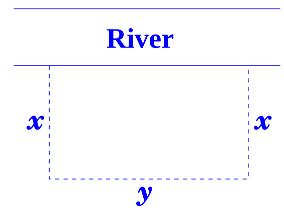
Section 4.4 – Optimization, Geometry, and Modeling

1. A farmer wants to fence a rectangular grazing area along a straight river (no fence is needed along the river). There are 1700 total feet of fencing available. What dimensions (length and width) will maximize the grazing area?

   Let x and y represent the dimensions of the rectangular pen, in feet, and let A represent its area.

   <u>Given</u>: $2x + y = 1700$

   <u>Find</u>: x and y that maximize A

   Since the pen is rectangular, we know that $A = xy$, and from our given information, we see that $y = 1700 - 2x$. Therefore,

   $$A = x(1700 - 2x) = 1700x - 2x^2.$$

   Next, we find the critical points of A; we have $A' = 1700 - 4x$, so

   $$\begin{aligned} 1700 - 4x &= 0 \\ 4x &= 1700 \\ x &= 425 \end{aligned}$$

   Therefore, $x = 425$ is the only critical point. Also, since 1700 feet is the maximum amount of fencing available, the largest value that x can have is $1700/2 = 850$. Therefore, the endpoints of our domain are $x = 0$ and $x = 850$. The table to the right confirms that the maximum value of A occurs when $x = 425$. Therefore, our final answer is $x = 425$ feet and $y = 1700 - 2(425) = 850$ feet.

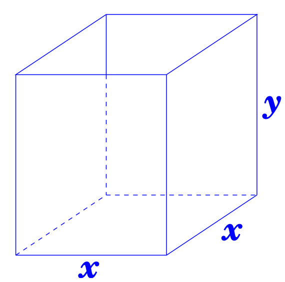
x	A
0	0
425	361,250
850	0

2. A box with an open top of fixed volume V with a square base is to be constructed. Find the dimensions of the box that minimize the amount of material used in its construction.

   Let x represent the length and width of the box, let y represent the height of the box, and let V represent the volume of the box (see diagram below).

   <u>Given</u>: $V = x^2 y$

   <u>Find</u>: x and y that minimize S,
   the surface area of the box.

   We begin by finding a formula for S, the surface area of the box. Since

   $$\begin{aligned} S &= (\text{Sum of the areas of the 4 sides}) \quad + \quad (\text{Area of the bottom}) \\ &= 4xy + x^2 \\ &= 4x(Vx^{-2}) + x^2 \\ &= 4Vx^{-1} + x^2, \end{aligned}$$

   we have $S' = -4Vx^{-2} + 2x = (2x^3 - 4V)/x^2$, so the critical points of S occur when

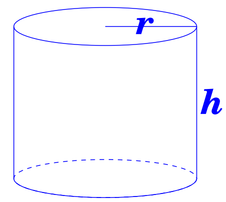
   $$\begin{aligned} 2x^3 - 4V &= 0 \\ x^3 &= 2V \\ x &= \sqrt[3]{2V}. \end{aligned}$$

Interval	Sign of S'
$0 < x < \sqrt[3]{2V}$	$-$
$x > \sqrt[3]{2V}$	$+$

Therefore, since the above sign chart confirms that S is decreasing for all x values to the left of $\sqrt[3]{2V}$ and increasing for all x values to the right of $\sqrt[3]{2V}$, we conclude that S has a global minimum at $x = \sqrt[3]{2V}$. In other words, the dimensions of the box that will minimize the amount of material used in the construction are $x = \sqrt[3]{2V}$ and $y = Vx^{-2} = V(2V)^{-2/3} = \sqrt[3]{V/4}$.

3. A metal can manufacturer needs to build cylindrical cans with volume 300 cubic centimeters. The material for the side of a can costs 0.03 cents per cm^2, and the material for the bottom and top of the can costs 0.06 cents per cm^2. What is the cost of the least expensive can that can be built?

Let r represent the radius of the can, let h represent the height of the can, and let C represent the cost of building the can, in cents.



Given: $\pi r^2 h = 300$

Find: Minimum Value of C

We begin by finding a formula for the cost of building the can. We have

$$\begin{aligned}
C &= \text{(Cost of the top and bottom)} \quad + \quad \text{(Cost of the outside)} \\
&= \text{(Area of Top and Bottom)(Cost Per Unit Area)} \quad + \quad \text{(Area of Outside)(Cost Per Unit Area)} \\
&= (2\pi r^2)(0.06) + (2\pi r h)(0.03) \\
&= 0.12\pi r^2 + 0.06\pi r\left(\frac{300}{\pi r^2}\right) \\
&= 0.12\pi r^2 + 18r^{-1},
\end{aligned}$$

so $C' = 0.24\pi r - 18r^{-2} = (0.24\pi r^3 - 18)/r^2$, which means that $C' = 0$ when $0.24\pi r^3 = 18$, or when $r = \sqrt[3]{75/\pi}$. Since the sign chart to the right confirms that C has a global minimum at $r = \sqrt[3]{75/\pi}$, we conclude that the cost of the least expensive can is given by

Interval	Sign of C'
$0 < r < \sqrt[3]{75/\pi}$	$-$
$r > \sqrt[3]{75/\pi}$	$+$

$$\begin{aligned}
C &= 0.12\pi(\sqrt[3]{75/\pi})^2 + 18(\sqrt[3]{75/\pi})^{-1} \\
&= \left(\frac{75}{\pi}\right)^{-1/3}\left(0.12\pi \cdot \frac{75}{\pi} + 18\right) \\
&= 27\sqrt[3]{\frac{\pi}{75}},
\end{aligned}$$

or about 9.38 cents.