Fundamental Theorem of Finitely Generated Abelian Groups

1. Fundamental Theorem of Finitely Generated Abelian Groups

Presentation of a group

Solutions to exercises

Presentation of a group

Definition 1

Suppose $\varphi : \mathbb{Z}^n \to \mathbb{Z}^m$ *is a homomorphism.*

(a) *The* **cokernel** *of* φ *is the quotient group* $\operatorname{coker}(\varphi) := \mathbb{Z}^m / \operatorname{image}(\varphi)$.

(b) *Suppose* φ *is given by an* $m \times n$ *matrix* A. *Write* $A\mathbb{Z}^n \triangleleft \mathbb{Z}^m$ *to denote* $\operatorname{image}(\varphi)$. *Any isomorphism*

$$\psi : \mathbb{Z}^m / A\mathbb{Z}^n \xrightarrow{\cong} G$$

is called a **presentation** *of a finitely generated abelian group* G*, and* A *is called a* **presentation matrix** *for* G.

Exercise 1

Let $\varphi : G \to H$ denote a group homomorphism. Prove the following:

(a) φ is injective if and only if $\ker \varphi = \{1_G\}$.

(b) φ is surjective if and only if $\operatorname{coker} \varphi = \{1_H\}$.

Question

Compute the determinants for the matrices in Exercises **??** and **??**. What is the pattern?

Exercise 2 (cf. Problem 83)

For any positive integer n, consider an $n \times n$ matrix A_n described by Pascal's triangle, exemplified by

$$A_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix}.$$

What finitely generated abelian group G_n is presented by the matrix A_n?

Solutions to exercises

Exercise 1

Solution:

(a) First, suppose φ is injective. Since $1_G \in \ker\varphi$, if $g \in \ker\varphi$ then $\varphi(g) = 1_H = \varphi(1_G)$ implies $g = 1_G$. On the other hand, if the kernel is trivial then suppose $\varphi(g) = \varphi(h)$. Multiply both sides by $\varphi(g)^{-1}$:

$$\begin{aligned} \varphi(g) &= \varphi(h) \\ \varphi(g)\varphi(g)^{-1} &= \varphi(h)\varphi(g)^{-1} \\ 1_H &= \varphi(hg^{-1}). \end{aligned}$$

This means $hg^{-1} \in \ker\varphi$ so $hg^{-1} = 1_G$. Multiply both sides by g:

$$\begin{aligned} hg^{-1} &= 1_G \\ hg^{-1}g &= 1_G g \\ h &= g, \end{aligned}$$

and so φ is injective. $\square$

(b) Say φ is surjective. Then $\varphi(G) = H$ implies $\operatorname{coker}\varphi = H/\varphi(G) = H/H = \{1_H\}$. Conversely, suppose the cokernel is trivial. By Lagrange's Theorem, the order of the subgroup $\varphi(G)$ must divide $|H|$ and since the cosets of a subgroup partition the group evenly, we must have

$$|H/\varphi(G)| \cdot |\varphi(G)| = |H|.$$

A trivial cokernel means $|H/\varphi(G)| = 1$ and it follows that $|\varphi(G)| = |H|$ and hence, $\varphi(G) = H$. Therefore, φ is surjective. $\square$

Exercise **??** (cf. Problem 79)

Solution: The Smith normal form of $A := \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right)$, which means A is a presentation matrix for a group isomorphic to $\boxed{\mathbb{Z}_3}$.

Exercise **??** (cf. Problem 80)

Solution: The Smith normal form of $\tilde{\Delta}$ from Problem **??** was $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 4 \end{smallmatrix}\right)$. Thus $\mathcal{S}(\Gamma) \cong \boxed{\mathbb{Z}_4}$.

Exercise ?? (cf. Problem 81)

Solution:

(a) $\begin{pmatrix} 5 & 0 & 0 \end{pmatrix} \xrightarrow{\cancel{C_2}\ \cancel{C_3}} (5)$ gives a presentation for a group isomorphic to $\boxed{\mathbb{Z}_5}$.

(b) $\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$ has 5 as an invariant factor, and free rank of 2. Therefore its cokernel is isomorphic to $\boxed{\mathbb{Z}_5 \oplus \mathbb{Z} \oplus \mathbb{Z}}$.

(c) Computing the Smith normal form

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow[C_3 \to C_3 - C_1]{C_2 \to C_2 - C_1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

shows the abelian group presented by the above matrix is isomorphic to $\boxed{\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2}$.

Exercise **??** (cf. Problem 82)

Solution: Via Sage, the Smith normal forms reveal the isomorphic abelian groups:

(a) $\begin{pmatrix} [r]3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ has cokernel $\boxed{\mathbb{Z}_4 \oplus \mathbb{Z}_4}$.

(b) $\begin{pmatrix} [r]3 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$ has cokernel $\boxed{\mathbb{Z}_3 \oplus \mathbb{Z}_{15}}$.

(c) $\begin{pmatrix} [r]3 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & -1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 0 & 11 \end{pmatrix}$ has cokernel $\boxed{\mathbb{Z}_{11} \oplus \mathbb{Z}_{11}}$.

Exercise 2 (cf. Problem 83)

Solution: When $n = 1$, the trivial group is presented. Suppose, for induction, the trivial group is presented by A_{n-1}. To clear the first column of A_n, subtract from each row R_i, for $i = 2, \ldots, n$, the row R_{i-1}. For $n \geq 2$ every entry a_{ij} not in the first row or column can be written

$$a_{ij} = a_{i,j-1} + a_{i-1,j}.$$

Thus upon clearing the first column, such an entry becomes $a_{ij} - a_{i-1,j} = a_{i,j-1}$.

Next, clear the first row by subtracting from each column C_j, for $j = 2, \ldots, n$, the column C_{j-1}. For $i, j \geq 2$, the ith entry in the jth column was replaced in the row operations by $a_{i,j-1}$. Thus when we clear the first row the (i,j)th entry, for $i, j \geq 2$, will become

$$\begin{aligned} a_{i,j-1} - a_{i,j-2} &= (a_{i-1,j-1} + a_{i,(j-1)-1}) - a_{i,j-2} \\ &= a_{i-1,j-1}. \end{aligned}$$

Having cleared the first row and the first column of A_n, the remaining $(n-1) \times (n-1)$ submatrix is a copy of A_{n-1} which, by the induction hypothesis, reduces to the identity matrix. It follows that A_n also reduces to the identity matrix, and therefore is a presentation for the trivial group. $\square$