MATH 236 (Calculus II)
Fall 2017

Name: SOLUTIONS

Thur 28 Sep 2017

Exam 1: Integrals (§4.5, 4.7, 5.1-5.5, 6.1-6.2, 6.4)

Exam Instructions: You have 75 minutes to complete this exam. Justification is required for all problems. Notation matters! You will also be penalized for missing units and rounding errors.

No electronic devices (phones, iDevices, computers, etc) except for a **basic scientific calculator**. On story problems, round to one decimal place. If you don't have a calculator, write "no calculator" and simplify as far as you can.

If you finish early then you may leave, UNLESS there are less than 5 minutes of class left. To prevent disruption, if you finish with less than 5 minutes of class remaining then please stay seated and quiet.

Your signature below indicates that you have read this page and agree to follow the Academic Honesty Policies of James Madison University.

Signature: **(1 pt)** ____________________

Good luck!

Formulas you may need:

$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$; domain of $\arcsin x$: $[-1,1]$; range of $\arcsin x$: $[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\frac{d}{dx}\arctan x = \frac{1}{x^2+1}$; domain of $\arctan x$: $(-\infty,\infty)$; range of $\arctan x$: $(-\frac{\pi}{2}, \frac{\pi}{2})$
$\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$; domain of $\operatorname{arcsec} x$: $(-\infty,-1]\cup[1,\infty)$; range of $\operatorname{arcsec} x$: $[0,\frac{\pi}{2}]\cup$ $[\frac{\pi}{2},\pi]$

$$\int \tan x\; dx = -\ln|\cos x| + C$$
$$\int \cot x\; dx = \ln|\sin x| + C$$
$$\int \sec x\; dx = \ln|\sec x \tan x| + C$$
$$\int \csc x\; dx = -\ln|\csc x \cot x| + C$$
$$\int \arcsin x\; dx = x\arcsin x + \sqrt{1-x^2} + C$$
$$\int \arctan x\; dx = x\arctan x - \frac{1}{2}\ln(x^2+1) + C$$

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$
$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$
$$\sin 2x = 2\sin x\cos x$$

Question	Points	Score
1	18	
2	20	
3	25	
4	18	
5	18	
Total:	99	

Formulas, cont.

mass = (density)(volume)
work = (force)(displacement)
hydrostatic force = (weight-density)(surface area)(depth of liquid)
weight-density of water = 62.4 lb/ft^3

Integral	Rewrite	Choose
$\int \sin^n x \, dx$, n odd	$\int$ (expression in $\cos x$) $\sin x$	$u = \cos x$
$\int \sin^m x \cos^n x \, dx$, n odd	$\int$ (expression in $\sin x$) $\cos x$	$u = \sin x$
$\int \sin^m x \cos^n x \, dx$, m odd	$\int$ (expression in $\cos x$) $\sin x$	$u = \cos x$
$\int \sec^m x \tan^n x \, dx$, m even	$\int$ (expression in $\tan x$) $\sec^2 x$	$u = \tan x$
$\int \sec^m x \tan^n x \, dx$, n odd	$\int$ (expression in $\sec x$) $\sec x \tan x$	$u = \sec x$

$$(A+B)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} A^k B^{n-k}$$

1. Compute the following derivatives.

(a) **(9 pts)** $\dfrac{d^2}{dx^2}\displaystyle\int_1^{e^x} f(t)g(t)\,dt$

$= \dfrac{d}{dx}\left(f(e^x)g(e^x)e^x\right)$

$= \left(f'(e^x)e^x \cdot g(e^x) + f(e^x)\cdot g'(e^x)e^x\right)e^x$
$\quad + f(e^x)g(e^x)e^x$

$\boxed{= \left(f'(e^x)g(e^x) + f(e^x)g'(e^x)\right)e^{2x} + f(e^x)g(e^x)e^x}$

(b) **(9 pts)** $\dfrac{d}{dx}\left(x^2\displaystyle\int_0^{\sin x}\sqrt{t}\,dt\right)$

$= 2x\displaystyle\int_0^{\sin x}\sqrt{t}\,dt + x^2\sqrt{\sin x}\,(\cos x)$

$= 2x\left(\frac{2}{3}t^{3/2}\right)\Big|_{0 \leftarrow \text{vanishes}}^{\sin x} + x^2\sqrt{\sin x}\,(\cos x)$

$= \frac{4}{3}x(\sin x)^{3/2} + x^2\sqrt{\sin x}\,(\cos x)$

$\boxed{= \left(\frac{4}{3}\sin x + x\cos x\right)x\sqrt{\sin x}}$

2. **(20 pts)** Evaluate: $\displaystyle\int_{-1}^{1} \frac{4(x^2-1)}{x^2-4}\,dx$

$$x^2-4 \,\overline{)\,x^2-1}$$
$$\underline{-(x^2-4)}$$
$$3$$

$$= 4\int_{-1}^{1} dx + 4\int_{-1}^{1} \frac{3}{\underset{(x+2)(x-2)}{x^2-4}}\,dx$$

$$= 4x\Big|_{-1}^{1} + 12\int_{-1}^{1}\left(\frac{A}{x+2} + \frac{B}{x-2}\right)dx$$

$$1 = A(x-2) + B(x+2)$$
$$= Ax - 2A + Bx - 2B$$
$$\Rightarrow 0x = Ax + Bx$$
$$A = -B$$
$$1 = -2A + 2B$$
$$= -2(-B) + 2B = 4B$$
$$\Rightarrow B = \frac{1}{4}, \quad A = -\frac{1}{4}$$

$$= 4(1) - 4(-1) + 12\left(-\frac{1}{4}\int_{-1}^{1}\frac{dx}{x+2} + \frac{1}{4}\int_{-1}^{1}\frac{dx}{x-2}\right)$$

$$= 8 - 3\ln|x+2|\Big|_{-1}^{1} + 3\ln|x-2|\Big|_{-1}^{1}$$

$$= 8 - \left(3\ln|3| - \cancel{3\ln|1|}\right) + \left(\cancel{3\ln|-1|} - 3\ln|-3|\right)$$

$$\boxed{= 8 - 6\ln 3}$$

3. **(25 pts)** Evaluate: $\int e^{2x}(1-e^{4x})^{\frac{3}{2}}\,dx$

$u=e^{2x}$ $\quad du = 2e^{2x}\,dx \Rightarrow e^{2x}\,dx = \frac{du}{2}$

$= \frac{1}{2}\int (1-u^2)^{3/2}\,du$

$w=\arcsin u$
$\Rightarrow u=\sin w$
$du = \cos w\,dw$

$= \frac{1}{2}\int \underbrace{(1-\sin^2 w)}_{\cos^2 w}{}^{3/2}\cos w\,dw$

$= \frac{1}{2}\int \cos^4 w\,dw = \frac{1}{2}\int\left(\frac{1}{2}(1+\cos 2w)\right)^2 dw$

$\cos^2 w = \frac{1}{2}(1+\cos 2w)$
$\Rightarrow 2\cos^2 w = 1+\cos 2w$
$2\cos^2 w - 1 = \cos 2w$

$= \frac{1}{8}\int\left(1 + 2\cos 2w + \cos^2 2w\right)dw$

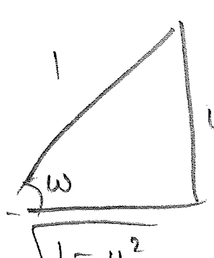
$= \frac{1}{8}(w + \sin 2w) + \frac{1}{8}\int \frac{1}{2}(1+\cos 4w)\,dw$

$= \frac{1}{8}w + \frac{1}{8}(2\sin w\cos w) + \frac{1}{16}w + \frac{1}{16}\left(\frac{1}{4}\sin 4w\right) + C$

$= \frac{3}{16}w + \frac{1}{4}\sin w\cos w + \frac{1}{64}(2\sin 2w\cos 2w) + C$

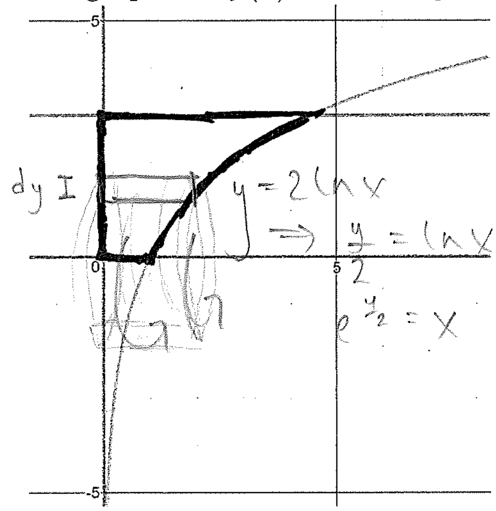
$= \frac{3}{16}\arcsin(e^{2x}) + \frac{1}{4}\sin(\cancel{\arcsin} e^{2x})\cos(\arcsin e^{2x})$

$+ \frac{1}{32}(2\sin w\cos w)(2\cos^2 w - 1) + C$



$$\boxed{= \frac{3}{16}\arcsin e^{2x} + \frac{1}{4}e^{2x}\sqrt{1-e^{4x}} + \frac{1}{16}e^{2x}\sqrt{1-e^{4x}}\left(2(1-e^{4x})-1\right) + C}$$

4. **(18 pts)** Find the volume of the solid of revolution given by the region bounded by the graphs of $f(x) = 2\ln x$, $y = 0$, $y = 3$, and $x = 0$, revolved around the x-axis.



(Discs/washers requires a piecewise integral)

Volume of one shell:

$= \pi y_{k+1}^2 x - \pi y_k^2 x$

$= \pi (y_{k+1}^2 - y_k^2) e^{y_k^*/2}$

$= \pi (y_{k+1} - y_k)(y_{k+1} + y_k) e^{y_k^*/2}$

$\underbrace{(y_{k+1} - y_k)}_{dy}$, $\underbrace{(y_{k+1} + y_k)}_{2y}$

Total Volume

$2\pi \int_0^3 y e^{y/2}\, dy$

Log poet

$u = y \qquad dv = e^{y/2}\, dy$

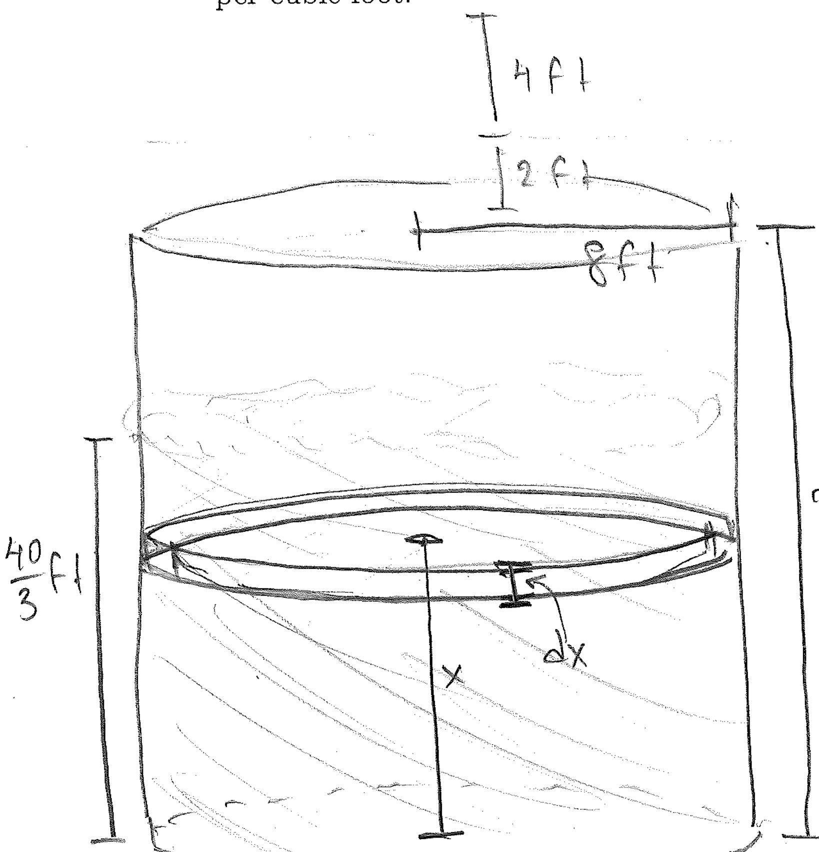
$du = dy \qquad v = 2e^{y/2}$

$= 2\pi \left(y(2e^{y/2}) \Big|_0^3 - \int_0^3 (2e^{y/2})\, dy \right) = 4\pi(3e^{3/2} - 0e^{0/2}) - 4\pi(2e^{y/2}) \Big|_0^3$

$= 12\pi e^{3/2} - 8\pi(e^{3/2} - e^0)$

$$\boxed{\begin{array}{l} = 4\pi e^{3/2} + 8\pi \\ \approx 81.5 \end{array}}$$

5. **(18 pts)** Find the work required to pump all the liquid out of the top of a tank and up to 4 feet above ground level, given that the tank is a giant upright cylinder with a radius of 8 feet and a height of 20 feet, buried so that its top is 2 feet below the surface. The tank is only two-thirds full and contains a strange liquid that weighs 71.8 pounds per cubic foot.



Work = (force)(displacement)

$= \omega$(volume)(displacement)

$= \omega(\pi 8^2 \Delta x)(26 - x_k^*)$

Total Work:

$$64\omega\pi \int_0^{40/3} (26 - x)\,dx$$

$$= 64\omega\pi \left(26x - \frac{x^2}{2}\right)\Bigg|_0^{40/3}$$

0 ← terms vanish

$$= 64\omega\pi \left(26\left(\frac{40}{3}\right) - \frac{\left(\frac{40}{3}\right)^2}{2}\right)$$

$$= (64\pi\ \text{ft}^2)(71.8\ \text{lb/ft}^3)\left((26\ \text{ft})\left(\frac{40}{3}\ \text{ft}\right) - \frac{\left(\frac{40}{3}\ \text{ft}\right)^2}{2}\right)$$

$\frac{4640}{18}$

$\boxed{\approx 3{,}720{,}000 \text{ ft-lb.}}$