

# First Isomorphism Theorem

## 1. First Isomorphism Theorem

First Isomorphism Theorem

Example: Special Linear Group

Finitely generated abelian groups

# First Isomorphism Theorem

There are naturally many homomorphisms of the form  $\bar{\varphi} : G \rightarrow G/\ker \varphi$ , and the bar typically denotes such homomorphisms.

A more general observation can be made, namely, that the image of any homomorphism is itself a quotient group – by the kernel!

## Theorem 1 (First Isomorphism Theorem)

*Suppose  $\varphi : G \rightarrow H$  is a group homomorphism. Then*

$$\varphi(G) = \text{image } \varphi \cong G/\ker \varphi.$$

## Proof.

To simplify notation, put  $K = \ker \varphi$  and use multiplicative notation to denote group operations. We shall exhibit an isomorphism; define

$$\begin{aligned}\psi : G/K &\rightarrow \varphi(G) \\ gK &\mapsto \varphi(g).\end{aligned}$$

Verifying  $\psi$  is an isomorphism is routine:

- **Claim.**  $\psi$  is well-defined.

**Proof:** To show  $\psi$  is a function, we must make sure image  $\psi \subset \varphi(G)$  – apparent in the definition of  $\psi$  – and that no element in  $G/K$  is mapped to two different elements in  $\varphi(G)$ .

Suppose  $gK = hK$ . By Proposition ??, there exists  $k \in K$  such that  $g = hk$ . By definition of the kernel,  $k \in K = \ker \varphi$  implies  $\varphi(k) = 1_H$ , the identity element in  $H$ . Then, since by hypothesis,  $\varphi$  is a homomorphism,

$$\psi(gK) = \varphi(g) = \varphi(hk) = \varphi(h)\varphi(k) = \varphi(h) \cdot 1_H = \varphi(h) = \psi(hK),$$

as required. □

- **Claim.**  $\psi$  is a homomorphism.

**Proof:** Choose  $gK, hK \in G/K$ . Then

$$\begin{aligned}\psi((gK)(hK)) &= \psi((gh)K) = \varphi(gh) \\ &= \varphi(g)\varphi(h) = \psi(gK)\psi(hK),\end{aligned}$$

as required. □

- **Claim.**  $\psi$  is one-to-one.

**Proof:** Suppose  $\psi(gK) = \psi(hK)$ . Then  $\varphi(g) = \varphi(h)$ . Multiply by  $\varphi(h)^{-1}$ :

$$\begin{aligned}\varphi(g)\varphi(h)^{-1} &= \varphi(h)\varphi(h)^{-1} = 1_H \\ \implies \varphi(g)\varphi(h)^{-1} &= \varphi(gh^{-1}) = 1_H \\ \implies gh^{-1} &\in \ker \varphi = K.\end{aligned}$$

Multiply by  $h$ :

$$g = gh^{-1}h \in Kh = hK,$$

and so by Proposition ??,  $gK = hK$ .



- **Claim.**  $\psi$  is onto.

**Proof:** Suppose  $h \in \varphi(G)$ . By definition of  $\varphi(G)$ ,  $h = \varphi(g)$  for some  $g \in G$ . Then  $h = \varphi(g) = \psi(gK)$ . □

This completes the proof of Theorem 1. □

## Example: Special Linear Group

### Example 1

Recall part (??) of Example ??, the group homomorphism

$$\delta : \mathrm{GL}(2, \mathbb{R}) \rightarrow \mathbb{R}^\times$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad - bc.$$

### Question

What is  $\ker \delta$  in Example 1?

The kernel in Example 1 is called the **special linear group** and is denoted  $SL(2, \mathbb{R})$ . It is the set of all  $2 \times 2$  matrices with real entries, whose determinant is 1.

### Exercise 1

Prove  $\delta$  in Example 1 is surjective.

It follows, from the First Isomorphism Theorem, that

$$GL(2, \mathbb{R}) / SL(2, \mathbb{R}) \cong \mathbb{R}^\times.$$

### Question

What are the cosets of  $SL(2, \mathbb{R})$  in  $GL(2, \mathbb{R})$ ?



## Finitely generated abelian groups

The following exercises motivate Theorem 2.

### Exercise 2 (cf. Problem 74)

Use the First Isomorphism Theorem to prove  $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$ .

Let us define a “basis” for the  $n$ -tuples in  $\mathbb{Z}^n$ . For each  $i = 1, \dots, n$ , let

$$\mathbf{e}_i := (e_{i1}, e_{i2}, \dots, e_{in}), \text{ where } e_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

We can think of each  $\mathbf{e}_i$  as a “component-wise identity” of  $\mathbb{Z}^n$ .

### Exercise 3 (cf. Problem 75)

Let  $m, n$  denote non-zero integers. Use the First Isomorphism Theorem to show

$$\mathbb{Z}^2 / \langle m\mathbf{e}_1, n\mathbf{e}_2 \rangle \cong \mathbb{Z}_m \times \mathbb{Z}_n.$$

### Question

How does Exercise 3 compare to Example ??? What about Proposition ???

Theorem 2 is the generalization:

## Theorem 2

Let  $H = \langle m_1 \mathbf{e}_1, \dots, m_n \mathbf{e}_n \rangle < \mathbb{Z}^n$ . Then

$$\mathbb{Z}^n / H \cong \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}.$$



## Proposition 1

*Every finitely generated abelian group is a quotient group of  $\mathbb{Z}^m$ , for some  $m \in \mathbb{N}$ .*

### Proof.

Suppose  $G$  is an abelian group with generators  $g_1, \dots, g_m$ . Then every element in  $G$  is of the form  $g_1^{a_1} \cdots g_m^{a_m}$  for integers  $a_1, \dots, a_m$ . Define a map

$$\begin{aligned}\phi : \mathbb{Z}^m &\rightarrow G \\ \mathbf{a} := (a_1, \dots, a_m) &\mapsto g_1^{a_1} \cdots g_m^{a_m}.\end{aligned}$$

If  $\phi$  is surjective then we may apply the First Isomorphism Theorem to conclude

$$\mathbb{Z}^m / \ker(\phi) \cong \phi(\mathbb{Z}^m) = G,$$

completing the proof.

First we show  $\phi$  is a group homomorphism. Suppose  $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^m$ .

$$\begin{aligned}\phi(\mathbf{a} + \mathbf{b}) &= \phi((a_1 + b_1, \dots, a_m + b_m)) \\ &= g_1^{a_1+b_1} \cdots g_m^{a_m+b_m} \\ &= (g_1^{a_1} g_1^{b_1}) \cdots (g_m^{a_m} g_m^{b_m}) \\ &= (g_1^{a_1} \cdots g_m^{a_m})(g_1^{b_1} \cdots g_m^{b_m}) \\ &= \phi((a_1, \dots, a_m)) \phi(b_1, \dots, b_m).\end{aligned}$$

Now we show  $\phi$  is surjective. Take  $g \in G$ , which again, can be written

$g = g_1^{a_1} \cdots g_m^{a_m}$  for some  $a_1, \dots, a_m \in \mathbb{Z}$ . By definition,

$\phi : (a_1, \dots, a_m) \mapsto g$ . So  $\phi$  is surjective.



The **Fundamental Theorem of Finitely Generated Abelian Groups**, which we state in Section ??, actually gives the precise structure of the quotient groups in Proposition 1.