DISCRETE MATHEMATICS

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Sets and Logic

Definition. *A* **set** *is a collection of objects called elements or members; order is not taken into account.*

Example.
$$A = \{a, b, c, d\}$$
is a set. Its elements are $a, b, c,$ and d.

Although the definition for set is very generic and vague, there are many less obvious examples of sets:

Example.

Definition. *Suppose A is a set. The non-negative integer*
$$|A| = \textit{number of elements in } A$$
is called the **cardinality** *of A.*

Definition. *The set with no elements is called the* **empty set** *(also the* **null set***, or the* **void set***) and is denoted $\emptyset$, or $\{\}$.*

Definition. *Two sets X and Y are* **equal** *means X and Y have the same cardinality. We write $X = Y$.*

Definition. *Suppose X, Y are sets. X is a* **subset** *of Y means every element of X is an element of Y. We write $X \subseteq Y$.*

Definition. *The set of all subsets of a set X is called the* **power set** *of X, denoted $\mathcal{P}(X)$, or 2^X.*

Definition. *Suppose $X \subseteq Y$. X is a* **proper subset** *of Y means, in addition, that X does not equal Y. We write $X \subset Y$. Note, some authors write $X \subsetneq Y$ to emphasize non-equality.*

Definition. *Let X, Y denote sets. The set*
$$X \cup Y = \{x \mid x \in X \textit{ or } x \in Y\}$$
is called the **union** *of X and Y.*

Definition. *Let X, Y denote sets. The set*

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$$

is called the **intersection** *of X and Y.*

Definition. *Let X, Y denote sets. The set*

$$X \setminus Y = \{x \mid x \in X \text{ and } x \notin Y\}$$

is called the **difference** *or* **relative complement**. *Some authors use $X - Y$.*

Definition. *The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of* **irrational numbers**.

Definition. *Sets X and Y are* **disjoint** *means $X \cap Y = \emptyset$.*

Definition. *A collection of sets $\mathcal{S}$ is* **pairwise disjoint** *means any two distinct sets in $\mathcal{S}$ are disjoint.*

Definition. *A* **universal set** *or* **universe** *is a set, usually inferred via context, whose subsets are those we are considering.*

Definition. *Given a universal set U with $X \subseteq U$, the set*

$$\overline{X} = U \setminus X$$

is called the **complement** *of X in U.*

Definition. *A* **Venn diagram** *is a pictorial view of sets drawn as follows: A rectangle depicts the universal set. Subsets of the universal set are drawn as circles. The inside of a circle represents the members of that set.*

Theorem. *Let U be a universal set and let $A, B, C \subseteq U$.*

Associative Laws:	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
Commutative Laws:	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Distributive Laws:	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
	$A \cup (B \cup C) = (A \cup B) \cap (A \cup C)$
Identity Laws:	$A \cup \emptyset = A$
	$A \cap U = A$
Complement Laws:	$A \cup \overline{A} = U$
	$A \cap \overline{A} = \emptyset$
Idempotent Laws:	$A \cup A = A$
	$A \cap A = A$
Bound Laws:	$A \cup U = U$
	$A \cap \emptyset = \emptyset$
Absorption Laws:	$A \cup (A \cap B) = A$
	$A \cap (A \cup B) = A$

Involution Laws:	$\overline{\overline{A}} = A$
0/1 *Laws:*	$\overline{\emptyset} = U$
	$\overline{U} = \emptyset$
De Morgan's Laws:	$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$
	$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

Proof. Left as an exercise. □

Definition. *A collection $\mathcal{S}$ of non-empty sets of X is a* **partition** *of the set X means every element in X belongs to exactly one member of $\mathcal{S}$.*

Definition. *An* **ordered pair** *of elements, written (a, b) is considered distinct from the ordered pair (b, a), unless $a = b$.*

Definition. *Say X, Y are sets. The set of ordered pairs*

$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}$$

is called the **Cartesian product** *of X and Y.*

Definition. *Ordered lists need not be restricted to two elements. An n*-**tuple***, written $(a_1, \ldots, a_n)$ takes order into account.*

Definition. *A non-zerointeger d* **divides** *an integer m means there exists an integer q, called the* **quotient***, such that $m = dq$. A positive integer is* **prime** *means the only positive integers that divide it are 1 and itself.*

Definition. *A sentence that is either true of false, but not both, is called a* **proposition***.*

Definition. *Suppose P, Q are propositions. The* **conjunction** *of P and Q is*

Conditional Propositions and Logical Equivalence

Definition. *Suppose P, Q are propositions. The statement "if P then Q" is called a* **conditional proposition***, denoted $P \to Q$. P is called the* **hypothesis***, or* **antecedent***. Q is called the* **conclusion***, or* **consequent***.*

Truth Tables

P	Q	$P \to Q$
true	true	true
false	true	true
false	false	true
true	false	false

When a conditional has a false antecedent, its truth value is always true. For this reason we sometimes say $P \to Q$ is *vacuously true* or *true by default.* In the order of operations, $\to$ is evaluated last.

Truth values can be difficult to parse when propositions are expressed in everyday conversation. The following statements mean the same thing:

1) "if P, then Q"

2) $P \to Q$
3) "Q only if P"
4) "When P, Q."
5) "If not Q, then not P." (called the *contrapositive* to the conditional proposition $P \to Q$)
6) Q is necessary for P.
7) P suffices for Q.

Definition. *Suppose $R = P \to Q$. The* **converse** *of R is $Q \to P$.*

Warning, the converse of a conditional and the contrapositive of a conditional are NOT THE SAME THING.

Definition. *A* **biconditional proposition***, denoted $P \leftrightarrow Q$, is defined by the truth table*

P	Q	$P \leftrightarrow Q$
true	*true*	*true*
true	*false*	*false*
false	*true*	*false*
false	*false*	*true*

Some equivalent statements:

1) "$P \leftrightarrow Q$"
2) "P if and only if Q"
3) "P is necessary and sufficient for Q."
4) "Q is necessary and sufficient for P."
5) "$Q \leftrightarrow P$"
6) "P iff Q"

Defining a proposition via truth table remedies the ambiguity that comes with trying to express a logical statement in lay-speak. Truth tables are very effective tools in writing proofs.

Definition. *Two propositions are* **logically equivalent** *means their truth tables are the same. The symbol for logical equivalence is $\equiv$.*

Example. De Morgan's Laws:

1) $\neg (P \vee Q) \equiv \neg P \wedge \neg Q$
2) $\neg (P \wedge Q) \equiv \neg P \vee \neg Q$

Proof. In each case, the propositions we are trying to show are logically equivalent are the proposition on the lefthand side (LHS) of $\equiv$ and the proposition on the righthand side (RHS) or $\equiv$.

1)

P	Q	$\neg (P \vee Q)$	$\neg P \wedge \neg Q$
true	true	false	false
true	false	false	false
false	true	false	false

false	false	true	true

2)

$\square$