

- Exam 2: Solutions posted as soon as make-ups are in. But we will go through them today. You'll get your test back in drill tomorrow and the curve will be posted.
- Midterm:
  - Covers **everything** up to §3.9. All the slides are up. We will work fast through them today, but solutions to the exercises in the slides will be posted.
  - Morning Section: Walker room 124  
Afternoon Section: Walker room 218  
You **must** take the test with your officially scheduled section.

# Mon 7 Mar (cont.)

- If you have questions about your exam conflicts, contact me NOW.
- Study guide is in MLP.
- Basic scientific calculator is allowed...? Yes.
- Sit in every other seat.
- 15 Questions, 10 points each.
- Don't expect a curve. :(

## §3.9 Derivatives of Logarithmic and Exponential Functions

The natural exponential function  $f(x) = e^x$  has an inverse function, namely  $f^{-1}(x) = \ln x$ . This relationship has the following properties:

1.  $e^{\ln x} = x$  for  $x > 0$  and  $\ln(e^x) = x$  for all  $x$ .
2.  $y = \ln x \iff x = e^y$
3. For real numbers  $x$  and  $b > 0$ ,

$$b^x = e^{\ln(b^x)} = e^{x \ln b}.$$

## Derivative of $y = \ln x$

Using 2. from the last slide, plus implicit differentiation, we can find  $\frac{d}{dx}(\ln x)$ . Write  $y = \ln x$ . We wish to find  $\frac{dy}{dx}$ . From 2.,

$$\frac{d}{dx}(x = e^y) \Rightarrow \frac{d}{dx}x = \frac{d}{dx}(e^y)$$

$$1 = e^y \left( \frac{dy}{dx} \right)$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\text{So } \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

## Derivative of $y = \ln |x|$

Recall, we can only take “ln” of a positive number. However:

- For  $x > 0$ ,  $\ln |x| = \ln x$ , so

$$\frac{d}{dx}(\ln |x|) = \frac{1}{x}.$$

- For  $x < 0$ ,  $\ln |x| = \ln(-x)$ , so

$$\frac{d}{dx}(\ln |x|) = \frac{d}{dx}(\ln(-x)) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}.$$

In other words, the absolute values do not change the derivative of natural log.

## Exercise

Find the derivative of each of the following functions:

- $f(x) = \ln(15x)$
- $g(x) = x \ln x$
- $h(x) = \ln(\sin x)$

## Derivative of $y = b^x$

What about other logs? Say  $b > 0$ . Since  $b^x = e^{\ln b^x} = e^{x \ln b}$  (by 3. on the earlier slide),

$$\begin{aligned}\frac{d}{dx}(b^x) &= \frac{d}{dx}(e^{x \ln b}) \\ &= e^{x \ln b} \cdot \ln b \\ &= b^x \ln b.\end{aligned}$$

## Exercise

Find the derivative of each of the following functions:

- $f(x) = 14^x$
- $g(x) = 45(3^{2x})$

## Exercise

Determine the slope of the tangent line to the graph

$$f(x) = 4^x \text{ at } x = 0.$$



## Story Problem Example

### Example

The energy (in Joules) released by an earthquake of magnitude  $M$  is given by the equation

$$E = 25000 \cdot 10^{1.5M}.$$

- (a) How much energy is released in a magnitude 3.0 earthquake?
- (b) What size earthquake releases 8 million Joules of energy?
- (c) What is  $\frac{dE}{dM}$  and what does it tell you?

## Derivatives of General Logarithmic Functions

The relationship  $y = \ln x \iff x = e^y$  applies to logarithms of other bases:

$$y = \log_b x \iff x = b^y.$$

Now taking  $\frac{d}{dx}(x = b^y)$  we obtain

$$1 = b^y \ln b \left( \frac{dy}{dx} \right)$$

$$\frac{dy}{dx} = \frac{1}{b^y \ln b}$$

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

## Example

The derivative of  $f(x) = \log_2(10x)$  is

- A.  $\frac{1}{10x}$
- B.  $\frac{1}{x \ln 2}$
- C.  $\frac{1}{x}$
- D.  $\frac{10}{x \ln 2}$

## Neat Trick: Logarithmic Differentiation

### Example

Compute the derivative of  $f(x) = \frac{x^2(x-1)^3}{(3+5x)^4}$ .

**Solution:** We can use logarithmic differentiation – first take the natural log of both sides and then use properties of logarithms.

$$\begin{aligned}\ln(f(x)) &= \ln\left(\frac{x^2(x-1)^3}{(3+5x)^4}\right) \\ &= \ln x^2 + \ln(x-1)^3 - \ln(3+5x)^4 \\ &= 2 \ln x + 3 \ln(x-1) - 4 \ln(3+5x)\end{aligned}$$

Now we take  $\frac{d}{dx}$  on both sides:

$$\frac{1}{f(x)} \left( \frac{df}{dx} \right) = 2 \left( \frac{1}{x} \right) + 3 \left( \frac{1}{x-1} \right) - 4 \left( \frac{1}{3+5x} \right) \quad (5)$$

$$\frac{f'(x)}{f(x)} = \frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}$$

Finally, solve for  $f'(x)$ :

$$\begin{aligned} f'(x) &= f(x) \left[ \frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x} \right] \\ &= \frac{x^2(x-1)^3}{(3+5x)^4} \left[ \frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x} \right] \end{aligned}$$

## Exercise

Use logarithmic differentiation to calculate the derivative of

$$f(x) = \frac{(x+1)^{\frac{3}{2}}(x-4)^{\frac{5}{2}}}{(5x+3)^{\frac{2}{3}}}.$$

## 3.9 Book Problems

9-29 (odds), 55-67 (odds)



# Midterm Review

- §2.1-2.2
  - **Material may not be explicitly tested, but the topics here are foundational to later sections.**
- §2.3 Techniques for Computing Limits
  - **Be able to do questions similar to 1-48.**
  - Know and be able to compute limits using analytical methods (e.g., limit laws, additional techniques).
  - Be able to evaluate one-sided and two-sided limits of functions.
  - Know the Squeeze Theorem and be able to use this theorem to determine limits.

# Midterm Review (cont.)

Exercise (problems from past midterm)

Evaluate the following limits:

- $\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x^2 - 9}$

- $\lim_{\theta \rightarrow 0} \frac{\sec \theta \tan \theta}{\theta}$

# Midterm Review (cont.)

- §2.4 Infinite Limits
  - **Be able to do questions similar to 17-30.**
  - Be able to use a graph, a table, or analytical methods to determine infinite limits.
  - Be able to use analytical methods to evaluate one-sided limits.
  - Know the **definition** of a vertical asymptote and be able to determine whether a function has vertical asymptotes.

# Midterm Review (cont.)

- §2.5 Limits at Infinity
  - **Be able to do questions similar to 9-30 and 38-46.**
  - Be able to find limits at infinity and horizontal asymptotes.
  - Know how to compute the limits at infinity of rational functions and algebraic functions.
  - Be able to list horizontal and/or vertical asymptotes of a function.

# Midterm Review (cont.)

## Exercise

Determine the horizontal asymptote(s) for the function

$$f(x) = \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$$

A.  $y = 2$

B.  $y = 0$

C.  $y = -2$

D.  $y = \pm 2$

# Midterm Review (cont.)

- §2.6 Continuity
  - **Be able to do questions similar to 9-44.**
  - Know the definition of continuity and be able to apply the continuity checklist.
  - Be able to determine the continuity of a function (including those with roots) on an interval.
  - Be able to apply the Intermediate Value Theorem to a function.

# Midterm Review (cont.)

Exercise (problem from past midterm)

Determine the value of  $k$  so the function is continuous on  $0 \leq x \leq 2$ .

$$f(x) = \begin{cases} x^2 + k & 0 \leq x \leq 1 \\ -2kx + 4 & 1 < x \leq 2 \end{cases}$$

# Midterm Review (cont.)

- §3.1 Introducing the Derivative
  - **Be able to do questions similar to 11-32.**
  - Know the definition of a derivative and be able to use this definition to calculate the derivative of a given function.
  - Be able to determine the equation of a line tangent to the graph of a function at a given point.
  - Know the 3 conditions for when a function is not differentiable at a point, and why these three conditions make a function not differentiable at the given point.



# Midterm Review (cont.)

- §3.2 Working with Derivatives
  - Be able to use the graph of a function to sketch the graph of its derivative, without computing derivatives
  - Know the 3 conditions for when a function is not differentiable at a point, and why these three conditions make a function not differentiable at the given point
  - Be able to determine where a function is not differentiable

# Midterm Review (cont.)

- §3.3 Rules for Differentiation
  - **Be able to do questions similar to 7-41.**
  - Be able to use the various rules for differentiation (e.g., constant rule, power rule, constant multiple rule, sum and difference rule) to calculate the derivative of a function.
  - Know the derivative of  $e^x$ .
  - Be able to find slopes and/or equations of tangent lines.
  - Be able to calculate higher-order derivatives of functions.

# Midterm Review (cont.)

## Exercise

Given that  $y = 3x + 2$  is tangent to  $f(x)$  at  $x = 1$  and that  $y = -5x + 6$  is tangent to  $g(x)$  at  $x = 1$ , write the equation of the tangent line to  $h(x) = f(x)g(x)$  at  $x = 1$ .

# Midterm Review (cont.)

- §3.4 The Product and Quotient Rules
  - **Be able to do questions similar to 7-42 and 47-52.**
  - Be able to use the product and/or quotient rules to calculate the derivative of a given function.
  - Be able to use the product and/or quotient rules to find tangent lines and/or slopes at a given point.
  - Know the derivative of  $e^{kx}$ .
  - Be able to combine derivative rules to calculate the derivative of a function.

**Note:** Functions are not always given by a formula. When faced with a problem where you don't know where to start, go through the rules first.

# Midterm Review (cont.)

## Exercise

Suppose you have the following information about the functions  $f$  and  $g$ :

$$f(1) = 6 \quad f'(1) = 2 \quad g(1) = 2 \quad g'(1) = 3$$

- Let  $F = 2f + 3g$ . What is  $F(1)$ ? What is  $F'(1)$ ?
- Let  $G = fg$ . What is  $G(1)$ ? What is  $G'(1)$ ?

# Midterm Review (cont.)

- §3.5 Derivatives of Trigonometric Functions

- **Be able to do questions similar to 1-55.**
- Know the two special trigonometric limits

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$$

and be able to use them to solve other similar limits.

- Know the derivatives of  $\sin x$ ,  $\cos x$ ,  $\tan x$ ,  $\cot x$ ,  $\sec x$ ,  $\csc x$ , and be able to use the quotient rule to derive the derivatives of  $\tan x$ ,  $\cot x$ ,  $\sec x$ , and  $\csc x$ .
- Be able to calculate derivatives (including higher order) involving trig functions using the rules for differentiation.

# Midterm Review (cont.)

## Exercise

Calculate the derivative of the following functions:

- $f(x) = (1 + \sec x) \sin^3 x$
- $g(x) = \frac{\sin x + \cot x}{\cos x}$

## Exercise

Evaluate  $\lim_{x \rightarrow -3} \frac{\sin(x+3)}{x^2 + 8x + 15}$ .

# Midterm Review (cont.)

- §3.6 Derivatives as Rates of Change
  - **Be able to do questions similar to 11-18.**
  - Be able to use the derivative to answer questions about rates of change involving:
    - Position and velocity
    - Speed and acceleration
    - Growth rates
    - Business applications



# Midterm Review (cont.)

- Be able to use a position function to answer questions involving velocity, speed, acceleration, height/distance at a particular time  $t$ , maximum height, and time at which a given height/distance is achieved.
- Be able to use growth models to answer questions involving growth rate and average growth rate, and cost functions to answer questions involving average and marginal costs.

# Midterm Review (cont.)

- §3.7 The Chain Rule
  - **Be able to do questions similar to 7-43.**
  - Be able to use both versions of the Chain Rule to find the derivative of a composition function.
  - Be able to use the Chain Rule more than once in a calculation involving more than two composed functions.
  - Know and be able to use the Chain Rule for Powers:

$$\frac{d}{dx} (f(x))^n = n (f(x))^{n-1} f'(x)$$

# Midterm Review (cont.)

## Exercise

Suppose  $f'(9) = 10$  and  $g(x) = f(x^2)$ . What is  $g'(3)$ ?

# Midterm Review (cont.)

- §3.8 Implicit Differentiation
  - **Be able to do questions similar to 5-26 and 33-46.**
  - Be able to use implicit differentiation to calculate  $\frac{dy}{dx}$ .
  - Be able to use the derivative found from implicit differentiation to find the slope at a given point and/or a line tangent to the curve at the given point.
  - Be able to calculate higher-order derivatives of implicitly defined functions.
  - Be able to calculate  $\frac{dy}{dx}$  when working with functions containing rational functions.

# Midterm Review (cont.)

## Exercise

Use implicit differentiation to calculate  $\frac{dz}{dw}$  for

$$e^{2w} = \sin(wz)$$

## Exercise

If  $\sin x = \sin y$ , then

- $\frac{dy}{dx} = ?$
- $\frac{d^2y}{dx^2} = ?$

# Midterm Review (cont.)

- §3.9 Derivatives of Logarithmic and Exponential Functions
  - Be able to compute derivatives involving  $\ln x$  and  $\log_b x$
  - Be able to compute derivatives of exponential functions of the form  $b^x$
  - Be able to use logarithmic differentiation to determine  $f'(x)$

## Running Out of Time on the Exam Plus other Study Tips

- Do practice problems completely, from beginning to end (as if it were a quiz). You might think you understand something but when it's time to write down the details things are not so clear.
- Find a buddy who understands concepts a little better than you and work on problems for 2-3 hours. Then find a buddy who is struggling and work with them 2-3 hours.
- Don't count on cookie cutter problems. If you are doing a practice problem where you've memorized all the steps, make sure you understand why each step is needed. The exam problems may have a small variation from homeworks and quizzes. If you're not prepared, it'll come as a "twist" on the exam...

## Running Out of Time on the Exam Plus other Study Tips (cont.)

- If you encounter an unfamiliar type of problem on the exam, relax, because it's most likely not a trick! The solutions will always rely on the information from the required reading/assignments. Take your time and do each baby step carefully.
- During the exam, do the problems you are most confident with first!
- During the exam, budget your time. Count the problems and divide by 50 minutes. The easier questions will take less time so doing them first leaves extra time for the harder ones. When studying, aim for 10 problems per hour (i.e., 6 minutes per problem).



## Running Out of Time on the Exam Plus other Study Tips (cont.)

- Always make sure you **answer the question**. This is also a good strategy if you're not sure how to start a problem, figure out what the question wants first.
- The exam is not a race. If you finish early take advantage of the time to check your work. You don't want to leave feeling smug about how quickly you finished only to find out next week you lost a letter grade's worth of points from silly mistakes.

## Other Study Tips

- Brush up on algebra, especially radicals, logs, common denominators, etc. Many times knowing the right algebra will simplify the problem!
- When in doubt, show steps.
- You will be punished for wrong notation. The slides for §3.1 show different notations for the derivative. Make sure whichever one you use in your work, that you are using it correctly.
- Read the question!
- Do the book problems.
- Look at the pictures in the book and the interactive applets on MLP.