Exam 4: Introduction to Integrals (§4.7, 4.9-5.4)

Exam Instructions: You have 50 minutes to complete this exam. Justification is required for all problems. Notation matters! You will also be penalized for missing units and rounding errors. No electronic devices (phones, iDevices, computers, etc) except for a **basic scientific calculator**. On story problems, round to one decimal place.

In addition, please provide the following data:

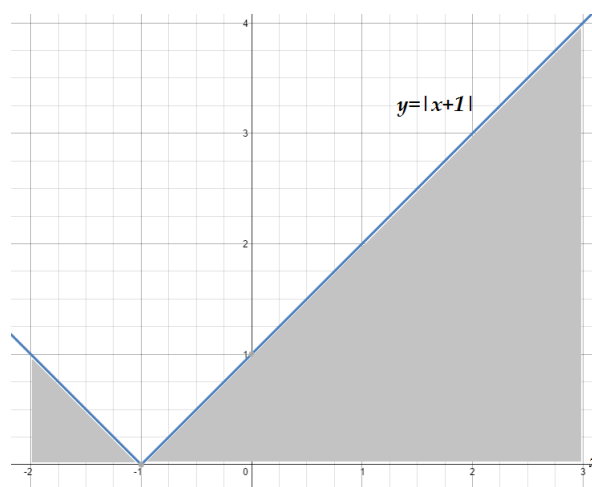Drill Instructor: _____

Drill Time: _____

Your signature below indicates that you have read this page and agree to follow the Academic Honesty Policies of the University of Arkansas.

Signature: **(1 pt)** _____

Good luck!

1. **(10 pts)** Use geometry to evaluate $\displaystyle\int_{-2}^{3} |x+1|\ dx$.

Solution:

$y=|x+1|$

The indefinite integral is the combined area of the shaded triangles:

$$\frac{1}{2}\,(1 \times 1) + \frac{1}{2}\,(4 \times 4) = \frac{1}{2} + 8 = \boxed{\frac{17}{2}}$$

2. *For this problem, you are not required to simplify answers.* Runners A and B are running a race in which Runner B has a head start. Their respective velocity functions and initial position functions are

$$\text{Runner A: } v(t) = 2at \text{ m/s}, \quad s(0) = 0 \text{ m},$$
$$\text{Runner B: } V(t) = b \text{ m/s}, \quad S(0) = c \text{ m},$$

where $a, b, c > 0$ are all constants, and t is in seconds.

(a) **(4 pts)** Find the initial velocity of each of the runners.

Solution:

$$\text{Runner A: } v(0) = 2a(0) = \boxed{0 \text{ m/s}}$$
$$\text{Runner B: } V(0) = b = \boxed{b \text{ m/s}}$$

(b) **(6 pts)** Find the position function of each of the runners.

> **Solution:**
> $$\text{Runner A: } s(t) = \int v(t) \ dt = \int 2at \ dt$$
> $$= at^2 + C$$
> $$S(0) = a(0)^2 + C = 0$$
> $$\implies C = 0$$
> $$\boxed{s(t) = at^2 \text{ m}}$$
> $$\text{Runner B: } S(t) = \int V(t) \ dt = \int b \ dt$$
> $$= bt + C$$
> $$S(0) = b(0) + C = c$$
> $$\implies C = c$$
> $$\boxed{S(t) = bt + c \text{ m}}$$

(c) **(3 pts)** Does Runner A ever catch up to Runner B? If so, at what time and what position?

> **Solution:** If so, then $at^2 = bt + c$. In other words,
> $$at^2 - bt - c$$
> To solve, use the quadratic formula:
> $$t = \frac{-(-b) \pm \sqrt{(-b)^2 - 4a(-c)}}{2a}$$
> $$= \frac{b \pm \sqrt{b^2 + 4ac}}{2a}$$
> Because $a, b, c > 0$, the inequality $b = \sqrt{b^2} < \sqrt{b^2 + 4ac}$ holds and so the solution with the minus sign does not fit the context of the problem. However, the solution with the plus sign does. So $\boxed{\text{yes,}}$ A catches up to B at
> $$\boxed{t = \frac{b + \sqrt{b^2 + 4ac}}{2a} \text{ seconds}} \text{ and at}$$
> $$S\left(\frac{b + \sqrt{b^2 + 4ac}}{2a}\right) = s\left(\frac{b + \sqrt{b^2 + 4ac}}{2a}\right)$$
> $$= \boxed{b\left(\frac{b + \sqrt{b^2 + 4ac}}{2a}\right) + c \text{ meters.}}$$

3. Suppose $f(x) \geq 0$ on $[0, 2]$, $f(x) \leq 0$ on $[2, 5]$, $\displaystyle\int_0^2 f(x) \, dx = 6$, and $\displaystyle\int_2^5 f(x) \, dx = -8$. Evaluate the following integrals.

(a) **(3 pts)** $\displaystyle\int_0^5 f(x) \, dx$

> **Solution:** $\displaystyle\int_0^5 f(x) \, dx = \int_0^2 f(x) \, dx + \int_2^5 f(x) \, dx = 6 + (-8) = \boxed{-2}$

(b) **(3 pts)** $\displaystyle\int_2^5 4|f(x)| \, dx$

> **Solution:** $\displaystyle\int_2^5 4|f(x)| \, dx = \int_2^5 4\left(-f(x)\right) \, dx$ because $f(x) \leq 0$ on $[2, 5]$
>
> $\displaystyle = -4 \int_2^5 f(x) \, dx = -4(-8) = \boxed{32}$

(c) **(3 pts)** $\displaystyle\int_0^5 \left(f(x) + |f(x)|\right) \, dx$

> **Solution:** $\displaystyle\int_0^5 \left(f(x) + |f(x)|\right) \, dx = \underbrace{\int_0^5 f(x) \, dx}_{-2, \text{ by (a)}} + \int_0^5 |f(x)| \, dx$
>
> $\displaystyle = -2 + \int_0^2 |f(x)| \, dx + \int_2^5 |f(x)| \, dx$
>
> $\displaystyle = -2 + \int_0^2 f(x) \, dx - \int_2^5 f(x) \, dx$
>
> $\displaystyle = -2 + 6 - (-8) = \boxed{12}$

4. Evaluate the following limits analytically. *(This includes L'Hôpital's Rule as a legitimate technique.)*

   (a) **(4 pts)** $\lim\limits_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)$

   Solution:

   $$\lim_{x\to\infty}\left(\underbrace{\sqrt{x^2+x+1}}_{\infty}-\underbrace{\sqrt{x^2-x}}_{\infty}\right)=\lim_{x\to\infty}\left(x\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}-x\sqrt{1-\frac{1}{x}}\right)$$

   $$=\lim_{x\to\infty}\underbrace{x}_{\infty}\underbrace{\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}-\sqrt{1-\frac{1}{x}}\right)}_{0}$$

   $$=\lim_{x\to\infty}\frac{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}-\sqrt{1-\frac{1}{x}}}{\frac{1}{x}}$$

   $$\text{put } t=\frac{1}{x}\longrightarrow\quad=\lim_{t\to0^+}\frac{\sqrt{1+t+t^2}-\sqrt{1-t}}{t}$$

   $$\text{L'Hôpital's Rule:}\quad=\lim_{t\to0^+}\frac{\frac{1}{2}(1+t+t^2)^{-\frac{1}{2}}(1+2t)-\frac{1}{2}(1-t)^{-\frac{1}{2}}(-1)}{1}$$

   $$=\frac{1}{2}(1+0+0^2)^{-\frac{1}{2}}(1+2(0))-\frac{1}{2}(1-0)^{-\frac{1}{2}}(-1)$$

   $$=\frac{1}{2}+\frac{1}{2}=\boxed{1}$$

   (b) **(4 pts)** $\lim\limits_{\theta\to0}\dfrac{3\sin8\theta}{8\sin3\theta}$

   Solution:

   $$\lim_{\theta\to0}\frac{\overbrace{3\sin8\theta}^{0}}{\underbrace{8\sin3\theta}_{0}}=\frac{3}{8}\lim_{\theta\to0}\frac{\sin8\theta}{\sin3\theta}$$

   $$\text{L'Hôpital's Rule:}\quad=\frac{3}{8}\lim_{\theta\to0}\frac{8\cos8\theta}{3\cos3\theta}$$

   $$=\frac{3}{8}\left(\frac{8\cos8(0)}{3\cos3(0)}\right)$$

   $$=\frac{24}{24}=\boxed{1}$$

(c) **(4 pts)** $\displaystyle\lim_{t\to 2} \frac{t^3 - t^2 - 2t}{t^2 - 4}$

Solution:

$$\lim_{t\to 2} \frac{\overbrace{t^3 - t^2 - 2t}^{t(t^2-t-2)}}{t^2 - 4} = \lim_{t\to 2} \frac{t(t-2)(t+1)}{(t-2)(t+2)}$$

$$= \lim_{t\to 2} \frac{t(t+1)}{t+2}$$

$$= \frac{2(2+1)}{2+2} = \frac{6}{4} = \boxed{\frac{3}{2}}$$

5. **(10 pts)** The surface of a water wave is described by $y = 5(1 + \cos x)$, for $-\pi \le x \le \pi$, where $y = 0$ corresponds to a trough of the wave (see figure). Find the average height of the wave above the trough on $[-\pi, \pi]$.
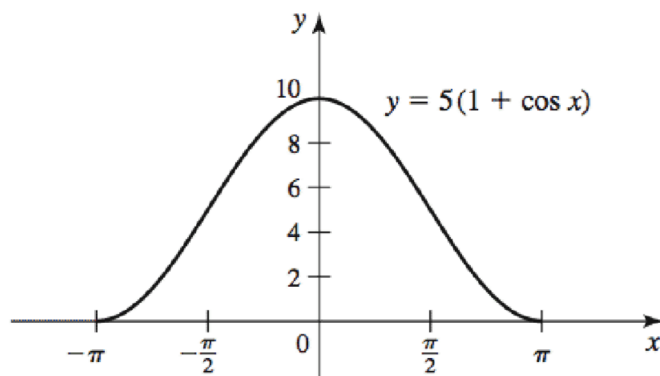
Briggs, et. al. *Calculus: Early Transcendentals, Second Edition.*

Solution: Let $f(x) = 5(1 + \cos x)$. The average value of f on $[-\pi, \pi]$ is

$$\bar{f} = \frac{1}{\pi - (-\pi)} \int_{-\pi}^{\pi} 5 \underbrace{(1 + \cos x)}_{\text{even function}} dx$$

$$= \frac{1}{2\pi} \left(2 \int_0^{\pi} 5(1 + \cos x)\ dx \right)$$

$$= \frac{5}{\pi} (x + \sin x) \Big|_0^{\pi}$$

$$= \frac{5}{\pi} (\pi + \sin \pi - (0 + \sin 0))$$

$$= \frac{5}{\pi}(\pi + 0 - 0 - 0) = \boxed{5}$$

The Fundamental Theorem of Calculus says:

Theorem (Fundamental Theorem of Calculus). *Suppose f is continuous on $[a, b]$.*

I. *The area function $A(x) = \displaystyle\int_a^x f(t)\ dt$, for $a \leq x \leq b$, is continuous on $[a, b]$ and differentiable on (a, b), and*

$$A'(x) = \frac{d}{dx} \int_a^x f(t)\ dt = f(x).$$

II. *If F is any antiderivative of f on $[a, b]$, then*

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

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6. Let $g(x) = \displaystyle\int_0^{\sin x} (t^2 + 1)\ dt$. Compute $g'(x)$ using

   (a) **(5 pts)** Part I of the Fundamental Theorem of Calculus.

   Solution:
   $$g'(x) = \frac{d}{dx} \int_a^{\sin x} (t^2 + 1)\ dt = \boxed{\left((\sin x)^2 + 1\right) \cos x}$$

   (b) **(5 pts)** Part II of the Fundamental Theorem of Calculus.

   Solution:
   $$\begin{aligned}
   g'(x) &= \frac{d}{dx} \int_a^{\sin x} (t^2 + 1)\ dt \\
   &= \frac{d}{dx} \left(\left(\frac{t^3}{3} + t\right) \Big|_0^{\sin x} \right) \\
   &= \frac{d}{dx} \left(\frac{(\sin x)^3}{3} + \sin x - \left(\frac{0^3}{3} - 0\right) \right) \\
   &= \frac{3}{3} (\sin x)^2 (\cos x) + \cos x \\
   &= \boxed{\left((\sin x)^2 + 1\right) \cos x}
   \end{aligned}$$

The following table contains relevant formulas for the next two questions:

Riemann sums with n rectangles for f on $[a,b]$	Σ-shortcuts
LEFT: $\displaystyle\sum_{k=1}^{n} f(a + (k-1)\Delta x)\Delta x$	$\displaystyle\sum_{k=1}^{n} c = cn \quad (c \text{ is a constant})$
RIGHT: $\displaystyle\sum_{k=1}^{n} f(a + k\Delta x)\Delta x$	$\displaystyle\sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$

7. **(10 pts)** Evaluate the left and right Riemann sums for f over the interval $[1,5]$ for $n = 8$.

x	1	1.5	2	2.5	3	3.5	4	4.5	5
f(x)	0	2	3	2	2	1	0	2	3

You are not required to use Σ-notation on this problem.

Solution: For both sums, $\Delta x = \dfrac{b-a}{n} = \dfrac{5-1}{8} = \left(\dfrac{1}{2}\right)$.

LEFT: $\quad (f(1) + f(1.5) + f(2) + f(2.5) + f(3) + f(3.5) + f(4) + f(4.5)) \left(\dfrac{1}{2}\right)$

$\qquad = (0 + 2 + 3 + 2 + 2 + 1 + 0 + 2)\left(\dfrac{1}{2}\right) = \dfrac{12}{2} = \boxed{6}$

RIGHT: $\quad$ LEFT $- f(1)\left(\dfrac{1}{2}\right) + f(5)\left(\dfrac{1}{2}\right)$

$\qquad = 6 - \dfrac{0}{2} + \dfrac{3}{2} = \boxed{\dfrac{15}{2}}$

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8. **EXTRA CREDIT!** Evaluate the following sums using Σ-notation.

   (a) **(2 pts)** $1 + 3 + 5 + 7 + \cdots + 99$

   > **Solution:**
   > $$\sum_{k=1}^{50}(2k - 1) = 2\sum_{k=1}^{50}k - \sum_{k=1}^{50}1$$
   > $$= 2\left(\frac{50(50 + 1)}{2}\right) - 1(50)$$
   > $$= 50(50) + 50 - 50 = \boxed{2500}$$

   (b) **(3 pts)** $4 + 9 + 14 + \cdots + 44$

   > **Solution:**
   > $$\sum_{k=1}^{9}(5k - 1) = 5\sum_{k=1}^{9}k - \sum_{k=1}^{9}1$$
   > $$= 5\left(\frac{9(10)}{2}\right) - 1(9)$$
   > $$= 25(9) - 1(9)$$
   > $$= 24(9) = \boxed{216}$$