Sections 3.4 and 3.5 – The Chain Rule and Trigonometric Functions

The Chain Rule (Version 1). If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

The Chain Rule (Version 2). Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

Notes:

Derivatives of Trigonometric Functions

$\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx} \cos x = -\sin x$

$\frac{d}{dx} \tan x = \sec^2 x$

$\frac{d}{dx} \cot x = \frac{d}{dx} \frac{1}{\tan x} = \frac{-\sec^2 x}{\tan^2 x} = -\csc^2 x$

$\frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x} = \frac{\sin x}{\cos^2 x} = \tan x \sec x$

$\frac{d}{dx} \csc x = \frac{d}{dx} \frac{1}{\sin x} = \frac{-\cos x}{\sin^2 x} = -\cot x \csc x$

1. Find the derivative of each of the following functions.

(a) $f(x) = \sqrt{\dfrac{x^2+9}{x+3}}$

$$f'(x) = \frac{1}{2}\left(\frac{x^2+9}{x+3}\right)^{-1/2}\left(\frac{(x+3)(2x)-(x^2+9)}{(x+3)^2}\right)$$

(c) $f(t) = \sin(t^2)$

$$f'(t) = (\cos(t^2)) \cdot 2t$$

(b) $y = x \cdot 2^{-x^2}$

$$y' = 2^{-x^2} + x(\ln 2) \cdot 2^{-x^2} \cdot (-2x)$$

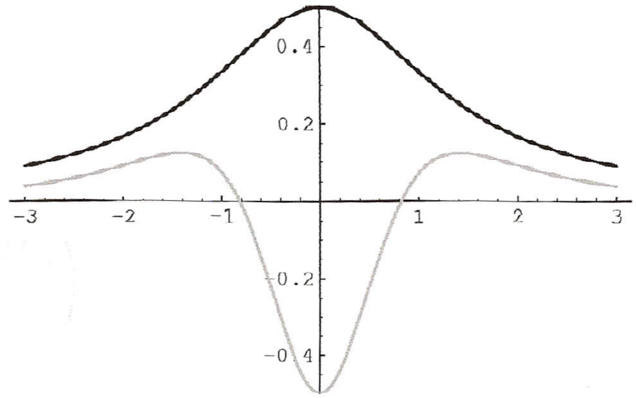
(d) $g(t) = \tan^2 t$

$$g'(t) = 2 \tan t \sec^2 t$$

(e) $f(x) = \sqrt{\cos(\sin^2 x)}$

$$f'(x) = \frac{1}{2}(\cos(\sin^2 x))^{-\frac{1}{2}} \cdot (-\sin(\sin^2 x)) \cdot 2 \sin x \cos x$$

2. Given to the right is the graph of the function $f(x) = (x^2+2)^{-1}$ (dark graph) and its second derivative $f''(x)$ (lighter graph).



(a) Find the equation of the tangent line to f at $x = 2$.

$f'(x) = -(x^2+2)^{-2} \cdot 2x$

$f'(2) = -(2^2+2)^{-2} \cdot 2(2)$

$= \frac{-4}{6^2} = \frac{-1}{9}$

$f(2) = \frac{1}{6}$

So, $y - \frac{1}{6} = \frac{-1}{9}(x-2)$

$y = \frac{-1}{9}x + \frac{7}{18}$

(b) Find a formula for $f''(x)$.

$f''(x) = 2(x^2+2)^{-3} \cdot 2x \cdot 2x - (x^2+2)^{-2} \cdot 2$

(c) Graphically estimate the interval on which f is concave down. Then, use your formula for the second derivative to find the exact interval on which f is concave down.

graphically: $x \in (-0.8, 0.8)$

$f''(x) = \frac{8x^2}{(x^2+2)^3} - \frac{2}{(x^2+2)^2} < 0$

$8x^2 - 2(x^2+2) < 0$

$6x^2 - 4 < 0$

$3x^2 < 2$

$x^2 < \frac{2}{3}$

$x \in \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right) \doteq (-0.816, 0.816)$

3. The population of the world, P (in billions) is well-modeled by the equation $P = 6e^{0.013t}$, where t is the number of years after the beginning of 1999. First, predict the population of the world in 2009. Then, predict the rate at which the world's population will be growing in 2009. Include units with your answers.

Pop. in 2009 $= P(10) = 6e^{0.013(10)}$

$\doteq 6.833$ billion

$P'(10) = (0.013) \cdot P(10)$

$\doteq 0.089$

$= 89$ million/year

4. Let f be a differentiable function, and let $g(x) = (f(\sqrt{x}))^3$.

(a) Calculate $g'(x)$.

$$g'(x) = 3(f(\sqrt{x}))^2 \cdot f'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

(b) Use the values in the table below to calculate $g'(4)$.

x	$f(x)$	$f'(x)$
2	1	-2
4	-3	4

$$g'(4) = 3(f(\sqrt{4}))^2 \cdot f'(\sqrt{4}) \cdot \frac{1}{2\sqrt{4}}$$

$$= 3 \cdot f(2)^2 \cdot f'(2) \cdot \frac{1}{2 \cdot 2}$$

$$= \frac{3 \cdot 1 \cdot (-2)}{4}$$

$$= -\frac{3}{2}$$