Fall 2013 Cal III

Exam 2 Review Examples

1) If possible, find the absolute maximum and minimum values of the following functions on the set R.

(a) $f(x,y) = x^2+y^2-4$; $R = \{(x,y) \mid x^2+y^2 < 4\}$

Solution. Find the critical points:

$$f_x = 2x = 0$$
$$f_y = 2y = 0.$$

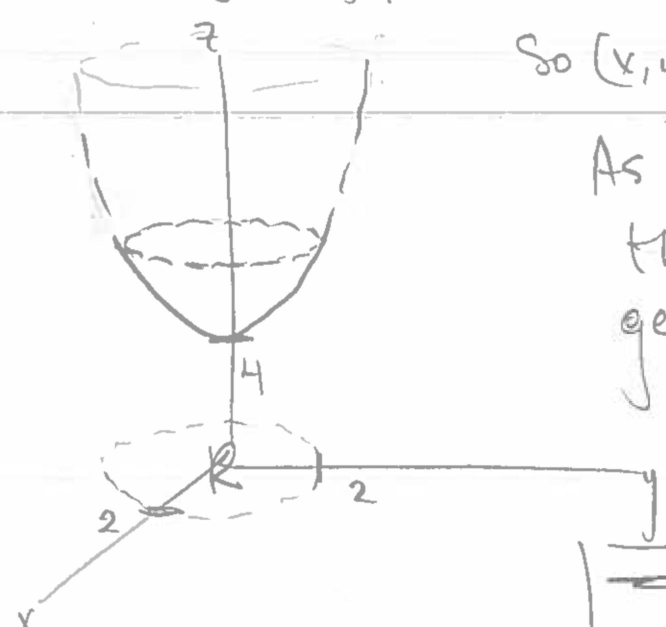
Both partials are defined everywhere and both equal zero at $(x,y) = (0,0)$. ✓

Use the discriminant to classify the point $(0,0)$:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = 2(2) - 0(0) = 4 > 0$$

and $f_{xx} = 2 > 0$,

So $(x,y) = (0,0)$ is a local min.



As (x,y)-values move closer to the boundary of R, $f(x,y)$ gets bigger, but since the boundary is not included, there is no absolute max.

$\boxed{\Longrightarrow (0,0) \text{ is the absolute min}}$

(b) $f(x,y) = 2e^{-x-y}$; $R = \{(x,y) \mid x \geq 0, y \geq 0\}$

$\boxed{\text{Solution.}}$ Find the critical points:

$$f_x = -2e^{-x-y} = 0$$

$$f_y = -2e^{-x-y} = 0$$

So no critical points.

On the other hand, $f(x,y)$ reaches its maximum (because it is a decreasing function) at $(0,0)$. As $(x,y) \to (\infty,\infty)$, $f(x,y) \to 0$.

The absolute max is $(0,0)$ and there is no absolute min.

2) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+2y}{x-2y} = ?$

$\boxed{\text{Solution.}}$ The limit cannot be evaluated by substituting $(x,y) = (0,0)$. On the other hand, the limit must exist when $(x,y) \to (0,0)$ from all directions (this is like in Cal I when we said the 2-sided limits must exist and be equal). If we approach $(0,0)$ from two different directions and get different answers, then the limit does not exist (this is the Two Path Test).

For these problems where $(x,y)\to(0,0)$, check the paths along a line $y=mx$ $(m\neq 0)$ first:

$$\lim_{(x,y)\to(0,0)} \frac{x+2(mx)}{x-2(mx)} = \lim_{(x,y)\to(0,0)} \frac{\cancel{x}(1+2m)}{\cancel{x}(1-2m)} = \frac{1+2m}{1-2m}$$

Since the limit changes values for different values of m, the limit $\boxed{\text{DNE}}$.

3) $$\lim_{(x,y)\to(0,0)} \frac{4xy}{3x^2+y^2} = ?$$

Solution. Again, since we can't plug in $(x,y)=(0,0)$ try $(x,y)\to(0,0)$ on the linear paths $y=mx$ $(m\neq 0)$.

$$\lim_{(x,y)\to(0,0)} \frac{4xy}{3x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{4x(mx)}{3x^2+(mx)^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{4m\cancel{x^2}}{(3+m^2)\cancel{x^2}} = \frac{4m}{3+m^2},$$

which varies depending on which line (which "m") $(x,y)\to(0,0)$.

So the limit $\boxed{\text{DNE}}$.

$\longrightarrow$

4) $\lim_{(x,y)\to(4,5)} \dfrac{\sqrt{x+y}-3}{x+y-9} = ?$

$\boxed{\text{Solution.}}$ This limit can be evaluated using an algebra technique related to the Difference of Squares Formula:

$$A^2 - B^2 = (A-B)(A+B)$$

Here, $A = \sqrt{x+y}$ and $B = 3$.

$$\lim_{(x,y)\to(4,5)} \frac{\sqrt{x+y}-3}{x+y-9}\left(\frac{\sqrt{x+y}+3}{\sqrt{x+y}+3}\right) = \lim_{(x,y)\to(4,5)} \frac{x+y-9}{(x+y-9)(\sqrt{x+y}+3)}$$

$$= \lim_{(x,y)\to(4,5)} \frac{1}{\sqrt{x+y}+3} = \frac{1}{\sqrt{4+5}+3} \boxed{= \frac{1}{6}}$$

5) Find the directions in the xy-plane in which

$$f(x,y) = e^{1-xy}$$

has zero change at the point $(1,0,e)$.

$\boxed{\text{Solution}}$ The goal is to find all possible unit vectors $\vec{u}$ so that the directional derivative $D_{\vec{u}}(1,0) = \nabla f(1,0) \cdot \vec{u} = 0$.

$f_x = -y e^{1-xy}$

$f_y = -x e^{1-xy}$

Solve for $\vec{u} = \langle u_1, u_2 \rangle$:

$$\nabla f(1,0) \cdot \vec{u} = \left\langle -(0)e^{1-(1)(0)}, -(1)e^{1-(1)(0)} \right\rangle \cdot \langle u_1, u_2 \rangle$$

$$= \langle 0, -e \rangle \cdot \langle u_1, u_2 \rangle$$

$$= 0(u_1) - e u_2 = 0.$$

$$\Rightarrow u_2 = 0.$$

But $\vec{u}$ must be a unit vector, so $\boxed{\vec{u} = \langle \pm 1, 0 \rangle.}$