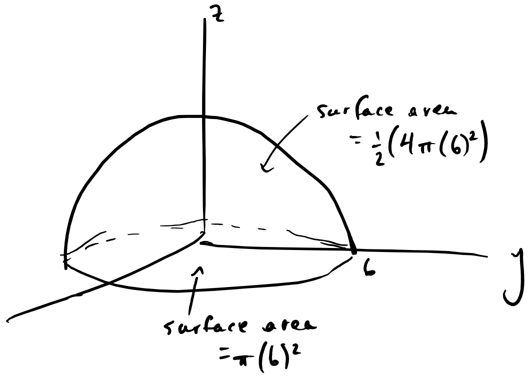
Quiz AA Solutions

1. S = upper hemisphere of radius 6

temperature $T(x,y,z) = x^2 + y^2$



top: $\vec{r}(u,v) = \langle 6\sin u\cos v, 6\sin u\sin v, 6\cos u\rangle$

$0 \le u \le \frac{\pi}{2}$

$0 \le v \le 2\pi$

$\vec{r}_u = \langle 6\cos u\cos v, 6\cos u\sin v, -6\sin u\rangle$

$\vec{r}_v = \langle -6\sin u\sin v, 6\sin u\cos v, 0\rangle$

$\vec{r}_u \times \vec{r}_v = \langle 0 - (-6\sin u)(6\sin u\cos v), -(0-(-6\sin u)(-6\sin u\sin v)), 6\cos u\cos v(6\sin u\cos v) - 6\cos u\sin v(-6\sin u\sin v)\rangle$

$= \langle 36\sin^2 u\cos v, 36\sin^2 u\sin v, 36\cos u\sin u(\cancel{\cos^2 v + \sin^2 v})\rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{(36\sin^2 u\cos v)^2 + (36\sin^2 u\sin v)^2 + (36\cos u\sin u)^2}$

$= 36\sqrt{\sin^4 u(\cancel{\cos^2 v + \sin^2 v}) + \cos^2 u\sin^2 u}$

$= 36\sin u\sqrt{\cancel{\sin^2 u + \cos^2 u}}$

* Also on the formula sheet —

Note, the $\hat{n}$ on the formula sheet is not the unit normal vector.

bottom: $\vec{r}(u,v) = \langle u\cos v, u\sin v, 0\rangle$

$0 \le u \le 6$ (polar coordinates)

$0 \le v \le 2\pi$

$\vec{r}_u = \langle \cos v, \sin v, 0\rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 0\rangle$

$\vec{r}_u \times \vec{r}_v = \langle 0, 0, u\cos^2 v - (-u\sin^2 v)\rangle$

$= \langle 0, 0, u\rangle$

$|\vec{r}_u \times \vec{r}_v| = u$

$$\iint_S T\,dS = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \underbrace{\overbrace{[(6\sin u\cos v)^2 + (6\sin u\sin v)^2]}^{(6\sin u)^2(\cancel{\cos^2 v + \sin^2 v})}(36\sin u)}_{36^2\sin^3 u = 36^2(1-\cos^2 u)(\sin u)}\,dv\,du + \int_0^6\int_0^{2\pi}\underbrace{[(u\cos v)^2 + (u\sin v)^2]}_{u^2}u\,dv\,du$$

Let $w = \cos u \rightsquigarrow \cos\frac{\pi}{2} = 0$

$dw = -\sin u\,du \quad \cos 0 = 1$

$$= \int_1^0 -36^2\int_0^{2\pi}(1-w^2)\,dv\,dw + \int_0^6\int_0^{2\pi} u^3\,dv\,du$$

$$= -36^2(2\pi)\int_1^0(1-w^2)\,dw + 2\pi\int_0^6 u^3\,du$$

$$= -36^2(2\pi)\left[w - \frac{w^3}{3}\Big|_1^{0 \leftarrow \text{terms vanish}}\right] + 2\pi\, \frac{u^4}{4}\Big|_{0 \leftarrow \text{terms vanish}}^{6} = -36^2(2\pi)\left[-\left(1 - \frac{1^3}{3}\right)\right] + \frac{\pi}{2}(6)^4 = \frac{4}{3}\pi(6)^4 + \frac{1}{2}\pi(6)^4$$

$$= \frac{8+3}{6}\pi(6^4) = 11\pi(6)^3$$

For the average, divide by the surface area

$$\frac{11\cancel{\pi}(6)^{\cancel{3}\,2}}{\cancel{3}\cancel{\pi}\,\cancel{(6)^2}} = \boxed{22}$$

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2. (a) $\vec{F} = \vec{r} = \langle x, y, z \rangle$,
   S = sphere of radius 1

* $\vec{r}$ is a radial vector field, not a parametrization

$$\text{flux} = \iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_D \underbrace{\operatorname{div}\vec{F}}_{1+1+1=3}\,dV \quad (\underline{\text{Divergence Theorem}})$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 3\rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= 3\int_0^{2\pi}\int_0^{\pi} \sin\varphi \, \frac{\rho^3}{3}\Big|_0^1 \, d\varphi\, d\theta$$

$$= \cancel{3}\left(\frac{1}{\cancel{3}}\right)\int_0^{2\pi} -\cos\varphi\Big|_0^{\pi}\, d\theta$$

$$= -(\underbrace{\cos\pi}_{-1} - \underbrace{\cos 0}_{1})(2\pi) = \boxed{4\pi}$$

* Note: Multiplying by volume only works if the divergence is a constant — you must say this if you want to use that shortcut.

OR, parametrize S:

$\vec{s}(u,v) = \langle \sin u\cos v, \sin u \sin v, \cos u \rangle \quad 0 \le u \le \pi \quad 0 \le v \le 2\pi$

* $\vec{r}$ is already taken

$\vec{s}_u \times \vec{s}_v = \langle \sin^2 u\cos v, \sin^2 u \sin v, \sin u\cos u \rangle$
(from the formula sheet)

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \int_0^{\pi}\int_0^{2\pi}\Big[\underbrace{(\sin u\cos v)(\sin^2 u\cos v)}_{\sin^3 u\cos^2 v} + \underbrace{(\sin u \sin v)(\sin^2 u \sin v)}_{\sin^3 u \sin^2 v} + \underbrace{\cos u(\sin u\cos u)}_{\cos^2 u \sin u}\Big]\,dv\,du$$

$$= \int_0^{\pi}\int_0^{2\pi}\left(\sin^3 u\,(\cancel{\cos^2 v + \sin^2 v}) + \cos^2 u \sin u\right)dv\,du$$

$\longrightarrow$

$$= \int_0^{\pi}\int_0^{2\pi} \sin u\,\cancel{(\sin^2 u + \cos^2 u)}\, dv\, du$$

$$= \int_0^{\pi} 2\pi \sin u\, du = 2\pi\left(-\cos u \Big|_0^{\pi}\right)$$

$$= 2\pi(-\overset{-1}{\cos\pi} + \overset{1}{\cos 0})$$

$$\boxed{= 4\pi}$$

(b) Same as (a):

Divergence Theorem:

$$\text{flux} = \iint_S \vec{F}\cdot\hat{n}\, dS = \iiint_D \underbrace{\operatorname{div}\vec{F}}_{1+1+1=3}\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} 3\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= 3\int_0\int_0^{\pi} \sin\varphi\ \frac{\rho^3}{3}\Big|_0^2\, d\varphi\, d\theta$$

$$= \cancel{3}\left(\frac{8}{\cancel{3}}\right)\int_0^{2\pi} -\cos\varphi \Big|_0^{\pi}\, d\theta$$

$$= -8(\underset{-1}{\cos\pi} - \underset{1}{\cos 0})(2\pi) = \boxed{32\pi}$$

Or parametrize:

$$\vec{s}(u,v) = \langle 2\sin u\cos v, 2\sin u \sin v, 2\cos u\rangle \quad 0\le u\le\pi \quad 0\le v\le 2\pi$$

$$\vec{s}_u \times \vec{s}_v = \langle 4\sin^2 u\cos v, 4\sin^2 u\sin v, 4\sin u\cos u\rangle$$

(from the formula sheet)

$$\iint_S \vec{F}\cdot\hat{n}\, dS = \int_0^{\pi}\int_0^{2\pi}\Big[\underbrace{(2\sin u\cos v)(4\sin^2 u\cos v)}_{8\sin^3 u\cos^2 v} + \underbrace{(2\sin u\sin v)(4\sin^2 u \sin v)}_{8\sin^3 u\sin^2 v} + \underbrace{2\cos u(4\sin u\cos u)}_{8\cos^2 u\sin u}\Big]\, dv\, du$$

$$= \int_0^{\pi}\int_0^{2\pi} 8\sin u \left(\cancel{\sin^2 u + \cos^2 u}\right) dv\, du$$

$$= \int_0^{\pi} 16\pi \sin u \, du = 16\pi\left(-\cos u \Big|_0^{\pi}\right)$$

$$= 16\pi\left(-\overset{-1}{\cos\pi} + \overset{1}{\cos 0}\right)$$

$$\boxed{= 32\pi}$$

(c) $$\vec{G} = \frac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}}$$

$$\operatorname{div}\vec{G} = (1)(x^2+y^2+z^2)^{-3/2} + (x)\left(\tfrac{-3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x)$$
$$+ (1)(x^2+y^2+z^2)^{-3/2} + (y)\left(\tfrac{-3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2y)$$
$$+ (1)(x^2+y^2+z^2)^{-3/2} + (z)\left(\tfrac{-3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2z)$$
$$= \frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{\cancel{2}}\,\frac{\cancel{2}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}$$
$$= 0$$

* Theorem 14.8 also gives divergence for radial vector fields:

$$\operatorname{div}\left(\frac{\vec{r}}{|\vec{r}|^p}\right) = \frac{3-p}{|\vec{r}|^p}$$

$$\text{flux} = \iiint_D \operatorname{div}\vec{G}\, dV \boxed{= 0}$$

(d) Same as part (c).

3. $\underset{\sim}{T} = e^{-x^2-y^2-z^2}$ $\quad$ (5

$\tilde{F} = -K\nabla T = -K\langle e^{-x^2-y^2-z^2}(-2x),\ e^{-x^2-y^2-z^2}(-2y),\ e^{-x^2-y^2-z^2}(-2z)\rangle$

$= 2Ke^{-x^2-y^2-z^2}\langle x,y,z\rangle$

S = sphere of radius a

flux $= \iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \operatorname{div}\vec{F}\,dV$

Product Rule for divergence (Theorem 14.11)

$= 2K\iiint_D \left[-2e^{-x^2-y^2-z^2}\underbrace{\langle x,y,z\rangle\cdot\langle x,y,z\rangle}_{x^2+y^2+z^2} + e^{-x^2-y^2-z^2}(1+1+1)\right]dV$

$= 2K\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\left(-2e^{-\rho^2}\rho^2+3e^{-\rho^2}\right)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

$= 2K\int_0^{2\pi}\int_0^{\pi}\sin\varphi\left(a^3e^{-a^2}\right)d\varphi\,d\theta$

Wolfram Alpha

$= 2Ka^3e^{-a^2}\int_0^{2\pi} -\cos\varphi\Big|_0^{\pi}$

$= 2Ka^3e^{-a^2}(2)(2\pi)$

$K=1$:

$\boxed{= 8\pi a^3e^{-a^2}}$

Or, to compute the surface integral directly,

Use Table 14.2:

$\vec{r}(u,v) = \langle a\sin u\cos v,\ a\sin u\sin v,\ a\cos u\rangle,\quad 0\le u\le\pi,\ 0\le v\le 2\pi$

$\vec{n} = \langle a^2\sin^2u\cos v,\ a^2\sin^2u\sin v,\ a^2\sin u\cos u\rangle$

$\vec{F} = 2Ke^{-a^2}\vec{r}$

$\iint_S \vec{F}\cdot\vec{n}\,dS = 2Ke^{-a^2}\int_0^{2\pi}\int_0^{\pi}\left(\underbrace{a^3\sin^3u\cos^2v + a^3\sin^3u\sin^2v + a^3\sin u\cos^2u}_{a^3\sin u}\right)du\,dv$

$= 2Ka^3e^{-a^2}\int_0^{2\pi} \underset{-(-1-1)}{-\cos u}\Big|_0^{\pi} = \boxed{8\pi a^3e^{-a^2}}$

4. $\vec{F} = \langle y^2, -z^2, x \rangle$

C = circle given by $\vec{r}(t) = \langle 3\cos t, 4\cos t, 5\sin t \rangle$
$(0 \le t \le 2\pi)$

Parametrize the disk via its radius:

$S = \langle 3u\cos v, 4u\cos v, 5u\sin v \rangle \Rightarrow \vec{F} = \langle (4u\cos v)^2, -(5u\sin v)^2, 3u\cos v \rangle$

$0 \le u \le 1$

$0 \le v \le 2\pi$

$t_u = \langle 3\cos v, 4\cos v, 5\sin v \rangle$

$t_v = \langle -3u\sin v, -4u\sin v, 5u\cos v \rangle$

$$\Rightarrow t_u \times t_v = \langle \overset{20u}{20u\cos^2 v - (-20)u\sin^2 v}, \overset{-15u}{(15u\cos^2 v - (-15u\sin^2 v))}, \underbrace{-12u\cos v\sin v - (-12u\cos v\sin v)}_{0} \rangle$$

Stokes' Theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F}) \cdot \hat{n}\, dS$$

$$= \iint_S \langle 0-(-2z), \underbrace{0-1}_{-1}, \underbrace{0-2y}_{-2y} \rangle \cdot \hat{n}\, dS$$

$$= \int_0^{2\pi}\int_0^1 [\underbrace{2(5u\sin v)(20u)}_{200u^2\sin v} \underbrace{-1(-15u)}_{+15u} - 2(\cancel{4u\cos v})(0)]\, du\, dv$$

$$= \int_0^{2\pi} \left[(200\sin v)\frac{u^3}{3} + 15\frac{u^2}{2}\right]\Bigg|_0^1 dv$$

$$= \frac{200}{3}\underbrace{(-\cos v)}_{-1+1} + \frac{15}{2}v \,\Bigg|_0^{2\pi} = \frac{15}{\cancel{2}}(\cancel{2}\pi) = \boxed{15\pi}$$

5. $\vec{F} = \langle 3x^2y, x^3 + 2yz^2, 2y^2z \rangle$

C = circle given by $\vec{r}(t) = \langle 3\cos t, 4\cos t, 5\sin t \rangle$

$$\operatorname{curl} \vec{F} = \langle \underset{h_y}{4yz} - \underset{g_z}{4yz}, \underset{f_z - h_x}{0 - 0}, \underset{g_x - f_y}{3x^2 - 3x^2} \rangle = \vec{0}$$

$\Rightarrow \vec{F}$ is <u>conservative</u>, so

$$\oint_C \vec{F} \cdot d\vec{r} = 0.$$