

## Exam 3: Transformations and line integrals (§13.7-14.5)

**Exam Instructions:** You have 50 minutes to complete this exam. Justification is required for all problems. No electronic devices (phones, iDevices, computers, etc) except for a **basic scientific calculator**. On story problems, round to one decimal place. If you finish early then you may leave, UNLESS there are less than 5 minutes of class left. To prevent disruption, if you finish with less than 5 minutes of class remaining then please stay seated and quiet.

In addition, please provide the following data:

Drill Instructor: \_\_\_\_\_

Drill Time: \_\_\_\_\_

Your signature below indicates that you have read this page and agree to follow the Academic Honesty Policies of the University of Arkansas.

Signature: (1 pt) \_\_\_\_\_

Good luck!

Formulas you may need:

<p><b>Jacobian Determinant of a Transformation of Two Variables</b></p> $J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}.$ <p><b>Jacobian Determinant of a Transformation of Three Variables</b></p> $J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$ <p>Recall that by expanding about the first row,</p> $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$	<p><b>Curl of a Vector Field</b></p> $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$ <p>← Unit vectors ← Components of <math>\nabla</math> ← Components of <math>\mathbf{F}</math></p> $= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}.$ <p><b>Divergence of a Vector Field</b></p> $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$
<p><b>Scalar Line Integrals in <math>\mathbb{R}^2</math></b></p> $\int_C f \, ds = \int_a^b f(x(t), y(t))  \mathbf{r}'(t)  \, dt = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} \, dt.$ <p><b>Scalar Line Integrals in <math>\mathbb{R}^3</math></b></p> $\int_C f \, ds = \int_a^b f(x(t), y(t), z(t))  \mathbf{r}'(t)  \, dt = \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt.$ <p><b>Green's Theorem—Circulation Form</b></p> <p>The circulation form of Green's Theorem is also called the <i>tangential</i>, or <i>curl</i>, form.</p> $\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{circulation}} = \oint_C f \, dx + g \, dy = \iint_R \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA.$	<p><b>Different Forms of Line Integrals of Vector Fields</b></p> <p>The line integral <math>\int_C \mathbf{F} \cdot \mathbf{T} \, ds</math> may be expressed in the following forms,</p> $\begin{aligned} \int_a^b \mathbf{F} \cdot \mathbf{r}'(t) \, dt &= \int_a^b (f(t)x'(t) + g(t)y'(t) + h(t)z'(t)) \, dt \\ &= \int_C f \, dx + g \, dy + h \, dz \\ &= \int_C \mathbf{F} \cdot d\mathbf{r}. \end{aligned}$ <p>For line integrals in the plane,</p> $\int_a^b \mathbf{F} \cdot \mathbf{r}'(t) \, dt = \int_a^b (f(t)x'(t) + g(t)y'(t)) \, dt = \int_C f \, dx + g \, dy = \int_C \mathbf{F} \cdot d\mathbf{r}.$
<p><b>Area of a Plane Region by Line Integrals</b></p> <p>Under the conditions of Green's Theorem, the area of a region <math>R</math> enclosed by a curve <math>C</math> is</p> $\oint_C x \, dy = - \oint_C y \, dx = \frac{1}{2} \oint_C (x \, dy - y \, dx).$	<p><b>Flux</b></p> <p>The flux of the vector field <math>\mathbf{F}</math> across <math>C</math> is</p> $\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_a^b (f(t)y'(t) - g(t)x'(t)) \, dt,$ <p>where <math>\mathbf{n} = \mathbf{T} \times \mathbf{k}</math> is the unit normal vector and <math>\mathbf{T}</math> is the unit tangent vector consistent with the orientation.</p>
<p><b>Green's Theorem, Flux Form</b></p> <p>The flux form of Green's Theorem is also called the <i>normal</i>, or <i>divergence</i>, form.</p> $\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n} \, ds}_{\text{outward flux}} = \oint_C f \, dy - g \, dx = \iint_R \left( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA$	

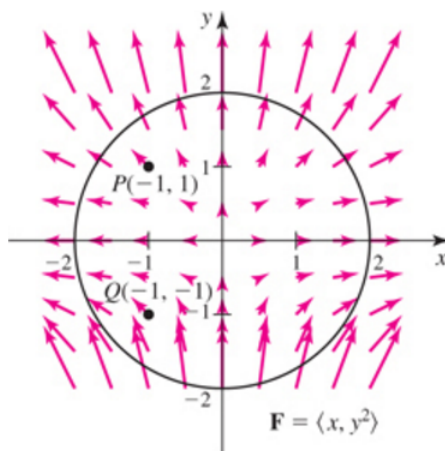
1. **(16 pts)** Compute the Jacobian,  $J(\rho, \varphi, \theta)$ , of the following transformation taking Cartesian to spherical coordinates:

$$x = \rho \sin \varphi \cos \theta \quad y = \rho \sin \varphi \sin \theta \quad z = \rho \cos \varphi$$

You must show your work and simplify.

2. **(16 pts)** Use Green's Theorem to find the area inside an ellipse with major and minor axes of length 12 and 8, respectively. In case you need them, the half-angle formulas are  $\cos^2 x = \frac{1 + \cos 2x}{2}$  and  $\sin^2 x = \frac{1 - \cos 2x}{2}$ .

3. The vector field  $\mathbf{F} = \langle x, y^2 \rangle$ , the circle  $C$  of radius 2 centered at the origin, and two points  $P = (-1, 1)$  and  $Q = (-1, -1)$ , are given in the figure below.



- (a) **(5 pts)** Without computing the divergence, does the graph suggest that the divergence is positive or negative at  $P$  and  $Q$ ? Justify your answer.
- (b) **(5 pts)** Compute the divergence of  $\mathbf{F}$  at  $P$  and  $Q$  to confirm your answer to part (a).
- (c) **(3 pts)** Label on the graph where the flux across  $C$  is outward.
- (d) **(8 pts)** Is the **net** outward flux across  $C$  positive or negative? You must justify your answer.

4. ( pts) Let  $\mathbf{F} = \langle 8xyz, 4x^2z, 4x^2y \rangle$ .

(a) (8 pts) What is the curl of  $\mathbf{F}$ ?

(b) (10 pts) What is the circulation of  $\mathbf{F}$  along  $C$ , where  $C$  is the closed curve formed by the square whose corners are the points  $(1, 1)$ ,  $(-1, 1)$ ,  $(-1, -1)$ , and  $(1, -1)$ ?

5. **(16 pts)** Evaluate the scalar line integral  $\int_C (x^2 + y^3) \, ds$ , where  $C$  is the line segment from  $(0, 0)$  to  $(5, 5)$ .

6. (12 pts) Match vector fields (a)-(d) with graphs (A)-(D).

(a)  $\mathbf{F} = \langle 0, x^2 \rangle$

(b)  $\mathbf{F} = \langle x - y, x \rangle$

(c)  $\mathbf{F} = \langle 2x, -y \rangle$

(d)  $\mathbf{F} = \langle y, x \rangle$

