MATH 2554 (Calculus I)
Summer 2015

Name: SOLUTIONS

Thurs 2 July 2015

Exam 3: Using Derivatives (∮3.10-4.6)

Version B

Exam Instructions: You have 50 minutes to complete this exam. Follow the directions and answer the question, using boss notation where appropriate. Justification is required for all problems.

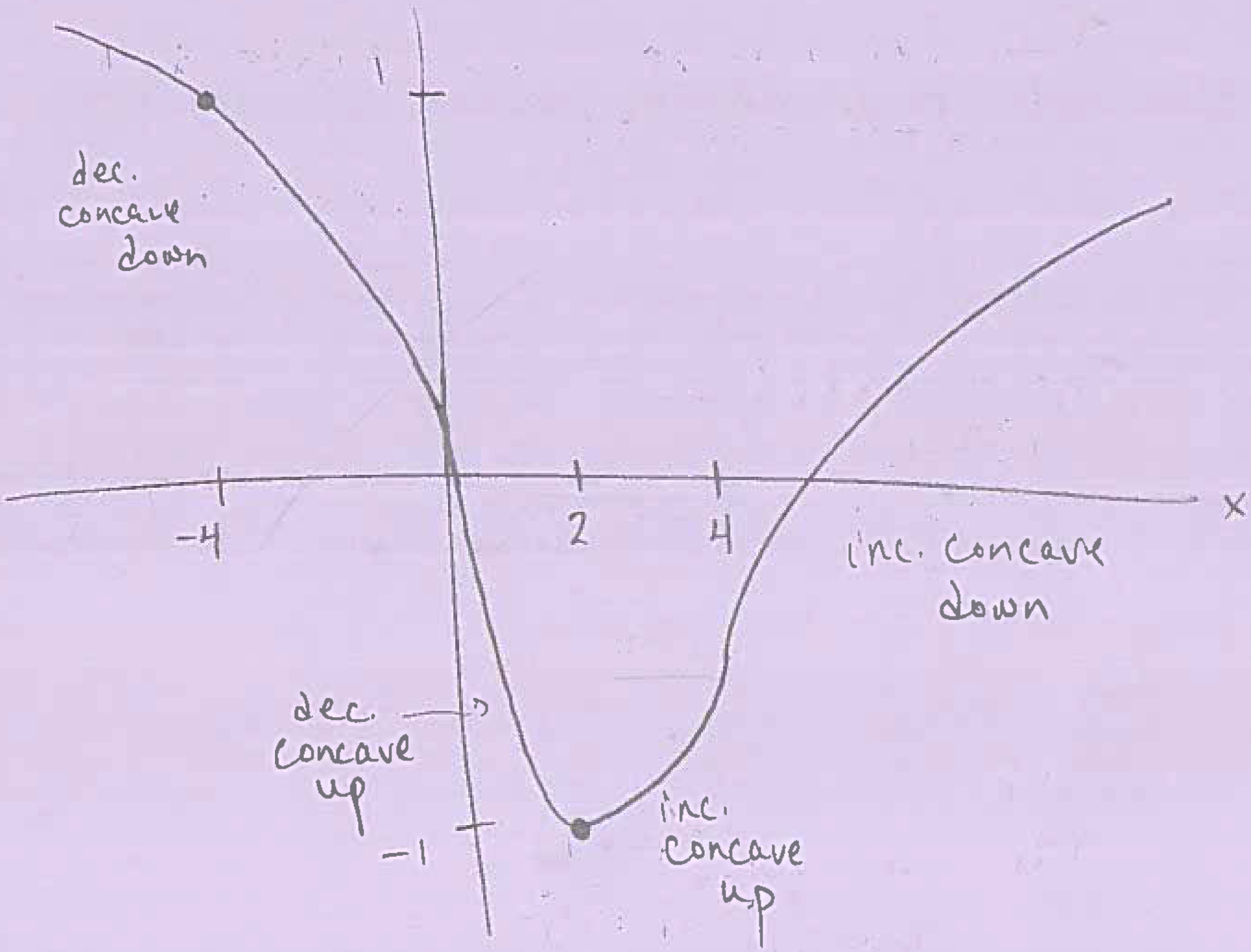
Your signature below indicates that you have read this page and agree to follow the Academic Honesty Policies of the University of Arkansas.

Signature: (1 pt) ______

Good luck!

1. **(20 pts)** Sketch a graph of a function $f(x)$, continuous on $(-\infty, \infty)$, that satisfies all of the following criteria:

- $f(-4) = 1$ and $f(2) = -1$
- $f'(x) < 0$ and $f''(x) < 0$ on $(-\infty, 0)$
- $f'(x) < 0$ and $f''(x) > 0$ on $(0, 2)$
- $f'(x) > 0$ and $f''(x) > 0$ on $(2, 4)$
- $f'(x) > 0$ and $f''(x) < 0$ on $(4, \infty)$



2. (a) **(9 pts)** What are the three hypotheses for Rolle's Theorem?

1. f is continuous on $[a,b]$
2. f is smooth on (a,b)
3. $f(a)=f(b)$

(b) **(7 pts)** Given the three hypotheses, what is the conclusion of Rolle's Theorem?

There exists a point c, between a and b, where $f'(c)=0$

(c) **(7 pts)** The *Mean Value Theorem* applies to $f(x) = x(x^2 - x - 2)$ on $[-1, 1]$. (You don't have to prove that.) Find the point(s) guaranteed to exist by the Mean Value Theorem.

$= x^3 - x^2 - 2x$

Slope of secant line:

$$\frac{f(1)-f(-1)}{1-(-1)} = \frac{1^3-1^2-2(1)-\left[(-1)^3-(-1)^2-2(-1)\right]}{1-(-1)}$$

$-1 \quad -1 \quad +2$

$$= \frac{-2-0}{2} = -1$$

Solve for c:

$$f'(c) = 3c^2-2c-2 = -1$$

$$3c^2-2c-1=0$$

$$(3c+1)(c-1)=0$$

$$\Rightarrow \boxed{c=\frac{-1}{3}} \quad 1$$

(in the interval)

3. **(7 pts ea)** Let $f(x) = \ln x - \sin(2-x)$.

(a) Write the equation for the linear approximation to $f(x)$ at $x = 2$.

$f(2) = \ln 2 - \sin(2-2) = \ln 2 - \sin 0 = \ln 2$

$f'(x) = \frac{1}{x} - \cos(2-x)(-1) = \frac{1}{x} + \cos(2-x)$

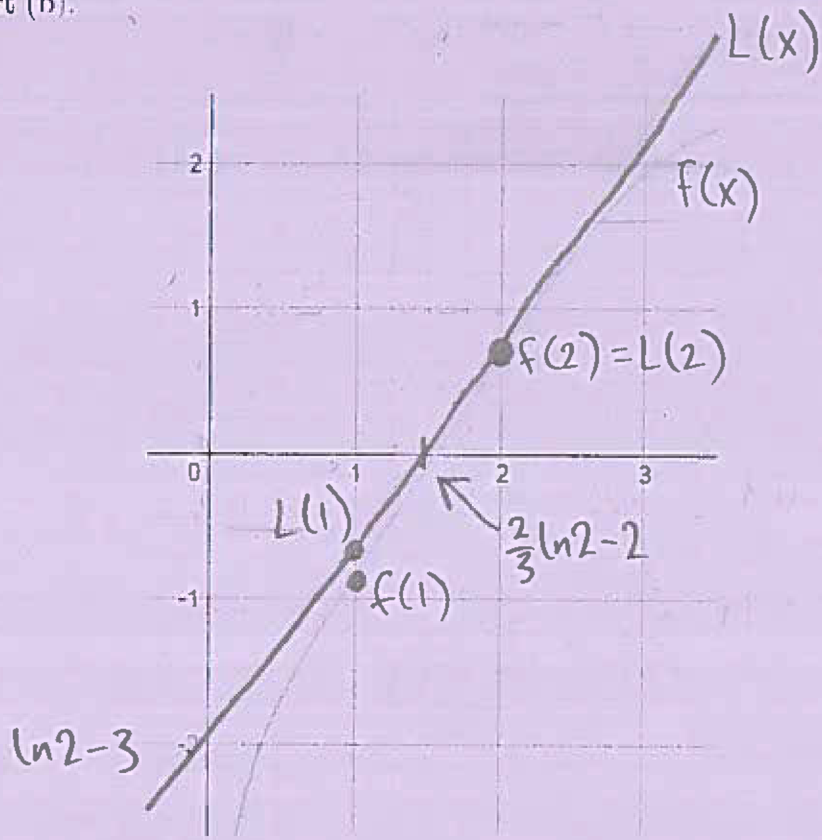
$f'(2) = \frac{1}{2} + \cos(2-2) = \frac{3}{2}$

$\boxed{L(x) = \ln 2 + \frac{3}{2}(x-2) = \frac{3}{2}x + (\ln 2 - 3)}$

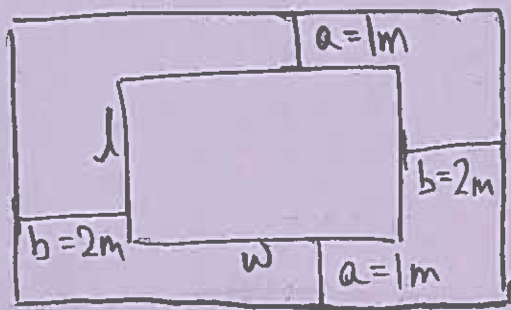
(b) Use your answer to (a) to approximate $f(1)$.

$f(1) \approx L(1) = \ln 2 + \frac{3}{2}(-1) = \boxed{\ln 2 - \frac{3}{2}}$

(c) Below is the graph of $f(x)$, drawn at the website `desmos.com/calculator`. On the same axis, draw your tangent line. Label both $f(1)$ and your approximation from part (b).



4. **(20 pts)** A rectangular flower garden with an area of 32 m^2 is surrounded by a grass border that is 1 m wide on the top and bottom, and 2 m wide on the other two sides. What dimensions of the garden minimize the combined area of the garden and borders? Use the 2nd Derivative Test to justify your answer.



Objective: Minimize Area

$A = (2a+l)(2b+w)$

$= \underbrace{4ab + 2bl + 2aw}_{\text{Constants}} + \underbrace{lw}_{=32}$

Constraint: $lw = 32\,m^2$

$\Rightarrow l = \frac{32}{w}$

Rewrite: $A(w) = 4ab + 2b\left(\frac{32}{w}\right) + 2aw + 32$

$A'(w) = -\frac{2b(32)}{w^2} + 2a = 0$

$2aw^2 = 2b(32)$

$w^2 = 2(32) \leftarrow$ plug in $a=1, b=2$

$\boxed{w = 8\text{ m}}$

2nd Derivative Test:

$A''(w) = -\frac{2b(32)(-2)}{w^3} > 0$

$\boxed{\Rightarrow l = \frac{32}{8} = 4\text{ m}}$

For all $w > 0$, the area function is concave up so $w = 8$m is a min.

5. **(10 pts ea)** Let $f(x)$ be a function, continuous on $(-\infty, \infty)$, such that

$$f'(x) = \frac{2-2x^2}{1+x^2} \quad \text{and} \quad f''(x) = \frac{-8x}{(1+x^2)^2}.$$

(a) Determine the intervals on which $f(x)$ is increasing and decreasing.

$f'(x) = \dfrac{2-2x^2}{1+x^2} > 0 \Rightarrow 2-2x^2 > 0$

always positive $\rightarrow 1+x^2$

$2 > 2x^2$

$1 > x^2$

$1 > |x|$

$\boxed{f \text{ is increasing on } (-1,1)}$

$f'(x) < 0 \Rightarrow 1 < |x|$

$\Rightarrow \boxed{f \text{ is decreasing on } (-\infty,-1) \text{ and } (1,\infty)}$

(b) Determine the intervals on which $f(x)$ is concave up and concave down.

$f''(x) = \dfrac{-8x}{(1+x^2)^2} > 0 \Rightarrow x < 0$

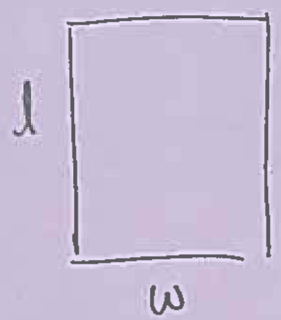
always positive $\rightarrow (1+x^2)^2$

$\boxed{f \text{ is concave up on } (-\infty,0)}$

$f''(x) < 0 \Rightarrow x > 0$

$\boxed{f \text{ is concave down on } (0,\infty)}$

6. **(20 pts)** A rectangle initially has dimensions 1 cm by 5 cm. All sides begin increasing in length at a rate of 2 cm/sec. At what rate is the area of the rectangle increasing after 20 sec?



Know: $\dfrac{dl}{dt} = \dfrac{dw}{dt} = 2 \text{ cm/sec}$

$l(0) = 5 \text{ cm}$ $\Rightarrow$ $l(t) = 5 + 2t$

$w(0) = 1 \text{ cm}$ $\quad$ $w(t) = 1 + 2t$

WTF: $\left.\dfrac{dA}{dt}\right|_{t=20\text{sec}}$ — area

$$A = lw = (5+2t)(1+2t)$$
$$= 5 + 2t + 10t + 4t^2$$
$$= 5 + 12t + 4t^2$$

$$\frac{dA}{dt} = 12 + 8t$$

$$\left.\frac{dA}{dt}\right|_{t=20\text{sec}} = 12 + 8(20) = \boxed{172 \text{ cm}^2/\text{sec}}$$