variables $x_1, \ldots, x_n$. Set $\mathcal{M} = \bigcup_{d \geq 0} \mathcal{M}_d$.

Proof. Equivalence of (i) and (ii) was the statement of an earlier theorem; (ii) implies (iv) is trivial. To prove (iii) implies (iv), define $\mathcal{M}$ as in (iv). For an element $u \in \mathcal{M}$, to check if its factors are also in $\mathcal{M}$, we can divide by one factor x_i at a time. The proof then reduces to showing if $u \in \mathcal{M}_{d+1}$, then for all $i = 1, \ldots, n$, $x_i^{-1}u \in \mathcal{M}_d$. Also, condition (iv) holds trivially if u is of degree 1, so we can assert $d \geq 1$.

First of all, observe, if $a \leq b$ are positive integers, then

$$\begin{aligned}\binom{b}{a} = \frac{b!}{(b-a)!a!} &\leq \frac{b!(b+1)}{(b-a)!a!(a+1)} \\ &\leq \frac{(b+1)!}{(b+1-(a+1))!(a+1)!} \\ &= \binom{b+1}{a+1}.\end{aligned}$$

Now apply this observation to condition (iii): for $d \geq 1$,

$$\begin{aligned}h(d+1) &\leq h(d)^{\langle d \rangle} \\ &\leq \left(h(d-1)^{\langle d-1 \rangle}\right)^{\langle d \rangle} \\ &\vdots \\ &\leq \left(\left(\cdots\left(h(1)^{\langle 1 \rangle}\right)^{\langle 2 \rangle \cdots}\right)^{\langle d-1 \rangle}\right)^{\langle d \rangle},\end{aligned}$$

and since $h(1) = n$ by definition, the inequality becomes

$$h(d+1) \leq \left(\left(\cdots\left(n^{\langle 1\rangle}\right)^{\langle 2\rangle\cdots}\right)^{\langle d-1\rangle}\right)^{\langle d\rangle}$$

$$\vdots$$

$$\leq \binom{n+d}{d+1}.$$

First consider the case where $h(d+1) < \binom{n+d}{d+1}$. The set $\mathcal{M}_{d+1}$ is finite and totally (revlex) ordered, so has a minimal element u_{d+1}. Then $\mathcal{M}_{d+1} = \mathcal{L}_{u_{d+1}}$. Similarly, write $\mathcal{M}_d = \mathcal{L}_{u_d}$. A previous proposition says $|\mathcal{L}_{x_1 u_d}| = |\mathcal{L}_{u_d}|^{\langle d\rangle}$. Along with the assumption of condition (iii), this gives the inequality

$$|\mathcal{L}_{u_{d+1}}| = |\mathcal{M}_{d+1}| = h(d+1) < h(d)^{\langle d\rangle} = |\mathcal{L}_{u_d}|^{\langle d\rangle} = |\mathcal{L}_{x_1 u_d}|.$$

Considering the revlex ordering, and the definition of $\mathcal{M}$, if a monomial $u \in \mathcal{M}_{d+1}$, then the above inequality implies

$$u \geq_{\text{rev}} u_{d+1} >_{\text{rev}} x_1 u_d >_{\text{rev}} \cdots >_{\text{rev}} x_n u_d$$

and hence

$$x_i^{-1} u >_{\text{rev}} u_d.$$

In other words, any degree d factor of u is rev larger than u_d, so must be in $\mathcal{L}_{u_d} = \mathcal{M}_d \subseteq \mathcal{M}$.

On the other hand, suppose $h(d+1) = \binom{n+d}{d+1}$, the maximum possible value. Then by (iii),

$$h(d+1) = \binom{n+d}{d+1} = h(d)^{\langle d\rangle} = \binom{n+d-1}{d}^{\langle d\rangle},$$

which implies $h(d)$ also equals its maximum possible value, $\binom{n+d-1}{d}$. By induction, the same will be true for all degrees smaller than d. Yet for a given degree d, the number of degree d monomials in n variables is exactly $\binom{n+d-1}{d}$. To see this, for a typical monomial expressed as $x_1^{d_1} \cdots x_n^{d_n}$ with $d_1 + \cdots + d_n = d$, write out each factor separately, along with a “1” each time the index increases by one. The resulting expression will have d factors consisting of the variables $x_1, \ldots, x_n$, and $n-1$ instances of the symbol “1”, a total of $d+n-1$ factors[3]. Thus enumerating the degree d monomials in n variables is the same as choosing which d of the $d+n-1$ factors are variables. The number of monomials is then $\binom{n+d-1}{d}$. Because of this, the sets $\mathcal{M}_i$, for $i = 1, \ldots, d+1$, each consist of all the possible degree i monomials. Hence if $u \in \mathcal{M}_{d+1}$ then any of its factors are also in $\mathcal{M} = \bigcup_i \mathcal{M}_i$, completing the proof that (iii) implies (iv).

[3] *Example.* Say $n = 4$, $d = 5$. Consider the monomial $x_1^2 x_3 x_4^2$. Rewriting as described above gives the expression $x_1 \cdot x_1 \cdot 1 \cdot 1 \cdot x_3 \cdot 1 \cdot x_4 \cdot x_4$.

The last and most difficult implication, (i) implies (iii), relies on a theorem by M. Green. The statement of the theorem first needs some terminology. For a positive integer a with Macaulay coefficients $k(d), \dots, k(1)$ define

$$a_{\langle d\rangle} = \binom{k(d)-1}{d} + \binom{k(d-1)-1}{d-1} + \cdots + \binom{k(1)-1}{1}.$$

In his proof Green uses the notion of a **general linear form** of the homogeneous k-algebra R. When k is infinite the affine k-space $\mathbb{V}(R_1)$ cannot be (nontrivially) partitioned into finitely many closed sets, so is irreducible. With every Zariski open set dense on $\mathbb{V}(R_1)$, a property holds on a general linear form means it holds on some open set in $\mathbb{V}(R_1)$.

Theorem (Green). *Let R be a homogeneous k-algebra, k an infinite field, and let $d \geq 1$ be an integer. Then*

$$H_{R/hR}(d) \leq H_R(d)_{\langle d\rangle}$$

for a general linear form h.

Proof. See [BrH]. □

A graded k-algebra is just a vector space, so tensoring with an infinite field extension of k will not affect the Hilbert function. Thus we can assert k is infinite and use Green's theorem to prove the rest of Macaulay's Theorem. For any linear form h in R, the exact sequence

$$0 \to hR_d \to R_{d+1} \to (R/hR)_{d+1} \to 0$$

yields the inequality

$$\begin{aligned} H_R(d+1) =& H_{hR}(d) + H_{R/hR}(d+1) \\ \leq& H_R(d) + H_{R/hR}(d+1). \end{aligned}$$

Therefore when h is a general linear form Green's Theorem (with d replaced by $d+1$) gives

$$H_R(d+1) \leq H_R(d) + H_R(d+1)_{\langle d+1\rangle}.$$

Let $k(d+1), \dots, k(1)$ denote the (d+1)st Macaulay coefficients for $H_R(d+1)$. Supposing (i) holds and $d \geq 1$, the inequality rearranged becomes

$$\begin{aligned} h(d) = H_R(d) &\geq H_R(d+1) - H_R(d+1)_{\langle d+1\rangle} \\ &= \left[\binom{k(d+1)}{d+1} + \binom{k(d)}{d} + \cdots + \binom{k(1)}{1}\right] \\ &\quad - \left[\binom{k(d+1)-1}{d+1} + \binom{k(d)-1}{d} + \cdots + \binom{k(1)-1}{1}\right] \\ &= \frac{d+1}{k(d+1)}\binom{k(d+1)}{d+1} + \frac{d}{k(d)}\binom{k(d)}{d} + \cdots + \frac{1}{k(1)}\binom{k(1)}{1} \\ &= \binom{k(d+1)-1}{d} + \binom{k(d)-1}{d-1} + \cdots + \binom{k(1)-1}{0}. \end{aligned}$$

If $k(1) = 0$, then the last term on the right hand side vanishes and upon applying ${}^{\langle d\rangle}$ to both sides, the inequality matches that in (iii).

The case where $k(1) \geq 1$ is trickier. By construction, the difference between successive Macaulay coefficients is always at least 1. Let j denote the largest index such that $k(j) - k(j-1) = 1$. By expanding this expression and rearranging terms, in particular we get $k(j) = k(1) + j - 1$. Now

$$\begin{aligned} H_R(d) &\geq \binom{k(d+1)-1}{d} + \cdots + \binom{k(j+1)-1}{j} + \sum_{i=0}^{j-1} \binom{k(i+1)-1}{i} \\ &= \binom{k(d+1)-1}{d} + \cdots + \binom{k(j+1)-1}{j} + \sum_{i=0}^{j-1} \binom{k(1)+i-1}{i}, \end{aligned}$$

where we can rewrite the binomials in the summation because of the choice of j.

Exercise. (Hint: Use induction.) For an nonnegative integer a,

$$\sum_{r=0}^{s} \binom{a+r-1}{r} = \binom{a+s}{s}.$$

Applying the exercise to the summation,

$$\begin{aligned} H_R(d) &\geq \binom{k(d+1)-1}{d} + \cdots + \binom{k(j+1)-1}{j} + \binom{k(1)+(j-1)}{j-1} \\ &= \binom{k(d+1)-1}{d} + \cdots + \binom{k(j+1)-1}{j} + \binom{k(j)}{j-1}. \end{aligned}$$

The choice of j ensures $k(j+1) - 1 > k(j)$, so the right hand side is a Macaulay expansion for some number. Thus,

$$\begin{aligned} H_R(d)^{\langle d\rangle} &\geq \binom{k(d+1)}{d+1} + \cdots + \binom{k(j+1)}{j+1} + \binom{k(j)+1}{j} \\ &= \binom{k(d+1)}{d+1} + \cdots + \binom{k(j+1)}{j+1} + \binom{k(1)+(j-1)+1}{j} \\ &= \binom{k(d+1)}{d+1} + \cdots + \binom{k(j+1)}{j+1} + \sum_{i=0}^{j} \binom{k(1)+i-1}{i}. \end{aligned}$$

Gathering terms, we get

$$\begin{aligned} h(d)^{\langle d\rangle} = H_R(d)^{\langle d\rangle} &\geq \binom{k(d+1)}{d+1} + \cdots + \binom{k(1)}{1} + \binom{k(1)-1}{0} \\ &= H_R(d+1) + 1 \\ &> H_R(d+1) = h(d+1), \end{aligned}$$

which is the conclusion of (iii). □

Cooper-Roberts Generalization

In order to make generalizations on the last section, Cooper-Roberts [CoRo] use classical (1969) combinatorial results of Clements-Lindström [ClLi]. Note the last section relied heavily on enumerating monomials. Clements-Lindström work strictly in terms of ordered n-tuples, but the bijection between monomials will be clear.

For a set of integers $1 \leq b_1 \leq b_2 \leq \cdots \leq b_n$, let F denote the set of all n-tuples whose ith component is at most b_i $(i = 1, \ldots, n)$. Then, for $d \leq b_1 + \cdots + b_n$ let F_d denote the subset of F whose elements' components add up to exactly d. Finally, for any subset $H \subset F$, let $H_d = H \cap F_d$. In terms of monomials, think of F as the set of monomials in n variables $x_1, \ldots, x_n$, such that the exponent on x_i is at most b_i $(i = 1, \ldots, n)$. F_d corresponds to the degree d monomials in F, and more generally, H_d corresponds to the degree d monomials in the subset $H \subset F$. Therefore, the results and notation in this section apply to n-tuples or monomials, depending on the context.

Definition. *Let $H \subseteq F$ and $d \leq b_1 + \cdots + b_n$.*

(1) The set of the lex first $|H_d|$ elements of F_d is called the **compression** *of H_d, denoted CH_d. The* **compression** *of H is denoted $CH = \bigcup_d CH_d$.*

(2) The set of the last $|H_d|$ elements of F_d is denoted by LH_d and called the **last compression** *of H_d.*

(3) Define the **closure** *of H by the function*

$$\Gamma(H) = \{(a_1 - 1, a_2, \ldots, a_n), (a_1, a_2 - 1, a_3, \ldots, a_n), \ldots, \\ (a_1, \ldots, a_{n-1}, a_n - 1) \mid (a_1, \ldots, a_n) \in H\}.$$

H is **closed** *means $\Gamma(H) \subset H$.*

A few more definitions are required to state the results, though Clements-Lindström do not name these terms: Define the multivalued function P by

$$P(H) = \{(a_1 + 1, a_2, \ldots, a_n), (a_1, a_2 + 1, a_3, \ldots, a_n), \ldots, \\ (a_1, \ldots, a_{n-1}, a_n + 1) \in H \mid (a_1, \ldots, a_n) \in H\}.$$

For an n-tuple $(a_1, \ldots, a_n)$, put

$$\alpha(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i$$
$$\alpha(H) = \sum_{a \in H} \alpha(a).$$

Finally, let S_r denote the set of the r first n-tuples of F.

Theorem (Clements-Lindström). *For* $H_d \subset F_d$*,*

$$\Gamma(CH_d) \subset C(\Gamma H_d).$$

Proof. See [ClLi]. □

Corollary (CL1). $P(LH_d) \subset L(PH_d)$*.* □

Corollary (CL2). *For* $H \subset F$*,* H *is closed implies* CH *is closed.* □

Corollary (CL3). *If* $|H| = r$ *and* H *is closed, then* $\alpha(H) \leq \alpha(S_r)$*.* □

In the context of monomials, the theorem is saying for a set H_d of degree d monomials in $H \subseteq F$, the degree $d-1$ factors of the lex first $|H_d|$ monomials are contained in the compression of the lex first $|\Gamma H_d|$ monomials, where ΓH_d is the set of the degree $d-1$ factors of all the elements of H_d. CL1 is a dual statement while CL2 follows immediately from the theorem. CL3 says if a set $H \subseteq F$ is closed with cardinality r, then the total degree of its elements is smaller than the total degree of the first r elements of F.

Cooper-Roberts (2008) use these results to generalize Macaulay's theorem to the **truncated polynomial ring**

$$S_e = k[x_1, \dots, x_n]/(x_1^{e_1+1}, \dots, x_n^{e_n+1}),$$

where $e_1 \geq \cdots \geq e_n \geq 1$. Notice S_e is not a homogeneous k-algebra, so the notions from the section on Macaulay's Theorem need modification. Again, "monomial" always means monic monomial.

Definition.

(1) An **order ideal of monomials** *is a nonempty set* $\mathcal{M}_e$ *of monomials in* S_e*, such that if* $u \in \mathcal{M}_e$ *and a monomial* $u' \in S_e$ *divides* u*, then* $u' \in \mathcal{M}_e$*.*
(2) A **revlexsegment order ideal** *is an order ideal of monomials* $\mathcal{M}_e$*, such that for each degree* d*, if* $u <_{\text{rev}} u'$ *are of degree* d*, and* $u \in \mathcal{M}_e$*, then* $u' \in \mathcal{M}_e$*.*

The following is a straightforward generalization:

Theorem. *Let* J *be a homogeneous ideal in* S_e*. Then there is an order ideal* $\mathcal{M}_e$ *whose canonical image in* $R_e = S_e/J$ *forms a* k*-basis of* R_e*.*

Proof. See [CoRo]. □

Cooper-Roberts apply the Clements-Lindström results with the set $1 \leq b_1 \leq \cdots \leq b_n$ replaced by the truncated polynomial integers $e_1 \geq \cdots \geq e_n \geq 1$, and with every instance of the lex order replaced by the revlex order.

Exercise. For an n-tuple $a \in \mathbb{N}^n$ let a^r denote the n-tuple whose coordinates are those of a, but in reverse order. Then $a >_{\text{rev}} b$ if and only if $a^r <_{\text{hlex}} b^r$.

Theorem. *Let R_e denote a quotient of S_e by a homogeneous ideal, let $\mathcal{S}$ denote the revlexsegment consisting of the $|H_{R_e}(d)|$ (rev) largest degree d monomials in S_e. Then $H_{R_e}(d-1)$ is at least equal to the number of degree $d-1$ factors of elements in $\mathcal{S}$.*

Proof. The theorem above says there is an order ideal $\mathcal{M}_e$ which forms a k-basis for R_e. Thus $\mathcal{S}$ is the compression of $(\mathcal{M}_e)_d$, the degree d subset of $\mathcal{M}_e$. Clements-Lindström says $\Gamma(C(\mathcal{M}_e)_d) \subseteq C(\Gamma(\mathcal{M}_e)_d)$, so

$$\begin{aligned} H_{R_e}(d-1) = |(M_e)_{d-1}| \geq |\Gamma((\mathcal{M}_e)_d)| &= |C(\Gamma((\mathcal{M}_e)_d)| \\ &\geq |\Gamma(\mathcal{S})|, \end{aligned}$$

which is exactly the number of degree $d-1$ factors of elements of $\mathcal{S}$. □

Theorem. *For R_e, a quotient of S_e by a homogeneous ideal, let $\mathcal{S}$ denote the revlexsegment consisting of the $|H_{R_e}(d)|$ (rev) largest degree d monomials in S_e ($d = 0, 1, \dots$). Then $H_{R_e}(d+1)$ is at most the number of degree $d+1$ monomials whose degree d factors all lie in $\mathcal{S}$.*

Proof. In this proof, all monomials are in S_e. Again, there exists an order ideal $\mathcal{M}_e$ which forms a k-basis for R_e. Let N_d denote the set of monomials not contained in $(\mathcal{M}_e)_d$. The set of degree $d+1$ multiples of monomials not in $\mathcal{S}$ is exactly $P(LN_d)$. CL2 says $P(LN_d) \subseteq L(P(N_d))$. Note by definition $L(P(N_d))$ has the same cardinality as $P(N_d) \subseteq N_{d+1}$. Therefore, by taking compliments in the set of monomials in S_e,

$$|(\mathcal{M}_e)_{d+1}| = H_{R_e}(d+1),$$

but $|(\mathcal{M}_e)_{d+1}|$ is at most the number of degree $d+1$ monomials whose degree d factors are all in $\mathcal{S}$. □

The Lex Plus Powers Conjecture

The EGH conjecture is stated as a weaker version of the lex plus powers conjecture, which, if true, will also generalize Macaulay's Theorem.

Definition. *An* **$\mathbb{A}$-regular sequence** *is a regular sequence $f_1, \dots, f_n$ satisfying* $\deg f_i = a_i$ $(i = 1, \dots, n)$.

Macaulay's Theorem says a quotient of a lex ideal has a bounded Hilbert function. A natural generalization would be that the same is true for a lex plus powers ideal. In fact, this is the following conjecture and it is equivalent to the EGH conjecture [Rich].

Conjecture (LPPH, or Lex Plus Powers Conjecture for Hilbert Functions). *Let $R = S/I$ where I contains an $\mathbb{A}$-regular sequence. Suppose there exists an $\mathbb{A}$-LPP ideal L such that $H_R(d) = H_{S/L}(d)$. Then*

$$H_R(d+1) \leq H_{S/(L_d + (a_1, \dots, a_n)S)}(d+1),$$

where L_d denotes the degree d part of L.

Definition. *An ideal I is an* **almost complete intersection** *means the number of generators for I is one more than the height of I.*

In 2003 Francisco proved the EGH Conjecture holds for almost complete intersections by proving a stronger form of the lex plus powers conjecture in that case:

Conjecture. *Let $L \subset S$ be an $\mathbb{A}$-LPP ideal. Suppose $I \subset S$ is a homogeneous ideal with the same Hilbert function that contains an $\mathbb{A}$-regular sequence. Then the graded Betti numbers for S/I are bounded above by those for S/L, in the sense that for all i, j, $\beta_{i,j}^{S/I} \leq \beta_{i,j}^{S/L}$.*

Later that year Richert proved both conjectures for $n = 2$. In 2008 Mermin-Peeva-Stillman proved the lex plus powers conjecture for ideals containing the squares of the variables and Mermin-Murai proved it for ideals containing a monomial regular sequence.

Conclusion

Not much else is known about the EGH conjecture. An interesting result was proved in 2006 by Caviglia-Maclagan: the EGH conjecture is true if $a_j > \sum_{i=1}^{j-1}(a_i - 1)$ for $j = 1, \ldots, n$. In 2008 Cooper then proved the "outstanding cases" when $n = 3$, namely the "tight degrees" $a_1 \leq a_2 \leq a_3 \leq a_1 + a_2 - 2$.

Bibliography

[BrH] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics 39, Cambridge University Press, Cambridge, 1993.

[ClLi] G.F. Clements and B. Lindström, *A Generalization of a Combinatorial Theorem of Macaualay*, Journal of Combinatorial Theory **7** (1969), 230-238.

[Coop] S.M. Cooper, *The Eisenbud-Green-Harris Conjecture for Ideals of Points*, ??? (2008).

[CoRo] S.M. Cooper and L.G. Roberts, *Algebraic Interpretation of a Theorem of Clements and Lindström*, Journal of Commutative Algebra ??? (2008).

[EGH1] D. Eisenbud, M. Green, J. Harris, *Higher Castelnuovo Theory*, Astérisque **218** (1993), 187-202.

[EGH2] D. Eisenbud, M. Green, J. Harris, *Cayley-Bacharach Theorems and Conjectures*, Bulletin of the American Mathematical Society **33(3)** (1996), 295-324.

[Rich] B.P. Richert, *A Study of the Lex Plus Powers Conjecture*, Journal of Pure and Applied Algebra **186(2)** (2004), 169-183.

[Stan] R.P. Stanley, *Hilbert Functions of Graded Algebras*, Advances in Mathematics **28** (1978), 57-83.