§2.4 Infinite Limits

We have examined a number of laws and methods to evaluate limits.

Question

Consider the following limit:

$$\lim_{x \to 0} \frac{1}{x}$$

How would you evaluate this limit?

In the next two sections, we examine limit scenarios involving infinity. The two situations are:

- **Infinite limits:** as x (i.e., the independent variable) approaches a finite number, y (i.e., the dependent variable) becomes arbitrarily large or small

$$\text{looks like: } \lim_{x \to \text{number}} f(x) = \pm\infty$$

- **Limits at infinity:** as x approaches an arbitrarily large or small number, y approaches a finite number

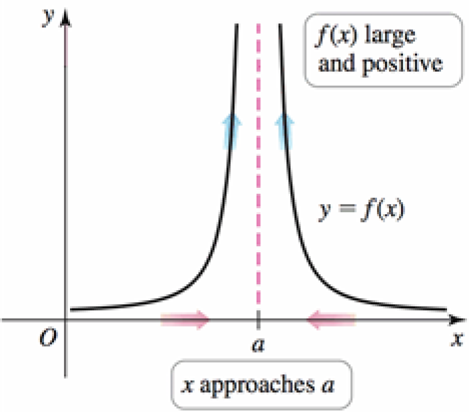
$$\text{looks like: } \lim_{x \to \pm\infty} f(x) = \text{number}$$

Definition (positively infinite limit)

Suppose f is defined for all x near a. If $f(x)$ grows arbitrarily large for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = \infty$$

and say **the limit of $f(x)$ as x approaches a is infinity**.

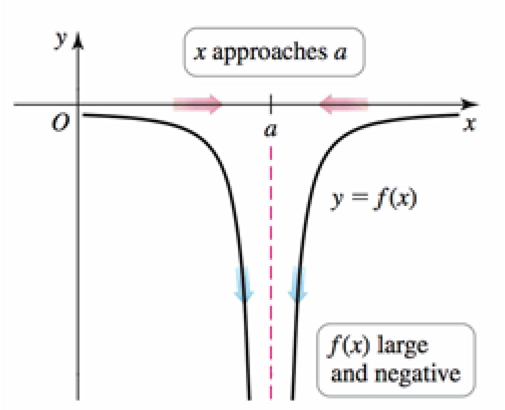

y
f(x) large
and positive
y = f(x)
O
a
x
x approaches a

Definition (negatively infinite limit)

Suppose f is defined for all x near a. If $f(x)$ is negative and grows arbitrarily large in magnitude for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = -\infty$$

and say **the limit of $f(x)$ as x approaches a is negative infinity**.


y
x approaches a
O
a
x
y = f(x)
f(x) large
and negative

The definitions work for one-sided limits, too.

Exercise

Using a graph and a table of values, given $f(x) = \dfrac{1}{x^2 - x}$, determine:

(a) $\lim_{x \to 0^+} f(x)$

(b) $\lim_{x \to 0^-} f(x)$

(c) $\lim_{x \to 1^+} f(x)$

(d) $\lim_{x \to 1^-} f(x)$

Definition

Suppose a function f satisfies at least one of the following:

- $\lim_{x\to a} f(x) = \pm\infty$,
- $\lim_{x\to a^+} f(x) = \pm\infty$
- $\lim_{x\to a^-} f(x) = \pm\infty$

Then the line $x = a$ is called a **vertical asymptote** of f.

Exercise

Given $f(x) = \dfrac{3x-4}{x+1}$, determine, analytically (meaning using "number sense" and without a table or a graph),

(a) $\lim\limits_{x \to -1^+} f(x)$

(b) $\lim\limits_{x \to -1^-} f(x)$

Here is a common way you can summarize your solutions involving limits:

"Since the numerator approaches (#) and the denominator approaches 0, and is (positive/negative), and since (analyze signs here), (insert limit problem)=($+\infty$ / $-\infty$)."

Remember to check for factoring –

Exercise

(a) What is/are the vertical asymptotes of

$$f(x) = \frac{3x^2 - 48}{x+4}?$$

(b) What is $\lim_{x \to -4} f(x)$? Does that correspond to your earlier answer?

2.4 Book Problems

7-10, 15, 17-23, 31-34, 44-45

§2.5 Limits at Infinity

Limits at infinity determine what is called the **end behavior** of a function.

Exercise

(a) Evaluate the following functions at the points $x = \pm 100, \pm 1000, \pm 10000;$

$$f(x) = \frac{4x^2 + 3x - 2}{x^2 + 2} \qquad g(x) = -2 + \frac{\cos x}{\sqrt[3]{x}}$$

(b) What is your conjecture about $\lim_{x \to \infty} f(x)$? $\lim_{x \to -\infty} f(x)$? $\lim_{x \to -\infty} g(x)$? $\lim_{x \to \infty} g(x)$?

Definition

If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write

$$\lim_{x \to \infty} f(x) = L.$$

The line $y = L$ is a **horizontal asymptote** of f.

The limit at negative infinity, $\lim_{x \to -\infty} f(x) = M$, is defined analogously and in this case, the horizontal asymptote is $y = M$.

Infinite Limits at Infinity

Question

Is it possible for a limit to be both an infinite limit and a limit at infinity? (Yes.)

If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write

$$\lim_{x\to\infty} f(x) = \infty.$$

(The limits $\lim\limits_{x\to\infty} f(x) = -\infty$, $\lim\limits_{x\to-\infty} f(x) = \infty$, and $\lim\limits_{x\to-\infty} f(x) = -\infty$ are defined similarly.)

Powers and Polynomials: Let n be a positive integer and let $p(x)$ be a polynomial.

- $n =$ even number: $\lim_{x \to \pm\infty} x^n = \infty$
- $n =$ odd number: $\lim_{x \to \infty} x^n = \infty$ and $\lim_{x \to -\infty} x^n = -\infty$

- (again, assuming n is positive)

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = \lim_{x\to\pm\infty} x^{-n} = 0$$

- For a polynomial, only look at the term with the highest exponent:

$$\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} (\text{constant}) \cdot x^n$$

The constant is called the **leading coefficient**, $\text{lc}(p)$. The highest exponent that appears in the polynomial is called the **degree**, $\deg(p)$.

Rational Functions: Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function.

- If $\deg(p) < \deg(q)$, i.e., the numerator has the smaller degree, then
$$\lim_{x \to \pm\infty} f(x) = 0$$
and $y = 0$ is a horizontal asymptote of f.

- If $\deg(p) = \deg(q)$, i.e., numerator and denominator have the same degree, then
$$\lim_{x \to \pm\infty} f(x) = \frac{\mathsf{lc}(p)}{\mathsf{lc}(q)}$$
and $y = \frac{\mathsf{lc}(p)}{\mathsf{lc}(q)}$ is a horizontal asymptote of f.

- If $\deg(p) > \deg(q)$, (numerator has the bigger degree) then

$$\lim_{x \to \pm\infty} f(x) = \infty \quad \text{or} \quad -\infty$$

  and f has no horizontal asymptote.

- Assuming that $f(x)$ is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeroes of q.

  (This is why it is a good idea to check for factoring and cancelling first!)

When evaluating limits at infinity for rational functions, it is not enough to use the previous rule to show the limit analytically.

To evaluate these limits, we divide both numerator and denominator by x^n, where n is the degree of the polynomial in the denominator.

Exercise

Determine the end behavior of the following functions (in other words, compute both limits, as $x \to \pm\infty$, for each of the functions):

1. $f(x) = \dfrac{x+1}{2x^2-3}$
2. $g(x) = \dfrac{4x^3-3x}{2x^3+5x^2+x+2}$
3. $h(x) = \dfrac{6x^4-1}{4x^3+3x^2+2x+1}$

Example

Determine the end behavior of the following functions.

1. $f(x) = \dfrac{4x^3}{2x^3 + \sqrt{9x^6 + 15x^4}}$ (radical signs appear)
2. $g(x) = \cos x$ (trig)
3. $h(x) = e^x$ (exponential)

Exercise

What are the vertical and horizontal asymptotes of $f(x) = \dfrac{x^2}{2x+1}$?

2.5 Book Problems

9-14, 15-33 (odds), 41-49 (odds), 53-59 (odds), 67