Survey of Calculus Final Exam Review — Fall 2016

SOLUTIONS

1. (a) $\lim\limits_{x\to 4} \frac{x-4}{\sqrt{x}-2}\left(\frac{\sqrt{x}+2}{\sqrt{x}+2}\right) = \lim\limits_{x\to 4} \frac{\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{x-4}} = \lim\limits_{x\to 4} \sqrt{x}+2$

$= \sqrt{4}+2 = \boxed{4}$

(b) $\lim\limits_{y\to 3} \frac{1}{y-3}\left(\frac{1}{y}-\frac{1}{3}\right) = \lim\limits_{y\to 3} \frac{1}{\cancel{y-3}}\left(\frac{\cancel{3-y}}{3y}\right) = \lim\limits_{y\to 3} \frac{-1}{3y} = \frac{-1}{3(3)}$

$= \boxed{\frac{-1}{6}}$

2. $f(x) = \frac{1}{5}x^2 - 2x$

(a) $\frac{f(10)-f(2)}{10-2} = \frac{\overset{20}{\frac{1}{5}(10)^2} \overset{-20}{-2(10)} - \left[\overset{\frac{4}{5}}{\frac{1}{5}(2)^2} \overset{-4}{-2(2)}\right]}{8}$

$= \frac{-\frac{4}{5}+4}{8} = \frac{-4+20}{5(8)} = \frac{16}{40} = \boxed{\frac{2}{5}}$

(b) Instantaneous: $f'(x) = \frac{2}{5}x - 2$

Average: $\frac{2}{5}$ (from (a))

$f'(x) = \frac{2}{5}x - 2 = \frac{2}{5}$

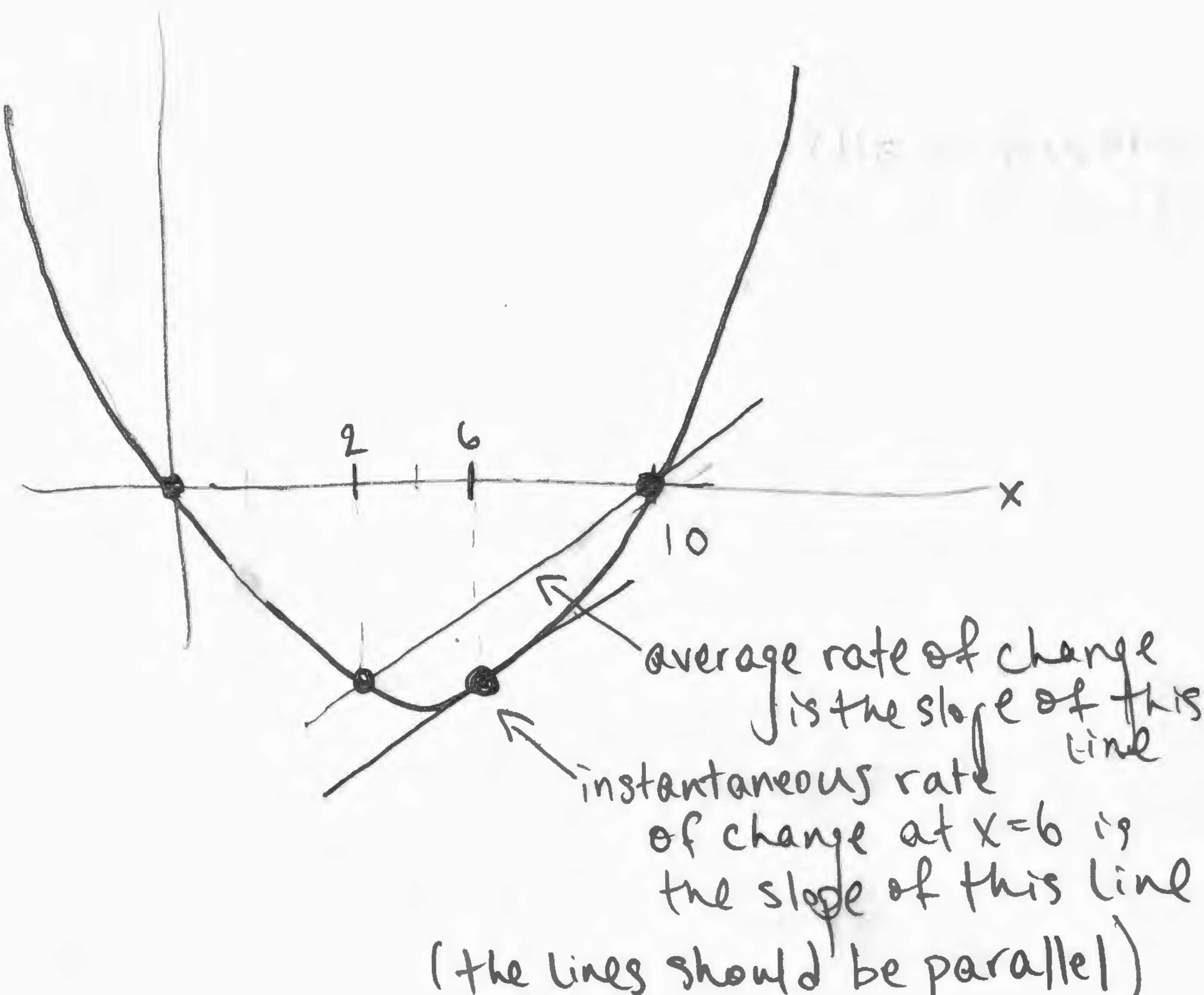
$$\Rightarrow \frac{2}{5}x = \frac{2}{5} + 2 = \frac{12}{5}$$

$$x = \frac{5}{2}\left(\frac{12}{5}\right) = \boxed{6}$$

(c) $f(x) = \frac{1}{5}x^2 - 2x$ ← parabola

$= x\left(\frac{1}{5}x - 2\right)$ ← find the zeros

$x=0$ $\quad$ $x=10$



* Graph is not to scale.

3. (a) $f(x) = \ln\sqrt{1+2x} = \ln\left((1+2x)^{1/2}\right)$

$$f'(x) = \frac{1}{\sqrt{1+2x}} \cdot \frac{1}{2}(1+2x)^{-1/2} \cdot (2)$$

$$= \frac{1}{\sqrt{1+2x}} \cdot \frac{1}{\sqrt{1+2x}} = \boxed{\frac{1}{1+2x}}$$

(b) $y = x^2 e^{x^3}$

$$y' = 2x\left(e^{x^3}\right) + x^2\left(e^{x^3} \cdot 3x^2\right)$$

$$= \boxed{2xe^{x^3} + 3x^4 e^{x^3}}$$

(c) $g(z) = (3z^2-4)^{97}$

$$g'(z) = \boxed{97(3z^2-4)(6z)}$$

(d) $\frac{d}{dx}\left[y = \frac{x^3-4x+5}{x^2+9}\right]$

$$\Rightarrow \frac{dy}{dx} = \boxed{\frac{(x^2+9)(3x^2-4) - (x^3-4x+5)(2x)}{(x^2+9)^2}}$$

$$= \frac{3x^4+27x^2-4x^2-36-6x^4+8x^2-10x}{(x^2+9)^2}$$

$$= \frac{-3x^4+31x^2-10x-36}{(x^2+9)^2}$$

$\longrightarrow$

(e) $k(t) = \frac{t}{\ln(t)}$

$$k'(t) = \frac{\ln(t)(1) - t\left(\frac{1}{t}\right)}{(\ln t)^2} = \boxed{\frac{\ln(t)-1}{(\ln(t))^2}}$$

4. The derivative tells how fast a function grows:

$f(t)=t^2$ $\quad g(t)=t^3$ $\quad h(t)=e^t$ $\quad k(t)=\ln(t)$

$f'(t)=2t$ $\quad g'(t)=3t^2$ $\quad h'(t)=e^t$ $\quad k'(t)=\frac{1}{t}$

All the derivatives are positive when $t>1$, so all the functions are growing. The larger the derivative, the faster the function grows.

However, derivatives are also curves, which may intersect, so that one curve may start out larger, but the other might become larger for larger t. For example, $f'(t)>g'(t)$ when $t<\frac{2}{3}$. But when $t>\frac{2}{3}$, $g'(t)>f'(t)$. In other words,

$$\lim_{t\to\infty} f'(t) < \lim_{t\to\infty} g'(t)$$

That is the same thing as saying

$$\frac{\lim_{t\to\infty} f'(t)}{\lim_{t\to\infty} g'(t)} = \lim_{t\to\infty} \frac{f'(t)}{g'(t)} < 1.$$

Since $\lim_{t\to\infty} \frac{f'(t)}{g'(t)} = \lim_{t\to\infty} \frac{2t}{3t^2} = \lim_{t\to\infty} \frac{2}{3t} = 0 < 1$, the function $g(t)$ grows faster than $f(t)$ for $t > \frac{2}{3}$ (and therefore for $t \gg 1$).

Compare the other functions:

$$\lim_{t\to\infty} \frac{f'(t)}{k'(t)} = \lim_{t\to\infty} \frac{2t}{\frac{1}{t}} = \lim_{t\to\infty} 2t^2 = \infty > 1$$

so $f(t)$ grows faster than $k(t)$ for $t \gg 1$

So far, $k(t) < f(t) < g(t)$ for $t \gg 1$.

$$\lim_{t\to\infty} \frac{g'(t)}{h'(t)} = \lim_{t\to\infty} \frac{3t^2}{e^\infty} = \frac{\infty}{\infty}.$$

Does $g'(t)$ or $h'(t)$ reach ∞ faster? The faster function also grows faster, so check the derivatives of $g'(t)$ and $h'(t)$:

$$\lim_{t\to\infty} \frac{g''(t)}{h''(t)} = \lim_{t\to\infty} \frac{6t}{e^t} = \frac{\infty}{\infty}.$$

Again, the function that reaches infinity faster must have a larger growth rate, so take derivatives again:

$$\lim_{t\to\infty} \frac{g'''(t)}{h'''(t)} = \lim_{t\to\infty} \frac{6}{e^t} = 0 < 1,$$

so $g'''(t) < h'''(t)$ when $t \gg 1$

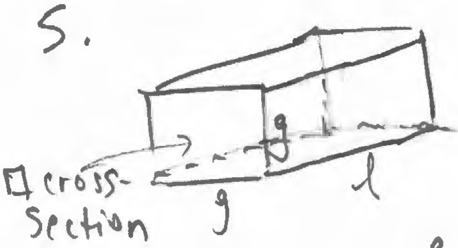
$\Rightarrow g''(t) < h''(t)$ when $t \gg 1$

$\Rightarrow g'(t) < h'(t)$ when $t \gg 1$

$\Rightarrow h(t)$ grows faster than $g(t)$ and

$$\boxed{k(t) < f(t) < g(t) < h(t) \text{ for } t \gg 1}$$

5.



l = length of box, > 0

g = length of sides in the cross-section, > 0

Constraint: $l + 4g \leq 100$

$$\Rightarrow l \leq 100 - 4g$$

Objective: Maximize Volume.

$$V = g^2 l \leq g^2(100 - 4g) = 100g^2 - 4g^3.$$

If the sum of the length and girth is less than 100, then $l < 100 - 4g$ means the volume is also smaller. So the critical point is given by $V(g) = 100g^2 - 4g^3 = 4g^2(25 - g)$

$$\Rightarrow V'(g) = 200g - 12g^2$$

$$= 4g(50 - 3g) = 0$$

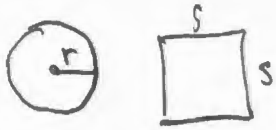
The endpoints are $[0, 100]$ $\Rightarrow g = 0, \frac{3}{50}$

$V(0) = 4(0)^2(25 - 0) = 0$

$V(100) = 4(100)^2(25 - 100) = 40000(-75) < 0$

$V\left(\frac{3}{50}\right) = 4\left(\frac{3}{50}\right)^2\left(25 - \frac{3}{50}\right) > 0 \leftarrow$ max

$\boxed{\approx 0.351 \text{ in}^3}$

6. 

s = side of square
r = radius of the circle
and $0 \leq s, r \leq 100$ cm.

<u>Constraint</u>: $2\pi r + 4s = 100$

$$\Rightarrow r = \frac{100-4s}{2\pi}$$

<u>Objective</u>: Maximize Area:

$$A = \pi r^2 + s^2 = \pi\left(\frac{100-4s}{2\pi}\right)^2 + s^2$$

$$A'(s) = \cancel{\pi}\cancel{(2)}\left(\frac{100-4s}{\cancel{2\pi}}\right)\left(-\frac{2}{\pi}s\right) = 0$$

$$\uparrow \quad s = 25 \qquad\qquad \uparrow \quad s = 0$$

$$A(0) = \pi\left(\frac{100-\cancel{4(0)}}{2\pi}\right)^2 + (0)^2 = \frac{50^2}{\pi}$$

$$A(25) = \pi\left(\frac{100-4(25)}{2\pi}\right)^2 + 25^2 = 25^2$$

$$A(100) = \pi\left(\frac{100-4(100)}{2\pi}\right)^2 + 100^2 = 100^2\left(\pi\left(\frac{-3}{2\pi}\right)^2 + 1\right) \leftarrow \text{max}$$

$$\boxed{\text{So the wire should not be cut. Use the whole wire for the circle.}}$$

7. If $f(x)=|x|$ is differentiable at $x=0$ then the 2-sided limit of the difference quotient must exist:

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{-\cancel{x}-0}{\cancel{x}} = -1$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{\cancel{x}-0}{\cancel{x}} = 1$$

Since $-1 \neq 1$, the limit of the difference quotient does not exist, and so $|x|$ is not differentiable at $x=0$.

8. (a) $\int x\sqrt{\underbrace{2-3x^2}_{u}}\,dx$. Put $u=2-3x^2$ ($x\,dx = -\frac{1}{6}du$)

$$\Rightarrow \frac{du}{dx} = -6x.$$

$$\Rightarrow -\frac{1}{6}du = x\,dx$$

$$= -\frac{1}{6}\int \sqrt{u}\,du = \frac{-\frac{1}{6}u^{3/2}}{3/2} + C$$

$$\boxed{= -\frac{1}{9}(2-3x^2)^{3/2} + C}$$

(b) $\int \frac{t}{\sqrt{2t^2+1}}\,dt$. Put $u = 2t^2+1$

$\Rightarrow \frac{du}{dt} = 4t$

$\Rightarrow \frac{1}{4}du = t\,dt$

$= \frac{1}{4}\int \frac{1}{\sqrt{u}}\,du$

$= \frac{\frac{1}{4}u^{1/2}}{1/2} + C = \boxed{\frac{1}{2}\sqrt{2t^2+1} + C}$

(c) $\int \frac{x^2-1}{\sqrt{x}}\,dx = \int\left(\frac{x^2}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right)dx$

$= \int\left(x^{3/2} - x^{-1/2}\right)dx$

$\boxed{= \frac{2}{5}x^{5/2} - 2x^{1/2} + C}$

(d) $\int \frac{3}{x+1}\,dx$ Put $u = x+1$

$\Rightarrow \frac{du}{dx} = 1$

$\Rightarrow du = dx$

$3\int \frac{1}{u}\,du = 3\ln|u| + C = \boxed{3\ln|x+1| + C}$

(e) $\int (e^{3x} - 5x^2)\,dx = \boxed{\frac{1}{3}e^{3x} - \frac{5}{3}x^3 + C}$

9. (a) $\int_0^3 \frac{e^x}{3+2e^x}\,dx$ Put $u = 3+2e^x$

$\frac{1}{2}du$

u

If $x=0$ then $u = 3+2e^0 = 5$

If $x=3$ then $u = 3+2e^3$

$\frac{du}{dx} = 2e^x \Rightarrow \frac{1}{2}du = e^x\,dx$

$\frac{1}{2}\int_5^{3+2e^3} \frac{1}{u}\,du = \frac{1}{2}\ln(u)\Big|_5^{3+2e^3} = \boxed{\frac{1}{2}\ln(3+2e^3) - \frac{1}{2}\ln(5)}$

(b) $\int_1^{\sqrt{5}} x(x-2)(x-4)\,dx = \int_1^{\sqrt{5}} (x^3 - 6x^2 + 8x)\,dx$

$= \frac{x^4}{4} - 2x^3 + 4x^2 \Big|_1^{\sqrt{5}}$

$= \frac{(\sqrt{5})^4}{4} - 2(\sqrt{5})^3 + 4(\sqrt{5})^2$

$25/4 \quad 10\sqrt{5} \quad 20$

$-\left[\frac{1^4}{4} - 2(1)^3 + 4(1)^2\right]$

$1/4 \quad -2 \quad 4$

$= \frac{25-1}{4} + 20 + 2 - 4 - 10\sqrt{5} = \boxed{24 - 10\sqrt{5}}$

(c) $\displaystyle\int_1^8 \left(bx^3 - cx^{1/3}\right)dx = \frac{b}{4}x^4 - \frac{3c}{4}x^{4/3}\Big|_1^8$

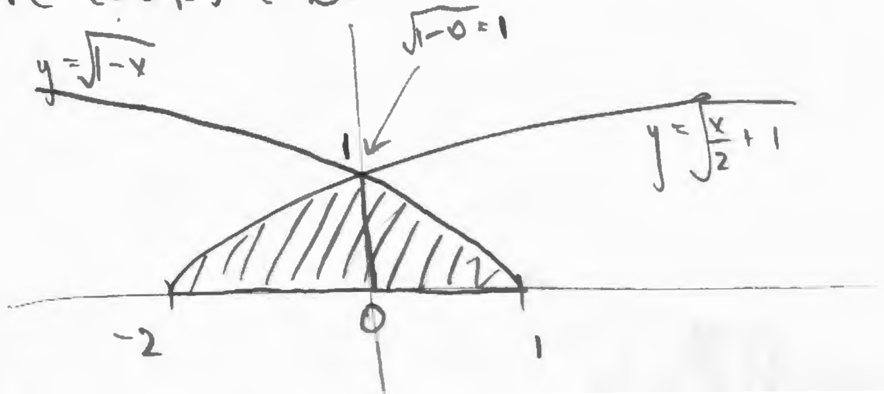
$$= \frac{b}{4}(8)^4 - \frac{3c}{4}(8)^{4/3} - \left[\frac{b}{4}(1)^4 - \frac{3c}{4}(1)^{4/3}\right]$$

$$\boxed{= 1024b - 6(8)^{1/3}c - \frac{b}{4} + \frac{3c}{4}}$$

10. (a) Find out where these curves intersect:

$\sqrt{\frac{x}{2}+1} = \sqrt{1-x}$	$\sqrt{\frac{x}{2}+1} = 0$	$\sqrt{1-x} = 0$
$\frac{x}{2} + \cancel{1} = \cancel{1} - x$	$\frac{x}{2} + 1 = 0$	$1 - x = 0$
$\frac{3}{2}x = 0$	$\frac{x}{2} = -1$	$\Rightarrow x = 1$
$\Rightarrow x = 0$	$\Rightarrow x = -2$	

Then $y = \sqrt{\frac{x}{2}+1}$ is a scaled, shifted root function and $y = \sqrt{1-x}$ is a backwards, shifted root function, so the picture looks like:



$\longrightarrow$

The area is

$$\int_{-2}^{0} \sqrt{\frac{x}{2}+1}\, dx + \int_{0}^{1} \sqrt{1-x}\, dx$$

Put $u = \frac{x}{2}+1$

$\Rightarrow \frac{du}{dx} = \frac{1}{2}$

$\Rightarrow 2\,du = dx$

If $x=-2$ then

$u = \frac{-2}{2}+1 = 0$

If $x=0$ then

$u = \frac{0}{2}+1 = 1$

Put $u = 1-x$

$\Rightarrow \frac{du}{dx} = -1$

$\Leftrightarrow -du = dx$

If $x=0$ then $u = 1-0 = 1$

If $x=1$ then $u = 1-1 = 0$

The area is

$$2\int_{0}^{1} \sqrt{u}\, du + -\int_{1}^{0} \sqrt{u}\, du$$

$$= \frac{2u^{3/2}}{3/2}\Big|_{0}^{1} + \frac{-u^{3/2}}{3/2}\Big|_{1}^{0}$$

$$= \frac{4}{3}(1)^{3/2} - \frac{4}{3}(0)^{3/2} + -\frac{2}{3}(0)^{3/2} - \left(-\frac{2}{3}(1)^{3/2}\right)$$

$$= \frac{4}{3} + \frac{2}{3} = \boxed{2}$$

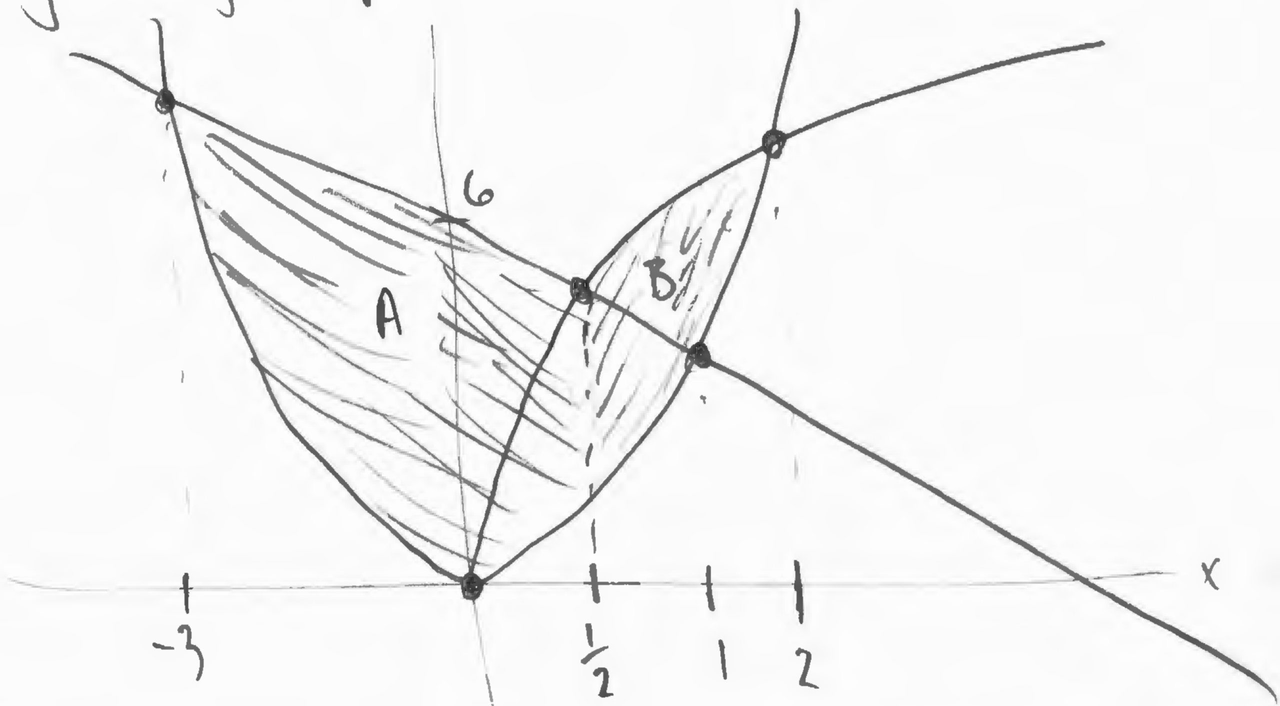
(b) Check where the curves intersect:

$$4\sqrt{2x} = 2x^2$$
$$16(2x) = 4x^4$$
$$0 = 4x^4 - 32x$$
$$= 4x(x^3 - 8)$$
$$\Rightarrow x = 0, 2$$

$$4\sqrt{2x} = -4x + 6$$
$$16(2x) = (-4x+6)^2$$
$$32x = 16x^2 - 48x + 36$$
$$0 = 16x^2 - 80x + 36$$
$$= 4(4x^2 - 20x + 9)$$
$$\Rightarrow x = \frac{-(-20) \pm \sqrt{\overbrace{(-20)^2 - 4(4)(9)}^{400 \; - \; 144}}}{2(4)}$$
$$= \frac{20 \pm 16}{8} = \frac{5 \pm 4}{2} = \frac{1}{2}, \underbrace{\cancel{\frac{9}{2}}}_{\text{extra root}}$$

$$2x^2 = -4x + 6$$
$$2x^2 + 4x - 6 = 0$$
$$2(x^2 + 2x - 3) = 0$$
$$2(x+3)(x-1) = 0$$
$$\Rightarrow x = -3, 1$$

The functions are a square root, a parabola, and a negatively sloped line:



The area of region A is

$$\int_{-3}^{\frac{1}{2}} (-4x+6-2x^2)\,dx = -2x^2+6x-\frac{2}{3}x^3 \Big|_{-3}^{\frac{1}{2}}$$

$$= -2\left(\tfrac{1}{2}\right)^2 + 6\left(\tfrac{1}{2}\right) - \frac{2}{3}\left(\tfrac{1}{2}\right)^3$$

$-\tfrac{1}{2} \quad 3 \quad -\tfrac{1}{6}$

$$-\left[-2(-3)^2+6(-3)-\frac{2}{3}(-3)^3\right]$$

$$-\left(-18 - \cancel{18} + \cancel{18}\right)$$

$$= -\frac{1}{2}+3-\frac{1}{6}+18 =$$

The area of region B is

$$\int_{\frac{1}{2}}^{2} (4\sqrt{2x}-2x^2)\,dx = \frac{4\sqrt{2}\,x^{3/2}}{3/2} - \frac{2}{3}x^3 \Big|_{\frac{1}{2}}^{2}$$

$$= \frac{8\sqrt{2}}{3}(2)^{3/2} - \frac{2}{3}(2)^3 - \left[\frac{8\sqrt{2}}{3}\left(\tfrac{1}{2}\right)^{3/2} - \frac{2}{3}\left(\tfrac{1}{2}\right)^3\right]$$

4

$$\frac{32}{3} \quad - \quad \frac{16}{3} \quad - \left(\frac{4}{3}-\frac{1}{12}\right)$$

$$= \frac{12}{3} - 12 = 4-12 = -8$$

<u>Total Area</u>: $-\frac{1}{2}+3-\frac{1}{6}+18-8 = 13-\frac{1}{3} = \boxed{\frac{38}{3}}$

$\frac{3}{6}$