Exam 4 Review Slides

Cal I Spring 2016

§4.7 L'Hôpital's Rule

(1) $\lim_{x\to\infty} x^2 \ln\left(\cos\frac{1}{x}\right) = \lim_{x\to\infty} \dfrac{\ln\left(\cos\frac{1}{x}\right)}{\frac{1}{x^2}}$ $\quad \frac{0}{0}$

$\infty \cdot 0$

Let $t = \frac{1}{x}$

$$= \lim_{t\to 0^+} \frac{\ln(\cos t)}{t^2}$$

$$\overset{LR}{=} \lim_{t\to 0^+} \frac{\frac{1}{\cos t}(\sin t)}{2t}$$

$$= \lim_{t\to 0^+} \frac{\tan t}{2t} \quad \frac{0}{0}$$

$$= \lim_{t\to 0^+} \frac{\sec^2 t}{2} = \frac{\sec^2(0)}{2} = \boxed{\frac{1}{2}}$$

(2) $\lim_{x\to\frac{\pi}{2}} (\pi - 2x)\tan x = \lim_{x\to\frac{\pi}{2}} \dfrac{\pi - 2x}{\frac{1}{\tan x}}$ $\quad \frac{0}{0}$

$0 \cdot \infty$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\pi - 2x}{\cot x}$$

$$\overset{LR}{=} \lim_{x\to\frac{\pi}{2}} \frac{-2}{-\csc^2 x}$$

$\longrightarrow$

$$= \lim_{x \to \frac{\pi}{2}} \frac{2}{\frac{1}{\sin^2 x}}$$

$$= 2\sin^2\left(\frac{\pi}{2}\right) = \boxed{2}$$

(3) $\lim_{x \to \infty} \left(x^2 e^{\frac{1}{x}} - x^2 - x\right)$ $= \lim_{x \to \infty} x^2\left(e^{1/x} - 1 - \frac{1}{x}\right)$

$\infty - \infty$ $\qquad \infty \cdot 0$

$$= \lim_{x \to \infty} \frac{e^{1/x} - 1 - \frac{1}{x}}{\frac{1}{x^2}}$$

Let $t = \frac{1}{x}$

$$= \lim_{t \to 0^+} \frac{e^t - 1 - t}{t^2}$$

$$\overset{LR}{=} \lim_{t \to 0^+} \frac{e^t - 1}{2t}$$

$$\overset{LR}{=} \lim_{t \to 0^+} \frac{e^t}{2} = \frac{e^0}{2} = \boxed{\frac{1}{2}}$$

(4) $\lim_{x \to 0^+} x^{\frac{1}{\ln x}}$ $\rightarrow$ Let $L = \ln\left(\lim_{x \to 0^+} x^{\frac{1}{\ln x}}\right)$

0^0

$$= \lim_{x \to 0^+} \ln\left(x^{\frac{1}{\ln x}}\right)$$

$\longrightarrow$

$$= \lim_{x\to 0^+} \frac{1}{\cancel{\ln x}} \cancel{\ln x}$$

$$= \lim_{x\to 0^+} 1 = 1$$

$$\Rightarrow \lim_{x\to 0^+} x^{\frac{1}{\ln x}} = e^1 = \boxed{e}$$

Show, using limits, that x^x grows faster than b^x as $x \to \infty$, for any $b > 1$.

$$\frac{\infty}{\infty} \quad \lim_{x\to\infty} \frac{x^x}{b^x} = \lim_{x\to\infty} \left(\frac{x}{b}\right)^x = e^L$$

$$\text{Let } L = \ln\left(\lim_{x\to\infty} \left(\frac{x}{b}\right)^x\right)$$

$$= \lim_{x\to\infty} \ln\left(\frac{x}{b}\right)^x$$

$$= \lim_{x\to\infty} x \ln\left(\frac{x}{b}\right) = \infty$$

$$\infty \cdot \infty$$

$$\Rightarrow \lim_{x\to\infty} \frac{x^x}{b^x} = e^\infty = \infty$$

and so the numerator, x^x, grows faster. →

§4.9 Antiderivatives

A payload is dropped at an elevation of 400m from a hot-air balloon that is descending at a rate of 10 m/s. Its acceleration due to gravity is $-9.8 m/s^2$.

(a) Find the velocity function for the payload.

The acceleration function is $a(t) = -9.8$, and the velocity function is

$$v(t) = \int a(t)\,dt = \int (-9.8)\,dt$$

$$= -9.8t + C$$

Since the hot-air balloon descends at 10 m/s,

$$v(0) = -10 = -9.8(0) + C$$

$$\Rightarrow C = -10$$

So $\boxed{v(t) = -9.8t - 10}$

(b) Find the position function for the payload.

The position function is

$$s(t) = \int v(t)\,dt = \int (-9.8t - 10)\,dt$$

$$= \frac{-9.8t^2}{2} - 10t + C$$

$\rightarrow$

The payload is dropped from 400m, so

$$s(0) = 400 = -4.9(0)^2 - 10(0) + C$$

$$\Rightarrow C = 400$$

and the position function is

$$\boxed{s(t) = -4.9t^2 - 10t + 400}$$

(c) Find the time when the payload strikes the ground.

The payload strikes the ground when

$$s(t) = -4.9t^2 - 10t + 400 = 0.$$

By the quadratic formula

$$t = \frac{10 \pm \sqrt{(-10)^2 - 4(-4.9)(400)}}{2(-4.9)}$$

$$\approx \cancel{-7.49} \quad \boxed{5.449 \text{ sec}}$$

§5.2 Definite Integrals

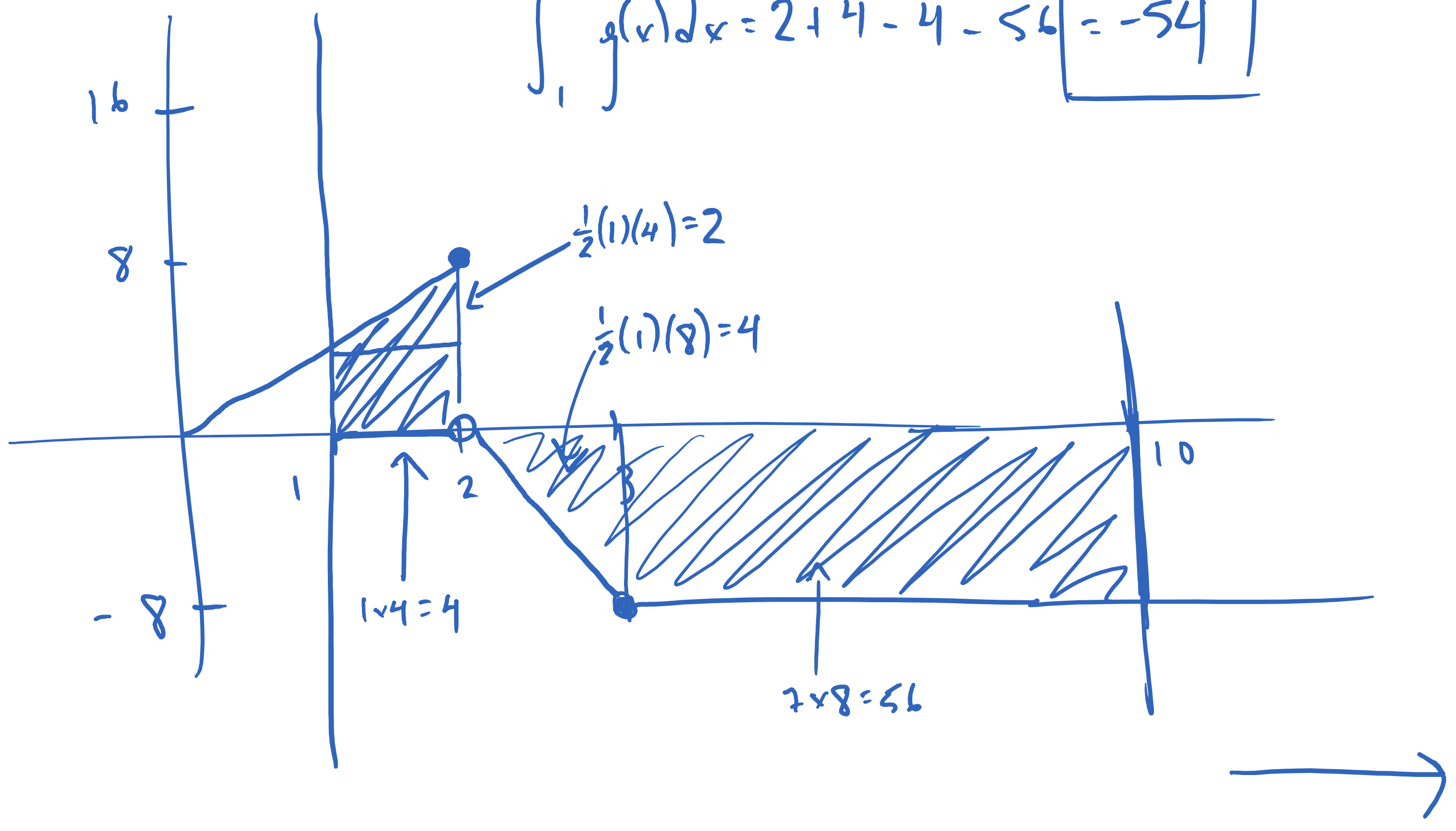
If f is continuous on $[a,b]$ and $\int_a^b |f(x)|\,dx = 0$, what can you conclude about f?

The net area under the curve $|f(x)|$ from a to b is 0, meaning its positive and negative components cancel. But $|f(x)|$ is never negative, so we must have $|f(x)|=0$, so $f=0$.

Use geometry to evaluate $\int_1^{10} g(x)\,dx$, where

$$g(x) = \begin{cases} 4x & 0 \le x \le 2 \\ -8x+16 & 2 < x \le 3 \\ -8 & x > 3 \end{cases}$$

$$\int_1^{10} g(x)\,dx = 2+4-4-56 \boxed{= -54}$$



§5.3 Fundamental Theorem of Calculus

Given $g(x)=\int_0^x (t^2+1)\,dt$, compute $g'(x)$ using

- FTOC I

$$g'(x)=\frac{d}{dx}\int_0^x (t^2+1)\,dt=\boxed{x^2+1}$$

- FTOC II

$$g'(x)=\frac{d}{dx}\left(\frac{t^3}{3}+t\,\Big|_0^x\right)$$

$$=\frac{d}{dx}\left(\frac{x^3}{3}+x-0\right)=\frac{\cancel{3}x^2}{\cancel{3}}+1$$

$$\boxed{=x^2+1}$$

§5.4 Working With Integrals

Find the point(s) at which the given function equals its average value on the given interval.

(1) $f(x)=e^x$ on $[0,2]$

average: $\bar{f}=\frac{1}{2-0}\int_0^2 e^x\,dx=\frac{1}{2}e^x\Big|_0^2=\frac{1}{2}(e^2-e^0)$

$$=\frac{1}{2}(e^2-1)$$

$\longrightarrow$

$$f(x) = e^x = \frac{1}{2}(e^2 - 1)$$

$$\boxed{x = \ln\left(\frac{1}{2}(e^2-1)\right)} \approx 1.161 \text{ (in } [0,2])$$

(2) $f(x) = \frac{\pi}{4}\sin x$ on $[0, \pi]$

$$\bar{f} = \frac{1}{\pi - \cancel{0}} \int_0^\pi \frac{\pi}{4} \sin x \, dx = \frac{1}{4}(-\cos x)\Big|_0^\pi$$

$$= -\frac{1}{4}(\underbrace{\cos\pi}_{-1} - \underbrace{\cos 0}_{1}) = \frac{1}{2}$$

$$f(x) = \frac{\pi}{4}\sin x = \frac{1}{2}$$

$$\sin x = \frac{2}{\pi}$$

$$\boxed{x = \arcsin\left(\frac{2}{\pi}\right)} \approx 0.69 \text{ (in } [0,\pi])$$

(3) $f(x) = \frac{1}{x}$ on $[1,4]$

$$\bar{f} = \frac{1}{4-1}\int_1^4 \frac{dx}{x} = \frac{1}{3}\ln|x|\Big|_1^4$$

$$= \frac{1}{3}(\ln 4 - \cancel{\ln 1}) = \frac{\ln 4}{3}$$

$$f(x) = \frac{1}{x} = \frac{\ln 4}{3}$$

$$\boxed{x = \frac{3}{\ln 4}} \approx 2.164 \text{ (in } [1,4])$$