MATH 2554 (Calculus I)

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Monday 9 March (Week 9)

- computer HW for $\oint 3.8 - 3.9$ is extended to Wednesday night
- midterm returned in drill
- Quiz #7 on Tues 10 Mar – covers $\oint 3.8 - 3.9$
- sub on Friday: Dr. Paulk

CEA REQUEST

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$\oint$ 3.8, cont.

The relationship $y = \ln x \iff x = e^y$ applies to logarithms of other bases:

$$y = \log_b x \quad \iff \quad x = b^y.$$

Now taking $\dfrac{d}{dx}\,(x = b^y)$ we obtain

$$\begin{aligned} 1 &= b^y \ln b \left(\frac{dy}{dx}\right) \\ \frac{dy}{dx} &= \frac{1}{b^y \ln b} \\ &= \frac{1}{x \ln b} \end{aligned}$$

So $\dfrac{d}{dx}(\log_b x) = \dfrac{1}{x \ln b}$.

Neat Trick: Logarithmic Differentiation

Example

Compute the derivative of $f(x) = \dfrac{x^2(x-1)^3}{(3+5x)^4}$.

We can use logarithmic differentiation: first take the natural log of both sides and then use properties of logarithms.

$$\ln(f(x)) = \ln\left(\frac{x^2(x-1)^3}{(3+5x)^4}\right)$$

$$= \ln x^2 + \ln (x-1)^3 - \ln (3+5x)^4$$

$$= 2\ln x + 3\ln(x-1) - 4\ln(3+5x)$$

Now we take $\dfrac{d}{dx}$ on both sides:

$$\frac{1}{f(x)}\left(\frac{df}{dx}\right) = 2\left(\frac{1}{x}\right) + 3\left(\frac{1}{x-1}\right) - 4\left(\frac{1}{3+5x}\right) \quad (5)$$

$$\frac{f'(x)}{f(x)} = \frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}$$

Finally, solve for $f'(x)$:

$$f'(x) = f(x)\left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}\right]$$

$$= \frac{x^2(x-1)^3}{(3+5x)^4}\left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}\right]$$

HW from Section 3.8

Do problems 9–27 odd, 31–37 odd, 41–47 odd (pp. 199–200 in textbook)

$\oint$ 3.9 Derivatives of Inverse Trigonometric Functions

Recall: If $y = f(x)$, then $f^{-1}(x)$ is the value of y such that $x = f(y)$.

Example

If $f(x) = 3x + 2$, then what is $f^{-1}(x)$?

NOTE: $f^{-1}(x) \neq f(x)^{-1} \left(= \frac{1}{f(x)}\right)$

Derivative of Inverse Sine

Remember, trig functions are functions, too. Just like with "f", there has to be something to "plug in". It makes no sense to just say $\sin$, without having $\sin($*something*$)$.

$$y = \sin^{-1} x \iff x = \sin y$$

The derivative of $y = \sin^{-1} x$ can be found using implicit differentiation:

$$\begin{aligned} x &= \sin y \\ \frac{d}{dx}(x) &= \frac{d}{dx}(\sin y) \\ 1 &= (\cos y)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\cos y} \end{aligned}$$

We still need to replace $\cos y$ with an expression in terms of x. We use the trig identity $\sin^2 y + \cos^2 y = 1$ (careful with notation: in this case we mean $(\sin y)^2 + (\cos y)^2 = 1$). Then

$$\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - x^2}.$$

The range of $y = \sin^{-1} x$ is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. In this range, cosine is never negative, so we can just take the positive portion of the square root.

Therefore,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}} \implies \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.$$

Exercise

Compute the following:

1. $\dfrac{d}{dx}\left(\sin^{-1}(4x^2 - 3)\right)$

2. $\dfrac{d}{dx}\left(\cos(\sin^{-1} x)\right)$

Derivative of Inverse Tangent

We use a similar method as with inverse sine. Let $y = \tan^{-1} x$ and use implicit differentiation:

$$\begin{aligned} x &= \tan y \\ \frac{d}{dx}(x) &= \frac{d}{dx}(\tan y) \\ 1 &= (\sec^2 y)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} \end{aligned}$$

Use the trig identity $\sec^2 y - \tan^2 y = 1$ to replace $\sec^2 y$ with $1 + x^2$:

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$$

Derivative of Inverse Secant

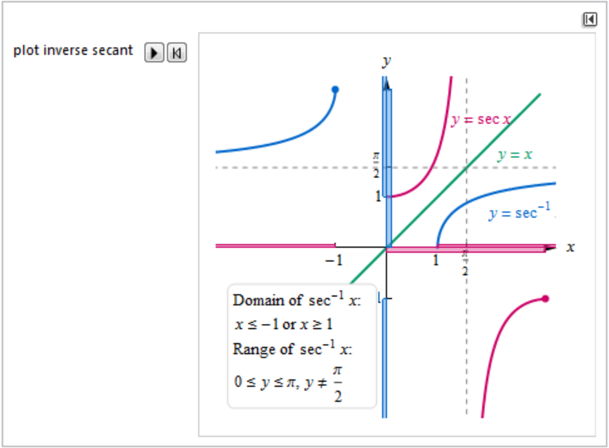
Again, use the same method as with inverse sine:

$$\begin{aligned} y &= \sec^{-1} x \\ x &= \sec y \\ \frac{d}{dx}(x) &= \frac{d}{dx}(\sec y) \\ 1 &= \sec y \tan y \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\sec y \tan y} \end{aligned}$$

Use the trig identity $\sec^2 y - \tan^2 y = 1$ again to get

$$\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}.$$

This time, the $\pm$ matters:



- If $x \geq 1$, then $0 \leq y < \dfrac{\pi}{2}$ and so $\tan y > 0$.
- If $x \leq -1$, then $\dfrac{\pi}{2} < y \leq \pi$ and so $\tan y < 0$.

Therefore,

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

Using other trig identities (which you do not need to prove)

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2} \quad \cot^{-1} x + \tan^{-1} x = \frac{\pi}{2} \quad \csc^{-1} x + \sec^{-1} x = \frac{\pi}{2}$$

we can get the rest of the inverse trig derivatives.

All other inverse trig derivatives

To summarize:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$(-1 < x < 1)$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$(-\infty < x < \infty)$$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$(|x| > 1)$$

Example

Compute the derivatives of $f(x) = \tan^{-1}\left(\frac{1}{x}\right)$ and $g(x) = \sin\left(\sec^{-1}(2x)\right)$.

Derivatives of Inverse Functions in General

Let f be differentiable and have an inverse on an interval I.
Let x_0 be a point in I at which $f'(x_0) \neq 0$.
Then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}$$

where $y_0 = f(x_0)$.

Example

Let $f(x) = \dfrac{1}{4}x^3 + x - 1$. Find $\left(f^{-1}\right)'(3)$.

HW from Section 3.9

Do problems 7–27 odd, 31–39 odd.

Wednesday 11 March (Week 9)

- computer HW for $\oint 3.8 - 3.9$ is extended to tonight
- sub on Friday: Dr. Paulk
- midterm returned in drill...
- Quiz schedule:
  - Quiz #8: $\oint 3.10$ Thurs 12 Mar, take-home
  - Quiz #9: $\oint 4.1, 4.2$ Thurs 19 Mar, due upon return from spring break
  - Quiz #10: $\oint 4.4$ in class Tues 31 Mar
- Exam #3: Friday 3 April

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$\oint 3.10$ Related Rates

In this section, we use our knowledge of derivatives to examine how variables change with respect to time.

The prime feature of these problems is that two or more variables, which are related in a known way, are themselves changing in time.

The goal of these types of problems is to determine the rate of change (i.e., the derivative) of one or more variables at a specific moment in time.

Problem

The edges of a cube increase at a rate of 2 cm/sec. How fast is the volume changing when the length of each edge is 50 cm?

- **Variables:** V (Volume of the cube) and x (length of edge)
- **How Variables are related:** $V = x^3$
- **Rates Known:** $\dfrac{dx}{dt} = 2$ cm/sec
- **Rate We Seek:** $\dfrac{dV}{dt}$ when $x = 50$ cm

Note that both V and x are functions of t (their respective sizes are dependent upon how much time has passed).

So we can write $V(t) = x(t)^3$ and then differentiate this with respect to t:

$$V'(t) = 3x(t)^2 \cdot x'(t).$$

Note that $x(t)$ is the length of the cube's edges at time t, and $x'(t)$ is the rate at which the edges are changing at time t.

We can rewrite the previous equation as

$$\frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt}.$$

So the rate of change of the volume when $x = 50$ cm is

$$\left.\frac{dV}{dt}\right|_{x=50} = 3 \cdot 50^2 \cdot 2 = 15000 \text{ cm}^3/\text{sec}.$$

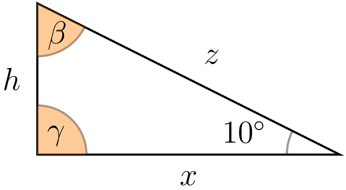
Steps for Solving Related Rates Problems

1. Read the problem carefully, making a sketch to organize the given information. Identify the rates that are given and the rate that is to be determined.
2. Write one or more equations that express the basic relationships among the variables.
3. Introduce rates of change by differentiating the appropriate equation(s) with respect to time t.
4. Substitute known values and solve for the desired quantity.
5. Check that the units are consistent and the answer is reasonable.

The Jet Problem

A jet ascends at a 10° angle from the horizontal with an airspeed of 550 miles/hr (its speed along its line of flight is 550 miles/hr). How fast is the altitude of the jet increasing? If the sun is directly overhead, how fast is the shadow of the jet moving on the ground?

Step 1: There are three variables: the distance the shadow has traveled (x), the altitude of the jet (h), and the distance the jet has actually traveled on its line of flight (z). We know that $\dfrac{dz}{dt} = 550$ miles/hr and we want to find $\dfrac{dx}{dt}$ and $\dfrac{dh}{dt}$. We also see that these variables are related through a right triangle:



Step 2: To answer how fast the altitude is increasing, we need an equation involving only h and z. Using trigonometry,

$$\sin(10^\circ) = \frac{h}{z} \implies h = \sin(10^\circ) \cdot z.$$

To answer how fast the shadow is moving, we need an equation involving only x and z. Using trigonometry,

$$\cos(10^\circ) = \frac{x}{z} \implies x = \cos(10^\circ) \cdot z.$$

Step 3: We can now differentiate each equation to answer each question:

$$h = \sin(10^\circ) \cdot z \implies \frac{dh}{dt} = \sin(10^\circ)\frac{dz}{dt}$$
$$x = \cos(10^\circ) \cdot z \implies \frac{dx}{dt} = \cos(10^\circ)\frac{dz}{dt}$$

Step 4: We know that $\frac{dz}{dt} = 550$ miles/hr. So

$$\frac{dh}{dt} = \sin(10^\circ) \cdot 550 \approx 95.5 \text{ miles/hr}$$
$$\frac{dx}{dt} = \cos(10^\circ) \cdot 550 \approx 541.6 \text{ miles/hr}$$

Step 5: Because both answers are in terms of miles/hr and both answers seem reasonable within the context of the problem, we conclude that the jet is gaining altitude at a rate of 95.5 miles/hr, while the shadow on the ground is moving at about 541.6 miles/hr.

Example

The sides of a cube increase at a rate of R cm$/$sec. When the sides have a length of 2 cm, what is the rate of change of the volume?

Example

Two boats leave a dock at the same time. One boat travels south at 30 miles/hr and the other travels east at 40 miles/hr. After half an hour, how fast is the distance between the boats increasing?

HW for Section 3.10

Do problems 5–12 all, 14–15, 17–18, 30–31.

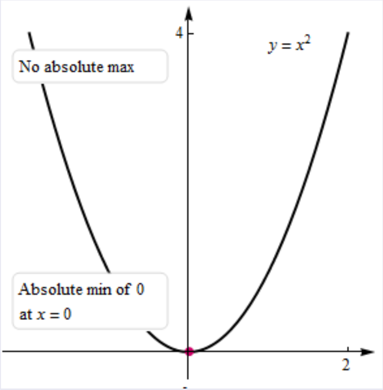
$\oint$ 4.1 Maxima and Minima

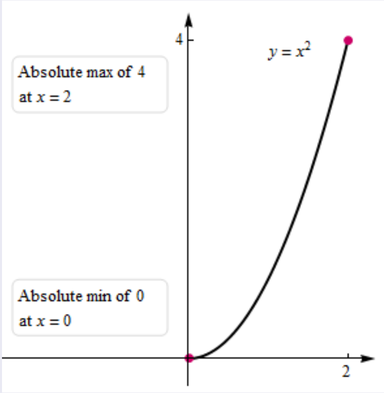
Definition

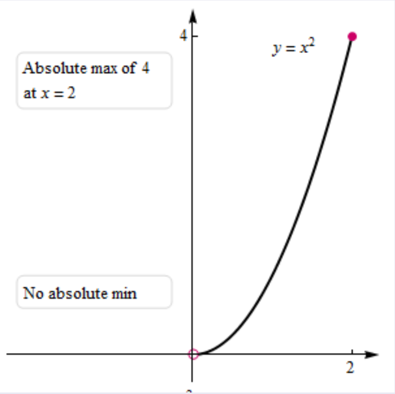
Let f be defined on an interval I containing c.

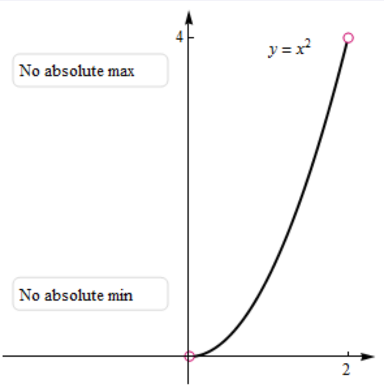
- f has an **absolute maximum** value on I at c means $f(c) \geq f(x)$ for every x in I.
- f has an **absolute minimum** value on I at c means $f(c) \leq f(x)$ for every x in I.

The existence and location of absolute extreme values depend on the function and the interval of interest:






Absolute max of 4 at $x = 2$
No absolute min
$y = x^2$
4
2


No absolute max
No absolute min
$y = x^2$
4
2

Extreme Value Theorem

Theorem (Extreme Value Theorem)

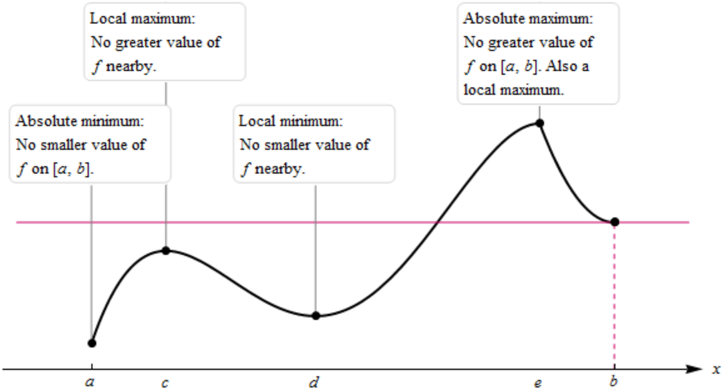
A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.

The EVT provides the criteria that ensures absolute extrema:

- the function must be continuous on the interval of interest;
- the interval of interest must be closed and bounded.

Local Maxima and Minima

Beyond absolute extrema, a graph may have a number of peaks and dips throughout its interval of interest:



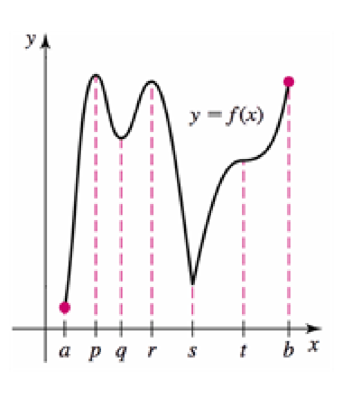
Definition

Suppose I is an interval on which f is defined and c is an interior point of I.

- If $f(c) \geq f(x)$ for all x in some open interval containing c, then $f(c)$ is a **local maximum** value of f.
- If $f(c) \leq f(x)$ for all x in some open interval containing c, then $f(c)$ is a **local minimum** value of f.

Exercise

Use the graph below to identify the points on the interval $[a, b]$ at which local and absolute extreme values occur.



Critical Points

Based on the previous graph, how is the derivative related to where the local extrema occur?

Local extrema occur where the derivative either does not exist or is equal to 0.

Definition

An interior point c of the domain of f at which $f'(c) = 0$ or $f'(c)$ fails to exist is called a **critical point** of f.

Local Extreme Point Theorem

Theorem (Local Extreme Point Theorem)

If f has a local minimum or maximum value at c and $f'(c)$ exists, then $f'(c) = 0$. **(Converse is not true!)**

It is possible for $f'(c) = 0$ or $f'(c)$ not to exist at a point, yet the point not be a local min or max. Therefore, critical points provide **candidates** for local extrema, but do not guarantee that the points are local extrema (see p. 227 immediately before Figure 4.9 for examples).

Locating Absolute Min and Max

Two facts help us in the search for absolute extrema:

- Absolute extrema in the interior of an interval are also local extrema, which occur at critical points of f.
- Absolute extrema may occur at the endpoints of f.

Procedure: Assume that the function f is continuous on $[a, b]$.

1. Locate the critical points c in (a, b), where $f'(c) = 0$ or $f'(c)$ does not exist. These points are **candidates** for absolute extrema.
2. Evaluate f at the critical points and at the endpoints of $[a, b]$.
3. Choose the largest and smallest values of f from Step 2 for the absolute max and min values, respectively.

NOTE: In this section, given an equation, we can identify critical points and absolute extrema, BUT NOT LOCAL EXTREMA. Techniques for locating local extrema come in later sections.

Exercise

Given $f(x) = (x+1)^{4/3}$ on $[-8, 8]$, determine the critical points and the absolute extreme values of f.

HW for Section 4.1

Do problems 11–25 odd, 31–45 odd (pp. 229–230 in textbook)