MATH 2574 (Calculus III)
Spring 2017

Name: ______________________________

Fri 21 Apr 2017

Exam 3: Transformations and line integrals (§13.7-14.5)

Exam Instructions: You have 50 minutes to complete this exam. Justification is required for all problems. No electronic devices (phones, iDevices, computers, etc) except for a **basic scientific calculator**. On story problems, round to one decimal place. If you finish early then you may leave, UNLESS there are less than 5 minutes of class left. To prevent disruption, if you finish with less than 5 minutes of class remaining then please stay seated and quiet.

In addition, please provide the following data:

Drill Instructor: ______________________________

Drill Time: ______________________________

Your signature below indicates that you have read this page and agree to follow the Academic Honesty Policies of the University of Arkansas.

Signature: **(1 pt)** ______________________________

Good luck!

Formulas you may need:

Jacobian Determinant of a Transformation of Two Variables

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Jacobian Determinant of a Transformation of Three Variables

$$J(u,v,w)=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}.$$

Recall that by expanding about the first row,

$$\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}=a_{11}(a_{22}a_{33}-a_{23}a_{32})-a_{12}(a_{21}a_{33}-a_{23}a_{31})+a_{13}(a_{21}a_{32}-a_{22}a_{31}).$$

Scalar Line Integrals in $\mathbb{R}^2$

$$\int_C f\,ds=\int_a^b f(x(t),y(t))|\mathbf{r}'(t)|dt=\int_a^b f(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\,dt.$$

Scalar Line Integrals in $\mathbb{R}^3$

$$\int_C f\,ds=\int_a^b f(x(t),y(t),z(t))|\mathbf{r}'(t)|\,dt=\int_a^b f(x(t),y(t),z(t))\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt.$$

Green's Theorem—Circulation Form

The circulation form of Green's Theorem is also called the *tangential*, or *curl*, form.

$$\underbrace{\oint_C \mathbf{F}\cdot d\mathbf{r}}_{\text{circulation}}=\underbrace{\oint_C f\,dx+g\,dy}_{\text{circulation}}=\iint_R\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dA.$$

Area of a Plane Region by Line Integrals

Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

$$\oint_C x\,dy=-\oint_C y\,dx=\frac{1}{2}\oint_C (x\,dy-y\,dx).$$

Green's Theorem, Flux Form

The flux form of Green's Theorem is also called the *normal*, or *divergence*, form.

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds}_{\text{outward flux}}=\underbrace{\oint_C f\,dy-g\,dx}_{\text{outward flux}}=\iint_R\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\right)dA$$

Curl of a Vector Field

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f & g & h\end{vmatrix}\begin{array}{l}\leftarrow \text{Unit vectors}\\ \leftarrow \text{Components of }\nabla\\ \leftarrow \text{Components of }\mathbf{F}\end{array}$$

$$=\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)\mathbf{i}+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)\mathbf{j}+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\mathbf{k}.$$

Divergence of a Vector Field

$$\text{div }\mathbf{F}=\nabla\cdot\mathbf{F}=\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}.$$

Different Forms of Line Integrals of Vector Fields

The line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ may be expressed in the following forms,

$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt=\int_a^b(f(t)x'(t)+g(t)y'(t)+h(t)z'(t))\,dt$$
$$=\int_C f\,dx+g\,dy+h\,dz$$
$$=\int_C \mathbf{F}\cdot d\mathbf{r}.$$

For line integrals in the plane,

$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt=\int_a^b(f(t)x'(t)+g(t)y'(t))\,dt=\int_C f\,dx+g\,dy=\int_C \mathbf{F}\cdot d\mathbf{r}.$$

Flux

The **flux** of the vector field **F** across C is

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds=\int_a^b(f(t)y'(t)-g(t)x'(t))\,dt,$$

where $\mathbf{n}=\mathbf{T}\times\mathbf{k}$ is the unit normal vector and **T** is the unit tangent vector consistent with the orientation.

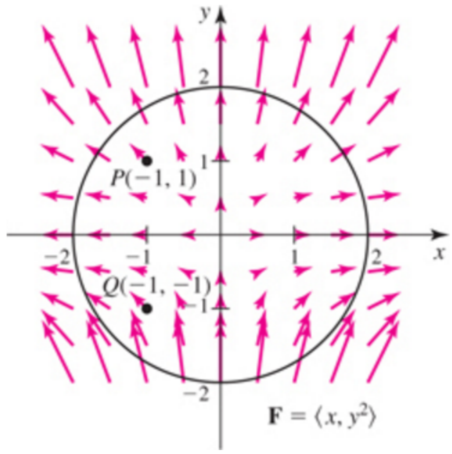
1. **(16 pts)** Compute the Jacobian, $J(\rho, \varphi, \theta)$, of the following transformation taking Cartesian to spherical coordinates:

$$x = \rho \sin\varphi \cos\theta \qquad y = \rho \sin\varphi \sin\theta \qquad z = \rho \cos\varphi$$

You must show your work and simplify.

2. **(16 pts)** Evaluate the scalar line integral $\int_C (x^3 + y^2)\, ds$, where C is the line segment from $(0,0)$ to $(4,4)$.

3. The vector field $\mathbf{F} = \langle x, y^2 \rangle$, the circle C of radius 2 centered at the origin, and two points $P = (-1, 1)$ and $Q = (-1, -1)$, are given in the figure below.



(a) **(5 pts)** Without computing the divergence, does the graph suggest that the divergence is positive or negative at P and Q? Justify your answer.

(b) **(5 pts)** Compute the divergence of $\mathbf{F}$ at P and Q to confirm your answer to part (a).

(c) **(3 pts)** Label on the graph where the flux across C is outward.

(d) **(8 pts)** Is the **net** outward flux across C positive or negative? You must justify your answer.

4. **(pts)** Let $\mathbf{F} = \langle 2xyz, x^2z, x^2y \rangle$.

   (a) **(8 pts)** What is the curl of $\mathbf{F}$?

   (b) **(10 pts)** What is the circulation of $\mathbf{F}$ along C, where C is the closed curve formed by the square whose corners are the points $(2, 2)$, $(-2, 2)$, $(-2, -2)$, and $(2, -2)$?

5. **(16 pts)** Use Green's Theorem to find the area inside an ellipse with major and minor axes of length 10 and 9, respectively. In case you need them, the half-angle formulas are $\cos^2 x = \dfrac{1+\cos 2x}{2}$ and $\sin^2 x = \dfrac{1-\cos 2x}{2}$.

6. **(12 pts)** Match vector fields (a)-(d) with graphs (A)-(D).

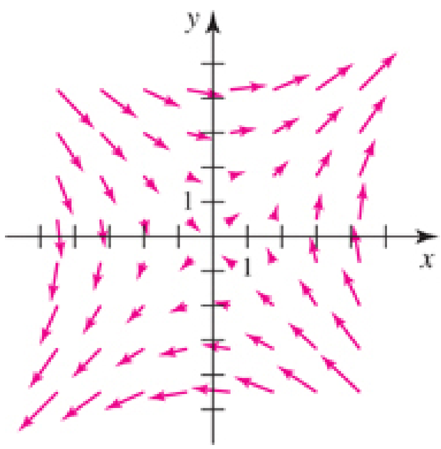
(a) $\mathbf{F} = \langle y, x \rangle$

(b) $\mathbf{F} = \langle x - y, x \rangle$

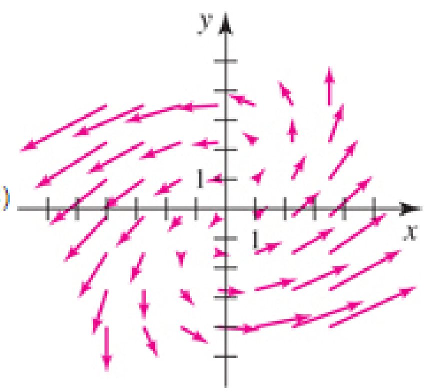
(c) $\mathbf{F} = \langle 2x, -y \rangle$

(d) $\mathbf{F} = \langle 0, x^2 \rangle$

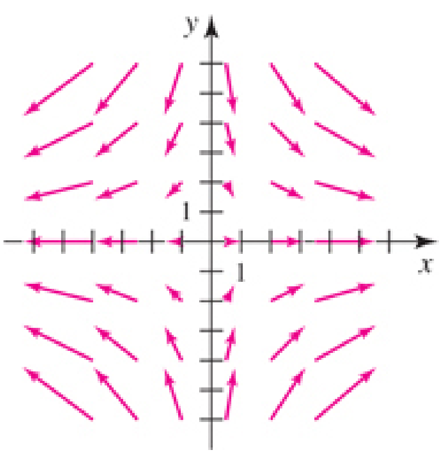
(A)



(C)



(B)



(D)