MATH 2574 (Calculus III)
Spring 2017

Name: SOLUTIONS

Fri 21 Apr 2017

Exam 3: Transformations and line integrals (§13.7-14.5)

Exam Instructions: You have 50 minutes to complete this exam. Justification is required for all problems. No electronic devices (phones, iDevices, computers, etc) except for a **basic scientific calculator**. On story problems, round to one decimal place. If you finish early then you may leave, UNLESS there are less than 5 minutes of class left. To prevent disruption, if you finish with less than 5 minutes of class remaining then please stay seated and quiet.

In addition, please provide the following data:

Drill Instructor: ______________________

Drill Time: ______________________

Your signature below indicates that you have read this page and agree to follow the Academic Honesty Policies of the University of Arkansas.

Signature: (1 pt) ______________________

Good luck!

Formulas you may need:

Jacobian Determinant of a Transformation of Two Variables

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Jacobian Determinant of a Transformation of Three Variables

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Recall that by expanding about the first row,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Scalar Line Integrals in $\mathbb{R}^2$

$$\int_C f\,ds = \int_a^b f(x(t),y(t))|\mathbf{r}'(t)|dt = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

Scalar Line Integrals in $\mathbb{R}^3$

$$\int_C f\,ds = \int_a^b f(x(t),y(t),z(t))|\mathbf{r}'(t)|\,dt = \int_a^b f(x(t),y(t),z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt.$$

Green's Theorem—Circulation Form

The circulation form of Green's Theorem is also called the *tangential*, or *curl*, form.

$$\underbrace{\oint_C \mathbf{F}\cdot d\mathbf{r}}_{\text{circulation}} = \underbrace{\oint_C f\,dx + g\,dy}_{\text{circulation}} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.$$

Area of a Plane Region by Line Integrals

Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

$$\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx).$$

Green's Theorem, Flux Form

The flux form of Green's Theorem is also called the *normal*, or *divergence*, form

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds}_{\text{outward flux}} = \underbrace{\oint_C f\,dy - g\,dx}_{\text{outward flux}} = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA$$

Curl of a Vector Field

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \begin{array}{l} \leftarrow \text{Unit vectors} \\ \leftarrow \text{Components of } \nabla \\ \leftarrow \text{Components of } \mathbf{F} \end{array}$$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}.$$

Divergence of a Vector Field

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

Different Forms of Line Integrals of Vector Fields

The line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ may be expressed in the following forms.

$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_a^b (f(t)x'(t) + g(t)y'(t) + h(t)z'(t))\,dt = \int_C f\,dx + g\,dy + h\,dz = \int_C \mathbf{F}\cdot d\mathbf{r}.$$

For line integrals in the plane,

$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_a^b (f(t)x'(t) + g(t)y'(t))\,dt = \int_C f\,dx + g\,dy = \int_C \mathbf{F}\cdot d\mathbf{r}.$$

Flux

The flux of the vector field $\mathbf{F}$ across C is

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b (f(t)y'(t) - g(t)x'(t))\,dt,$$

where $\mathbf{n} = \mathbf{T}\times\mathbf{k}$ is the unit normal vector and $\mathbf{T}$ is the unit tangent vector consistent with the orientation.

1. **(16 pts)** Compute the Jacobian, $J(\rho, \varphi, \theta)$, of the following transformation taking Cartesian to spherical coordinates:

$$x = \rho \sin\varphi \cos\theta \qquad y = \rho \sin\varphi \sin\theta \qquad z = \rho \cos\varphi$$

You must show your work and simplify.

$$J(\rho,\varphi,\theta) = \frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}$$

$$= \sin\varphi\cos\theta\left(0 + \rho\sin\varphi\cos\theta(\rho\sin\varphi)\right)$$
$$+ \rho\cos\varphi\cos\theta\left(0 + \rho\sin\varphi\cos\theta(\cos\varphi)\right)$$
$$+ -\rho\sin\varphi\sin\theta\left(\sin\varphi\sin\theta(-\rho\sin\varphi) - \rho\cos\varphi\sin\theta(\cos\varphi)\right)$$

$$= \rho^2\sin^3\varphi\cos^2\theta + \rho^2\sin\varphi\cos^2\varphi\cos^2\theta + \rho^2\sin^3\varphi\sin^2\theta + \rho^2\sin\varphi\cos^2\varphi\sin^2\theta$$

$$= \rho^2\sin\varphi\left(\sin^2\varphi\cos^2\theta + \cos^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi\sin^2\theta\right)$$

$$= \rho^2\sin\varphi\left[(\sin^2\varphi + \cos^2\varphi)\cos^2\theta + (\sin^2\varphi + \cos^2\varphi)\sin^2\theta\right]$$

$$= \rho^2\sin\varphi\left(\cos^2\theta + \sin^2\theta\right) = \boxed{\rho^2\sin\varphi}$$

2. **(16 pts)** Use Green's Theorem to find the area inside an ellipse with major and minor axes of length 12 and 8, respectively. In case you need them, the half-angle formulas are $\cos^2 x = \dfrac{1+\cos 2x}{2}$ and $\sin^2 x = \dfrac{1-\cos 2x}{2}$.

Parametrize $\vec{r}(t) = \langle 12\cos t, 8\sin t\rangle \quad 0 \le t \le 2\pi$

$$\text{Area} = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

$$= \int_0^{2\pi} 12\cos t(8\cos t)\,dt$$

$$= 96\int_0^{2\pi} \cos^2 t\,dt$$

$$= 96\int_0^{2\pi} \left(\frac{1+\cos 2t}{2}\right)dt$$

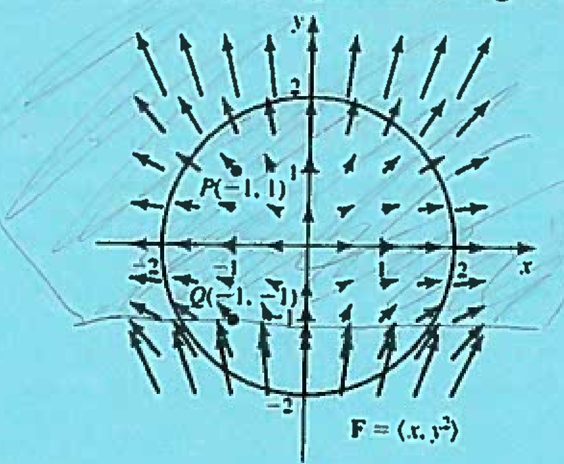
$$= 96\left(\frac{1}{2}\left(t + \frac{1}{2}\sin 2t\right)\right)\Bigg|_0^{2\pi}$$

vanishes

$$= 96\left(\frac{1}{2}\right)(2\pi)$$

$$\boxed{= 96\pi}$$

3. The vector field $\mathbf{F} = \langle x, y^2 \rangle$, the circle C of radius 2 centered at the origin, and two points $P = (-1, 1)$ and $Q = (-1, -1)$, are given in the figure below.



(a) **(5 pts)** Without computing the divergence, does the graph suggest that the divergence is positive or negative at P and Q? Justify your answer.

positive @ P, since all arrows point outward; negative @ Q, since the longer arrows point inward

(b) **(5 pts)** Compute the divergence of $\mathbf{F}$ at P and Q to confirm your answer to part (a).

$\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = 1 + 2y$

@P: $1 + 2(1) = 3 > 0$

@Q: $1 + 2(-1) = -1 < 0$

(c) **(3 pts)** Label on the graph where the flux across C is outward.

–shaded region–

(d) **(8 pts)** Is the **net** outward flux across C positive or negative? You must justify your answer.

Parametrize $C: \vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$, $0 \le t \le 2\pi$

net flux $= \iint_R \operatorname{div}\vec{F}\, dA$ (by Green's Theorem)

$$= \int_0^{2\pi}\int_0^2 (1 + 2(r\sin\theta))\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 (r + 2r^2\sin\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2} + \frac{2}{3}r^3\sin\theta\right)\Big|_0^2 d\theta$$

4. (**pts**) Let $\mathbf{F} = \langle 8xyz, 4x^2z, 4x^2y \rangle$.

(a) **(8 pts)** What is the curl of $\mathbf{F}$?

$\text{curl}\,\vec{F} = \langle 4x^2 - 4x^2, 8xy - 8xy, 8xz - 8xz \rangle$

$\boxed{= \vec{0}}$

(b) **(10 pts)** What is the circulation of $\mathbf{F}$ along C, where C is the closed curve formed by the square whose corners are the points $(1,1)$, $(-1,1)$, $(-1,-1)$, and $(1,-1)$?

-pretend there is a z-coordinate —

It does not matter because from part (a), $\text{curl}\,\vec{F} = \vec{0} \Rightarrow \vec{F}$ is conservative.

$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} \boxed{= 0}$

(where $\vec{r}$ is any parametrization of C)

5. **(16 pts)** Evaluate the scalar line integral $\int_C (x^2 + y^3)\, ds$, where C is the line segment from $(0, 0)$ to $(5, 5)$.

Write $\vec{r}(t) = \langle t, t \rangle \quad 0 \le t \le 5$

$\Rightarrow \vec{r}\,'(t) = \langle 1, 1 \rangle$ and $|\vec{r}\,'(t)| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$$\int_C (x^2 + y^3)\, ds = \int_0^5 (t^2 + t^3)(\sqrt{2})\, dt$$

$$= \sqrt{2}\left(\frac{t^3}{3} + \frac{t^4}{4}\right)\Bigg|_0^5$$ ← terms vanish

$$= 5^3\sqrt{2}\underbrace{\left(\frac{1}{3} + \frac{5}{4}\right)}_{\frac{4}{12} + \frac{15}{12} = \frac{19}{12}}$$

$$= 125\sqrt{2}\left(\frac{19}{12}\right)$$

$$\boxed{= \frac{2375\sqrt{2}}{12}}$$

6. **(12 pts)** Match vector fields (a)-(d) with graphs (A)-(D).

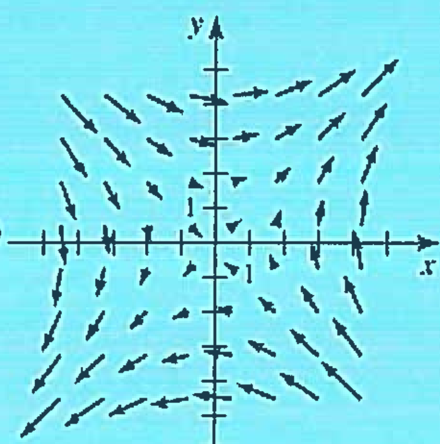
(a) $\mathbf{F} = \langle 0, x^2 \rangle \longrightarrow$ D

(b) $\mathbf{F} = \langle x - y, x \rangle \longrightarrow$ C

(c) $\mathbf{F} = \langle 2x, -y \rangle \longrightarrow$ B

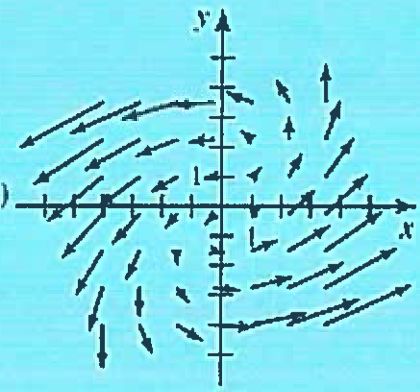
(d) $\mathbf{F} = \langle y, x \rangle \longrightarrow$ A

(A)



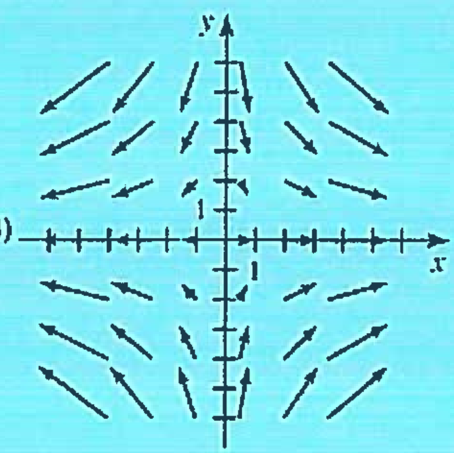
(d)

(C)



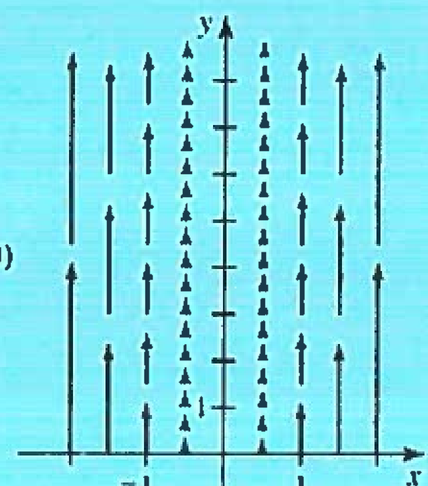
(b)

(B)



(c)

(D)



(a)