Topology Solutions - 8 May 2010
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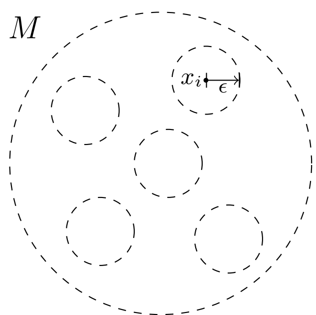
Morning Session

1. If M is a manifold with boundary, then the *double* of M is defined by identifying two copies of M along their boundaries by the identity map. Let $M = D^2 - \cup_i D_\epsilon(x_i)$ where $\{D_\epsilon(x_i)\}$ are n mutually disjoint open discs of radius ϵ in the interior of D^2 centered at $\{x_i\}$. Let W be the double of M. Determine the fundamental group and Euler characteristic of W.



Solution. In general, given two compact manifolds X, Y identified on a closed connected submanifold A, the Euler characteristic is $\chi(X) + \chi(Y) - \chi(A)$. The Euler characteristic of a closed disc is 1, while the Euler characteristic of a circle is 0. Considering D^2 as the union of M and n closed discs, which intersect on n circles,

$$\chi(D^2) = \chi(M) + n - 0$$

implies

$$\begin{aligned}\chi(W) &= \chi(M) + \chi(M) - 0 \\ &= 2 - 2n.\end{aligned}$$

$\square$

2. Let $U_1, U_2, \ldots$ be a countable open covering of a metric space X. A *refinement* is an open covering $V_1, V_2, \ldots$ of X such that for each i, $V_i \subset U_j$ for some j. Show that there exists a refinement $V_1, V_2, \ldots$ which is **star-finite** i.e., for each i, $V_i \cap V_j \neq \emptyset$ for at most finitely many values of j.

Solution. X is a metric space so in particular is paracompact Hausdorff. Thus there exists a partition of unity $\{\phi_n\}_n$ on X dominated by $\{U_n\}_n$. By definition, the family $\{\operatorname{Supp}\phi_n\}$ is locally finite. Therefore for each n let V_n denote the interior of $\{\operatorname{Supp}\phi_n\} \subset U_n$. $\square$

3. Let M be a compact smooth manifold of dimension n, and let $f : M \to \mathbf{R}^n$ be a smooth map. Show that f has a singular point.

Solution. Assume f has no singular points. In other words, $df_x : T_xM \to \mathbb{R}^n$ is an isomorphism for all $x \in M$. So f is a local diffeomorphism. Then since every $x \in M$ has a neighborhood mapping diffeomorphically to $\mathbb{R}^n$, $f(M)$ must be open. Note M is compact implies $f(M)$ is compact, so $f(M) \neq \mathbb{R}^n$. But $f(M)$ is also closed, hence clopen (and non-empty), a contradiction. Therefore f must have a singular point. $\square$

4. Let $\mathbb{R}P^2$ and T denote, in this order, the real projective plane and the torus $S^1 \times S^1$. Prove that any map
$$f : \mathbb{R}P^2 \to T,$$
is homotopic to a constant map.

Solution. A map $f : \mathbb{R}P^2 \to T$ induces a homomorphism on the fundamental groups $f_* : \pi_1(\mathbb{R}P^2) \to \pi_1(T)$, or, $f_* : \mathbb{Z}_2 \to \mathbb{Z} \times \mathbb{Z}$. Since $\mathbb{Z}_2$ cannot inject into $\mathbb{Z} \times \mathbb{Z}$, f_* must be the zero map. Particularly, $f_* \circ \pi_1(\mathbb{R}P^2) = 0$ implies f factors through the universal cover of T, which is $\mathbb{R} \times \mathbb{R}$.

$$\begin{tikzcd} & \mathbb{R} \times \mathbb{R} \arrow[d] \\ \mathbb{R}P^2 \arrow[ur, dashed] \arrow[r, "f"'] & T \end{tikzcd}$$

The plane deformation retracts to the origin, which then maps to a single point in T. Therefore composing with the lift gives a homotopy to a constant map. $\square$

5. Consider the covering map
$$f : S^2 \to \mathbf{R}P^2.$$
Let X be the homotopy pushout of the diagram

$$\begin{tikzcd} S^2 \arrow[r, "f"] \arrow[d, "f"'] & \mathbf{R}P^2 \\ \mathbf{R}P^2. & \end{tikzcd}$$

Calculate the homology groups of X. (Recall that a homotopy pushout of a diagram $\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \\ Z & \end{tikzcd}$ is $(X \times [0,1]) \coprod Y \coprod Z / \sim$ with the quotient topology, where $\sim$ is the smallest equivalence relation satisfying $(x, 0) \sim f(x), (x, 1) \sim g(x)$ for every $x \in X$.)

Solution. The resulting space is a thickened sphere, where the boundaries are both projective planes. Thus removing any intermdiate sphere $S^2 \times \{c\}$, with $c \in (0, 1)$, results in two disjoint projective planes, since the thickened part can now retract to the boundary. Let R_1 denote the outer projective plane when $S^2 \times \{\frac{1}{4}\}$ is removed, R_2 denote the inner projective plane when $S^2 \times \{\frac{3}{4}\}$ is removed, and S denote the open set $S^2 \times \left(\frac{1}{4}, \frac{3}{4}\right)$. To find the homology groups of X, use a Mayer-Vietoris sequence:

$$\cdots \to 0 \to H_2(S) \to H_2(R_1) \oplus H_2(R_2) \to$$
$$\to H_2(X) \to H_1(S) \to H_1(R_1) \oplus H_1(R_2) \to H_1(X) \to 0$$

becomes

$$0 \to \mathbb{Z} \to 0 \oplus 0 \to H_2(X) \to 0 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to H_1(X) \to 0.$$

Therefore $H_2(X) \cong 0$ and $H_1(X) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The higher homology groups vanish. Finally, X is path connected so $H_0(X) \cong 0$. $\square$

Afternoon Session

1. Let X be obtained by gluing two solid tori $D^2 \times S^1$ along their boundary via the map $f : \partial D^2 \times S^1 \to \partial D^2 \times S^1$ given by $f(x, y) = (y^p x, y)$ where p is a fixed positive integer.
   (a) For which values of p can X be given the structure of a topological manifold?
   (b) Compute $\pi_1(X)$.

   Solution.
   (a) Define $\partial D^2 \times S^1 \to \partial D^2 \times S^1$ by $(u, v) \mapsto (v^{-p}u, v)$. Then $g \circ f$ and $f \circ g$ are both the identity, so f is a homeomorphism. Therefore, for all values of p, X can be given the structure of a topological manifold. □
   (b) Use VanKampen's Theorem. Each solid torus has fundamental group $\mathbb{Z}$; let a, b be their respective generators. Then for (x, y) on the boundary the inclusion of the first coordinate in each of the tori is trivial, while for the second is the identity. Therfore

$$\pi_1(X) = \frac{a\mathbb{Z} * b\mathbb{Z}}{ab^{-1}\mathbb{Z}} \cong \mathbb{Z}.$$

   □

2. Consider the space

$$O_{n+1,2} = \{(x_1, x_2)| < x_1, x_2 >= 0\} \subset S^n \times S^n$$

   where S^n is the unit sphere in the Euclidean space $\mathbb{R}^{n+1}$ with standard inner product. Denote by $p : O_{n+1,2} \to S^n$ the projection on the first factor. Prove that there is a section $s : S^n \to O_{n+1,2}$ (i.e. a continuous map s such that $ps = Id$) if and only if n is odd.

   Solution. Suppose a section $s : S^n \to O_{n+1,2}$ exists. Since $\langle x, x\rangle \neq 0$ for any $x \in S^n$, s followed by projection to the second factor gives a continuous map $S^n \to S^n$ with no fixed points. When n is even, this is not possible since the map would induce a non-vanishing vector field on S^n, which is not possible.

   Conversely, if n is odd, define

$$s : (x_1, x_2, \dots, x_{n+1}) \mapsto (x_2, -x_1, x_3, -x_4, \dots, x_{n+1}, -x_n)$$

   where coordinates are given in the ambient space $\mathbb{R}^{n+1}$. □

3. Let X, Y be topological spaces with Y compact. Let $p : X \times Y \to X$ be the projection to the first factor. Show that p maps each closed subset of $X \times Y$ onto a closed subset of X.

   Solution. Let W be a closed set in $X \times Y$, and assume $u \in X$ is a limit point for $p(W)$, with $u \notin p(W)$. Then any neighborhood of u meets $p(W)$ and in fact, while $\{u\} \times Y$ is disjoint from W, $(U \times Y) \cap W \neq \emptyset$, for any neighborhood U of u. For each $(u, y) \in \{u\} \times Y$, there is a basic neighborhood $U \times V$ which is disjoint from W; otherwise (u, y) is a limit point of W, hence contained in W, a contradiction. A cover of $\{u\} \times Y$ with such neighborhooods has a finite subcover $\bigcup_{i=1}^{n}(U_i \times V_i)$ because $\{u\} \times Y$ is compact.

But

$$\left(\bigcap_{i=1}^{n} U_i\right) \times Y \subset \bigcup_{i=1}^{n} (U_i \times V_i)$$

is a neighborhood of $\{u\} \times Y$ which does not intersect W, and that is a contradiction. Conclude if $u \notin p(W)$, then u cannot be a limit point for $p(W)$, so $p(W)$ is closed. $\square$

4. Let X be the union of the three coordinate axes in $\mathbb{R}^3$. Calculate the homology of $\mathbb{R}^3 - X$.

Solution. Write $Y = \mathbb{R}^3 - X$. Since the origin is removed, Y retracts to a sphere with six discs removed, or, a disc with five discs removed. Consider D^2 as the union of Y with five disjiont discs; the intersection is on five disjoint circles. Use a Mayer-Vietoris sequence to calculate homology:

$$\cdots \to 0 \to H_2(Y) \oplus 0 \to 0 \to \mathbb{Z}^5 \to H_1(Y) \oplus 0 \to 0$$

implies $H_i(Y) = 0$ for $i \geq 2$, $H_1(Y) \cong \mathbb{Z}^5$. Finally, Y is path connected, so $H_0(Y) \cong \mathbb{Z}$. $\square$

5. Let $S^2 \subset \mathbb{R}^3$ be the standard unit sphere and

$$X = \{(x, y, z) \in S^2 : y^2 z = x^3 - xz^2\}$$

Is X a smooth submanifold of $\mathbb{R}^3$?

Solution. Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$(x, y, z) \mapsto (x^2 + y^2 + z^2, y^2 z - x^3 + xz^2).$$

Then X is a smooth submanifold of $\mathbb{R}^3$ if $(1, 0)$ is a regular value. The Jacobian is

$$df_{(x,y,z)} = \begin{pmatrix} 2x & 2y & 2z \\ -3x^2 + z^2 & 2yz & y^2 + 2xz \end{pmatrix}$$

which has rank < 2 when the 2×2 minors vanish, i.e.,

$$\begin{aligned} 4xyz + 6x^2 y - 2yz^2 &= 0 \\ 2xy^2 + 10x^2 z - 2z^3 &= 0 \\ 2y^3 + 4xyz - 4yz^2 &= 0. \end{aligned}$$

By factoring equations in the system, deduce from the third equation if $y = 0$, then $x = \pm\sqrt{\frac{1}{6}}, z = \pm\sqrt{\frac{5}{6}}$. If $x = -z \neq 0$ then $z = \pm\sqrt{\frac{1}{6}}, y = \pm\sqrt{\frac{2}{3}}$ and if $x = -z = 0$ then $y = \pm 1$. Finally, from the third equation $3x = z$ implies $y = 0$, which was already considered. In the first equation, if $4xz + 6x - 2z^2 = 0$ then $x = 0$ implies $y = \pm\sqrt{\frac{2}{3}}, z = \pm\sqrt{\frac{1}{3}}$, or $x = 1$ implies $y = z = 0$. And, if the second equation vanishes then using the defining map for X, the system

$$\begin{aligned} z(z^2 - 5x^2) &= x(1 - x^2 - z^2) \\ x(x^2 - z^2) &= z(1 - x^2 - z^2) \end{aligned}$$

implies $z = \pm\sqrt{3}x, y = \pm\sqrt{2\sqrt{3}}x$. The condition that the point lie on $\mathbb{S}^2$ then gives $x = \pm\sqrt{\frac{2\pm\sqrt{3}}{8}}, y = \pm\sqrt{\frac{2\sqrt{3}+3}{4}}, z = \pm\sqrt{\frac{6\pm\sqrt{3}}{8}}$.

Now, each of the above points must satisfy the entire system

$$\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ y^2 z &= x^3 - xz^2 \\ 4xyz + 6x^2y - 2yz^2 &= 0 \\ 2xy^2 + 10x^2z - 2z^3 &= 0 \\ 2y^3 + 4xyz - 4yz^2 &= 0. \end{aligned}$$

But they don't. Conclude the Jacobian of f always has maximum rank, in particular at points in $f^{-1}(1,0)$. Therefore $(1,0)$ is a regular value, which implies X is a submanifold of $\mathbb{R}^3$. $\square$