MATH 2554 (Calculus I)

Dr. Ashley K. Wheeler

University of Arkansas

March 6, 2015

Table of Contents

Monday 2 March (Week 8)

- computer HWs: Don't wait for Sunday or else you'll get server crashes.
- Exam #2 returned in drill
- MIDTERM Tues 3 March
  - NO notecard
  - cumulative – see the Midterm Study Guide for content
  - 6-7:30p
  - location: SCEN 403
  - 5:30p drills that day: go to an earlier drill.
  - Chemistry conflict: SCEN 404 at 5PM. Email me if I haven't replied to you yet about the conflict, and be sure to stay in contact with your Chemistry prof. If you've scheduled for the exam in CEA's office then you may also take it in SCEN 404 at 5p, unless you have reduced distractions.
- Quiz #7 on Thurs 5 Mar – covers § 3.8 – 3.9

$\oint$ 2.3 Techniques for Computing Limits

Be able to do questions similar to 1-48, p.73

- Know and be able to compute limits using analytical methods (e.g., limit laws, additional techniques)
- Be able to evaluate one-sided and two-sided limits of functions
- Know the Squeeze Thm and be able to use this theorem to determine limits

Note: Material from sections 2.1 and 2.2 are foundational to the chapter. The material may not be explicitly tested, but the topics in these sections are foundational to later sections.

Midterm Review

(problems from past midterm)

Exercise

Evaluate the following limits:

- $$\lim_{x\to 3} \frac{x^2 - x - 6}{x^2 - 9}$$
- $$\lim_{\theta\to 0} \frac{\sec\theta \tan\theta}{\theta}$$

∮ 2.4 Infinite Limits

Be able to do questions similar to 17-30, p. 83

- Be able to use a graph, a table, or analytical methods to determine infinite limits
- Be able to use analytical methods to evaluate one-sided limits
- Know the definition of a vertical asymptote and be able to determine whether a function has vertical asymptotes

$\oint$ 2.5 Limits at Infinity

Be able to do questions similar to 9-30 and 38-46, p. 92

- Be able to find limits at infinity and horizontal asymptotes
- Know how to compute the limits at infinity of rational functions and algebraic functions
- Be able to list horizontal and/or vertical asymptotes of a function

$\oint$ 2.6 Continuity

Be able to do questions similar to 9-44, p. 103-104

- Know the definition of continuity and be able to apply the continuity checklist
- Be able to determine the continuity of a function (including those with roots) on an interval
- Be able to apply the Intermediate Value Thm to a function

Midterm Review

(problem from past midterm)

Exercise

Determine the value of k so the function is continuous on $0 \leq x \leq 2$.

$$f(x) = \begin{cases} x^2 + k & 0 \leq x \leq 1 \\ -2kx + 4 & 1 < x \leq 2 \end{cases}$$

∮ 3.1 Introducing the Derivative

Be able to do questions similar to 11-32, p. 132

- Know the definition of a derivative and be able to use this definition to calculate the derivative of a given function.
- Be able to determine the equation of a line tangent to the graph of a function at a given point
- Know the 3 conditions for when a function is not differentiable at a point, and why these three conditions make a function not differentiable at the given point

∮ 3.2 Rules for Differentiation

Be able to do questions similar to 7-41, p. 142-143

- Be able to use the various rules for differentiation (e.g., constant rule, power rule, constant multiple rule, sum and difference rule) to calculate the derivative of a function
- Know the derivative of e^x
- Be able to find slopes and/or equations of tangent lines

Midterm Review

Exercise

Given that $y = 3x + 2$ is tangent to $f(x)$ at $x = 1$ and that $y = -5x + 6$ is tangent to $g(x)$ at $x = 1$, write the equation of the tangent line to $h(x) = f(x)g(x)$ at $x = 1$.

$\oint$ 3.3 The Product and Quotient Rules

Be able to do questions similar to 7-42 and 47-52, p. 152-153

- Be able to use te product and/or quotient rules to calculate the derivative of a given function
- Be able to use the product and/or quotient rules to find tangent lines and/or slopes at a given point
- Know the derivate of e^{kx}
- Be able to combine derivative rules to calculate a derivative of a function

Midterm Review

$\oint$ 3.4 Derivatives of Trigonometric Functions

Be able to do questions similar to 1-55, p. 161-162

- Know the two special trigonometric limits

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \text{and} \qquad \lim_{x\to 0}\frac{\cos x-1}{x}=0$$

  and be able to use them to solve other similar limits
- Know the derivatives of $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$, and be able to use the quotient rule to derive the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$
- Be able to calculate derivatives (including higher order) involving trig functions using the rules for differentiation

Midterm Review

Exercise

Calculate the derivative of the following functions:

- $f(x) = (1 + \sec x)\sin^3 x$
- $g(x) = \dfrac{\sin x + \cot x}{\cos x}$

Midterm Review

$\oint$ 3.5 Derivatives as Rates of Change

Be able to do questions similar to 11-18, p. 171-172

- Be able to use the derivative to answer questions about rates of change involving:
  - Position and velocity;
  - Speed and acceleration;

Midterm Review

$\oint$ 3.6 The Chain Rule

Be able to do questions similar to 7-43, p. 180-181

- Be able to use both versions of the Chain Rule to find the derivative of a composition function
- Know and be able to use the Chain Rule for Powers:

$$\frac{d}{dx}(f(x))^n = n(f(x))^{n-1} f'(x)$$

- Be able to use the Chain Rule more than once in a calculation involving more than two composition functions

$\oint$ 3.7 Implicit Differentiation

Be able to do questions similar to 5-26 and 33-46, p. 189

- Be able to use implicit differentiation to calculate $\dfrac{dy}{dx}$.
- Be able to use the derivative found from implicit differentiation to find the slope at a given point and/or a line tangent to the curve at the given point.
- Be able to calculate higher-order derivatives of implicitly defined functions.
- Be able to calculate $\dfrac{dy}{dx}$ when working with functions containing rational functions

Exercise

Use implicit differentiation to calculate $\dfrac{dz}{dw}$ for

$$e^{2w} = \sin(wz)$$

Exercise

If $\sin x = \sin y$, then

- $\dfrac{dy}{dx} = ?$
- $\dfrac{d^2y}{dx^2} = ?$

Running out of Time on the Exam

- Do practice problems completely, from beginning to end (as if it were a quiz). You might think you understand something but when it's time to write down the details things are not so clear.
- Find a buddy who understands concepts a little better than you and work on problems for 2-3 hours. Then find a buddy who is struggling and work with them 2-3 hours. Explaining to someone else tests how deeply you really know the material. This strategy also helps reduce stress because it doesn't require you to devote a full day or night of studying, just 2-3 hours at a time of productive work.
- Don't count on cookie cutter problems. If you are doing a practice problem where you've memorized all the steps, make sure you understand why each step is needed. The exam problems may have a small variation from homeworks and quizzes. If you're not prepared, it'll come as a "twist" on the exam...

Running out of Time on the Exam, cont.

- If you encounter an unfamiliar type of problem on the exam, relax, because it's most likely not a trick. The solutions will always rely on the information from the required reading/assignments. Take your time and do each baby step carefully.
- During the exam, do the problems you are most confident with first! Different people will find different problems easier.
- During the exam, budget your time. Count the problems and divide by 50 minutes. The easier questions will take less time so doing them first leaves extra time for the harder ones. When studying, aim for 10 problems per hour (i.e., 6 minutes per problem).
- The exam is not a race. If you finish early take advantage of the time to check your work. You don't want to leave feeling smug about how quickly you finished only to find out next week you lost a letter grade's worth of points from silly mistakes.

Other Study Tips

- Brush up on algebra, especially radicals, logs, common denominators, etc. Many times knowing the right algebra will simplify the problem!
- When in doubt, show steps. See the document camera notes to get an idea of what's expected.
- You will be punished for wrong notation. The slides for $\oint 3.1$ show different notations for the derivative. Make sure whichever one you use in your work, that you are using it correctly.
- Read the question!
- Do the book problems.
- Look at the pictures in the book and the interactive applets on MLP.

Midterm Review

Friday 6 March (Week 8)

- Wednsday was a snow day. We will cover $\oint 3.8$ today and try to start $\oint 3.9$. Next week we start $\oint 3.10$, which is one of the harder sections of the course. Please stay ahead on the reading so you don't get lost in lecture.
- MIDTERM is still being graded – stand by

$\oint$ 3.8 Derivatives of Logarithmic and Exponential Functions

The natural exponential function $f(x) = e^x$ has an inverse function, namely $f^{-1}(x) = \ln x$. This relationship has the following properties:

1. $e^{\ln x} = x$ for $x > 0$ and $\ln(e^x) = x$ for all x.
2. $y = \ln x \quad \Longleftrightarrow \quad x = e^y$
3. For real numbers x and $b > 0$,
$$b^x = e^{\ln(b^x)} = e^{x \ln b}.$$

Derivative of $y = \ln x$

Using 2. from the last slide, plus implicit differentiation, we can find $\dfrac{d}{dx}(\ln x)$. Write $y = \ln x$. We wish to find $\dfrac{dy}{dx}$. From 2.,

$$\begin{aligned}\frac{d}{dx}\left(x = e^y\right) \Rightarrow \frac{d}{dx}x &= \frac{d}{dx}(e^y) \\ 1 &= e^y\left(\frac{dy}{dx}\right) \\ \frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x}\end{aligned}$$

So $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$.

Derivative of $y = \ln|x|$

Remember, you can only take $\ln$ of a positive number.

- For $x > 0$, $\ln|x| = \ln x$, so
$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}.$$
- For $x < 0$, $\ln|x| = \ln(-x)$, so
$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln(-x)) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}.$$

In other words, the absolute values do not change the derivative:
$$\frac{d}{dx}(\ln x) = \frac{d}{dx}(\ln|x|) = \frac{1}{x}.$$

Midterm Review

Exercise

Find the derivative of each of the following functions:

- $f(x) = \ln(15x)$
- $g(x) = x \ln x$
- $h(x) = \ln(\sin x)$

Derivative of $y = b^x$

What about other logs? Say $b > 0$. Since $b^x = e^{\ln b^x} = e^{x \ln b}$ (by 3. on the earlier slide),

$$\begin{aligned}\frac{d}{dx}(b^x) &= \frac{d}{dx}(e^{x \ln b}) \\ &= e^{x \ln b} \cdot \ln b \\ &= b^x \ln b.\end{aligned}$$

So for $b > 0$, $\dfrac{d}{dx}(b^x) = b^x \ln b$.

Exercise

Find the derivative of each of the following functions:

- $f(x) = 14^x$
- $g(x) = 45(3^{2x})$

Exercise

Determine the slope of the tangent line to the graph $f(x) = 4^x$ at $x = 0$.

Midterm Review

Story Problem Example

Exercise

The energy (in Joules) released by an earthquake of magnitude M is given by the equation

$$E = 25000 \cdot 10^{1.5M}.$$

1. How much energy is released in a magnitude 3.0 earthquake?
2. What size earthquake releases 8 million Joules of energy?
3. What is $\dfrac{dE}{dM}$ and what does it tell you?

Derivatives of General Logarithmic Functions

The relationship $y = \ln x \iff x = e^y$ also applies to logarithms of other bases:

$$y = \log_b x \quad \iff \quad x = b^y.$$

Now taking $\frac{d}{dx}(x = b^y)$ we obtain

$$\begin{aligned} 1 &= b^y \ln b \left(\frac{dy}{dx}\right) \\ \frac{dy}{dx} &= \frac{1}{b^y \ln b} \\ &= \frac{1}{x \ln b} \end{aligned}$$

So $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$.

Neat Trick: Logarithmic Differentiation

Example

Compute the derivative of $f(x) = \dfrac{x^2(x-1)^3}{(3+5x)^4}$.

We can use logarithmic differentiation: first takee the natural log of both sides and then use properties of logarithms.

$$\ln(f(x)) = \ln\left(\frac{x^2(x-1)^3}{(3+5x)^4}\right)$$
$$= \ln\left(x^2\right) + \ln\left(x-1\right)^3 - \ln\left(3+5x\right)^4$$
$$= 2\ln x + 3\ln(x-1) - 4\ln(3+5x)$$

Now we take $\dfrac{d}{dx}$ on both sides:

$$\frac{1}{f(x)}\left(\frac{df}{dx}\right) = 2\left(\frac{1}{x}\right) + 3\left(\frac{1}{x-1}\right) - 4\left(\frac{1}{3+5x}\right) \quad (5)$$

$$\frac{f'(x)}{f(x)} = \frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}$$

Finally, solve for $f'(x)$:

$$f'(x) = f(x)\left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}\right]$$

$$= \frac{x^2(x-1)^3}{(3+5x)^4}\left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}\right]$$

HW from Section 3.8

Do problems 9–27 odd, 31–37 odd, 41–47 odd (pp. 199–200 in textbook)

3.9 Derivatives of Inverse Trigonometric Functions

Recall that if $y = f(x)$, then $f^{-1}(x)$ is the value of y such that $x = f(y)$.

Example: If $f(x) = 3x + 2$, then $f^{-1}(x) = \dfrac{x-2}{3}$.

NOTE: $f^{-1}(x)$ is not the same as $\dfrac{1}{f(x)}$.

$$(f(x))^{-1} = \frac{1}{f(x)}$$

Derivative of Inverse Sine

$y = \sin^{-1} x \iff x = \sin y$. So the derivative of $y = \sin^{-1} x$ can be found by applying $\dfrac{d}{dx}$ to both sides of $x = \sin y$ and then finding $\dfrac{dy}{dx}$:

$$x = \sin y$$
$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$$
$$1 = (\cos y)\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Since $\sin^2 y + \cos^2 y = 1$, then
$\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - x^2}$.

Because $-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$ (this is the range of $y = \sin^{-1} x$), we have that
$\cos y \geq 0 \implies \cos y = \sqrt{1 - x^2}$.

Therefore,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}} \implies \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.$$

Midterm Review

Compute the following:

1. $$\frac{d}{dx}\left[\sin^{-1}(4x^2-3)\right]$$

2. $$\frac{d}{dx}\left[\cos(\sin^{-1}x)\right]$$

Midterm Review

Derivative of Inverse Tangent

We use a similar method as with inverse sine:

$$\begin{aligned} y &= \tan^{-1} x \\ x &= \tan y \\ \frac{d}{dx}(x) &= \frac{d}{dx}(\tan y) \\ 1 &= (\sec^2 y)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} \end{aligned}$$

Since $\sec^2 y = 1 + \tan^2 y$, we have $\sec^2 y = 1 + x^2$.
So

Derivative of Inverse Secant

We use a similar method as with inverse sine:

$$\begin{aligned} y &= \sec^{-1} x \\ x &= \sec y \\ \frac{d}{dx}(x) &= \frac{d}{dx}(\sec y) \\ 1 &= \sec y \tan y \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\sec y \tan y} \end{aligned}$$

Since $\sec^2 y = 1 + \tan^2 y$, then
$\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$.

If $x \geq 1$, then $0 \leq y < \dfrac{\pi}{2}$ and so $\tan y > 0$.

If $x \leq -1$, then $\dfrac{\pi}{2} < y \leq \pi$ and so $\tan y < 0$. So

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

Midterm Review

All other inverse trig derivatives

Using the facts that $\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$, $\cot^{-1} x + \tan^{-1} x = \frac{\pi}{2}$, and $\csc^{-1} x + \sec^{-1} x = \frac{\pi}{2}$, we can differentiate these identities to obtain all inverse trig derivatives:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2 -}}$$

Compute the derivatives of $f(x) = \tan^{-1}(1/x)$ and $g(x) = \sin\left[\sec^{-1}(2x)\right]$.

Derivatives of Inverse Functions in General

Let f be differentiable and have an inverse on an interval I. Let x_0 be a point in I at which $f'(x_0) \neq 0$.

Then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}$$

where $y_0 = f(x_0)$.

Example: Let $f(x) = \frac{1}{4}x^3 + x - 1$. Find $\left(f^{-1}\right)'(3)$.

HW from Section 3.9

Do problems 7–27 odd, 31–39 odd.