Take Home Quiz #7

Math 235 (Calc I)
Fall 2017

$\boxed{\text{SOLUTIONS}}$

1. (a) Let f = SAT scores. Then $f' < 0$ ("declining") and $f'' > 0$ ("at a slower rate").

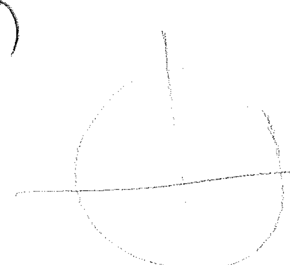
(b) i. It is getting hotter, and faster!

ii. It is getting hotter, but at a slower rate.

iii. It is cooling down, and faster.

iv. It is cooling down at a slower rate.

2. (a)



Looking at the unit circle, sine is positive for positive angles.

The domain for $f(x)$ is

$$\ldots (-5\pi, -4\pi), (-3\pi, -2\pi), (-\pi, 0), (0, \pi), (2\pi, 3\pi), (4\pi, 5\pi), \ldots$$

or $\boxed{(2n\pi, (2n+1)\pi)}$.

(b) $f(x)$ is a composition of trig functions. Its period is 2π, since

$$f(x+2\pi) = \sin(x+2\pi)^{\sin(x+2\pi)} = \sin x^{\sin x} = f(x).$$

$\longrightarrow$

(c) $\lim_{x\to 0^+} \sin x^{\sin x} \rightsquigarrow \ln\left(\lim_{x\to 0^+} \sin x^{\sin x}\right)$

0^0

$$= \lim_{x\to 0^+} \ln\left(\sin x^{\sin x}\right)$$

$$= \lim_{x\to 0^+} \sin x \ln(\sin x)$$

$0 \cdot -\infty$

$$= \lim_{x\to 0^+} \frac{\ln(\sin x)}{\frac{1}{\sin x}}$$

$$\overset{LR}{=} \lim_{x\to 0^+} \frac{\frac{\cancel{\cos x}}{\cancel{\sin x}}}{\frac{\cancel{\cos x}}{-\sin^{\cancel{2}} x}}$$

$$= \lim_{x\to 0^+} -\sin x = 0$$

and $\lim_{x\to \pi^-} -\sin x = 0$

$$\Rightarrow \lim_{x\to 0^+} \sin x^{\sin x} = \lim_{x\to \pi^-} \sin x^{\sin x} = e^0 = \boxed{1}$$

(d) i. $\ln(f(x)) = \ln\left(\sin x^{\sin x}\right) = \sin x \ln(\sin x)$

$$\Rightarrow \frac{d}{dx}\left[\ln f(x)\right] = \frac{d}{dx}\left(\sin x \ln(\sin x)\right)$$

$$\frac{f'(x)}{f(x)} = \cos x \ln(\sin x) + \cancel{\sin x}\left(\frac{\cos x}{\cancel{\sin x}}\right)$$

$$\Rightarrow f'(x) = f(x)\left(\cos x \ln(\sin x) + \cos x\right)$$

$$= \sin x^{\sin x} \cos x\left(\ln(\sin x) + 1\right)$$ ✓

ii: The only way for $f(x) = 0$ is if the base of the exponential equals zero. But $\sin x = 0$ when x is a multiple of π, and these points are not in the domain.

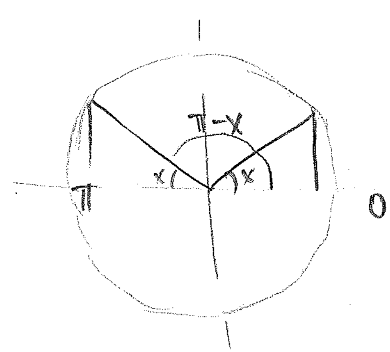
iii. $\cos x = 0$ at (integer) multiples of $\frac{\pi}{2}$.

iv. $\ln(\sin x) = -1$

$\Rightarrow \sin x = e^{-1} = \frac{1}{e}$

v. One of the critical points is $x = \frac{\pi}{2}$. The other two are given by the formula $\sin x = \frac{1}{e}$. When x is between 0 and π, $\sin x = \sin(\pi - x)$. So the critical points are:

$$\boxed{\begin{aligned} &x = 0, \\ &\arcsin\tfrac{1}{e} \approx 0.377, \\ &\pi - \arcsin\tfrac{1}{e} \approx 2.765 \end{aligned}}$$



(e) $f''(x) = \frac{d}{dx} \sin x^{\sin x} \cos x(\ln(\sin x) + 1)$

$$= \left(\sin x^{\sin x} \cos x(\ln(\sin x)+1)\right)\left(\cos x\ln(\sin x)+1\right)$$

$$+ \sin x^{\sin x}\left(-\sin x(\ln(\sin x)+1) + \cos x \cdot \frac{\cos x}{\sin x}\right)$$

$$= \sin x^{\sin x}\left(\cos^2 x(\ln(\sin x)+1)^2 - \sin x(\ln(\sin x)+1) + \frac{\cos^2 x}{\sin x}\right)$$

*This means there was a typo on the assignment.

$$f''\left(\tfrac{\pi}{2}\right) = \underbrace{\left(\sin\tfrac{\pi}{2}\right)^{\sin\frac{\pi}{2}}}_{=1}\left(\cancel{\left(\cos^2\tfrac{\pi}{2}\right)\left(\ln\left(\sin\tfrac{\pi}{2}\right)+1\right)^2} - \underbrace{\sin\tfrac{\pi}{2}}_{=1}\left(\ln\underbrace{\left(\sin\tfrac{\pi}{2}\right)}_{=1}+1\right) + \cancel{\frac{\cos^2\frac{\pi}{2}}{\sin\frac{\pi}{2}}}\right)$$

$= -1 \Rightarrow$ $\boxed{\text{concave down, max at } x = \frac{\pi}{2}}$.

Since $\sin\left(\arcsin\frac{1}{e}\right) = \frac{1}{e}$,

$$f''\left(\arcsin\tfrac{1}{e}\right) = \left(\tfrac{1}{e}\right)^{\frac{1}{e}}\left(\cancel{\cos^2\left(\arcsin\tfrac{1}{e}\right)\left(\ln\left(\tfrac{1}{e}\right)+1\right)^2}^{0} - \cancel{\tfrac{1}{e}\left(\ln\tfrac{1}{e}+1\right)}^{0} + \frac{\cos^2\left(\arcsin\frac{1}{e}\right)}{\frac{1}{e}}\right)$$

$$= \frac{\left(\frac{1}{e}\right)^{\frac{1}{e}}\left(\sqrt{1-\left(\frac{1}{e}\right)^2}\right)^2}{\frac{1}{e}} > 0 \Rightarrow \boxed{\text{concave up, min at } x = \arcsin\tfrac{1}{e}}$$

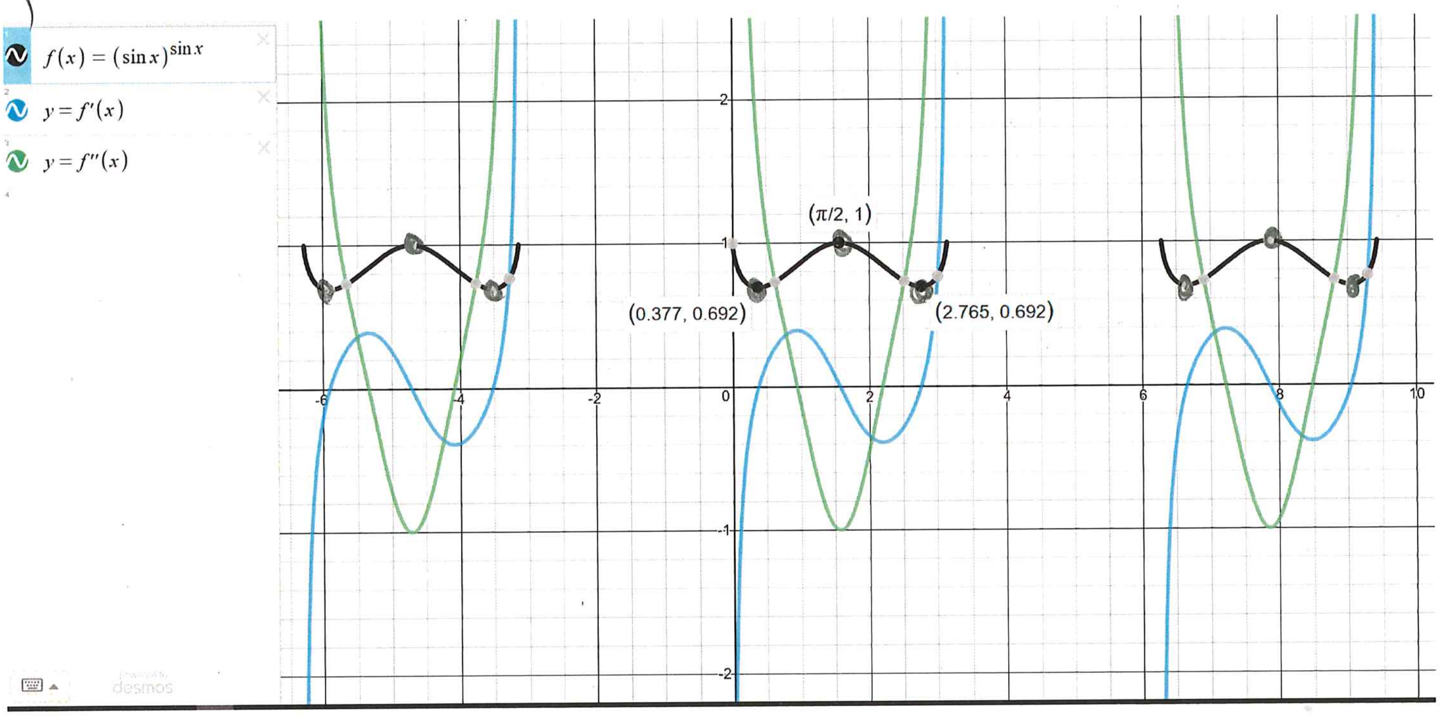
→

$$f''\left(\arcsin\left(\pi - \tfrac{1}{e}\right)\right) = \frac{\left(\frac{1}{e}\right)^{\frac{1}{e}} \cos^2\left(\pi - \arcsin \frac{1}{e}\right)}{\frac{1}{e}} > 0$$

$$\Rightarrow \boxed{\text{concave up, min at } x = \pi - \arcsin \tfrac{1}{e}}$$

*Because of the typo fix, the 2^{nd} Derivative Test works on all the critical points.

(f)



3. (a) For the domain of $f(x)$,

$$x^2 + c > 0.$$

$x^2 > c$ (when $c<0$)

$-\sqrt{-c}$ $\quad$ $\sqrt{c}$

- $c > 0 \Rightarrow$ domain $= \mathbb{R}$
- $c = 0 \Rightarrow$ domain $= \mathbb{R} \setminus \{0\}$
- $c < 0 \Rightarrow$ domain $= (-\infty, -\sqrt{-c}), (\sqrt{c}, \infty)$.

(b) i. $f(0) = \ln(0^2 + c)$

$= \ln c = y$ unless

$c \le 0$, since in that case $\ln c$ is undefined.

ii. $0 = \ln(x^2 + c)$

$\Rightarrow e^0 = x^2 + c$

$1 - c = x^2$

Solutions exist when $c \le 1$; the x-intercepts in that case are $x = \pm\sqrt{1-c}$.

(c) i. $f(-x) = \ln((-x)^2 + c) = \ln(x^2 + c)$

$\Rightarrow f$ is even.

ii. Since f does not involve any trig functions, it is not periodic.

(d) i. Since f is even,

$\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x)$

$= \lim_{x \to \pm\infty} \ln(x^2 + c) = \infty$

ii. Vertical asymptotes are at $x = \pm\sqrt{-c}$, so we must have $c < 0$.

Again, since f is even,

$$\lim_{x \to -\sqrt{-c}^-} f(x) = \lim_{x \to \sqrt{-c}^+} f(x)$$

$$= \lim_{x^2 \to -c} \ln(x^2+c) = -\infty$$

The one-sided limits are necessary because of the domain restriction, $x \in (-\infty, -\sqrt{-c}) (\sqrt{-c}, \infty)$.

iii. There is no function $y = mx+b$ such that

$$\lim_{x \to \pm\infty} (f(x) - mx - b) = 0.$$

(e) ii. $f'(x) = \frac{2x}{x^2+c}$

is undefined when $x^2 = -c$. If $c > 0$ then no such points exists. If $c \le 0$, then $x = \pm\sqrt{-c}$, but these points are not in the domain.

On the other hand, if $c > 0$ then $f'(x) = 0 \Rightarrow x = 0$ is a critical point. If $c < 0$ then $x=0$ is not in the domain, so there are no critical points. If $c = 0$ then there are no critical points

→

ii. When $c > 0$,

$$f'(-c) = \frac{2(-c)}{(-c)^2+c} = \frac{-2c}{c^2+c} = -\frac{2}{c} < 0 \Rightarrow f \text{ is dec.}$$

$-c \qquad \phi \qquad c$

$-c \qquad 0 \qquad c$

$$f'(c) = \frac{2c}{c^2+c} = \frac{2}{c} > 0 \Rightarrow f \text{ is inc.}$$

Since f changes from decreasing to increasing, $x=0$ is a local min

$$2x^2+2c \qquad -4x^2$$

iii. $$f''(x) = \frac{(x^2+c)(2) - 2x(2x)}{(x^2+c)^2} = \frac{2c-2x^2}{(x^2+c)^2}$$

When $c>0$, $f''(0) = \dfrac{2c-2(0)^2}{(0^2+c)^2} > 0 \Rightarrow$ concave up and $x=0$ gives a local min.

iv. If $c>0$ then the range is $[\ln(c), \infty)$, since all values going into the natural log will be larger than c.

Otherwise the range is $\mathbb{R}$.

This means f will have a global min at $x=0$ when $c>0$. The y coordinate for the min is $f(0) = \ln c$.

$\longrightarrow$

(f) • $c>0$. From part (e)ii., f is decreasing on $(-\infty, 0)$ and increasing on $(0,\infty)$.

• $c=0$. There is a vertical asymptote at $x=0$. When $x<0$, $f'(x) = \frac{2x}{x^2} < 0$, and so f is decreasing. Similarly, f is increasing on $(0,\infty)$.

• $c<0$. There are no critical points but we must check each interval in the domain:

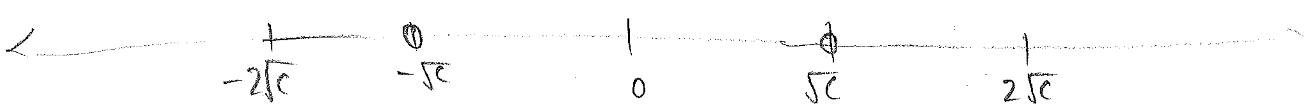
On $(-\infty, -\sqrt{-c})$, $f'(x) = \frac{2x}{x^2+c} < 0$, since $x^2 > -c$.

$\Rightarrow f$ is decreasing.

On $(\sqrt{-c}, \infty)$, $f'(x) = \frac{2x}{x^2+c} > 0 \Rightarrow f$ is increasing.

(g) • $c>0$. $f''(x) = \frac{2c-2x^2}{(x^2+c)^2} = 0 \Rightarrow x = \pm\sqrt{c}$ are possible inflection points.

$f''(0) = \frac{2}{c} > 0$



conc

$f''(-2\sqrt{c}) = \frac{2c-2(-2\sqrt{c})^2}{(x^2-2\sqrt{c})^2} < 0$

$f''(2\sqrt{c}) = \frac{2c-2(2\sqrt{c})^2}{(x^2+2\sqrt{c})^2} < 0$

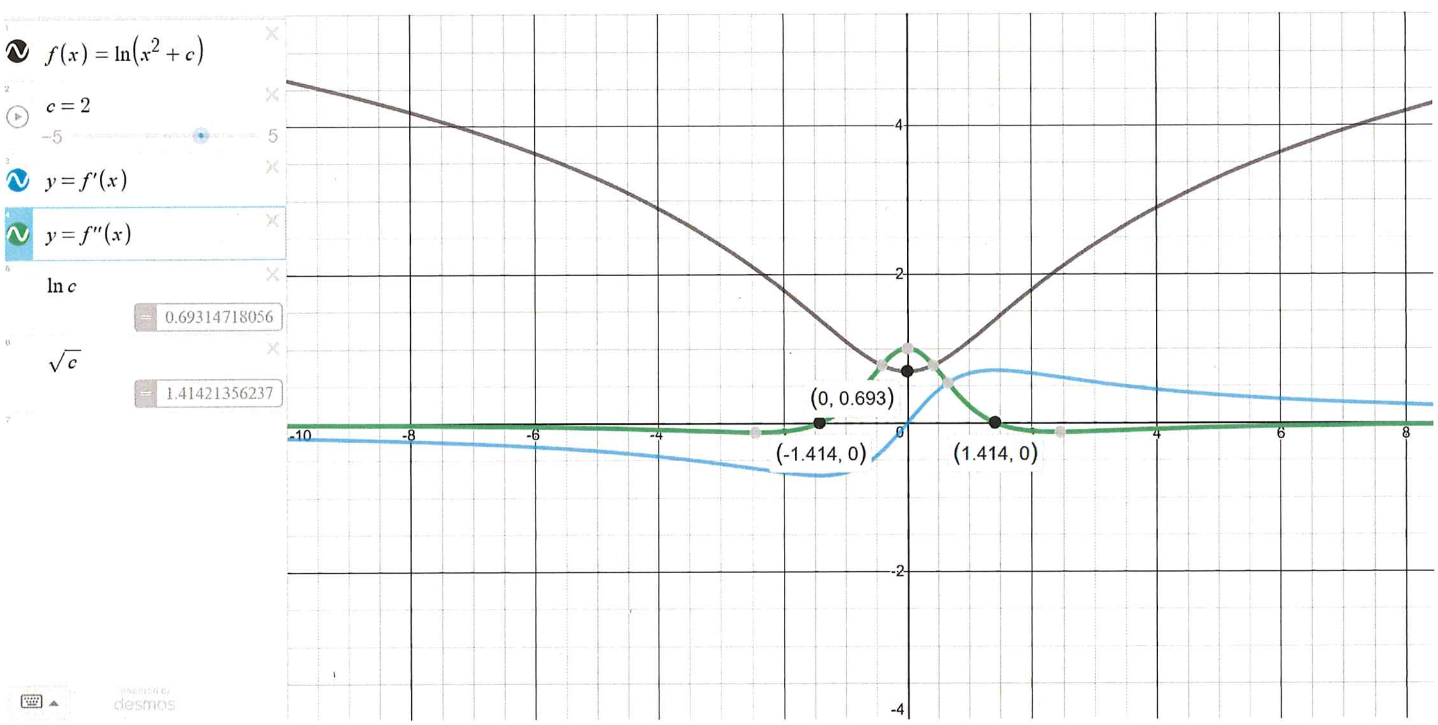
Concave up: $(-\sqrt{c}, \sqrt{c})$ — Inflection pts: $x = \pm\sqrt{c}$

Concave down: $(-\infty, -\sqrt{c}), (\sqrt{c}, \infty)$

• $c = 0$. $f''(x) = \frac{-2x^2}{x^4} = \frac{-2}{x^2} < 0$ so f is always concave down.

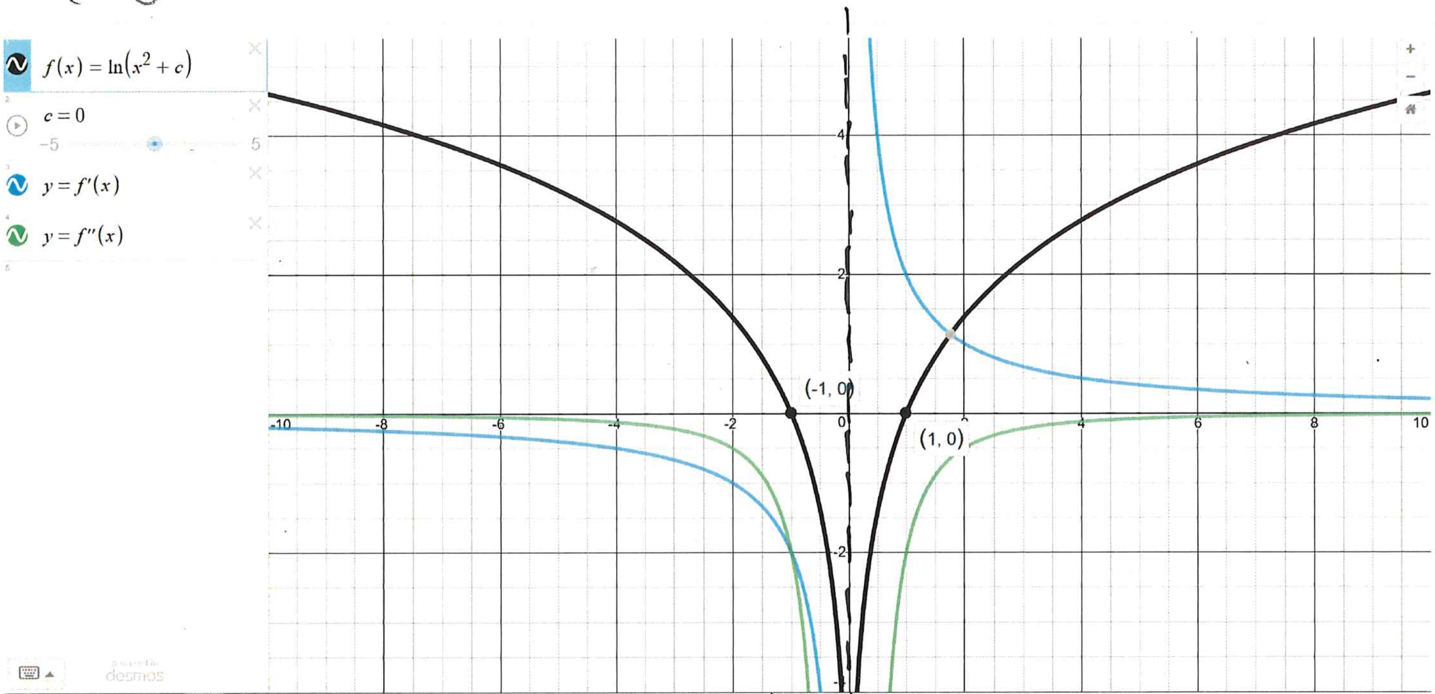
• $c < 0$. $f''(x) = \frac{2c - 2x^2}{(x^2+c)^2} < 0$ for all x so f is always concave down.

(h) • $c > 0$.



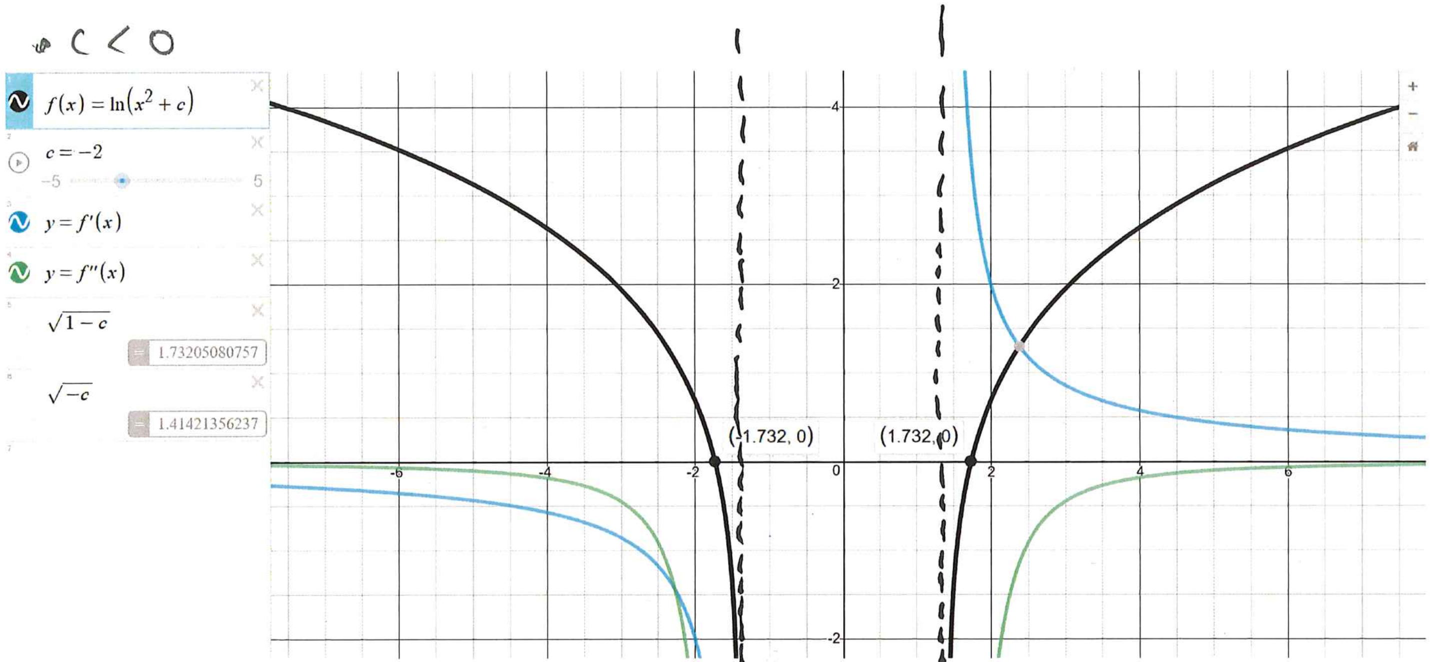
domain $= \mathbb{R}$

• $c = 0$



domain: $x \neq 0$

• $c < 0$



$x = -\sqrt{2}$ $\quad x = \sqrt{2}$

$(-\infty, -\sqrt{2})$ $\quad (\sqrt{2}, \infty)$