MATH 2554 (Calculus I)

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January 30, 2015

Table of Contents

Monday 26 January (Week 3)

- MAA quiz results
  - posted on MLP soon
  - originally out of 25
  - grading: your raw score taken out of 15, then scaled to out of 10
  - If your raw score was 15/25 or higher then you got 10/10.
- Quizzes: Hold on to your old quizzes, for studying, computing your grade, etc.
- Thurs 29 Jan Quiz 3
- 3rd WebHW is live.
- EXAM #1: Friday 6 February
  - in class
  - covers up to and including $\oint 3.1$

(recall, from $\oint 2.5$ Limits at Infinity)

Rational Functions: Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function.

1. If $\deg(p) < \deg(q)$, i.e., the numerator has the smaller degree, then
$$\lim_{x\to\pm\infty} f(x) = 0$$
(also, $y = 0$ is a horizontal asymptote of f).

2. If $\deg(p) = \deg(q)$, i.e., numerator and denominator have the same degree, then
$$\lim_{x\to\pm\infty} f(x) = \frac{\mathrm{lc}(p)}{\mathrm{lc}(q)},$$
and $y = \dfrac{\mathrm{lc}(p)}{\mathrm{lc}(q)}$ is a horizontal asymptote of f.

(recall, from $\oint 2.5$)

3. If $\deg(p) > \deg(q)$, (numerator has the bigger degree) then

$$\lim_{x \to \pm\infty} f(x) = \infty \quad \text{or} \quad -\infty$$

and f has no horizontal asymptote.

4. Assuming that $f(x)$ is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeroes of q.

(This is why it is a good idea to check for factoring and cancelling first!)

(recall, from $\oint 2.5$)

Pneumonic for limits at infinity for rational functions:

BOB0

Bigger On Bottom 0

BOTN

Bigger On Top Neither

BETC (Betsy)

Bottom Equals Top Coefficient

(recall, from $\oint 2.5$)

Algebraic and Transcendental Functions:
Determine the end behavior of the following functions.

- $f(x) = \dfrac{4x^3}{2x^3 + \sqrt{9x^6 + 15x^4}}$ (radical signs appear)
- $g(x) = \cos x$ (trig)
- $h(x) = e^x$ (exponential)

HW from Section 2.5

Do problems 9–10, 13–35 odds, 39, 43, 45, 53 (pp. 92–93 in textbook)

∮ 2.6 Continuity

Informal Def.: A function f is continuous at $x = a$ means near $x = a$ the graph can be drawn without lifting a pencil. In other words, no holes, breaks, asymptotes, etc.

Formal Def.: A function f is continuous at a means

$$\lim_{x \to a} f(x) = f(a).$$

If f is not continuous at a, then a is a point of discontinuity.

Continuity Checklist

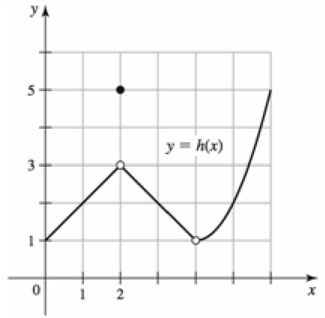
In order to claim something is continuous, you must verify all three:

1. $f(a)$ is defined (i.e., a is in the domain of f – no holes, asymptotes).

2. $\lim_{x \to a} f(x)$ exists. You must check both sides and make sure they equal the same number.

3. $\lim_{x \to a} f(x) = f(a)$ (i.e., the value of f equals the limit of f at a). What is an example of a function that satisfies this condition?

Where are the points of discontinuity of the function below? Which aspects of the checklist fail?



Checklist:

1. function is defined
2. the two-sided limit exists
3. 2. = 1.

Continuity Rules: If f and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\frac{f}{g}$, provided $g(a) \neq 0$
6. $[f(x)]^n$

From the rules above, we can deduce:

1. Polynomials are continuous for all $x = a$.
2. Rational functions are continuous at all $x = a$ except for the points where the denominator is zero.
3. If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

Wednesday 28 January (Week 3)

- Quizzes:
  - MAA quiz results ...
  - Double check the solutions for Quiz 1 – corrections are posted.
  - Hold on to your old quizzes, for studying, computing your grade, etc.
  - See me (OH) if you have questions about quizzes or computing your grade.
- Thurs 29 Jan Quiz 3
- EXAM #1: Friday 6 February
  - in class
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- See the course schedule: Monday is $\oint 3.1$ but if possible we will start it on Friday. Wednesday is review for the exam.

($\oint$ 2.6)

Continuity on an Interval: Consider the cases where f is not defined past a certain point. (e.g., $\ln x$)

A function f is continuous from the left (or left-continuous) at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

A function f is continuous from the right (or right-continuous) at a if

$$\lim_{x \to a^+} f(x) = f(a).$$

$(\oint 2.6)$

A function f is continuous on an interval I means it is continuous at all points of I.

Notation: Intervals are usually written

$$[a, b]\,,\ (a, b]\,,\ [a, b)\,,\ \text{or}\ (a, b)\,.$$

When I contains its endpoints, "continuity on I" means continuous from the right or left at the endpoints.

Example

Let $f(x) = \begin{cases} x^3 + 4x + 1 & \text{if } x \leq 0 \\ 2x^3 & \text{if } x > 0. \end{cases}$

1. Use the continuity checklist to show that f is not continuous at 0.
2. Is f continuous from the left or right at 0?
3. State the interval(s) of continuity.

Continuity of Functions with Roots

(assuming m and n are positive integers and n/m is in lowest terms)

- If m is odd, then $[f(x)]^{n/m}$ is continuous at all points at which f is continuous.
- If m is even, then $[f(x)]^{n/m}$ is continuous at all points a at which f is continuous and $f(a) \geq 0$.

Question

Where is $f(x) = \sqrt[4]{4 - x^2}$ continuous?

Continuity of Transcendental Functions

Trig Functions: The basic trig functions are all continuous at all points IN THEIR DOMAIN. Note there are points of discontinuity where the functions are not defined – for example, $\tan x$ has asymptotes at multiples of π.

Exponential Functions: The exponential functions b^x and e^x are continuous on all points of their domains.

Inverse Functions: If a continuous function f has an inverse on an interval I (meaning if $x \in I$ then $f^{-1}(y)$ passes the vertical line test), then its inverse f^{-1} is continuous on the interval J, which is defined as all the numbers $f(x)$, given x is in I.

Intermediate Value Theorem

Theorem (Intermediate Value Theorem)

Suppose f is continuous on the interval $[a, b]$ and L is a number satisfying

$$f(a) < L < f(b) \quad \text{or} \quad f(b) < L < f(a).$$

Then there is at least one number $c \in (a, b)$, i.e., $a < c < b$, satisfying

$$f(c) = L.$$

Example: Let $f(x) = -x^5 - 4x^2 + 2\sqrt{x} + 5$. Use IVT to show that $f(x) = 0$ has a solution in the interval $(0, 3)$.

HW from Section 2.6

Do problems 9–23 odds, 29–37 odds, 45, 49, 51, 53 (pp. 103–105 in textbook)

Friday 30 January (Week 3)

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  - in class
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- See the course schedule: Monday is $\oint 3.1$ but if possible we will start it today. Wednesday is review for the exam.

∮ 2.7 Precise Definitions of Limits

Assume that $f(x)$ exists for all x in some open interval (open means: neither of the endpoints not included) containing a, except possibly at a.

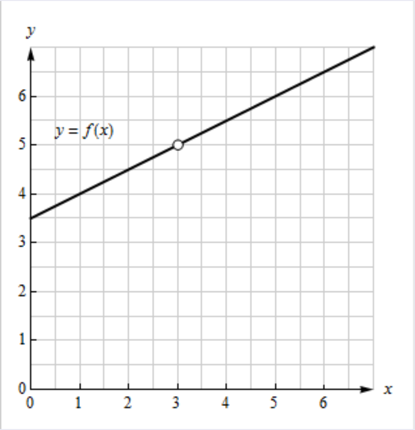
"The limit of $f(x)$ as x approaches a is L", i.e.,

$$\lim_{x \to a} f(x) = L,$$

means for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Seeing ϵs and δs on a Graph



Problem:

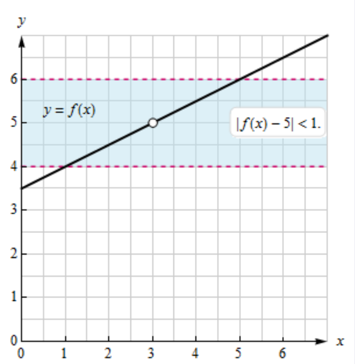
Using the graph, for each $\epsilon > 0$, determine a value of $\delta > 0$ to satisfy the statement

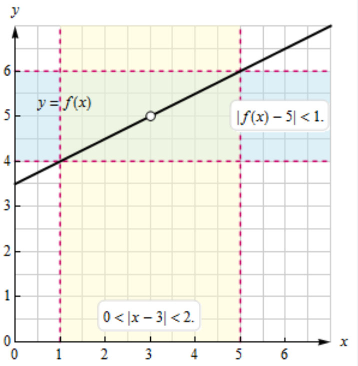
$$|f(x) - 5| < \epsilon \quad \text{whenever} \quad 0 < |x - 3| < \delta.$$

- $\epsilon = 1$
- $\epsilon = 0.5.$

Seeing ϵs and δs on a Graph

When $\epsilon = 1$:

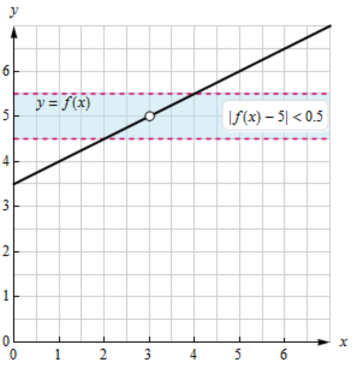


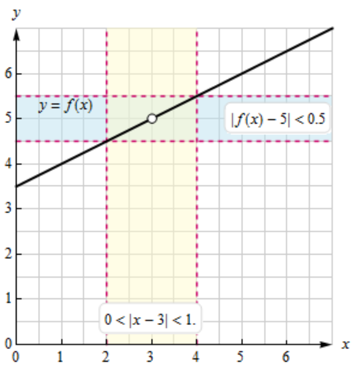


$\ldots\ \delta = 2$

Seeing ϵs and δs on a Graph

When $\epsilon = 0.5$:



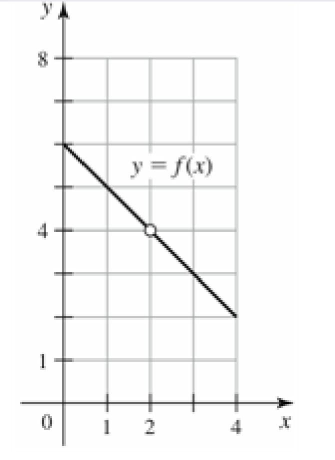


$\ldots \delta = 1$

The ϵs and δs give a way to visualize computing the limit, and proving it exists. As the ϵs get smaller and smaller, we want there to always be a δ.

In this example,

$$\lim_{x \to 3} f(x) = 5.$$



Exercise

Using the graph, for each $\epsilon > 0$, determine a value of $\delta > 0$ to satisfy the statement

$$|f(x) - 4| < \epsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

- $\epsilon = 1$
- $\epsilon = 0.5.$

Exercise: Let $f(x) = x^2 - 4$. For $\epsilon = 1$, find a value for $\delta > 0$ so that

$$|f(x) - 12| < \epsilon \quad \text{whenever} \quad 0 < |x - 4| < \delta.$$

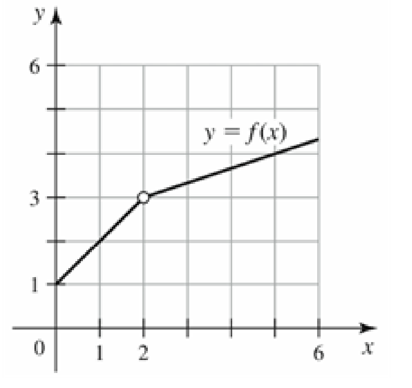
In this example, $\lim\limits_{x \to 4} f(x) = 12$.

Finding a Symmetric Interval

Question

When finding an interval $(a-\delta, a+\delta)$ around the point a, what happens if you compute two different δs?

Answer: To obtain a symmetric interval around a, use the smaller of the two δs as your distance around a.



Exercise

The graph below shows $f(x)$ with

$$\lim_{x\to 2} f(x) = 3.$$

For $\epsilon = 1$, find the corresponding value of $\delta > 0$ so that

$$|f(x) - 3| < \epsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

HW from Section 2.7

Do problems 1–7, 9–18 (pp. 115–116)

∮ 3.1 Introducing the Derivative

Recall from Ch 2: We said that the slope of the tangent line at a point is the limit of the slopes of the secant lines as the points get closer and closer.

- slope of secant line: $\dfrac{f(x) - f(a)}{x - a}$ (avg. rate of change)
- slope of tangent line: $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ (instantaneous rate of change)

Use the relationship between secant lines and tangent lines, specifically the slope of the tangent line, to find the equation of a line tangent to the curve $f(x) = x^2 + 2x + 2$ at the point $P = (1, 5)$.