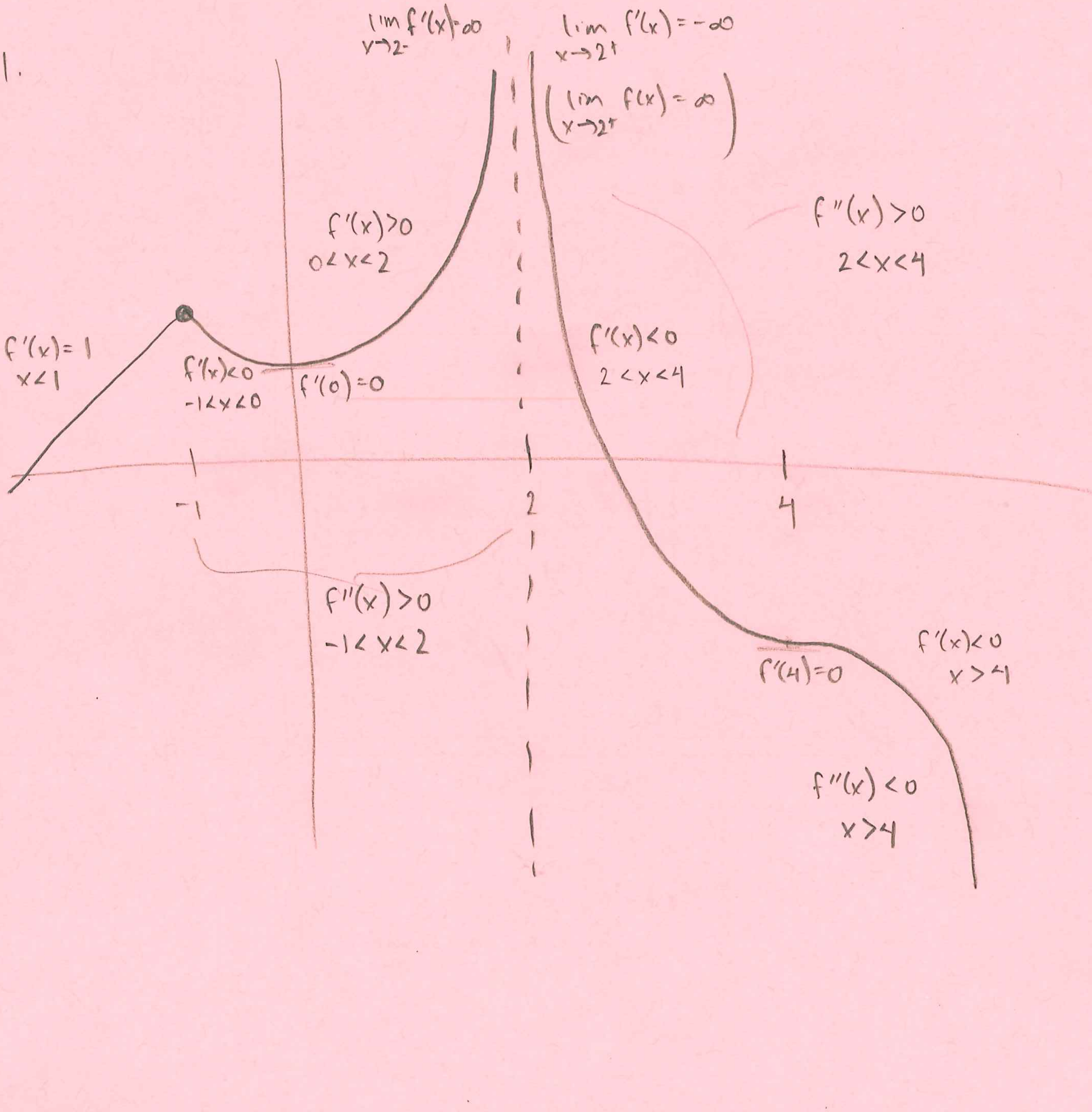
Math 235 (Calc I)
Fall 2017

Exam 3: Graphing with derivatives §3.10, 4.1–4.6

SOLUTIONS

1.



2. $f(x) = xe^{\frac{1}{x}}$

(a) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} xe^{\frac{1}{x}}$ $\quad 0 \cdot e^{\frac{1}{0}} = 0 \cdot \infty$

$= \lim_{x\to 0^-} \frac{e^{\frac{1}{x}}}{\frac{1}{x}}$ $\quad$ let $y = \frac{1}{x}$. As $x \to 0^-$, $y \to -\infty$

$= \lim_{y\to -\infty} \frac{e^y}{y}$ $\quad \frac{e^{-\infty}}{-\infty} = \frac{0}{-\infty}$

$= \boxed{0.}$

$\lim_{x\to 0^+} xe^{\frac{1}{x}} = \lim_{x\to 0^+} \frac{e^{\frac{1}{x}}}{\frac{1}{x}}$ $\quad 0 \cdot \infty$ $\quad$ As $x \to 0^+$, $y \to \infty$

$= \lim_{y\to \infty} \frac{e^y}{y}$ $\quad \frac{\infty}{\infty}$

L'Hôpital's Rule:

$= \lim_{y\to \infty} \frac{e^y}{1} = \boxed{\infty}$

(b) $\mathbb{R} \setminus \{0\}$

(c) x-int: Let $y=0$.

$0 = x e^{\frac{1}{x}}$ $\Rightarrow$ $x=0$ ← not in the domain

$\uparrow$ never 0

$\Rightarrow$ $\boxed{\text{no x-intercepts}}$

y-int: Let $x=0$ ... can't because $x=0$ is not in the domain

$\Rightarrow$ $\boxed{\text{no y-intercepts}}$

(d) $f(-x) = -x e^{\frac{1}{-x}} \neq f(x)$

$\neq -f(x)$

$\Rightarrow$ $\boxed{\text{neither even nor odd}}$

(e) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} x e^{\frac{1}{x}} = \boxed{\infty}$

$\infty \cdot e^{\frac{1}{\infty}} = \infty \cdot e^{0}$

$\lim\limits_{x\to-\infty} x e^{\frac{1}{x}} = \boxed{-\infty}$

$-\infty \cdot e^{\frac{1}{-\infty}}$

(f) The only point not in the domain is $x=0$.

From part (e), $\lim\limits_{x\to 0^+} f(x) = \infty$ so there is a

$\boxed{\text{vertical asymptote at } x=0.}$

⟶

(g) $f'(x) = (1)e^{\frac{1}{x}} + \cancel{x}e^{\frac{1}{x}}\left(\frac{-1}{x^{\cancel{2}}}\right)$

$= e^{\frac{1}{x}}\left(1 - \frac{1}{x}\right)$ is undefined when $x=0$, but that point is not in the domain.

$f'(x) = 0 \Rightarrow 1 - \frac{1}{x} = 0$, since $e^{\frac{1}{x}} \neq 0$

$1 = \frac{1}{x} \Rightarrow \boxed{x = 1}$

(h) $f''(x) = e^{\frac{1}{x}}\left(\frac{-1}{x^2}\right)\left(1 - \frac{1}{x}\right) + e^{\frac{1}{x}}\left(\frac{1}{x^2}\right)$

$= \cancel{\frac{-e^{\frac{1}{x}}}{x^2}} + \frac{e^{\frac{1}{x}}}{x^3} + \cancel{\frac{e^{\frac{1}{x}}}{x^2}} = \frac{e^{\frac{1}{x}}}{x^3}$

2nd Derivative Test:

$f''(1) = \frac{e^{\frac{1}{1}}}{1^3} > 0 \Rightarrow$ concave up $\cup$

so $x=1$ is a min.

However, since the range, image(f), is all real numbers (from part (e) about end behavior),

$\boxed{x=1 \text{ is a local min.}}$

→

(i) Check the intervals $(-\infty, 0)$, $(0, 1)$, $(1, \infty)$

break in the domain — critical point

$-1 \quad \emptyset \quad \tfrac{1}{2} \quad \emptyset \quad 2$

$0 \qquad 1$

$f'(-1) = e^{\frac{1}{-1}}\left(1 - \frac{1}{-1}\right) = 2e^{-1} > 0$

$f'\left(\frac{1}{2}\right) = e^{\frac{1}{\frac{1}{2}}}\left(1 - \frac{1}{\frac{1}{2}}\right) = -e^{2} < 0$

$f'(2) = e^{\frac{1}{2}}\left(1 - \frac{1}{2}\right) = \frac{1}{2}e^{\frac{1}{2}} > 0$

$\Rightarrow$ increasing on $(-\infty, 0)$, $(1, \infty)$
decreasing on $(0, 1)$

(j) Look for any possible points of inflection

$f''(x) = \dfrac{e^{\frac{1}{x}}}{x^3} = 0$ has no solution so

no inflection points

Check the intervals $(-\infty, 0)$, $(0, \infty)$

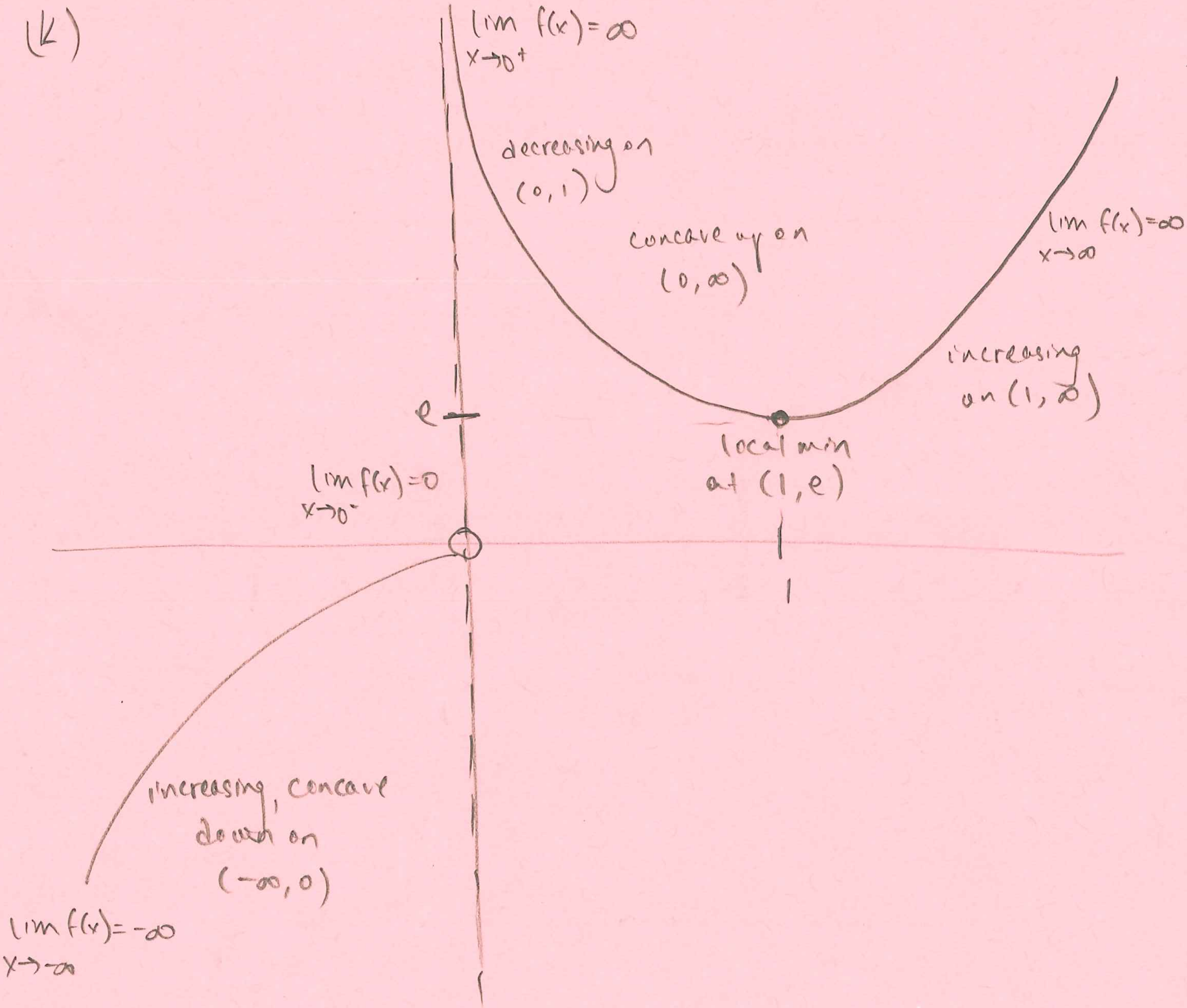
$-1 \qquad 0 \qquad 1$

$f''(-1) = \dfrac{e^{\frac{1}{-1}}}{(-1)^3} = -e^{-1} < 0$

$f''(1) = \dfrac{e^{\frac{1}{1}}}{1^3} = e$

$\Rightarrow$ concave up on $(-\infty, 0)$, concave down on $(0, \infty)$

(k)


$\lim_{x \to 0^+} f(x) = \infty$
decreasing on $(0,1)$
concave up on $(0,\infty)$
$\lim_{x \to \infty} f(x) = \infty$
increasing on $(1,\infty)$
e
local min at $(1,e)$
$\lim_{x \to 0^-} f(x) = 0$
1
increasing, concave down on $(-\infty,0)$
$\lim_{x \to -\infty} f(x) = -\infty$

3. $f(x)=(1+x)^{-3}=\frac{1}{(1+x)^3}$

(a) $L(x)=f(a)+f'(a)(x-a)$

$a=0$

$f'(x)=-3(1+x)^{-2}$

$\Rightarrow L(x)=f(0)+f'(0)(x-0)$

$=(1+0)^{-3}-3(1+0)^{-2}x$

$\boxed{=1-3x}$

(b) $dy=f'(x)\,dx=\boxed{-3(1+x)^{-2}\,dx}$

(c) $\Delta x=\frac{1}{3}\Rightarrow x=a+\Delta x=\frac{1}{3}$

$\Delta y=f(x)-f(a)$

$=f\left(\frac{1}{3}\right)-f(0)$

$=\left(1+\frac{1}{3}\right)^{-3}-(1+0)^{-3}$

$=\left(\frac{4}{3}\right)^{-3}-1^{-3}$

$\boxed{\begin{aligned}&=\left(\frac{3}{4}\right)^{3}-1\\&\approx-0.578\end{aligned}}$

$dy=f'(a)\,\underset{\Delta x}{dx}$

$=-3\cancel{(1+0)^{-2}}\left(\frac{1}{3}\right)$

$\boxed{=-1}$

(d)