Fri 26 June

- Quiz 9 Monday covers Optimization ($\oint$4.4) with Related Rates sprinkled in. Closed book. Not collaborative.
- Next week: $\oint$4.5-4.7. Exam 3 on Thursday is 50 minutes, covers $\oint$3.10-4.7.

1 Week 5: 22-26 June

- Friday 26 June

$\oint$4.4 Optimization Problems

- Essential Feature of Optimization Problems
- Guidelines for Optimization Problems
- Book Problems

$\oint$ 4.4 Optimization Problems

In many scenarios, it is important to find a maximum or minimum value under given constraints. Given our use of derivatives from the previous sections, optimization problems follow directly from what we have studied.

Example

Given all nonnegative real numbers x and y between 0 and 50 such that their sum is 50 (i.e., $x + y = 50$), which pair has the maximum product?

This is a basic optimization problem. In this problem, we are given a **constraint** $(x + y = 50)$ and asked to maximize an **objective function** $(A = xy)$.

The first step is to express the objective function $A = xy$ in terms of a single variable by using the constraint:

$$y = 50 - x$$
$$\implies A(x) = x(50 - x).$$

Then to maximize A, we find the critical points:

$$A'(x) = 50 - 2x = 0$$
$$\text{means} \quad x = 25 \quad \text{is a critical point.}$$

The domain of $A(x)$ is $[0, 50]$. To maximize A we evaluate A at the endpoints of the domain and at the critical point:

$$\begin{aligned} A(0) &= 0(50-0) = 0 \\ A(25) &= 25(50-25) = 625 \\ A(50) &= 50(50-0) = 0 \end{aligned}$$

So 625 is the maximum value of A and A is maximized when $x = 25$ (which means $y = 25$).

To answer the question, the pair of nonnegative numbers summing to 50 with the maximum product is 25 and 25.

Essential Feature of Optimization Problems

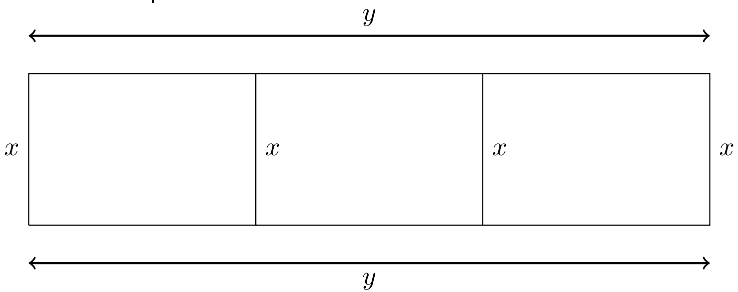
All optimization problems take the following form:

What is the maximum (or minimum) value of an objective function subject to the given constraint(s)?

Most optimization problems have the same basic structure as the previous problem: An objective function (possibly with several variables and/or constraints) with methods of calculus used to find the maximum/minimum values.

Example

Suppose you wish to build a rectangular pen with two interior parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen?



From looking at the picture, we can identify the constraints:

$$2y + 4x = 500 \implies y = -2x + 250.$$

The objective function is what we must maximize. In this case it is the area, $A = xy$. So we write

$$A(x) = x(-2x + 250) = -2x^2 + 250x.$$

Taking the derivative and setting it to zero, we get

$$\begin{aligned} A'(x) &= -4x + 250 = 0 \\ &\implies x = 62.5 \quad \text{is a critical point.} \end{aligned}$$

We use the picture to identify the domain. Since we have 500 ft of fencing available we must have $0 \leq x \leq 125$. Now we find the max:

$$\begin{aligned} A(0) &= 0(-2(0)+250) = 0 \\ A(62.5) &= 62.5(-2(62.5)+250) = 7812.5 \\ A(125) &= 125(-2(125)+250) = 0 \end{aligned}$$

The maximum area is 7812.5 ft^2. The pen's dimensions (answer the question!) are $x = \boxed{62.5 \text{ ft}}$ by $y = -2(62.5) + 250 = \boxed{125 \text{ ft}}$.

Guidelines for Optimization Problems

1. READ THE PROBLEM carefully, identify the variables, and organize the given information with a picture.
2. Identify the objective function (i.e., the function to be optimized). Write it in terms of the variables of the problem.
3. Identify the constraint(s). Write them in terms of the variables of the problem.
4. Use the constraint(s) to eliminate all but one independent variable of the objective function.
5. With the objective function expressed in terms of a single variable, find the interval of interest for that variable.
6. Use methods of calculus to find the absolute maximum or minimum value of the objective function on the interval of interest. If necessary, check the endpoints.

Exercise

An open rectangular box with square base is to be made from 48 ft^2 of material. What dimensions will result in a box with the largest possible volume?

Exercise

Find the dimensions of the rectangle of largest area which can be inscribed in the closed region bounded by the x-axis, y-axis, and the graph of $y = 8 - x^3$.

4.4 Book Problems

5-13, 18-20, 26