Topology Solutions - 4 Jan 2010
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Morning Session

1. Let $f : K \to T$ be a covering where K is a compact connected oriented 2-manifold and T is the torus. Prove that K is homeomorphic to the torus.

   Solution. Since K is compact, the degree d of f must be finite. The relationship between Euler characteristics is $\chi(K) = d\chi(T) = 0$. By classification of compact connected oriented 2-manifolds (K does not have boundary becasue T doesn't), K must be homeomorphic to the torus. $\square$

2. Prove that the sphere S^n has a nowhere vanishing vector field if and only if n is odd.

   Solution. Suppose a nonvanishing (normalized, without loss of generality) vector field $v : S^n \to S^n$ exists, i.e., v has no fixed points. S^n is compact and orientable and v is smooth, by definition, so, applying the Lefschetz Fixed-Point Theorem, v has no fixed points implies the Lefschetz number $L(v) = 0$. Defining $h : [0,1] \times v(S^n) \to S^n$ by

$$h\left(t, v(x)\right) = \frac{(1-t)v(x) + tx}{\|(1-t)v(x) + tx\|}$$

   gives a homotopy of v to the identity. Therefore $L(v) = \chi(S^n)$, the Euler characteristic of S^n. But $S^n = 2$ when n is even. To see this, consider the ranks of the homology groups,

$$\operatorname{rank} H_k(S^n) = \begin{matrix} 1 & \text{if } k = 0, n \\ 0 & \text{otherwise.} \end{matrix}$$

   Then

$$\begin{aligned} \chi(S^n) &= \sum_{k=0}^{\infty} (-1)^k \operatorname{rank} H_k(S^n) \\ &= \begin{matrix} 0 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{matrix} \end{aligned}$$

   Conclude n cannot be even.

   Conversely, suppose n is odd. Then define $v : S^n \subset \mathbb{R}^{n+1} \to S^n$ by $(x_1, \ldots, x_n) \mapsto (x_2, -x_1, x_4, -x_3, \ldots, -x_n)$. This is nonzero everywhere, apparently smooth, and

$$\begin{aligned} (x_1, \ldots, x_n) \cdot (x_2, \ldots, -x_n) &= x_1x_2 - x_1x_2 + \cdots - x_nx_{n-1} \\ &= 0 \end{aligned}$$

   means it is a vector field. $\square$

3. Suppose that X is a normal topological space, and E, F are countable union of closed subsets of X and $\overline{E} \cap F = E \cap \overline{F} = \emptyset$. (The bar denotes the closure in X). Prove that E, F have disjoint open neighborhoods in X.

Solution. By normality of X, for each n there exists an open set U_n such that

$$\bigcup_{k=1}^{n} E_k \subset U_n \subset \overline{E}.$$

Then $U = \cup_n U_n$ is open, and

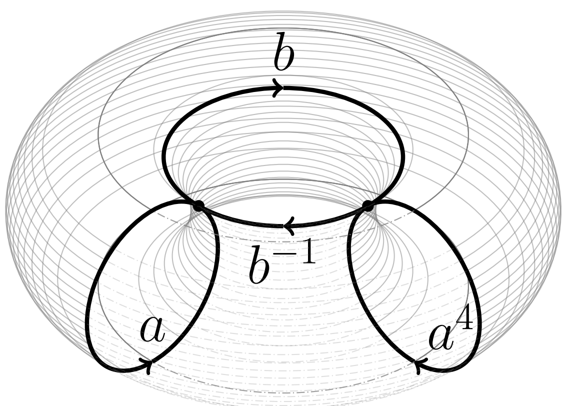
$$E \subset U \subset \overline{E} \cap F = \emptyset.$$

By symmetry, constructing an open set V such that

$$F \subset V \subset \overline{F} \cap E = \emptyset$$

gives the desired disjoint open neighborhoods, U, V. $\square$

4. Let $S^1 = \{z \in \mathbb{C} \,|\, \|z\| = 1\}$. Let X be obtained from the space $S^1 \times [0,1]$ by identifying every point $(z,0)$ with the point $(z^4,1)$ (with the quotient topology). Compute the fundamental group and homology of X.



Solution. Let a denote the loop corresponding to the circle $S^1 \times \{0\}$, and set the origin as the base point, x_0. As the cylinder is homotopy equivalent to the circle, $\pi_1(S^1 \times [0,1]) \cong \mathbb{Z}$. Consider $S^1 \times [0,1]$ with an edge joining x_0 to $(1,0)$. Note the other boundary $S^1 \times \{1\}$ corresponds to a^4. The result is homotopy equivalent to a wedge of two circles; let b denote the second generator, and the fundamental group is $\mathbb{Z} * \mathbb{Z}$. Finally, to get X, attach an open rectangle to the two boundaries and the edge joining them. This adds the relation $a = ba^4b^{-1}$. Therefore,

$$\pi_1 X \cong \frac{a\mathbb{Z} * b\mathbb{Z}}{\langle a^{-1}ba^4b^{-1} \rangle}.$$

For the homology groups, X is compact and orientable so

$$H_2(X) \cong \mathbb{Z}$$

and in the reduced homology

$$\widetilde{H}_0(X) = 0$$

(so $H_0(X) \cong \mathbb{Z}$). Let σ denote the rectangle $[0,1] \times [0,1]$ used in constructing X. Then the boundary map gives $\partial\sigma = 3a$. Therefore,

$$H_1(X) \cong \frac{a\mathbb{Z} \oplus b\mathbb{Z}}{\langle 3a \rangle}.$$

□

5. Suppose that $f : M \to \mathbb{R}$ is a smooth function. Recall that $x \in M$ is a critical point if $df(x) : T_xM \to \mathbb{R}$ is zero, x is a nondegenerate critical point if the Hessian $H_x = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ has nonzero determinant for any coordinate system $x_1, \ldots, x_n$, and f is a Morse function if every critical point is nondegenerate. Let S^n be the unit sphere of $\mathbb{R}^{n+1}$ and $f = x_{n+1} : S^n \to \mathbb{R}$ be the "height function". Prove that f is a Morse function.

   Solution. Write

$$f : (x_1, \cdots, x_n) \mapsto x_{n+1} := \pm\sqrt{1 - \sum_{k=1}^{n} x_k^2}.$$

   Then

$$df(x_1, \cdots, x_n) = \begin{pmatrix} \frac{\pm x_1}{\sqrt{1-\sum_{k=1}^n x_k^2}} \\ \vdots \\ \frac{\pm x_n}{\sqrt{1-\sum_{k=1}^n x_k^2}} \end{pmatrix}$$

   is zero if and only if $x_1 = \cdots x_n = 0$. Also,

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \begin{matrix} \frac{\mp x_j}{\left(1-\sum_{k=1}^n x_k^2\right)^{-3/2}} \pm \frac{1}{\sqrt{1-\sum_{k=1}^n x_k^2}} & i = j \\ 0 & i \neq j \end{matrix}.$$

   At $x_1 = \cdots = x_n = 0$, this is a matrix with zeros off the diagonal and 1 or -1 entries on the diagonal. □

Afternoon Session

1. Prove that (a) the space of $n \times n$ matrices $M_{n\times n}$ is a smooth manifold. (b) the determinant function $det : M_{n\times n} \to \mathbb{R}$ is a Morse function iff $n = 1, 2$.

   Solution. (a) By total ordering the matrix entries, $M_{n\times n}$ is $\mathbb{R}^{n^2}$.

2. Let D^2 be a closed 2-disk, S^1 its boundary. Is the space $X = S^1 \times S^1$ a retract of the space $Y = D^2 \times S^1$, i.e., does there exist a continuous map $Y \to X$ which is the identity on $X \subset Y$?

   Solution. Suppose such a map f exists. Note $\dim X = 2$, $\dim Y = 3$, and $\partial Y = X$, where ∂ denotes the boundary. Since f is smooth Sard's theorem guarantees a regular value $x \in X$ for f. Furthermore, x is a regular value for $f|_X$, the identity on X. So $f^{-1}(x)$ is a smooth (3-2)-manifold with boundary $f^{-1}(x) \cap \partial Y$. And because $f^{-1}(x)$ is compact, it must be a disjoint union of circles and line segments. But that implies $\partial f^{-1}(x)$ consists of an even number of points.

$$\begin{aligned} \partial f^{-1}(x) &= f^{-1}(x) \cap \partial Y \\ &= f^{-1}(x) \cap X \\ &= \{x\} \end{aligned}$$

is a contradiction. Therefore no such map f exists. $\Box$.

3. Suppose that X is a first countable, Hausdorff space and A is a subset of X which intersects each compact subset of X in a set closed in X. Prove that A is closed.

   Solution. Suppose x is a limit point of A. A countable basis $\{\mathcal{B}_n\}_n$ for x gives a sequence $S = \{x_n\}_n$, with $x_n \in \mathcal{B}_n$ for all n, which converges to x. Then $S \cup \{x\}$ is compact. To see this, suppose $\mathcal{U} = \{U_m\}_m$ is an open cover. If $x \in U_m$ for some m, then there exists N such that $x_n \in U_m$ for all $n \geq N$. Finitely many points of S remain, so apparently have a finite subcover of $\mathcal{U}$, which combined with U_m give the desired finite subcover for $S \cup \{x\}$.

   By hypothesis, A intersects $S \cup \{x\}$ in a closed set, C. Since X is Hausdorff and $S \cup \{x\}$ is compact, $S \cup \{x\}$ is closed.

4. Let $f : M \to N$ be a smooth map between compact manifolds M, N with $\dim M = \dim N$ and N is path connected. Let $z \in N$ be a regular value. We define the $\mod (2)$ degree, $deg_2(f, z)$, to be the ($\mod (2)$) number of points of $f^{-1}(z)$. Prove that $deg_2(f, z)$ is independent of z.

   Solution.

5. Give an example of a (path-connected) covering space which is not a regular covering space.

   Solution.