Take-Home Quiz 7: Manipulating Taylor series (§8.2-8.4)

Directions: This quiz is due on November 15, 2017 at the beginning of lecture. You may use whatever resources you like – e.g., other textbooks, websites, collaboration with classmates – to complete it **but YOU MUST DOCUMENT YOUR SOURCES**. Acceptable documentation is enough information for me to find the source myself. Rote copying another's work is unacceptable, regardless of whether you document it.

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1. The **second-order differential equation** (differential equation involving y'')

$$x^2y'' + xy' + (x^2 - p^2) = 0,$$

where p is a nonnegative integer, arises in many applications in physics and engineering, including one model for the vibration of a beaten drum. The solution to this differential equation is called the **Bessel function of order** p and is denoted $J_p(x)$. The Bessel function has the power series expansion

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+p)!2^{2k+p}} x^{2k+p}.$$

(a) Use the fact that

$$J_p'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k(2k+p)}{k!(k+p)!2^{2k+p}} x^{2k+p-1} \quad \text{and}$$
$$J_p''(x) = \sum_{k=0}^{\infty} \frac{(-1)^k(2k+p)(2k+p-1)}{k!(k+p)!2^{2k+p}} x^{2k+p-2}.$$

to verify that $y = J_p(x)$ satisfies the differential equation given above.

(b) **§8.2 #68** Recall, from §7.7, the **Ratio Test for Absolute Convergence** says a series $\sum_{k=0}^{\infty} b_k$ converges absolutely if $\lim_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right| < 1$. Use the Ratio Test for Absolute Convergence to find the interval of convergence for $J_p(x)$.

(c) To get an idea of what the Bessel function looks like, you can use `desmos.com/calculator`. Type in "$p = 0$" to get a slider. Then add the equations

$$S_0 = \frac{(-1)^0}{0!(0+p)! \cdot 2^{2\cdot 0+p}} x^{2\cdot 0+p}$$
$$S_1 = S_0 + \frac{(-1)^1}{1!(1+p)! \cdot 2^{2\cdot 1+p}} x^{2\cdot 1+p}$$
$$S_2 = S_1 + \frac{(-1)^2}{2!(2+p)! \cdot 2^{2\cdot 2+p}} x^{2\cdot 2+p}$$
$$\vdots$$

up to the partial sum S_9. Sketch or print the graph of S_9 for $p = 0, 1, 2, 3$.

2. **§8.3 #48** Let $f(x) = \sqrt{x}$.

(a) Find the 4th order Taylor polynomial, $P_4(x)$, for f centered at $x = 1$.

(b) Using part (a) as a guide, write the Taylor series, $T(x)$, for f centered at $x = 1$, in summation form.

(c) **Taylor's Theorem** (Theorem 8.9) says the nth remainder for f at $x = 1$ is

$$f(x) - P_n(x) = R_n(x) = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\cdots\left(\frac{1}{2} - n + 1\right)}{n!} \int_1^x (x-t)^n t^{\frac{1}{2}-n}\, dt.$$

However, **Lagrange's Form for the Remainder** (Theorem 8.10) states that there is at least one number c between 1 and x where

$$R_n(x) = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\cdots\left(\frac{1}{2}-n\right)}{(n+1)!} c^{\frac{1}{2}-n-1}(x-1)^{n+1}.$$

Using that fact, compute $\lim_{n\to\infty} R_n(x)$, assuming $x \in \left(\frac{1}{2}, \frac{3}{2}\right)$.

Your answer should confirm that $f(x)$ equals its Taylor series centered at $x = 1$ on the interval $\left(\frac{1}{2}, \frac{3}{2}\right)$.

3. **§8.3 #56** Recall, the **geometric series**

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{when } x \in (-1, 1).$$

Use this fact to compute

(a) the Maclaurin series for $f(x) = \frac{x}{9-x^2} = x\left(\frac{1}{\frac{1}{9}\left(1-\frac{x^2}{9}\right)}\right) = 9x\left(\frac{1}{1-\frac{x^2}{9}}\right)$

(b) and the interval of convergence.

4. **§8.4 #56** The Maclaurin series for $\ln(1+x)$ is given by

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k \qquad \text{when } x \in (-1, 1].$$

(a) Use this fact to compute

- the Maclaurin series for $\ln(4+x^2) = \ln\left(\frac{1}{4}\left(1+\frac{x^2}{4}\right)\right) = \ln\left(\frac{1}{4}\right) + \ln\left(1+\frac{x^2}{4}\right)$
- and the interval of convergence.

(b) Compute $\int_{0.5}^{1} \ln(4+x^2)\, dx$ by substituting your answer from part (a) into the integrand.