Topology QR Solutions – 1 Sep 2008

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Morning Session

1. Let X be $\prod_{i=1}^{\infty} \mathbb{R}_i$, where each $\mathbb{R}_i$ is the Euclidean real line. Generate the topology on X from basis sets of the form $\prod_{i=1}^{\infty} U_i$, where each U_i is open in $\mathbb{R}_i$.

   (a) Is X Hausdorff?

   (b) Is X connected?

   (c) Is X locally compact?

   (d) Does X have a countable dense subset?

   Justify your answer.

   Solution.

   (a) Yes. Choose $x = (x_1, x_2, \dots) \neq (y_1, y_2, \dots) = y \in X$. Then there exists i such that $x_i \neq y_i$. By Hausdorffness of $\mathbb{R}_i$, there are disjoint neighborhoods $U_i, V_i \subset \mathbb{R}_i$, around x_i and y_i, respectively. Now put $U_j = V_j = \mathbb{R}_j$ for all $j \neq i$ to get disjoint neighborhoods around x, y, respectively. □

   (b) No. Let A denote the set of all bounded sequences in $\mathbb{R}$. This set is nonempty and open, since perturbing each element by ϵ keeps the sequence bounded. Similarly, $X \setminus A$ is nonempty and open. So X has a separation. □

   (c) Yes. A basis element is an infinite product of open intervals and its closure, an infinite product of closed intervals, is compact by Tychnoff's Theorem. □

   (d) No. In the box topology note
$$\overline{\prod_{i=1}^{\infty} A_i} = \prod_{i=1}^{\infty} \overline{A_i}.$$

*with additional input from M. Hochster, G.P. Scott, and others from the U of M Mathematics Department

If A is a countable dense subset then its projection to any coordinate must also be countably dense. But a countable product of a countable set is not countable. $\square$

2. Let $X = S^1 \vee S^1$, the wedge of two circles, i.e., "figure eight". If $\phi : X \to S^3$ is an embedding of X into a 2-sphere in the 3-sphere S^3, compute $H_*(S^3 \smallsetminus \phi(X), \mathbb{Z})$.

Solution. Put $Y := S^3 \smallsetminus \phi(X)$. Then $S^3 = Y \cup S^2$ and the intersection is $S^2 \smallsetminus \phi(X)$, which is homotopy equivalent to a circle. Use a Mayer-Vietoris sequence to compute homology:

$$\begin{aligned} 0 \to H_3(S^1) \to H_3(Y) \oplus H_3(S^2) &\to H_3(S^3) \to H_2(S^1) \\ \to H_2(Y) \oplus H_2(S^2) &\to H_2(S^3) \to H_1(S^1) \\ \to H_1(Y) \oplus H_1(S^2) &\to H_1(S^3) \to \cdots \to 0 \end{aligned}$$

gives

$$0 \to H_3(Y) \oplus 0 \to \mathbb{Z} \to 0 \to H_2(Y) \oplus \mathbb{Z} \to 0 \to \mathbb{Z} \to H_1(Y) \oplus 0 \to 0.$$

This implies $H_3(Y) \simeq H_1(Y) \simeq \mathbb{Z}$; $H_0(Y) \simeq \mathbb{Z}$ since Y is connected. To find $H_2(Y)$, use the exact sequence

$$0 \to H_3(Y) \to H_2(Y) \to H_1(Y) \to 0$$

to get $H_2(Y) \simeq \mathbb{Z}^2$. The higher homology groups vanish.

3. If we express S^3 as the union of two connected nonempty open subsets X, Y, then $X \cap Y$ is always connected.

Solution. X and Y are homotopy equivalent to 3-balls; write a Mayer-Vietoris sequence

$$\begin{aligned} 0 \to H_3(X \cap Y) \to H_3(X) \oplus H_3(Y) \to H_3(S^3) \to \\ H_2(X \cap Y) \to H_2(X) \oplus H_2(Y) \to H_2(S^3) \to H_1(X \cap Y) \\ \to H_1(X) \oplus H_1(Y) \to H_1(S^3) \to \cdots \end{aligned}$$

Ultimately, the sequence becomes

$$0 \to H_3(S^3) \to H_2(X \cap Y) \to 0.$$

Then $H_1(X \cap Y) = 0$ and $H_2(X \cap Y) \simeq H_3(S^3) \simeq \mathbb{Z}$. Higher homology groups vanish, so by classification of surfaces $X \cap Y$ must be connected, which implies $H_0(X \cap Y) \simeq \mathbb{Z}$. $\square$

4. Assume that M and N are smooth manifolds without boundary, that M is compact, and that N is non-compact and connected. Show that every smooth map $f : M \to N$ has at least one critical point.

   Solution. Assume f has no critical points. In other words, $df_x : T_xM \to T_{f(x)}N$ is surjective for all $x \in M$. So f is a local diffeomorphism onto its image. Then since every $x \in M$ has a neighborhood mapping diffeomorphically to $f(M)$, $f(M)$ must be open. Note M is compact implies $f(M)$ is compact, so $f(M) \neq N$. But $f(M)$ is also closed, hence clopen (and non-empty), a contradiction. Therefore f must have a critical point. $\square$

5. Let S_g denote the oriented surface of genus $g \geq 0$. Let $f : \S_g \to S_g$ be a map which is homotopic to the identity. For each g, does f necessarily have a fixed point? Your answer can depend on g.

   Solution. Since f is homotopic the identity, its Lefshetz number is equal to the Euler characteristic of S_g. So if $\chi(S_g) \neq 0$ then f necessarily has a fixed point. This happens when $g \neq 1$, since $\chi(S_g) = 2 - 2g$. $\square$

Afternoon Session

1. Let X and Y be Hausdorff topological spaces such that Y is compact. Let $p : X \times Y \to X$ be the projection onto the first factor. Show that p maps each closed subset of $X \times Y$ to a closed subset of X.

   Solution. Let W be a closed set in $X \times Y$, and assume $u \in X$ is a limit point for $p(W)$, with $u \notin p(W)$. Then any neighborhood of u meets $p(W)$ and in fact, while $\{u\} \times Y$ is disjoint from W, $(U \times Y) \cap W \neq \emptyset$, for any neighborhood U of u. For each $(u, y) \in \{u\} \times Y$, there is a basic neighborhood $U \times V$ which is disjoint from W – otherwise (u, y) is a limit point of W, hence contained in W, a contradiction. A cover of $\{u\} \times Y$ with such neighborhooods has a finite subcover $\{U_i \times V_i\}_{i=1}^n$ because Y is compact implies $\{u\} \times Y$ is compact. But

   $$(\cap_{i=1}^n U_i) \times Y \subset \cup_{i=1}^n U_i \times V_i$$

   is a neighborhood of $\{u\} \times Y$ which does not intersect W, which is a contradiction. Conclude if $u \notin p(W)$, then u cannot be a limit point for $p(W)$, so $p(W)$ is closed. $\square$

2. Using covering space technique, find all the subgroups of index two of F_2, the free group of rank two.

Solution.

3. Let $i : S^1 \times D^3 \to S^1 \times S^3$ be a smooth embedding (not necessarily standard). Consider the identification space

$$M = \left(S^1 \times S^3 - i(S^1 \times \mathrm{Int}(D^3))\right) \cup_h (D^2 \times S^2),$$

where h is the identity map of

$$S^1 \times S^2 = \partial(D^2) \times S^2 = \partial\left((S^1 \times S^3 - i(S^1 \times \mathrm{Int}(D^3))\right).$$

Show that $\pi_1(M, *)$ is cyclic.

Solution.

4. Let X and Y be metric spaces and let X be compact. Let f be an isometry of X onto a subspace of Y and let g be an isometry of Y onto a subspace of X. Show that f is onto.

Solution.

5. Compute the homology of the space formed as the union of the unit sphere $\left\{(x, y, z) : x^2 + y^2 + z^2 = 1\right\}$ and the closed interval along the z-axis from $(0, 0, -1)$ to $(0, 0, 1)$.

Solution. Let X denote the space in question. Then X is homotopy equivalent to a sphere and a circle with a point in common. Compute the homology using a Mayer-Vietoris sequence:

$$0 \to \mathbb{Z} \oplus 0 \to H_2(X) \to 0 \to 0 \oplus \mathbb{Z} \to H_1(X) \to \cdots \to 0.$$

So $H_2(X) \simeq H_1(X) \simeq H_0(X) \simeq \mathbb{Z}$, since X is also path connected. The rest of the homology groups are zero. $\square$