MATH 2574 (Calculus III)
Spring 2017

Name: SOLUTIONS

Wed 15 Mar 2017

Exam 2: Multivariate Derivatives and Multiple Integrals
(§12.3-12.9, 10.1-10.3, 13.1-13.5)

Exam Instructions: You have 50 minutes to complete this exam. Justification is required for all problems. No electronic devices (phones, iDevices, computers, etc) except for a **basic scientific calculator**. On story problems, round to one decimal place. If you finish early then you may leave, UNLESS there are less than 5 minutes of class left. To prevent disruption, if you finish with less than 5 minutes of class remaining then please stay seated and quiet.

In addition, please provide the following data:

Drill Instructor: ______

Drill Time: ______

Your signature below indicates that you have read this page and agree to follow the Academic Honesty Policies of the University of Arkansas.

Signature: **(1 pt)** ______

Good luck!

1. **(6 pts)** The density of a thin circular plate of radius 2 is given by $\rho(x, y) = 4 + xy$. The edge of the plate is described by the parametric equations $x = \cos t$, $y = \sin t$, for $0 \leq t \leq 2\pi$. Find the rate of change of the density with respect to t on the edge of the plate.

$$\rho(x(t), y(t)) = 4 + (\cos t)(\sin t)$$

$$= 4 + \cos t \sin t$$

$$\boxed{\frac{d\rho}{dt} = -\sin^2 t + \cos^2 t}$$

2. Evaluate (or show non-existence of) the following limits:

(a) **(5 pts)** $\lim\limits_{(x,y,z)\to(\ln 2,3,1)} (1 + x)\, l\cancel{n}\, \cancel{e}^{yz}$

$$= (1 + \ln 2)(3)(1)$$

$$\boxed{= 3 + 3\ln 2}$$

(b) **(5 pts)** $\lim\limits_{(u,v)\to(0,0)} \frac{|uv|}{uv} = \lim\limits_{(u,mu)\to(0,0)} \frac{|u(mu)|}{u(mu)}$

Use 2-Path Test along the line $v = mu$

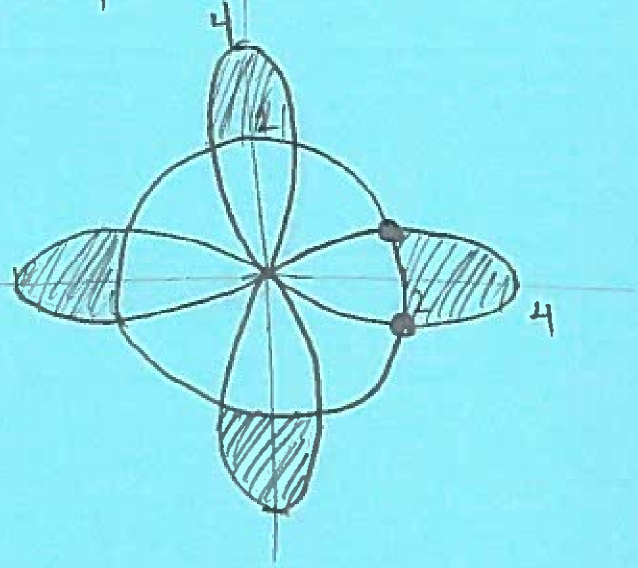
$$= \begin{cases} 1 & \text{if } m = 1 \\ -1 & \text{if } m = -1 \end{cases}$$

$\Rightarrow$ does not exist.

3. **(10 pts)** Find the area of the region inside the rose $r = 4\cos 2\theta$ and outside the circle $r = 2$. *(In case you need it, the half-angle formula is* $\cos^2 x = \frac{1+\cos 2x}{2}$*.)*

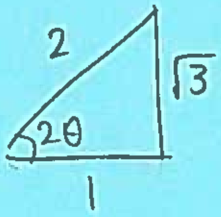
By symmetry, use one petal
and multiply by 4:

Bounds of integration:



circle → $2 = 4\cos 2\theta$ ← rose

$\frac{1}{2} = \cos 2\theta$



$\Rightarrow 2\theta = \frac{\pi}{3}$

$\theta = \frac{\pi}{6}, \frac{-\pi}{6}$ ⇝ Use more symmetry:

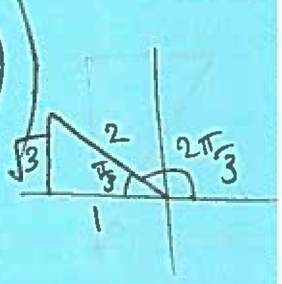
$$8\int_0^{\pi/6}\int_2^{4\cos 2\theta} r\,dr\,d\theta$$

$$= 8\int_0^{\pi/6}\left(\frac{\overset{8}{\cancel{16}}\cos^2 2\theta}{\cancel{2}} - \frac{2^{\cancel{2}}}{\cancel{2}}\right)d\theta$$

$$= 8\int_0^{\pi/6}\underbrace{\left(\overset{4}{\cancel{8}}\left(\frac{1+\cos 4\theta}{\cancel{2}}\right) - 2\right)}_{2+4\cos 4\theta}d\theta$$

$$= 8(2\theta + \sin 4\theta)\Big|_0^{\pi/6}$$ 0 ← terms vanish

$$= 8\left(2\left(\frac{\pi}{6}\right) + \sin\left(\frac{2\pi}{3}\right)\right)$$



$$\boxed{\begin{gathered}= 8\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)\\ \approx 15.3\end{gathered}}$$

4. **(12 pts)** Find the absolute maximum and minimum values of the function

$$f(x, y) = x^2 + y^2 - 2x - 2y$$

on the closed region R, bounded by the triangle with vertices (0, 0), (2, 0), (0, 2).

$f_x = 2x - 2 = 0 \Rightarrow x = 1$

$f_y = 2y - 2 = 0 \Rightarrow y = 1$ CP

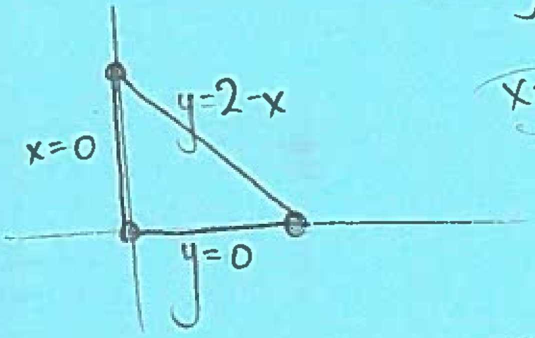
Discriminant:

$\begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix} = 2(2) - 0(0) = 4 > 0$;

$f_{xx}(1,1) = 2 > 0$

$\Rightarrow$ min.

On the boundary:



$x=0$: $f(0,y) = y^2 - 2y$

$\frac{d}{dy} f(0,y) = 2y - 2 = 0 \Rightarrow y = 1$

$y=0$: $f(x,0) = x^2 - 2x$

$\frac{d}{dx} f(x,0) = 2x - 2 = 0 \Rightarrow x = 1$

$y = 2-x$:

$f(x,2-x) = x^2 + \underbrace{(2-x)^2}_{\cancel{4} - 4x + x^2} \cancel{-2x} \underbrace{- 2(2-x)}_{\cancel{-4} + \cancel{2x}}$

$= 2x^2 - 4x$

$\frac{d}{dx} f(x,2-x) = 4x - 4 = 0$

$\Rightarrow x = 1$

$\Rightarrow y = 2 - 1 = 1$

Compare:

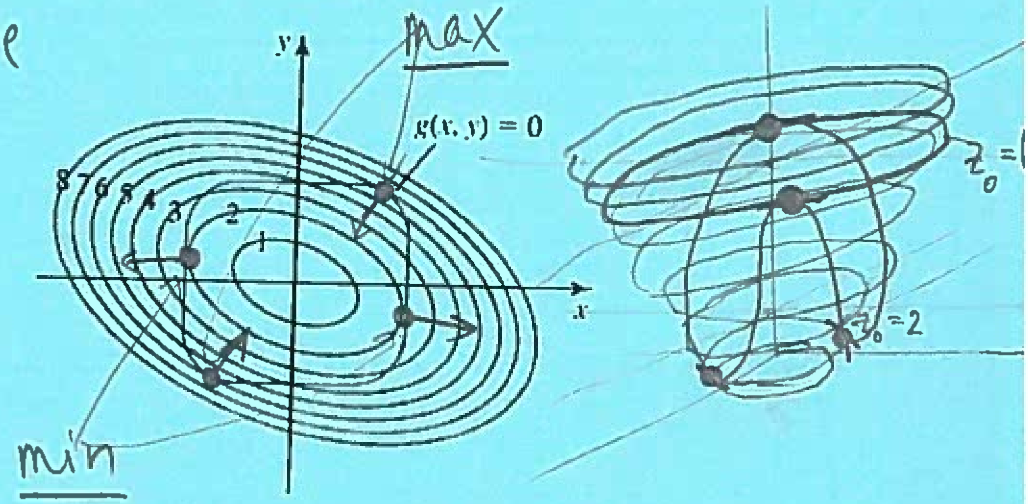
$f(1,1) = 1^2 + 1^{\cancel{2}} - 2 - 2 = \boxed{-2 \leftarrow \text{min @ } (x,y) = (1,1)}$

$f(0,1) = 1^2 - 2 = -1$

$f(1,0) = 1^2 - 2 = -1 \leftarrow$ max @ $(x,y) = (0,1), (1,0)$

5. **(8 pts)** The following figure shows the level curves for various $z = z_0$ of the function f, along with the constraint curve $g(x, y) = 0$. Estimate the maximum and minimum values of f subject to the constraint. At each point where an extreme value occurs, indicate the direction of ∇f and the direction of ∇g.

Look for where $g(x,y)=0$ is tangent to the level curves.



6. **(6 pts)** Compute the directional derivative of

$$g(x, y) = \sin(\pi(2x - y))$$

at the point $P = (-1, -1)$ in the direction of $\mathbf{u} = \langle \frac{12}{13}, -\frac{5}{13} \rangle$.

$$D_{\vec{u}}\, g(-1,-1) = \nabla g(-1,-1) \circ \left\langle \frac{12}{13}, \frac{-5}{13} \right\rangle$$

$$\nabla g = \left\langle \cos(\pi(2x-y))(2\pi), \cos(\pi(2x-y)(-\pi)) \right\rangle$$

$$\Rightarrow D_{\vec{u}}\, g(-1,-1) = 2\pi \cos(\pi\underbrace{(-2-(-1))}_{-1})\left(\frac{12}{13}\right) + (-\pi)\cos(\pi\underbrace{(-2-(-1))}_{-1})\left(\frac{-5}{13}\right)$$

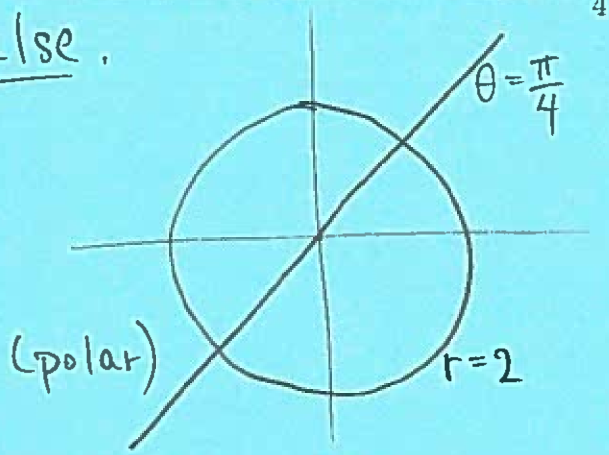
$$= 2\pi(-1)\left(\frac{12}{13}\right) + \pi(-1)\left(\frac{5}{13}\right)$$

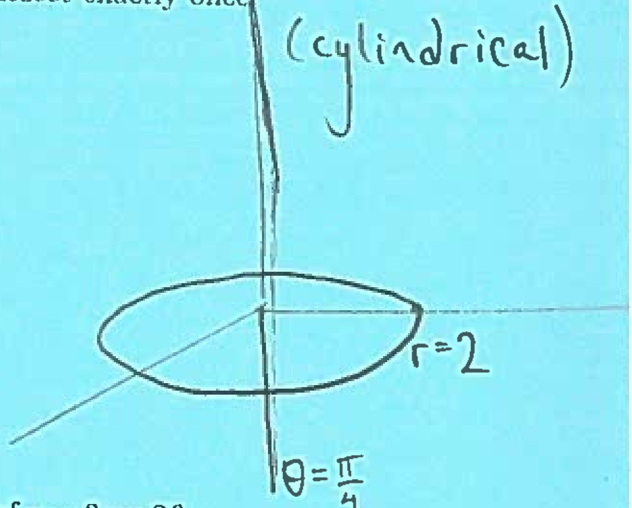
$$= \frac{-24}{13}\pi - \frac{5}{13}\pi = \boxed{\frac{-29}{13}\pi}$$

7. Determine whether the following statements are true or false. You must justify your answer.

(a) **(4 pts)** The graphs of $r = 2$ and $\theta = \frac{\pi}{4}$ intersect exactly once.

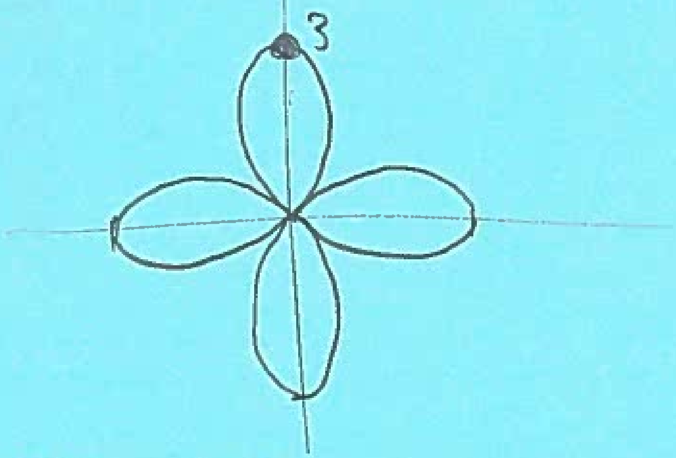
False.





(b) **(4 pts)** The point $\left(3, \frac{\pi}{2}\right)$ lies on the graph of $r = 3\cos 2\theta$.

True.



(c) **(4 pts)** The graphs of $r = 2\sec\theta$ and $r = 3\csc\theta$ are lines.

True

$$\underbrace{r\cos\theta}_{x} = 2$$

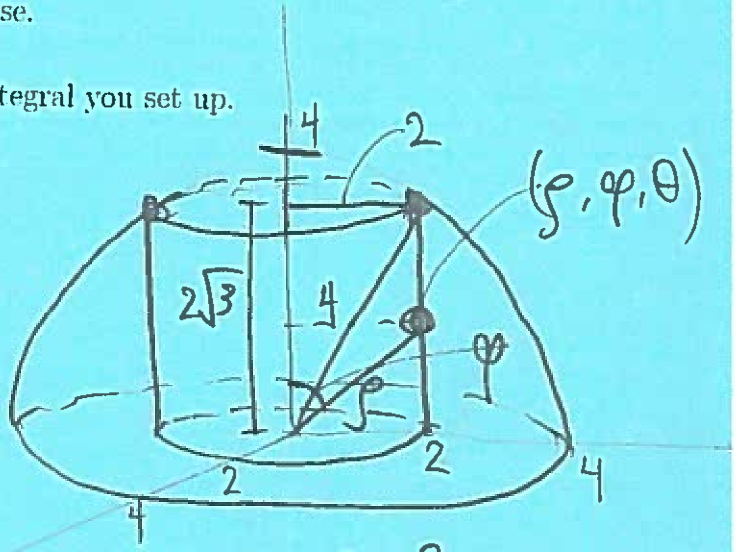
$$\underbrace{r\sin\theta}_{y} = 3$$

8. **(10 pts)** Set up, but **do not evaluate**, the integral for the volume of material remaining in a hemisphere of radius 4 after a cylindrical hole of radius 2 is drilled through the center of the hemisphere perpendicular to its base.

ExTrA cReDiT (5pts) Evaluate the integral you set up.

$4 = 2\csc\varphi \Rightarrow \sin\varphi = \frac{1}{2}$

$\rightarrow \varphi = \frac{\pi}{6}$



$\sin\varphi = \frac{2}{\rho}$

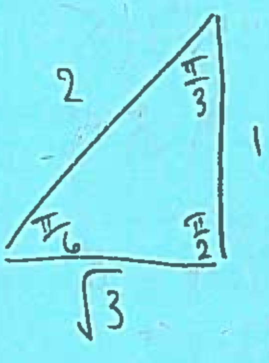
$\Rightarrow \rho = 2\csc\varphi$

Volume:

$$\boxed{\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_{2\csc\varphi}^{4} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta}$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin\varphi\left(\frac{4^3}{3} - \frac{2^3\csc^3\varphi}{3}\right) d\varphi\, d\theta$$

$$\frac{64}{3}\sin\varphi - \frac{8}{3}\csc^2\varphi$$



$$= \int_0^{2\pi}\left(-\frac{64}{3}\cos\varphi\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} + \frac{8}{3}\cot\varphi\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}\right) d\theta$$

$$= \left(-\frac{\overset{32}{\cancel{64}}}{3}\left(0 - \frac{\sqrt{3}}{\cancel{2}}\right) + \frac{8}{3}\left(0 - \sqrt{3}\right)\right)(2\pi)$$

$$= \frac{\overset{24}{(32-8)}\sqrt{3}}{3}(2\pi) \boxed{= 16\sqrt{3}\pi \approx 87.7}$$