Take-Home Quiz 7: Geometric applications of derivatives (§4.3-4.6)

Directions: This quiz is due on November 15, 2017 at the beginning of lecture. You may use whatever resources you like – e.g., other textbooks, websites, collaboration with classmates – to complete it **but YOU MUST DOCUMENT YOUR SOURCES**. Acceptable documentation is enough information for me to find the source myself. Rote copying another's work is unacceptable, regardless of whether you document it.

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1. **Interpreting 1st and 2nd derivatives**

   (a) **§4.3 #66** On an episode of *The Simpsons*, Homer reads from a newspaper and announces "Here's good news! According to this eye-catching article, SAT scores are declining at a slower rate." Interpret Homer's statement in terms of a function and its first and second derivatives.

   (b) **§4.3 #68** Let $f(t)$ be the temperature in Moody 201, in degrees Celsius, t minutes after the Calc I Exam 3 has begun. At $t = 3$ you feel uncomfortably hot. How do you feel about the given data in each of the following cases?

      i. $f'(3) = 0.2$, $f''(3) = 0.4$
      ii. $f'(3) = 0.2$, $f''(3) = -0.4$
      iii. $f'(3) = -0.2$, $f''(3) = 0.4$
      iv. $f'(3) = -0.2$, $f''(3) = -0.4$

2. **§4.6 #26** The function $f(x) = (\sin x)^{\sin x}$ is weird.

   (a) Because of the exponential, f is only defined when $\sin x > 0$. (*Helpful hint: To see this, on* `desmos` *graph it on the same axes as* $y = \sin x$.) Write down the domain of $f(x)$ using interval notation. *Hint: For what values of* x *is* $\sin x > 0$*?*

   (b) Explain why $f(x)$ is periodic. What is its period?

   (c) Find $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to\pi^-} f(x)$ using L'Hôpital's Rule.

   (d) i. Use logarithmic differentiation to show

   $$f'(x) = (\sin x)^{\sin x} \cos x \left(\ln(\sin x) + 1\right).$$

      ii. Why doesn't $f(x)$ ever equal 0?
      iii. For what value(s) of x does $\cos x = 0$?
      iv. For what value(s) of $\sin x$ does $\ln(\sin x) = -1$?
      v. Use your answers to parts i.-iv. to identify the critical points for f in the interval $(0, \pi)$. *Hint: There are three. You may have to look up information about the function* $\arcsin x$ *to properly identify them.*

   (e) The second derivative of f is

   $$f''(x) = (\sin x)^{\sin x} \left(\ln(\sin x) + 1\right) \left(\cos^2 x \left(\ln(\sin x) + 1\right) + \frac{\cos^2 x}{\sin x} - \sin x\right)$$

   (if you wish, you may verify this in private). Use the 2nd Derivative Test to classify whether or not the critical points are local minima or maxima. *Hint: You can save some time by using part (d)iv.* If the test is inconclusive, say so.

   (f) Sketch the graph of $f(x)$ (you will need `desmos`). Label the critical points and make sure your graph is consistent with your answers to parts (a)-(e).

3. **§4.6 #30** The graph of $f(x) = \ln(x^2 + c)$ varies as c varies. In this problem, we go through the procedures in §4.5 to investigate how.

Helpful hint: In `desmos`*, type in "$f(x) = ln(x^2 + c)$" and add the slider for c. Click on the wrench icon in the top right corner of the page to change the graphing window to $-10 \le x \le 10$, $-4 \le y \le 6$. Click on the endpoints that come with the slider to change them to $-5 \le c \le 5$. There will be a play button with the slider; press it.*

(a) **Domain** Find the domain of f (*Hint: Compare it to the domain of* $\ln x$.) when
- $c > 0$.
- $c = 0$.
- $c < 0$. *Hint: Draw a number line and shade the values that satisfy the inequality* $|x| > \sqrt{-c}$.

(b) **Intercepts**

i. y-intercepts: Even though f is a logarithmic function, in this example it will have y-intercepts for certain values of c. To find them you must evaluate $f(0)$.
- What are the y-intercepts for the values of c, when f does have them?
- For what values of c does f not have any y-intercepts?

ii. x-intercepts: To find the x-intercepts, set $f(x) = 0$. *Hint: Recall that to solve logarithmic equations, you must "e" both sides.*
- For what values of c does f have x-intercepts?
- What are the x-intercepts, when f has them?

(c) **Symmetry**

i. Simplify $f(-x)$. Is f an even or odd function (or neither)?

ii. Rather than checking $f(x+p) = f(x)$ as in the textbook, explain in words why f isn't periodic in this example.

(d) **Asymptotes**

i. Horizontal asymptotes: Horizontal asymptotes are found by checking the **end behavior** (the limit as x approaches positive infinity and negative infinity). If f is symmetric, you can save time by only computing $\lim_{x\to\infty} f(x)$, since

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) \quad \text{when } f(x) \text{ is even and}$$
$$\lim_{x\to-\infty} f(x) = -\lim_{x\to\infty} f(x) \quad \text{when } f(x) \text{ is odd.}$$

Find the horizontal asymptotes for f. If you use symmetry, then say so.

ii. Vertical asymptotes: For the values of c where f has them, the vertical asymptotes of f are at $x = \pm\sqrt{-c}$.
- For what values of c does f have vertical asymptotes? How do you know?
- In the case where f does have vertical asymptotes, evaluate

$$\lim_{x\to-\sqrt{-c}^-} f(x) \quad \text{and} \quad \lim_{x\to\sqrt{-c}^+} f(x)$$

(if you use symmetry, explain how). Why are the one-sided limits necessary?

iii. Slant asymptotes: Explain in words why f will not have any slant asymptotes.

(e) **Extrema** *Helpful hint: Type "$y = f'(x)$" into* `desmos`*, using the same browser window as your graph for $f(x)$.*

i. Critical points:
- Compute $f'(x)$.
- Identify points ***in the domain*** where f' does not exist. Be specific, since your answer depends on the different possible values of c.

- Identify points ***in the domain*** where $f'(x) = 0$. Be specific, since your answer depends on the different possible values of c.

ii. 1st Derivative Test: Use it to identify local extrema. If you are using symmetry as a shortcut, explain how. Be specific about which values of c you are considering and why.

iii. 2nd Derivative Test: *Helpful hint: Type "$y = f''(x)$" into* `desmos`*, using the same browser window as your graphs for $f(x)$ and $f'(x)$.*

- Find $f''(x)$.
- Use the 2nd Derivative Test to identify local extrema. Be specific about which values of c you are considering and why.

iv. Global extrema:

- What is the range of $f(x)$? Be specific, since it depends on the values of c.
- For what values of c does $f(x)$ have global extrema? What are the global extrema in those cases (give the x- and y-coordinates for each extremum, and whether it is a max or min).

(f) **Intervals of Increase or Decrease** Identify the intervals where f is increasing and decreasing when

- $c > 0$.
- $c = 0$.
- $c < 0$.

(g) **Concavity and Points of Inflection** Identify the intervals where f is concave up and concave down, as well as any inflection points, when

- $c > 0$.
- $c = 0$.
- $c < 0$.

(h) **Sketch the Curve** Sketch a graph of f in each of the following cases. Make sure in every case your curve is consistent with your answers to parts (a)-(g) by labelling the domain, the intercepts, the asymptotes, the extrema, and the inflection points.

- $c > 0$
- $c = 0$
- $c < 0$