The base for these slides was done by Dr. Shannon Dingman, later encoded into LaTeX by Dr. Brad Lutes and modified/formatted by Dr. Ashley K. Wheeler.

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§4.4 Optimization Problems

In many scenarios, it is important to find a maximum or minimum value under given constraints. Given our use of derivatives from the previous sections, optimization problems follow directly from what we have studied.

Question

Given all nonnegative real numbers x and y between 0 and 50 such that their sum is 50 (i.e., $x + y = 50$), which pair has the maximum product?

This is a basic optimization problem. In this problem, we are given a **constraint** $(x + y = 50)$ and asked to maximize an **objective function** $(A = xy)$.

The first step is to express the objective function $A = xy$ in terms of a single variable by using the constraint:

$$y = 50 - x \implies A(x) = x(50 - x).$$

To maximize A, we find the critical points:

$$A'(x) = 50 - 2x \text{ which has a critical point at } x = 25.$$

Since $A(x)$ has domain $[0, 50]$, to maximize A we evaluate A at the endpoints of the domain and at the critical point:

$$A(0) = A(50) = 0 \text{ and } A(25) = 625.$$

So 625 is the maximum value of A and A is maximized when $x = 25$ (which means $y = 25$).

Essential Feature of Optimization Problems

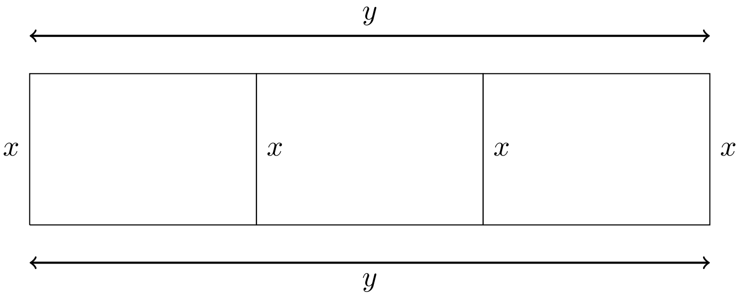
All optimization problems take the following form:

What is the maximum (or minimum) value of an objective function subject to the given constraint(s)?

Most optimization problems have the same basic structure as the previous problem: An objective function (possibly with several variables and/or constraints) with methods of calculus used to find the maximum/minimum values.

Exercise

Suppose you wish to build a rectangular pen with two interior parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen?



By the picture, $2y + 4x = 500$ which implies $y = -2x + 250$. We are maximizing $A = xy$. So write

$$A(x) = x(-2x + 250) = -2x^2 + 250x.$$

Taking the derivative, $A'(x) = -4x + 250 = 0$, A has a critical point at $x = 62.5$.

From the picture, since we have 500 ft of fencing available we must have $0 \leq x \leq 125$. To find the max we must examine the points $x = 0, 62.5, 125$:

$$A(0) = A(125) = 0 \text{ and } A(62.5) = 7812.5$$

We see that

$$\boxed{\text{the maximum area is } 7812.5 \text{ ft}^2.}$$

The pen's dimensions (answer the question!) are $\boxed{x = 62.5 \text{ ft}}$ and

$$\boxed{y = -2(62.5) + 250 = 125 \text{ ft.}}$$

Guidelines for Optimization Problems

1. READ THE PROBLEM carefully, identify the variables, and organize the given information with a picture.
2. Identify the objective function (i.e., the function to be optimized). Write it in terms of the variables of the problem.
3. Identify the constraint(s). Write them in terms of the variables of the problem.
4. Use the constraint(s) to eliminate all but one independent variable of the objective function.
5. With the objective function expressed in terms of a single variable, find the interval of interest for that variable.
6. Use methods of calculus to find the absolute maximum or minimum value of the objective function on the interval of interest. If necessary, check the endpoints.

Question

The sum of a pair of positive real numbers that have a product of 9 is

$$S(x) = x + \frac{9}{x},$$

where x is one of the numbers. This sum $S(x)$ has a minimum when:

A. $x = 9$

B. $x = 3$

C. $x = 6$

D. none of the above

Exercise

An open rectangular box with square base is to be made from 48 ft^2 of material. What dimensions will result in a box with the largest possible volume?

Exercise

Find the dimensions of the rectangle of largest area which can be inscribed in the closed region bounded by the x-axis, y-axis, and the graph of $y = 8 - x^3$.

4.4 Book Problems

5-16, 19-20, 24, 26

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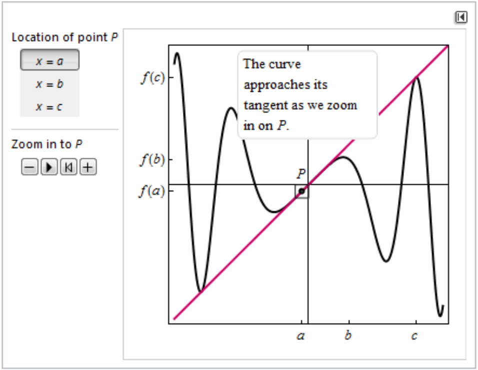
§4.5 Linear Approximation and Differentials

Suppose f is a function such that f' exists at some point P. If you zoom in on the graph, the curve appears more and more like the tangent line to f at P.



Linear Approximation

This idea – that smooth curves (i.e., curves without corners) appear straighter on smaller scales – is the basis of linear approximations.

One of the properties of a function that is differentiable at a point P is that it is locally linear near P (i.e., the curve approaches the tangent line at P.)

Therefore, it makes sense to approximate a function with its tangent line, which matches the value and slope of the function at P.

This is why you've had to do so many "find the equation for the tangent line to the given point" problems!

Definition

Suppose f is differentiable on an interval I containing the point a. The **linear approximation** to f at a is the linear function

$$L(x) = f(a) + f'(a)(x - a) \qquad \text{for } x \text{ in } I.$$

Remarks: Compare this definition to the following: At a given point $P = (a, f(a))$, the slope of the line tangent to the curve at P is $f'(a)$. So the equation of the tangent line is

$$y - f(a) = f'(a)(x - a).$$

(Yes, it is the same thing!)

Exercise

Write the equation of the line that represents the linear approximation to

$$f(x) = \frac{x}{x+1} \qquad \text{at } a = 1.$$

Then *use* the linear approximation to estimate $f(1.1)$.

Solution: First compute

$$f'(x) = \frac{1}{(x+1)^2}, \quad f(a) = \frac{1}{2}, \quad f'(a) = \frac{1}{4}$$

$$L(x) = \frac{1}{2} + \frac{1}{4}(x-1) = \frac{1}{4}x + \frac{1}{4}.$$

Solution (continued):

Because $x = 1.1$ is near $a = 1$, we can estimate $f(1.1)$ using $L(1.1)$:

$$f(1.1) \approx L(1.1) = 0.525$$

Note that $f(1.1) = 0.5238$, so the error in this estimation is

$$\frac{0.525 - 0.5238}{0.5238} \times 100 = 0.23\%.$$

Exercise

(a) The linear approximatioln to $f(x) = \sqrt{1+x}$ at the point $x = 0$ is (choose one):

A. $L(x) = 1$

B. $L(x) = 1 + \frac{x}{2}$

C. $L(x) = x$

D. $L(x) = 1 - \frac{x}{2}$

(b) What is an approximation for $f(0.1)$?

Our linear approximation $L(x)$ is used to approximate $f(x)$ when a is fixed and x is a nearby point:

$$f(x) \approx f(a) + f'(a)(x - a)$$

When rewritten,

$$f(x) - f(a) \approx f'(a)(x - a)$$

$$\implies \Delta y \approx f'(a)\Delta x.$$

A change in y can be approximated by the corresponding change in x, magnified or diminished by a factor of $f'(a)$.

This is another way to say that $f'(a)$ is the rate of change of y with respect to x!

$$\Delta y \approx f'(a)\Delta x$$

$$\frac{\Delta y}{\Delta x} \approx f'(a)$$

So if f is differentiable on an interval I containing the point a, then the change in the value of f (the Δy), between two points a and $a + \Delta x$ in I, is approximately $f'(x)\Delta x$.

We now have two different, but related quantities:

- The change in the function $y = f(x)$ as x changes from a to $a + \Delta x$ (which we call Δy).
- The change in the linear approximation $y = L(x)$ as x changes from a to $a + \Delta x$ (called the differential, dy).

$$\Delta y \approx dy$$

When the x-coordinate changes from a to $a + \Delta x$:

- The function change is **exactly** $\Delta y = f(a + \Delta x) - f(a)$.
- The linear approximation change is

$$\begin{aligned}\Delta L &= L(a + \Delta x) - L(a) \\ &= \big(f(a) + f'(a)(a + \Delta x - a)\big) - \big(f(a) + f'(a)(a - a)\big) \\ &= f'(a)\Delta x\end{aligned}$$

and this is dy.

We define the differentials dx and dy to distinguish between the change in the function (Δy) and the change in the linear approximation (ΔL):

- dx is simply the change in x, i.e. Δx.
- dy is the change in the linear approximation, which is $\Delta L = f'(a)\Delta x$.

SO:

$$\Delta L = f'(a)\Delta x$$

$$dy = f'(a)dx$$

$$\frac{dy}{dx} = f'(a) \quad \text{(at } x = a\text{)}$$

Definition

Let f be differentiable on an interval containing x.

- A small change in x is denoted by the **differential** dx.
- The corresponding change in $y = f(x)$ is approximated by the **differential** $dy = f'(x)dx$; that is,

$$\Delta y = f(x + \Delta x) - f(x)$$
$$\approx dy = f'(x)dx.$$

The use of differentials is critical as we approach integration.

Example

Use the notation of differentials $[dy = f'(x)dx]$ to approximate the change in $f(x) = x - x^3$ given a small change dx.

Solution: $f'(x) = 1 - 3x^2$, so $dy = (1 - 3x^2)dx$.
A small change dx in the variable x produces an approximate change of $dy = (1 - 3x^2)dx$ in y.

For example, if x increases from 2 to 2.1, then $dx = 0.1$ and

$$dy = \left(1 - 3(2)^2\right)(0.1) = -1.1.$$

This means as x increases by 0.1, y decreases by 1.1.

4.5 Book Problems

13-20, 35-50

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§4.6 Mean Value Theorem

In this section, we examine the Mean Value Theorem, one of the "big ideas" that provides the basis for much of calculus.

Before we get to the mean Value Theorem, we examine Rolle's Theorem, where the property $f(a) = f(b)$ holds, for some function $f(x)$ defined on an interval $[a, b]$.

Question

If you have two points $(a, f(a))$ and $(b, f(b))$, with the property that $f(a) = f(b)$, what might this look like?

Theorem (Rolle's Theorem)

Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. Then there is at least one point c in (a, b) such that $f'(c) = 0$.

Essentially what Rolle's Theorem concludes is that at some point(s) between a and b, f has a horizontal tangent.

Question

Note the hypotheses in this theorem: f is continuous on $[a, b]$ and differentiable on (a, b). Why are these important?

Exercise

Determine whether Rolle's Theorem applies to the function $f(x) = x^3 - 2x^2 - 8x$ on the interval $[-2, 0]$.

- If it doesn't, find an interval for which Rolle's Thm could apply to that function.
- If it does, what is the "c" value so that $f'(c) = 0$?

Theorem (Mean Value Theorem (MVT))

If f is continuous on a closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

See Figure 4.68 on p. 276 for a visual justification of MVT.

The slope of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$ is

$$\frac{f(b) - f(a)}{b - a}.$$

MVT says that there is a point c on f where the tangent line at c (whose slope is $f'(c)$) is parallel to this secant line.

Question

Suppose you leave Fayetteville for a location in Fort Smith that is 60 miles away. If it takes you 1 hour to get there, what can we say about your speed? If it takes you 45 minutes to get there, what can we say about your speed?

Example

Let $f(x) = x^2 - 4x + 3$.

1. Determine whether the MVT applies to $f(x)$ on the interval $[-2, 3]$.
2. If so, find the point(s) that are guaranteed to exist by the MVT.

Example

How many points c satisfy the conclusion of the MVT for $f(x) = x^3$ on the interval $[-1, 1]$? Justify your answer.

Theorem (Zero Derivative Implies Constant Function)

If f is differentiable and $f'(x) = 0$ at all points of an interval I, then f is a constant function on I.

Theorem (Functions with Equal Derivatives Differ by a Constant)

If two functions have the property that $f'(x) = g'(x)$ for all x of an interval I, then $f(x) - g(x) = C$ on I, where C is a constant.

Theorem (Intervals of Increase and Decrease)

Suppose f is continuous on an interval I and differentiable at all interior points of I.

- *If $f'(x) > 0$ at all interior points of I, then f is increasing on I.*
- *If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.*

4.6 Book Problems

7-14, 17-24