MATH 2574 (Calculus III)
Spring 2017

Name: SOLUTIONS

Wed 15 Mar 2017

Exam 2: Multivariate Derivatives and Multiple Integrals (§12.3-12.9, 10.1-10.3, 13.1-13.5)

Exam Instructions: You have 50 minutes to complete this exam. Justification is required for all problems. No electronic devices (phones, iDevices, computers, etc) except for a **basic scientific calculator**. On story problems, round to one decimal place. If you finish early then you may leave, UNLESS there are less than 5 minutes of class left. To prevent disruption, if you finish with less than 5 minutes of class remaining then please stay seated and quiet.

In addition, please provide the following data:

Drill Instructor: ______________________

Drill Time: ______________________

Your signature below indicates that you have read this page and agree to follow the Academic Honesty Policies of the University of Arkansas.

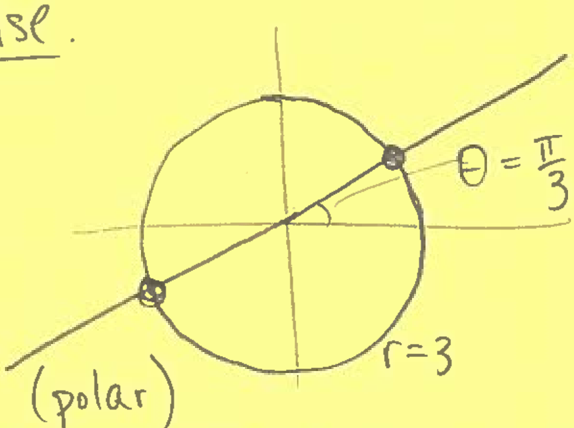
Signature: **(1 pt)** ______________________

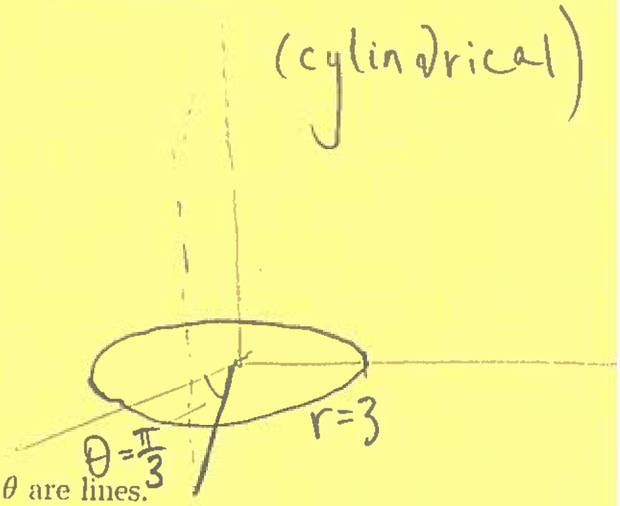
Good luck!

1. Determine whether the following statements are true or false. You must justify your answer.

   (a) **(4 pts)** The graphs of $r = 3$ and $\theta = \frac{\pi}{3}$ intersect exactly once.

False.





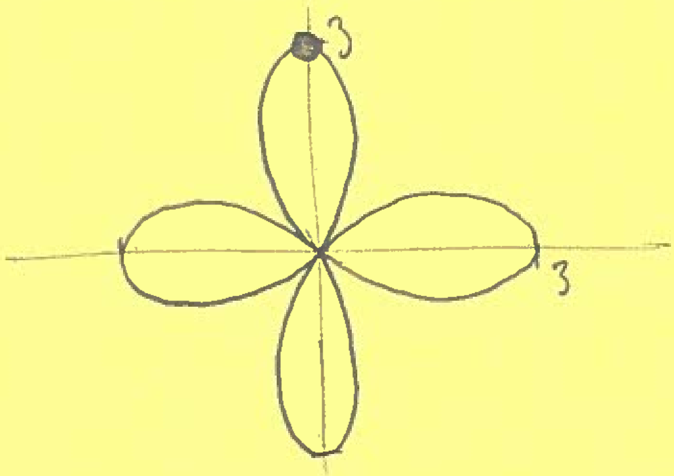
   (b) **(4 pts)** The graphs of $r = 4\sec\theta$ and $r = \csc\theta$ are lines.

True.

$\underbrace{r\cos\theta}_{x} = 4$ $\qquad$ $\underbrace{r\sin\theta}_{y} = 1$

   (c) **(4 pts)** The point $\left(3, \frac{\pi}{2}\right)$ lies on the graph of $r = 3\cos 2\theta$.

True.



2. **(12 pts)** Find the absolute maximum and minimum values of the function

$$f(x, y) = x^2 + y^2 - 2x - 2y$$

on the closed region R, bounded by the triangle with vertices $(0,0)$, $(2,0)$, $(0,2)$.

$f_x = 2x - 2 = 0 \Rightarrow x = 1$

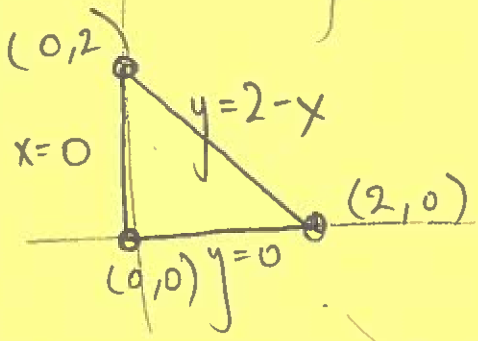
$f_y = 2y - 2 = 0 \Rightarrow y = 1$ CP

Discriminant:

$$\begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix} = 2(2) - 0(0) = 4 > 0; \quad f_{xx}(1,1) = 2 > 0$$

$\Rightarrow$ min

Boundary:



$x=0$: $f(0,y) = y^2 - 2y$; $\frac{d}{dy} f(0,y) = 2y - 2 = 0 \Rightarrow y = 1$

$y=0$: $f(x,0) = x^2 - 2x$; $\frac{d}{dx} f(x,0) = 2x - 2 = 0 \Rightarrow x = 1$

$y=2-x$: $f(x,2-x) = x^2 + \underbrace{(2-x)^2}_{\cancel{4}-4x+x^2} \cancel{-2x} \underbrace{-2(2-x)}_{\cancel{-4}+\cancel{2x}}$

$= 2x^2 - 4x$; $\frac{d}{dx} f(x,2-x) = 4x - 4$

$\Rightarrow x = 1,$

$y = 2 - 1 = 1$

Compare:

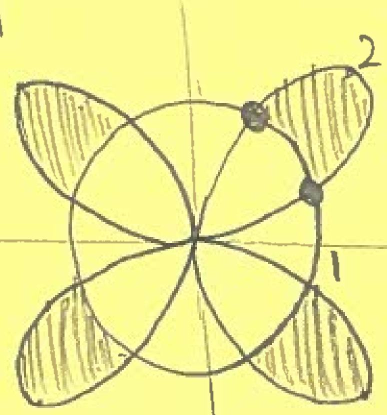
$f(1,1) = 1^2 + 1^2 - 2 - 2 = -2 \leftarrow$ min
@$(x,y) = (1,1)$

$f(0,1) = 1^2 - 2 = -1$

$f(1,0) = 1 - 2 = -1$ $\leftarrow$ max
@$(x,y) = (0,1), (1,0)$

3. **(10 pts)** Find the area of the region inside the rose $r = 2\sin 2\theta$ and outside the circle $r = 1$. *(In case you need it, the half-angle formula is $\cos^2 x = \dfrac{1+\cos 2x}{2}$.)*

By symmetry use one petal and multiply by 4.



Bounds of integration:

circle $\to 1 = 2\sin 2\theta \leftarrow$ rose

$\frac{1}{2} = \sin 2\theta$

$\Rightarrow 2\theta = \frac{\pi}{6}$

$\theta = \frac{\pi}{12}, \frac{\pi}{2} - \frac{\pi}{12} = \frac{5\pi}{12}$

$\frac{\pi}{2} = \frac{6\pi}{12}$

Integral:

$$4\int_{\pi/12}^{5\pi/12}\int_{1}^{2\sin 2\theta} r\,dr\,d\theta$$

Solution:

$$\boxed{2\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right) \approx 3.8}$$

$$= \overset{2}{\cancel{4}}\int_{\pi/12}^{5\pi/12}\left(\frac{(2\sin 2\theta)^2}{\cancel{2}} - \frac{1^2}{\cancel{2}}\right)d\theta$$

$4\sin^2 2\theta - 1$

$$\cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2} + \frac{\cos 2x}{2}$$

$$1-\sin^2 x \Rightarrow 1 - \frac{1}{2} - \frac{\cos 2x}{2} = \sin^2 x$$

(b/c $\cos^2 x + \sin^2 x = 1$)

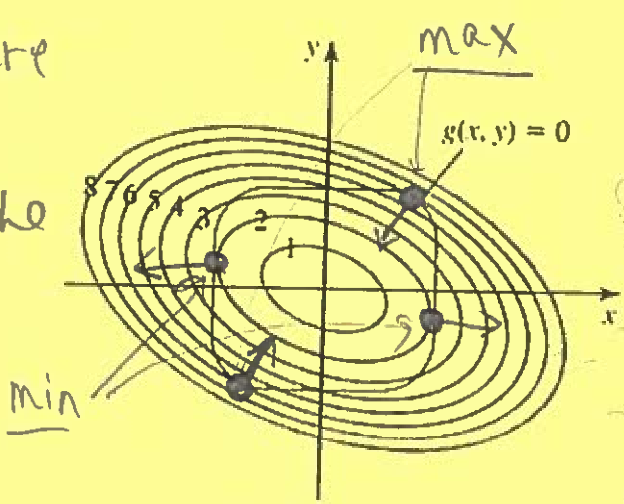
$$= 2\int_{\pi/12}^{5\pi/12}\left(\overset{2}{\cancel{4}}\left(\frac{1}{\cancel{2}} - \frac{\cos 4\theta}{\cancel{2}}\right) - 1\right)d\theta = 2\int_{\pi/12}^{5\pi/12}(1 - 2\cos 4\theta)\,d\theta = 2\left(\theta - 2\left(\frac{1}{\cancel{4}_2}\sin 4\theta\right)\right)\Bigg|_{\pi/12}^{5\pi/12}$$

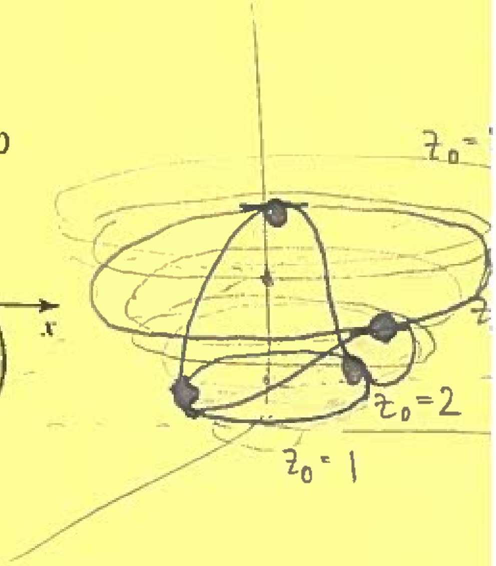
$$= 2\left[\left(\frac{5\pi}{12} - \frac{\pi}{12}\right) - \frac{1}{2}\left(\sin\frac{5\pi}{3} - \sin\frac{\pi}{3}\right)\right]$$

$\frac{\pi}{3}$, $\frac{\sqrt{3}}{2}$, $\frac{\sqrt{3}}{2}$

4. **(8 pts)** The following figure shows the level curves for various $z = z_0$ of the function f, along with the constraint curve $g(x, y) = 0$. Estimate the maximum and minimum values of f subject to the constraint. At each point where an extreme value occurs, indicate the direction of ∇f and the direction of ∇g.

Look for where $g(x,y)=0$ is tangent to the level curves.





5. **(6 pts)** Compute the directional derivative of

$$g(x, y) = \sin\left(\pi(2x - y)\right)$$

at the point $P = (-1, -1)$ in the direction of $\mathbf{u} = \langle \frac{5}{13}, -\frac{12}{13} \rangle$.

$$D_{\hat{u}} g(-1,-1) = \nabla g(-1,-1) \cdot \left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle$$

$$\nabla g = \left\langle \cos(\pi(2x-y))(2\pi), \cos(\pi(2x-y))(-\pi) \right\rangle$$

$$\Rightarrow D_{\hat{u}} g(-1,-1) = 2\pi \cos(\pi(\underbrace{-2-(-1)}_{-1}))\left(\frac{5}{13}\right) + (-\pi)\cos(\pi(\underbrace{-2-(-1)}_{-1}))\left(-\frac{12}{13}\right)$$

$$= -\frac{10\pi}{13} - \frac{12\pi}{13} = \boxed{-\frac{22\pi}{13}}$$

6. Evaluate (or show non-existence of) the following limits:

(a) **(5 pts)** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{|xy|} = \lim_{(x,mx)\to(0,0)} \frac{x(mx)}{|x(mx)|}$

Use 2-Path Test

along the line $y = mx$

$$= \begin{cases} 1 & \text{if } m=1 \\ -1 & \text{if } m=-1 \end{cases}$$

$\Rightarrow$ Does not exist.

(b) **(5 pts)** $\displaystyle\lim_{(x,y,z)\to(1,\ln 2,3)} (1+y)\cancel{\ln}\,\cancel{e}^{xz}$

$= (1+\ln 2)(1)(3)$

$\boxed{= 3 + 3\ln 2}$

7. **(6 pts)** The density of a thin circular plate of radius 2 is given by $\rho(x,y) = 3 + xy$. The edge of the plate is described by the parametric equations $x = 2\cos t$, $y = 2\sin t$, for $0 \le t \le 2\pi$. Find the rate of change of the density with respect to t on the edge of the plate.

$\rho(x(t), y(t)) = 3 + (2\cos t)(2\sin t)$

$= 3 + 4\cos t\sin t$

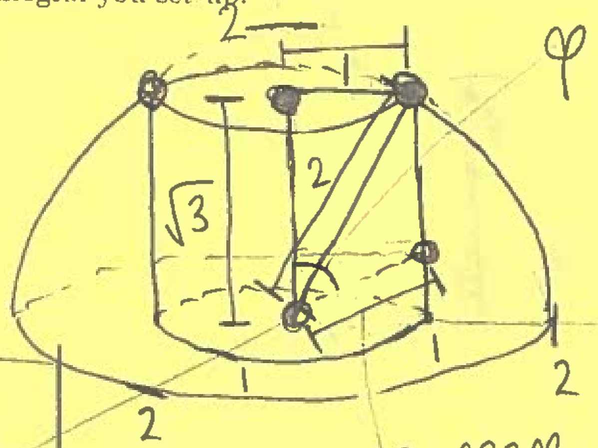
$\boxed{\dfrac{d\rho}{dt} = -4\sin^2 t + 4\cos^2 t}$

8. **(10 pts)** Set up, but **do not evaluate**, the integral for the volume of material remaining in a hemisphere of radius 2 after a cylindrical hole of radius 1 is drilled through the center of the hemisphere perpendicular to its base.

ExTrA cReDiT (5 pts) Evaluate the integral you set up.

$2 = \csc\varphi \Rightarrow \sin\varphi = \frac{1}{2}$

$\Rightarrow \varphi = \frac{\pi}{6}$



$\rho = \csc\varphi$

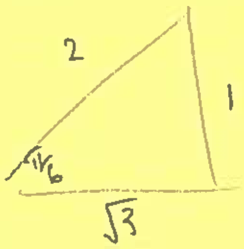
$\left(\text{b/c } \sin\varphi = \frac{1}{\rho}\right)$

Volume:

$$\boxed{\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_{\csc\varphi}^{2} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta}$$

$$= \int_0^{2\pi}\int_{\pi/6}^{\pi/2} \sin\varphi\left(\frac{2^3}{3} - \frac{\csc^3\varphi}{3}\right) d\varphi \, d\theta$$

$\frac{8}{3}\sin\varphi - \frac{1}{3}\csc^2\varphi$



$$= \frac{8}{3}\int_0^{2\pi}\left((-\cos\varphi)\Big|_{\pi/6}^{\pi/2} + \frac{1}{3}\cot\varphi\Big|_{\pi/6}^{\pi/2}\right) d\theta$$

$$= \left[\frac{\cancel{8}^{\,4}}{3}\left(-\left(0-\frac{\sqrt{3}}{\cancel{2}}\right)\right) + \frac{1}{3}\left(0-\sqrt{3}\right)\right](2\pi) = \left(\frac{4\sqrt{3}}{3} - \frac{\sqrt{3}}{3}\right)(2\pi)$$

$$\boxed{= \frac{\cancel{3}\sqrt{3}}{\cancel{3}}(2\pi) \approx 10.9}$$