MATH 2574 (Calculus III)
Fall 2015

Name: SOLUTIONS

Drill: ——

Tues 1 Dec 2015

Quiz 12: Divergence, Curl, and Green's Theorem (§14.4-14.5)

1. **(4 pts)** $\mathbf{F} = \langle e^{-x+y}, e^{-y+z}, e^{-z+x} \rangle$ is a vector field in $\mathbb{R}^3$.

   (a) curl $\mathbf{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{-x+y} & e^{-y+z} & e^{-z+x} \end{vmatrix}$

   $= \left\langle \frac{\partial}{\partial y}(e^{-z+x}) - \frac{\partial}{\partial z}(e^{-y+z}), -\frac{\partial}{\partial x}(e^{-z+x}) + \frac{\partial}{\partial z}(e^{-x+y}), \frac{\partial}{\partial x}(e^{-y+z}) - \frac{\partial}{\partial y}(e^{-x+y}) \right\rangle$

   $\boxed{= \langle -e^{-y+z}, -e^{-z+x}, -e^{-x+y} \rangle}$

   (b) div $\mathbf{F} = \nabla \cdot \vec{F}$

   $= \frac{\partial}{\partial x}(e^{-x+y}) + \frac{\partial}{\partial y}(e^{-y+z}) + \frac{\partial}{\partial z}(e^{-z+x})$

   $\boxed{= -e^{-x+y} - e^{-y+z} - e^{-z+x}}$

2. **(2 pts) Green's Theorem:** Let C be a simple closed smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Let $\mathbf{F} = \langle f(x,y), g(x,y) \rangle$ denote a vector field, where f and g have continuous first partial derivatives on R. Then

   (a) Green's Theorem says the circulation of $\mathbf{F}$ on R is (write the equation):

   $$\oint_C \vec{F} \cdot d\vec{r} = \oint_C f\,dx + g\,dy = \iint_R \operatorname{curl} \vec{F}\, dA = \iint_R (g_x - f_y)\, dA$$

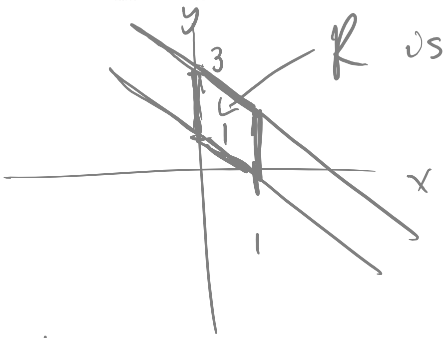
   (b) and that the flux of $\mathbf{F}$ across the boundary of R is (write the equation):

   $$\oint_C \vec{F} \cdot \hat{n}\, ds = \oint_C f\,dy - g\,dx = \iint_R \operatorname{div} \vec{F}\, dA = \iint_R (f_x + g_y)\, dA$$

3. **(4 pts)** Let $\mathbf{F} = \langle \overset{f}{x - y}, \overset{g}{-x + 2y} \rangle$ denote a vector field on the paralleogram

$$R = \{(x, y) \mid 1 - x \leq y \leq 3 - x, 0 \leq x \leq 1\}.$$

Compute (a) the circulation of $\mathbf{F}$ on R and (b) the outward flux of $\mathbf{F}$ across the boundary of R.



R is connected and simply connected, so instead of trying to parametrize its boundary, use Green's Theorem

(a) $$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\underset{-1}{g_x} - \underset{-1}{f_y})\, dA = \int_0^1 \int_{1-x}^{3-x} (-1-(-1))\, dy\, dx$$

$$\boxed{=0}$$

(b) $$\oint_C \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_{1-x}^{3-x} (f_x + g_y)\, dy\, dx$$

$$= \int_0^1 \int_{1-x}^{3-x} (1 + 2)\, dy\, dx$$

$$= \int_0^1 3y \Big|_{1-x}^{3-x} dx$$

$$= \int_0^1 3(3-x) - 3(1-x)\, dx = 6x \Big|_0^1 \boxed{= 6}$$

$$-\cancel{3x} - 3 + \cancel{3x}$$