**Math 2554 Exam 1: Limits**
**Fri 19 Sep 2014**

Name: SOLUTIONS

# Calculus I
## Exam 1 (Chapter 2: Limits)

Please provide the following data:

Drill Instructor: ______________________________

Drill Time: ______________________________

Student ID or clicker #: ______________________________

**Exam Instructions:** Sit with your drill section, according to the map shown on the projector. You have 50 minutes to complete this exam. One 3 × 5 inch notecard, one side only, is allowed. No graphing calculators. No programmable calculators. No electronic devices except for the approved calculators (so no phones, iDevices, computers, etc). If you finish early then you may leave, UNLESS there are less than 5 minutes of class left. To prevent disruption, if you finish with less than 5 minutes of class remaining then please stay seated and quiet.

Your signature below indicates that you have read this page and agree to follow the Academic Honesty Policies of the University of Arkansas.

Signature: (1 pt) ____________________________________________

Good luck!

1. Fill in the blanks for the following limit laws.

(a) **Quotients of Functions**

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

provided (1 pt) $\lim_{x\to a} g(x) \neq 0$.

(1 pt) If $g$ is a polynomial, then why is it OK to just require $g(a) \neq 0$ (instead of the limit at $a$)?

Since $g$ is a polynomial, $\lim_{x\to a} g(x) = g(a)$.

(b) **Fractional Powers**

$$\lim_{x\to a} f(x)^{\frac{m}{n}} = \left(\lim_{x\to a} f(x)\right)^{\frac{m}{n}}$$

provided $m, n > 0$ are integers and $\frac{m}{n}$ is in lowest terms. If $n$ is even, then we also need (2 pts) $f(x) \geq 0$ for (1 pt) $x$ sufficiently close to $a$.

Rewrite the rule for one-sided limits:

$$\lim_{x\to a^+} f(x)^{\frac{m}{n}} = \text{(1 pt)} \left(\lim_{x\to a^+} f(x)\right)^{\frac{m}{n}}$$

provided $m, n > 0$ are integers and $\frac{m}{n}$ is in lowest terms. If $n$ is even, then we also need (2 pts) $f(x) \geq 0$ for (2 pts) $x$ sufficiently close to $a$ and $x > a$

$$\lim_{x\to a^-} f(x)^{\frac{m}{n}} = \text{(1 pt)} \left(\lim_{x\to a^-} f(x)\right)^{\frac{m}{n}}$$

provided $m, n > 0$ are integers and $\frac{m}{n}$ is in lowest terms. If $n$ is even, then we also need (2 pts) $f(x) \geq 0$ for (2 pts) $x$ sufficiently close to $a$ and $x < a$.

(c) **End Behavior and Asymptotes of Rational Functions**

Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function, and we write $p, q$ as

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_2 x^2 + a_1 x + a_0$$
$$q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_2 x^2 + b_1 x + b_0$$

with $a_m, b_n \neq 0$.

If $m < n$, then $\lim_{x\to\pm\infty} f(x) =$ (1 pt) $0$, and (2 pts) $y = 0$ is a horizontal asymptote for $f$.

If $m = n$, then $\lim_{x\to\pm\infty} f(x) =$ (1 pt) $\frac{a_m}{b_n}$, and (2 pts) $y = \frac{a_m}{b_n}$ is a horizontal asymptote of $f$.

If $m > n$, then $\lim_{x\to\pm\infty} f(x) =$ (1 pt) $\infty$ or (1 pt) $-\infty$.

2. For the following questions, if a one-sided limit is computed, your justification must involve determining the sign of the numerator and denominator for $x$-values sufficiently close to 0.

$$\text{Suppose } g(x) = \begin{cases} \dfrac{x-5}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Evaluate:

(a) (4 pts) $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \dfrac{\overset{-\text{ near } 0}{x-5}}{\underset{+\text{ near } 0}{x}}$

$= -\infty$

(b) (4 pts) $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \dfrac{\overset{-\text{ near } 0}{x-5}}{\underset{+\text{ near } 0}{x}}$

$= \infty$

(c) (3 pts) $\lim_{x\to 0} g(x)$ does not exist, because the one-sided limits are not equal (or finite)

(d) (1 pt) $g(0) = 0$

(2 pts) Does $g$ have a vertical asymptote at the line $x = 0$? Explain why or why not.

Yes, since $\lim_{x \to 0^+} g(x) = -\infty$ and since $\lim_{x \to 0^-} g(x) = \infty$

Now suppose $h(x) = \begin{cases} g(x) & x < 0 \\ 0 & x \geq 0 \end{cases}$.

Evaluate:

(a) (2 pts) $\lim_{x \to 0^+} h(x) = \lim_{x \to 0^+} 0 = 0$.

(b) (3 pts) $\lim_{x \to 0^-} h(x) = \lim_{x \to 0^+} g(x) = \infty$ (previous page)

(c) (2 pts) $\lim_{x \to 0} h(x)$ does not exist, because the one-sided limits are not equal

(d) (1 pt) $h(0) = 0$

(2 pts) Does $h$ have a vertical asymptote at the line $x = 0$? Explain why or why not.

Yes, since $\lim_{x \to 0^-} h(x) = \infty$.

3. (8 pts) Using Figure 1 as a guide, explain how the Squeeze Theorem can be used to compute

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right),$$

and then say what the limit is.

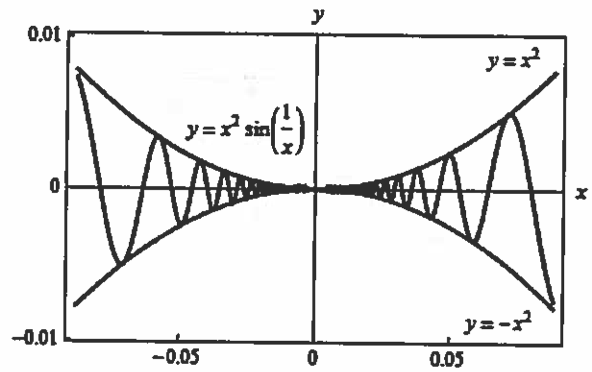


Figure 1: (Briggs, W. and Cochran, L. *Calculus: Early Transcendentals*)

From the picture, and since

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1,$$

we have the inequality

$$-x^2 \le x^2\sin\left(\frac{1}{x}\right) \le x^2$$ for all x close to zero.

Therefore

$$\underbrace{\lim_{x\to 0} -x^2}_{=0} \le \lim_{x\to 0} x^2\sin\left(\frac{1}{x}\right) \le \underbrace{\lim_{x\to 0} x^2}_{=0}.$$

By the Squeeze Theorem, we must have

$$\lim_{x\to 0} x^2\sin\left(\frac{1}{x}\right) = 0.$$

4. Given the graph of $f$ in the following figures, find the slope of the secant line that passes through $(0,0)$ and $(h, f(h))$, in terms of $h$. Then calculate the limit of that slope as $h \to 0^+$ and as $h \to 0^-$.

(a) (5 pts) $f(x) = x^{\frac{1}{3}}$

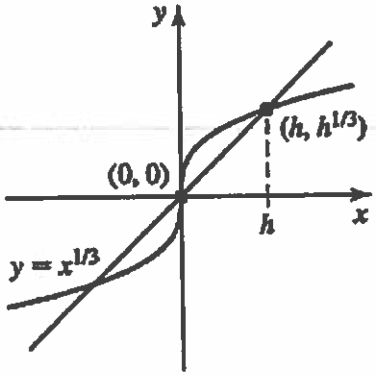


Figure 2: (Briggs, W. and Cochran, L. *Calculus: Early Transcendentals*)

secant: $\dfrac{f(h)-f(0)}{h-0} = \dfrac{h^{1/3}-0}{h} = \dfrac{h^{1/3}}{h} = h^{-2/3} = \dfrac{1}{h^{2/3}}$.

$\lim_{h\to 0^+} \dfrac{1}{h^{2/3}} = \infty$ ← pos

+ for $h>0$

$\lim_{h\to 0^-} \dfrac{1}{h^{2/3}} = \infty$ ← pos

pos (b/c a square)

(c) (4 pts) In both parts (a) and (b), what does your answer tell you about the tangent line to the curve at $(0,0)$? Why doesn't $f$ itself have a vertical asymptote in either of these cases?

The tangent has infinite slope, i.e., is vertical.

The reason f itself doesn't have a vertical asymptote at $x=0$ is because

$$\lim_{x\to 0} f(x) = f(0) = 0$$

for both (a) and (b).

5. (ChAlLeNgE pRoBlEm) Suppose $\lim_{x\to 1} f(x) = 4$. What is $\lim_{x\to -1} f(x^2)$?

When $x \to -1$, $x^2 \to 1$.

Let $y = x^2$. So $\lim_{x\to -1} f(x^2) = \lim_{x^2\to 1} f(x^2)$

$$= \lim_{y\to 1} f(y) = 4.$$

(b) (5 pts) $f(x) = x^{\frac{2}{3}}$

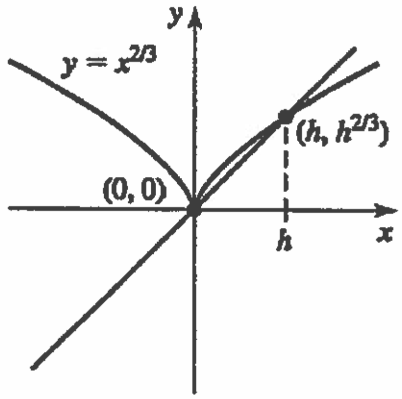


Figure 3: (Briggs, W. and Cochran, L. *Calculus: Early Transcendentals*)

secant:

$$\frac{f(h)-f(0)}{h-0} = \frac{h^{2/3}-0}{h-0} = \frac{1}{h^{1/3}}.$$

$$\lim_{h\to 0^+} \frac{1}{h^{1/3}} = \infty \qquad \lim_{h\to 0^-} \frac{1}{h^{1/3}} = -\infty$$

$+$

$+$ for $h>0$

$+$

$-$ for $h<0$

6. For each function $f(x)$, evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, and then identify any horizontal asymptotes. Next, find the vertical asymptotes. For each vertical asymptote $x = a$, evaluate $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$. Justify your answers.

(a) (7 pts) $f(x) = \dfrac{x^2 - 4x + 3}{x - 1} = \dfrac{(x-3)\cancel{(x-1)}}{\cancel{x-1}}$

$\lim_{x\to\infty} x-3 = \infty$

$\lim_{x\to-\infty} x-3 = -\infty$

no horizontal asymptotes.

There are no vertical asymptotes

(b) (7 pts) $f(x) = \dfrac{x^2 - 4}{x(x - 2)} = \dfrac{(x+2)(x-2)}{x(x-2)} = \dfrac{x+2}{x}$

$\lim_{x\to\infty} \dfrac{x+2}{x} = 1$ (rational function limit law)

$\lim_{x\to-\infty} \dfrac{x+2}{x} = 1$

means there is a vertical asymptote at $y=1$.

A vertical asymptote appears at $x=0$ (but not $x=2$):

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{x+2}{x} = -\infty$ (pos / neg)

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \dfrac{x+2}{x} = \infty$ (pos / pos)

7. (5 pts) Let $f(x) = x^2 + 5x - 3$. Evaluate, analytically:

$$\lim_{x \to 4} \frac{f(x) - f(4)}{x - 4}$$

$$= \lim_{x \to 4} \frac{(x^2 + 5x - \cancel{3}) - (4^2 + 5(4) - \cancel{3})}{x - 4}$$

$$= \lim_{x \to 4} \frac{x^2 + 5x - 36}{x - 4} = \lim_{x \to 4} \frac{(x+9)\cancel{(x-4)}}{\cancel{x-4}}$$

$$= \lim_{x \to 4} x + 9 = 13$$

(3 pts) Use your answer above to write the equation of the tangent line to $f(x)$ at $x = 4$.

Use point-slope form:

$$f(4) = 4^2 + 5(4) - 3 = 33$$

slope $= 13$

$$y - 33 = 13(x - 4)$$

8. (6 pts) Does the function

$$f(x) = 2x^5 - 8x^3 + 5x^2 + 3x - 5$$

cross the horizontal line $y = -4$ for some $x$ in the interval $[0,1]$? (Yes, it does.) Justify your answer, and in particular, mention any important theorems you use and why they apply in this situation.

If $-4$ is between $f(0)$ and $f(1)$ then since $f$ is continuous (b/c it's a polynomial) we can apply the Intermediate Value Theorem:

$$f(0) = 2(0)^5 - 8(0)^3 + 5(0)^2 + 3(0) - 5$$
$$= -5$$

$$f(1) = 2(1)^5 - 8(1)^3 + 5(1)^2 + 3(1) - 5$$
$$= 2 - 8 + \cancel{5} + 3 - \cancel{5}$$
$$= -3$$

Since $f(0) = -5 \leq -4 \leq -3 = f(1)$, $f$ must cross $y = -4$ at some $x \in [0,1]$.