Exer-slides from Wed 15 July 2015:

$\boxed{\S 4.4}$

- What two non-negative real numbers a and b whose sum is 23 will
  (a) minimize a^2+b^2?
  (b) maximize a^2+b^2?

$\boxed{\text{Solution}}$

(a) Objective: Minimize

$$A = a^2+b^2$$

Constraint: $a+b=23 \Rightarrow a=23-b$.

$a, b \geq 0$

Interval of Interest:

$0 \leq b \leq 23$

Rewrite:

$$A(b) = (23-b)^2 + b^2$$

$$= 23^2 - 2(23)b + b^2 + b^2$$

$$= 23^2 - 2(23)b + 2b^2$$

$$A'(b) = -2(23) + 4b = 0$$

$\Rightarrow b = \frac{23}{2} = 11.5$ is the critical point.

$$A(0) = (23-0)^2 + 0^2 = 23^2$$

$$A\left(\frac{23}{2}\right) = \left(23 - \frac{23}{2}\right)^2 + \left(\frac{23}{2}\right)^2 = 2\left(\frac{23}{2}\right)^2 = \frac{23^2}{2}$$

$\longrightarrow$

$A(23) = (23-23)^2 + 23^2 = 23^2$.

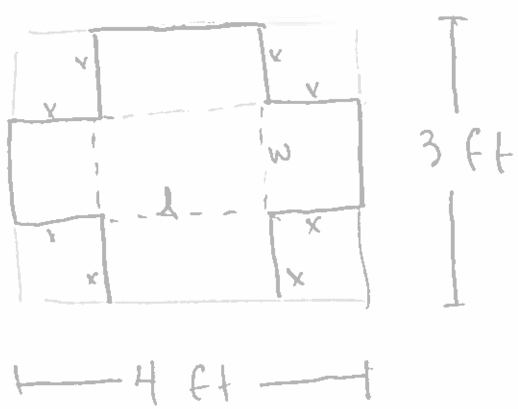
The min is attained when $\boxed{b = a = \frac{23}{2} = 11.5}$.

(b) The max is attained when

$\boxed{b=0,\ a=23 \quad \text{or} \quad b=23,\ a=0}$.

• Squares with sides of length x are cut out of each corner of a 3ft × 4ft cardboard rectangle. The resulting piece of cardboard is then folded into a box without a lid. Find the volume of the largest box that can be formed in this way.

Solution.



Objective: Maximize Volume

$V = lwh$, where

$l = 4-2x$

$w = 3-2x$

$h = x$.

(constraints are given in the picture.)

$V = (4-2x)(3-2x)x$

$= (12 - 6x - 8x + 4x^2)x = 12x - 14x^2 + 4x^3$

$$V'(x) = 12 - 28x + 12x^2 = 0$$

$$4(3 - 7x + 3x^2) = 0$$

$$\Rightarrow x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(3)}}{2(3)}$$

$$= \frac{7 \pm \sqrt{13}}{6} \leftarrow \text{critical points}$$

<u>Interval of Interest:</u>

$0 \leq x \leq \frac{3}{2}$ (see picture)

Only $\frac{7-\sqrt{13}}{6} \approx 0.56$ is in the interval.

$$V(0) = (4 - 2(0))(3 - 2(0))(0) = 0$$

$$V\left(\frac{7-\sqrt{13}}{6}\right) = \overset{\text{pos.}}{\left(4 - 2\left(\frac{7-\sqrt{13}}{6}\right)\right)} \overset{\text{pos.}}{\left(3 - 2\left(\frac{7-\sqrt{13}}{6}\right)\right)} \overset{\text{pos.}}{\left(\frac{7-\sqrt{13}}{6}\right)} > 0$$

$$V\left(\frac{3}{2}\right) = \left(4 - 2\left(\frac{3}{2}\right)\right)\left(3 - 2\left(\frac{3}{2}\right)\right)\left(\frac{3}{2}\right) = 0$$

So the maximum volume is

$$V\left(\frac{7-\sqrt{13}}{6}\right) \boxed{\approx 3.03 \text{ ft}^3}$$

$\longrightarrow$

$\boxed{§4.6}$

Determine whether the Mean Value Theorem (or Rolle's Theorem) applies to the following functions. If it does, then find the point(s) guaranteed by the theorem to exist.

(1) $f(x) = \sin(2x)$ on $[0, \frac{\pi}{2}]$

- continuous on $[0, \frac{\pi}{2}]$ ✓
- Smooth on $(0, \frac{\pi}{2})$ ✓
- $f(0) = \sin(2(0)) = 0$
  $f(\frac{\pi}{2}) = \sin(2(\frac{\pi}{2})) = 0$ / Rolle's Theorem applies.

Find "c":

$f'(c) = 2\cos(2c) = 0$

$\cos(2c) = 0$

$\Rightarrow 2c = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$c = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}, \ldots$

in the interval: $\boxed{c = \frac{\pi}{4}}$

$\longrightarrow$

(2) $g(x) = \ln(2x)$ on $[1, e]$

- continuous on $[1, e]$ ✓
- smooth on $(1, e)$ ✓

MVT applies

- $g(1) = \ln(2(1)) = \ln 2$

  $g(e) = \ln(2(e)) = \ln 2 + 1$

<u>Slope of secant line</u>:

$$\frac{g(e) - g(1)}{e - 1} = \frac{\ln 2 + 1 - \ln 2}{e - 1} = \frac{1}{e-1}$$

<u>Find "c"</u>:

$$g'(c) = \frac{1}{2c} \cdot 2 = \frac{1}{c} = \frac{1}{e-1}$$

$$\Rightarrow \boxed{c = e - 1} \approx 1.72$$

(in the interval)

(3) $h(x) = 1 - |x|$ on $[-1, 1]$

- Continuous on $[-1, 1]$ ✓
- smooth on $(-1, 1)$ ✗

  b/c $h'(0)$ is undefined.

$\boxed{\text{MVT does not apply.}}$

(4) $j(x) = x + \frac{1}{x}$ on $[1,3]$

- continuous on $[1,3]$ ✓
- smooth on $(1,3)$ ✓

} MVT applies

- $j(1) = 1 + \frac{1}{1} = 2$

  $j(3) = 3 + \frac{1}{3}$

Slope of secant line:

$$\frac{j(3) - j(1)}{3-1} = \frac{3 + \frac{1}{3} - 2}{2} = \frac{1 + \frac{1}{3}}{2} = \frac{\frac{4}{3}}{2} = \frac{4}{6} = \frac{2}{3}$$

Find "c":

$$j'(c) = 1 + \left(\frac{-1}{c^2}\right) = \frac{2}{3}$$

$$1 - \frac{2}{3} = \frac{1}{c^2}$$

$$c^2 = 3$$

$$c = \pm\sqrt{3}$$

neg. is not in the interval

$$\Rightarrow \boxed{c = \sqrt{3}}$$

⟶

(5) $k(x) = \dfrac{x}{x+2}$ on $[-1,2]$

- Continuous on $[-1,2]$ ✓
- Smooth on $(-1,2)$ ✓

MVT applies

$$k(-1) = \frac{-1}{-1+2} = -1$$

$$k(2) = \frac{2}{2+2} = \frac{1}{2}$$

Slope of secant line:

$$\frac{k(2)-k(-1)}{2-(-1)} = \frac{\frac{1}{2}-(-1)}{3} = \frac{\frac{3}{2}}{3} = \frac{1}{2}$$

Find "c":

$$k'(c) = \frac{(c+2)(1)-c(1)}{(c+2)^2} = \frac{2}{(c+2)^2} = \frac{1}{2}$$

$$4 = (c+2)^2$$

$$2 = |c+2|$$

$c+2=2$ $\quad$ $-c-2=2$

$\boxed{\Rightarrow c=0}$ $\quad$ $c=-4$

$\longrightarrow$

$\boxed{\S 5.2}$

Suppose $\int_1^4 f(x)\,dx = 8$ and $\int_1^6 f(x)\,dx = 5$.

Evaluate:

(a) $\int_1^4 (-3f(x))\,dx = -3\int_1^4 f(x)\,dx = -3(8)\boxed{=-24}$

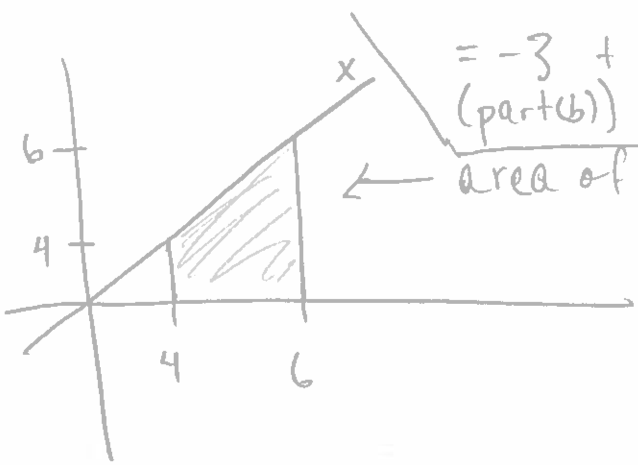
(b) $\int_6^4 12f(x)\,dx = -\int_4^6 12f(x)\,dx$

$= -12\int_4^6 f(x)\,dx = -12\left(\int_1^6 f(x)\,dx - \int_1^4 f(x)\,dx\right)$

$= -12(5-8)\boxed{=36}$

(c) $\int_4^6 (f(x)+3x)\,dx = \int_4^6 f(x)\,dx + 3\int_4^6 x\,dx$

$= \underset{\text{(part (b))}}{-3} + \underset{\text{(see picture)}}{3(10)} = -3+30\boxed{=27}$



area of trapezoid

$= \frac{2(4+6)}{2} = 10$

$\longrightarrow$

§5.3

(a) $\displaystyle\int_0^{\ln 8} e^x\,dx = e^x \Big|_0^{\ln 8} = e^{\ln 8} - e^0 = 8-1 = \boxed{7}$

(b) $\displaystyle\frac{d}{dx}\int_x^0 \frac{dp}{p^2+1} = -\frac{d}{dx}\int_0^x \frac{dp}{p^2+1} = -\frac{d}{dx}\int_0^x \frac{1}{p^2+1}\,dp = \boxed{\frac{-1}{x^2+1}}$

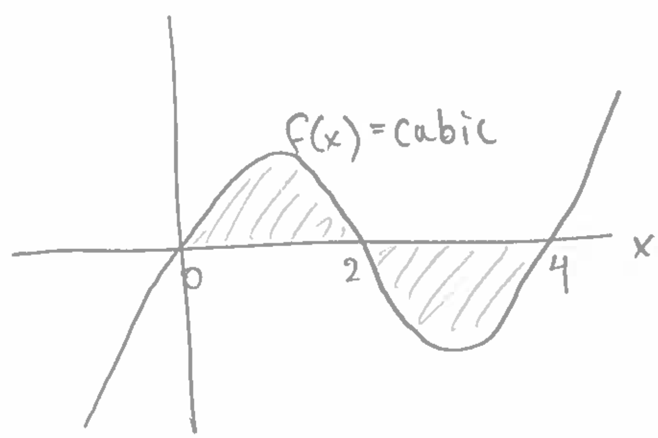
(c) net area of the region bounded between $f(x) = x(x-2)(x-4)$ and the x-axis

Solution

The bounds on the definite integral will be the zeros of $f(x)$: $x(x-2)(x-4)=0$

$\Rightarrow x = 0, 2, 4$

$\displaystyle\int_0^4 x(x-2)(x-4)\,dx$



$\displaystyle= \int_0^4 x(x^2-6x+8)\,dx$

$\displaystyle= \int_0^4 (x^3-6x^2+8x)\,dx$

$\displaystyle= \frac{x^4}{4} - 2x^3 + 4x^2 \Big|_0^4 = \frac{4^4}{4} - 2(4^3) + 4(4^2) = 4^3 - 2(4^3) + 4^3 = \boxed{0}$

OR (alternate Solution)

Let $u = x-2 \Rightarrow x = u+2$ (and $du = dx$)

Then $f(x) = x(x-2)(x-4)$

$= (u+2)(u)(u-2)$;

the zeros are $u = 0, \pm 2$ so the integral

becomes $\displaystyle\int_{-2}^{2} (u+2)(u)(u-2)\,du$

$= \displaystyle\int_{-2}^{2} (u^3 - 4u)\,du = \boxed{0}$

odd function

(d) $\displaystyle\frac{d}{dy}\int_2^{y^3} (t^2+t+1)\,dt = \left((y^3)^2 + y^3 + 1\right)\cdot 3y^2$

$= \boxed{3y^8 + 3y^5 + 3y^2}$

$\longrightarrow$

$\boxed{\S\ 5.4}$

Find the point(s) at which the function

$$f(x) = 1 - |x|$$

equals its average value on the interval $[-1, 1]$. Then draw a picture of $f(x)$, labelling the points and the average value you computed.

Solution.

$$\bar{f} = \frac{1}{1-(-1)} \int_{-1}^{1} (1 - |x|)\, dx = \frac{1}{\cancel{2}} \cdot \cancel{2} \int_0^1 (1-x)\, dx = x - \frac{x^2}{2} \Big|_0^1$$

even function

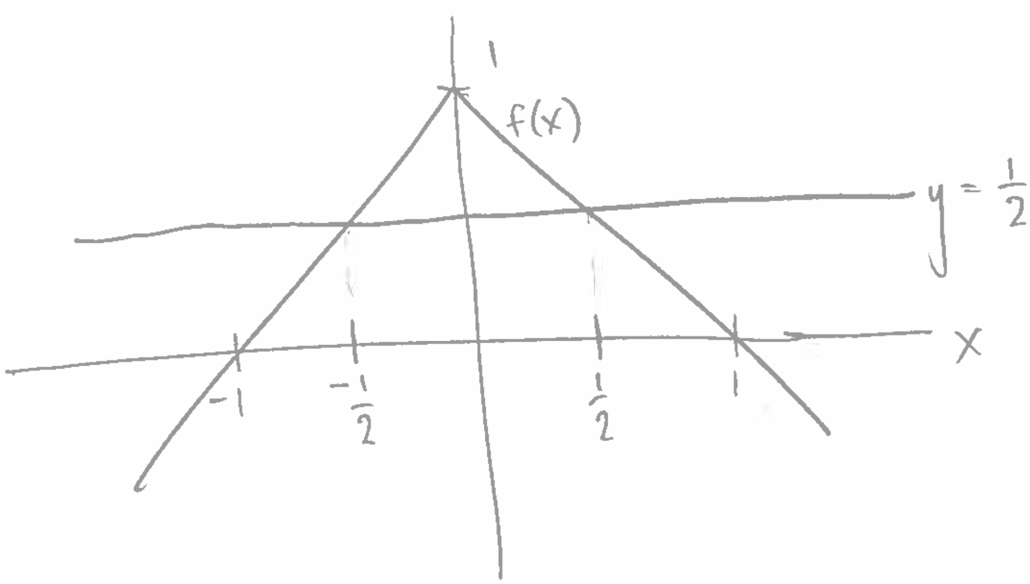
$$= 1 - \frac{1^2}{2} - \left(0 - \frac{0^2}{2}\right)$$

$$= \frac{1}{2}$$

Find "c":

$$f(c) = 1 - |c| = \frac{1}{2}$$

$$\frac{1}{2} = |c| \Rightarrow \boxed{c = \pm \frac{1}{2}}$$



$\boxed{\S 5.5}$

1. $\displaystyle\int \frac{y}{\sqrt{y-4}}\,dy \qquad u = y-4 \rightarrow y = u+4$
$\qquad du = dy$

$$= \int \frac{u+4}{\sqrt{u}}\,du = \int \left(u^{1/2} + 4u^{-1/2}\right) du$$

$$= \frac{u^{3/2}}{\frac{3}{2}} + 4\,\frac{u^{1/2}}{\frac{1}{2}} + C$$

$$\boxed{= \frac{2}{3}(y-4)^{3/2} + 8(y-4)^{1/2} + C}$$

2. $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx \qquad u = e^x + e^{-x}$
$\qquad du = (e^x - e^{-x})\,dx$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$\boxed{= \ln\left|e^x + e^{-x}\right| + C}$$

3. $\int_0^1 2x(4-x^2)\,dx$ $\quad u = 4-x^2 \quad \Rightarrow u(0) = 4-0^2 = 4$

$du = -2x\,dx \quad u(1) = 4-1^2 = 3$

$-du = 2x\,dx$

$$= -\int_4^3 u\,du = -\frac{u^2}{2}\Big|_4^3 = -\left(\frac{3^2}{2} - \frac{4^2}{2}\right)$$

$$\boxed{= \frac{7}{2}}$$

OR (alternate solution)

$$\int_0^1 2x(4-x^2)\,dx = \int_0^1 (8x - 2x^3)\,dx$$

$$= \frac{8x^2}{2} - \frac{2x^4}{4}\Big|_0^1 = 4(1^2) - \frac{1}{2}(1^4) - 0$$

$$= 4 - \frac{1}{2} \boxed{= \frac{7}{2}}$$

4. $\int_1^{e^2} \frac{\ln x}{x}\,dx$ $\quad u = \ln x \quad \Rightarrow u(1) = \ln(1) = 0$

$du = \frac{1}{x}\,dx \quad u(e^2) = \ln(e^2) = 2$

$$= \int_0^2 u\,du = \frac{u^2}{2}\Big|_0^2 = \frac{2^{\cancel{2}}}{\cancel{2}} - \frac{0^2}{2} \boxed{= 2}$$

Easter Egg-xercises

1. Find the 101^{st} derivative of $y=\cos(7x)$ at $x=0$.

Solution

$y' = -7\sin 7x$

$y'' = -49\cos 7x$

$y''' = (-7)(-49)\sin 7x$

$y^{(4)} = 7^4 \cos 7x$

$y^{(100)} = 7^{100}\cos 7x$

$y^{(101)} = -7^{101}\sin 7x$

$y^{(101)}(0) = -7^{101}\sin(7(0))$

$\boxed{=0}$

2. For what values of the constants a and b is $(-1,2)$ a point of inflection on the curve
$y = ax^3 + bx^2 - 8x + 2$?

Solution.

Set up a system of equations:

$2 = a(-1)^3 + b(-1)^2 - 8(-1) + 2$

$0 = -a + b + 8$

$\Rightarrow a = b + 8$

$y' = 3ax^2 + 2bx - 8$

$y'' = 6ax + 2b$

$y''(-1) = 6a(-1) + 2b = 0$

$-6(b+8) + 2b = 0$

$-6b - 48 + 2b = 0$

$-4b = 48$

$\Rightarrow b = -12$

$\boxed{a = b + 8 = -12 + 8 = -4}$

Check $x=-1$ is an inflection point:

$y'' = 6(-4)x + 2(-12) > 0$

$-24x - 24 > 0$

$-24 > 24x$

$\Rightarrow y$ is concave up when $x < -1$

and concave down when $x > -1$.

3. $\int_u^v (\cos t) g'(\sin t)\,dt$

$w = \sin t$

$dw = \cos t\,dt$

$\Rightarrow w(u) = \sin u$

$w(v) = \sin v$

$= \int_{\sin u}^{\sin v} g'(w)\,dw$

$\boxed{= g(\sin v) - g(\sin u)}$