Math 116 Quiz 2: $\oint$ 6.4, 7.1-7.2 **Name:** SOLUTIONS
Tue 18 Sep 2012

You have 30 minutes to complete this quiz. Eyes on your own paper and good luck!

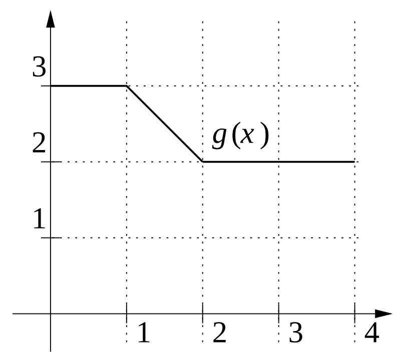
1. **Definitions/Concepts.** (1 pt) Fill in the blank:

   The method of substitution reverses the Chain Rule. The method of integration by parts reverses the Product Rule.

2. **Questions/Problems.** (4 pts) Suppose $g(x)$ is given by the graph to the right, below. Find $\int_0^4 x\,g'(x)\,dx$.

   Method 1: Notice $g'(x) = 0$ everywhere except when $1 < x < 2$, in which case $g'(x) = -1$. Therefore

$$\int_0^4 x\,g'(x)\,dx = \int_1^2 x\,(-1)\,dx$$
$$= -\frac{1}{2}x^2\Big|_1^2$$
$$= -2 + \frac{1}{2} = -\frac{3}{2}.$$



   Method 2: Use integration by parts; let $u = x$ and $dv = g'(x)dx$, so that $du = dx$ and $v = g(x)$. Then

$$\int_0^4 x\,g'(x)\,dx = x \cdot g(x)|_0^4 - \int_0^4 g(x)dx.$$

   The integral on the righthand side is the area under $g(x)$ for $x \in [0,4]$. Looking at the graph this area is 9.5. Also, from the graph we have $g(4) = 2$. So the original integral becomes

$$\begin{aligned}\int_0^4 x\,g'(x)\,dx &= x \cdot g(x)|_0^4 - \int_0^4 g(x)dx \\ &= (4 \cdot g(4) - 0 \cdot g(0)) - 9.5 \\ &= 4 \cdot 2 - 9.5 = -1.5.\end{aligned}$$

3. **Computations/Algebra.**

   (a) (3 pts) Using the 2nd FTOC and the Chain Rule, calculate $\frac{d}{dt}\int_{e^t}^{t^4} \sqrt{8 + x^2}\,dx$.

   In order to apply the 2nd FTOC, we need to put the integral in the appropriate form. First, we need the lower limit to be a constant. By using parts 1 and 2 of Theorem 5.2, the expression becomes

$$\frac{d}{dt}\left(-\int_a^{e^t}\sqrt{8+x^2}dx+\int_a^{t^4}\sqrt{8+x^2}dx\right).$$

It does not really matter which constant we choose for a, as long as it is a constant. Using the linearity property of derivatives, we apply the Chain Rule to each term separately. First, consider

$$\frac{d}{dt}\left(-\int_a^{e^t}\sqrt{8+x^2}dx\right)\overset{\text{(by linearity)}}{=}-\frac{d}{dt}\int_a^{e^t}\sqrt{8+x^2}dx.$$

If we let

$$F(u)=\int_a^{u}\sqrt{8+x^2}dx,$$

then the Chain Rule says

$$\frac{d}{dt}F(u)=F'(u)\cdot\frac{du}{dt}.$$

In this case, $u=e^t$. So the first term becomes

$$-\left(\sqrt{8+(e^t)^2}\right)\cdot e^t.$$

Similarly, the second term is $\left(\sqrt{8+(t^4)^2}\right)\cdot 4t^3$. So the entire derivative is

$$\frac{d}{dt}\int_{e^t}^{t^4}\sqrt{8+x^2}dx=-e^t\sqrt{8+e^{2t}}+4t^3\sqrt{8+t^8}.$$

(b) (3 pts ea) Evaluate the integrals. **Then check by differentiating your answer.**

i. $\int x^2\sin x dx=-x^2\cos x+2x\sin x+2\cos x+C$

Proof. Use the method of integration by parts. Using the pneumonic "LogPoET", set $u=x^2$ and $dv=\sin x dx$. Then $\frac{du}{dx}=2x$ implies $du=2xdx$, and $v=-\cos x$ is an antiderivative[1] for $\sin x$. Using the "Ultra-Violet Voodoo" formula, we get

$$\begin{aligned}\int x^2\sin x dx&=uv-\int vdu\\&=(x^2)(-\cos x)-\int(-\cos x)(2xdx)\\&=-x^2\cos x+2\int x\cos x dx.\end{aligned}$$

[1] We are really integrating both sides of the equation $dv=\sin x dx$, to get $\int dv=\int\sin x dx$. To see why in this case we do not need to acknowledge the constant C, see p. 342 in section 7.2.

Apply the integration by parts method again to evaluate $\int x \cos x dx$. This time, set $u = x$ and $dv = \cos x dx$. Then $du = dx$ and $v = \sin x$, and the full expression becomes

$$\begin{aligned}\int x^2 \sin x dx =& -x^2 \cos x + 2 \int x \cos x dx \\ =& -x^2 \cos x + 2 \left(uv - \int v du \right) \\ =& -x^2 \cos x + 2 \left((x)(\sin x) - \int (\sin x)(dx) \right) \\ =& -x^2 \cos x + 2 \left(x \sin x - \int \sin x dx \right) \\ =& -x^2 \cos x + 2 \left(x \sin x - (-\cos x + C) \right) \\ =& -x^2 \cos x + 2x \sin x + 2 \cos x + C,\end{aligned}$$

where we replace $2C$ with the symbol C again to express the solution as a family of functions varying with the constant C. □

Proof by differentiation. Differentiate with respect to x:

$$\begin{aligned}\frac{d}{dx} \left(-x^2 \cos x + 2x \sin x + 2 \cos x + C \right) =& \left((-2x)(\cos x) + (-x^2)(-\sin x) \right) \\ &+ \left((2)(\sin x) + (2x)(\cos x) \right) + (-2 \sin x) \\ &+ (0) \\ =& -2x \cos x + x^2 \sin x + 2 \sin x + 2x \cos x \\ &- 2 \sin x \\ =& x^2 \sin x,\end{aligned}$$

which is exactly the expression in the original integrand. □

ii. $\int (\alpha^2 + 3)^2 d\alpha = \frac{1}{5}\alpha^5 + 2\alpha^3 + 9\alpha + C$

Proof. It is enough to expand the squared expression and then apply the power rule:

$$\begin{aligned}\int (\alpha^2 + 3)^2 d\alpha =& \int (\alpha^4 + 6\alpha^2 + 9) d\alpha \\ =& \frac{\alpha^5}{5} + \frac{6\alpha^3}{3} + 9\alpha + C\end{aligned}$$

which simplifies to the expression above. □

Proof by differentiation. Differentiate with respect to α:

$$\begin{aligned}\frac{d}{d\alpha} \left(\frac{1}{5}\alpha^5 + 2\alpha^3 + 9\alpha + C \right) =& \alpha^4 + 6\alpha^2 + 9 \\ =& (\alpha^4 + 3). \ \square\end{aligned}$$

MORE QUIZ ON THE BACK –>

iii. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \ln(e^x + e^{-x}) + C$

Proof. Use the method of substitution:

$$\begin{aligned} u &= e^x + e^{-x} \\ du &= (e^x - e^{-x})dx \end{aligned}$$

Then

$$\begin{aligned} \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx &= \int \frac{du}{u} \\ &= \ln(|u|) + C \\ &= \ln\left(|e^x + e^{-x}|\right) + C \\ &= \ln(e^x + e^{-x}) + C; \end{aligned}$$

the absolute value signs may be removed since $e^x + e^{-x}$ is always positive and nonzero. □

Proof by differentiation. Verify by differentiating with respect to x:

$$\frac{d}{dx}\left(\ln(e^x + e^{-x}) + C\right) = \left(\frac{1}{e^x + e^{-x}}\right) \cdot (e^x - e^{-x})$$

by the Chain Rule. □

iv. $\int (\ln t)^2 dt = t(\ln t)^2 - 2t \ln t + 2t + C$

Proof. Apply integration by parts twice:

$$\begin{aligned} \int (\ln t)^2 dt =& uv - \int v du \\ =& \left((\ln t)^2\right)(t) - \int (t)\left(\frac{2\ln t}{t} dt\right) \\ =& t(\ln t)^2 - 2\int \ln t dt \\ =& t(\ln t)^2 - 2\left(uv - \int v du\right) \\ =& t(\ln t)^2 - 2\left((\ln t)(t) - \int (t)\left(\frac{1}{t} dt\right)\right) \\ =& t(\ln t)^2 - 2t\ln t + 2\int dt \\ =& t(\ln t)^2 - 2t\ln t + 2t + C \quad \square \end{aligned}$$

Proof by differentiation. Take the derivative with respect to t:

$$\begin{aligned}
\frac{d}{dt}\left(t(\ln t)^2 - 2t\ln t + 2t + C\right) &= \left((1)\left((\ln t)^2\right) + (t)\left(2\ln t \cdot \frac{1}{t}\right)\right) \\
&\quad + \left((-2)(\ln t) + (-2t)\left(\frac{1}{t}\right)\right) + 2 \\
&= (\ln t)^2 + 2\ln t - 2\ln t - 2 + 2 \\
&= (\ln t)^2 \quad \square
\end{aligned}$$

ChAlLeNgE pRoBlEm: Derive the following formula:

$$\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$$

Using integration by parts, put $u = x^n$ and $dv = e^x dx$. Then

$$\begin{aligned}
\int x^n e^x \, dx &= (x^n)(e^x) - \int (e^x)(nx^{n-1} dx) \\
&= x^n e^x - n \int x^{n-1} e^x \, dx.
\end{aligned}$$