# §14.8: Divergence Theorem

Monday, December 7, 2015 11:14 AM

recall: Stokes' Theorem is the circulation form of Green's Theorem, with a dimension added:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (\text{curl}\,\vec{F})\,dA$$

in $\mathbb{R}^3$

$$= \iint_R (\nabla\times\vec{F})\cdot\hat{k}\,dA$$

Stokes' Theorem:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$$

Flux Form of Green's Theorem:

$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R (\text{div}\,\vec{F})\,dA$$

inc. dim. by 1

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_D \underbrace{\nabla\cdot\vec{F}}_{\text{div}\,\vec{F}}\,dV$$

where: D is connected, simply connected region in $\mathbb{R}^3$, enclosed by

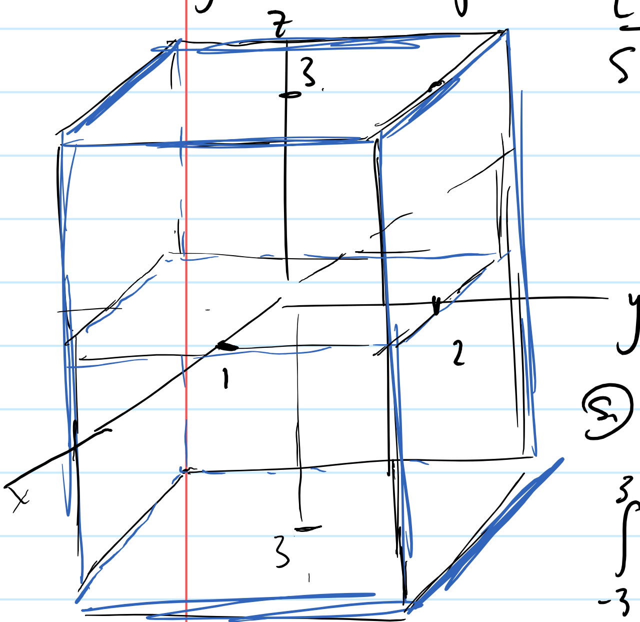
S = smooth, oriented

ex: # 12

$\vec{F} = \langle x^2, y^2, z^2 \rangle$

$D = \{ (x,y,z) \mid |x| \le 1, |y| \le 2, |z| \le 3 \}$

Verify the Divergence Theorem




LHS:

S = surface of the cube;

parametrize one side at a time.

the normal vectors are parallel to the axes:

$S_1$

$x = 1 \qquad \vec{n} = \hat{i}$

$$\int_{-3}^{3} \int_{-2}^{2} \langle 1^2, y^2, z^2 \rangle \cdot \hat{i} \, dy \, dz$$

$$= \int_{-3}^{3} y \Big|_{-2}^{2} dz$$

$$= 4z \Big|_{-3}^{3} = 24$$

$S_2$

$x = -1 \Rightarrow \vec{n} = -\hat{i}$

$$\int_{-3}^{3} \int_{-2}^{2} \langle (-1)^2, y^2, z^2 \rangle \cdot (-\hat{i}) \, dy \, dz$$

$$= -24$$

and similar cancelling happens for $S_3, \ldots, S_6$.

RHS: $\nabla \cdot \vec{F} = 2x + 2y + 2z$

$$2 \int_{-3}^{3} \int_{-2}^{2} \int_{-1}^{1} (x + y + z) \, dx \, dy \, dz$$

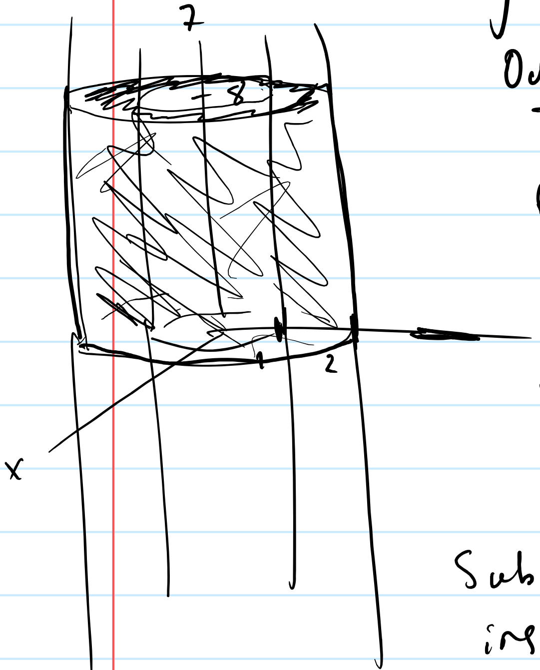
odd function each time, so $= 0$.

## Hollow Regions (So D is "technically not simply connected)

ex: #30

$\vec{F} = \langle x, 2y, 3z \rangle$ D = region between the two cylinders $x^2+y^2=1$, $x^2+y^2=4$, $0 \le z \le 8$

Find the net outward flux of $\vec{F}$ across the boundary of D.




Outside cylinder:

$\vec{n} = \langle x, y, 0 \rangle$ points out

flux = $\iint_R \langle x, 2y, 3z \rangle \cdot \langle x, y, 0 \rangle \, dA$

$= \int_0^{2\pi} \int_0^2 (r^2\cos^2\theta + 2r^2\sin^2\theta) r \, dr \, d\theta$

Subtract the flux for the inside cylinder:

$\int_0^{2\pi} \int_0^1 (r^3 + r^3\sin^2\theta) \, dr \, d\theta$

Div Thm

$= \int_0^8 \int_0^{2\pi} \int_1^2 \nabla \cdot \vec{F} \, dV$

$= \int_0^8 \int_0^{2\pi} \int_1^2 6r \, dr \, d\theta \, dz = \int_0^8 \int_0^{2\pi} \frac{\cancel{6}^{3} r^2}{\cancel{2}} \Big|_1^2 \, d\theta \, dz$

$= \int_0^8 \int_0^{2\pi} \underbrace{(3(4) - 3)}_{9} \, d\theta \, dz$

$= 9(2\pi)(8) = \boxed{144\pi}$

# Stokes' Thm

ex: #16 Find $\oint_C \vec{F} \cdot d\vec{r}$

$\vec{F} = \langle 2xy\sin z, x^2 \sin z, x^2 y \cos z \rangle$

C = boundary of the plane
$z = 8 - 2x - 4y$ in the first octant

$\nabla \times \vec{F} =$

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#121 $\vec{F} = \langle x^2 - y^2, z^2 - x^2, y^2 - z^2 \rangle$

C = boundary of $|x| \le 1$ $y \le |1|$ $z = 0$
(square)

param square: $\langle u, v, 0 \rangle$
$0 \le u \le 1$ normal: $\langle 0, 0, 1 \rangle$
$0 \le v \le 1$