Wed 13 Apr

- Exam 3: Stay tuned for the data.
- No (scheduled) office hours Friday. I will be in $\tilde{1}$220p.
- Quiz 9 on Thursday (tomorrow) covers §4.7, 4.9.
- ALL MLPs are open now.
- April 22: Last day to drop with a "W".

Examining Growth Rates

We can use l'Hôpital's Rule to examine the rate at which functions grow in comparison to one another.

Definition

Suppose f and g are functions with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Then **$f$ grows faster than g** as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \text{ or } \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty.$$

$g \ll f$ means that f grows faster than g as $x \to \infty$.

Definition

The functions f and g have **comparable growth rates** if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M, \text{ where } 0 < M < \infty.$$

Pitfalls in Using l'Hôpital's Rule

1. L'Hôpital's Rule says that $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$. NOT

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \left[\frac{f(x)}{g(x)}\right]' \text{ or } \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \left[\frac{1}{g(x)}\right]' f'(x)$$

   (i.e., don't confuse this rule with the Quotient Rule).
2. Be sure that the limit with which you are working is in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
3. When using l'Hôpital's Rule more than once, simplify as much as possible before repeating the rule.
4. If you continue to use l'Hôpital's Rule in an unending cycle, another method must be used.

4.7 Book Problems

13-59 (odds), 69-79 (odds)

§4.9 Antiderivatives

With differentiation, the goal of problems was to find the function f' given the function f.

With antidifferentiation, the goal is the opposite. Here, given a function f, we wish to find a function F such that the derivative of F is the given function f (i.e., $F' = f$).

Definition

A function F is called an **antiderivative** of a function f on an interval I provided $F'(x) = f(x)$ for all x in I.

Example

Given $f(x) = 4$, an antiderivative of $f(x)$ is $F(x) = 4x$.

NOTE: Antiderivatives are not unique!

They differ by a constant (C):

Theorem

*Let F be any antiderivative of f. Then **all** the antiderivatives of f have the form $F + C$, where C is an arbitrary constant.*

Recall: $\dfrac{d}{dx} f(x) = f'(x)$ is the derivative of $f(x)$.

Now: $\int f(x)\, dx = F + C$ *is* the antiderivative of $f(x)$. It doesn't matter which F you choose, since writing the C will show you are talking about all the antiderivatives at once. The C is also why we call it the *indefinite* integral.

Example

Find the antiderivatives of the following functions:

(1) $f(x) = -6x^{-7}$

(2) $g(x) = -4\cos 4x$

(3) $h(x) = \csc^2 x$

Indefinite Integrals

Example

$\int 4x^3 \, dx = x^4 + C$, where C is the **constant of integration**.

The dx is called the **differential** and it is the same dx from Section 4.5. Like the $\frac{d}{dx}$, it shows which variable you are talking about. The function written between the $\int$ and the dx is called the **integrand**.

Rules for Indefinite Integrals

Power Rule: $\int x^p \, dx = \dfrac{x^{p+1}}{p+1} + C$

(p is any real number except -1)

Constant Multiple Rule: $\int cf(x) \, dx = c \int f(x) \, dx$

Sum Rule: $\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx$

Exercise

$$\int (5x^4 + 2x + 1)\, dx =$$

A. $20x^3 + 2 + C$

B. $x^5 + x^2 - x + C$

C. $x^5 + x^2 + C$

D. $x^5 + 2x^2 - x + C$

Exercise

Evaluate the following indefinite integrals:

(1) $\int (3x^{-2} - 4x^2 + 1)\ dx$

(2) $\int 6\sqrt[3]{x}\ dx$

(3) $\int 2\cos(2x)\ dx$

Indefinite Integrals of Trig Functions

Table 4.9 (p. 322) provides us with rules for finding indefinite integrals of trig functions.

1. $\dfrac{d}{dx}(\sin ax) = a\cos ax \longrightarrow \displaystyle\int \cos ax\ dx = \frac{1}{a}\sin ax + C$
2. $\dfrac{d}{dx}(\cos ax) = -a\sin ax \longrightarrow \displaystyle\int \sin ax\ dx = -\frac{1}{a}\cos ax + C$
3. $\dfrac{d}{dx}(\tan ax) = a\sec^2 ax \longrightarrow \displaystyle\int \sec^2 ax\ dx = \frac{1}{a}\tan ax + C$
4. $\dfrac{d}{dx}(\cot ax) = -a\csc^2 ax \longrightarrow \displaystyle\int \csc^2 ax\ dx = -\frac{1}{a}\cot ax + C$
5. $\dfrac{d}{dx}(\sec ax) = a\sec ax\tan ax \longrightarrow \displaystyle\int \sec ax\tan ax\ dx = \frac{1}{a}\sec ax + C$
6. $\dfrac{d}{dx}(\csc ax) = -a\csc ax\cot ax \longrightarrow \displaystyle\int \csc ax\cot ax\ dx = -\frac{1}{a}\csc ax + C$

Example

Evaluate the following indefinite integral: $\int 2\sec^2 2x \; dx$.

Solution: Using rule 3, with $a = 2$, we have

$$\int 2\sec^2 2x \; dx = 2\int \sec^2 2x \; dx = 2\left[\frac{1}{2}\tan 2x\right] + C = \tan 2x + C.$$

Exercise

Evaluate $\int 2\cos(2x) \; dx$.

Other Indefinite Integrals

7. $\dfrac{d}{dx}(e^{ax}) = ae^{ax} \longrightarrow \displaystyle\int e^{ax}\ dx = \frac{1}{a}e^{ax} + C$

8. $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x} \longrightarrow \displaystyle\int \frac{dx}{x} = \ln|x| + C$

9. $\dfrac{d}{dx}\left(\sin^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{1}{\sqrt{a^2 - x^2}} \longrightarrow \displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

10. $\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{a}{a^2 + x^2} \longrightarrow \displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

11. $\dfrac{d}{dx}\left(\sec^{-1}\left|\dfrac{x}{a}\right|\right) = \dfrac{a}{x\sqrt{x^2 - a^2}} \longrightarrow \displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

Initial Value Problems

In some instances, you have enough information to determine the value of C in the antiderivative. These are often called **initial value problems**. Finding $f(x)$ is often called **finding the solution**.

Example

If $f'(x) = 7x^6 - 4x^3 + 12$ and $f(1) = 24$, find $f(x)$.

Solution: $f(x) = \int (7x^6 - 4x^3 + 12)\, dx = x^7 - x^4 + 12x + C$. Now find out which C gives $f(1) = 24$:

$$24 = f(1) = 1 - 1 + 12 + C,$$

so $C = 12$. Hence, $f(x) = x^7 - x^4 + 12x + 12$.

Exercise

Find the function f that satisfies $f''(t) = 6t$ with $f'(0) = 1$ and $f(0) = 2$.

4.9 Book Problems

11-45 (odds), 59-73 (odds), 83-93 (odds)

Advice: To solve 83-93 (odds), read pages 325-326, focusing on Example 8.