Take-Home Quiz 1

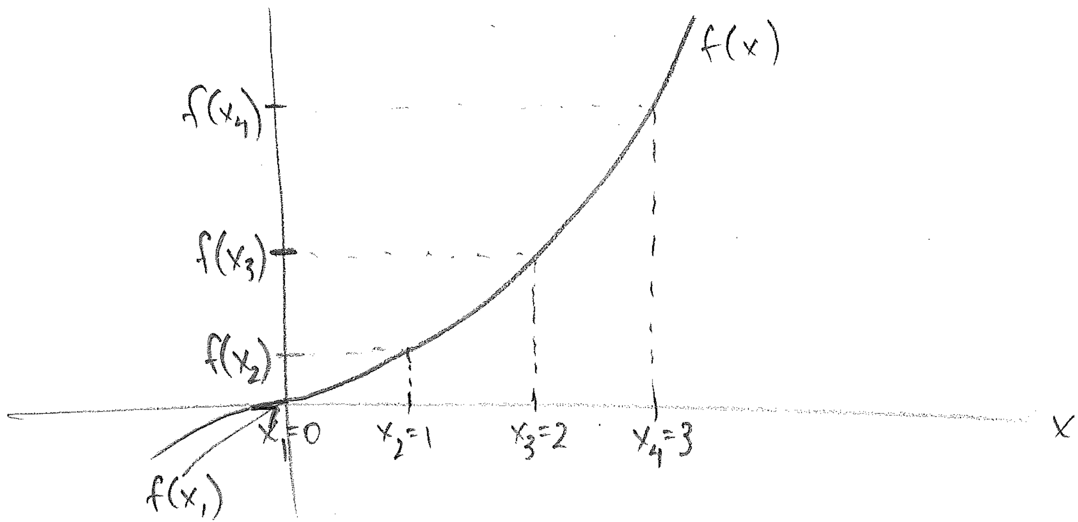
SOLUTIONS

Math 236 (Calc II)
Fall 2017

1. (a) For each $k = 1, 2, 3$, choose x_k^* so that

$$f'(x_k^*) = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}.$$

In this case, $f(x) = x^3$.



i. $f'(x_1^*) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \Rightarrow 3(x_1^*)^2 = \dfrac{x_2^3 - x_1^3}{x_2 - x_1}$

$$= \frac{1^3 - 0^3}{1 - 0} = 1$$

$$\Rightarrow (x_1^*)^2 = \frac{1}{3}$$

$$\boxed{x_1^* = \sqrt{\frac{1}{3}} \approx 0.58}$$

(positive square root, since $0 \le x_1^* \le 1$)

$\longrightarrow$

$$3(x_2^*)^2 = \frac{2^3 - 1^3}{2-1} = \frac{7}{1}$$

$$\Rightarrow \boxed{x_2^* = \sqrt{\frac{7}{3}} \approx 1.53}$$

$$3(x_3^*)^2 = \frac{3^3 - 2^3}{3-2} = 27 - 8 = 19$$

$$\Rightarrow \boxed{x_3^* = \sqrt{\frac{19}{3}} \approx 2.52}$$

ii. $$\int_0^3 x^3\,dx \approx \sum_{k=1}^{3} (x_k^*)^3 (1) = \left(\sqrt{\frac{1}{3}}\right)^3 + \left(\sqrt{\frac{7}{3}}\right)^3 + \left(\sqrt{\frac{19}{3}}\right)^3$$

$$= \frac{1^{3/2} + 7^{3/2} + 19^{3/2}}{3^{3/2}}$$

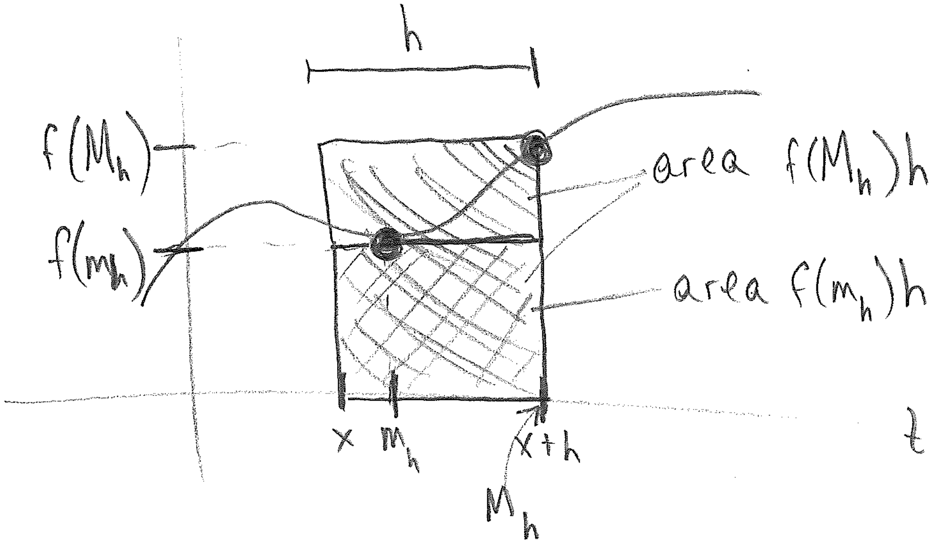
$$\boxed{\approx 19.7}$$

iii. $$\int_0^3 x^3\,dx = \frac{x^4}{4}\Big|_0^3$$ ← terms vanish

$$= \frac{3^4}{4} = \boxed{\frac{81}{4} = 20.25}$$

→

(b) The definite integral $\int_x^{x+h} f(t)\,dt$ is, by definition, the signed area between $f(t)$ and the t-axis on the interval $[x, x+h]$. On that interval, $f(t)$ takes on a minimal value for some $t = m_h$, and a maximal value for some $t = M_h$. The numbers m_h and M_h depend on h.



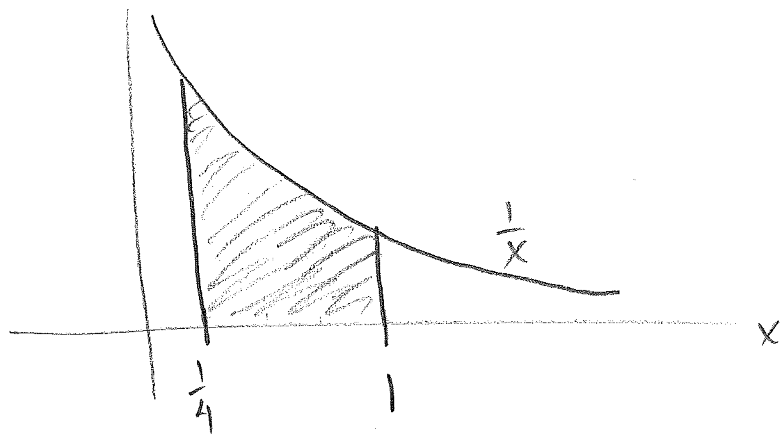
The area $f(m_h)h$ is the lowest estimate for $\int_x^{x+h} f(t)\,dt$ while the area $f(M_h)h$ is the highest estimate. Thus, we get the inequality

$$f(m_h)h \leq \int_x^{x+h} f(t)\,dt \leq f(M_h)h .$$

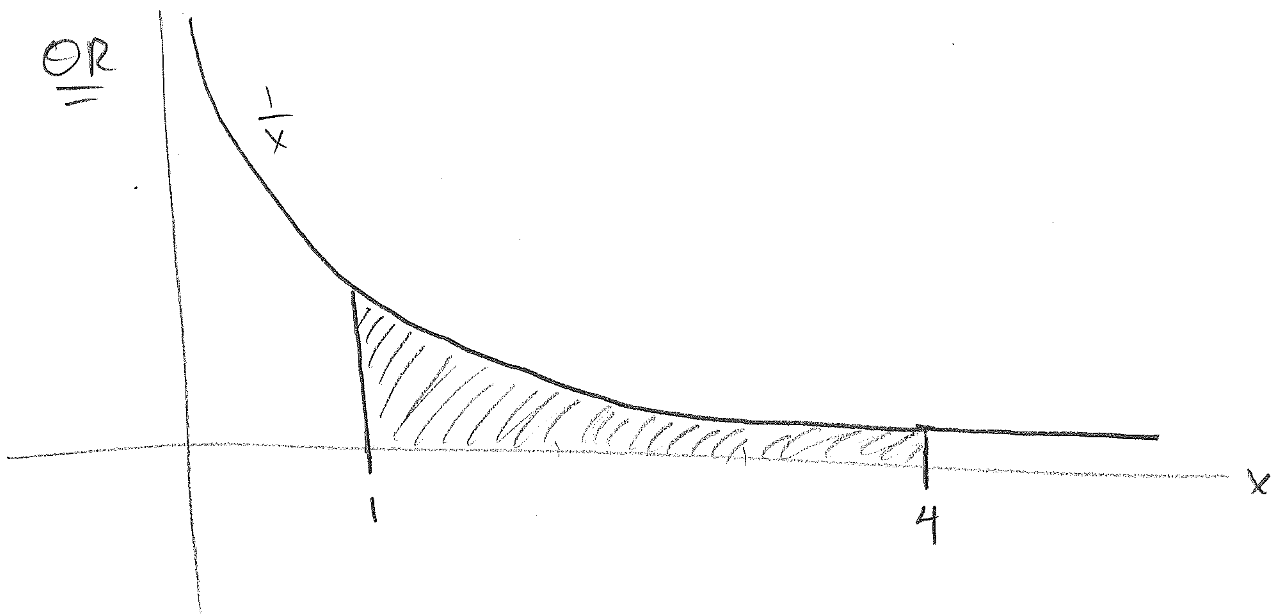
2. (a) In general, $\ln x = \int_1^x \frac{1}{t}\,dt$. The signed area under $\frac{1}{x}$ on the interval $[\frac{1}{4}, 1]$ is

$$\int_{\frac{1}{4}}^{1} \frac{1}{t}\,dt = -\int_1^{\frac{1}{4}} \frac{1}{t}\,dt = -\ln\left(\tfrac{1}{4}\right) = \boxed{\ln 4}$$

(b)



<u>OR</u>



(c) $-\ln\frac{1}{4} = \ln 4 \boxed{\approx 1.4}$

3. (a) At time t hours after midnight the total water flow is

$$\int w(t)\,dt = 15\int\left(1 + \cos\left(\frac{2\pi(t-16)}{24}\right)\right)dt.$$

let $u = \frac{\pi}{12}(t-16) \Rightarrow du = \frac{\pi}{12}\,dt$

$\Rightarrow dt = \frac{12}{\pi}\,du$

$$\overset{\vee}{=} 15\left(\frac{12}{\pi}\right)\int(1 + \cos u)\,du$$

$$= \frac{180}{\pi}(u + \sin u) + C.$$

At 4p, no water has trickled into the pot; $t = 16 \Rightarrow u = \frac{\pi}{12}(16-16) = 0$. This is the initial condition used to determine C:

$$0 = \frac{180}{\pi}(0 + \sin(0)) + C \Rightarrow C = 0.$$

The amount of water that has trickled into the pot is

$$\frac{180}{\pi}(u + \sin u) = \frac{180}{\pi}\left(\frac{\pi}{12}(t-16) + \sin\left(\frac{\pi}{16}(t-16)\right)\right).$$

The pot is 2 quarts $= \frac{1}{2}$ gallon. The pot becomes full at

$$\frac{1}{2} = \frac{180}{\pi}\left(\frac{\pi}{12}(t-16) + \sin\left(\frac{\pi}{12}(t-16)\right)\right).$$

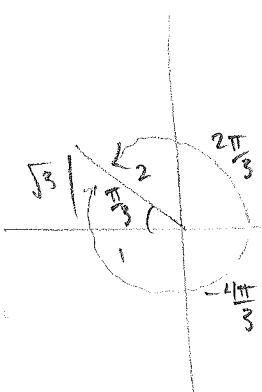
To solve, use a graph to find the intersection of $y = \frac{1}{2}$ and $y = \frac{180}{\pi}\left(\frac{\pi}{12}(t-16) + \sin\left(\frac{\pi}{12}(t-16)\right)\right)$. From desmos.com/calculator, the intersection is $t \approx 16.017$ hours after midnight. Since $0.017 \text{ hours}\left(\frac{60 \text{ min}}{\text{hour}}\right) = 1.02$ min it takes $\boxed{\sim 1.0 \text{ minutes.}}$ The time is $\boxed{4{:}01\text{p.}}$

(b) Using the 24-hour period from midnight to midnight, the total water flow is

$$\int_0^{24} w(t)\,dt = \frac{180}{\pi}\left(\frac{\pi}{12}(t-16) + \sin\left(\frac{\pi}{12}(t-16)\right)\right)\Bigg|_0^{24}$$

$$= \frac{180}{\pi}\left(\frac{\pi}{12}\left(\overset{24}{(24)-16-(0-16)}\right) + \sin\left(\frac{\pi}{12}\underset{8}{(24-16)}\right) - \sin\left(\frac{\pi}{12}\underset{-16}{(0-16)}\right)\right)$$

$$= \frac{180}{\pi}\left(2\pi + \cancel{\sin\left(\frac{2\pi}{3}\right)} - \cancel{\sin\left(-\frac{4\pi}{3}\right)}\right) = 360$$



$\boxed{\approx 360.0 \text{ gallons}}$

(c) Between starting time t_0 and time t the flow is

$$\boxed{\int_{t_0}^{t} w(T)\,dT = \frac{180}{\pi}\left(\frac{\pi}{12}(t-t_0) + \sin\left(\frac{\pi}{12}(t-16)\right) - \sin\left(\frac{\pi}{12}(t_0-16)\right)\right)}$$

(d) At 5a, $t_0 = 5$. Solve

$$\frac{1}{2} = \frac{180}{\pi}\left(\frac{\pi}{12}(t-5) + \sin\left(\frac{\pi}{12}(t-16)\right) - \sin\left(\frac{\pi}{12}\underbrace{(5-16)}_{}\right)\right).$$

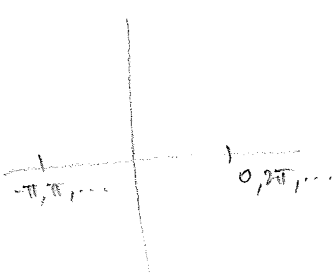
Via desmos.com/calculator, $t \approx 5.581$.

Using $0.581\,\text{hr}\left(\frac{60\text{ min}}{\text{hour}}\right) \approx \boxed{34.9 \text{ min}}$, it will be $\boxed{5{:}34\text{a.}}$

(e) The rate of flow is $w(t)$. Find its critical points:

$$w'(t) = 15\left(-\sin\left(\frac{\pi}{12}(t-16)\right)\right)\left(\frac{\pi}{12}\right) = 0$$

$$\Rightarrow \sin\left(\frac{\pi}{12}(t-16)\right) = 0$$



$\frac{\pi}{12}(t-16) =$ integer multiples of π

$t-16 =$ integer multiple of 12

$t = (\text{integer multiple of } 12) + 16$

$= \ldots, -8, 4, 16, 28, \ldots$

So water flows fastest at either 4a or 4p.

$$w(4) = 15\left(1 + \cos\left(\underbrace{\tfrac{\pi}{12}(4-16)}_{-\pi}\right)\right) = 0 \text{ gal/hour}$$

$$w(16) = 15\left(1 + \cos\left(\tfrac{\pi}{12}(0)\right)\right) = 30 \text{ gal/hour.}$$

$\Rightarrow$ water flows fastest at $\boxed{4p.}$

(f) The flow rate is periodic, so for the pot to fill the fastest, place it under the trickle so that at 4p it is half full. Then

$$\int_{t_0}^{16} w(t)\,dt = \frac{1}{4} = \frac{180}{\pi}\left(\frac{\pi}{12}(16-t_0) + \cancel{\sin\tfrac{\pi}{12}(16-16)} - \sin\left(\tfrac{\pi}{12}(t_0-16)\right)\right)$$

$\Rightarrow t_0 \approx 15.992$ (desmos)

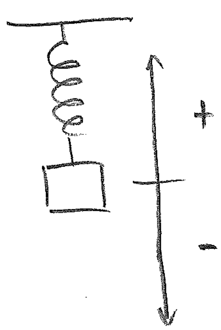
and $0.992 \text{ hours}\left(\frac{60 \text{ min}}{\text{hr}}\right) = 59.52 \text{ min}$

$0.52 \text{ min}\left(\frac{60 \text{ sec}}{\text{min}}\right) = 31.2 \text{ sec}$,

So place the pot under the trickle at $\boxed{3{:}59{:}31p.}$

It will be full at $\boxed{4{:}00{:}29p.}$

4.



The mass' position at $t=4$ is

$$\int_0^4 v(t)\,dt = \int_0^4 \left(3\cos\left(\frac{3t}{\sqrt{2}}\right) - 3\sqrt{2}\sin\left(\frac{3t}{\sqrt{2}}\right)\right)dt.$$

$$u = \frac{3t}{\sqrt{2}} \Rightarrow du = \frac{3}{\sqrt{2}}\,dt$$

$$\Rightarrow 3\,dt = \sqrt{2}\,du$$

$$u(4) = \frac{3(4)}{\sqrt{2}} = \frac{12}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{12\sqrt{2}}{2} = 6\sqrt{2}$$

$$u(0) = \frac{3(0)}{\sqrt{2}} = 0$$

$$= \sqrt{2}\int_0^{6\sqrt{2}} \cos u\,du - 2\int_0^{6\sqrt{2}} \sin u\,du$$

$$= \left(\sqrt{2}\sin u + 2\cos u\right)\Big|_0^{6\sqrt{2}}$$

$$= \sqrt{2}\sin(6\sqrt{2}) + 2\cos(6\sqrt{2})$$

$$- \sqrt{2}\cancel{\sin 0} - 2\cos 0$$

$$\approx -2.0$$

$$\Rightarrow \boxed{\text{below at } t=4 \text{ sec}}$$

The mass' position at $t=5$ is

$$\int_0^5 v(t)\,dt$$

$$= \sqrt{2}\sin u + 2\cos u\Big|_0^{t=5}$$

$$u(5) = \frac{3(5)}{\sqrt{2}} = \frac{15}{\sqrt{2}}$$

$$= \sqrt{2}\sin\left(\frac{15}{\sqrt{2}}\right) + 2\cos\left(\frac{15}{\sqrt{2}}\right)$$

$$-2$$

$$\approx -4.1$$

$$\Rightarrow \boxed{\text{below at } t=5 \text{ sec}}$$

5. (a) $\int \arcsin x \, dx$

$u = \arcsin x \Rightarrow du = \frac{dx}{\sqrt{1-x^2}}$

$dv = dx \Rightarrow v = x$

$= (\arcsin x)(x) - \int x \frac{dx}{\sqrt{1-x^2}}$

$= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx$ $\quad w = 1 - x^2$

$\Rightarrow \frac{dw}{dx} = -2x \Rightarrow x\,dx = -\frac{1}{2} dw$

$= -\frac{1}{2} \int \frac{1}{\sqrt{w}} dw = -\frac{1}{\cancel{2}}(\cancel{2} w^{1/2}) + C = \left(-\sqrt{1-x^2} + C\right)$

$= x \arcsin x + \sqrt{1-x^2} + C.$

(b) $\frac{d}{dx}\left(\sqrt{1-x^2} + x \arcsin x\right)$

$= \frac{1}{\cancel{2}}(1-x^2)^{-1/2}(-\cancel{2}x) + (1)\arcsin x + x \frac{1}{\sqrt{1-x^2}}$

$= \cancel{\frac{-x}{\sqrt{1-x^2}}} + \arcsin x + \cancel{\frac{x}{\sqrt{1-x^2}}}$

$= \arcsin x.$