Jacobian Determinant of a Transformation of Two Variables

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Jacobian Determinant of a Transformation of Three Variables

$$J(u,v,w)=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}.$$

Recall that by expanding about the first row,

$$\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}=a_{11}(a_{22}a_{33}-a_{23}a_{32})-a_{12}(a_{21}a_{33}-a_{23}a_{31})+a_{13}(a_{21}a_{32}-a_{22}a_{31}).$$

Scalar Line Integrals in $\mathbb{R}^2$

$$\int_C f\,ds=\int_a^b f(x(t),y(t))|\mathbf{r}'(t)|dt=\int_a^b f(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\,dt.$$

Scalar Line Integrals in $\mathbb{R}^3$

$$\int_C f\,ds=\int_a^b f(x(t),y(t),z(t))|\mathbf{r}'(t)|\,dt=\int_a^b f(x(t),y(t),z(t))\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt.$$

Green's Theorem—Circulation Form

The circulation form of Green's Theorem is also called the *tangential*, or *curl*, form.

$$\underbrace{\oint_C \mathbf{F}\cdot d\mathbf{r}}_{\text{circulation}}=\underbrace{\oint_C f\,dx+g\,dy}_{\text{circulation}}=\iint_R\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dA.$$

Area of a Plane Region by Line Integrals

Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

$$\oint_C x\,dy=-\oint_C y\,dx=\frac{1}{2}\oint_C (x\,dy-y\,dx).$$

Green's Theorem, Flux Form

The flux form of Green's Theorem is also called the *normal*, or *divergence*, form.

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds}_{\text{outward flux}}=\underbrace{\oint_C f\,dy-g\,dx}_{\text{outward flux}}=\iint_R\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\right)dA$$

Curl of a Vector Field

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f & g & h\end{vmatrix}\begin{array}{l}\leftarrow \text{Unit vectors}\\ \leftarrow \text{Components of } \nabla\\ \leftarrow \text{Components of } \mathbf{F}\end{array}$$

$$=\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)\mathbf{i}+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)\mathbf{j}+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\mathbf{k}.$$

Divergence of a Vector Field

$$\operatorname{div}\mathbf{F}=\nabla\cdot\mathbf{F}=\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}.$$

Different Forms of Line Integrals of Vector Fields

The line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ may be expressed in the following forms,

$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt=\int_a^b (f(t)x'(t)+g(t)y'(t)+h(t)z'(t))\,dt$$
$$=\int_C f\,dx+g\,dy+h\,dz$$
$$=\int_C \mathbf{F}\cdot d\mathbf{r}.$$

For line integrals in the plane,

$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt=\int_a^b (f(t)x'(t)+g(t)y'(t))\,dt=\int_C f\,dx+g\,dy=\int_C \mathbf{F}\cdot d\mathbf{r}.$$

Flux

The **flux** of the vector field **F** across C is

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds=\int_a^b (f(t)y'(t)-g(t)x'(t))\,dt,$$

where $\mathbf{n}=\mathbf{T}\times\mathbf{k}$ is the unit normal vector and **T** is the unit tangent vector consistent with the orientation.