Name: _____

MATH 2574 (Calculus III)
Fall 2015

Drill: _____

Tues 3 Nov 2015

Quiz 9: The Jacobian (§13.4-13.5, 13.7)

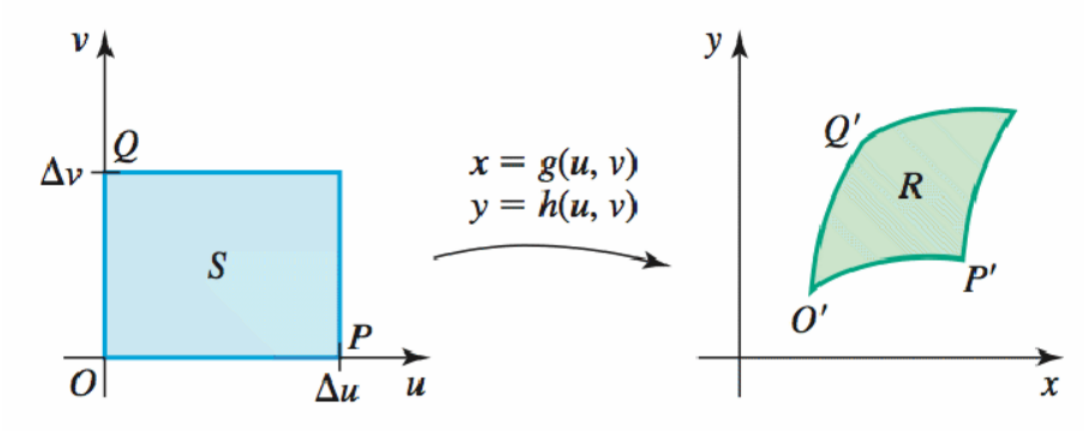
The Jacobian is a magnification (or reduction) factor that relates the area of a small region, or neighborhood, near a point (u, v) in $\mathbb{R}^2$ (uv-plane), to the area of the *preimage*[*] of that region near the point (x, y) in $\mathbb{R}^2$ (xy-plane), where

$$T : x = g(u, v) \quad \text{and} \quad y = h(u, v)$$

is a one-to-one transformation.

Suppose S is a little rectangle in the uv-plane with vertices $(0, 0), (\Delta u, 0), (\Delta u, \Delta v), (0, \Delta v)$. The preimage of S under the transformation given above is a small region R in the xy-plane. The arrows ($\mapsto$) below, and the picture, indicate the respective preimages of each of the following points:

$$(0, 0) = O \mapsto O'$$
$$(\Delta u, 0) = P \mapsto P'$$
$$(0, \Delta v) = Q \mapsto Q'$$



(a) **(3 pts)** Write down the coordinates for each of the points O', P', Q'.

$O' =$

$P' =$

$Q' =$

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[*]The text instead uses the term *image* and writes $T(S) = R$. The reason for the discrepancy is delicate and relevant to the field of Algebraic Geometry. The transformation T can be described in two ways:

T : { unknowns in the xs and ys } $\to$ { unknowns in the us and vs } *(Algebraic Paradigm)*

or

T : { known values in the us and vs } $\to$ { corresponding x- and y-values } *(Geometric Paradigm)*

(b) The linear approximation of $g(u, v)$ near the point $O = (0, 0)$ is:

$$g(u, v) \approx g(0, 0) + g_u(0, 0) \cdot u + g_v(0, 0) \cdot v$$

i. **(1 pt)** Write down the linear approximation of $h(u, v)$ near O.

ii. **(2 pts)** The points P and Q are close to the point O; use the linear approximations for g and h to compute

$g(\Delta u, 0) \approx$ $\qquad$ $h(\Delta u, 0) \approx$

$g(0, \Delta v) \approx$ $\qquad$ $h(0, \Delta v) \approx$

iii. **(2 pts)** Use the approximations in ii. to find the area of the parallelogram with sides given by the vectors $\overrightarrow{O'P'}$ and $\overrightarrow{O'Q'}$. *(Hint: Use the cross product by first adding a third variable and setting its direction equal to zero.)*

iv. **(2 pts)** What is the approximate ratio of the area of R to the area of S?