Take Home Quiz #1

SOLUTIONS

Math 2574 (Cal III)
Spring 2017

1. Write $\langle 4,-8\rangle = c_1\langle 1,1\rangle + c_2\langle -1,1\rangle$.

$$\Rightarrow 4 = c_1 - c_2 \quad \Rightarrow c_1 = 4 + c_2$$
$$-8 = c_1 + c_2 \qquad -8 = (4+c_2)+c_2$$
$$-12 = 2c_2.$$
$$\Rightarrow c_2 = -6,\ c_1 = 4+(-6) = -2$$

And so $\boxed{\langle 4,-8\rangle = -2\vec{u} - 6\vec{v}}$

2. Complete the square:

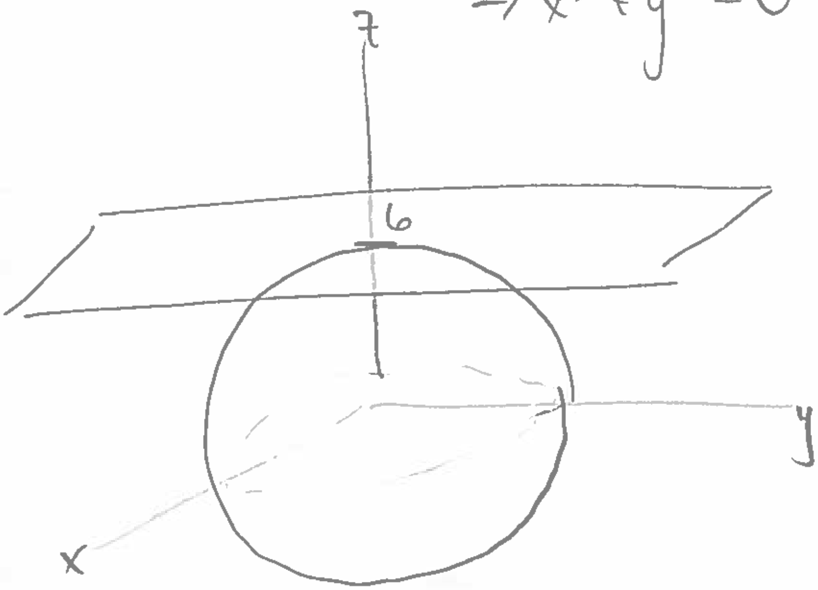
$$x^2 - 8x + \left(\frac{-8}{2}\right)^2 + y^2 + 14y + \left(\frac{14}{2}\right)^2 + z^2 - 18z + \left(\frac{-18}{2}\right)^2 \geq 65 + 4^2 + 7^2 + 9^2$$

$$(x-4)^2 + (y+7)^2 + (z-9)^2 \geq 211$$

The solutions make up a sphere of radius $\sqrt{211}$ centered at the point $(4,-7,9)$, and all points outside of the sphere.

3. When $z=6$, $x^2+y^2+(6)^2=36$

$$\Rightarrow x^2+y^2=0 \quad \boxed{\text{is a point, at } (0,0,6)}$$



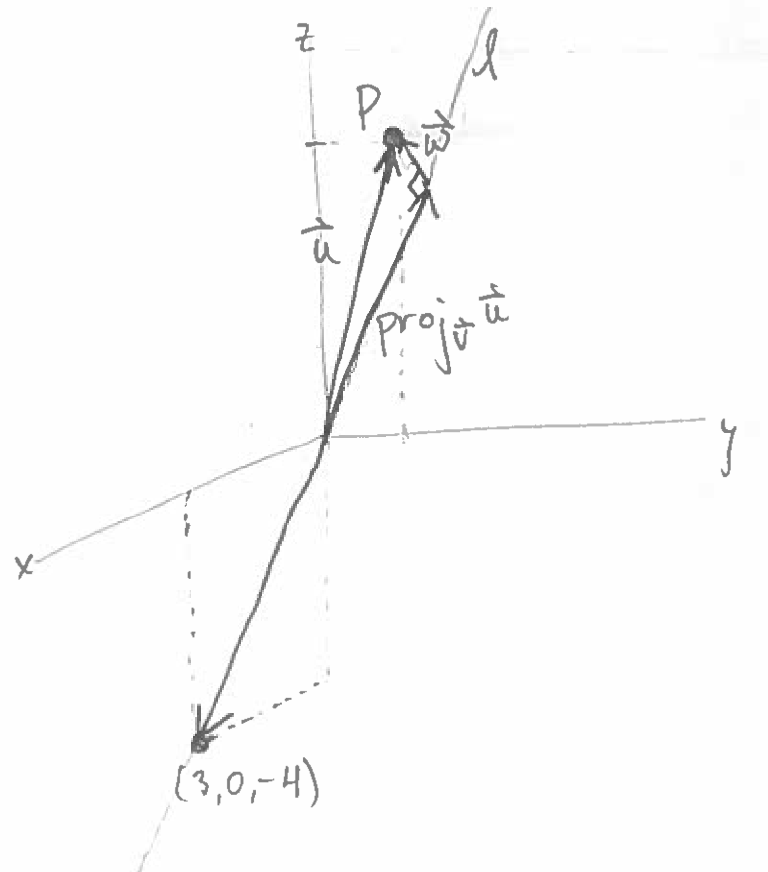
4. (a) $\vec{v}=\langle 3,0,-4\rangle$

(b) $\vec{u}=\langle 0,2,6\rangle$

(c) $\text{proj}_{\vec{v}}\vec{u}=\dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$

$$=\frac{(0)(3)+(2)(0)+(6)(-4)}{3^2+0^2+(-4)^2}\vec{v}$$

$$=\frac{-24}{25}\langle 3,0,-4\rangle$$



(d) Orthogonal:

$$\vec{w}\cdot\vec{v}=\vec{u}\cdot\vec{v}-(\text{proj}_{\vec{v}}\vec{u})\cdot\vec{v}$$

$$=\vec{u}\cdot\vec{v}-\left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\right)\cdot\vec{v}=\vec{u}\cdot\vec{v}-\vec{u}\cdot\vec{v}=0$$

The magnitude of $\vec{w}$ is also the distance between P and the point $\frac{-24}{25}(3,0,-4)$, which is on the line l.

(e) $\vec{w} = \vec{u} - \text{proj}_{\vec{v}}\vec{u}$

$$= \langle 0,2,6\rangle - \left(\frac{-24}{25}\right)\langle 3,0,-4\rangle$$

$$= \left\langle 0 + \frac{72}{25}, 2+0, 6-\frac{96}{25}\right\rangle$$

$$= \left\langle \frac{72}{25}, 2, \frac{54}{25}\right\rangle$$

$$|\vec{w}| = \sqrt{\left(\frac{72}{25}\right)^2 + 2^2 + \left(\frac{54}{25}\right)^2} \approx 4.12$$

is the least distance from P to l because $\vec{w}$ is orthogonal to l.

5. In a magnetic field, $\vec{F} = q(\vec{v} \times \vec{B})$

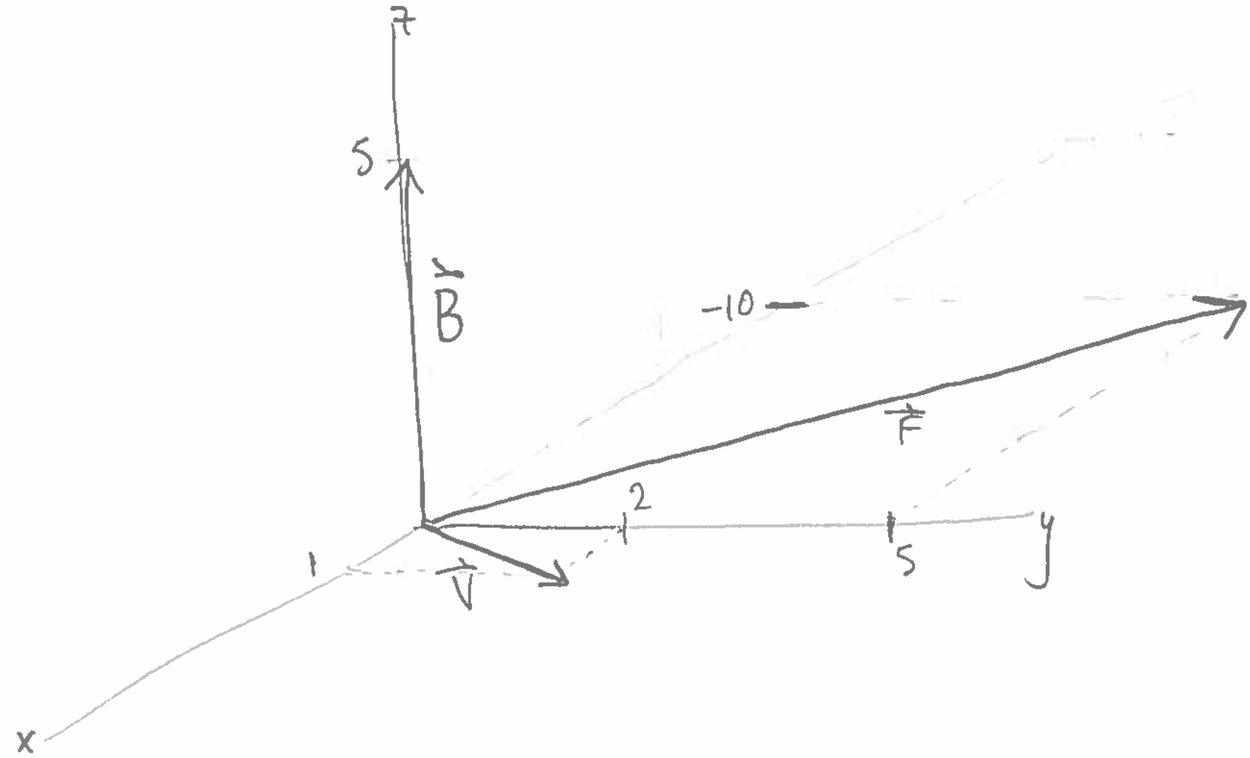
$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 0 & 0 & 5 \end{vmatrix} = ((2)(5)-0)\hat{i} - ((1)(5)-0)\hat{j} + 0\hat{k}$$

$$= 10\hat{i} - 5\hat{j}$$

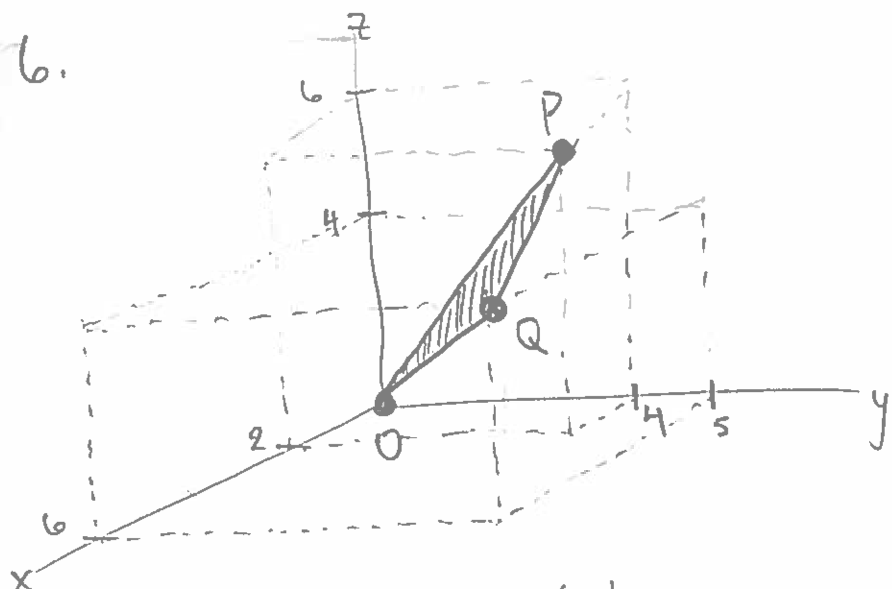
$$\Rightarrow \vec{F} = -1\langle 10,-5,0\rangle = \boxed{\langle -10,5,0\rangle}$$

$$|\vec{F}| = \sqrt{(-10)^2 + 5^2 + 0^2} = \sqrt{125} = \boxed{5\sqrt{5}}$$

$\longrightarrow$)



6.



$$\text{area} = \frac{1}{2}\left|\vec{OP} \times \vec{OQ}\right|$$

$$= \frac{1}{2}\left|\langle 2, \overset{2}{4}, \overset{3}{6}\rangle \times \langle 6,5,4\rangle\right| = \left|\langle 1,2,3\rangle \times \langle 6,5,4\rangle\right|$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 6 & 5 & 4 \end{vmatrix} = \big(\overset{8}{2(4)} - \overset{-15}{3(5)}\big)\hat{i} - \big(\overset{4}{1(4)} - \overset{-18}{3(6)}\big)\hat{j} + \big(\overset{5}{1(5)} - \overset{-12}{2(6)}\big)\hat{k}$$

$$= \langle -7, 14, -7\rangle$$

$$\Rightarrow \text{area} = \sqrt{(-7)^2 + \underbrace{(14)^2}_{= 4(7^2)} + (-7)^2} = \sqrt{6(7)^2} = \boxed{7\sqrt{6}}$$

7\. (a) $\vec{r}$ and $\vec{R}$ intersect if

$$\langle 2+2t, 8+t, 10+3t\rangle = \langle 6+s, 10-2s, 16-s\rangle$$

$$2+2t = 6+s \Rightarrow s = -4+2t$$

$$\begin{aligned} 8+t &= 10-2s \\ &= 10-2(-4+2t) \\ &= 10+8-4t \end{aligned}$$

$$5t = 10 \Rightarrow t = 2 \text{ and } s = -4+2(2) = 0$$

Check the other equation:

$$10+3(2) \stackrel{?}{=} 16-0$$

$$10+6 = 16 \checkmark$$

$\boxed{\text{Yes}}$ $\vec{r}$ and $\vec{R}$ intersect when $t=2, s=0$. The coordinates of that point are given by the position vector $\vec{R}(0) = \langle 6+0, 10-2(0), 16-0\rangle \boxed{= \langle 6, 10, 16\rangle}$

(or $\vec{r}(2) = \langle 2+2(2), 8+2, 10+3(2)\rangle = \langle 6, 10, 16\rangle$)

(b) The particles collide if there is some value $T > 0$, where $\vec{r}(T) = \vec{R}(T)$.

$$2+2T = 6+T \Rightarrow T = 4$$

$$8+(4) = 10-2(4) \Rightarrow 12 = 2 \quad \times$$

So no such T exists, and the $\boxed{\text{particles never collide.}}$

8. $\vec{r}\,'(t) = \langle \cos t, -\sin t, -e^{-t} \rangle$

$|\vec{r}\,'(t)| = \sqrt{\cos^2 t + (-\sin t)^2 + (-e^{-t})^2}$

$= \sqrt{1 + e^{-2t}}$

$\vec{T}(0) = \dfrac{\vec{r}(0)}{|\vec{r}(0)|} = \dfrac{\langle \cos(0), -\sin(0), -e^{-(0)} \rangle}{\sqrt{1 + e^{-2(0)}}}$

$= \dfrac{\langle 1, 0, -1 \rangle}{\sqrt{2}} = \boxed{\left\langle \dfrac{1}{\sqrt{2}}, 0, -\dfrac{1}{\sqrt{2}} \right\rangle}$

9. $\vec{r}\,'(t) = \left\langle \dfrac{1}{2\sqrt{t}}, 0, 1 \right\rangle$

If $\vec{r}, \vec{r}\,'$ are orthogonal, then

$\vec{r}(t) \cdot \vec{r}\,'(t) = 0$

$= \sqrt{t}\left(\dfrac{1}{2\sqrt{t}}\right) + 1(0) + t(1) \implies \dfrac{1}{2} + t = 0$

$t = -\dfrac{1}{2}$

OR $|\vec{r}(t) \times \vec{r}\,'(t)| = |\vec{r}(t)||\vec{r}\,'(t)| \sin \underset{\uparrow}{\theta}$

$= \dfrac{\pi}{2}$ for orthogonality.

$\vec{r} \times \vec{r}\,' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sqrt{t} & 1 & t \\ \frac{1}{2\sqrt{t}} & 0 & 1 \end{vmatrix} = (1-0)\hat{i} - \left(\sqrt{t} - \dfrac{t}{2\sqrt{t}}\right)\hat{j} + \left(\dfrac{-1}{2\sqrt{t}}\right)\hat{k}$

$\longrightarrow$

$$|\vec{r}\times\vec{r}\,'| = \sqrt{1^2+\left(\frac{-\sqrt{t}}{2}\right)^2+\left(\frac{-1}{2\sqrt{t}}\right)^2}$$

$$= |\vec{r}||\vec{r}\,'| = \sqrt{(\sqrt{t})^2+1^2+t^2}\sqrt{\left(\frac{1}{2\sqrt{t}}\right)^2+0^2+1^2}$$

Square both sides!

$$\Rightarrow \cancel{1}+\cancel{\frac{t}{4}}+\cancel{\frac{1}{4t}} = (t+1+t^2)\left(\frac{1}{4t}+1\right)$$

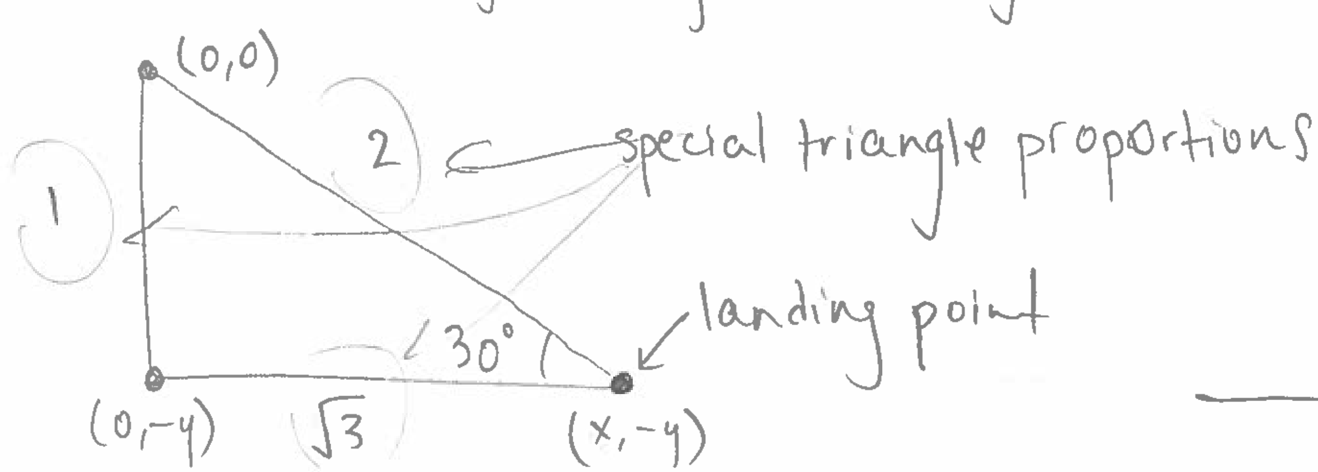
$$= \frac{1}{4}+\cancel{\frac{1}{4t}}+\cancel{\frac{t}{4}}+t+\cancel{1}+t^2$$

$$0 = \frac{1}{4}+t+t^2 = \left(t+\frac{1}{2}\right)^2 \Rightarrow t=-\frac{1}{2}$$

However, $-\frac{1}{2}$ is not in the domain for t (because $t>0$). So $\boxed{\text{the vectors are never orthogonal.}}$

10. (a) The length of the jump is

$$L = \sqrt{x(\text{time of flight})^2 + y(\text{time of flight})^2}$$



From the triangle, $x = -\sqrt{3}y \Rightarrow y = \frac{-1}{\sqrt{3}}x$.

To find the position vector, use

$\vec{a} = \langle 0, -9.8 \rangle \, m/s^2 \leftarrow$ acceleration due to gravity

$\vec{v}_0 = \langle 40, 0 \rangle \, m/s \leftarrow$ initial velocity

$\Rightarrow \vec{v}(t) = \int \vec{a}\,dt = \langle 0, -9.8t \rangle + \vec{C} \leftarrow$ velocity vector

$\vec{v}(0) = \cancel{\langle 0, -9.8(0) \rangle} + \vec{C} = \langle 40, 0 \rangle$

and so $\vec{v}(t) = \langle 0, -9.8t \rangle + \langle 40, 0 \rangle$

$= \langle 40, -9.8t \rangle \, m/s$

The initial position is $\vec{r}_0 = \langle 0, 8 \rangle$ m.

$\Rightarrow \vec{r}(t) = \int \vec{v}\,dt = \left\langle 40t, -\frac{9.8}{2}t^2 \right\rangle + \vec{C}$

$\vec{r}(0) = \cancel{\langle 40(0), -4.9(0)^2 \rangle} + \vec{C} = \langle 0, 8 \rangle$

and so $\vec{r}(t) = \langle 40t, -4.9t^2 + 8 \rangle$.

The skier lands when $y = \frac{-1}{\sqrt{3}}x$

$\Rightarrow -4.9t^2 + 8 = -\frac{1}{\sqrt{3}}(40t)$

$0 = 4.9t^2 - \frac{40}{\sqrt{3}}t - 8$

$\longrightarrow$

$$t = \frac{-\left(\frac{-40}{\sqrt{3}}\right) \pm \sqrt{\left(\frac{-40}{\sqrt{3}}\right)^2 - 4(4.9)(-8)}}{2(4.9)}$$

$$= \frac{\frac{40}{\sqrt{3}} \pm \sqrt{\frac{1600}{3} + 156.8}}{9.8} \approx 5.04 \text{ sec}$$

only take the positive solution

and so the length of the jump is

$$L = \sqrt{\left(40\left(\frac{\frac{40}{\sqrt{3}} + \sqrt{\frac{1600}{3}+156.8}}{9.8}\right)\right)^2 + \left(-4.9\left(\frac{\frac{40}{\sqrt{3}} + \sqrt{\frac{1600}{3}+156.8}}{9.8}\right)^2 + 8\right)^2}$$

$$\boxed{\approx 232.66 \text{ m}}$$

(b) The position vector changes:

$\vec{a} = \langle -0.15, -9.8 \rangle \text{ m/s}^2$

$\Rightarrow \vec{v} = \langle -0.15t, -9.8t \rangle + \langle 40, 0 \rangle = \langle -0.15t + 40, -9.8t \rangle$

$\Rightarrow \vec{r} = \left\langle -\frac{0.15}{2}t^2 + 40t, -4.9t^2 \right\rangle + \langle 0, 8 \rangle = \langle -0.075t^2 + 40t, -4.9t^2 + 8 \rangle$

$\longrightarrow$

The time of flight is given by

$$-4.9t^2+8=\left(\frac{-1}{\sqrt{3}}\right)\left(-0.075t^2+40t\right)$$

$$0=\left(4.9+\frac{0.075}{\sqrt{3}}\right)t^2-\frac{40}{\sqrt{3}}t-8$$

$\Rightarrow T \approx 25.75$ sec
(exact solution is stored
in the calculator)

$$\Rightarrow L=|\vec{r}(T)| \approx \boxed{3{,}340.08\text{ m}}$$

(c) Find the position vector:

$$\vec{a}=\langle 0,-9.8\rangle$$

$$\Rightarrow \vec{v}=\int \vec{a}\,dt=\langle 0,-9.8t\rangle+\vec{c}$$

$$\vec{v}_0=\vec{v}(0)=\langle 0,-9.8(0)\rangle+\vec{c}=\langle 40\cos\theta,40\sin\theta\rangle$$

$$\Rightarrow \vec{v}(t)=\langle 40\cos\theta,-9.8t+40\sin\theta\rangle$$

$$\vec{r}=\int \vec{v}\,dt=\langle 40\cos\theta\, t,-4.9t^2+40\sin\theta\, t\rangle+\vec{c}$$

$$\vec{r}(0)=\langle 40\cos\theta(0),-4.9(0)^2+40\sin\theta(0)\rangle+\vec{c}=\langle 0,8\rangle$$

$$\Rightarrow \vec{r}(t)=\langle 40\cos\theta\, t,-4.9t^2+40\sin\theta\, t+8\rangle$$

$\longrightarrow$

Find time of flight:

$$-4.9t^2+40\sin\theta t+8=-\frac{1}{\sqrt{3}}(40\cos\theta)t$$

$$0=4.9t^2+\left(-\frac{1}{\sqrt{3}}40\cos\theta-40\sin\theta\right)t-8$$

$$T=\frac{\frac{1}{\sqrt{3}}40\cos\theta+40\sin\theta \pm\sqrt{\left(\frac{1}{\sqrt{3}}40\cos\theta+40\sin\theta\right)^2-4(4.9)(-8)}}{2(4.9)}$$

Let $\Theta=\frac{1}{\sqrt{3}}40\cos\theta+40\sin\theta\approx 23.09\cos\theta+40\sin\theta$

so $T=\dfrac{\Theta+\sqrt{\Theta^2+156.8}}{9.8}$

The length of the jump is maximized when T is maximized!

$$T'=\frac{1}{9.8}\left(\Theta'+\frac{1}{2}(\Theta^2+156.8)^{-1/2}(2\Theta)(\Theta')\right)=0$$

$$\Theta'=-\frac{40}{\sqrt{3}}\sin\theta+40\cos\theta$$

$$\Rightarrow \frac{1}{9.8}\left(-\frac{40}{\sqrt{3}}\sin\theta+40\cos\theta+\frac{\left(\frac{40}{\sqrt{3}}\cos\theta+40\sin\theta\right)\left(-\frac{40}{\sqrt{3}}\sin\theta+40\cos\theta\right)}{\sqrt{\left(\frac{40}{\sqrt{3}}\cos\theta+40\sin\theta\right)^2+156.8}}\right)=0$$

→

Factor!

$$\underbrace{\cancel{\frac{1}{9.8}}\left(-\frac{40}{\sqrt{3}}\sin\theta+40\cos\theta\right)}\underbrace{\left(1+\frac{\frac{40}{\sqrt{3}}\cos\theta+40\sin\theta}{\sqrt{\left(\frac{40}{\sqrt{3}}\cos\theta+40\sin\theta\right)^2+156.8}}\right)}=0$$

$$\Rightarrow 40\cos\theta=\frac{40}{\sqrt{3}}\sin\theta$$

$$\sqrt{3}=\tan\theta$$

$$\Rightarrow \boxed{\theta=60^\circ} \leftarrow$$ maximizes the jump.

$$\Rightarrow \frac{40}{\sqrt{3}}\cos\theta+40\sin\theta=-\sqrt{\left(\frac{40}{\sqrt{3}}\cos\theta+40\sin\theta\right)^2+156.8}$$

$$\Rightarrow \left(\frac{40}{\sqrt{3}}\cos\theta+40\sin\theta\right)^2=\left(\frac{40}{\sqrt{3}}\cos\theta+40\sin\theta\right)^2+156.8$$

no solution.