Mon 22 Sep 2014

- Exams returned in drill tomorrow.
- Check first that your points are added correctly. There should be 2 extra points added to your raw score, plus the point given for your signature.
- If you dispute the grading, write down your SPECIFIC concerns over the SPECIFIC question(s) on a separate sheet of paper to submit, along with your test, to your drill instructor.
- Once you leave drill with your exam, you may not ask for points back.

3.2: Rules for Differentiation

If every derivative had to be computed using the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

computing derivatives would be extremely time consuming. Fortunately, using this definition, we can generate a number of rules that will assist us in finding derivatives.

Constant Functions

The constant function $f(x) = c$ is a horizontal line with a slope of 0 at every point.

Looking at a graph of a constant function, we see the instantaneous rate of change is 0 at every point.

Using the definition of a derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$

Therefore, for constant functions, $f'(x) = 0$

Power Rule

Consider the power function of the form $f(x) = x^n$.

Using the definition of a derivative:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a}$$

$$= \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1})}{x - a}$$

$$= (a^{n-1} + a^{n-2}a + \ldots + a \cdot a^{n-2} + a^{n-1}) = na^{n-1}$$

So for power functions, $\frac{d}{dx}(x^n) = nx^{n-1}$

Constant Multiple Rule

Consider the function of the form $cf(x)$.

Using the definition of a derivative:

$$\frac{d}{dx}[cf(x)] = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h}$$

$$= \lim_{h \to 0} \frac{c[f(x+h) - f(x)]}{h} = c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= cf'(x)$$

So if *f* is differentiable at *x*, $\frac{d}{dx}[cf(x)] = cf'(x)$

Sum Rule

Consider the function of the form $F(x) = f(x) + g(x)$.

Using the definition of a derivative:

$$\frac{d}{dx}[f(x)+g(x)] = \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)]}{h}+\frac{g(x+h)-g(x)}{h}\right]$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= f'(x)+g'(x)$$

Sum Rule

So if *f* and *g* are differentiable at *x*,

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$$

The Sum Rule can be generalized for more than two functions to include *n* functions.

Note: Using the Sum Rule and the Constant Multiple Rule produces the Difference Rule:

$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$$

Example

Using the Differentiation Rules we have discussed, calculate the derivatives of the following functions. Note which rule(s) you are using.

1. $y = x^5$
2. $y = 4x^3 - 2x^2$
3. $y = -1500$
4. $y = 3x^2 - 2x + 4$

Derivatives of Exponentials

Consider the exponential functions $y = 2^x$ and $y = 3^x$

If we examine the behavior of any exponential function

$f(x) = b^x$ around x = 0, we see that

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{b^h - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}$$

Note the behavior if b=2 or b=3

Derivatives of Exponentials

Given that *$f'(0) < 1$* when *$b=2$* and *$f'(0)>1$* when *$b=3$*, there must be a *b* such that *$f'(0)=1$.*

$$\lim_{h \to 0} \frac{b^h - 1}{h} = 1$$

The *b* that satisfies this is the natural exponential function $f(x) = e^x$, where *e* = 2.718281…

The exponential function has a tangent line with slope 1 at x=0.

Derivatives of Exponentials

Therefore, $\lim\limits_{h \to 0} \dfrac{e^h - 1}{h} = 1$, which allows us to find the derivative of the exponential function $f(x) = e^x$

$$\frac{d}{dx}(e^x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} = e^x \cdot \lim_{h \to 0} \frac{(e^h - 1)}{h} = e^x \cdot 1 = e^x$$

So $\dfrac{d}{dx}(e^x) = e^x$

Finding slopes of tangent lines

Now that we have a number of differentiation rules, we can use the derivative to find the slope of the line tangent to that curve at a specific point.

Exercise: Find the slope of the line tangent to the curve

$f(x) = x^3 + 2x - 4$ at the point (2, 8).

Where does this curve have a horizontal tangent?

The slope of the line tangent to $y = x(2x^2 + 1)$ at x = 1 is

A. 5

B. 3

C. 7

D. 6

Higher-order Derivatives

Because the derivative of *f* is a function, we can take the derivative of *f'*. This is called the second derivative of f, and denoted *f''*. In general, we can differentiate *f* as often as needed.

The *n*th derivative of *f* is $f^{(n)}(x) = \dfrac{d^n f}{dx^n} = \dfrac{d}{dx}[f^{(n-1)}(x)]$

HW from section 3.2

- Do problems 3-45 multiples of 3 (e.g., 3, 6, 9, 12, 15, …) (pgs. 142-145 in textbook)

NOTE: In this section, we do not yet know the product and quotient rules. Use only those rules we have derived so far.

Wed 24 Sep 2014

- Exam 1: You may rework problems for 1/3 points back. Due Monday during lecture. Solutions posted on Monday after lecture.

- MIDTERM (see syllabus for date/time) location TBA

More Rules for Differentiation

We have developed a number of differentiation rules focused on the use of the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We continue with a few more rules, namely the product rule and the quotient rule.

Recall the Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

and the Difference rule: $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Product Rule

A common error in calculus classes is to assume that the derivative of the product of two functions is just the product of the derivatives. For example, let $f(x) = x^2$ and $g(x) = x^3$. What is $\frac{d}{dx}[f(x) \cdot g(x)]$ if you:

1. Take the product of the individual derivatives? (e.g., $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g'(x)$)
2. Multiply the two functions together, then take the derivative using the power rule?

Product Rule

From the previous example, we see that

$$\frac{d}{dx}[f(x)\cdot g(x)] \neq f'(x)\cdot g'(x)$$

The correct version of the Product Rule is:

$$\frac{d}{dx}[f(x)\cdot g(x)] = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

Derivation of the Product Rule

As with the rules derived in section 3.2, the product rule stems from the definition of the derivative:

$$\frac{d}{dx}[f(x)\cdot\ g(x)] = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) + [-f(x)g(x+h) + f(x)g(x+h)] - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h\to 0}\frac{f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} g(x+h)\frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} f(x)\frac{g(x+h) - g(x)}{h}$$

$$= g(x)f'(x) + f(x)g'(x)$$

Exercise

Use the product rule to find the derivative of the function $(x^2+3x)(2x-1)$

Quotient Rule

Just as with the product rule, the derivative of the quotient of two functions is not the quotient of the derivative of each function (e.g., $\frac{d}{dx}[\frac{f(x)}{g(x)}] \neq \frac{f'(x)}{g'(x)}$)

Consider the function $q(x) = \frac{f(x)}{g(x)}$, which translates to the product $f(x) = q(x) \cdot g(x)$

Using the product rule, we have $f'(x) = q'(x) \cdot g(x) + q(x) \cdot g'(x)$

which when solved for q'(x): $q'(x) = \frac{f'(x) - q(x) \cdot g'(x)}{g(x)}$

Exercise

Use the quotient rule to find the derivative of

$$\frac{4x^3 + 2x - 3}{x + 1}$$

Use of the Product and Quotient Rules

The Product and Quotient Rules can be used to find the derivative of increasingly complex functions and enhance rules we have already derived.

EX: Find the slope of the line tangent to the curve $f(x) = \dfrac{2x-3}{x+1}$ at the point (4,1).

The Quotient Rule also allows us to extend the Power Rule to negative numbers:

If *n* is any integer, then $\dfrac{d}{dx}\left[x^n\right] = nx^{n-1}$

Derivative of ekx

The product rule allows us to find the derivative of ekx.

$$\frac{d}{dx}(e^{kx}) = ke^{kx}$$

Exercise: What is the derivative of $x^2 e^{3x}$?

Rates of Change

The derivative provides information about the instantaneous rate of change of the function being differentiated.

For example, suppose that the population of a culture can be modeled by the function $p(t) = \dfrac{800}{1 + 7e^{-0.2t}}$.

We can find the instantaneous growth rate of the population at any time $t \geq 0$ as well as the steady-state population.

Combining Derivative Rules

Some situations call for using more than one differentiation rules:

EX: If $f(x)=\dfrac{x(3-x)}{2x^2}$, find f'(x)

HW from section 3.3

- Do problems 6-51 multiples of 3 (pgs. 152-154 in textbook)

Fri 26 Sep 2014

- Reminder: Stay on top of Webwork!
- Exam 1: You may rework problems for 1/3 points back. Due Monday during lecture. Solutions posted on Monday after lecture.
- Tuesday: Quiz(?) in drill. Study Ch. 3 materials up to today (section 3.1-3.5)
- Exam 2 is AFTER the midterm (compare to other Cal I sections) ;)

Wednesday Oct 15: Midterm Exam 6:30-8p, Location TBD

Derivatives of Trig Functions

Trig functions are commonly used to model cyclic or periodic behavior in everyday settings. Therefore it is important to know how these functions change across time.

Before we proceed with the limits of the six trig functions, we need two important limits that will allow us to show this.

Trig Limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

Evaluating trig limits

Exercise: Evaluate $\lim_{x \to 0} \frac{\sin 9x}{x}$ and $\lim_{x \to 0} \frac{\sin 9x}{\sin 5x}$

Derivatives of sin and cos

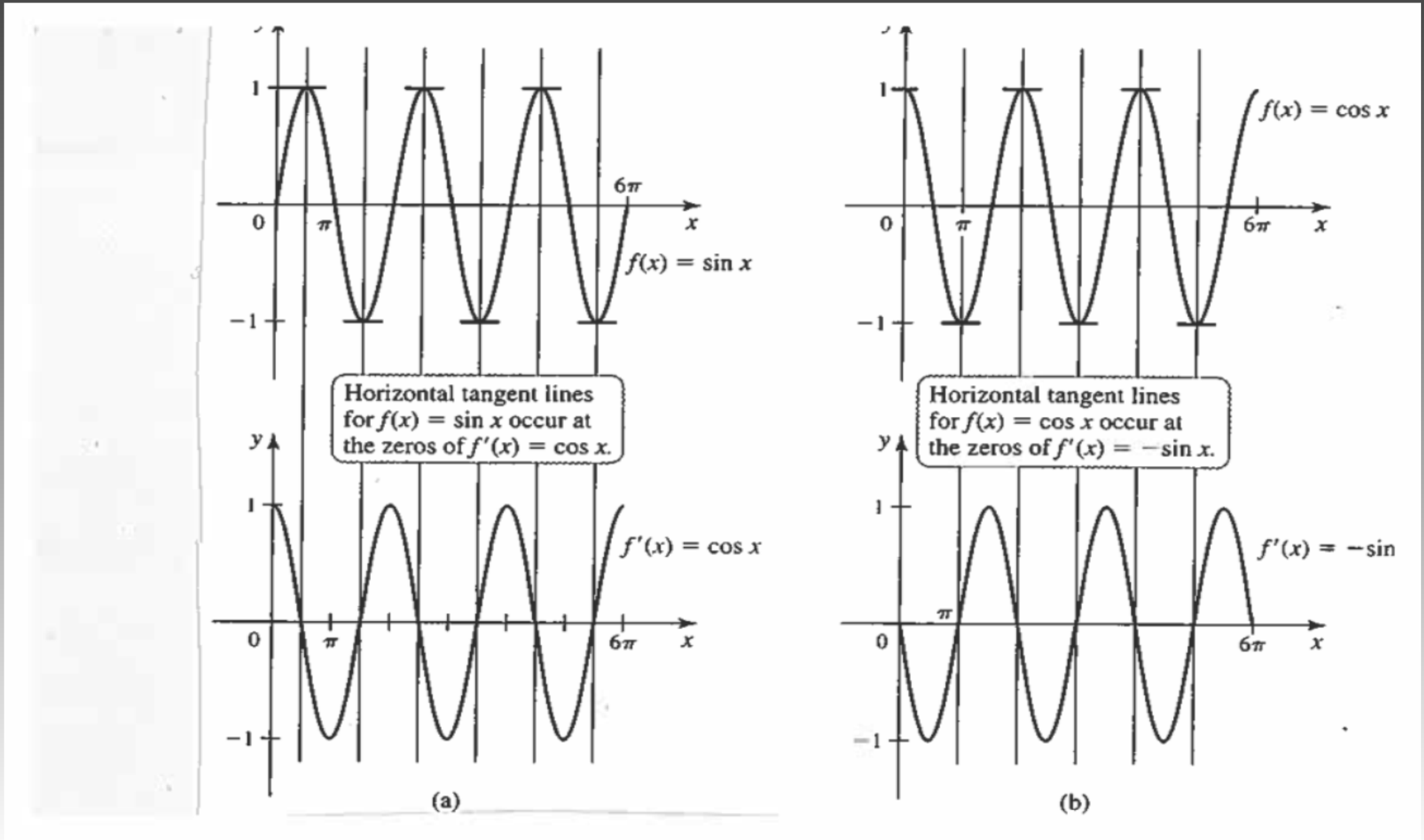
Using the previous limits and the definition of the derivative, we now gain the derivative of sine and cosine:

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

Derivatives of sine and cosine

Examining the graphs of sine and cosine illustrate the relationship between the functions and their derivatives.



Higher Order Trig Derivatives

Just as we have previously discussed, because the derivative is a function itself, *f(x)* can be differentiated as many times as needed, even with trig functions. However, there is a cyclic relationship between the higher order derivatives of sin and cos.

$$f(x) = \sin x \qquad g(x) = \cos x$$

$$f'(x) = \cos x \qquad g'(x) = -\sin x$$

$$f''(x) = -\sin x \qquad g''(x) = -\cos x$$

$$f'''(x) = -\cos x \qquad g'''(x) = \sin x$$

$$f''''(x) = \sin x \qquad g''''(x) = \cos x$$

Use the difference and product rules to find the derivative of the function $y = \cos x - x \sin x$

A. $-\sin x + x\cos x$

B. $x\cos x$

C. $-2\sin x - x\cos x$

D. $x\cos x - 2\sin x$

Derivatives of other Trig functions

Knowing the derivatives of sin x and cos x, we can use these to find the derivatives of the other 4 trig functions.

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

So $\frac{d}{dx}(\tan x) = \sec^2 x$

Trig Identities You Should Know

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$\cot x = \frac{1}{\tan x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\csc x = \frac{1}{\sin x}$$

Exercise

Compute the derivative of the following functions:

$$f(x) = \frac{\tan x}{1 + \tan x}$$

$$g(x) = \sin x \cos x$$

Higher-order trig derivatives

Just as we have previously discussed, because the derivative is a function itself, *f(x)* can be differentiated as many times as needed, even with trig functions. However, there is a cyclic relationship between the higher order derivatives of sin and cos.

$$f(x) = \sin x$$
$$f'(x) = \cos x$$
$$f''(x) = -\sin x$$
$$f'''(x) = -\cos x$$
$$f''''(x) = \sin x$$

$$g(x) = \cos x$$
$$g'(x) = -\sin x$$
$$g''(x) = -\cos x$$
$$g'''(x) = \sin x$$
$$g''''(x) = \cos x$$

HW from section 3.4

- Do problems 7, 13, 17, 21-27, 33, 35, 44-46, 53-55 (pgs. 161-162 in textbook)

Derivatives as Rates of Change

Up to now in Chapter 3 we have seen a number of methods and rules to compute derivatives. We now look at the question "When would we ever need this?"

Today we look at four areas where the derivative assists us with determining the rate of change in various contexts.

Position and Velocity

Recall from section 2.1 the definitions of average velocity and instantaneous velocity. In section 2.1, we often were given position functions *s = f(t)*, where the position *s* of an object was measured at any time *t*.

In 2.1, we measured the distance away from a given point in terms of *a* and *a+h*. Now we look at the displacement of the object between $t = a$ and $t = a + \Delta t$ is

$$\Delta s = f(a + \Delta t) - f(a)$$

Here Δt represents how much time has elapsed.

Speed and Acceleration

Continuing to use the position function $s = f(t)$, we can find the speed and acceleration of the object as well.

Question: How are speed and velocity related?

A rock is dropped off a bridge and its distance *s* (in feet) from the bridge after *t* seconds is $s(t) = 16t^2 + 4t$. The velocity of the rock at *t = 2* is ______ and the acceleration at *t = 2* is ________.

A. 64 ft/s; 16 ft/s2

B. 68 ft/s; 32 ft/s2

C. 64 ft/s; 32 ft/s2

D. 68 ft/s; 16 ft/s2

Growth Models

Suppose *p = f(t)* is a function of the growth of some quantity of interest (e.g., population, prices, etc.). The average growth rate of *p* between times *t = a* and a later time *t = a + Δt* is the change in *p* divided by the elapsed time *Δt*. So:

$$\frac{\Delta p}{\Delta t} = \frac{f(a+\Delta t) - f(a)}{\Delta t}$$

Average and Marginal Cost

The final example stems from the world of business.

Suppose a company produces a large amount of a particular quantity. Associated with manufacturing the quantity is a **cost function** *C(x)* that gives the cost of manufacturing *x* items. This cost may include a **fixed cost** to get started as well as a **unit cost** (or **variable cost**) in producing one item.

HW from section 3.5

- Do problems 9-12, 17-18, 22-23, 27-37 odd (pgs. 171-175 in textbook)