Mon 11 Apr

- Exam 3: expect Thursday in drill. Stay tuned for the solutions. Protocol for appeals.
- No (scheduled) office hours Friday.

§4.7 L'Hôpital's Rule

In Ch. 2, we examined limits that were computed using analytical techniques. Some of these limits, in particular those that were indeterminate, could not be computed with simple analytical methods.

For example,

$$\lim_{x\to 0} \frac{\sin x}{x} \quad \text{and} \quad \lim_{x\to 0} \frac{1-\cos x}{x}$$

are both limits that can't be computed by substitution, because plugging in 0 for x gives $\frac{0}{0}$.

Theorem (L'Hôpital's Rule ($\frac{0}{0}$))

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is $\pm\infty$).

(The rule also applies if $x \to a$ is replaced by $x \to \pm\infty$, $x \to a^+$ or $x \to a^-$.)

Example

Evaluate the following limit:

$$\lim_{x \to -1} \frac{x^4 + x^3 + 2x + 2}{x + 1}.$$

Solution: By direct substitution, we obtain 0/0. So we must apply l'Hôpital's Rule (LR) to evaluate the limit:

$$\begin{aligned}
\lim_{x \to -1} \frac{x^4 + x^3 + 2x + 2}{x + 1} &\overset{\text{LR}}{=} \lim_{x \to -1} \frac{\frac{d}{dx}\left(x^4 + x^3 + 2x + 2\right)}{\frac{d}{dx}(x + 1)} \\
&= \lim_{x \to -1} \frac{4x^3 + 3x^2 + 2}{1} \\
&= -4 + 3 + 2 = 1
\end{aligned}$$

Theorem (L'Hôpital's Rule ($\frac{\infty}{\infty}$))

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is $\pm\infty$).

(The rule also applies if $x \to a$ is replaced by $x \to \pm\infty$, $x \to a^+$ or $x \to a^-$.)

Exercise

Evaluate the following limits using l'Hôpital's Rule:

- $\displaystyle \lim_{x\to\infty} \frac{4x^3 - 2x^2 + 6}{\pi x^3 + 4}$
- $\displaystyle \lim_{x\to 0} \frac{\tan 4x}{\tan 7x}$

L'Hôpital's Rule in disguise

Other indeterminate limits in the form $0 \cdot \infty$ or $\infty - \infty$ cannot be evaluated directly using l'Hôpital's Rule.

For $0 \cdot \infty$ cases, we must rewrite the limit in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. A common technique is to divide by the reciprocal:

$$\lim_{x\to\infty} x^2 \sin\left(\frac{1}{5x^2}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{5x^2}\right)}{\frac{1}{x^2}}$$

Exercise

Compute $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$.

For $\infty - \infty$, we can divide by the reciprocal as well as use a change of variables:

Example

Find $\lim_{x\to\infty} x - \sqrt{x^2+2x}$.

Solution:

$$\begin{aligned}
\lim_{x\to\infty} x - \sqrt{x^2+2x} &= \lim_{x\to\infty} x - \sqrt{x^2(1+\frac{2}{x})} \\
&= \lim_{x\to\infty} x - x\sqrt{1+\frac{2}{x}} \\
&= \lim_{x\to\infty} x\left(1-\sqrt{1+\frac{2}{x}}\right) \\
&= \lim_{x\to\infty} \frac{1-\sqrt{1+\frac{2}{x}}}{\frac{1}{x}}
\end{aligned}$$

This is now in the form $\frac{0}{0}$, so we can apply l'Hôpital's Rule and evaluate the limit.

In this case, it may even help to change variables. Let $t = \frac{1}{x}$:

$$\lim_{x\to\infty} \frac{1-\sqrt{1+\frac{2}{x}}}{\dfrac{1}{x}} = \lim_{t\to 0^+} \frac{1-\sqrt{1+2t}}{t}.$$

Other Indeterminate Forms

Limits in the form 1^∞, 0^0, and ∞^0 are also considered indeterminate forms, and to use l'Hôpital's Rule, we must rewrite them in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Here's how:

Assume $\lim\limits_{x\to a} f(x)^{g(x)}$ has the indeterminate form 1^∞, 0^0, or ∞^0.

1. Evaluate $L = \lim\limits_{x\to a} g(x) \ln f(x)$. This limit can often be put in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, which can be handled by l'Hôpital's Rule.
2. Then $\lim\limits_{x\to a} f(x)^{g(x)} = e^L$. Don't forget this step!

Example

Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Solution: This is in the form 1^∞, so we need to examine

$$\begin{aligned}
L &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \\
&= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \\
&\overset{\text{LR}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1.
\end{aligned}$$

NOT DONE! Write

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e^{L}=e^{1}=e.$$