Ideals Generated by Principal Minors

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The motivation for my thesis topic comes from analogous questions about various ideals defined using generic matrices. By a generic matrix X, we mean we let

$$X = X_{r\times s} = \begin{pmatrix} x_{11} & \cdots & x_{1s} \\ \vdots & \ddots & \vdots \\ x_{r1} & \cdots & x_{rs} \end{pmatrix}$$

denote an $r \times s$ matrix of indeterminates. Throughout the talk, K shall always denote an arbitrary algebraically closed field. Let $K[X]$ denote the polynomial ring over K in the entries (x_{ij}). Imposing conditions on the matrix X yields ideals in $K[X]$.

Historical Examples

All of these examples have origins in Invariant Theory.

1. Determinantal Ideals

$$\mathrm{I}_t = \mathrm{I}_t(X) = \text{ideal in } K[X] \text{ generated by size } t \text{ minors of } X$$

• (Eagon+Hochster 1971) are normal and have height $(r-t+1)(s-t+1)$.
• (Hochster+Roberts 1974) are Cohen-Macaulay.
• (Svanes 1974) are Gorenstein if and only if $r = s$.

2. Pfaffian Ideals – We require the matrix X to be square, say $n \times n$, and alternating (skew symmetric with zeros on the main diagonal).

$$\mathrm{Pf}_t = \mathrm{Pf}_t(X) = \text{ideal in } K[X] \text{ generated by square roots of size } t \text{ principal minors of } X$$

A minor is **principal** means it is symmetric about the main diagonal. It is not hard to see Pfaffian ideals are the zero ideal when t is odd.

• (Room 1938) $\mathrm{ht}(\mathrm{Pf}_{2h}) = \binom{n-2h+2}{2}$
• (Kleppe+Laksov 1980) Pfaffian ideals are normal and Gorenstein.
• (Buchsbaum+Eisenbud 1977) *Structure Theorem for height 3 Gorenstein ideals in a regular local ring.* The result was partially motivated by Serre's 1960 result that every Gorenstein ideal of codimension 2 is a complete intersection.

Theorem [BuEi]**.** *An ideal of codimension 3 in a regular local ring is Gorenstein if and only if it is the ideal of $(n-1)$th order Pfaffians of some $n \times n$ alternating matrix of rank $n-1$.*

3. Grassmannians – a.k.a **Grassmann varieties**; $\mathrm{Grass}(r,n) \hookrightarrow \mathbb{P}^{\binom{n}{r}-1}$ is the projective subvariety parametrized by the r-dimensional vector subspaces of K^n.

$$K[\wedge^r X] = \text{homogeneous coordinate ring for } \mathrm{Grass}(r,n) \text{ under Plücker embedding}$$

The matrix $\wedge^r X$ is the $\binom{n}{r} \times \binom{n}{r}$ matrix of size r-minors of the $n \times n$ generic matrix X (where the basis is induced by the standard basis on K^n). Its entries are called Plücker coordinates.

Principal Minors

From now on X shall denote a size n square matrix. For fixed $t \leq n$, we define

$$\mathfrak{P}_t = \mathfrak{P}_t(X) = \text{ideal in } K[X] \text{ generated by size } t \text{ principal minors of } X = X_{n\times n}$$

Example. $K[X]/\mathfrak{P}_1 \cong$ polynomial ring in $n^2 - n$ variables over K.

Example. $K[X_{2\times 2}]/\mathfrak{P}_2$ is a homogeneous coordinate ring for $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ (Segre embedding).

Theorem (–). *For all n, $K[X_{n\times n}]/\mathfrak{P}_2$ is*
- *a domain,*
- *normal,*
- *toric, i.e., can be generated by monomials in the Laurent ring*

$$K[x_{ij}, x_{ij}^{-1} \mid x_{ij} \textit{ is an entry of } X], \textit{ and}$$

- *a complete intersection*
- *of codimension $\binom{n}{2}$.*

We may abuse notation and refer to an ideal and the quotient by that ideal interchangeably. For example, we may say "$\mathfrak{P}_2$ is normal" to mean $K[X]/\mathfrak{P}_2$ is normal.

Corollary (–). *$\mathfrak{P}_2$ is Gorenstein and (in the characteristic p case) strongly F-regular.*

Proof. Since $\mathfrak{P}_2$ is toric, the resulting quotient ring is a direct summand of $K[X]$. Together with normality, by a result from [HoHu], this implies weakly F-regular. Complete intersections are Gorenstein, in which case weakly F-regular is equivalent to strongly F-regular. □

The key to the proving the theorem is using the fact that the the generators for $\mathfrak{P}_2$ are binomial. Details are in the thesis.

$t > 2$ Strategy

Unfortunately, once $t > 2$ the generators for $\mathfrak{P}_t$ are no longer binomial and another strategy is needed. It turns out the irreducible components of $\mathcal{V}(\mathfrak{P}_t)$ are stratified, according to the rank of the matrices contained in them. Thus we define

$$\mathcal{Y}_{n,r,t} = \text{locally closed set of } n \times n \text{ matrices of rank } r, \text{ whose principal } t\text{-minors vanish}$$

and try to study the components of the Zariski closures of these sets. The first useful theorem that pops out is

Theorem (–). *In the localized ring $K[X]\left[\frac{1}{\det X}\right]$, the K-map*

$$X \mapsto X^{-1}$$

induces an isomorphism

$$\mathcal{Y}_{n,n,t} \cong \mathcal{Y}_{n,n,n-t}.$$

This theorem is really just a direct result of a classical determinantal identity. Before stating it, we introduce some notation. Suppose A is an invertible $n \times n$ matrix. Then we write $\operatorname{adj} A$ to denote the classical adjoint of A, i.e.,

$$\operatorname{adj} A = (\det A)A^{-1}.$$

Theorem [M]**.** *Suppose A is an invertible matrix. Let*

$$\mu = \textit{size } t \textit{ minor of } \operatorname{adj} A \textit{ indexed by}$$

$$\underline{i} = i_1, \ldots, i_t \textit{ rows}$$

$$\underline{j} = j_1, \ldots, j_t \textit{ columns}$$

Then

$$\mu = (\det A)^{t-1} A_{\underline{i},\underline{j}}$$

where $A_{\underline{i},\underline{j}}$ is the size $n - t$ minor of A obtained by omitting the rows $\underline{i}$ and columns $\underline{j}$.

As a consequence, we get another important result.

Theorem (–). *For $n \geq 4$,*

$$\operatorname{rad}(\mathfrak{P}_{n-1}) = \mathrm{I}_{n-1} \cap \mathfrak{Q}_{n-1}$$

where $\mathfrak{Q}_{n-1}$ is the height n defining ideal for the closure of $\mathcal{Y}_{n,n,n-1}$.

Remark. For $n \geq 4$, $\operatorname{ht} \mathrm{I}_{n-1} = n^2 - (n(n-2) + 2(n-2)) = 4$ and so $\operatorname{ht} \mathfrak{P}_{n-1} = 4$.

Remark. Also, $\mathfrak{P}_{n-1}$, for $n > 4$, can never be a complete intersection because it is of mixed height.

Corollary (–). *$\mathfrak{P}_3(X_{4\times 4})$ is a complete intersection.*

Special Case: When $n = 4$

In this section n shall always equal 4. We saw $\mathfrak{P}_{n-1} = \mathfrak{P}_3$ is a complete intersection, with two components. A natural question to ask is, are the two components linked? If so, then since I_3 is known to be Gorenstein we can deduce some properties about $\mathfrak{Q}_3$ as well. For example, by Peskine+Szpiro (1974) linkage will imply $\mathfrak{Q}_3$ is Cohen-Macaulay. And, by a remark in Huneke+Ulrich (1987), $\mathfrak{Q}_3$ is an **almost complete intersection**, i.e., the number of generators for $\mathfrak{Q}_3$ is one more than its height. If $\mathfrak{P}_3$ is radical, then linkage will follow.

Theorem (–). $\mathfrak{P}_3 = \mathrm{I}_3 \cap \mathfrak{Q}_3$.

Conjecture. $\mathfrak{P}_{n-1}$ is reduced for all n.

Corollary (–). *I_3 and $\mathfrak{Q}_3$ are* **algebraically linked**, *i.e.*

$$\mathfrak{P}_3 :_{K[X]} \mathrm{I}_3 = \mathfrak{Q}_3$$

$$\mathfrak{P}_3 :_{K[X]} \mathfrak{Q}_3 = \mathrm{I}_3.$$

Corollary (–). *$\mathfrak{Q}_3$ is Cohen-Macaulay and an almost complete intersection. Furthermore, we can write*

$$\mathfrak{Q}_3 = \mathfrak{P}_3 :_{K[X]} \det X.$$

Remark. In general, for $n \geq 4$,

$$\mathfrak{Q}_{n-1} = \mathfrak{P}_{n-1} :_{K[X]} (\det X)^{\infty},$$

as a result of contracting $\ker(X \mapsto X^{-1})/\mathfrak{P}_1$. See the thesis for the details of the construction of $\mathfrak{Q}_{n-1}$.

$n > 4$: Focus on the sets $\mathcal{Y}_{n,r,t}$

$$\mathcal{Z}_{n,r} = \text{open set of rank } r \text{ matrices in } \operatorname{Spec} K[X]$$

Then we can write $\mathcal{Y}_{n,r,t} = \mathcal{Z}_{n,r} \cap \mathcal{V}(\mathfrak{P}_t)$. We define a map

$$\Theta : \mathcal{Z}_{n,r} \to \mathrm{Grass}(r,n) \times \mathrm{Grass}(r,n)$$

$$A \mapsto (\mathrm{row}\, A, \mathrm{col}, A)$$

which gives a bundle whose fibres are isomorphic to $\mathrm{GL}(r,K)$. A matrix $A \in \mathcal{Z}_{n,r}$ is in $\mathcal{V}(\mathfrak{P}_t)$ if and only if the diagonal entries of $\wedge^t A$ vanish (using the standard basis on K^n).

We state some results for the special case $r = t$. Choose $A \in \mathcal{Z}_{n,t}$ and factor

$$A = B \cdot A' \cdot C,$$

where B is the identity submatrix in rows $\underline{i}$, C is the identity submatrix in columns $\underline{j}$, and A' is the size t submatrix of A given by rows $\underline{i}$, columns $\underline{j}$. When we "wedge" both sides, we get

$$\wedge^t A = \wedge^t B \cdot \wedge^t A' \cdot \wedge^t C.$$

Note how $\wedge^t A'$ is a scalar, and $\boldsymbol{u} = \wedge^t B$, $\boldsymbol{v} = \wedge^t C$ are vectors. Therefore, $A \in \mathcal{Y}_{n,t,t}$ if and only if the component-wise multiplication of $\boldsymbol{u}$ and $\boldsymbol{v}$ give zero. Equivalently, since B and C both give representatives of elements in $\mathrm{Grass}(t,n)$, we get a condition on the Plücker coordinates, namely, that the products of the respective Plücker coordinates of B and C all equal 0.

Theorem (–). $\dim \mathcal{Y}_{n,n-2,n-2} = n^2 - 4 - n.$

Studying the components of $\mathcal{Y}_{n,t,t}$ is equivalent to studying locally closed subsets of $\mathrm{Grass}(t,n) \times \mathrm{Grass}(t,n)$ given by vanishing of Plücker coordinates. This is NP-hard!!![1] We defer to the theory of **matroid varieties**. See Pitsoulis *Topics in Matroid Theory* (2013) for an introduction to matroids. A particular well-behaved class of matroid varieties are the **positroid varieties**.

Theorem (Knutson+Lam+Speyer 2013). *Positroid varieties are*

- *normal,*
- *Cohen-Macaulay,*
- *have rational singularities*[2]*, and*
- *are always defined by Plücker coordinates.*

Theorem (–). *Every irreducible component of every subscheme of* $\mathrm{Grass}(n-2,n)$ *defined by Plücker coordinates is a positroid variety.*

It thus follows that any component of $\mathcal{V}(\mathfrak{P}_{n-2})$ arising as the closure of a component of $\mathcal{Y}_{n,n-2,n-2}$ is a product of positroid subvarieties of $\mathrm{Grass}(n-2,n)$.

What's Next

- Appeal to matroid theory for a description of components of $\mathcal{Y}_{n,t,t}$.
- Understand $\mathcal{Y}_{n,n-1,n-2}$ to completely understand $\mathcal{V}(\mathfrak{P}_{n-2})$.
- Describe $\mathfrak{Q}_{n-1}$.

[1] Follows from a result by Shor (1991).

[2] a geometric condition that happens to be stronger than Cohen-Macaulay

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