MATH 2554 (Calculus I)

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February 13, 2015

Table of Contents

Monday 9 February (Week 5)

- Quiz #4 tomorrow DURING drill
- Quiz #5 distributed Thurs
- Exam #1 results:
    - Exams returned tomorrow during drill.
    - Median raw scores for each problem are on the webpage, along with the curved grading scale.
    - Out of 75 points, including your signature on the cover page.

- Points back:
  - The number written on the cover page is your raw score. Make sure it is correct.
  - If you have a dispute with the grading, write down in words, on a separate sheet of paper, exactly what your reasoning in solving the problem was. It is not enough to say "I deserve partial credit", or "The grading should have been worth x points." Remember, you are trying to convince me you understand the material. What you write on that separate sheet of paper should reflect that.
  - This must be done by the end of drill. Return the exam, along with your list of appeals, to your drill instructor.

∮ 3.2 Rules of Differentiation

Recall the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(as a function of x, i.e., a formula).

And, for any particular point a, we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Constant Functions

The constant function $f(x) = c$ is a horizontal line with a slope of 0 at every point. This is consistent with the definition of the derivative:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{c - c}{h} \\ &= \lim_{h\to 0} 0 = 0. \end{aligned}$$

Therefore, for constant functions, $f'(x) = 0$.

Fact: For any positive integer n, we can factor

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

For example, when $n = 2$, we get

$$x^2 - a^2 = (x-a)(x+a),$$

which is the difference of squares formula.

Power Rule

Suppose $f(x) = x^n$ where n is a positive integer. Then at a point a,

$$
\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{x^n - a^n}{x - a} \\
&= \lim_{x\to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{x - a} \\
&= (a^{n-1} + a^{n-2}\cdot a + \cdots + a \cdot a^{n-2} + a^{n-1}) = na^{n-1}.
\end{aligned}
$$

Using the formula for the derivative as a function of x, one can show $\dfrac{d}{dx}(x^n) = nx^{n-1}$.

Constant Multiple Rule

Consider a function of the form $cf(x)$, where c is a constant. Just like with limits, we can factor out the constant:

$$\begin{aligned}
\frac{d}{dx}[cf(x)] &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} \\
&= \lim_{h\to 0}\frac{c[f(x+h)-f(x)]}{h} = c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= cf'(x)
\end{aligned}$$

Therefore, $\dfrac{d}{dx}[cf(x)] = cf'(x)$.

Sum Rule

Sums of functions also behave under the same limit laws when we differentiate:

$$\begin{aligned}\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}\\ &= \lim_{h\to 0}\left[\frac{[f(x+h)-f(x)]}{h}+\frac{[g(x+h)-g(x)]}{h}\right]\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)+g'(x)\end{aligned}$$

Example

Using the differentiation rules we have discussed, calculate the derivatives of the following functions. Note which rule(s) you are using.

1. $y = x^5$
2. $y = 4x^3 - 2x^2$
3. $y = -1500$
4. $y = 3x^3 - 2x + 4$

Exponential Functions

Let $f(x) = b^x$, where $b > 0$, $b \neq 1$. To differentiate at 0, we write

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{b^x - b^0}{x} = \lim_{x\to 0} \frac{b^x - 1}{x}.$$

It is not obvious what this limit should be. However, consider the cases $b = 2$ and $b = 3$. By constructing a table of values, we can see that

$$\lim_{x\to 0} \frac{2^x - 1}{x} \approx 0.693 \quad \text{and} \quad \lim_{x\to 0} \frac{3^x - 1}{x} \approx 1.099.$$

So, $f'(0) < 1$ when $b = 2$ and $f'(0) > 1$ when $b = 3$. As it turns out, there is a particular number b, with $2 < b < 3$, whose graph has a tangent line with slope 1 at $x = 0$.
In other words, such a number b has the property that

$$\lim_{x \to 0} \frac{b^x - 1}{x} = 1.$$

Question

What number is it?

Ans: This number is $e = 2.718281828459\ldots$ (known as the Euler number). The function $f(x) = e^x$ is called the *natural exponential function*.

Now, using $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$, we can find the formula for $\frac{d}{dx}(e^x)$:

$$\begin{aligned}
\frac{d}{dx}(e^x) &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} \\
&= e^x \left(\lim_{h \to 0} \frac{e^h - 1}{h} \right) \\
&= e^x \cdot 1 = e^x
\end{aligned}$$

Exercise

- Find the slope of the line tangent to the curve $f(x) = x^3 - 4x - 4$ at the point $(2, -4)$.
- Where does this curve have a horizontal tangent?

Higher-Order Derivatives

If we can write the derivative of f as a function of x, then we can take its derivative, too. The derivative of the derivative is called the *second derivative* of f, and is denoted f''.

In general, we can differentiate f as often as needed. If we do it n times, the nth derivative of f is

$$f^{(n)}(x) = \frac{d^n f}{dx^n} = \frac{d}{dx}[f^{(n-1)}(x)].$$

HW from Section 3.2

Do problems 3-45 (x3) (pp. 142–145 in textbook)

For these problems, use only the rules we have derived so far.

Wednesday 11 February (Week 5)

- Exam curve: How to adjust your score to fit the syllabus points
- Thurs 12 Feb usual weekly take-home quiz (Quiz 5)

∮ 3.3 The Product and Quotient Rules

Issue: Derivatives of products and quotients do NOT behave like they do for limits. As an example, consider

$$f(x) = x^2 \quad \text{and} \quad g(x) = x^3.$$

We can try to differentiate their product in two ways:

- $$\begin{aligned}\frac{d}{dx}[f(x)g(x)] &= \frac{d}{dx}\left(x^5\right)\\ &= 5x^4\end{aligned}$$
- $$\begin{aligned}f'(x)g'(x) &= (2x)(3x^2)\\ &= 6x^3\end{aligned}$$

Question

Which answer is the correct one?

Product Rule

If f and g are any two functions that are differentiable at x, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x).$$

In the example from the previous slide, we have

$$\begin{aligned}\frac{d}{dx}[x^2 \cdot x^3] &= \frac{d}{dx}(x^2) \cdot (x^3) + x^2 \cdot \frac{d}{dx}(x^3)\\ &= (2x) \cdot (x^3) + x^2 \cdot (3x^2)\\ &= 2x^4 + 3x^4\\ &= 5x^4\end{aligned}$$

Derivation of the Product Rule

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)g(x+h) + [-f(x)g(x+h) + f(x)g(x+h)] - f(x)g(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h}\right) + \left(\lim_{h\to 0}\frac{f(x)g(x+h) - f(x)g(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(g(x+h)\frac{f(x+h) - f(x)}{h}\right) + \left(\lim_{h\to 0} f(x)\frac{g(x+h) - g(x)}{h}\right)$$

$$= g(x)f'(x) + f(x)g'(x)$$

Derivation of Quotient Rule

Question

Let $q(x) = \dfrac{f(x)}{g(x)}$. What is $\dfrac{d}{dx}q(x)$?

We can write $f(x) = q(x)g(x)$ and then use the Product Rule:

$$f'(x) = q'(x)g(x) + g'(x)q(x)$$

and now solve for $q'(x)$:

$$q'(x) = \frac{f'(x) - q(x)g'(x)}{g(x)}.$$

Then, to get rid of $q(x)$, plug in $\frac{f(x)}{g(x)}$:

$$q'(x) = \frac{f'(x) - g'(x)\frac{f(x)}{g(x)}}{g(x)}$$

$$= \frac{g(x)\left(f'(x) - g'(x)\frac{f(x)}{g(x)}\right)}{g(x) \cdot g(x)}$$

$$= \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

"LO-D-HI minus HI-D-LO over LO squared"

Exercise

Use the Quotient Rule to find the derivative of

$$\frac{4x^3 + 2x - 3}{x + 1}.$$

Exercise

Find the slope of the tangent line to the curve

$$f(x) = \frac{2x - 3}{x + 1} \text{ at the point } (4, 1).$$

The Quotient Rule also allows us to extend the Power Rule to negative numbers:

If n is any integer, then $\dfrac{d}{dx}\left[x^n\right] = nx^{n-1}$.

Question

How?

Friday 13 February (Week 5)

- possible snow day on Monday, so stay caught up and read $\oint 3.5$
- Wednesday: $\oint 3.6$: The Chain Rule

Exercise

If $f(x) = \dfrac{x(3-x)}{2x^2}$, find $f'(x)$.

Derivative of e^{kx}

For any real number k,

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}.$$

Exercise

What is the derivative of $x^2 e^{3x}$?

Rates of Change

The derivative provides information about the instantaneous rate of change of the function being differentiated (compare to the limit of the slopes of the secant lines from $\oint 2.1$).

For example, suppose that the population of a culture can be modeled by the function $p(t)$. We can find the instantaneous growth rate of the population at any time $t \geq 0$ by computing $p'(t)$ as well as the *steady-state population* (also called the *carrying capacity* of the population). The steady-state population equals

$$\lim_{t \to \infty} p(t).$$

HW from Section 3.3

Do problems 6–51 (x3) (pp. 152–154 in textbook).

$\oint$ 3.4 Derivatives of Trigonometric Functions

Derivative formulas for sine and cosine can be derived using the following limits:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

Exercise

Evaluate $\lim\limits_{x\to 0} \dfrac{\sin 9x}{x}$ and $\lim\limits_{x\to 0} \dfrac{\sin 9x}{\sin 5x}$.

Derivatives of sine and cosine functions

Using the previous limits and the definition of the derivative, we obtain

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$

Trig Identities you should know

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$\cot x = \frac{1}{\tan x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\csc x = \frac{1}{\sin x}$$

Derivatives of other Trig functions

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \cos x - (-\sin x)\sin x}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}$$

So $\dfrac{d}{dx}(\tan x) = \sec^2 x$.

By using trig identities and the Quotient Rule, we obtain

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\csc x \cot x$$
$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \sec x \tan x$$
$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\tan x}\right) = -\csc^2 x$$

Exercise

Compute the derivative of the following functions:

$$f(x) = \frac{\tan x}{1 + \tan x} \qquad g(x) = \sin x \cos x$$

Higher-order trig derivatives

There is a cyclic relationship between the higher order derivatives of $\sin x$ and $\cos x$:

$$
\begin{aligned}
f(x) &= \sin x \\
f'(x) &= \cos x \\
f''(x) &= -\sin x \\
f^{(3)}(x) &= -\cos x \\
f^{(4)}(x) &= \sin x
\end{aligned}
$$

$$
\begin{aligned}
g(x) &= \cos x \\
g'(x) &= -\sin x \\
g''(x) &= -\cos x \\
g^{(3)}(x) &= \sin x \\
g^{(4)}(x) &= \cos x
\end{aligned}
$$

HW from Section 3.4

Do problems 7, 13, 17, 21–27, 33, 35, 44–46, 53–55 (pp. 161–162 in textbook)