MATH 235 (Calculus I)
Fall 2017

Name: SOLUTIONS

Tues 26 Sep 2017

Exam 1: Functions and limits, v.B (§1.1-1.5, 2.2-2.6)

Exam Instructions: You have 75 minutes to complete this exam. Justification is required for all problems. Notation matters! You will also be penalized for missing units and rounding errors. No electronic devices (phones, iDevices, computers, etc). If you finish early then you may leave, UNLESS there are less than 5 minutes of class left. To prevent disruption, if you finish with less than 5 minutes of class remaining then please stay seated and quiet.

Your signature below indicates that you have read this page and agree to follow the Academic Honesty Policies of James Madison University.

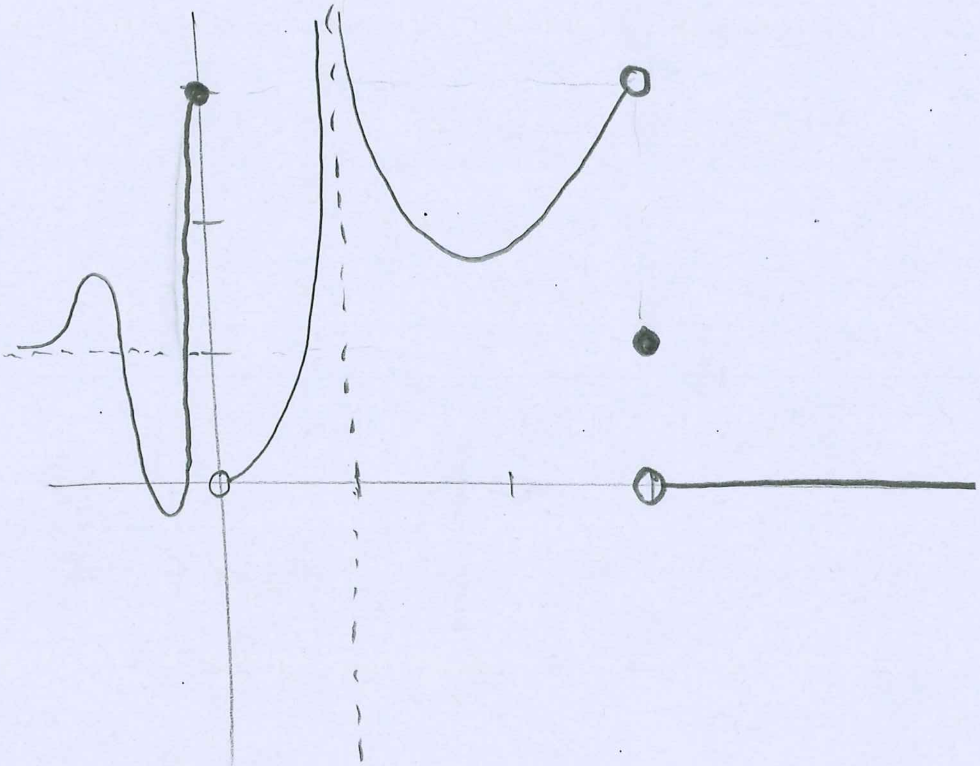
Signature: **(1 pt)** ______________________

Good luck!

Question	Points	Score
1	18	
2	10	
3	15	
4	20	
5	16	
6	12	
7	8	
Total:	99	

1. **(18 pts)** Sketch the graph of an example of a function f that satisfies all of the given conditions.

- $\lim_{x\to 0^+} f(x) = 0$
- $\lim_{x\to 1} f(x) = \infty$
- $f(0) = 3$
- $\lim_{x\to 0^-} f(x) = 3$
- $\lim_{x\to -\infty} f(x) = 1$
- $\lim_{x\to \infty} f(x) = 0$
- $\lim_{x\to 3^-} f(x) = 3$
- $f(3) = 1$
- $\lim_{x\to 3^+} f(x) = 0$



2. A spherical balloon is being inflated and the radius of the balloon is increasing at a rate of 3 cm/s.

(a) **(4 pts)** Express the radius r of the balloon as a function of the time t (in seconds).

$r(t) = 3t + r_0$

radius at $t=0$

(b) **(6 pts)** If $V = \frac{4}{3}\pi r^3$ is the volume of the balloon as a function of the radius, find $V \circ r$ and interpret it.

$$V\circ r = \frac{4}{3}\pi\left(r(t)\right)^3 = \frac{4}{3}\pi(3t+r_0)^3$$

$$= \frac{4}{\cancel{3}}\pi\left(\overset{9}{\cancel{27}}t^3 + \cancel{3}(9t^2r_0) + \cancel{3}(3tr_0) + \frac{1}{3}r_0^3\right)$$

$$\boxed{36\pi t^3} + 36\pi r_0 t^2 + 12\pi r_0 t + \frac{4}{3}\pi r_0^3$$

(if $r_0 = 0$)

V∘r is volume as a function of time.

3. **(15 pts)** The formal definition of a limit says: $\lim\limits_{x \to a} f(x) = L$ means for every $\epsilon > 0$, there exists $\delta > 0$, such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Prove, using the formal definition of a limit, that

$$\lim_{x \to -1} (2x + 1) = -1.$$

Suppose $|f(x) - (-1)| < \varepsilon$ for some $\varepsilon > 0$.

Then: $|f(x) + 1| = |(2x+1) + 1|$

$$= |2x + 2| < \varepsilon$$

$$\Rightarrow 2|x+1| < \varepsilon$$

$$|x+1| < \frac{\varepsilon}{2}$$

$$|x - (-1)| < \frac{\varepsilon}{2}.$$

Take $\delta = \frac{\varepsilon}{2}$.

∎

4. **(20 pts)** Using the continuity checklist, find the values of a and b that make f continuous at 2 and 3.

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x < 3 \\ 2x - a + b & \text{if } x \ge 3 \end{cases}$$

Continuity ✓list:

a=2

- $f(2) = a(2)^2 - b(2) + 3$

$$= 4a - 2b + 3$$

is defined ✓

- $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \dfrac{x^2-4}{x-2}$

$$= \lim_{x\to 2^-} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}$$

$$= \lim_{x\to 2^-} (x+2) = 2+2$$

$$= 4$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (ax^2 - bx + 3)$$

$$= 4a - 2b + 3$$

$$\Rightarrow \lim_{x\to 2} f(x) = 4 = 4a - 2b + 3$$

- $f(2) = 4 = \lim\limits_{x\to 2} f(x) = 4a - 2b + 3$

$$4 = 4\left(\frac{3+4b}{10}\right) - 2b + 3$$

$$= \frac{6}{5} + \frac{8}{5}b - 2b + 3 \Rightarrow \overset{-\frac{1}{5}}{4 - 3 - \frac{6}{5}} = \overset{-\frac{2}{5}}{\left(\frac{8}{5} - 2\right)} b \Rightarrow \boxed{b = \frac{1}{2}}$$

a-3

- $f(3) = 2(3) - a + b$

$$= 6 - a + b$$

- $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (ax^2 - bx + 3)$

$$= a(3)^2 - b(3) + 3$$

$$= 9a - 3b + 3$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (2x - a + b)$$

$$= 6 - a + b$$

$$\Rightarrow \lim_{x\to 3} f(x) = 9a - 3b + 3$$

- " " $= 6 - a + b = f(3)$

$$\Rightarrow 10a = 3 + 4b$$

$$a = \frac{3+4b}{10}$$

$$= \frac{3 + 4(\frac{1}{2})}{10} \Rightarrow \boxed{a = \frac{1}{2}}$$

5. Evaluate the limits. You must show work.

(a) **(4 pts)** $\lim\limits_{u\to-4} \frac{\sqrt{u^2+9}-5}{u+4}\left(\frac{\sqrt{u^2+9}+5}{\sqrt{u^2+9}+5}\right)$

$$= \lim_{u\to-4} \frac{u^2+\overset{-16}{9-25}}{(u+4)(\sqrt{u^2+9}+5)} = \lim_{u\to-4}\frac{\cancel{(u+4)}(u-4)}{\cancel{(u+4)}(\sqrt{u^2+9}+5)}$$

$$= \frac{-4-4}{\sqrt{(-4)^2+9}+5} = -\frac{8}{10} = \boxed{-\frac{4}{5}}$$

(b) **(4 pts)** $\lim\limits_{u\to-2}\sqrt{u^4+3u+6}$

$$= \sqrt{\lim_{u\to-2}(u^4+3u+6)}$$

$$= \sqrt{\underset{16}{(-2)^4}+\underset{-6}{3(-2)}+\underset{+6}{6}} = \sqrt{16} = \boxed{4}$$

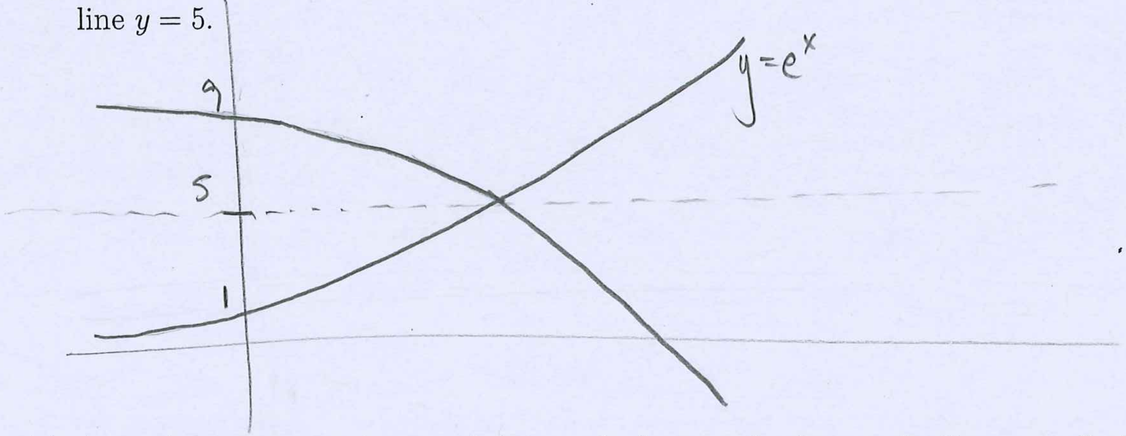
(c) **(4 pts)** $\lim\limits_{x\to-\infty} \frac{\sqrt{3x^2+1}}{2x-5}\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}$ $\quad x<0 \Rightarrow x=-\sqrt{x^2}$

$$= \lim_{x\to-\infty}\frac{-\sqrt{\frac{3\cancel{x^2}}{\cancel{x^2}}+\overset{\to 0}{\frac{1}{x^2}}}}{\frac{2\cancel{x}}{\cancel{x}}-\underset{\to 0}{\frac{5}{x}}} = \boxed{\frac{-\sqrt{3}}{2}}$$

(d) **(4 pts)** $\lim\limits_{x\to\infty} \frac{1-x^2}{x^3-x+1}\frac{\left(\frac{1}{x^3}\right)}{\left(\frac{1}{x^3}\right)}$ 0

$$= \lim_{x\to\infty}\frac{\overset{\to 0}{\frac{1}{x^3}}-\overset{\to 0}{\frac{\cancel{x}}{x}}}{\frac{x^3}{x^3}-\underset{\downarrow 0}{\frac{1}{x^2}}+\underset{\downarrow 0}{\frac{1}{x^3}}} = \frac{0}{1} = \boxed{0}$$

6. (a) **(5 pts)** Sketch, on the same axes, the graph $y = e^x$ and its reflection about the line $y = 5$.



(b) **(7 pts)** Find the equation of the graph that results from reflecting $y = e^x$ about the line $y = 5$.

$y = -e^x + 10$

7. Simplify each expression.

(a) **(2 pts)** $e^{\ln \ln e^x} = \ln(e^x) = \boxed{x}$

(b) **(2 pts)** $\ln(1/e^y) = \ln(1) - \ln(e^y) = \boxed{-y}$

(c) **(2 pts)** $3 \ln A + 2 \ln B - \ln C = \boxed{\ln\left(\frac{A^3B^2}{C}\right)}$

(d) **(2 pts)** $e^{-\ln z} = e^{\ln(z^{-1})}$

$= z^{-1} = \boxed{\frac{1}{z}}$