Finiteness of associated primes of local cohomology modules over Stanley-Reisner rings

joint w/ Roberto Barrera and Jeffrey Madsen

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Thank-you for the invitation to speak!

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Little is known about local cohomology modules.

Local cohomology modules

- $R =$ commutative Noetherian ring with 1
- $I =$ ideal in R
- M=R-module (may or may not be Noetherian or finitely generated)
- $j =$ non-negative integer

The **jth local cohomology module of M with support in I** is defined as the following direct limit of Ext modules:

$$H_I^j(M) = \varinjlim_t \mathrm{Ext}_R^j(R/I^t, M).$$

It is the right derived functor of $H^0_I(?)$:

$$
\begin{aligned}
H^0_I(M) :=& \bigcup_t \operatorname{Ann}_M I^t \\
=& \{u \in M \mid uI^t = 0 \text{ for some } t\} \\
=& \varinjlim_t \operatorname{Hom}_R(R/I^t, M) (= \varinjlim_t \operatorname{Ext}^0_R(R/I^t, M))
\end{aligned}
$$

the global sections of the sheaf $\tilde{M}$ with support on the closed subscheme $\operatorname{Spec} R/I \subset \operatorname{Spec} R$.

- $H^1_I(M)$ measures the obstruction to extending a section of a sheaf to a global section; put $\mathcal{X} = \operatorname{Spec} R$ and $\mathcal{U} = \mathcal{X} \setminus \operatorname{Spec}(R/I)$

$$0 \to H^0_I(M) \to H^0(\mathcal{X}, \tilde{M}) \to H^0(\mathcal{U}, \tilde{M}|_{\mathcal{U}}) \to H^1_I(M) \to 0$$

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- If $(R, \mathfrak{m})$ is a local ring and M is finite generated, then $H^j_{\mathfrak{m}}(M)$ can detect regular sequences, compute depth, and reveal the Cohen-Macaulay and Gorenstein properties.

In practice, $H^j_I(M)$ is the jth cohomology module of the **Čech complex**

$$0 \to M \to \bigoplus_i M_{f_i} \to \bigoplus_{i<j} M_{f_i f_j} \to \cdots \to M_{f_1 \cdots f_s} \to 0$$

where:

- $f_1, \ldots, f_s \in R$ generate I up to radical
- given any $f \in R$ and any R-module N, $N_f = N \otimes_R R_f$, and $R_f = R[\frac{1}{f}]$ is the **localization of R at f**
- the maps in the complex are the natural localization maps $u \mapsto \frac{u}{1}$

Localization

Localizing at a prime ideal gives the **stalk at a point** in the **Zariski topolgy**:

$$\begin{aligned}
\operatorname{Spec} R &= \{\text{prime ideals in } R\} \leftarrow \text{ topological space} \\
\mathcal{V}(J) &= \{\text{primes containing the ideal } J\} \leftarrow \text{ its closed sets} \\
&= \operatorname{Spec} R/J
\end{aligned}$$

R **localized at** P is given by $R_P = R\left[\frac{1}{f} \mid f \in R \setminus P\right]$ and $N_P = N \otimes_R R_P$

Localization is **flat**; as a result, many questions can be reduced to the local case (**local-global principle**).

(statement about an R-module N is true)

$$\Leftrightarrow$$

(same statement about N_P is true for all $P \in \operatorname{Ass}_R N$)

- $\operatorname{Ass}_R N =$ **assassinator of N**, set of all primes associated to N
- **P is associated to N** means $P = \operatorname{Ann}_R(u)$, the set of ring elements that annihilate some element $u \in N$; equivalently, $P \in \operatorname{Ass}_R N$ if and only if R/P is isomorphic to a submodule of N.

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Local cohomology is the local-global analogue to **sheaf cohomology**.

Finiteness of associated primes

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Do the local cohomology modules over a Noetherian ring R have finitely many associated primes?

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(Answer: No.)

Counterexamples

- A. Singh 2000: $R = \dfrac{\mathbb{Z}[u,v,w,x,y,z]}{(ux+vy+wz)} \quad \Longrightarrow \quad |\operatorname{Ass}_R(H^3_{(x,y,z)}R)| = \infty$

  Reason: This local cohomology module has p-torsion for all primes $p \in \mathbb{Z}$.

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- M. Katzman 2002:

$$R = \frac{K[s, t, u, v, x, y]}{(su^2x^2 - (s+t)uvxy + tv^2y^2)} \quad (K = \text{ any field})$$

$$\implies |\operatorname{Ass}_R(H^2_{(x,y)}R)| = \infty$$

  Also shows torsion for infinitely many ring elements. Unlike in Singh's example, this ring is local.

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Affirmatives

- M. Hellus 2001: $M = R$ is Cohen-Macaulay and
  - $\operatorname{Ass}_R(H^3_{(x,y)}R)$ is finite for every $x, y \in R$
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- T. Marley 2001: $(R, \mathfrak{m})$ is local, M is finitely generated and
  - $\dim R \leq 3$
  - $\dim R = 4$ and R is regular on the punctured spectrum ($\operatorname{Spec} R \setminus \mathfrak{m}$ is smooth)
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  - $\dim M \leq 3$
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  - $\dim M/IM \leq 2$ and M satisfies Serre's condition $S_{\dim M - 3}$
  - $\dim M/IM \leq 3$, $\operatorname{Ann}_R M = 0$, R is unramified, and M satisfies $S_{\dim M - 3}$

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Proving anything more broad has been HARD!

Regular in characteristic p

Theorem (Huneke and R. Sharp 1993)

Yes, when R is a regular ring containing a field of characteristic $p > 0$.

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Why characteristic p?

To prove things in characteristic p: use the **Frobenius map**

$$\begin{aligned} F^e : R &\to R \\ r &\mapsto r^{p^e}. \end{aligned}$$

- In characteristic p it's a ring map! $(F^e(r+s) = (r+s)^{p^e} = r^{p^e} + s^{p^e} = F^e(r) + F^e(s))$
- $F^e(R) \cong R$ as rings
- When R is regular, it is locally free as a module over itself.

Huneke & Sharp: It suffices to show for R local, and so write $R \cong F^e(R)^{\oplus m_e} (\cong R^{\oplus m_e})$ as modules.

Then using the Ext definition of local cohomology:

$$\begin{aligned}
\mathrm{Ass}_R(H^j_I M) &= \mathrm{Ass}_{F^e(R)}\left((H^j_I M)\otimes_R F^e(R)\right)\\
&= \mathrm{Ass}_{F^e(R)}\left(\varinjlim_t \mathrm{Ext}^j_R(R/I^t, M)\otimes_R F^e(R)\right)\\
&= \mathrm{Ass}_{F^e(R)}\left(\varinjlim_e \mathrm{Ext}^j_R(R/I^{p^e}, M)\otimes_R F^e(R)\right)\\
&= \mathrm{Ass}_{F^e(R)}\left(\varinjlim_e \mathrm{Ext}^j_{F^e(R)}(R/I^{p^e}\otimes_R F^e(R), M\otimes_R F^e(R))\right)\\
&= \mathrm{Ass}_{F^e(R)}\left(\varinjlim_e \mathrm{Ext}^j_{F^e(R)}(F^e(R)/IF^e(R), M\otimes_R F^e(R))\right)\\
&= \mathrm{Ass}_R\left(\varinjlim_e \mathrm{Ext}^j_R(R/I, M\otimes_R R\right)\\
&= \mathrm{Ass}_R\left(\varinjlim_e \mathrm{Ext}^j_R((R/I)^{\oplus m_e}, M\right)\\
&= \mathrm{Ass}_R\left(\varinjlim_e \mathrm{Ext}^j_R(R/I, M)^{\oplus m_e}\right)\\
&\subseteq \mathrm{Ass}_R\left(\mathrm{Ext}^j_R(R/I, M)\right)
\end{aligned}$$

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Actually proved a stronger result: *For R regular containing a field of characteristic 0, and any maximal ideal $\mathfrak{m}$ in R, the number of associated primes of $H_I^j(M)$ contained in $\mathfrak{m}$ is finite.*

J.-E. Björk 1979: Over a formal power series ring in finitely many variables over a field of characteristic 0, there exists a class of **holonomic** $\mathcal{D}$-modules, to which local cohomology modules belong.

An associated prime in $\mathfrak{m}$ is the restriction of a prime in the completion $\hat{R}$ of R with respect to $\mathfrak{m}$, that is associated to $H^j_{I\hat{R}}(M \otimes_R \hat{R}) \cong H^j_I(M) \otimes_R \hat{R}$.

Cohen's Structure Theorem: $\hat{R} \cong R/\mathfrak{m}[[x_1, \ldots, x_n]]$
$\implies H^j_I(M) \otimes_R \hat{R} = H^j_{\hat{I}}(\hat{M})$ is holonomic.

Holonomic modules are semisimple $\implies$ finitely many associated primes.

Equicharacteristic

Later, Lyubeznik somewhat reconciled the characteristic p and 0 cases.

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For R regular containing a field of characteristic p or 0, and any maximal ideal $\mathfrak{m}$ in R, the number of associated primes of $H^j_I(M)$ contained in $\mathfrak{m}$ is finite.

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The proof reduces to one or the other characteristic, after which the techniques are different.

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Cohen's Structure Theorem: $\hat{R} \cong R/\mathfrak{m}[[x_1, \ldots, x_n]]$.

Björk 1979: The localizations $\hat{R}_f$ have finite $\mathcal{D}$-length in characteristic 0.

Lyubeznik 1997: The localizations $\hat{R}_f$ are F-finite in characteristic p;

F-finite $\implies$ finite F-length.

finite F-length $\implies$ finite $\mathcal{D}$-length.

Mixed characteristic

Theorem (Lyubeznik 2000)

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Proof reduces to the known results in equicharacteristic.

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Key ingredient: Updated notion of holonomicity by V. Bavula (2009).

Our main result

We use methods very similar to Lyubeznik to show the following:

Theorem (BMW 2015)

If R is a Stanley-Reisner ring over a field and its associated simplicial complex is a T-space, then the set of associated primes of any local cohomology module over R is finite.

Stanley-Reisner rings

- $S = K[x_1, \ldots, x_n]$, the polynomial ring over a field K
- $\Delta =$ simplicial complex with vertices labelled by the variables $x_1, \ldots, x_n$
- $I_\Delta = (x_{i_1} \cdots x_{i_t} \mid \{x_{i_1}, \ldots, x_{i_t}\} \notin \Delta)S$ is called the **face ideal of Δ over K**

$K[\Delta] = S/I_\Delta$ is called the **Stanley-Reisner ring of Δ over K**.

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In fact, the minimal primes (minimal in $\operatorname{Ass}_R R/(0)$ with respect to containment) are in bijection with the facets of Δ.

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A simplicial complex Δ is a T-**space** means for every face $F \in \Delta$, if $x \notin F$ then there exists a facet in Δ containing F but not x.

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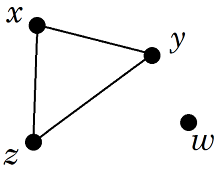
Example



$$\begin{aligned}\Delta &= \{\{x,y\},\{x,z\},\{y,z\},\{w\},\{x\},\{y\},\{z\}\}\\ I_\Delta &= (xw, yw, zw, xyz, xyw, xzw, yzw)S\\ R &= S/I_\Delta\\ &= \frac{K[x,y,z,w]}{(z,w)\cap(y,w)\cap(x,w)\cap(x,y,z)}\end{aligned}$$

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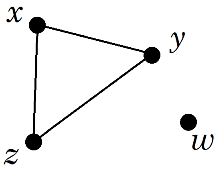

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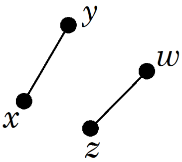

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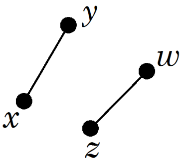

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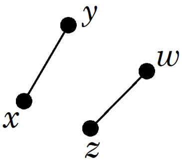

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Is it a T-space? (No.) In fact, a graph is a T-space if and only if none of its vertices have degree 1.

Finite length

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Common approach: Construct a **filtration** of R-submodules

$$0 = N_0 \subset N_1 \subset \cdots \subset N_l = N$$

in such a way that each of the factors N_i/N_{i-1} has finitely many associated primes.

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Problem: When N is not finitely generated (e.g., $N = H_I^j M$) it is HARD! to prove it has finite length.

Strategy: Show finite length over a larger ring, an R-algebra. For example, $\mathcal{D}$.

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Lyubeznik 2000: To show local cohomology modules have finite $\mathcal{D}$-length it is enough to show R_f, for any $f \in R$, has finite $\mathcal{D}$-length.

- consequence of the Čech complex definition of local cohomology
- Proving R_f has finite $\mathcal{D}$-length is still HARD! (recall Björk's result from earlier)

Rings of differential operators

Local cohomology modules are $\mathcal{D}$-modules.

- $K =$ field
- $R = K$-algebra
- $\mathcal{D} = D(R; K)$ is the set of "derivatives" we are allowed to take in R and coefficients are in the field K; includes multiplication by elements in R

$\mathcal{D}$ stands for the **ring of operators of R over K**. The operators include multiplication by elements in R
$\implies$ $\mathcal{D}$ is an R-algebra, i.e., $\mathcal{D}$-modules are R-modules.

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$\Longrightarrow \mathcal{D} = D(R; K) = \frac{\mathcal{I}(J)}{J\mathcal{D}_S}$, where $\mathcal{I}(J)$ denotes the **idealizer of J**, the set of operators $\delta \in \mathcal{D}_S$ such that $\delta(J) \subseteq J$

- $\partial_i^t = K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$-linear maps $\frac{1}{t!}\frac{\partial^t}{\partial x_i^t} : R \to R$ where $x_i^u \mapsto \binom{u}{t} x_i^{t-u}$, called **divided powers**
- monomial notation $\mathbf{x}^{\mathbf{a}}\underline{\partial}^{\mathbf{t}} = x_1^{a_1} \cdots x_n^{a_n} \partial_1^{t_1} \cdots \partial_n^{t_n}$

Theorem (BMW 2015)

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We show: T-space $\implies$ the converse

Holonomicity

Lyubeznik modified and applied Bavula's definition of holonomicity to characteristic-freely prove the local cohomology modules over a polynomial ring over a field have finitely many associated primes:

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Lyubeznik modified and applied Bavula's definition of holonomicity to characteristic-freely prove the local cohomology modules over a polynomial ring over a field have finitely many associated primes:

$\mathcal{D}_S$ has a filtration of K-vector spaces $K = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ where for each $j \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{F}_j = K \cdot \{\mathbf{x}^{\mathbf{a}} \underline{\partial}^{\mathbf{t}} \mid a_1 + \cdots + a_n + t_1 + \cdots t_n \leq j\},$$

called the **Bernstein filtration**.

Recall: If $R = S/J$ then $\mathcal{D} = \frac{\mathcal{I}(J)}{J\mathcal{D}_S}$
$\implies \mathcal{G}_j = \frac{\mathcal{F}_j \cap \mathcal{I}(J)}{J\mathcal{D}_S}$ gives a filtration $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots$ on $\mathcal{D}$.

For $R = S/I_\Delta$, $\mathcal{D} = R\langle x_i\partial_i^t \mid 1 \leq i \leq n, t \geq 0\rangle$

$$\implies \mathcal{G}_j = K \cdot \{\mathbf{x}^{\mathbf{a}}\underline{\partial}^{\mathbf{t}} \mid a_1 + \cdots + a_n + t_1 + \cdots t_n \leq j \text{ and for each } i,\ a_i \geq t_i\};$$

we call $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots$ the **Bernstein filtration on R**.

Definition (Bavula 2009; Lyubeznik 2010; BMW 2015)

A $\mathcal{D}$-module N is **holonomic** means there exists an ascending chain of K-modules $N_0 \subset N_1 \subset \cdots$ (called a ***K*-filtration**) satisfying

(i) $\cup_i N_i = N$ and

(ii) for all i and j, $\mathcal{G}_j N_i \subset N_{i+j}$,

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Theorem (BMW 2015)

Every holonomic $\mathcal{D}$-module has finite length.

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Suppose R is a Stanley-Reisner ring over a field and its simplicial complex is a T-space. Then for all $f \in R$, the localized ring R_f is holonomic.

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Corollary

The local cohomology modules $H^j_I M$ over R have finitely many associated primes.

More questions

(1) Does K have to be a field?

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Questions from the audience?

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