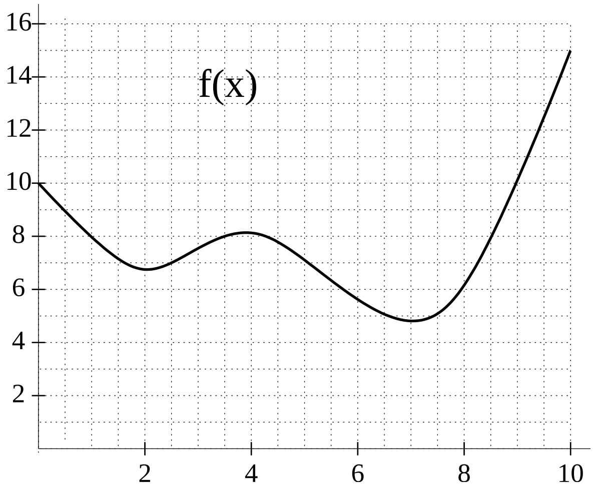
Section 4.2 – Optimization

Some Definitions. Let f be a function.

1. f has a *global maximum* at $x = p$ if $f(p)$ is greater than or equal to all output values of f.
2. f has a *global minimum* at $x = p$ if $f(p)$ is less than or equal to all output values of f.
3. *Optimization* refers to the process of finding the global maximum or global minimum of a function.

Example. For the function f given below, locate all local and global maxima and minima on the interval $[0, 10]$.



$f(2) = 6.8$ and $f(7) = 4.9$ are local minima.

$f(3.9) = 8.1$ is a local maximum.

$f(7) = 4.9$ is a global minimum.

$f(10) = 15$ is a global maximum.

General Rule. To find the global maximum and the global minimum of a continuous function on a closed interval (i.e., an interval that contains its endpoints), compare the output values of the function at the following locations:

1. critical points
2. endpoints

Exercises

1. Find the global maximum and global minimum value of $f(x) = x + \frac{3}{x}$ on the interval $[1, 4]$.

   We have
   $$f'(x) = 1 - \frac{3}{x^2} = \frac{x^2 - 3}{x^2},$$
   so our critical points occur when $x^2 - 3 = 0$, or when $x = \pm\sqrt{3}$. Since the endpoints of our interval are $x = 1$ and $x = 4$, the only relevant critical point is $x = \sqrt{3}$. Calculating the value of f at these three points, we have
   $$\begin{aligned} f(1) &= 1 + \frac{3}{1} = 4 \\ f(4) &= 4 + \frac{3}{4} = 4.75 \\ f(\sqrt{3}) &= \sqrt{3} + \frac{3}{\sqrt{3}} = 2\sqrt{3} \approx 3.46 \end{aligned}$$
   Therefore, $f(\sqrt{3}) = 2\sqrt{3}$ is the global minimum and $f(4) = 4.75$ is the global maximum.

2. (Taken from *Hughes-Hallett, et. al.*) When you cough, your windpipe contracts. The speed, v, at which the air comes out depends on the radius, r, of your windpipe. If R is the normal (rest) radius of your windpipe, then for $0 \leq r \leq R$, the speed is given by $v = a(R-r)r^2$, where a is a positive constant. What value of r maximizes the speed?

Since $v = a(R-r)r^2 = aRr^2 - ar^3$, we have

$$\frac{dv}{dr} = 2arR - 3ar^2 = ar(2R - 3r),$$

so $dv/dr = 0$ when $r = (2/3)R$, meaning that $r = (2/3)R$ is the only critical point of our speed function. Since $r = 0$ and $r = R$ are the endpoints of our interval of consideration, we can calculate and compare the values of v at the relevant three points.

$$\begin{aligned} v|_{r=0} &= a(R-0)(0)^2 = 0 \\ v|_{r=R} &= a(R-R)R^2 = 0 \\ v|_{r=(2/3)R} &= a\left(R - \frac{2R}{3}\right)\left(\frac{2R}{3}\right)^2 \\ &= a \cdot \frac{R}{3} \cdot \frac{4R^2}{9} \\ &= \frac{4aR^3}{27} \end{aligned}$$

Therefore, the maximum coughing speed occurs when $r = (2/3)R$, that is, when the radius of the windpipe is two-thirds of its normal (rest) radius.

3. (Taken from *Hughes-Hallett, et. al.*) The potential energy, U, of a particle moving along the x-axis is given by

$$U = b\left(\frac{a^2}{x^2} - \frac{a}{x}\right),$$

where a and b are positive constants and $x > 0$. What value of x minimizes the potential energy?

First, we have

$$U'(x) = b \cdot \frac{d}{dx}\left(\frac{a^2}{x^2} - \frac{a}{x}\right) = b \cdot \frac{d}{dx}\left(\frac{a^2 - ax}{x^2}\right) = ab\left(\frac{x - 2a}{x^3}\right),$$

so $U'(x) = 0$ when $x - 2a = 0$, or when $x = 2a$. Therefore, $x = 2a$ is the only critical point. Also, since we can see from our formula for $U'(x)$ that $U'(x) < 0$ for $0 < x < 2a$ and $U'(x) > 0$ for $x > 2a$, we see that U is decreasing everywhere to the left of $x = 2a$ and increasing everywhere to the right of $x = 2a$. It follows that U has a global minimum at $x = 2a$, meaning that $2a$ is the value of x that minimizes the potential energy.

4. Let $f(x) = xe^{-x^2}$.

   (a) Locate all local maximum and all local minimum values of f.

   We have

   $$f'(x) = xe^{-x^2} \cdot (-2x) + e^{-x^2} \cdot 1 = e^{-x^2}(1 - 2x^2),$$

   so $f'(x) = 0$ when $1 - 2x^2 = 0$, that is, when $x = \pm\sqrt{1/2}$. From the sign chart to the right, we see that f has a local minimum at $x = -1/\sqrt{2}$ and a local maximum at $x = 1/\sqrt{2}$.

Interval	Sign of $f'(x)$
$x < -\sqrt{1/2}$	$-$
$-\sqrt{1/2} < x < \sqrt{1/2}$	$+$
$x > \sqrt{1/2}$	$-$

(b) Find the global maximum and the global minimum values of f on the interval $[0, 2]$.

To determine the global maximum and minimum values of f on $[0, 2]$, we begin by comparing the value of f at the endpoints of our interval and the one critical point that lies within the interval.

$$\begin{aligned} f(0) &= 0e^{-0^2} = 0 \\ f(2) &= 2e^{-2^2} = 2e^{-4} \approx 0.037 \\ f\left(\frac{1}{\sqrt{2}}\right) &= \frac{1}{\sqrt{2e}} \approx 0.43 \end{aligned}$$

Therefore, $f(0) = 0$ is the global minimum value of f and $f(1/\sqrt{2}) = 1/\sqrt{2e}$ is the global maximum value of f on $[0, 2]$.

5. Give an example of a function that does not have a global maximum or a global minimum value.

Since the function f defined by $f(x) = x^3$ can get arbitrarily large and arbitrarily small on the interval $(-\infty, \infty)$, we conclude that f has no global maximum and no global minimum value on this interval.