$\oint$ 5.5 Substitution Rule

Idea: Suppose we have $F(g(x))$, where F is an antiderivative of f. Then

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$$

and $\displaystyle\int f(g(x)) \cdot g'(x)\, dx = F(g(x)) + C$

Let $u = g(x)$, where g' is continuous on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

u-Substitution is the Chain Rule, backwards.

Example

Evaluate $\displaystyle\int 8x\cos(4x^2+3)\;dx.$

Solution: Look for a function whose derivative also appears.

$$u(x) = 4x^2 + 3$$

$$\text{and } u'(x) = \frac{du}{dx} = 8x$$

$$\Longrightarrow \; du = 8x\;dx.$$

Now rewrite the integral and evaluate. Replace u at the end with its expression in terms of x.

$$\begin{aligned}
\int 8x\cos(4x^2+3)\ dx &= \int \cos(\underbrace{4x^2+3}_{u})\ \underbrace{8x\ dx}_{du} \\
&= \int \cos u\ du \\
&= \sin u + C \\
&= \sin(4x^2+3) + C
\end{aligned}$$

We can check the answer – by the Chain Rule,

$$\frac{d}{dx}\left(\sin\left(4x^2+3\right)+C\right) = 8x\cos\left(4x^2+3\right).$$

Procedure for Substitution Rule (Change of Variables)

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\ dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

Warning: Not all integrals yield to the Substitution Rule.

Exercise

Evaluate the following integrals. Check your work by differentiating each of your answers.

1. $\displaystyle\int \sin^{10} x \cos x \; dx$
2. $\displaystyle -\int \frac{\csc x \cot x}{1 + \csc x} \; dx$
3. $\displaystyle\int \frac{1}{(10x-3)^2} \; dx$
4. $\displaystyle\int (3x^2 + 8x + 5)^8 (3x+4) \; dx$

There are times when the u-substitution is not obvious or that more work must be done.

Example

Evaluate $\displaystyle\int \frac{x^2}{(x+1)^4}\,dx.$

Solution: Let $u = x+1$. Then $x = u-1$ and $du = dx$. Hence,

$$\begin{aligned}\int \frac{x^2}{(x+1)^4}\,dx &= \int \frac{(u-1)^2}{u^4}\,du \\ &= \int \frac{u^2-2u+1}{u^4}\,du\end{aligned}$$

$$= \int \left(u^{-2} - 2u^{-3} + u^{-4}\right)\ du$$
$$= \frac{-1}{u} + \frac{1}{u^2} + \frac{-1}{3u^3} + C$$
$$= \frac{-1}{x+1} + \frac{1}{(x+1)^2} - \frac{1}{3(x+1)^3} + C$$

Exercise

Check it.

This type of strategy works, usually, on problems where u can be written as a linear function of x.

Substitution Rule for Definite Integrals

We can use the Substitution Rule for Definite Integrals in two different ways:

1. Use the Substitution Rule to find an antiderivative F, and then use the Fundamental Theorem of Calculus to evaluate $F(b) - F(a)$.
2. Alternatively, once you have changed variables from x to u, you may also change the limits of integration and complete the integration with respect to u. Specifically, if $u = g(x)$, the lower limit $x = a$ is replaced by $u = g(a)$ and the upper limit $x = b$ is replaced by $u = g(b)$.

The second option is typically more efficient and should be used whenever possible.

Example

Evaluate $\displaystyle\int_0^4 \frac{x}{\sqrt{9+x^2}}\,dx$.

Solution: Let $u = 9 + x^2$. Then $du = 2x\,dx$. Because we have changed the variable of integration from x to u, the limits of integration must also be expressed in terms of u. Recall, u is a function of x (the $g(x)$ in the Chain Rule). For this example,

$$\begin{aligned} x = 0 \quad &\Longrightarrow \quad u(0) = 9 + 0^2 = 9 \\ x = 4 \quad &\Longrightarrow \quad u(4) = 9 + 4^2 = 25 \end{aligned}$$

We had $u = 9 + x^2$ and $du = 2x\ dx \implies \frac{1}{2}du = x\ dx$. So:

$$\int_0^4 \frac{x}{\sqrt{9+x^2}}\ dx = \frac{1}{2}\int_9^{25} \frac{du}{\sqrt{u}}$$

$$= \frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right)\Bigg|_9^{25}$$

$$= \sqrt{25} - \sqrt{9}$$

$$= 5 - 3 = 2.$$

Exercise

Evaluate $\displaystyle\int_0^2 \frac{2x}{(x^2+1)^2}\, dx.$

5.5 Book Problems

9-39 (odds), 53-63 (odds)

Advice for the FINAL

- Review your notes and the slides first, particularly problems we did in class, then review Quizzes, before visiting outside resources.
- Review the Midterm for an idea of the type questions the coordinator likes to ask and how they are graded.
- $+C$s, dxs, $\lim$, units, etc. should be included in your answers *or else*. Don't try to round answers unless it is for a story problem, in which case, you should say "approximately".

Advice for the FINAL (cont.)

- Practice limits and l'Hôpital's Rule so you know which is the quickest technique.
- "Mean Value Theorem for Derivatives" $=$ MVT from $\oint$4.6.
- $\arctan = \tan^{-1}$, etc.
- Know the difference between 1st and 2nd Derivative Tests. Also, only plug numbers into the number line for the 1st Derivative Test.

Final (Exam #4) Review

- $\oint$3.10 Related Rates
  - Know the steps to solving related rates problems, and be able to use them to solve problems given variables and rates of change.
  - Be able to solve related rates problems. If, while doing the HW (paper or computer), you were provided a formula in order to solve the problem, then I will do the same. If you were not provided a formula while doing the HW (paper or computer), then I also will not provide the formula.

Final (Exam #4) Review (cont.)

Exercise

An inverted conical water tank with a height of 12 ft and a radius of 6 ft is drained through a hole in the vertex at a rate of 2 ft^3/sec. What is the rate of change of the water depth when the water depth is 3 ft?

Final (Exam #4) Review (cont.)

- $\oint$4.1 Maxima and Minima
  - Know the definitions of maxima, minima, and what makes these points local or absolute extrema (both analytically and graphically).
  - Know how to find critical points for a function.
  - Given a function on a given interval, be able to find local and/or absolute extrema.
  - Given specified properties of a function, be able to sketch a graph of that function.

Final (Exam #4) Review (cont.)

- $\oint$4.2 What Derivatives Tell Us
  - Be able to use the first derivative to determine where a function is increasing or decreasing.
  - Be able to use the First Derivative Test to identify local maxima and minima. Be able to explain in words how you arrived at your conclusion.
  - Be able to find critical points, absolute extrema, and inflection points for a function.
  - Be able to use the second derivative to determine the concavity of a function.
  - Be able to use the Second Derivative Test to determine whether a given point is a local max or min. Be able to explain in words how you arrived at your conclusion.
  - Know your Derivative Properties!!! (see Figure 4.36 on p. 242)

Final (Exam #4) Review (cont.)

- $\oint$4.3 Graphing Functions
  - Be able to find specific characteristics of a function that are spelled out in the Graphing Guidelines on p. 248 (e.g., know how to find x- and y-intercepts, vertical/horizontal asymptotes, critical points, inflection points, intervals of concavity and increasing/decreasing, etc.).
  - Be able to use these specific characteristics of a function to sketch a graph of the function.
  - TIP: If you are looking for intervals for increasing/decreasing or concave up/down, you should *treat* the naughty points as critical points and/or possible points of inflection.

Final (Exam #4) Review (cont.)

- $\oint$4.4 Optimization Problems
  - Be able to solve optimization problems that maximize or minimize a given quantity.
  - Be able to identify and express the constraints and objective function in an optimization problem.
  - Be able to determine your interval of interest in an optimization problem (e.g., what range of x-values are you searching for your extreme points?)
  - **As to formulas, the same comment made above with respect to formulas for related rates problems applies here as well.**

Final (Exam #4) Review (cont.)

Exercise

What two nonnegative real numbers a and b whose sum is 23 will

(a) minimize $a^2 + b^2$?

(b) maximzie $a^2 + b^2$?

Final (Exam #4) Review (cont.)

- $\oint$4.5 Linear Approximation and Differentials
  - Be able to find a linear approximation for a given function.
  - Be able to use a linear approximation to estimate the value of a function at a given point.
  - Be able to use differentials to express how the change in x (dx) impacts the change in y (dy).

Final (Exam #4) Review (cont.)

- $\oint$4.6 Mean Value Theorem (for Derivatives)
  - Know and be able to state Rolle's Thm and the Mean Value Thm, including knowing the hypotheses and conclusions for both.
  - Be able to apply Rolle's Thm to find a point in a given interval.
  - Be able to apply the MVT to find a point in a given interval.
  - Be able to use the MVT to find equations of secant and tangent lines.

Final (Exam #4) Review (cont.)

Exercise (s)

Determine whether the Mean Value Theorem (or Rolle's Theorem) applies to the following functions. If it does, then find the point(s) guaranteed by the theorem to exist.

(1) $f(x) = \sin(2x)$ on $\left[0, \frac{\pi}{2}\right]$

(2) $g(x) = \ln(2x)$ on $[1, e]$

(3) $h(x) = 1 - |x|$ on $[-1, 1]$

Final (Exam #4) Review (cont.)

Exercise (s)

(4) $j(x) = x + \frac{1}{x}$ on $[1, 3]$

(5) $k(x) = \frac{x}{x+2}$ on $[-1, 2]$

Final (Exam #4) Review (cont.)

- $\oint$4.7 L'Hôpital's Rule
  - Know how to use L'Hôpital's Rule, including knowing under what conditions the Rule works.
  - Be able to apply L'Hôpital's Rule to a variety of limits that are in indeterminate forms (e.g., 0/0, ∞/∞, $0 \cdot \infty$, $\infty - \infty$, 1^∞, 0^0, ∞^0).
  - Be able to use L'Hôpital's Rule to determine the growth rates of two given functions.
  - Be aware of the pitfalls in using L'Hôpital's Rule.
  - PRACTICE THESE. Some of the book problems have non-obvious algebra tricks that simplify an otherwise crazy problem.

Final (Exam #4) Review (cont.)

- $\oint$4.8 Antiderivatives
  - Know the definition of an antiderivative and be able to find one or all antiderivatives of a function.
  - Be able to evaluate indefinite integrals, including using known properties of indefinite integrals (i.e., Power Rule, Constant Multiple Rule, Sum Rule).
  - Know how to find indefinite integrals of the six trig functions, of e^{ax}, of $\ln x$, and of the three inverse trig functions listed in the notes.
  - Be able to solve initial value problems to find specific antiderivatives.
  - Be able to use antiderivatives to work with motion problems.

Final (Exam #4) Review (cont.)

- $\oint$5.1 Approximating Areas under Curves
  - Be able to use rectangles to approximate area under the curve for a given function.
  - Know how to calculate left Riemann sums, right Riemann sums, and midpoint Riemann sums for a function.
  - Be able to sum a series of numbers written in sigma notation. You need to know these common sums:

$$\sum_{k=1}^{n} c = cn \quad \text{and} \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

  - Be able to identify whether a given Riemann sum written in sigma notation is a left, right, or midpoint sum.

Final (Exam #4) Review (cont.)

- $\oint$5.2 Definite Integrals
  - Be able to compute left, right, or midpoint Riemann sums for curves that have negative components, and understand the concept of net area.
  - Be able to evaluate a definite integral using geometry or a given graph.
  - Know the properties of definite integrals and be able to use them to evaluate a definite integral.

Final (Exam #4) Review (cont.)

Exercise

Suppose

$$\int_1^4 f(x)\ dx = 8 \qquad \text{and} \qquad \int_1^6 f(x)\ dx = 5.$$

Evaluate the following integrals:

(a) $\displaystyle\int_1^4 (-3f(x))\ dx$

(b) $\displaystyle\int_6^4 12f(x)\ dx$

(c) $\displaystyle\int_4^6 (f(x) + 3x)\ dx$

Final (Exam #4) Review (cont.)

- $\oint$5.3 Fundamental Theorem of Calculus
  - Understand the concept of an area function, and be able to evaluate an area function as x changes.
  - Know the two parts of the Fundamental Theorem of Calculus and its significance (i.e., the inverse relationship between differentiation and integration).
  - Use the FTC to evaluate definite integrals or simplify given expressions.

Final (Exam #4) Review (cont.)

Exercise

Evaluate each:

(a) $\displaystyle\int_0^{\ln 8} e^x \, dx$

(b) $\displaystyle\frac{d}{dx}\int_x^0 \frac{dp}{p^2+1}$

(c) the net area of the region bounded between the x-axis and the function $f(x) = x(x-2)(x-4)$

(d) $\displaystyle\frac{d}{dy}\int_2^{y^3} (t^2+t+1) \, dt$

Final (Exam #4) Review (cont.)

- $\oint$5.4 Working with Integrals
  - Be able to integrate even and odd functions knowing the "shortcuts" provided by these functions' characteristics.
  - Be able to find the average value of a function.
  - Know the Mean Value Theorem for Integrals and be able to use it to find points associated with the average value of a function.

Final (Exam #4) Review (cont.)

Exercise

Find the point(s) at which the function

$$f(x) = 1 - |x|$$

equals its average value on the interval $[-1, 1]$. Then draw the picture of $f(x)$, labelling the points and the average value you computed.

Final (Exam #4) Review (cont.)

- $\oint$5.5 Substitution Rule
  - Definite integrals.
  - Indefinite integrals.
  - Change of variables.

Exercise (s)

Evaluate, using substitution:

1. $\displaystyle\int \frac{y}{\sqrt{y-4}}\,dy$

Final (Exam #4) Review (cont.)

Exercise (s)

2. $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\ dx$
3. $\displaystyle\int_0^1 2x(4 - x^2)\ dx$
4. $\displaystyle\int_1^{e^2} \frac{\ln x}{x}\ dx$

Tips for Studying Efficiently and Effectively

- Given today's lists of materials you should know for the exam, if you see a topic you don't know then go back to the slides covering that topic first.
- Review slides for days you missed.
- Redo the quizzes until you can get a perfect score without looking at the key.
- Book problems. Do those problems with the same attention and care you put into Exam #3.
- If you spent 10 hours on Exam #3 then spend at least that much time studying for the final.

Exercise (s)

1. Find the 101st derivative of $y = \cos 7x$ at $x = 0$.
2. For what values of the constants a and b is $(-1, 2)$ a point of inflection on the curve $y = ax^3 + bx^2 - 8x + 2$?