Mon 15 Feb

- Expect Exam back on Thursday.
- Quizzes:
  - Include drill instructor and time.
  - Don't turn in the Quiz sheet with your work.
  - Drill Exercise Tues 16 Feb and Quiz 4 Thurs 18 Feb.

Mon 15 Feb (cont.)

- Announcement:
  A student in this class requires a note-taker. If you are willing to upload your notes and plan to attend class on a REGULAR basis, please sign up via the CEA Online Services on the Center for Educational Access (CEA) website `http://cea.uark.edu`. On the CEA Online Services login screen, click on "Sign Up as a Note-taker". At the end of the semester you will receive verification of 48 community service hours OR a $50 gift card for providing class notes. All interested students are encouraged to sign up; preference may be given to volunteers seeking community service in an effort engage U of A students in community service opportunities. Please contact the Center for Educational Access at ceanotes@uark.edu if you have any questions.

§3.2 Graphing the Derivative

Recall: The graph of the derivative is essentially the graph of the collection of slopes of the tangent lines of a graph. If you just have a graph (without an equation for the graph), the best you can do is approximate the graph of the derivative.

Simple Checklist:

1. Note where $f'(x) = 0$.
2. Note where $f'(x) > 0$.

Question

What does this look like?

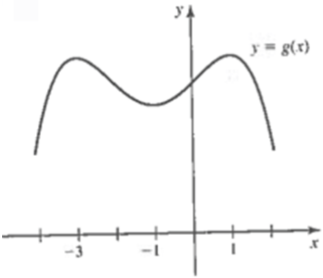
3. Note where $f'(x) < 0$.

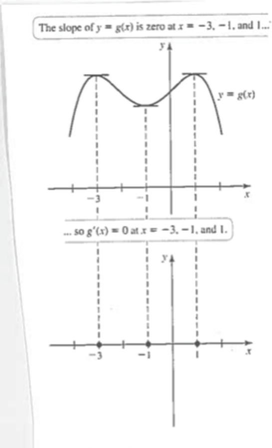
Question

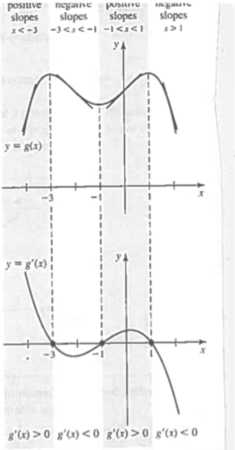
What does this look like?

Example

Given the graph of $g(x)$, sketch the graph of $g'(x)$.

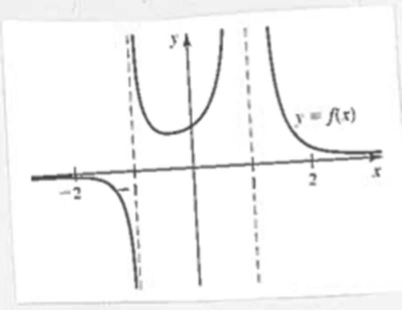


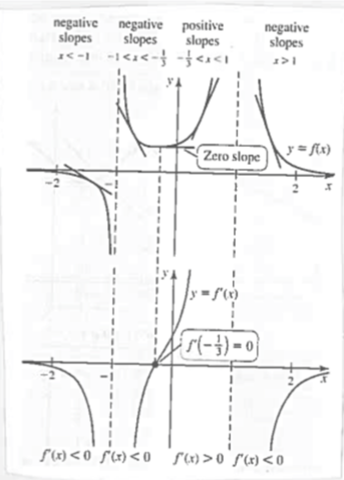

The slope of $y = g(x)$ is zero at $x = -3, -1$, and 1...
$y = g(x)$
... so $g'(x) = 0$ at $x = -3, -1$, and 1.


positive slopes $x < -3$
negative slopes $-3 < x < -1$
positive slopes $-1 < x < 1$
negative slopes $x > 1$
$y = g(x)$
$y = g'(x)$
$g'(x) > 0$
$g'(x) < 0$
$g'(x) > 0$
$g'(x) < 0$

Example (With Asymptotes)

Given the graph of $f(x)$, sketch the graph of $f'(x)$.




negative slopes
$x < -1$
negative slopes
$-1 < x < -\frac{1}{3}$
positive slopes
$-\frac{1}{3} < x < 1$
negative slopes
$x > 1$
Zero slope
$y = f(x)$
$y = f'(x)$
$f'(-\frac{1}{3}) = 0$
$f'(x) < 0$
$f'(x) < 0$
$f'(x) > 0$
$f'(x) < 0$

Recall the relationship between differentiability and continuity.

Exercise

If a function g is not continuous at $x = a$, then g

A. must be undefined at $x = a$.

B. is not differentiable at $x = a$.

C. has an asymptote at $x = a$.

D. all of the above.

E. A. and B. only.

3.2 Book Problems

5-14

§3.3 Rules of Differentiation

Recall the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(as a function of x, i.e., a formula).
And, for any particular point a, we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Constant Functions

The constant function $f(x) = c$ is a horizontal line with a slope of 0 at every point. This is consistent with the definition of the derivative:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{c - c}{h} \\ &= \lim_{h \to 0} 0 = 0. \end{aligned}$$

Therefore, for constant functions, $\frac{d}{dx} c = 0$.

Fact: For any positive integer n, we can factor

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

For example, when $n = 2$, we get

$$x^2 - a^2 = (x-a)(x+a),$$

which is the difference of squares formula.

Power Rule, cont.

Suppose $f(x) = x^n$ where n is a positive integer. Then at a point a,

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a} \\ &= \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{x - a} \\ &= (a^{n-1} + a^{n-2} \cdot a + \cdots + a \cdot a^{n-2} + a^{n-1}) = na^{n-1}. \end{aligned}$$

Using the formula for the derivative as a function of x, one can show $\frac{d}{dx}(x^n) = nx^{n-1}$.

Constant Multiple Rule

Consider a function of the form $cf(x)$, where c is a constant. Just like with limits, we can factor out the constant:

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}\\ &= \lim_{h\to 0}\frac{c[f(x+h)-f(x)]}{h} = c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= cf'(x)\end{aligned}$$

Therefore, $\frac{d}{dx}[cf(x)] = cf'(x)$.

Sum Rule

Sums of functions also behave under the same limit laws when we differentiate:

$$
\begin{aligned}
\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0} \left[\frac{[f(x+h)-f(x)]}{h} + \frac{[g(x+h)-g(x)]}{h} \right] \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= f'(x)+g'(x)
\end{aligned}
$$

So if f and g are differentiable at x,

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x).$$

The Sum Rule can be generalized for more than two functions to include n functions.

Note: Using the Sum Rule and the Constant Multiple Rule produces the Difference Rule:

$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x).$$

Exercise

Using the differentiation rules we have discussed, calculate the derivatives of the following functions. Note which rule(s) you are using.

1. $y = x^5$
2. $y = 4x^3 - 2x^2$
3. $y = -1500$
4. $y = 3x^3 - 2x + 4$

Let $f(x) = b^x$, where $b > 0$, $b \neq 1$. To differentiate at 0, we write

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{b^x - b^0}{x} = \lim_{x \to 0} \frac{b^x - 1}{x}.$$

It is not obvious what this limit should be. However, consider the cases $b = 2$ and $b = 3$. By constructing a table of values, we can see that

$$\lim_{x \to 0} \frac{2^x - 1}{x} \approx 0.693 \quad \text{and} \quad \lim_{x \to 0} \frac{3^x - 1}{x} \approx 1.099.$$

So, $f'(0) < 1$ when $b = 2$ and $f'(0) > 1$ when $b = 3$. As it turns out, there is a particular number b, with $2 < b < 3$, whose graph has a tangent line with slope 1 at $x = 0$. In other words, such a number b has the property that

$$\lim_{x \to 0} \frac{b^x - 1}{x} = 1.$$

Question

What number is it?

Answer: This number is $e = 2.718281828459\ldots$ (known as the Euler number). The function $f(x) = e^x$ is called the **natural exponential function**.

Now, using $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$, we can find the formula for $\dfrac{d}{dx}(e^x)$:

$$
\begin{aligned}
\frac{d}{dx}(e^x) &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} \\
&= e^x \left(\lim_{h\to 0} \frac{e^h - 1}{h} \right) \\
&= e^x \cdot 1 = e^x
\end{aligned}
$$

Exercise

(a) Find the slope of the line tangent to the curve $f(x) = x^3 - 4x - 4$ at the point $(2, -4)$.

(b) Where does this curve have a horizontal tangent?

If we can write the derivative of f as a function of x, then we can take *its* derivative, too. The derivative of the derivative is called the **second derivative** of f, and is denoted f''.

In general, we can differentiate f as often as needed. If we do it n times, the nth derivative of f is

$$f^{(n)}(x) = \frac{d^n f}{dx^n} = \frac{d}{dx}[f^{(n-1)}(x)].$$

3.3 Book Problems

9-48 (every 3rd problem), 51-53, 58-60

- For these problems, use only the rules we have derived so far.