MATH 2554 (Calculus I)

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Monday 16 March (Week 10)

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- Quiz #10 in class Tues 31 Mar on $\oint 4.4$
- Exam #3 Friday 3 April – expect up to $\oint 4.5$

CEA REQUEST

A student in this class requires a note-taker. If you are willing to upload your notes and plan to attend class on a REGULAR basis, please sign up via the CEA Online Services on the Center for Educational Access (CEA) website http://cea.uark.edu. On the CEA Online Services login screen, click on "Sign Up as a Note-taker". At the end of the semester you will receive verification of 48 community service hours OR a $50 gift card for providing class notes. All interested students are encouraged to sign up; preference may be given to volunteers seeking community service in an effort engage U of A students in community service opportunities. Please contact the Center for Educational Access at ceanotes@uark.edu if you have any questions.

$\oint$ 4.2 What Derivatives Tell Us

Definition

Suppose a function f is defined on an interval I.

- We say that f is **increasing** on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.
- We say that f is **decreasing** on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.

How is it related to the derivative?

Suppose f is continuous on an interval I and differentiable at every interior point of I.

- If $f'(x) > 0$ for all interior points of I, then f is increasing on I.
- If $f'(x) < 0$ for all interior points of I, then f is decreasing on I.

Example

Sketch a function that is continuous on $(-\infty, \infty)$ that has the following properties:

- $f'(-1)$ is undefined;
- $f'(x) > 0$ on $(-\infty, -1)$;
- $f'(x) < 0$ on $(-1, \infty)$.

Example

Find the intervals on which

$$f(x) = 3x^3 - 4x + 12$$

is increasing and decreasing.

First Derivative Test

Suppose that f is continuous on an interval that contains a critical point c and assume f is differentiable on an interval containing c, except perhaps at c itself.

- If f' changes sign from positive to negative as x increases through c, then f has a **local maximum** at c.
- If f' changes sign from negative to positive as x increases through c, then f has a **local minimum** at c.
- If f' does not change sign at c (from positive to negative or vice versa), then f has **no** local extreme value at c.

NOTE: The First Derivative Test does NOT test for increasing/decreasing, only local max/min. Use it on critical points.

Exercise

If $f(x) = 2x^3 + 3x^2 - 12x + 1$, identify the critical points on the interval $[-3, 4]$, and use the First Derivative Test to locate the local maximum and minimum values. What are the absolute max and min?

Absolute extremes on any interval

The Extreme Value Theorem (cf., Section 4.1) stated that we were guaranteed extreme values only on closed intervals.

However: Suppose f is continuous on an interval I that contains only one local extremum at $(x =)c$.

- If it is a local minimum, then $f(c)$ is the absolute minimum of f on I.
- If it is a local maximum, then $f(c)$ is the absolute maximum of f on I.

Derivative of the derivative tells us:

Just as the first derivative f' told us whether the function f was increasing or decreasing, the second derivative f'' also tells us whether f' is increasing or decreasing.

Definition

Let f be differentiable on an open interval I.

- If f' is increasing on I, then f is **concave up** on I.
- If f' is decreasing on I, then f is **concave down** on I.

Definition

If f is continuous at c and f changes concavity at c (from up to down, or vice versa), then f has an **inflection point** at c.

Test for Concavity

Suppose that f'' exists on an interval I.

- If $f'' > 0$ on I, then f is concave up on I.
- If $f'' < 0$ on I, then f is concave down on I.
- If c is a point of I at which $f''(c) = 0$ and f'' changes signs at c, then f has an inflection point at c.

Second Derivative Test

Suppose that f'' is continuous on an open interval containing c with $f'(c) = 0$.

- If $f''(c) > 0$, then f has a local minimum at c.
- If $f''(c) < 0$, then f has a local maximum at c.
- If $f''(c) = 0$, then the test is inconclusive.

See the Recap of Derivative Properties (Figure 4.36 on p. 242) for a summary.

HW from Section 4.2

Do problems 11–35 odd, 47–61 odd, 67 (pp. 243–245 in textbook)

Wednesday 18 March (Week 10)

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$\oint$ 4.2, cont.

Exercise

Let $f(x) = 2x^3 - 6x^2 - 18x$.

1. Determine the intervals on which f is concave up or down, and identify any inflection points.
2. Locate the critical points, and use the Second Derivative Test to determine whether they correspond to local minima or maxima, or whether the test is inconclusive.

$\oint$ 4.3 Graphing Functions

Graphing Guidelines:

1. Identify the domain or interval of interest.
2. Exploit symmetry.
3. Find the first and second derivatives.
4. Find critical points and possible inflection points.
5. Find intervals on which the function is increasing or decreasing, and concave up/down.
6. Identify extreme values and inflection points.
7. Locate vertical/horizontal asymptotes and determine end behavior.
8. Find the intercepts.
9. Choose an appropriate graphing window and make a graph.

Exercise

According to the graphing guidelines, sketch a graph of

$$f(x) = \frac{x^2}{x^2 - 4}.$$

HW from Section 4.3

Do problems 7, 9, 13–19 odd, 23, 29, 43, 45 (pp. 254–255 in textbook)

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$\oint 4.4$ Optimization Problems

In many scenarios, it is important to find a maximum or minimum value under given constraints. Given our use of derivatives from the previous sections, optimization problems follow directly from what we have studied.

Question

Given all nonnegative real numbers x and y between 0 and 50 such that their sum is 50 (i.e., $x + y = 50$), which pair has the maximum product?

This is a basic optimization problem. In this problem, we are given a **constraint** ($x + y = 50$) and asked to maximize an **objective function** ($A = xy$).

The first step is to express the objective function $A = xy$ in terms of a single variable by using the constraint:

$$y = 50 - x \implies A(x) = x(50 - x).$$

To maximize A, we find the critical points:

$$A'(x) = 50 - 2x \text{ which has a critical point at } x = 25.$$

Since $A(x)$ has domain $[0, 50]$, to maximize A we evaluate A at the endpoints of the domain and at the critical point:

$$A(0) = A(50) = 0 \text{ and } A(25) = 625.$$

So 625 is the maximum value of A and A is maximized when $x = 25$ (which means $y = 25$).

Essential Feature of Optimization Problems

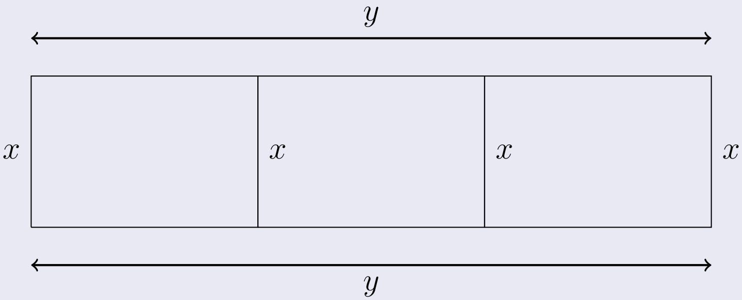
All optimization problems take the following form:

What is the maximum (or minimum) value of an objective function subject to the given constraint(s)?

Most optimization problems have the same basic structure as the previous problem: An objective function (possibly with several variables and/or constraints) with methods of calculus used to find the maximum/minimum values.

Exercise

Suppose you wish to build a rectangular pen with two interior parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen?



By the picture, $2y + 4x = 500$ which implies $y = -2x + 250$. We are maximizing $A = xy$. So write

$$A(x) = x(-2x + 250) = -2x^2 + 250x.$$

Taking the derivative, $A'(x) = -4x + 250 = 0$, A has a critical point at $x = 62.5$.

From the picture, since we have 500 ft of fencing available we must have $0 \leq x \leq 125$. To find the max we must examine the points $x = 0, 62.5, 125$:

$$A(0) = A(125) = 0 \text{ and } A(62.5) = 7812.5$$

We see that

$$\boxed{\text{the maximum area is } 7812.5 \text{ ft}^2.}$$

The pen's dimensions (answer the question!) are $\boxed{x = 62.5 \text{ ft}}$ and

$$\boxed{y = -2(62.5) + 250 = 125 \text{ ft.}}$$

Guidelines for Optimization Problems

1. READ THE PROBLEM carefully, identify the variables, and organize the given information with a picture.
2. Identify the objective function (i.e., the function to be optimized). Write it in terms of the variables of the problem.
3. Identify the constraint(s). Write them in terms of the variables of the problem.
4. Use the constraint(s) to eliminate all but one independent variable of the objective function.
5. With the objective function expressed in terms of a single variable, find the interval of interest for that variable.
6. Use methods of calculus to find the absolute maximum or minimum value of the objective function on the interval of interest. If necessary, check the endpoints.

Exercise

An open rectangular box with square base is to be made from 48 ft^2 of material. What dimensions will result in a box with the largest possible volume?

Exercise

Find the dimensions of the rectangle of largest area which can be inscribed in the closed region bounded by the x-axis, y-axis, and the graph of $y = 8 - x^3$.

HW from Section 4.4

Do problems 5–13 all, 18–20 all, 26 (pp. 261–263 in textbook)