Wed 2 Mar

- Exam 2:
  - Friday 4 Mar. Covers up to §3.8.
  - Spring 2015 Practice Exam. Also look for quizzes on the old webpages for more problems.
  - For more problems study the evens in each of the sections covered.
  - Basic scientific calculator is allowed.

Wed 2 Mar (cont.)

- Midterm:
    - Tuesday 8 March. Covers everything up to §3.9.
    - Morning Section: Walker room 124
      Afternoon Section: Walker room 218
      You must take the test with your officially scheduled section.
    - Stay tuned for conflict resolutions. If you haven't emailed me already regarding a conflict, do it NOW.
    - Stay tuned for a study guide.
    - Basic scientific calculator is allowed....? Stay tuned.

Wed 2 Mar (cont.)

- Quiz 6 next Thurs. Only some of the quiz problems are graded now. You are always welcome to my office for feeback on your work.

§3.7 The Chain Rule

The rules up to now have not allowed us to differentiate composite functions

$$f \circ g(x) = f(g(x)).$$

Example

If $f(x) = x^7$ and $g(x) = 2x - 3$, then $f(g(x)) = (2x - 3)^7$. To differentiate we could mulitply the polynomial out... but in general we should use a much more efficient strategy to emply to composition functions.

Example

Suppose that Yvonne (y) can run twice as fast as Uma (u). Then write $\frac{dy}{du} = 2$.

Suppose that Uma can run four times as fast as Xavier (x). So $\frac{du}{dx} = 4$.

How much faster can Yvonne run than Xavier? In this case, we would take both our rates and multiply them together:

$$\frac{dy}{du} \cdot \frac{du}{dx} = 2 \cdot 4 = 8.$$

Version 1 of the Chain Rule

If g is differentiable at x, and $y = f(u)$ is differentiable at $u = g(x)$, then the composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Guidelines for Using the Chain Rule

Assume the differentiable function $y = f(g(x))$ is given.

1. Identify the outer function f, the inner function g, and let $u = g(x)$.
2. Replace $g(x)$ by u to express y in terms of u:

$$y = f(g(x)) \implies y = f(u)$$

3. Calculate the product $\dfrac{dy}{du} \cdot \dfrac{du}{dx}$
4. Replace u by $g(x)$ in $\dfrac{dy}{du}$ to obtain $\dfrac{dy}{dx}$.

Example

Use Version 1 of the Chain Rule to calculate $\frac{dy}{dx}$ for $y = (5x^2 + 11x)^{20}$.

- inner function: $u = 5x^2 + 11x$
- outer function: $y = u^{20}$

We have $y = f(g(x)) = (5x^2 + 11x)^{20}$. Differentiate:

$$\begin{aligned} \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} &= 20u^{19} \cdot (10x + 11) \\ &= 20(5x^2 + 11x)^{19} \cdot (10x + 11) \end{aligned}$$

Exercise

Use the first version of the Chain Rule to calculate $\frac{dy}{dx}$ for

$$y = \left(\frac{3x}{4x+2}\right)^5 .$$

Exercise

Use the first version of the Chain Rule to calculate $\frac{dy}{dx}$ for

$$y = \cos(5x+1).$$

A. $y' = -\cos(5x+1) \cdot \sin(5x+1)$
B. $y' = -5\sin(5x+1)$
C. $y' = 5\cos(5x+1) - \sin(5x+1)$
D. $y' = -\sin(5x+1)$

Notice if $y = f(u)$ and $u = g(x)$, then $y = f(u) = f(g(x))$, so we can also write:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= f'(u) \cdot g'(x) \\ &= f'(g(x)) \cdot g'(x). \end{aligned}$$

Example

Use Version 2 of the Chain Rule to calculate $\frac{dy}{dx}$ for $y = (7x^4 + 2x + 5)^9$.

- inner function: $g(x) = 7x^4 + 2x + 5$
- outer function: $f(u) = u^9$

Then

$$f'(u) = 9u^8 \implies f'(g(x)) = 9(7x^4 + 2x + 5)^8$$
$$g'(x) = 28x^3 + 2.$$

Putting it together,

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x) = 9(7x^4 + 2x + 5)^8 \cdot (28x^3 + 2)$$

Exercise

Use Version 2 of the Chain Rule to calculate $\frac{dy}{dx}$ for

$$y = \tan\left(5x^5 - 7x^3 + 2x\right).$$

If g is differentiable for all x in the domain and n is an integer, then

$$\frac{d}{dx}\left[(g(x))^n \right] = n(g(x))^{n-1} \cdot g'(x).$$

Example

$$\frac{d}{dx}\left[(1-e^x)^4\right] = ?$$

Answer:

$$\begin{aligned}\frac{d}{dx}\left[(1-e^x)^4\right] &= 4(1-e^x)^3 \cdot (-e^x)\\ &= -4e^x(1-e^x)^3\end{aligned}$$

Example

Compute $\dfrac{d}{dx}\left[\sqrt{(3x-4)^2+3x}\right]$.

Composition of 3 or More Functions (cont.)

Answer:

$$\begin{aligned}
\frac{d}{dx}\left[\sqrt{(3x-4)^2+3x}\right] &= \frac{1}{2}\big((3x-4)^2+3x\big)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left[(3x-4)^2+3x\right] \\
&= \frac{1}{2\sqrt{\left((3x-4)^2+3x\right)}} \cdot \left[2(3x-4)\frac{d}{dx}(3x-4)+3\right] \\
&= \frac{1}{2\sqrt{\left((3x-4)^2+3x\right)}} \cdot \left[2(3x-4)\cdot 3+3\right] \\
&= \frac{18x-21}{2\sqrt{\left((3x-4)^2+3x\right)}}
\end{aligned}$$

3.7 Book Problems

7-33 (odds), 38, 45-67 (odds)

§3.8 Implicit Differentiation

Up to now, we have calculated derivatives of functions of the form $y = f(x)$, where y is defined **explicitly** in terms of x. In this section, we examine relationships between variables that are **implicit** in nature, meaning that y either is not defined explicitly in terms of x or cannot be easily manipulated to solve for y in terms of x.

The goal of **implicit differentiation** is to find a single expression for the derivative directly from an equation of the form $F(x, y) = 0$ without first solving for y.

Example

Calculate $\frac{dy}{dx}$ directly from the equation for the circle

$$x^2 + y^2 = 9.$$

Solution: To remind ourselves that x is our independent variable and that we are differentiating with respect to x, we can replace y with $y(x)$:

$$x^2 + (y(x))^2 = 9.$$

Now differentiate each term with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}((y(x))^2) = \frac{d}{dx}(9).$$

By the Chain Rule, $\frac{d}{dx}((y(x))^2) = 2y(x)y'(x)$ (Version 2), or $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$ (Version 1). So

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ \implies \frac{dy}{dx} &= \frac{-2x}{2y} \\ &= -\frac{x}{y}. \end{aligned}$$

The derivative is a function of x and y, meaning we can write it in the form

$$F(x,y) = -\frac{x}{y}.$$

To find slopes of tangent lines at various points along the circle we just plug in the coordinates. For example, the slope of the tangent line at (0,3) is

$$\left.\frac{dy}{dx}\right|_{(x,y)=(0,3)} = -\frac{0}{3} = 0.$$

The slope of the tangent line at $(1, 2\sqrt{2})$ is

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,2\sqrt{2})} = -\frac{1}{2\sqrt{2}}.$$

The point is that, in some cases it is difficult to solve an implicit equation in terms of y and then differentiate with respect to x. In other cases, although it may be easier to solve for y in terms of x, you may need two or more functions to do so, which means two or more derivatives must be calculated (e.g., circles).

The goal of implicit differentiation is to find one single expression for the derivative directly given $F(x, y) = 0$ (i.e., some equation with xs and ys in it), without solving first for y.

Question

The following functions are implicitly defined:

- $x + y^3 - xy = 4$
- $\cos(x - y) + \sin y = \sqrt{2}$

For each of these functions, how would you find $\frac{dy}{dx}$?

Exercise

Find $\frac{dy}{dx}$ for $xy + y^3 = 1$.

Exercise

Find an equation of the line tangent to the curve $x^4 - x^2y + y^4 = 1$ at the point $(-1, 1)$.

Example

Find $\frac{d^2y}{dx^2}$ if $xy + y^3 = 1$.

Exercise

If $\sin x = \sin y$, then $\frac{dy}{dx} = ?$ and $\frac{d^2y}{dx^2} = ?$

A. $\dfrac{\cos y}{\cos x}$; $\quad \dfrac{\tan y \cos^2 x - \sin x \cos y}{\cos^2 x}$

B. $\dfrac{\cos x}{\cos y}$; $\quad \dfrac{\tan y \cos^2 x - \sin x \cos y}{\cos^2 y}$

C. $\dfrac{\cos x}{\cos y}$; $\quad \dfrac{\cos y(\sin x - \sin y)}{\cos^2 y}$

D. $\dfrac{\cos y}{\cos x}$; $\quad \dfrac{\cos y(\sin x - \sin y)}{\cos^2 x}$

Implicit differentiation also allows us to extend the power rule to rational exponents: Assume p and q are integers with $q \neq 0$. Then

$$\frac{d}{dx}(x^{\frac{p}{q}}) = \frac{p}{q}x^{\frac{p}{q}-1}$$

(provided $x \geq 0$ when q is even and $\frac{p}{q}$ is in lowest terms).

Exercise

Prove it.

3.8 Book Problems

5-25 (odds), 31-49 (odds)