Take-Home Quiz #5

Math 2574 (Cal III)
Spring 2017

SOLUTIONS

1. (a) $x = t^2 - 2$, $y = t^3 - t$ $\longrightarrow$ (D) (not periodic)

(b) $x = \cos(t + \sin 50t)$, $y = \sin(t + \cos 50t)$ $\longrightarrow$ (B) (both coordinates are bounded by 1)

(c) $x = t + \cos 2t$, $y = t - \sin 4t$ $\longrightarrow$ (A) (increasing as t increases)

(d) $x = 2\cos t + \cos 20t$, $y = 2\sin t + \sin 20t$ $\longrightarrow$ (C) (periodic and between circles of radius 1 and 3 centered at the origin)

2. (a) $r = 2 + 2\sin\theta$

$$\Rightarrow x = (2 + 2\sin\theta)\cos\theta = 2\cos\theta + 2\sin\theta\cos\theta$$

$$y = (2 + 2\sin\theta)\sin\theta = 2\sin\theta + 2\sin^2\theta$$

Slope of the tangent line is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\cos\theta + 4\sin\theta\cos\theta}{-2\sin\theta + 2\cos^2\theta - 2\sin^2\theta}$$

$\longrightarrow$

$$= \frac{\cos\theta(1+2\sin\theta)}{-\sin\theta + \cos 2\theta} \leftarrow \text{trig id}$$

Horizontal tangent: $\cos\theta(1+2\sin\theta)=0$

$\Rightarrow \cos\theta = 0 \Rightarrow \theta = (\text{odd }\#)\frac{\pi}{2}$

or

$1+2\sin\theta = 0$

$\sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$ ± multiples of 2π

$r = 2+2\sin\left((\text{odd }\#)\frac{\pi}{2}\right)$

$= 0, 4$

$\Rightarrow (x,y) = \boxed{\cancel{(0,0)}, (4,0)}$

$r = 2+2\sin\left(\frac{7\pi}{6} \text{ or } \frac{11\pi}{6} \pm \text{multiples of } 2\pi\right)$

$= 2+2\left(-\frac{1}{2}\right) = 1 \Rightarrow (x,y) = \boxed{\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)}$

Vertical tangent: $-\sin\theta + \cos 2\theta = 0$

$\Rightarrow \cos 2\theta = \sin\theta$

trig id

$\sin\left(2\theta + \frac{\pi}{2}\right) = \sin\theta$

$2\theta + \frac{\pi}{2} = \theta \Rightarrow \theta = -\frac{\pi}{2}$

$r = 2+2\sin\left(-\frac{\pi}{2}\right)$

$= 2+2(-1) = 0$

$\Rightarrow (x,y) = \cancel{(0,0)}$

(b) $r = 3 + 6\sin\theta$

$\Rightarrow x = 3\cos\theta + 6\sin\theta\cos\theta \qquad \frac{dx}{d\theta} = -3\sin\theta + 6\cos^2\theta - 6\sin^2\theta$

$y = 3\sin\theta + 6\sin^2\theta \qquad \frac{dy}{d\theta} = 3\cos\theta + 12\sin\theta\cos\theta$

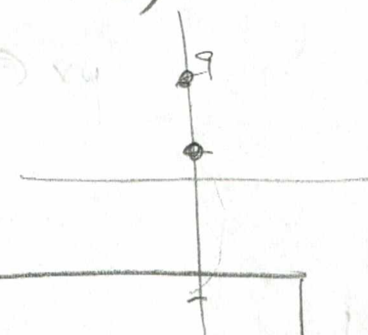
$$\frac{dy}{dx} = \frac{\cancel{3}\cos\theta(1+4\sin\theta)}{-\cancel{3}(\sin\theta - 2\cos 2\theta)}$$

Horizontal:

$\cos\theta = 0 \Rightarrow \theta = (\text{odd }\#)\frac{\pi}{2} \Rightarrow r = 3 + 6\sin\left((\text{odd}\#)\frac{\pi}{2}\right)$

$= -3, 9$

$\theta = -\frac{\pi}{2} \qquad \theta = \frac{\pi}{2}$

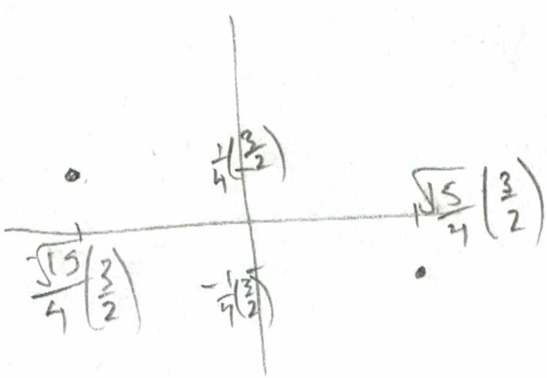


$1 + 4\sin\theta = 0$

$\sin\theta = \frac{-1}{4}$

$\Rightarrow r = 3 + \cancel{6}^{3}\left(\frac{-1}{\cancel{4}_{2}}\right) = \frac{3}{2}$

$$\Rightarrow \boxed{(x,y) = (0,3), (0,9), \left(\frac{3\sqrt{15}}{8}, \frac{-3}{8}\right), \left(\frac{-3\sqrt{15}}{8}, \frac{3}{8}\right)}$$



Vertical:

$\sin\theta - 2\cos 2\theta = 0$

$\Rightarrow \theta \approx 0.635, 2.507, -1.003, -2.139 \pm 2\pi$ (graphing calculator)

$r \approx 3 + 6(0.593, -0.843)$

$\approx 6.559, -2.059$

Using $x = r\cos\theta$, $y = r\sin\theta$

$$\boxed{\Rightarrow (x,y) \approx (5.268, 3.55), \quad (-1.107, 1.736)}$$

(c) $r = \sec\theta$

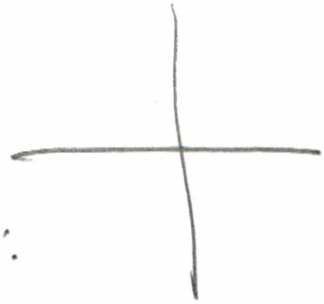
$\Rightarrow x = \sec\theta\cos\theta = 1 \quad \frac{dx}{d\theta} = 0$

$y = \sec\theta\sin\theta = \tan\theta \quad \frac{dy}{d\theta} = \sec^2\theta = \frac{1}{\cos^2\theta}$

$x=1$: no horizontal tangent, vertical tangent everywhere

3. Points of possible discontinuity:

$\{(x,y) \mid xy = 0\}$ = x-axis and y-axis



Use the definition of continuity:

① f is defined at $(x_0,0)$ and $(0,y_0)$ ✓

$f(x_0,0) = f(0,y_0) = a$ ✓

② $\lim\limits_{(x,y)\to(x_0,0)} f(x,y)$ and $\lim\limits_{(x,y)\to(0,y_0)} f(x,y)$ exist:

$$\lim_{(x,y)\to(x_0,0)} \frac{1+2xy-\cos xy}{xy}$$

Let $t = xy$. Then as $(x,y)\to(x_0,0)$ or $(0,y_0)$, $t \to 0$

Then $\lim\limits_{t\to 0} \dfrac{1+2t-\cos t}{t}$

$= \lim\limits_{t\to 0} \dfrac{1+2+\sin t}{1}$ (L'Hôpital's Rule)

$\lim\limits_{t\to 0}(3+\sin t) = 3$

③ $f(x_0,0) = f(0,y_0) = \lim\limits_{(x,y)\to(x_0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,y_0)} f(x,y)$

$\Rightarrow \boxed{a=3}$

4. Average: $\bar{f} = \frac{1}{\text{area}(R)} \iint_R f(x,y)\,dA$

$$= \frac{1}{a^2}\int_0^a \int_0^a (x+y-8)\,dx\,dy$$

$$= \frac{1}{a^2}\int_0^a \left(\frac{x^2}{2}+xy-8x\right)\Big|_0^a dy$$ terms vanish

$$= \frac{1}{a^2}\int_0^a \left(\frac{a^2}{2}+ay-8a\right)dy$$

$$= \frac{1}{a^2}\left(\frac{a^2}{2}y+\frac{ay^2}{2}-8ay\right)\Big|_0^a$$

$$= \frac{1}{\cancel{a^2}}\left(\frac{a^3}{2}+\frac{a^3}{2}-8\cancel{a^2}\right)$$

$$= a-8=0$$

$$\Rightarrow \boxed{a=8}$$

5. (a) $r=-1+\sin\theta \longrightarrow$ (A) $(|-1|=|1| \Rightarrow$ cardiod)

(b) $r=2+\sin\theta \longrightarrow$ (B) (oval/dimple)

(c) $r=1+2\sin\theta \longrightarrow$ (E) (inner loop)

(d) $r=-1+2\cos\theta \longrightarrow$ (C) (inner loop and $(r,\theta)=(1,0)$ is on the graph)

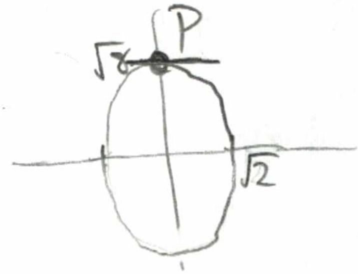
(e) $r=1-2\cos\theta \longrightarrow$ (D) (reflection of (d))

(f) $r=1+\frac{2}{3}\sin\theta \longrightarrow$ (F) (dimple)

6. Slope of tangent line to $f(x,y)=\frac{1}{\sqrt{2}}$ at $P=(0,\sqrt{8})$

$$\sqrt{1-\frac{x^2}{4}-\frac{y^2}{16}}=\frac{1}{\sqrt{2}}$$

$$1-\frac{x^2}{4}-\frac{y^2}{16}=\frac{1}{2} \Rightarrow \frac{1}{2}=\frac{x^2}{4}+\frac{y^2}{16}$$



Implict differentiation:

$$1=\frac{x^2}{2}+\frac{y^2}{8}$$

$$0=\frac{1}{2}(2x)+\frac{1}{4}y\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{-x}{\frac{1}{4}y} \longrightarrow \left.\frac{dy}{dx}\right|_{(x,y)=(0,\sqrt{8})}=\frac{-0}{\frac{1}{4}\sqrt{8}}=\boxed{0}$$

Direction of tangent line: Parametrize the ellipse

$$\vec{r}(t)=\langle\sqrt{2}\cos t,\sqrt{8}\sin t\rangle \quad 0\le t\le\pi$$

$$\Rightarrow \vec{r}'(t)=\langle-\sqrt{2}\sin t,\sqrt{8}\cos t\rangle$$

at P, $t=\frac{\pi}{2} \Rightarrow \vec{r}'\left(\frac{\pi}{2}\right)=\langle-\sqrt{2},0\rangle$

Gradient: $\nabla f=\left\langle\frac{1}{2}\left(1-\frac{x^2}{4}-\frac{y^2}{16}\right)^{-1/2}\left(-\frac{2x}{4}\right),\frac{1}{2}\left(1-\frac{x^2}{4}-\frac{y^2}{16}\right)^{-1/2}\left(-\frac{2y}{16}\right)\right\rangle$

$$=\left\langle\frac{-x}{4f(x,y)},\frac{-y}{16f(x,y)}\right\rangle$$

Check orthogonality (at P): $\langle-\sqrt{2},0\rangle\cdot\left\langle\frac{-0}{4f(0,\sqrt{8})},\frac{-\sqrt{8}}{16f(0,\sqrt{8})}\right\rangle=0$ ✓

7. Write $z = f(x,y) = 2 + 2x^2 + \frac{y^2}{2}$

Normal direction to the surface is

$\langle f_x, f_y, 1\rangle = \langle 4x, y, 1\rangle$

$\Rightarrow \vec{n} = \langle f_x(-\tfrac{1}{2}, 1), f_y(-\tfrac{1}{2}, 1), 1\rangle = \langle -2, 1, 1\rangle$

Verify $(-\frac{1}{2}, 1, 3)$ is on the surface:

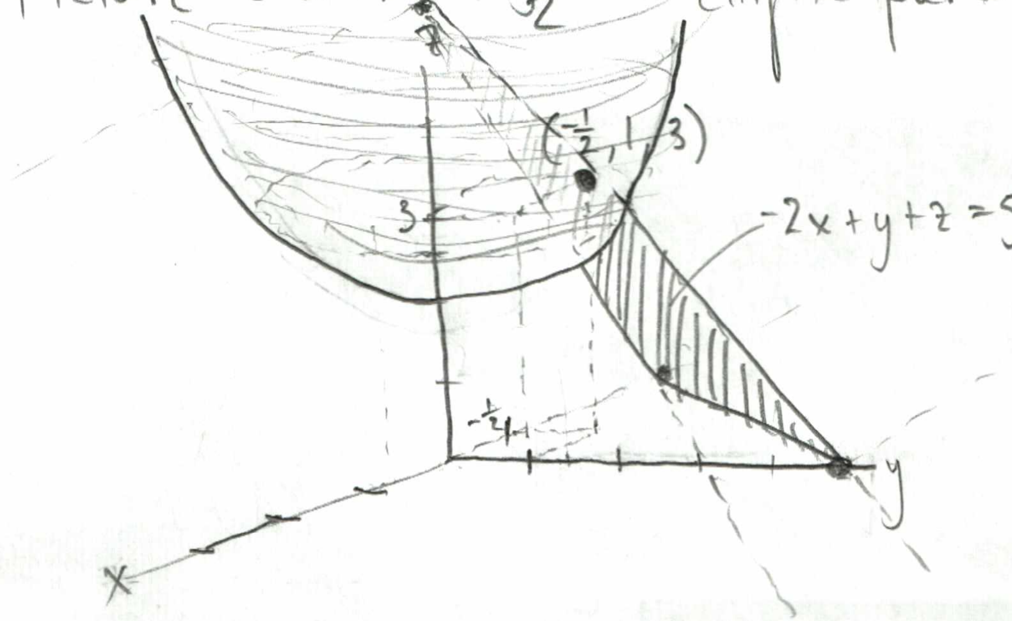
$$2 + 2\left(\tfrac{-1}{2}\right)^2 + \frac{1^2}{2} = 2 + \frac{1}{2} + \frac{1}{2} = 3 \checkmark$$

Tangent plane:

$$\langle -2, 1, 1\rangle \cdot \langle x + \tfrac{1}{2}, y - 1, z - 3\rangle = 0$$

$$\boxed{\begin{array}{c} -2(x + \tfrac{1}{2}) + (y - 1) + (z - 3) = 0 \\ \text{or} \\ -2x + y + z = 5 \end{array}}$$

Picture: $z = 2 + 2x^2 + \frac{y^2}{2}$ ← elliptic paraboloid shift up by 2



8. $V = \frac{\pi r^2 h}{3} \Rightarrow L(r,h) = \frac{2}{3}\pi ab(r-a) + \frac{\pi}{3}a^2(h-b) + \frac{\pi a^2 b}{3}$

is the linear approximation at (a,b).

$\Rightarrow \Delta L = \frac{2\pi}{3}ab(r-a) + \frac{\pi}{3}a^2(h-b)$

(a) $a = 6.5,\ b = 4.20$

$\Delta L = \frac{\pi}{3}(6.5)\left(2(4.20)(0.1) + (6.5)(-0.05)\right) \boxed{\approx 3.505}$

(b) $a = 5.40,\ b = 12.0$

$\Delta L = \frac{\pi}{3}(5.40)\left(2(12.0)(-0.03) + (5.40)(-0.04)\right) \boxed{\approx -5.293}$

9. (a) $1 \le r \le 2 \Rightarrow 1 \le \rho \sin\varphi \le 2$

$\csc\varphi \le \rho \le 2\csc\varphi$

$$\text{Volume} = \int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_{\csc\varphi}^{2\csc\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/6}^{\pi/3} \sin\varphi \left.\frac{\rho^3}{3}\right|_{\csc\varphi}^{2\csc\varphi} d\varphi \, d\theta = \int_0^{2\pi}\int_{\pi/6}^{\pi/3} \sin\varphi \underbrace{\left(\frac{8\csc^3\varphi}{3} - \frac{\csc^3\varphi}{3}\right)}_{\frac{7}{3}\csc^3\varphi} d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/6}^{\pi/3} \frac{7}{3}\csc^2\varphi \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} -\frac{7}{3}\cot\varphi \Big|_{\pi/6}^{\pi/3} d\theta = \int_0^{2\pi} -\frac{7}{3}\underbrace{\left(\frac{1}{\sqrt{3}} - 2\right)}_{\frac{1-2\sqrt{3}}{\sqrt{3}}} d\theta$$

$$= \int_0^{2\pi} \frac{-1+14\sqrt{3}}{3\sqrt{3}}\, d\theta$$

$$= \frac{-1+14\sqrt{3}}{3\sqrt{3}} \cdot 2\pi \; \boxed{\approx 3.505}$$

(b) $z = \sqrt{x^2+y^2}$

$\Rightarrow \rho\cos\varphi = \sqrt{\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta}$

$= \rho\sin\varphi$

$\rightarrow \tan\varphi = 1 \rightarrow \varphi = \frac{\pi}{4}$

$1 \le z \le 2$

$1 \le \rho\cos\varphi \le 2 \Rightarrow \sec\varphi \le \rho \le 2\sec\varphi$

Volume: $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\varphi}^{2\sec\varphi} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$

$$= \int_0^{2\pi}\int_0^{\pi/4} \sin\varphi\left(\frac{8}{3}\sec^3\varphi - \frac{1}{3}\sec^3\varphi\right) d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{7}{3}\tan\varphi\sec^2\varphi\, d\varphi\, d\theta \quad \left(\begin{array}{l}\text{Use } u = \tan\varphi \\ du = \sec^2\varphi\, d\varphi\end{array}\right)$$

$\longrightarrow$

$$= \int_0^{2\pi} \frac{7}{3} \int_{\tan 0}^{\tan \frac{\pi}{4}} u \, du \, d\theta = \frac{7}{3} \int_0^{2\pi} \frac{u^2}{2} \Bigg|_0^{\frac{\sqrt{2}}{2}} d\theta$$

$$= \frac{7}{3} \left(\frac{\left(\frac{\sqrt{2}}{2}\right)^2}{\cancel{2}} \right) \cdot \cancel{2}\pi = \boxed{\frac{7}{6}\pi}$$