Math 2574C | Exam 3 | Fall 2015

Directions: No calculators, phones or other electronic aids are allowed. Show all your work. If you use a formula from memory, write it down. *Clearly indicate your final answer.* You will be graded not only on your final answer, but on the clarity of your solutions.

Name SOLUTIONS

TA Name: ______
Drill Time: ______

GRADE	
Problem 1	/ 20
Problem 2	/ 10
Problem 3	/ 10
Problem 4	/ 25
Problem 5	/ 20
Problem 6	/ 15
Total	/100

1. (20 pts) Evaluate the following integral exactly as written.

$$\int_0^8 \int_0^{\ln 4} \int_0^{\ln 2} 2ze^{-x-y}\,dx\,dy\,dz$$

$$= 2\int_0^8 z \int_0^{\ln 4} e^{-y} \int_0^{\ln 2} e^{-x}\,dx\,dy\,dz$$

$$= 2\int_0^8 z \int_0^{\ln 4} e^{-y} \left(-e^{-x} \Big|_0^{\ln 2} \right) dy\,dz$$

$$= -\cancel{\left(\underset{\frac{1}{2}}{e^{-\ln 2}} - \underset{-1}{e^{-0}}\right)(2)} \int_0^8 z \left(-e^{-y} \Big|_0^{\ln 4} \right) dz$$

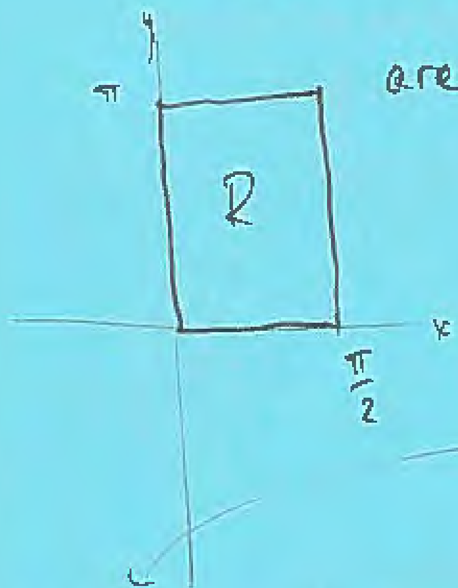
$$= -\left(\underset{\frac{1}{4}}{e^{-\ln 4}} - \underset{-1}{e^{-0}}\right) \frac{z^2}{2} \Bigg|_0^8$$

$0 \leftarrow$ terms vanish

$$= \frac{3}{\cancel{4}}\,\frac{8^{\cancel{2}}}{\cancel{2}} = \boxed{24}$$

2. **(10 pts)** Compute the average value of $g(x,y) = \cos x \sin y$ over the region

$$R = \{(x,y) : 0 \le x \le \tfrac{\pi}{2},\ 0 \le y \le \pi\}.$$



$$\text{area}(R) = \pi\left(\frac{\pi}{2}\right) = \frac{\pi^2}{2}$$

$$\bar{g} = \frac{1}{\frac{\pi^2}{2}} \int_0^{\pi/2} \int_0^{\pi} \cos x \sin y \, dy\, dx$$

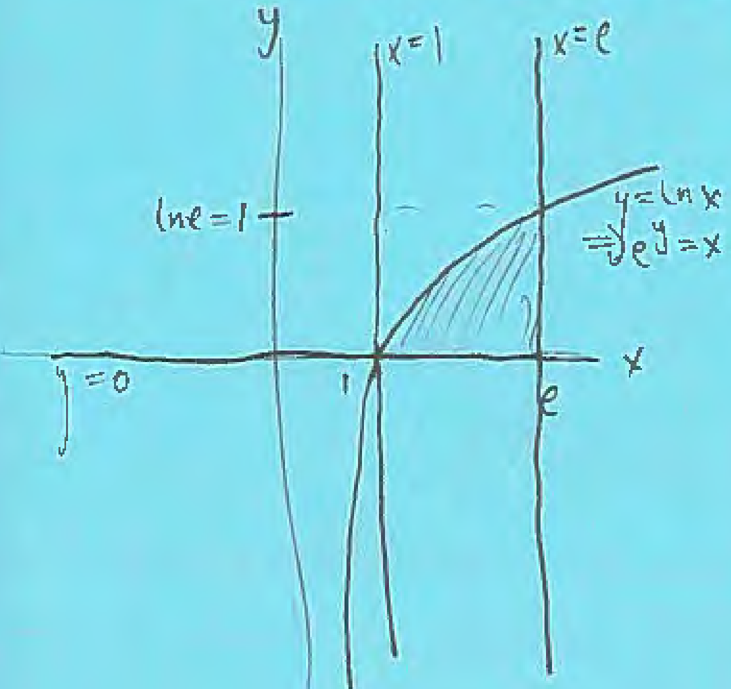
$$= \frac{2}{\pi^2} \int_0^{\frac{\pi}{2}} \cos x \left(-\cos y \Big|_0^{\pi} \right) dx$$

$$= -(\underset{-1}{\cos\pi} - \underset{1}{\cos 0})\left(\frac{2}{\pi^2}\right) \sin x \Big|_0^{\frac{\pi}{2}} = \frac{4}{\pi^2}\left(\underset{1}{\sin\frac{\pi}{2}} - \sin 0\right) = \boxed{\frac{4}{\pi^2}}$$

3. **(10 pts)** Consider the integral

$$\int_1^e \int_0^{\ln x} f(x,y)\, dy\, dx.$$

Sketch the region of integration and then rewrite the integral in the order $dx\, dy$.



$$\int_0^1 \int_{e^y}^{e} f(x,y)\, dx\, dy$$

4. A spherical fish tank of radius 2 ft is filled with water to a level 1 ft from the top.

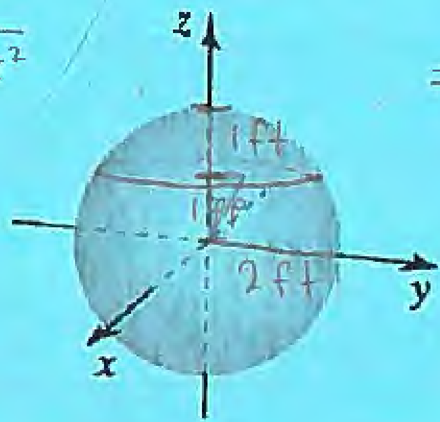
(a) **(4 pts)** On the sphere below, draw and label the tank's radius and water level, with units included.

Cylindrical:
$\sqrt{r^2+z^2}=4 \Rightarrow z=\sqrt{4-r^2}$

$\sqrt{4-r^2}=1$

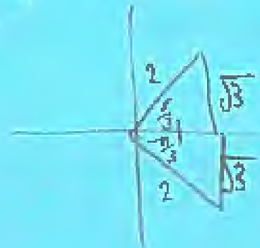
$\Rightarrow 4-r^2=1$

$r^2=3\ ;\ r=\pm\sqrt{3}$



$\sec\varphi=2$

$\Rightarrow \cos\varphi=\frac{1}{2} \Rightarrow \varphi=\pm\frac{\pi}{3}$



(b) **(2 pts)** Write the equation for your sphere, in spherical coordinates.

$\rho=2$

(c) **(9 pts)** Write down a triple integral that will give the volume of the *empty space* in the fish tank.

$$\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\sec\varphi}^{2}\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

Or:

$$\int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta$$

(d) **(7 pts)** Evaluate the integral from (c).

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\sin\varphi\left(\frac{\rho^3}{3}\Bigg|_{\sec\varphi}^{2}\right)d\varphi\, d\theta$$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\left(\frac{2^3}{3}\sin\varphi-\frac{\sec^3\varphi\sin\varphi}{3}\right)d\varphi\, d\theta$$

$\sec^3\varphi\sin\varphi$

$=\frac{\sin\varphi}{\cos^3\varphi}$

$=\tan\varphi\sec^2\varphi$

$u=\tan\varphi$

$\Rightarrow du=\sec^2\varphi\, d\varphi$

$$=\frac{1}{3}\int^{2\pi}\left(-8\cos\varphi\Big|_0^{\frac{\pi}{3}}-\frac{\tan^2\varphi}{2}\Bigg|_0^{\frac{\pi}{3}}\right)d\theta \leftarrow \text{terms vanish}$$

$$=-\frac{2\pi}{3}\left(8\left(\cos\tfrac{\pi}{3}-\cos 0\right)+\frac{\tan^2\left(\frac{\pi}{3}\right)}{2}\right)=-\frac{2\pi}{3}\left(-\frac{8}{2}+\frac{3}{2}\right)=\frac{2\pi}{3}\left(\frac{5}{2}\right)=\boxed{\frac{5}{3}\pi}$$

$$=\int_0^{2\pi}\int_0^{\sqrt{3}}\left(r\sqrt{4-r^2}-r\right)dr\, d\theta \qquad u=4-r^2,\quad du=-2r\,dr$$

$$=\int_0^{2\pi}\left(\int_{4-0^2}^{4-(\sqrt{3})^2}-\frac{1}{2}\sqrt{u}\,du-\frac{r^2}{2}\Bigg|_0^{\sqrt{3}}\right)d\theta \leftarrow \text{terms vanish}$$

$$=\int_0^{2\pi}\left(-\frac{1}{2}\left(\frac{u^{3/2}}{\frac{3}{2}}\Bigg|_4^1\right)-\frac{(\sqrt{3})^2}{2}\right)d\theta$$

$$=\left(\frac{2}{2}\cdot\frac{1}{3}\left(1-4^{3/2}\right)-\frac{3\cdot 3}{2\cdot 3}\right)2\pi=\left(\frac{14-9}{6}\right)2\pi=\boxed{\frac{5}{3}\pi}$$

(e) **(3 pts)** What is the volume of the water in the tank?

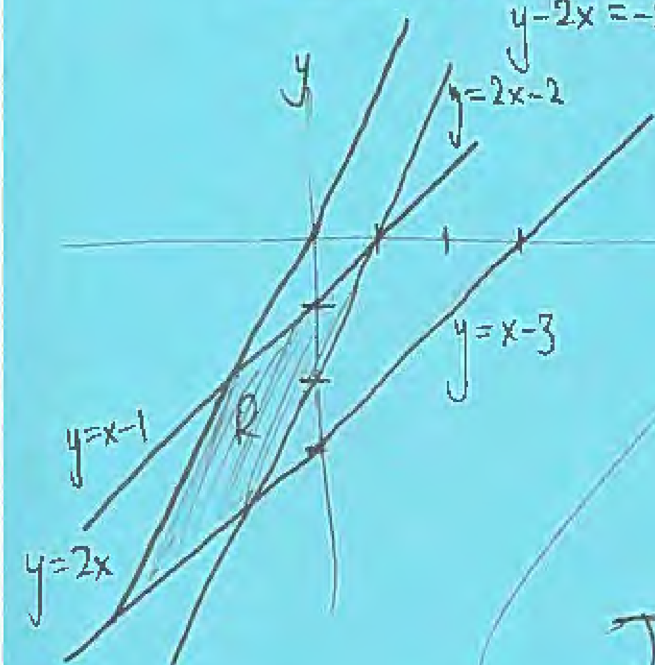
$$\frac{4}{3}\pi\left(2^3\right)-\frac{5}{3}\pi=\frac{32-5}{3}\pi=\frac{27}{3}\pi=\boxed{9\pi\ ft^3}$$

5. **(20 pts)** Evaluate the following integral using a change of variables of your choice. Sketch the original and new regions of integration, R and S.

$$\iint_R \underbrace{(x-y)}_{u}\sqrt{\underbrace{y-2x}_{v}}\,dA$$

R is bounded by the lines $y = 2x+2$, $y = 2x$, $y = x-3$, and $y = x-1$.

$y-2x=-2 \quad y-2x=0 \quad x-y=3 \quad x-y=1$



let $\boxed{\begin{aligned} u &= x-y \\ v &= y-2x \end{aligned}}$ $\begin{aligned} &\Rightarrow 1 \le u \le 3 \\ &\Rightarrow -2 \le v \le 0 \end{aligned}$

Then $y = x-u$

$v = x-u-2x$

$= -u-x$

$\cancel{x} = -u-v \leftarrow g(u,v)$

$y = -u-v-u = -2u-v \leftarrow h(u,v)$

$$J(u,v) = \begin{vmatrix} g_u & h_u \\ g_v & h_v \end{vmatrix} = \begin{vmatrix} -1 & -2 \\ -1 & -1 \end{vmatrix}$$

$= \underbrace{(-1)(-1)}_{1} - \underbrace{(-2)(-1)}_{2}$

$= -1$

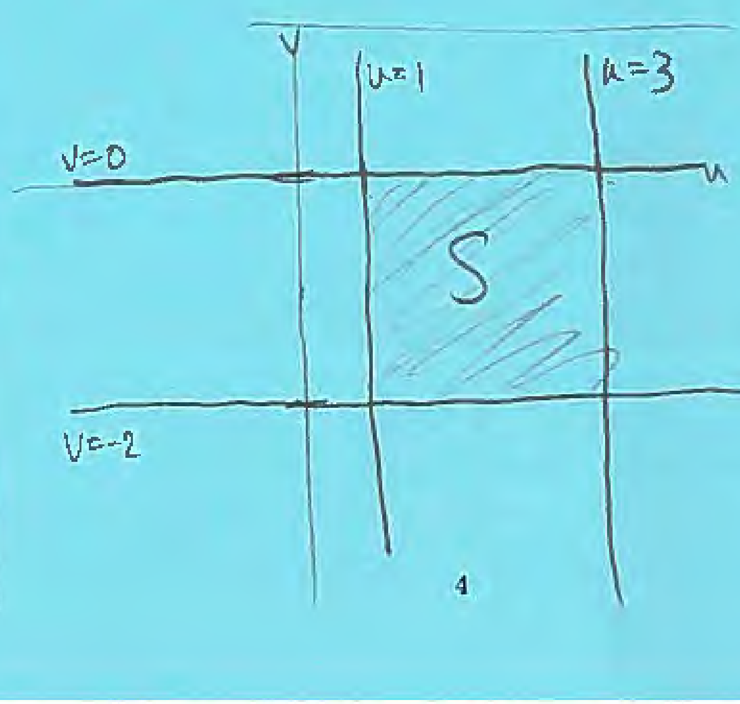
Integral becomes

$$|-1|\int_1^3\int_{-2}^0 u\sqrt{v}\,dv\,du$$

$$= \int_1^3 u\left(\frac{v^{3/2}}{\frac{3}{2}}\Bigg|_{-2}^{\overset{\text{vanish}}{0}}\right)du$$

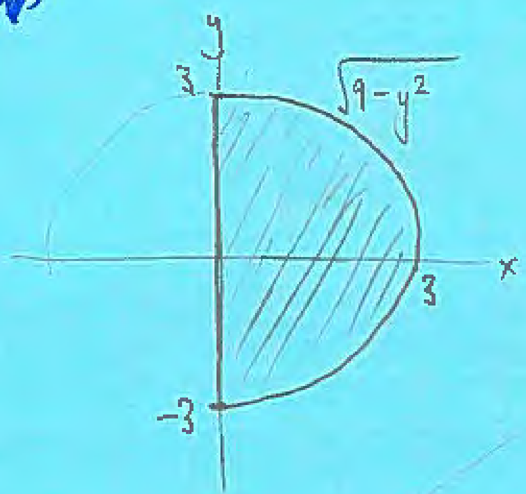
$$= \frac{3}{2}\left(-(-2)^{3/2}\right)\cdot\frac{u^2}{2}\Bigg|_1^3$$

$$= -\frac{3}{4}(-2)^{3/2}\underbrace{(3^2-1^2)}_{8} = \boxed{-6(-2)^{3/2}}$$



6. **(15 pts)** For the integral below, sketch the region of integration and evaluate the integral using polar coordinates.

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}} (9 - x^2 - y^2)\, dx\, dy$$



$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{3} (9 - r^2)\, r\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{9r^2}{2}\Big|_0^3 - \frac{r^4}{4}\Big|_0^3 \right) d\theta$$

$\leftarrow$ terms vanish

$$= \left(\frac{2}{2}\,\frac{9(3)^2}{2} - \frac{3^4}{4} \right) \Big(\underbrace{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)}_{\pi} \Big)$$

$$\frac{2(3^4) - 3^4}{4}$$

$$= \frac{3^4}{4}\pi = \boxed{\frac{81}{4}\pi}$$