Wed 30 Mar

- Exam 3: next Friday. Covers §3.10-4.6
- Algebra Seminar: today at 3p in SCEN 322.
  "The talk will be given by our own Ashley Wheeler on the Title: Local cohomology of Stanley-Reisner rings." (from the department email).

§4.4 Optimization Problems

In many scenarios, it is important to find a maximum or minimum value under given constraints. Given our use of derivatives from the previous sections, optimization problems follow directly from what we have studied.

Question

Given all nonnegative real numbers x and y between 0 and 50 such that their sum is 50 (i.e., $x + y = 50$), which pair has the maximum product?

This is a basic optimization problem. In this problem, we are given a **constraint** $(x + y = 50)$ and asked to maximize an **objective function** $(A = xy)$.

The first step is to express the objective function $A = xy$ in terms of a single variable by using the constraint:

$$y = 50 - x \implies A(x) = x(50 - x).$$

To maximize A, we find the critical points:

$$A'(x) = 50 - 2x \text{ which has a critical point at } x = 25.$$

Since $A(x)$ has domain $[0, 50]$, to maximize A we evaluate A at the endpoints of the domain and at the critical point:

$$A(0) = A(50) = 0 \text{ and } A(25) = 625.$$

So 625 is the maximum value of A and A is maximized when $x = 25$ (which means $y = 25$).

Essential Feature of Optimization Problems

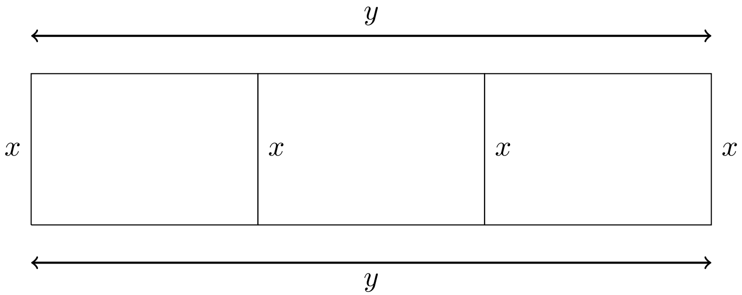
All optimization problems take the following form:

What is the maximum (or minimum) value of an objective function subject to the given constraint(s)?

Most optimization problems have the same basic structure as the previous problem: An objective function (possibly with several variables and/or constraints) with methods of calculus used to find the maximum/minimum values.

Exercise

Suppose you wish to build a rectangular pen with two interior parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen?



By the picture, $2y + 4x = 500$ which implies $y = -2x + 250$. We are maximizing $A = xy$. So write

$$A(x) = x(-2x + 250) = -2x^2 + 250x.$$

Taking the derivative, $A'(x) = -4x + 250 = 0$, A has a critical point at $x = 62.5$.

From the picture, since we have 500 ft of fencing available we must have $0 \leq x \leq 125$. To find the max we must examine the points $x = 0, 62.5, 125$:

$$A(0) = A(125) = 0 \text{ and } A(62.5) = 7812.5$$

We see that

$$\boxed{\text{the maximum area is } 7812.5 \text{ ft}^2.}$$

The pen's dimensions (answer the question!) are $\boxed{x = 62.5 \text{ ft}}$ and

$$\boxed{y = -2(62.5) + 250 = 125 \text{ ft.}}$$

Guidelines for Optimization Problems

1. READ THE PROBLEM carefully, identify the variables, and organize the given information with a picture.
2. Identify the objective function (i.e., the function to be optimized). Write it in terms of the variables of the problem.
3. Identify the constraint(s). Write them in terms of the variables of the problem.
4. Use the constraint(s) to eliminate all but one independent variable of the objective function.
5. With the objective function expressed in terms of a single variable, find the interval of interest for that variable.
6. Use methods of calculus to find the absolute maximum or minimum value of the objective function on the interval of interest. If necessary, check the endpoints.

Question

The sum of a pair of positive real numbers that have a product of 9 is

$$S(x) = x + \frac{9}{x},$$

where x is one of the numbers. This sum $S(x)$ has a minimum when:

A. $x = 9$

B. $x = 3$

C. $x = 6$

D. none of the above

Exercise

An open rectangular box with square base is to be made from 48 ft^2 of material. What dimensions will result in a box with the largest possible volume?

Exercise

Find the dimensions of the rectangle of largest area which can be inscribed in the closed region bounded by the x-axis, y-axis, and the graph of $y = 8 - x^3$.

4.4 Book Problems

5-16, 19-20, 24, 26