MATH 2574 (Calculus III)
due Mon 27 Feb 2017

Take-Home Quiz 3: Derivatives in multivariables (§12.4-12.8)

Directions: This quiz is due on February 27, 2017 at the beginning of lecture. You may use whatever resources you like – e.g., other textbooks, websites, collaboration with classmates – to complete it **but YOU MUST DOCUMENT YOUR SOURCES**. Acceptable documentation is enough information for me to find the source myself. Rote copying another's work is unacceptable, regardless of whether you document it.

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1. **12.4 #78** Traveling waves (for example, water waves or electromagnetic waves) exhibit periodic motion in both time and position. In one dimension, some types of wave motion are governed by the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ is the height or displacement of the wave surface at position x and time t, and c is the constant speed of the wave.

Show that $u(x,t) = 5\cos(2(x+ct)) + 3\sin(x-ct)$ satisfies the wave equation.

2. **12.4 #88** The formal definition of differentiability is given as follows (p. 902):

The function $z = f(x,y)$ is **differentiable at** (a,b) means $f_x(a,b)$ and $f_y(a,b)$ exist and the change $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$ equals

$$f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where

$$\begin{aligned} \varepsilon_1 &= \frac{f(a+\Delta x, b) - f(a,b)}{\Delta x} - f_x(a,b) \longrightarrow 0 \quad \text{as} \quad \Delta x \to 0 \quad \text{and} \\ \varepsilon_2 &= \frac{f(a, b+\Delta y) - f(a,b)}{\Delta y} - f_y(a,b) \longrightarrow 0 \quad \text{as} \quad \Delta y \to 0. \end{aligned}$$

For the function $z = f(x,y) = 2x + 3y^2$, let $(a,b) = (0,0)$. Find:

(a) Δz *(Hint: Your answer should be in terms of Δx and Δy.)*

(b) $f_x(a,b)$ and $f_y(a,b)$

(c) ε_1 and ε_2

According to the definition of differentiable given above, is the function $z = f(x,y) = 2x + 3y^2$ differentiable at $(0,0)$?

3. **12.5 #30** Use a tree diagram (explained on p. 908) to write the required Chain Rule formula for $\dfrac{\partial u}{\partial z}$, given that

$$\begin{gathered} u = f(v,w,x) \\ v = g(r,s,t), \quad w = h(r,s,t), \quad x = p(r,s,t) \\ r = F(z). \end{gathered}$$

4. **12.5 #64** Cartesian coordinates (x, y) and **polar coordinates** (r, θ) are related through the following transformation equations

$$x = r\cos\theta \qquad y = r\sin\theta$$

and

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}.$$

   (a) Evaluate the partial derivatives x_r, y_r, x_θ, y_θ.

   (b) Evaluate the partial derivatives r_x, r_y, θ_x, θ_y (assume $r \geq 0$).

   (c) For a function $z = f(x, y)$, write the expressions for z_r and z_θ.

   (d) For a function $w = g(r, \theta)$, write the expressions for w_x and w_y.

5. **12.6 #40** Consider the function $f(x, y) = -4 + 6x^2 + 3y^2$ and the point $P = (-1, -2)$ in $\mathbb{R}^2$. Sketch the xy-plane showing P and the level curve of f through P. Then indicate the directions of maximum increase, maximum decrease, and no change for f.

6. **12.6 #74** The surface

$$f(x, y, z) = xy + xz - yz - 1$$

is a level surface of the function $w = f(x, y, z)$ (for $w = 0$).

   (a) Find the gradient of f and evaluate it at the point $P = (1, 1, 1)$.

   (b) The set of all vectors orthogonal to the gradient with their tails at P forms the **tangent plane** of f to P. Find an equation of that plane.

7. **12.7 #60** In general, real numbers (which usually have infinite decimal expansions) cannot be represented exactly in a computer by floating-point numbers (which have finite decimal expansions). *(In fact, that was the point of the Taylor estimations from Cal II!)*

   Suppose that floating-point numbers on a particular computer carry an error of at most 10^{-16}. Estimate the maximum error that is committed in doing the following arithmetic operations. Express the error in absolute (dz for $z = f(x, y)$ or dw for $w = F(x, y, z)$) and relative $\left(\left| \frac{\Delta z - dz}{\Delta z} \right| \text{ or } \left| \frac{\Delta w - dw}{\Delta w} \right| \right)$ terms.

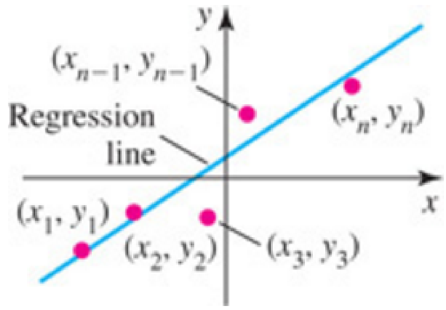
   (a) $f(x, y) = xy$

   (b) $f(x, y) = \frac{x}{y}$

   (c) $F(x, y, z) = xyz$

   (d) $F(x, y, z) = \frac{\frac{x}{y}}{z}$

8. **12.8 #70** Suppose you collect data for two variables x and y (e.g., height and shoe size) in the form of pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ (so there are n data points). This data may be plotted as a scatterplot in the xy-plane, as shown in the figure:



The technique known as **linear regression** asks the question: What is the equation of the line that "best fits" the data (the blue line in the figure)? The **least squares criterion** for best fit requires that the sum of the squares of the vertical distances between the line and the data points is a minimum. The equation of the best fit (or **regression**) line has the form $y = mx + b$, where the slope m and the y-intercept b are determined by the least squares condition.

Suppose you are given the three data points $(x_1, y_1) = (1, 2)$, $(x_2, y_2) = (3, 4)$, and $(x_3, y_3) = (5, 6)$.

(a) The sum of the squares of the vertical distances between the regression line and the data points is a function of m and b:

$$F(m, b) = ((m + b) - 2)^2 + (3m + b) - 5)^2 + ((4m + b) - 6)^2$$

Briefly explain where this formula comes from.

(b) Find the critical points of F and the values of m and b that minimize F.

(c) Graph the three data points and the regression line.