Mon 6 July

- MLP homework 4.6-4.7 is due Tuesday (tomorrow) night
- Exam 3 feedback
  - median $= 69/125$ ($\approx 55.2\%$)
  - You may redo problems for up to 90% of points back. With this setup, everyone has the potential to still get an A.
  - You may collaborate and use resources, including office hours.
  - The deadline to submit is 4pm Friday. NO EXCEPTIONS.
  - If you don't redo your problems, the grade sticks.

Mon 6 July (cont.)

- Quiz feedback:
  - Quiz 8 median $= 20/25$
  - Quiz 9 median $= 16/20$
- Expect 2 quizzes this week and one next week.
- Almost done!!! Exam 3's topics covered applications of the derivative (L'Hôpital's Rule is also an application). The remainder of the course is a segue into integration, the primary topic of Cal II.

$\oint$ 4.7 L'Hôpital's Rule

In Ch. 2, we examined limits that were computed using analytical techniques. Some of these limits, in particular those that were indeterminate, could not be computed with simple analytical methods. For example,

$$\lim_{x\to 0} \frac{\sin x}{x} \quad \text{and} \quad \lim_{x\to 0} \frac{1-\cos x}{x}$$

are both limits that can't be computed by substitution, because plugging in 0 for x gives $\frac{0}{0}$.

Theorem (L'Hôpital's Rule $\left(\frac{0}{0}\right)$)

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is $\pm\infty$).

(The rule also applies if $x \to a$ is replaced by $x \to \pm\infty$, $x \to a^+$ or $x \to a^-$.)

Example

Evaluate the following limit:

$$\lim_{x\to -1} \frac{x^4+x^3+2x+2}{x+1}.$$

Solution: By direct substitution, we obtain $\frac{0}{0}$. So we must apply l'Hôpital's Rule (LR) to evaluate the limit:

$$\begin{aligned}
\lim_{x\to -1} \frac{x^4+x^3+2x+2}{x+1} &\overset{\text{LR}}{=} \lim_{x\to -1} \frac{\frac{d}{dx}\left(x^4+x^3+2x+2\right)}{\frac{d}{dx}(x+1)} \\
&= \lim_{x\to -1} \frac{4x^3+3x^2+2}{1} \\
&= -4+3+2 = 1
\end{aligned}$$

Theorem (L'Hôpital's Rule ($\frac{\infty}{\infty}$))

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is $\pm\infty$).

(The rule also applies if $x \to a$ is replaced by $x \to \pm\infty$, $x \to a^+$ or $x \to a^-$.)

Exercise

Evaluate the following limits using l'Hôpital's Rule:

1. $\lim\limits_{x\to\infty} \dfrac{4x^3 - 2x^2 + 6}{\pi x^3 + 4}$
2. $\lim\limits_{x\to 0} \dfrac{\tan 4x}{\tan 7x}$

Other indeterminate limits in the form $0 \cdot \infty$ or $\infty - \infty$ cannot be evaluated directly using l'Hôpital's Rule. For $0 \cdot \infty$ cases, we must rewrite the limit in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. A common technique is to divide by the reciprocal:

$$\lim_{x \to \infty} x^2 \sin\left(\frac{1}{5x^2}\right) = \lim_{x \to \infty} \frac{\sin\left(\frac{1}{5x^2}\right)}{\frac{1}{x^2}}$$

Exercise

Compute $\displaystyle\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$.

For $\infty - \infty$, we can divide by the reciprocal as well as use a change of variables:

Example

Find $\lim_{x\to\infty} x - \sqrt{x^2 + 2x}$.

Solution:

$$\begin{aligned}
\lim_{x\to\infty} x - \sqrt{x^2+2x} &= \lim_{x\to\infty} x - \sqrt{x^2(1+\tfrac{2}{x})} \\
&= \lim_{x\to\infty} x - x\sqrt{1+\tfrac{2}{x}} \\
&= \lim_{x\to\infty} x\left(1-\sqrt{1+\tfrac{2}{x}}\right) \\
&= \lim_{x\to\infty} \frac{1-\sqrt{1+\frac{2}{x}}}{\frac{1}{x}}
\end{aligned}$$

This is now in the form $\frac{0}{0}$, so we can apply l'Hôpital's Rule and evaluate the limit. In this case, it may even help to change variables. Let $t = \frac{1}{x}$:

$$\lim_{x\to\infty} \frac{1 - \sqrt{1 + \frac{2}{x}}}{\frac{1}{x}} = \lim_{t\to 0^+} \frac{1 - \sqrt{1+2t}}{t}.$$

Other Indeterminate Forms

Limits in the form 1^∞, 0^0, and ∞^0 are also considered indeterminate forms, and to use l'Hôpital's Rule, we must rewrite them in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Here's how: Assume $\lim_{x \to a} f(x)^{g(x)}$ has the indeterminate form 1^∞, 0^0, or ∞^0.

1. Evaluate $L = \lim_{x \to a} g(x) \ln f(x)$. This limit can often be put in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, which can be handled by l'Hôpital's Rule.
2. Then $\lim_{x \to a} f(x)^{g(x)} = e^L$. Don't forget this step!

Example

Evaluate $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$.

Solution: This is in the form 1^{∞}, so we need to examine

$$\begin{aligned}
L &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \\
&= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \\
&\overset{\text{LR}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1.
\end{aligned}$$

NOT DONE! Write

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^L = e^1 = e.$$

We can use l'Hôpital's Rule to examine the rate at which functions grow in comparison to one another.

Definition

Suppose f and g are functions with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Then $\boldsymbol{f}$ **grows faster than** $\boldsymbol{g}$ as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \text{ or } \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty.$$

$g \ll f$ means that f grows faster than g as $x \to \infty$.

Definition

The functions f and g have **comparable growth rates** if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M, \text{ where } 0 < M < \infty.$$

Pitfalls in Using l'Hôpital's Rule

1. L'Hôpital's Rule says that $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$.

   NOT $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \left[\dfrac{f(x)}{g(x)}\right]'$ or $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \left[\dfrac{1}{g(x)}\right]' f'(x)$

   (i.e., don't confuse this rule with the Quotient Rule).

2. Be sure that the limit with which you are working is in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Pitfalls in Using l'Hôpital's Rule (cont.)

3. When using l'Hôpital's Rule more than once, simplify as much as possible before repeating the rule.
4. If you continue to use l'Hôpital's Rule in an unending cycle, another method must be used.

4.7 Book Problems

13-39 (odds), 43-51 (odds), 63-69 (odds)

$\oint$ 4.8 Antiderivatives

With differentiation, the goal of problems was to find the function $f'(x)$, given the function $f(x)$.

With antidifferentiation, the goal is the opposite; in this case we wish to find a function F whose derivative is f, i.e.,

$$F'(x) = f(x).$$

Definition

A function F is called an **antiderivative** of a function f on an interval I means

$$F'(x) = f(x) \quad \text{for all } x \text{ in } I.$$

Example

Given $f(x) = 4$, an antiderivative of $f(x)$ is $F(x) = 4x$.

NOTE: Antiderivatives are not unique!

They differ by a constant (C):

Theorem

Let F be any antiderivative of f. Then all the antiderivatives of f have the form $F + C$, where C is an arbitrary constant.

Recall: $\frac{d}{dx} f(x) = f'(x)$ is the derivative of $f(x)$.

Now: $\displaystyle\int f(x)dx = F(x) + C$ is the antiderivative of $f(x)$.

It doesn't matter which F you choose, since writing the C will show you are talking about all the antiderivatives at once. The C is also why we use the term **indefinite integral**.

Example

$$\int 4x^3 \; dx = x^4 + C.$$

The C is called the **constant of integration**. The dx is called the **differential** (and it is the same dx from $\oint$4.5). Like the $\frac{d}{dx}$, it shows which variable we are talking about. The function written between the $\int$ and the dx is called the **integrand**.

Power Rule: $\displaystyle\int x^p dx = \frac{x^{p+1}}{p+1} + C$
(p is any real number except -1)

Constant Multiple Rule: $\displaystyle\int cf(x)dx = c\int f(x)dx$

Sum Rule: $\displaystyle\int (f(x) + g(x))\, dx = \int f(x)dx + \int g(x)dx$

Exercise

Evaluate the following indefinite integrals:

1. $$\int \left(3x^{-2} - 4x^2 + 1\right) dx$$
2. $$\int 6\sqrt[3]{x}\ dx$$

Table 4.5 (in the text) provides us with rules for finding indefinite integrals of trig functions.

1. $\dfrac{d}{dx}(\sin ax) = a\cos ax \quad \longrightarrow \quad \displaystyle\int \cos ax \; dx = \frac{1}{a}\sin ax + C$

2. $\dfrac{d}{dx}(\cos ax) = -a\sin ax \quad \longrightarrow \quad \displaystyle\int \sin ax \; dx = -\frac{1}{a}\cos ax + C$

3. $\dfrac{d}{dx}(\tan ax) = a\sec^2 ax \quad \longrightarrow \quad \displaystyle\int \sec^2 ax \; dx = \frac{1}{a}\tan ax + C$

Indefinite Integrals of Trig Functions (cont.)

$$4.\ \frac{d}{dx}(\cot ax) = -a\csc^2 ax \quad \longrightarrow \quad \int \csc^2 ax\ dx = -\frac{1}{a}\cot ax + C$$

$$5.\ \frac{d}{dx}(\sec ax) = a\sec ax\tan ax \quad \longrightarrow \quad \int \sec ax\tan ax\ dx = \frac{1}{a}\sec ax + C$$

$$6.\ \frac{d}{dx}(\csc ax) = -a\csc ax\cot ax \quad \longrightarrow \quad \int \csc ax\cot ax\ dx = -\frac{1}{a}\csc ax + C$$

Table 4.6 provides us with rules for finding other indefinite integrals.

7. $\frac{d}{dx}(e^{ax}) = ae^{ax} \longrightarrow \int e^{ax}\ dx = \frac{1}{a}e^{ax} + C$

8. $\frac{d}{dx}(\ln|x|) = \frac{1}{x} \longrightarrow \int \frac{dx}{x} = \ln|x| + C$

9. $\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2 - x^2}} \longrightarrow \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

Other Indefinite Integrals (cont.)

$$10.\ \frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{a^2+x^2} \quad \longrightarrow \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$11.\ \frac{d}{dx}\left(\sec^{-1}\left|\frac{x}{a}\right|\right) = \frac{a}{x\sqrt{x^2-a^2}} \longrightarrow \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$

Example

Evaluate the following indefinite integral: $\int 2\sec^2 2x \; dx$.

Solution: Using Rule 3., with $a = 2$, we have

$$\int 2\sec^2 2x \; dx = 2\int \sec^2 2x \; dx = 2\left[\frac{1}{2}\tan 2x\right] + C = \tan 2x + C.$$

Exercise

Evaluate $\int 2\cos(2x) \; dx$.

In some instances, we have enough information to determine the value of C in the antiderivative. These are often called **initial value problems**.

Example

If $f'(x) = 7x^6 - 4x^3 + 12$ and $f(1) = 24$, find $f(x)$.

Solution: $f(x) = \int (7x^6 - 4x^3 + 12)\ dx = x^7 - x^4 + 12x + C$. Now find out which C gives $f(1) = 24$:

$$24 = f(1) = 1 - 1 + 12 + C,$$

so $C = 12$. Hence, $f(x) = x^7 - x^4 + 12x + 12$.

Exercise

Find the function f that satisfies $f''(t) = 6t$ with $f'(0) = 1$ and $f(0) = 2$.

4.8 Book Problems

11-45 (odds), 55-59 (odds), 63, 65

- To solve 55-59 (odds), 63, and 65, look through the section, focusing in on Example 7.