Mon 27 Oct 2014

- Quiz Thurs 30 Oct covers 4.1-4.2
- Quiz Tues 4 Nov covers 4.2-4.4
- EXAM 2 Friday 7 Nov covers 3.9-4.5

What Derivatives Tell Us

In section 4.1, we used the derivative to identify critical points related to maxima and minima.

In this section, we use derivatives to further examine the behavior of functions.

Definition of Increasing & Decreasing Functions

Suppose a function f is defined on an interval I.

We say that f is **increasing** on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 in I and $x_2 > x_1$.

We say that f is **decreasing** on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 in I and $x_2 > x_1$.

Connection of Derivative to Increasing/Decreasing Functions

Recall that the derivative gives us the slopes of tangent lines. Therefore:

Suppose f is continuous on an interval I and differentiable at every *interior* point of I.

If $f'(x) > 0$ for all interior points of I, then f is increasing on I.

If $f'(x) < 0$ for all interior points of I, then f is decreasing on I.

Example

Sketch a function that is continuous on $(-\infty, \infty)$ that has the following properties:

- $f'(-1)$ is undefined;
- $f'(x) > 0$ on $(-\infty, -1)$;
- $f'(x) < 0$ on $(-1, \infty)$

Exercise

Find the intervals on which *f* is increasing and decreasing.

$$f(x) = 3x^3 - 4x + 12$$

If you graph *f(x)* and *f '(x)* on the same graph, what do you notice?

Using Derivatives to Identify Local Maxima/Minima

The derivative can also assist us in determining whether critical points are local maxima or minima, based on our knowledge of increasing and decreasing functions.

Note the graphs on pg. 235

First Derivative Test

Suppose that f is continuous on an interval that contains a critical point c and assume f is differentiable on an interval containing c, except perhaps at c itself.

- If f' changes sign from positive to negative as x increases through c, then f has a local maximum at c.
- If f' changes sign from negative to positive as x increases through c, then f has a local minimum at c.
- If f' does not change sign at c (from positive to negative or vice versa), then f has **no** local extreme value at c.

Problem

If $f(x) = 2x^3 + 3x^2 - 12x + 1$, identify the critical points on the interval [-3, 4], and use the First Derivative Test to locate the local maximum and minimum values.

What are the absolute max and min?

Absolute Extreme Values on Any Interval

Previously we had stated that in the Extreme Value Thm (section 4.1) that we were guaranteed extreme values only on closed intervals. However:

Suppose *f* is continuous on an interval *I* that contains one local extremum at *c*.

- If a local minimum occurs at c, then *f(c)* is the absolute minimum of *f* on *I.*
- If a local maximum occurs at c, then *f(c)* is the absolute maximum of *f* on *I.*

Fri 31 Oct 2014

- Quiz Tues 4 Nov covers 4.2-4.4
- Monday in lecture we cover 4.5
- EXAM 2 Friday 7 Nov covers 3.9-4.5

Uses of the Second Derivative

Just as the first derivative f' told us whether the function f was increasing or decreasing, the second derivative f'' also tells us whether f' is increasing or decreasing.

Question: What does it mean for f' to be increasing or decreasing?

The graph of a function *f* that satisfies $f'(x) > 0$ and $f''(x) < 0$ on $(0, \infty)$ is:

A. Increasing at an increasing rate

B. Increasing at a decreasing rate

C. Decreasing at an increasing rate

D. Decreasing at a decreasing rate

Concavity and Inflection Points

Let f be differentiable on an open interval I.

- If f' is increasing on I, then f is concave up on I.
- If f' is decreasing on I, then f is concave down on I.

If f is continuous at c and f changes concavity at c (from up to down, or vice versa), then f has an inflection point at c.

Test for Concavity

Suppose that f'' exists on an interval *I.*

- If $f'' > 0$ on *I*, then *f* is concave up on *I*.
- If $f'' < 0$ on *I*, then *f* is concave down on *I*.
- If *c* is a point of *I* at which $f''(c) = 0$ and f'' changes signs at *c*, then *f* has an inflection point at *c*.

Question

What would a function with the following properties look like?

- $f'(x) > 0$ and $f''(x) > 0$
- $f'(x) > 0$ and $f''(x) < 0$
- $f'(x) < 0$ and $f''(x) > 0$
- $f'(x) < 0$ and $f''(x) < 0$

Second Derivative Test

Suppose that f'' is continuous on an open interval containing c with $f'(c) = 0$.

- If $f''(c) > 0$, then f has a local minimum at c.
- If $f''(c) < 0$, then f has a local maximum at c.
- If $f''(c) = 0$, then the test is inconclusive.

Note: See the Recap of Derivative Properties on p. 242 for a summary of all points

Problem

Given $f(x) = 2x^3 - 6x^2 - 18x$,

1. Determine the intervals on which it is concave up or down, and identify any inflection points.

2. Locate the critical points, and use the 2nd Derivative Test to determine whether they correspond to local minima or maxima, or whether the test is inconclusive.

Homework from Section 4.2

Do problems 11-35 odd, 47-61 odd, and 67 (pgs. 243-245).

Graphing Functions

Question: Have you ever attempted to graph a function using your graphing calculator, only to have either the calculator fail to produce the graph (e.g., due to incorrect window settings, insufficient domain/range, etc.) or the graph either not appear as you thought?

The various tests and tools we now have examined can go a long way in helping us draw and analyze functions

Graphing Functions

Message of the section: Graphing utilities are valuable for exploring functions, producing preliminary graphs, and checking your work. But they should not be relied on exclusively because **they cannot explain *why* a graph has its shape.**

In this section, we will use the tools and our knowledge of functions to produce graphs and explain why their shape is how it is.

Graphing Guidelines

1. Identify the domain or interval of interest.
2. Exploit symmetry.
3. Find the first and second derivatives.
4. Find critical points and possible inflection points.
5. Find intervals on which the function is increasing or decreasing, and concave up/down.
6. Identify extreme values and inflection points.
7. Locate vertical/horizontal asymptotes and determine end behavior.

Example

According to the graphing guidelines, sketch a graph of

$$f(x) = 2x^6 - 3x^4$$

Example

According to the graphing guidelines, sketch a graph of

$$f(x) = 2x^6 - 3x^4$$

1. **Identify the domain or interval of interest:** *f(x)* is a polynomial, so the domain is (-∞,∞).
2. **Exploit symmetry:** *f(x)* is an even function (e.g., *f(-x)=f(x)*). So the graph is symmetric across the y-axis.
3. **Find the first and second derivatives:**

$$f'(x) = 12x^5 - 12x^3 = 12x^3(x^2 - 1)$$

$$f''(x) = 60x^4 - 36x^2 = 12x^2(5x^2 - 3)$$

Example

According to the graphing guidelines, sketch a graph of

$$f(x) = 2x^6 - 3x^4$$

4. **Find critical points and possible inflection points:** *f'* is a polynomial, so *f'* is defined everywhere. *f'(x) = 0* implies that the critical points are at x = -1, 0, 1.

   f'' is a polynomial and thus defined everywhere. *f''(x)=0* implies potential inflection points at 0 and

   $\pm\sqrt{\frac{3}{5}}$

Example

According to the graphing guidelines, sketch a graph of

$$f(x) = 2x^6 - 3x^4$$

5. **Find intervals on which the function is increasing/decreasing and concave up/down:**

   Critical points at x=-1, 0, 1: *f'(-2) < 0; f'(-.5) > 0;*

   f'(.5) < 0; f'(2) > 0; So *f* is increasing on (-1, 0) and (1, ∞) and decreasing on (-∞, -1) and (0, 1).

   Possible Inflection points at 0 and $\pm\sqrt{\frac{3}{5}}$: *f''(-1) > 0; f''(-.5) < 0; f''(.5) < 0; f''(1) > 0;* So *f* is concave up on $\left(-\infty, -\sqrt{\frac{3}{5}}\right)$ and $\left(\sqrt{\frac{3}{5}}, \infty\right)$ and is concave down on $\left(-\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}\right)$

Example

According to the graphing guidelines, sketch a graph of

$$f(x) = 2x^6 - 3x^4$$

6. **Identify extreme values and inflection points:** Using the first derivative test, we have a local max at x=0 [(0, 0)] and local min's at x=-1 [(-1, -1)] and x=1 [(1, -1)].

   Using the second derivative test, we see that we do not have an inflection point at x=0 (no sign change) but do have inflection points at x=$\pm\sqrt{\frac{3}{5}}$:

$$\left(\sqrt{\frac{3}{5}}, -\frac{81}{125}\right) \qquad \left(-\sqrt{\frac{3}{5}}, -\frac{81}{125}\right)$$

Example

According to the graphing guidelines, sketch a graph of

$$f(x) = 2x^6 - 3x^4$$

7. **Locate vertical/horizontal asymptotes and determine end behavior:** There are no vertical or horizontal asymptotes, and $\lim_{x \to \infty} f(x) = \infty, \lim_{x \to -\infty} f(x) = \infty$
8. **Find the intercepts:** When x=0, y=0. So the y-intercept is (0, 0).

   When *f(x)* = 0, x=0 and $\pm\sqrt{\frac{3}{2}}$. So the x-intercepts are (0, 0), $\left(-\sqrt{\frac{3}{2}},0\right)$, and $\left(\sqrt{\frac{3}{2}},0\right)$

Example

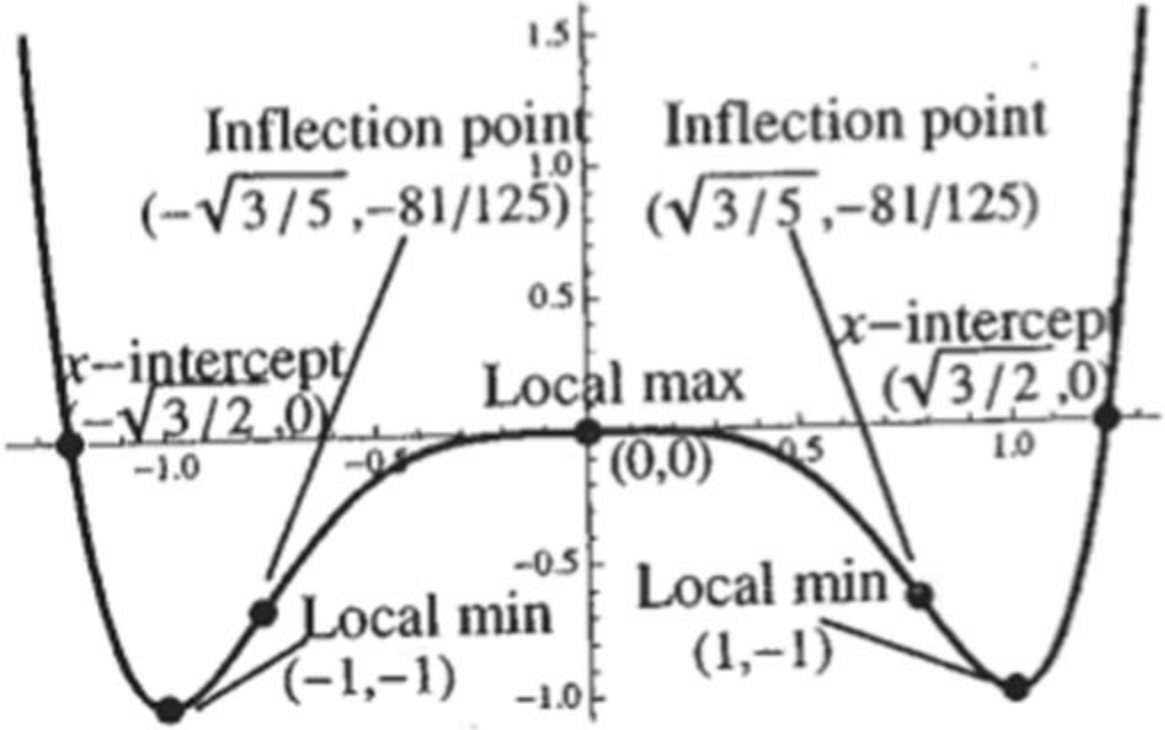
According to the graphing guidelines, sketch a graph of

$$f(x) = 2x^6 - 3x^4$$

9. **Choose an appropriate graphing window and make a graph:** All of the critical points, inflection points, and intercepts occur relatively close to 0. So we can choose a window of [-2, 2] x [-2, 2] to graph this function.

So graphing the function with these pieces of information, we find:

Graph of f(x)



1.5
1.0
0.5
−0.5
−1.0
−1.0
1.0
Inflection point
(−√3/5 ,−81/125)
Inflection point
(√3/5 ,−81/125)
x−intercept
(−√3/2 ,0)
x−intercept
(√3/2 ,0)
Local max
(0,0)
Local min
(−1,−1)
Local min
(1,−1)

Exercise

According to the graphing guidelines, sketch a graph of

$$f(x) = \frac{x^2}{x^2 - 4}$$

The function $f(x)=\dfrac{x-2}{x^2+2}$ has:

A. A zero, a vertical asymptote, and a horizontal asymptote

B. A zero and a vertical, but not a horizontal, asymptote

C. A vertical and horizontal asymptote, but no zero

D. A zero and a horizontal, but no vertical, asymptote

Homework from Section 4.3

Do problems 7, 9, 13-19 odd, 23, 29, 43, 45 (pgs. 254-255).

Optimization

In many scenarios, it is important to find a maximum or minimum value under given constraints. Given our use of derivatives from previous sections, optimization problems follow directly from what we have studied.

Problem

Given all nonnegative real numbers x and y between 0 and 50 such that their sum is 50 (e.g., $x + y = 50$), which pair has the maximum product?

This is a basic optimization problem. In this problem, we are given a **constraint** (e.g., $x + y = 50$) and asked to maximize an **objective function** (e.g., $A = xy$)

Problem

Given all nonnegative real numbers x and y between 0 and 50 such that their sum is 50 (e.g., $x + y = 50$), which pair has the maximum product?

The first step is to express the objective function $A = xy$ in terms of a single variable by using the constraint:

$y = 50 – x$ gives us $A = x\,(50 – x)$

To maximize A, we find the critical points: $A'(x)=50 – 2x$ which has a critical value at $x = 25$

Problem

Given all nonnegative real numbers x and y between 0 and 50 such that their sum is 50 (e.g., $x + y = 50$), which pair has the maximum product?

Then to maximize the function, we examine the endpoints [0, 50] and the critical point [25]. In this case, $A(0) = A(50) = 0$, so $A(25)=625$ is the maximum value. Therefore, $x = y = 25$ maximizes $A = xy$.

Essential Feature of Optimization Problems

All optimization problems take the following form:

What is the maximum (or minimum) value of an objective function subject to the given constraint(s)?

Most optimization problems have the same basic structure of the previous maximization problem: An objective function (possibly with several variables and/or constraints), with methods of calculus used to find the maximum/minimum values.

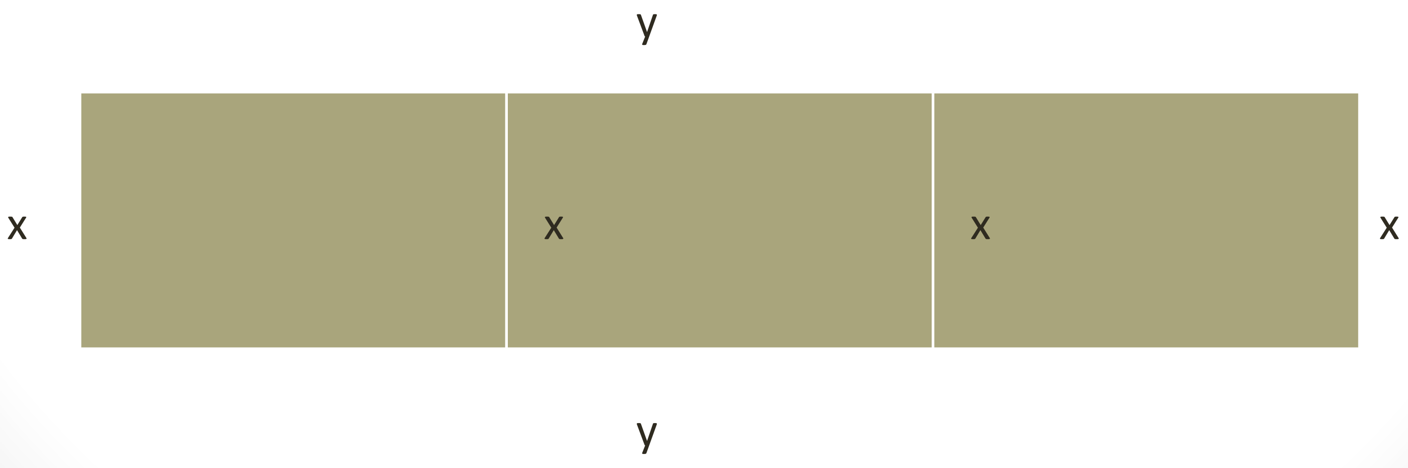
Example

Suppose you wish to build a rectangular pen with three parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen?

Example

Suppose you wish to build a rectangular pen with three partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen?

In many instances, a picture may be of considerable help:



Example

Based on the picture, we see that $2y + 4x = 500$, which simplifies to $y = -2x + 250$.

We are maximizing $A = xy$. So $A = x(-2x+250) = -2x^2+250x$.

$A'(x) = -4x + 250$. So $A'(x) = 0$ implies that $x = 62.5$. Since $0 \leq x \leq 125$, we examine the points 0, 62.5, and 125.

$A(0)=A(125)=0$, so $A(62.5) = 7812.5$ ft^2 maximizes the area. So the pen's dimensions are 62.5 ft x 125 ft.

Guidelines for Optimization Problems

1. Read the problem carefully, identify the variables, and organize the given information with a picture.
2. Identify the objective function (e.g., the function to be optimized). Write it in terms of the variables of the problem.
3. Identify the constraint(s). Write them in terms of the variables of the problem.
4. Use the constraint(s) to eliminate all but one independent variable of the objective function.
5. With the objective function expressed in terms of a single variable, find the interval of interest for that variable.
6. Use methods of calculus to find the absolute maximum or minimum value of the objective function on the interval of interest. If necessary, check the endpoints.

Problem

An open rectangular box with a square base is to be made from 48 ft^2 of material. What dimensions will result in a box with the largest possible volume?

Problem

Find the dimensions of the rectangle of largest area which can be inscribed in the closed region bounded by the x-axis, y-axis, and graph of $y = 8 - x^3$.

Homework from Section 4.4

Do problems 5-13 all, 18-20 all, 26 (pgs. 261-263).