MATH 2554 (Calculus I)
Spring 2016

Name: SOLUTIONS

Fri 8 Apr 2016

Exam 3: Applications and Story Problems (§3.10-4.6)

Exam Instructions: You have 50 minutes to complete this exam. Justification is required for all problems. Notation matters! You will also be penalized for missing units and rounding errors. No electronic devices (phones, iDevices, computers, etc) except for a **basic scientific calculator**. On story problems, round to one decimal place.

In addition, please provide the following data:

Drill Instructor: ____________________

Drill Time: ____________________

Your signature below indicates that you have read this page and agree to follow the Academic Honesty Policies of the University of Arkansas.

Signature: **(1 pt)** ____________________

Good luck!

1. (a) **(5 pts)** Explain why Rolle's Theorem cannot be applied to the function $f(x) = |x|$ on the interval $[-a, a]$ for any $a > 0$.

f(x) is not smooth at x=0, which is in the interval (−a,a).

(b) **(3 pts)** Is there any interval where Rolle's Theorem applies to $f(x)$? Why or why not?

No – for any interval [a,b] not containing 0, either a,b<0 or a,b>0. In either case, f(a) ≠ f(b).

(c) **(3 pts)** Is there any interval where the Mean Value Theorem applies to $f(x)$? Why or why not?

Yes. As long as [a,b] does not contain 0, f is continuous on [a,b] and smooth on (a,b).

2. **(12 pts)** Suppose you own a tour bus and you book groups of 20 to 70 people for a day tour. The cost per person is \$30 minus 25¢ for every ticket sold. If gas and other miscellaneous costs are \$200, how many tickets should you sell to maximize your profit?

x = # of tickets sold domain = [20,70]

Profit: $p(x) = \underbrace{(30 - 0.25x)}_{\text{ticket price}} \underbrace{x}_{\text{\# of tickets}} - \underbrace{200}_{\text{costs}}$

To maximize, write

$$p(x) = 30x - 0.25x^2 - 200$$

$$p'(x) = 30 - 0.5x = 0$$

$$\Rightarrow x = \boxed{60 \text{ tickets}}$$

$p(20) =$

$p(60) =$

$p(70) =$

$L(x)=f(a)+f'(a)(x-a)$

3. (a) **(3 pts)** Write the equations for the linear approximation to $g(x) = \sin x$ and $h(x) = \cos x$ at $x = 0$.

$\sin x \approx \sin(0) + \cos(0)(x-0) = x$

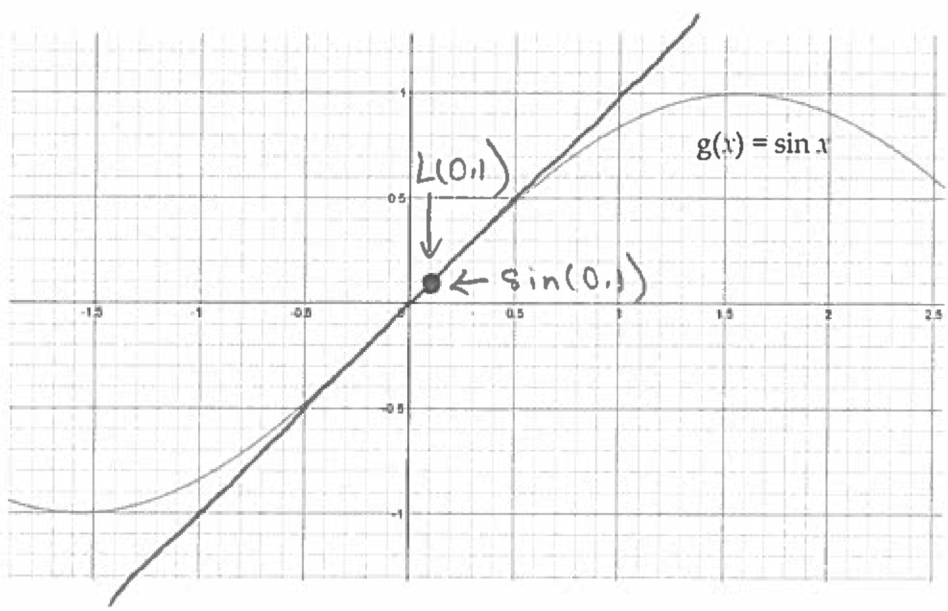
$\cos x \approx \cos(0) - \cancel{\sin(0)(x-0)} = 1$

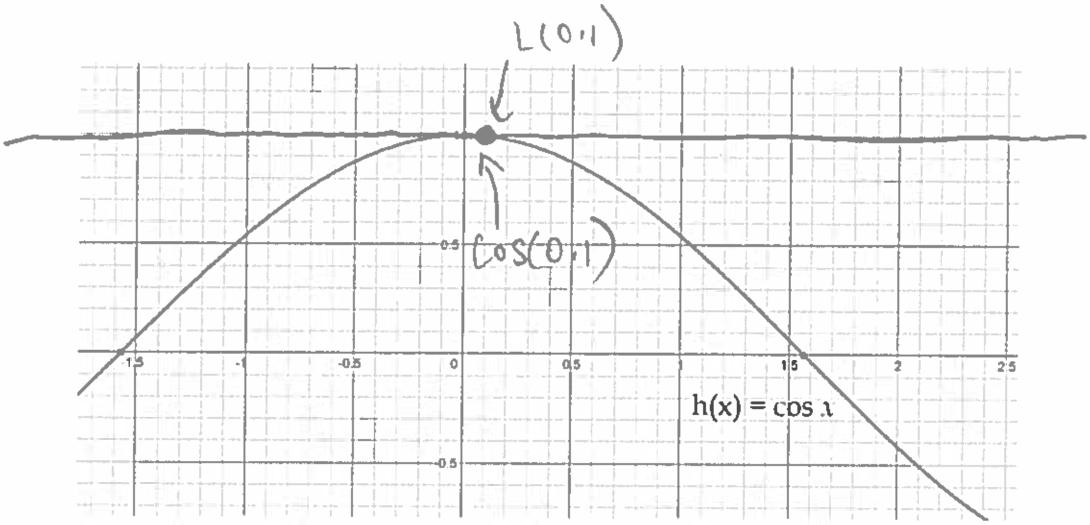
(b) **(3pts)** Use your answers to (a) to approximate $\sin(0.1)$ and $\cos(0.1)$.

$\sin(0.1) \approx 0.1$

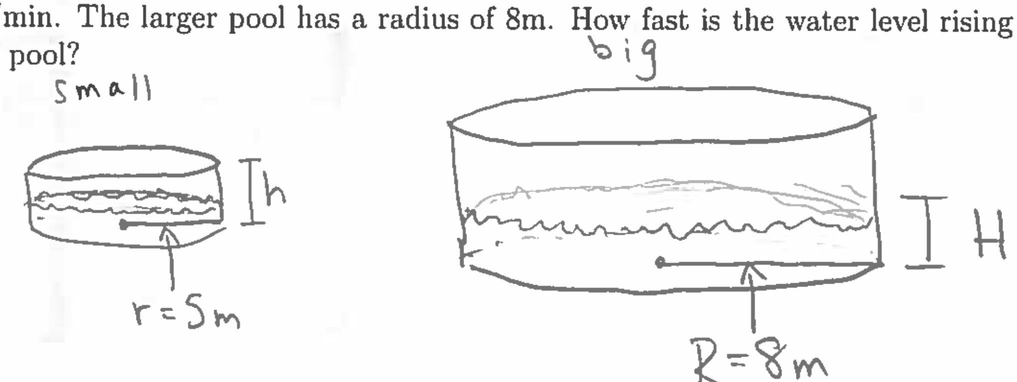
$\cos(0.1) \approx 1$

(c) **(5 pts)** The following are the respective graphs of $\sin x$ and $\cos x$, drawn at the website `desmos.com/calculator`. On each graph, draw your tangent line. Label both your approximations from (b) and the exact values $\sin(0.1)$ and $\cos(0.1)$.





4. **(12 pts)** Two cylindrical swimming pools are being filled simultaneously at the same rate (in m^3/min). The smaller pool has a radius of 5m, and the water level rises at a rate of 0.5m/min. The larger pool has a radius of 8m. How fast is the water level rising in the larger pool?



Know: $\frac{dh}{dt} = 0.5\ m/min$

WTF: $\frac{dH}{dt}$

constants

$V_{\text{small pool}} = \pi r^2 h$ $\qquad V_{\text{big pool}} = \pi R^2 H$

$$\frac{dV_{small}}{dt} = \pi r^2 \frac{dh}{dt} = \frac{dV_{big}}{dt} = \pi R^2 \frac{dH}{dt}$$

$$\Rightarrow \frac{dH}{dt} = \frac{\cancel{\pi} r^2 \frac{dh}{dt}}{\cancel{\pi} R^2} = \frac{(5m)^2}{(8m)^2}(0.5\ m/min) \approx \boxed{0.2\ m/min}$$

5. The goal of this problem is to produce a graph of the function

$$f(x) = \tan^{-1} x^3$$

from scratch.

(a) **(2 pts)** Find the domain for $f(x)$.

$(-\infty, \infty)$

(b) **(2pts)** Is $f(x)$ even, odd, or neither? You must justify your answer.

$f(-x) = \tan^{-1}((-x)^3) = \tan^{-1}(-x^3)$ b/c x^3 is odd

$= -\tan^{-1}(x^3)$ b/c $\tan x$ is odd $\Rightarrow \tan^{-1} x$ is odd

odd

(c) **(4 pts)** Find $f'(x)$ and $f''(x)$. You are not required to simplify.

$f'(x) = \dfrac{1}{1+(x^3)^2} \cdot 3x^2 = \boxed{\dfrac{3x^2}{1+x^6}}$

$6x \quad + 6x^7 \quad - 18x^7$

$f''(x) = \dfrac{(1+x^6)(6x) - 3x^2(6x^5)}{(1+x^6)^2}$

$= \dfrac{6x - 12x^7}{(1+x^6)^2} = \boxed{\dfrac{6x(1-2x^6)}{(1+x^6)^2}}$

(d) **(3 pts)** Find the critical points. If there are none, then say why.

$f'(x) = \dfrac{3x^2}{1+x^6} = 0 \Rightarrow \boxed{x=0}$

(f' is defined everywhere)

(e) **(3 pts)** Find the possible inflection points. If there are none, then say why.

$f''(x) = \dfrac{6x(1-2x^6)}{(1+x^6)^2} = 0 \Rightarrow \boxed{x=0}$

or $1-2x^6=0$

$\Rightarrow \boxed{x = \pm\sqrt[6]{\frac{1}{2}}}$

(f) **(3 pts)** What are the intervals where $f(x)$ is increasing? What are the intervals where $f(x)$ is decreasing?

$f'(x) > 0$ when $3x^2 > 0$, since the denominator is positive.

increasing: $(-\infty, 0), (0, \infty)$ (since $x=0$ is a CP)

decreasing: never

(g) **(3 pts)** What are the intervals where $f(x)$ is concave up? What are the intervals where $f(x)$ is concave down?

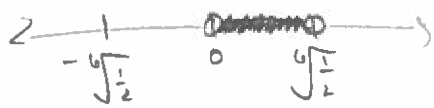
$f''(x) > 0$ when $6x(1-2x^6) > 0$, since the denominator is always positive.

Either $x > 0$ and $1-2x^6 > 0$

$-2x^6 > -1$

$x^6 < \frac{-1}{-2}$

$\Rightarrow |x| < \sqrt[6]{\frac{1}{2}}$



or

$x < 0$ and $|x| > \sqrt[6]{\frac{1}{2}}$

Concave up: $(-\infty, -\sqrt[6]{\frac{1}{2}}), (0, \sqrt[6]{\frac{1}{2}})$

concave down: $(-\sqrt[6]{\frac{1}{2}}, 0), (\sqrt[6]{\frac{1}{2}}, \infty)$

(h) **(4 pts)** Find the local extrema and inflection points. You must justify your answers. If there are no extrema or inflection points you should also say why.

Since f' is never negative, there are no local extrema.

The inflection points are at $x = 0, \pm\sqrt[6]{\frac{1}{2}}$, since by part (g), there are points where f'' changes sign.

(i) **(4 pts)** Use all the information above, plus the following facts, to draw a well-labeled graph of $f(x)$. Your picture should be consistent with your answers to (a)-(h).

- $f(x)$ has no vertical asymptotes, but it has horizontal asymptotes at $y = \pm\frac{\pi}{2}$.
- $f(x)$ intersects the origin.