MATH 2554 (Calculus I)
Spring 2016

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Exam 2: Derivatives (§3.2-3.8)

Exam Instructions: You have 50 minutes to complete this exam. Justification is required for all problems. No electronic devices (calculators, phones, iDevices, computers, etc). If you finish early then you may leave, UNLESS there are less than 5 minutes of class left. To prevent disruption, if you finish with less than 5 minutes of class remaining then please stay seated and quiet.

In addition, please provide the following data:

Drill Instructor: ______________________________

Drill Time: ______________________________

Your signature below indicates that you have read this page and agree to follow the Academic Honesty Policies of the University of Arkansas.

Signature: **(1 pt)** ____

Good luck!

1. **(3.8 M 2 min)** Find $\frac{dy}{dx}$, given $x^2(y-2) - e^y = 0$.
    - **implicit**
    - e^x
    - **prod rule**
2. **(3.8 M 1 min)** Find $\frac{dw}{dx}$, given $w^2 - 3zw = 2$.
    - **other vars**
    - **implicit**
    - **prod rule**
3. **(3.8 M 4 min)** Find the slope of the curve $5\sqrt{x} - 10\sqrt{y} = \sin x$ at the point $(4\pi, \pi)$.
    - **implicit**
    - **plug-in**
    - **trig**
    - **alg fct**
4. **(3.8 M 3 min)** Find $\frac{dv}{du}$, given $\sqrt{3u^7 + v^2} = \sin^2 v + 100uv$.
    - **other vars**
    - **alg fct**
    - **implicit**
    - **trig**
    - **prod rule**
5. **(3.8 M 1 min)** The volume of a torus with an inner radius of a and an outer radius of b is
$$V = \tfrac{1}{4}\pi^2(b+a)(b-a).$$
Find $\frac{db}{da}$ for a torus with a volume of $64\pi^2$.
    - **implicit**
    - **context/other vars**
    - **prod rule**
6. **(3.8 H 3 min)** The lateral surface area of a cone of radius r and height h (the surface area excluding the base) is
$$A = \pi r\sqrt{r^2 + h^2}.$$
Find $\frac{dr}{dh}$ for a cone with a lateral surface area of 1500π.

- **implicit**
- **alg fct**
- **prod rule**
- **context/other vars**

7. **(3.8 M 4 min)** Find y'', given $x^4 + y^4 = 64$.

   - **higher order**
   - **implicit**

8. **(3.8 H 5 min)** Find y'', given $\sin x + x^2 y = 10$.

   - **higher order**
   - **implicit**
   - **trig**
   - **prod rule**

9. **(3.7 M 3 min)** Recall that a function f is **even** means for all x, $f(-x) = f(x)$, i.e, the graph is symmetric about the y-axis. A function f is **odd** means for all x, $f(-x) = -f(x)$, or, equivalently, $-f(-x) = f(x)$. In that case the graph is symmetric about the origin.

   (a) If f is even and differentiable, then is f' even, odd, or neither?

   (b) If f is odd and differentiable, then is f' even, odd, or neither?

   - **chain**
   - **abst fct**
   - **context**

10. **(3.7 H 1 min)** Find $\frac{d^2}{dx^2}(f(g(x)))$.

    - **higher order**
    - **chain**
    - **abst fct**

11. **(3.7 H 10 min)** A mechanical oscillator (such as a mass on a spring or a pendulum) subject to frictional forces satisfies the equation (called a **differential equation**)

$$y''(t) + 2y'(t) + 5y(t) = 0,$$

where y is the displacement of the oscillator from its equilibrium position. Verify by substitution that the function

$$y(t) = e^{-t}(\sin 2t - 2\cos 2t)$$

satisfies this equation.

- **trig**
- **higher order**
- **prod rule**
- **context/other vars**
- **chain**

12. **(3.7 M 2 min)** Suppose f is differentiable on $[-2, 2]$ with $f'(0) = 3$ and $f'(1) = 5$. Let $g(x) = f(\sin x)$. Evaluate the following expressions.

    (a) $g'(0)$

    (b) $g'\left(\frac{\pi}{2}\right)$

    (c) $g'(\pi)$

    - **chain**
    - **plug in**
    - **trig**
    - **abst fct**

13. **(3.7 H 2 min)** Assume f is a differentiable function whose graph passes through the point $(1, 4)$. Suppose $g(x) = f(x^2)$ and the line tangent to the graph of f at $(1, 4)$ is $y = 3x + 1$. Determine each of the following.

    (a) $g(1)$

    (b) $g'(x)$

    (c) $g'(1)$

    - **chain**
    - **tan line given**
    - **abst fct**
    - **plug in**

14. **(3.7 M 1 min)** Suppose f and g are differentiable and non-negative at x. Find the derivative of $y = \sqrt{f(x)g(x)}$.

    - **chain**
    - **abst fct**
    - **alg fct**
    - **prod rule**

15. **(3.7 H 1 min)** Suppose f and g are differentiable for all real numbers, and m and n are integers. Find y', given $y = f(g(x^m))^n$.

   - **chain**
   - **proof**

16. **3.7** For each of the following, find y'.

   (a) **(M 2 min)** $y = \left(\dfrac{e^x}{x+1}\right)^8$
      - **qrule**
      - **chain**
      - e^x

   (b) **(H 1 min)** $y = \tan(xe^x)$
      - **other trig**
      - **prod rule**
      - e^x
      - **chain**

   (c) **(M 1 min)** $y = \left(\dfrac{e^x}{4x+2}\right)^5$
      - **chain**
      - **qrule**
      - e^x

   (d) **(M 2 min)** $y = e^{2x}(2x-7)^5$
      - **prod rule**
      - **chain**
      - e^x

   (e) **(M 1 min)** $y = \dfrac{te^t}{t+1}$
      - **prod rule**
      - **other vars**
      - e^x
      - **qrule**

   (f) **(M 1 min)** $y = (z+4)^3 \tan z$
      - **prod rule**
      - **other vars**
      - **chain**
      - **other trig**

(g) **(M 2 min)** $y = \sqrt{(3x-4)^2 + 3x}$

- **chain x2**
- **alg fct**

(h) **(M 1 min)** $y = \sin^2\left(3^{3x+1}\right)$

- **trig**
- **chain x3**
- b^x

(i) **(M 1 min)** $y = \cos^4\left(7x^3\right)$

- **trig**
- **chain x2**

(j) **(E 1 min)** $y = (1 - e^{-0.05x})^{-1}$

- **neg exp**
- e^x
- **chain x2**

(k) **(H 2 min)** $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

- **chain x2**
- **alg fct**

17. **(3.7 M 5 min)** Let $h(x) = f(g(x))$ and $k(x) = g(g(x))$. Use the table

x	1	2	3	4	5
f′(x)	-6	-3	8	7	2
g(x)	4	1	5	2	3
g′(x)	9	7	3	-1	-5

to compute the following derivatives.

(a) $h'(1)$

(b) $h'(2)$

(c) $h'(3)$

(d) $k'(3)$

(e) $k'(1)$

(f) $k'(5)$

- **chain**
- **table derivs**
- **abst fct**

18. **(3.6 M 7 min)** A cost function of the form $C(x) = \frac{1}{2}x^2$ reflects **diminishing returns to scale**. Find and graph the cost, average cost, and marginal cost functions. Interpret the graphs and explain the idea of diminishing returns.

   - **graph**
   - **interpret**
   - $C(s)$

19. **(3.6 CALCULATOR 11 min)** Two stones are thrown vertically upward with matching initial velocities of 48 ft/s at time $t = 0$. One stone is thrown from the edge of a bridge that is 32 ft above the ground and the other stone is thrown from gorund level. The height of the stone thrown from the bridge after t secconds is

$$f(t) = -16t^2 + 48t + 32$$

and the height of the stone thrown from the ground after t seconds is

$$g(t) = -16t^2 + 48t.$$

   (a) Show that the stones reach their heigh points at the same time.

   (b) How much higher does the stone thrown from the bridge go than the stone thrown from the ground?

   (c) When do the stones strike the ground and with what velocities?

20. **(3.6 M 1 min)** On the moon, a feather will fall to the ground at the same rate as a heavy stone. Suppose a feather is dropped from a height of 40 m above the surface of the moon. Then its height s (in meters) above the ground after t seconds is $s = 40 - 0.8t^2$. Determine the velocity and acceleration of the feather the moment it strikes the surface of the moon.

   - **story**

21. **(3.6 CALCULATOR 7 min)** Suppose a stone is thrown vertically upward from the edge of a cliff on Mars (where the acceleration due to gravity is only about 12 ft/s^2) with an initial velocity of 64 ft/s from a height of 192 ft above the ground. The height s of the stone above the ground after t seconds is given by

$$s = -6t^2 + 64t + 192.$$

   (a) Determine the velocity v of the stone after t seconds.

   (b) When does the stone reach its highest point?

   (c) What is the height of the stone at the highest point?

   (d) When does the stone strike the ground?

(e) With what velocity does the stone strike the ground?

22. **(3.5 M 3 min)** Verify the following derivative formula using the Quotient Rule:

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

- **qrule**
- **other trig**
- **proof**

23. **3.5** Evaluate the following limits:

(a) **(E 1 min)** $\lim_{x\to 0} \frac{\sin 5x}{3x}$

- **sin lim**

(b) **(M 4 min)** $\lim_{x\to 0} \frac{\sin 3x}{\tan 4x}$

- **sin lim**
- **other trig fcts**

(c) **(M 2 min)** $\lim_{\theta\to 0} \frac{\cos^2\theta - 1}{\theta}$

- **cos lim**
- **factoring**
- **other vars**

(d) **(M 2 min)** $\lim_{\theta\to 0} \frac{\sec\theta - 1}{\theta}$

- **cos lim**
- **other trig fcts**
- **other vars**

(e) **(M 2 min)** $\lim_{x\to -3} \frac{\sin(x+3)}{x^2 + 8x + 15}$

- **sin lim**
- **factoring**
- **change of vars**

24. **3.4** One of several **Leibniz Rules** in calculus deals with higher-order derivatives of products. Let $(fg)^{(n)}$ denote the nth derivative of the product fg, for $n \geq 1$.

   (a) **(M 1 min)** Prove that $(fg)^{(2)} = f''g + 2f'g' + fg''$.

      - **prod rule**
      - **proof**

   (b) **(NO)** Prove that, in general,

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)},$$

   where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ are **binomial coefficients**.

25. **(3.4 CALCULATOR 6 min)** The magnitude of the gravitaional force between two objects of mass M and m is given by $F(x) = -\frac{GMm}{x^2}$, where x is the distance between the centers of mass of the objects and $G = 6.7 \times 10^{11}$ N-m^2/kg^2 is the gravitational constant (N stands for newton, the unit of force; the negative sign indicates an attractive force).

   (a) Find the instantaneous rate of change of the force with respect to the distance between the objects.

   (b) For two identical objects of mass $M = m = 0.1$ kg, what is the instantaneous rate of change of the force at a separation of $x = 0.01$ m?

   (c) Does the instantaneous rate of change of the force increase or decrease with the separation? Explain.

26. **(3.4 H 5 min)** Suppose the line tangent to the graph of f at $x = 2$ is $y = 4x + 1$ and suppose $y = 3x - 2$ is the line tangent to the graph of g at $x = 2$. Find an equation of the line tangent to the following curves at $x = 2$.

   (a) $y = f(x)g(x)$

   (b) $y = \dfrac{f(x)}{g(x)}$

   - **only tan line is given**
   - **eqn of tan line**
   - **prod rule**
   - **qrule**
   - **abst fcts**

27. **(3.4 M 2 min)** Use the following table to find the given derivatives.

x	1	2	3	4
f(x)	5	4	3	2
f′(x)	3	5	2	1
g(x)	4	2	5	3
g′(x)	2	4	3	1

(a) $\left.\dfrac{d}{dx}\left(f(x)g(x)\right)\right|_{x=4}$

(b) $\left.\dfrac{d}{dx}\left(xf(x)\right)\right|_{x=3}$

(c) $\left.\dfrac{d}{dx}\left(\dfrac{xf(x)}{g(x)}\right)\right|_{x=4}$

- **abst fct w/** x
- **prod rule**
- **qrule**
- **table derivs**

28. **3.4** Find the derivative of each of the following functions.

(a) **(H 6 min)** $f(x) = \dfrac{4 - x^2}{x - 2}$

- **qrule**
- **factoring**
- **undef pt**

(b) **(M 1 min)** $f(z) = z^2(e^3 z + 4) - \dfrac{2z}{z^2 + 1}$

- **gnarly**
- **prod rule**
- **qrule**
- **constant** e

(c) **(NO)** $y = \dfrac{x - a}{\sqrt{x} - \sqrt{a}}$, where a is a positive constant

29. **3.4** Suppose $f(2) = 2$ and $f'(2) = 3$. Let $g(x) = x^2 \cdot f(x)$ and $h(x) = \dfrac{f(x)}{x - 3}$.

(a) **M 1 min)** Find an equation of the line tangent to $y = g(x)$ at $x = 2$.

- **prod rule**

- **abst fct w/ x**
- **eqn of tan line**
- **plug in**

(b) **(M 1 min)** Find an equation of the line tangent to $y = h(x)$ at $x = 2$.

- **qrule**
- **eqn of tan line**
- **abst fct w/ x**

30. **(3.4 E 1 min)** Given $f(x) = \frac{1}{x}$, find $f'(x)$, $f''(x)$, and $f'''(x)$.

- **higher order**
- **neg exp**

31. **(3.3 E 2 min)** Use the table to find the following derivatives.

x	1	2	3	4	5
f′(x)	3	5	2	1	4
g′(x)	2	4	3	1	5

(a) $\left.\dfrac{d}{dx}\left(f(x)+g(x)\right)\right|_{x=1}$

(b) $\left.\dfrac{d}{dx}\left(2x-3g(x)\right)\right|_{x=4}$

- **table derivs**
- **abst fct w/ x**

32. **(3.3 M 2 min)** Find the equation of the line tangent to the curve $y = x + \sqrt{x}$ that has slope 2.

- **eqn of tan line but solve**
- **alg fct**

33. **3.3** Suppose $f(3) = 1$ and $f'(3) = 4$. Let $g(x) = x^2 + f(x)$ and $h(x) = 3f(x)$.

(a) **(E 1 min)** Find an equation of the line tangent to $y = g(x)$ at $x = 3$.

- **eqn of tan line**
- **abst fct w/ x**

(b) **(E 1 min)** Find an equation of the line tangent to $y = h(x)$ at $x = 3$.

- **eqn of tan line**
- **abst fct**

34. **3.3** Find $f'(x), f''(x)$, and $f'''(x)$ for the following functions.

   (a) **(E 1 min)** $f(x) = 3x^3 + 5x^2 + 6x$
      - **polynomial**
      - **higher order**

   (b) **(E 1 min)** $f(x) = 3x^2 + 5e^x$
      - e^x
      - **polynomial**
      - **higher order**

   (c) **(E 1 min)** $f(x) = 10e^x$
      - e^x
      - **higher order**

35. **(3.2 E 3 min)** A common model for population growth uses the logistic (or **sigmoid**) curve. Consider the logistic curve in the figure, where $P(t)$ is the population at time $t \geq 0$.

   (a) At approximately what time is the rate of growth P' the greatest?

   (b) Is P' positive or negative for $t \geq 0$?

   (c) Is P' an increasing or decreasing function of time (or neither)?

   (d) Sketch the graph of P' on the same axes.

   - **context**

36. **(3.2 M 2 min)** Create the graph of a continuous function f such that

$$f'(x) = \begin{cases} 1 & x < 0 \\ 0 & 0 < x < 1 \\ -1 & x > 1. \end{cases}$$

   Is more than one graph possible?

   - **graph**