**MATH 2554 (Calculus I)**
**Spring 2016**

Name: SOLUTIONS

Fri 4 Mar 2016

# Exam 2: Derivatives (§3.2-3.8)

**Exam Instructions:** You have 50 minutes to complete this exam. Justification is required for all problems. Notation matters! You will also be penalized for missing units and rounding errors. No electronic devices (phones, iDevices, computers, etc) except for a **basic scientific calculator**.

In addition, please provide the following data:

Drill Instructor: ______________________

Drill Time: ______________________

**Your signature below indicates that you have read this page and agree to follow the Academic Honesty Policies of the University of Arkansas.**

Signature: **(1 pt)** ______________________

Good luck!

1. **(2 pts ea)** Assume $f$ is a differentiable function whose graph passes through the point $(1,4)$. Suppose $g(x) = f(x^2)$ and the line tangent to the graph of $f$ at $(1,4)$ is $y = 3x+1$. Determine each of the following:

   (a) $g(1) = f(1^2) = f(1) = 4$

   (b) $g'(x) = f'(x^2)\cdot 2x = 2xf'(x^2)$

   (c) $g'(1) = 2(1)f'(1^2) = 2\cdot 3 = 6$

2. **(4 pts)** Find $y'$, given $y = \sin\left(\tan^3\left(xe^{5x+2}\right)\right)$. You do not need to simplify.

$$y' = \cos\left(\tan^3\left(xe^{5x+2}\right)\right)\cdot 3\tan^2\left(xe^{5x+2}\right)\cdot \sec^2\left(xe^{5x+2}\right)\cdot\left(e^{5x+2} + 5xe^{5x+2}\right)$$

3. **(6 pts)** Find the equation of the line tangent to the curve $y = x + \sqrt{x}$ that has slope 2.

$$y' = 1 + \tfrac{1}{2}x^{-1/2} = 2$$

$$\Rightarrow \frac{1}{2\sqrt{x}} = 1$$

$$\frac{1}{2} = \sqrt{x}$$

$$\Rightarrow x = \frac{1}{4}$$

$$y = \frac{1}{4} + \sqrt{\frac{1}{4}}$$

$$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

line:

$$\boxed{y - \frac{3}{4} = 2\left(x - \frac{1}{4}\right)}$$

4. Two stones are thrown vertically upward with matching initial velocities of 48 ft/s at time $t = 0$. One stone is thrown from the edge of a bridge that is 32 ft above the ground and the other stone is thrown from ground level. The height of the stone thrown from the bridge after $t$ seconds is

$$f(t) = -16t^2 + 48t + 32$$

and the height of the stone thrown from the ground after $t$ seconds is

$$g(t) = -16t^2 + 48t.$$

(a) **(6 pts)** Show that the stones reach their high points at the same time.

The high point occurs at the vertex of the parabola, when the derivative is zero.

$f'(t) = -32t + 48 = 0$

$\Rightarrow t = \frac{-48}{-32} = \frac{3}{2}$ sec

$g'(t) = -32t + 48$

$\Rightarrow t = \frac{3}{2}$ sec.

(b) **(6 pts)** How much higher does the stone thrown from the bridge go than the stone thrown from the ground?

$$f\left(\tfrac{3}{2}\right) - g\left(\tfrac{3}{2}\right) = \cancel{-16\left(\tfrac{3}{2}\right)^2} + \cancel{48\left(\tfrac{3}{2}\right)} + 32 - \left(\cancel{-16\left(\tfrac{3}{2}\right)^2} + \cancel{48\left(\tfrac{3}{2}\right)}\right)$$

$$= \boxed{32 \text{ feet}}$$

(c) **(12 pts)** When do the stones strike the ground and with what velocities? Include the exact numerical answers in your work, then round your final answer to three decimal places.

$f(t) = -16t^2 + 48t + 32 = 0$

$-16(t^2 - 3t - 2) = 0$

$\Rightarrow t = \dfrac{3 \pm \sqrt{3^2 - 4(1)(-2)}}{2(1)}$

$= \dfrac{3 \pm \sqrt{17}}{2}$ ; since $\sqrt{17} > 3$, take the $\oplus$ solution

$\boxed{\approx 3.562 \text{ sec}}$

$f'\left(\dfrac{3+\sqrt{17}}{2}\right) = -32\left(\dfrac{3+\sqrt{17}}{2}\right) + 48$

$\boxed{\approx -65.970 \text{ ft/sec}}$

$g(t) = -16t^2 + 48t = 0$

$-16t(t-3) = 0$

$\Rightarrow \boxed{t = 3 \text{ sec}}$

$g'(3) = -32(3) + 48$

$\boxed{= -48 \text{ ft/sec}}$

5. **(5 pts)** ~~Evaluate~~ $\lim_{\theta \to 0} \dfrac{\cos^2\theta - 1}{\theta}$. $= \lim_{\theta \to 0} \dfrac{-\sin^2\theta}{\theta} = -\left(\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta}\right)\left(\lim_{\theta \to 0} \sin\theta\right) = 0.$

$\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$, $\lim_{\theta \to 0} \sin\theta = \sin 0 = 0$

OR

$= \lim_{\theta \to 0} \dfrac{(\cos\theta - 1)(\cos\theta + 1)}{\theta} = \left(\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta}\right)\left(\lim_{\theta \to 0} \cos\theta + 1\right) = 0.$

$0$ $\qquad \cos 0 + 1 = 2$

★ ***EXTRA CREDIT*** **(2 pts)** One of several **Leibniz Rules** in calculus deals with higher-order derivatives of products. Let $(fg)^{(n)}$ denote the $n$th derivative of the product $fg$, for $n \geq 1$. Prove that $(fg)^{(2)} = f''g + 2f'g' + fg''$.

$(fg)^{(2)} = ((fg)')' = (f'g + fg')'$

$= (f''g + f'g') + (f'g' + fg'')$

$= f''g + 2f'g' + fg''$

6. **(12 pts)** Find $\frac{d^2w}{dz^2}$, given $\sin z + z^2 w = 10$. You do not have to simplify but your answer should only contain the quantities $z$ and $w$ – i.e., no derivatives.

$$\frac{d}{dz}\left(\sin z + z^2 w = 10\right)$$

$$= \cos z + 2zw + z^2 \frac{dw}{dz} = 0$$

$$\Rightarrow \frac{dw}{dz} = \frac{-\cos z - 2zw}{z^2}.$$

$$\Rightarrow \frac{d^2w}{dz^2} = \frac{z^2\left(-(-\sin z) - \left(2w + 2z\frac{dw}{dz}\right)\right) - \left(-\cos z - 2zw\right)(2z)}{(z^2)^2}$$

$$= \boxed{\frac{z^2\left(\sin z + 2w - 2z\left(\frac{-\cos z - 2zw}{z^2}\right)\right) + 2z(\cos z + 2zw)}{z^4}}$$

7. **(3 pts ea)** Use the following table to find the given derivatives.

| x | f(x) | f′(x) | g(x) | g′(x) |
|---|---|---|---|---|
| 1 | 5 | 4 | 3 | 2 |
| 2 | 2 | 4 | 3 | 1 |

(a) $\left.\frac{d}{dx}\left(x^2 + f(x)\right)\right|_{x=2} = 2(2) + f'(2)$

$\quad = 2x + f'(x)$

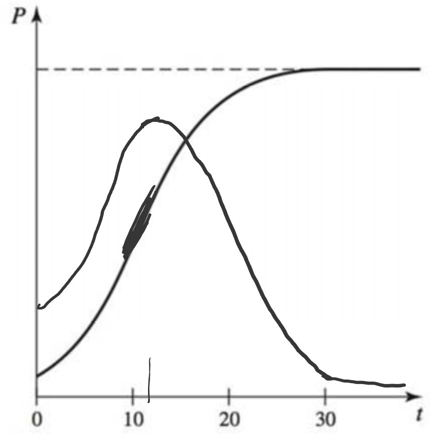
$= 4 + 4 = \boxed{8}$

(b) $\left.\frac{d}{dx}\left(xg(x)\right)\right|_{x=1} = g(1) + (1)g'(1)$

$\quad = g(x) + xg'(x)$

$= 3 + 2 = \boxed{5}$

8. A common model for population growth uses the logistic (or **sigmoid**) curve. Consider the logistic curve in the figure, where $P(t)$ is the population at time $t \geq 0$.




**Logistic growth** (p. 143 *Calculus: Early Transcendentals*, Briggs, et al., 2nd Edition).

(a) **(2 pts)** At approximately what time is the rate of growth $P'$ the greatest?

$t \approx 11$

(b) **(2 pts)** Is $P'$ positive or negative for $t \geq 0$?

positive

(c) **(2 pts)** Is $P'$ an increasing or decreasing function of time (or neither)?

neither

(d) **(5 pts)** Sketch the graph of $P'$ on the same axes. You do not need to worry about a vertical scale.

★ ***EXTRA CREDIT* (3 pts)** Suppose $f$ and $g$ are differentiable for all real numbers, and $m$ and $n$ are integers. Find $y'$, given $y = f(g(x^m))^n$.

$y' = nf(g(x^m))^{n-1} \cdot f'(g(x^m)) \cdot g'(x^m) \cdot mx^{m-1}$