Defining equations for matroid varieties – using the Grassmann-Cayley algebra

Virtual Inspiring Talks Series

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Mount Holyoke College

Fall 2020

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Thank-you for the invitation to speak!

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About me:

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About me:

- Biracial, 2nd generation college student

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About the project:

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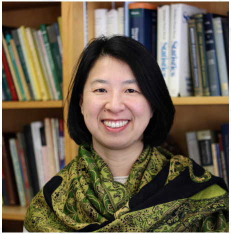
- Joint w/ Jessica Sidman (MHC) and Will Traves (USNA).



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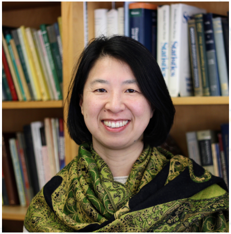

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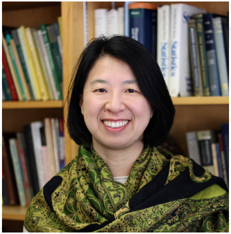


- Paper accepted to *Journal of Combinatorial Theory, Series A* this year (and we've already got a citation!).
- Goal: To find defining equations for matroid varieties.

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Outline:

- Front matter
- Introduction: What is a matroid?
- Problem: Defining equations for matroid varieties
- Paradigm shift: Matroids as point configurations
- Tool: Grassmann-Cayley algebra
- Back matter

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Advice: Refer to Ravi Vakil's *3 Things* when viewing this (or any other) math talk!

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Introduction

What is a matroid?

Q: Which collections of 3 columns form a basis for the column space of A?

$$A = \begin{pmatrix} -1 & 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 0 & -2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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How can we tell?

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Could row reduce A first.

$$\begin{pmatrix} -1 & 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 0 & -2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \to \operatorname{rref} A = \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

background from *Euclidea*

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Preferred way: Look for non-zero **3-minors** of A (or of rref A).

Fact: Columns of an $r \times r$ matrix are linearly independent if and only if the determinant of the matrix is non-zero.

Number the columns of A 1 through 5. Here are the 3-minors, according to their column indices:

$$\begin{array}{ccc}
\{1,2,3\} \to 0 & \{1,2,4\} \to -6 & \{1,2,5\} \to -12 \\
\{1,3,4\} \to -9 & \{1,3,5\} \to -18 & \{1,4,5\} \to -9 \\
\{2,3,4\} \to 3 & \{2,3,5\} \to -6 & \{2,4,5\} \to -11 \\
\{3,4,5\} \to 0 & &
\end{array}$$

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The **matroid** $\mathcal{M}_A$ on A is given by the set:

$\mathcal{B} = \{$all 3-tuples except $\{1,2,3\}$ and $\{3,4,5\}\} \subset \{1,\ldots,5\}$

The sets in $\mathcal{B}$ are called **bases** for the matroid $\mathcal{M}_A$.

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Matroids are a generalization of the collections of linearly independent columns of a matrix.

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Applications: Where are matroids used?

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- *Combinatorial optimization:* artificial intelligence, machine learning, software engineering; example: travelling salesman problem

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- *Coding theory:* error correcting, data compression, cryptography

background from *Euclidea*

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Applications: Where are matroids used?

- *Combinatorial optimization:* artificial intelligence, machine learning, software engineering; example: travelling salesman problem
- *Coding theory:* error correcting, data compression, cryptography
- *Network theory:* particle physics, biology, social networks; example: bridges of Königsberg problem

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Problem

Defining equations for matroid varieties

Q: Given a matroid on a matrix A, what other matrices have the same matroid as A? (e.g., rref A)

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Defining equations for matroid varieties

Q: Given a matroid on a matrix A, what other matrices have the same matroid as A? (e.g., rref A)

Consider this **generic matrix**:

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \end{pmatrix}$$

We want all possible x-values that will give the same matroid (i.e. same columns as bases) as A.

Defining equations are a system of equations in the x's whose solution set minimally cuts out all of the matrices with matroid $\mathcal{M}_A$.

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Defining equations are a system of equations in the x's whose solution set minimally cuts out all of the matrices with matroid $\mathcal{M}_A$.

Call this solution set $\mathcal{V}_A$, the **matroid variety** on A.

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Mnëv (1985), Sturmfels (1989), Knutson, Lam, & Speyer (2013): **All "hell" breaks loose when we try to find defining equations for $\mathcal{V}_A$.**

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Why?

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background from *Euclidea*

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Why? Two reasons:

- The *Zariski topology* (beyond the scope of this talk).

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Why? Two reasons:

- The *Zariski topology* (beyond the scope of this talk).
- The equations are not obvious (as we shall see).

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Paradigm shift

Matroids as point configurations

Idea: Think of the columns of A as points in the plane.

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The columns of the matrix A are vectors in 3-space, that span lines through the origin.

If we cut these lines with a plane then the lines become points in the plane.

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Idea: Think of the columns of A as points in the plane.

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Why would we do this?

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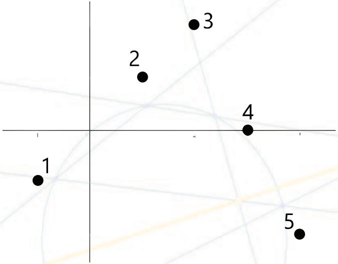

Q: What do you notice about the points in relation to the matroid $\mathcal{M}_A$?

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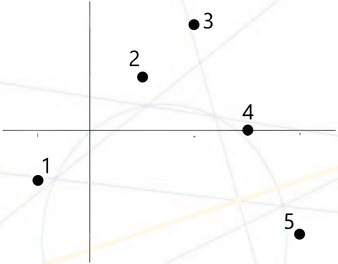

Q: What do you notice about the points in relation to the matroid $\mathcal{M}_A$?

Points 1, 2, and 3 are collinear if and only if $\{1, 2, 3\}$ is not a basis for $\mathcal{M}_A$. We say the **bracket** $[123]$ vanishes, or equals 0.

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The bracket is shorthand for determinant:

$$[123] = 0 \iff \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = 0$$

This is a defining equation in the x's!

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Are there others?

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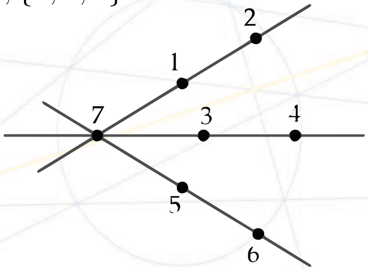
Are there others? ???

background from *Euclidea*

Example (Ford 2013):

$\mathcal{M}_{\text{pencil}}$ = matroid on a 3×7 matrix with *non*bases $\{1,2,7\},\{3,4,7\},\{5,6,7\}$



Q: Which brackets vanish?

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All hell breaks loose: $[134][256] - [234][156] = 0$ is also a defining equation for $\mathcal{V}_{\text{pencil}}$.

background from *Euclidea*

All hell breaks loose: $[134][256] - [234][156] = 0$ is also a defining equation for $\mathcal{V}_{\text{pencil}}$. How could we have known this?!

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Ans: The **Grassmann-Cayley algebra**.

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Tool

Grassmann-Cayley algebra

The Grassmann-Cayley algebra was invented to do *synthetic projective geometry*.

We do arithmetic on the points (vectors) themselves, and obtain expressions in the brackets.

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Tool

Grassmann-Cayley algebra

The Grassmann-Cayley algebra was invented to do *synthetic projective geometry*.

We do arithmetic on the points (vectors) themselves, and obtain expressions in the brackets.

There are two operations:

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join ($\vee$): refers to the line passing through points

The line joining 1 and 2 is $1 \vee 2$, or 12.

background from *Euclidea*

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The line joining 1 and 2 is $1 \vee 2$, or 12.

Three points joined makes a bracket, and three collinear points make the bracket vanish.

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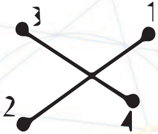
meet ($\wedge$): refers to the intersection of two lines



The meet of the lines $1 \vee 2$ and $3 \vee 4$ is $(1 \vee 2) \wedge (3 \vee 4)$, or $12 \wedge 34$.

background from *Euclidea*

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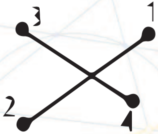

The meet of the lines $1 \vee 2$ and $3 \vee 4$ is $(1 \vee 2) \wedge (3 \vee 4)$, or $12 \wedge 34$.

The meet operation uses *shuffle products*, which also produce expressions in the brackets.

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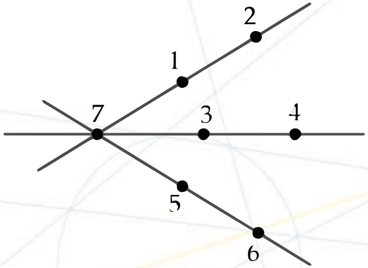

We have $((1 \vee 2) \wedge (3 \vee 4)) \vee 5 \vee 6 = 0$, because these three points are collinear.

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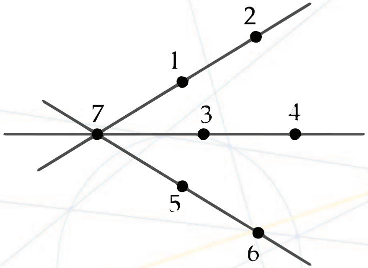

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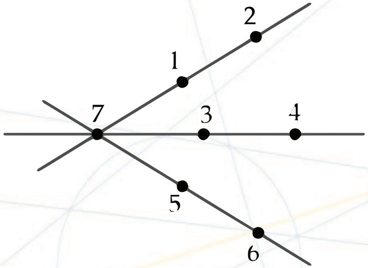
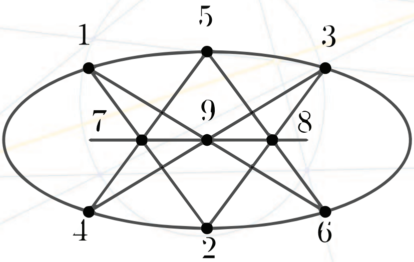


We have $((1 \vee 2) \wedge (3 \vee 4)) \vee 5 \vee 6 = 0$, because these three points are collinear. Here's how to apply the shuffles:

$$\begin{aligned}(12 \wedge 34) \vee 56 &= ([134]2 - [234]1)56 \\ &= [134][256] - [234][156] = 0\end{aligned}$$

background from *Euclidea*

Example: **Pascal's theorem** says if six points on a conic are joined in the way illustrated below, then the resulting intersection points (7, 8, and 9) are collinear.



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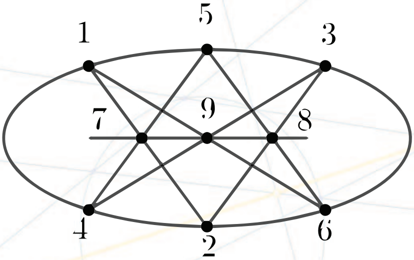

We have a matroid $\mathcal{M}_{\mathsf{Pascal}}$ corresponding to this configuration of points.

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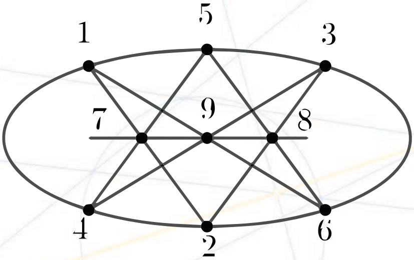

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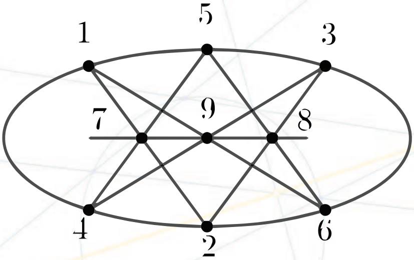

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Q: Which brackets vanish?

background from *Euclidea*

Theorem (Sidman, Traves, W)

The defining equations for the matroid variety $\mathcal{V}_{Pascal}$ include at least one quartic, three independent cubics, and three independent quadrics in the brackets, besides the vanishing brackets $[127], [238], [349], [457], [568], [169], [789]$.

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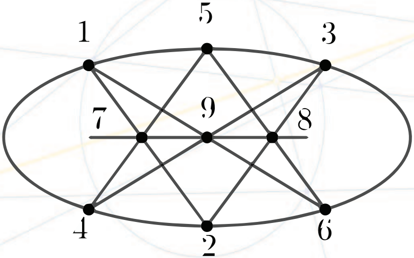
Finding the quartic: The statement of Pascal's theorem says we have $(12 \wedge 45) \vee (23 \wedge 56) \vee (34 \wedge 61) = 0$.



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Apply the shuffle products:

$$
\begin{aligned}
&(12 \wedge 45) \vee (23 \wedge 56) \vee (34 \wedge 61) \\
&= ([145]2 - [245]1) \vee ([256]3 - [356]2) \vee ([361]4 - [461]3)
\end{aligned}
$$

background from *Euclidea*

Now “foil”:

$$
\begin{aligned}
&(([145]2 - [245]1) \vee ([256]3 - [356]2)) \vee ([361]4 - [461]3) \\
&= ([145][256]23 - [145][356]22 - [245][256]13 + [245][356]12) \\
&\qquad \vee ([361]4 - [461]3)
\end{aligned}
$$

background from *Euclidea*

Finish "foiling":

$$
\begin{aligned}
&([145][256]23 - [145][356]22 - [245][256]13 + [245][356]12) \\
&\quad \vee ([361]4 - [461]3) \\
&= [145][256][361][234] - [145][256][461][233] \\
&\quad - [145][356][361][224] + [145][356][461][223] \\
&\quad - [245][256][361][134] + [245][256][461][133] \\
&\quad + [245][356][361][124] - [245][356][461][123]
\end{aligned}
$$

background from *Euclidea*

Finish "foiling":

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&([145][256]23 - [145][356]22 - [245][256]13 + [245][356]12) \\
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&\quad - [245][256][361][134] + [245][256][461][133] \\
&\quad + [245][356][361][124] - [245][356][461][123]
\end{aligned}
$$

Q: What happens when two numbers share a bracket?

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$$[233] \leftrightarrow \begin{vmatrix} x_{12} & x_{13} & x_{13} \\ x_{22} & x_{23} & x_{23} \\ x_{32} & x_{33} & x_{33} \end{vmatrix}$$

Q: What happens to a matrix with two repeated columns?

$$[233] \leftrightarrow \begin{vmatrix} x_{12} & x_{13} & x_{13} \\ x_{22} & x_{23} & x_{23} \\ x_{32} & x_{33} & x_{33} \end{vmatrix}$$

Q: What happens to a matrix with two repeated columns?

It means we get a bunch of cancelling...

$$
\begin{aligned}
&[145][256][361][234] - \cancel{[145][256][461][233]} \\
&\quad - \cancel{[145][356][361][224]} + \cancel{[145][356][461][223]} \\
&\quad - [245][256][361][134] + \cancel{[245][256][461][133]} \\
&\quad + [245][356][361][124] - [245][356][461][123] \\
&= [145][256][361][234] - [245][256][361][134] \\
&\quad + [245][356][361][124] - [245][356][461][123] = 0
\end{aligned}
$$

background from *Euclidea*

$$
\begin{aligned}
&[145][256][361][234] - \cancel{[145][256][461][233]} \\
&\quad - \cancel{[145][356][361][224]} + \cancel{[145][356][461][223]} \\
&\quad - [245][256][361][134] + \cancel{[245][256][461][133]} \\
&\quad + [245][356][361][124] - [245][356][461][123] \\
&= [145][256][361][234] - [245][256][361][134] \\
&\quad + [245][356][361][124] - [245][356][461][123] = 0
\end{aligned}
$$

The statement of Pascal's theorem gives us a non-obvious defining equation for $\mathcal{V}_{\text{Pascal}}$!

background from *Euclidea*

Conclusion:

background from *Euclidea*

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- Finding defining equations for matroid varieties is HARD.

background from *Euclidea*

Conclusion:

- Finding defining equations for matroid varieties is HARD.
- The Grassmann-Cayley algebra lets us find *some* of the defining equations, by using the geometry of the matroids when thought of as point configurations.

background from *Euclidea*

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Further work:

background from *Euclidea*

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Further work:

- Are our equations the only defining equations for $\mathcal{V}_{\mathrm{Pascal}}$?

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- Defining equations for matroid varieties given by other point configurations (e.g., Pappus' theorem)?

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Further work:

- Are our equations the only defining equations for $\mathcal{V}_{\text{Pascal}}$?
- Defining equations for matroid varieties given by other point configurations (e.g., Pappus' theorem)?
- Defining equations for matroid varieties given by wheel graphs?

background from *Euclidea*

Q & A:

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(1) What do you love most about mathematics?

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(2) What was your favorite math topic as an undergraduate?
(3) How did you prepare for graduate school?
(4) What was the most challenging part of grad school and how did you handle it?

background from *Euclidea*

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Thank-you!