Fall 2015 Cal III Final Review Lecture

(Chapter 14 Review Problems)

#22 Is the vector field $\vec{F} = \langle \overset{f}{e^x\cos y}, \overset{g}{-e^x\sin y}\rangle$ conservative? If so, then find a potential function.

$f_y = -e^x\sin y$

$g_x = -e^x\sin y$ ✓ Yes, $\vec{F}$ is conservative.

Let $\varphi = \int f\,dx = \int e^x\cos y\,dx$.

$= e^x\cos y + \underbrace{C(y)}_{\text{stuff with only y's}}$

Set $\varphi_y = g$, so

$-e^x\sin y + \frac{\partial C}{\partial y} = -e^x\sin y$

$\Rightarrow \frac{\partial C}{\partial y} = 0$ and $C(y) =$ constant

and a potential function is

$\varphi = e^x\cos y$

Check: $\nabla\varphi = \langle e^x\cos y, -e^x\sin y\rangle = \vec{F}$ ✓

OR

Let $\varphi = \int g\,dy = \int -e^x\sin y\,dy$

$= e^x\cos y + \underbrace{C(x)}_{\text{stuff with x's}}$

Set $\varphi_x = f$ so

$e^x\cos y + \frac{\partial C}{\partial x} = e^x\cos y$

$\Rightarrow \frac{\partial C}{\partial x} = 0$ and $C(x) =$ constant

and a potential function is

$\boxed{\varphi = e^x\cos y.}$

Check: $\nabla\varphi = \langle e^x\cos y, -e^x\sin y\rangle = \vec{F}$ ✓

#26 Find $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \langle \overset{f}{x}, \overset{g}{-y}\rangle$ and C is the square with vertices $(\pm 1, \pm 1)$ (oriented counterclockwise).

Quickest Way:

Green's Theorem says

$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (\text{curl}(\vec{F}))\,dA$

where R is the interior of C.

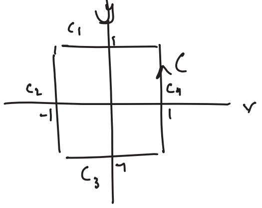
$\text{curl}\,\vec{F} = g_x - f_y$

$= 0 - 0$ so the integral is $\boxed{0}$

Equivalently, $f_y = g_x = 0$ means $\vec{F}$ is conservative, so by FTOC for line integrals,

$\oint_C \vec{F}\cdot d\vec{r} = \boxed{0.}$

OR even without the quicker ways you can always try to evaluate the integral directly.



Parametrize C:

$\vec{r}_1(t) = \langle 1,1\rangle + t(\underbrace{\langle -1,1\rangle - \langle 1,1\rangle}_{\langle -2,0\rangle})$ $\quad 0 \le t \le 1$

$= \langle 1-2t, 1\rangle$

$\vec{r}_1{}'(t) = \langle -2, 0\rangle$

$\vec{r}_2(t) = \langle -1,1\rangle + t(\underbrace{\langle -1,-1\rangle - \langle -1,1\rangle}_{\langle 0,-2\rangle})$ $\quad 0 \le t \le 1$

$= \langle -1, 1-2t\rangle$

$\vec{r}_2{}'(t) = \langle 0, -2\rangle$

$\vec{r}_3(t) = \langle -1,-1\rangle + t(\overbrace{\langle 1,-1\rangle - \langle -1,-1\rangle}^{\langle 2,0\rangle})$

$= \langle -1+2t, -1\rangle \quad 0 \le t \le 1$

$\vec{r}_3{}'(t) = \langle 2, 0\rangle$

$\vec{r}_4(t) = \langle 1,-1\rangle + t(\underbrace{\langle 1,1\rangle - \langle 1,-1\rangle}_{\langle 0,2\rangle})$ $\quad 0 \le t \le 1$

$= \langle 1, -1+2t\rangle$

$\vec{r}_4{}'(t) = \langle 0, 2\rangle.$

And $\oint_C \vec{F}\cdot d\vec{r} = \int_0^1 \underbrace{\overset{(1-\cancel{2t)(-2})}{\langle 1-2t, -1\rangle\cdot\langle -2,0\rangle}}_{C_1}\,dt$

$+ \int_0^1 \underbrace{\overset{-\cancel{(1-2t)(-2)}}{\langle -1, -(1-2t)\rangle\cdot\langle 0,-2\rangle}}_{C_2}\,dt + \int_0^1 \underbrace{\overset{(-1+\cancel{2t)(2)}}{\langle -1+2t, -(-1)\rangle\cdot\langle 2,0\rangle}}_{C_3}\,dt$

$+ \int_0^1 \underbrace{\overset{-\cancel{(-1+2t)(2)}}{\langle 1, -(-1+2t)\rangle\cdot\langle 0,2\rangle}}_{C_4}\,dt = \boxed{0}.$

#34 Find the area of the region $\overset{R}{\text{region}}$ bounded by

$\vec{r}(t) = \langle \cos^3 t, \sin^3 t \rangle$,

$0 \le t \le 2\pi$

$= \iint_R dA$ but what is R?

Use Green's Theorem (Life Hack):

$$\iint_R \underbrace{(\text{curl}\,\vec{F})}\,dA = \oint_C \vec{F}\cdot d\vec{r}$$

Choose $\vec{F} = \langle 0, x\rangle$ (or $\langle -y, 0\rangle$)

to make $\text{curl}\,\vec{F} = g_x - f_y = 1$.

$$= \int_0^{2\pi} \langle 0, \cos^3 t\rangle \cdot \langle 3\cos^2 t(-\sin t), 3\sin^2 t(\cos t)\rangle\,dt$$

$$= \int_0^{2\pi} 3\cos^4 t \sin^2 t\,dt \leftarrow \text{HARD! Integral}$$

Try $\vec{F} = \langle -y, 0\rangle$:

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle -\sin^3 t, 0\rangle \cdot \langle 3\cos^2 t(-\sin t), 3\sin^2 t(\cos t)\rangle\,dt$$

$$= \int_0^{2\pi} 3\cos^2 t\sin^4 t\,dt \leftarrow \text{also HARD.}$$

$x\,dy - y\,dx$ (see Section 14.4 in text)

Use area $= \frac{1}{2}\oint_C$

$$= \frac{1}{2}\int_0^{2\pi} (3\cos^4 t\sin^2 t + 3\cos^2 t\sin^4 t)\,dt$$

$$= \frac{3}{2}\int_0^{2\pi} \cos^2 t\sin^2 t\left(\cancel{\cos^2 t + \sin^2 t}\right)dt$$

$$= \frac{3}{2}\int_0^{2\pi} \underbrace{\left(\frac{1+\cos 2t}{2}\right)\left(\frac{1-\cos 2t}{2}\right)}_{\frac{1}{4}(1-\cos^2(2t)) = \left(\frac{1}{2}\sin(2t)\right)^2 = \frac{1}{4}\left(\frac{1-\cos 4t}{2}\right)}dt \quad \text{(Half-Angle formulas)}$$

$$= \frac{3}{2}\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\int_0^{2\pi} (1 - \cos 4t)\,dt$$

$$= \frac{3}{16}\left(t - \cancel{\frac{1}{4}\sin 4t}\right)\Big|_0^{2\pi} = \boxed{\frac{3\pi}{8}}$$

L3

#48 Use a surface integral to find the area of S $f(x,y)=\sqrt{2}xy$ above the region $\{(r,\theta) \mid 0 \le r \le 2, 0 \le \theta \le 2\pi\}$.

S is already parametrized w/ 2 parameters x and y.

$\vec{r}(x,y) = \langle x, y, \sqrt{2}xy \rangle$

and $|\vec{r}_x \times \vec{r}_y| = \sqrt{z_x^2 + z_y^2 + 1}$

$$= \sqrt{(\sqrt{2}y)^2 + (\sqrt{2}x)^2 + 1}$$

$$\text{area} = \iint_S dS = \int_0^{2\pi}\int_0^2 \sqrt{2r^2+1}\, r\,dr\,d\theta$$

$u = 2r^2 + 1$

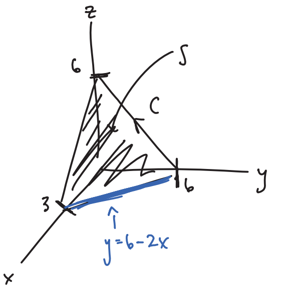
$du = 4r\,dr$

$$= \int_0 \frac{1}{4}\frac{(2r^2+1)^{3/2}}{\frac{3}{\cancel{2}}}\Bigg|_0^2 d\theta$$

$$= \frac{1}{6}\int_0^{2\pi} \left(\big(\underset{9}{2(2)^2+1}\big)^{3/2} - (2(0)^2+1)^{3/2}\right) d\theta$$

$$= \frac{1}{\underset{3}{\cancel{6}}}(26)(\cancel{2}\pi) = \boxed{\frac{26\pi}{3}}$$

#58 Evaluate $\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = \langle x^2-y^2, x, 2yz \rangle$ and C is the boundary of the plane $z = 6-2x-y$ in the first octant



Stokes' Theorem.

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot\hat{n}\,dS$$

$\text{curl}\,\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & x & 2yz \end{vmatrix}$$

$$= \langle 2z-0, -(0-0), 1-(-2y) \rangle$$

S is already parametrized in x and y:

$\vec{r}(x,y) = \langle x, y, 6-2x-y \rangle$ $0 \le x \le 3$, $0 \le y \le 6-2x$

so $\vec{r}_x \times \vec{r}_y = \pm\langle -z_x, -z_y, 1 \rangle$ ← choose the positive and assume C is oriented counterclockwise.

$= \langle 2, 1, 1 \rangle$

and
$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^3\int_0 \left[2(6-2x-y)(2) + \cancel{(0)(1)} + \underset{1+2y}{(1+2y)(1)}\right] dy\,dx$$

$$= \int_0^3 \left((25-8x)y - \frac{\cancel{2}y^2}{\cancel{2}}\right)\Bigg|_0^{6-2x} dx$$ ← terms vanish

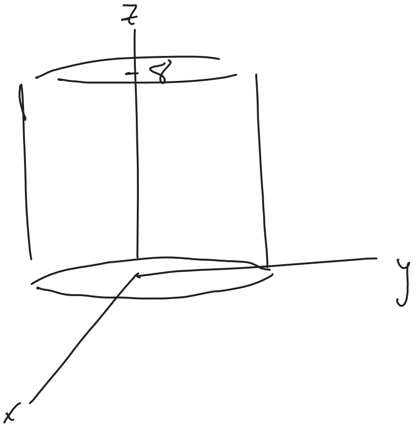
$$= \int_0^3 \left[25(6) - \overset{24}{\cancel{48}}x - 50x + 16x^2 - (36 - 24x + 4x^2)\right] dx$$

$19(6) - 74x + 12x^2$

$$= 19(6)x - \frac{\overset{37}{\cancel{74}}x^2}{\cancel{2}} + \frac{\overset{4}{\cancel{12}}x^3}{\cancel{3}}\Bigg|_0^3 = 19\overset{38}{(2)}(3)^2 - 37(3)^2 + 4(3)^3$$ ← terms vanish

$$\boxed{= 9(37)}$$

#64 Compute the outward flux of the vector field $\vec{F} = \langle x^2, y^2, z^2 \rangle$ across the cylinder $S = \{(x,y,z) \mid x^2+y^2=4, 0 \le z \le 8\}$



Divergence Theorem:

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D (\operatorname{div}\vec{F})\,dV$$

where D = region enclosed by S

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F}$$

$$= 2x + 2y + 2z$$

Cylindrical coordinates:

$$\iiint_D \operatorname{div}\vec{F}\,dV = 2\int_0^8\int_0^{2\pi}\int_0^2 (r\cos\theta + r\sin\theta + z)\,r\,dr\,d\theta\,dz$$

$$= 2\int_0^8\int_0^{2\pi}\left(\frac{r^3}{3}(\cos\theta+\sin\theta) + rz\right)\Bigg|_0^2 d\theta\,dz$$

0 ← terms vanish

$$= 2\int_0^8\left(\frac{2^3}{3}\cancel{(\sin\theta-\cos\theta)} + 2z\theta\right)\Bigg|_0^{2\pi} dz$$

$$= 2\left(\cancel{2}(2\pi)\frac{z^2}{\cancel{2}}\Bigg|_0^8\right) = \boxed{256\pi}$$