MATH 2574 (Calculus III)
Fall 2015

SOLUTIONS

Tues 15 Sep 2015

Quiz 3: Trajectories and Arc Length ($\oint$11.7-11.8)

Note: The typo in the course number as well as the typo in the half-angle formula have been fixed.

Directions: You have 30 minutes to complete this quiz. You may collaborate.

1. **(2 pts)** A cycloid is the path traced by a point on a rolling circle (think of a light on the rim of a moving bicycle wheel). The cycloid generated by a circle of radius a is given by the parametric equation

$$x = a(t - \sin t) \quad y = a(1 - \cos t).$$

(a) The parameter range $0 \leq t \leq 2\pi$ produces one arch of the cycloid. Compute its length. **Hint:** You might need the half-angle formula

$$\sin^2\theta = \tfrac{1}{2}\left(1 - \cos 2\theta\right).$$

Solution: The length of the arch is given by the arc length formula; substitute the given values in the problem and simplify:

$$\begin{aligned}
\text{length of arch} &= \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{\left(a(1-\cos t)\right)^2 + (a\sin t)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{a^2\left(1 - 2\cos t + \cos^2 t\right) + a^2\sin^2 t\, dt} \\
&= a\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt.
\end{aligned}$$

To evaluate the integral, first use the given half-angle formula,

$$\begin{aligned}
\sin^2\theta &= \tfrac{1}{2}\left(1 - \cos 2\theta\right) \\
2\sin^2\theta &= 1 - \cos 2\theta \\
4\sin^2\theta &= 2 - 2\cos 2\theta \\
4\sin^2\left(\tfrac{t}{2}\right) &= 2 - 2\cos t,
\end{aligned}$$

having set $t = 2\theta$. Then the length of one arch is

$$\begin{aligned} a\int_0^{2\pi} \sqrt{2-2\cos t}\, dt &= a\int_0^{2\pi} \sqrt{4\sin^2\left(\tfrac{t}{2}\right)}\, dt \\ &= a\int_0^{2\pi} 2\sin\left(\tfrac{t}{2}\right)\, dt \\ &= a\left(-4\cos\left(\tfrac{t}{2}\right)\Big|_0^{2\pi}\right) \\ &= a\left[-4\cos\pi - (-4\cos 0)\right] \\ &= a\left[-4(-1) - (-4)\right] \\ &\boxed{= 8a.} \end{aligned}$$

(b) Draw a well-labelled graph of the arch of the cycloid.

<u>**Solution**</u>**:** In order to draw the graph without a calculator, use techniques from Cal I (see $\oint$4.3 in the text for more information).

First examine the derivatives. For the entire domain $0 \leq t \leq 2\pi$,

$$x'(t) = a(1-\cos t) \geq 0,$$

i.e., $x(t)$ increases as t increases. This means the graph will be drawn from left to right. Furthermore, $x'(t)$ is symmetric about the point $t = \pi$ so the rate at which the x-coordinate is drawn will also be symmetric about $x(\pi) = a\pi$. Similarly, the y-coordinate has derivative

$$y'(t) = a\sin t,$$

which is antisymmetric about the point $t = \pi$, so the y-coordinate increases for $0 < t < \pi$ and decreases $\pi < t < 2\pi$. Its maximum value is $y(\pi) = 2a$.

Now examine the derivative of y as a function of x:

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{a\sin t}{a(1-\cos t)}$$

(The factor a in both the numerator and denominator can be disregarded.) Considering the given domain $0 \leq t \leq 2\pi$, the derivative is undefined for $t = 0, 2\pi$ and zero for $t = \pi$. From the information above about $x'(t)$ and $y'(t)$, the derivative $\frac{dy}{dx}$ is positive for $0 < t < \pi$ (so y is increasing as a function of x) and negative for $\pi < t < 2\pi$ (so y

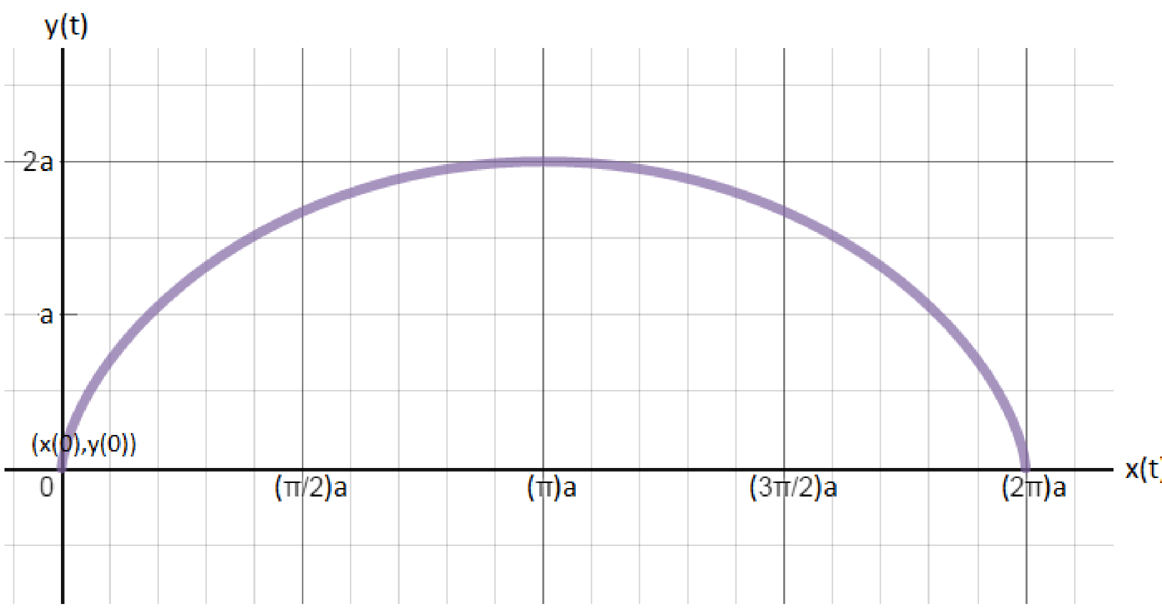
is decreasing as a function of x). The second derivative,

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{\sin t}{1-\cos t}\right) \\
&= \frac{\frac{d}{dt}\left(\frac{\sin t}{1-\cos t}\right)}{\frac{dx}{dt}} \\
&= \frac{\frac{(1-\cos t)(\cos t)-\sin t(\sin t)}{(1-\cos t)^2}}{a(1-\cos t)} \\
&= \frac{\cos t-(\cos^2 t+\sin^2 t)}{a(1-\cos t)^3} \\
&= \frac{-(1-\cos t)}{a(1-\cos t)^3} \\
&= \frac{-1}{a(1-\cos t)^2},
\end{aligned}$$

is always negative, so the graph will be concave down.

Finally, compute some key coordinates $(x(t), y(t))$, using values in the domain $0 \leq t \leq 2\pi$:

$$\begin{aligned}
x(0) &= a(0-\sin 0) = 0 & y(0) &= a(1-\cos 0) = 0 \\
x(2\pi) &= a(2\pi-\sin 2\pi) = 2a\pi & y(2\pi) &= a(1-\cos 2\pi) = 0
\end{aligned}$$



2. A golf ball has an initial position

$$\vec{r}(0) = \langle x_0, y_0 \rangle = \langle 0, 0 \rangle = 0\hat{\imath} + 0\hat{\jmath} \text{ ft}$$

when it is hit at an angle of $30°$ from the ground and with an initial speed of 150 ft/s. For the following, neglect air resistance and assume gravity is a constant $g = 32$ ft/s^2. **You must include units in your answers to receive credit.**

(a) **(1 pt)** The golf ball's acceleration vector is: $\vec{a}(t) =$

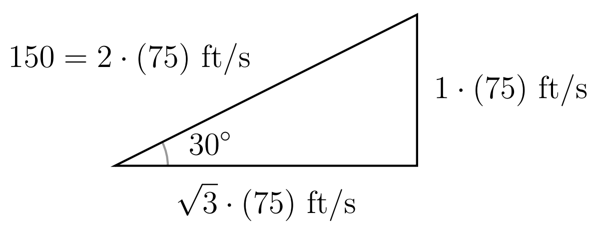
Solution. $\boxed{\langle 0, -32 \rangle \text{ ft/s}^2}$, because there is no air resistance, and the only other force on the ball given is gravity, which is negative, according to the given coordinate system.

(b) **(1 pt)** Its initial speed is: $|\vec{v}(0)| =$

Solution. $\boxed{150 \text{ ft/s}}$ (it was given in the problem).

(c) **(1 pt)** Its initial velocity is: $\vec{v}(0) =$

Solution. $\boxed{\langle 75\sqrt{3}, 75 \rangle \text{ ft/s.}}$ To see why, use the initial angle of the golf ball's trajectory. The initial speed is the hypotenuse of a special triangle:



The base of the triangle is in the $\hat{\imath}$-direction and the height of the triangle is in the $\hat{\jmath}$-direction.

(d) **(1 pt)** The golf ball's velocity vector is $\vec{v}(t) =$

Solution. $\boxed{\langle 75\sqrt{3}, 75 - 32t \rangle \text{ ft/s.}}$ The velocity vector is one of the solutions of the indefinite integral

$$\int \vec{a}(t)\, dt = \int \langle 0, -32 \rangle\, dt = \langle 0, -32t \rangle + \vec{C}.$$

The initial value computed in (c) gives the correct constant vector $\vec{C}$:

$$\vec{v}(0) = \langle 0, -32 \cdot (0) \rangle + \vec{C} = \langle 75\sqrt{3}, 75 \rangle$$

(e) **(1 pt)** The golf ball's position vector is $\vec{r}(t) =$

Solution. $\boxed{\langle 75\sqrt{3}t, 75t - 16t^2\rangle \text{ ft}}$ is found using the same procedure as in (d):

$$\int \vec{v}(t)\ dt = \int \langle 75\sqrt{3}, 75 - 32t\rangle\ dt = \langle 75\sqrt{3}t, 75t - 32\left(\tfrac{1}{2}t^2\right)\rangle + \vec{C}$$
$$\vec{r}(0) = \langle 75\sqrt{3}\cdot(0), 75\cdot(0) - 32\left(\tfrac{1}{2}\cdot(0)^2\right)\rangle + \vec{C} = \langle x_0, y_0\rangle = \langle 0, 0\rangle$$

(f) **(1 pt)** Determine the golf ball's time of flight.

Solution. The vertical component of the golf ball's trajectory is given by a parabola $y(t) = 75t - 16t^2$ whose zeros are exactly when the golf ball hits the ground.

$$\begin{aligned} y(t) = 0 &= 75t - 16t^2 \\ &= t(75 - 16t) \\ \implies t &= 0, \tfrac{75}{16} \end{aligned}$$

The solution $t = 0$ is when the golf ball is hit. The next time the ball is on the ground is after $\boxed{t = \frac{75}{16}\text{ s.}}$

(g) **(1 pt)** How far does the golf ball travel?

Solution. The distance is exactly the x-component of the position vector, evaluated at the time from (f):

$$\begin{aligned} x(t) &= 75\sqrt{3}t \\ x\left(\tfrac{75}{16}\right) &= 75\sqrt{3}\cdot\left(\tfrac{75}{16}\right) \\ &\boxed{= \tfrac{75^2\sqrt{3}}{16}\text{ ft.}} \end{aligned}$$

(h) **(1 pt)** What is the maximum height of the golf ball?

Solution. The maximum height is the vertex of the parabola in the y-component of $\vec{r}(t)$. There are several ways to compute it, one is to evaluate at the zero of the derivative:

$$\begin{aligned} y'(t) = 0 &= 75 - 32t \\ \implies t &= \tfrac{75}{32} \end{aligned}$$

is when $y(t)$ attains its maximum;

$$y\left(\tfrac{75}{32}\right) = 75\cdot\left(\tfrac{75}{32}\right) + 16\cdot\left(\tfrac{75}{32}\right)^2 = \tfrac{75^2}{32} - \tfrac{1}{2}\left(\tfrac{75^2}{32}\right) = \tfrac{1}{2}\left(\tfrac{75^2}{32}\right)\boxed{= \tfrac{75^2}{64}\text{ ft.}}$$