Math 116: Extra Credit on Parametric Equations ($\oint$ 8.2) **due: Tue 2 Oct 2012**

Find the length of the following parametric curves:

1. #17: $x = 3 + 5t,\ y = 1 + 4t$ for $1 \le t \le 2$. Explain why your answer is reasonable.

   Solution: Use the arc length formula from page 403. First, compute the derivatives:

$$x'(t) = 5 \quad y'(t) = 4$$

   Then from the formula the arc length is

$$\begin{aligned}\int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt &= \int_1^2 \sqrt{5^2 + 4^2} dt \\ &= 2 \cdot \sqrt{41} - 1 \cdot \sqrt{41} \\ &= \sqrt{41} \approx 6.403.\end{aligned}$$

   To see why this answer makes sense observe we also could have used the Pythagorean Theorem to compute arc length. The parametrized curve is a line and the bounds $1 \le t \le 2$ mean we are computing the length from the point $(8, 5)$ to $(13, 9)$. The arc length is then given by

$$\begin{aligned}\sqrt{\Delta x^2 + \Delta y^2} &= \sqrt{(13-8)^2 + (9-5)^2} \\ &= \sqrt{5^2 + 4^2} \\ &= \sqrt{41}.\end{aligned}$$

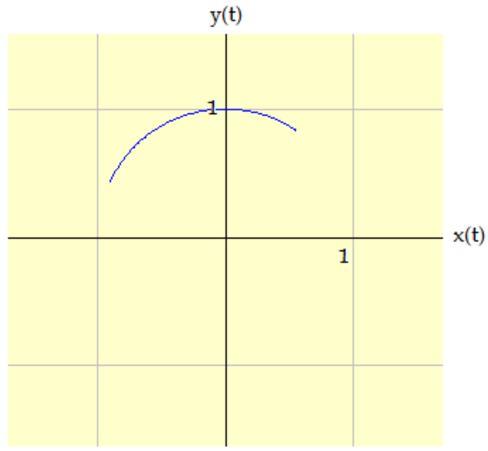
2. #18: $x = \cos(e^t),\ y = \sin(e^t)$ for $0 \le t \le 1$. Explain why your answer is reasonable.

   Solution: We have

$$x'(t) = -e^t \sin(e^t) \quad y'(t) = e^t \cos(e^t)$$

   and

$$\begin{aligned}\int_0^1 \sqrt{(-e^t \sin(e^t))^2 + (e^t \cos(e^t))^2}\, dt &= \int_0^1 \sqrt{e^{2t}(\sin^2(e^t) + \cos^2(e^t))}\, dt \\ &= \int_0^1 e^t\, dt \\ &= e^t|_0^1 \\ &= e - 1 \approx 1.718.\end{aligned}$$



The picture above is the curve we are parametrizing; in this case x and y parametrize the unit circle. The bounds $0 \leq t \leq 1$ give a portion of the circle somewhere between the angles 0 and π. Therefore we would expect the arc length to be a little less than π, so the answer is reasonable.

3. #19: $x = \cos(3t),\ y = \sin(5t)$ for $0 \leq t \leq 2\pi$.

   Solution: This is a straightforward computation, using the arc length formula, then a calculator to evaluate the integral:

$$\int_0^{2\pi} \sqrt{(-3\sin(3t))^2 + (5\cos(5t))^2}\, dt \approx 24.603.$$

4. #20 $x = \cos^3 t,\ y = \sin^3 t$, for $0 \leq t \leq 2\pi$.

   Solution: The arc in this case is a closed curve, but if we integrate by a distance of $\frac{\pi}{2}$ at a time and add the results we will get a positive number:

$$\begin{aligned}\int_0^{2\pi} \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2}\, dt &= \int_0^{2\pi} \sqrt{9\cos^2 t \sin^2 t}\, dt \\ &= \int_0^{2\pi} 3\cos t \sin t\, dt \\ &= 6.\end{aligned}$$