Math 115 Quiz 8: $\oint$ 4.3, 4.4 **Mon 15 November 2010**
Optimization **Name:** ____________________________

You have 30 minutes to complete this quiz. Make your variables clear and consistent (so if you want to say, for example, $\frac{dy}{dx}$, you should also mention $y = f(x)$, or "y is a function of x"). Calculators are OK.

1. **Definitions/Concepts.** (1 pt each)

   (a) Give an example of a family of functions, $f(x)$, depending on a parameter a, such that each member of the family has exactly one critical point.

   $$f(x) = ax^2$$

   (b) TRUE or **FALSE**: If the radius of a circle is increasing at a constant rate, then so is the area.

   The radius is a function of time, $r(t)$, hence so is the area $A(t) = \pi r(t)^2$. Let $r'(t) = c$, the constant. Then

   $$\begin{aligned} A'(t) &= 2\pi r(t) \cdot r'(t) \\ &= 2c\pi r(t) \end{aligned}$$

   is not constant, since r is increasing.

2. **Questions/Problems.** Verify your extrema using either a well-labeled graph or the 2nd derivative.

   (a) (2pts) What point(s) on the graph of $y = \frac{1}{x^2}$ is closest to $(0, 0)$?

   Let R denote the square of the distance between the graph and the origin. Then minimize R:

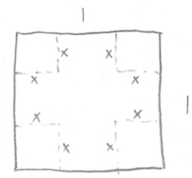
   $$\begin{aligned} R &= y^2 + x^2 \\ &= \left(\frac{1}{x^2}\right)^2 + x^2 \\ \frac{dR}{dx} = 0 &= \frac{-4}{x^5} + 2x \\ &= -4 + 2x^6 \\ 2 &= x^6 \\ \pm 2^{\frac{1}{6}} &= x \end{aligned}$$

   Investigate the nature of the critical points:

   $$\begin{aligned} \frac{d^2R}{dx^2} &= \frac{20}{x^6} + 2 \\ &> 0 \text{ for all } x \end{aligned}$$

so both are minima. Therefore the points on the graph closest to the origin are $\left(\pm 2^{\frac{1}{6}}, 2^{-\frac{1}{3}}\right) \approx (\pm 1.122, 0.794)$.

(b) (3 pts) I have a square piece of sheet metal, 1 meter on a side. I plan to cut equal squares from the four corners and fold up the sides to make a box (with no top). How big should the cut-off squares be in order to maximize the volume of the box?



From the figure, the volume of the box is

$$V = x(1-2x)^2.$$

Maximize the volume:

$$\begin{aligned} V &= x(1-2x)^2 \\ &= x - 4x^2 + 4x^3 \\ \frac{dV}{dx} &= 1 - 8x + 12x^2 = 0 \end{aligned}$$

The quadratic formula gives $x = \frac{8 \pm \sqrt{4}}{24}$ so $x = \frac{1}{2}, \frac{1}{6}$ are the critical points.

$$\frac{d^2V}{dx^2} = -8 + 24x$$

is positive when $x = \frac{1}{2}$ and negative when $x = \frac{1}{6}$ so the cutoff squares should be $\frac{1}{6} \times \frac{1}{6} \approx 1.167 \times 1.167\ \text{m}^2$ (if the lengths were half a meter, then there would be no metal left).

turn over →

3. **Computations/Algebra.** (1 pt each) Find the critical points.

   (a) $f(x) = (x-a)^2 + b$

$$\begin{aligned} f'(x) = 0 &= 2(x-a) \\ x &= a \end{aligned}$$

   (b) $g(y) = y^3 - ay^2 + b$

$$\begin{aligned} g'(y) = 0 &= 3y^2 - 2ay \\ &= y(3y - 2a) \\ y &= 0, \frac{2a}{3} \end{aligned}$$

   (c) $h(z) = az(z-b)^2$

$$\begin{aligned} h'(z) = 0 &= a(z-b)^2 + 2az(z-b) \\ &= ((z-b) + 2z)(z-b) \\ &= (3z-b)(z-b) \\ z &= \frac{b}{3}, b \end{aligned}$$

ChAlLeNgE PrObLeM: Give an example of a function f which satisfies the following. If no such function exists, say why. Assume f'' exists everywhere.

$$f(x)f'(x)f''(x)f'''(x) < 0 \text{ for all } x$$

(This problem was on the 1998 Putnam Mathematics Competition. I took the solution from University of Hawaii's math department website.)

No such exists.

Proof. By replacing $f(x)$ with $-f(x)$ if necessary, we can assume $f(x) > 0$. Similarly, by replacing $f(x)$ with $f(-x)$ if necessary, we can assume $f'(x) > 0$. Now to satisfy the condition f'' and f''' must have different signs. There are two cases:
Case 1: $f''(x) > 0$ and $f'''(x) < 0$.
Since $f'''(x) < 0$, the graph of $f'(x)$ is concave down. In particular, the graph of $f'(x)$ lies below its tangent line at $x = 0$. Thus,

$$f'(x) \leq f'(0) + f''(0)x$$

and this implies

$$\begin{aligned} f'\left(\frac{-f'(0)}{f''(0)}\right) &\leq f'(0) + f''(0) \cdot \left(\frac{-f'(0)}{f''(0)}\right) \\ &= 0 \end{aligned}$$

contradicting $f'(x)$ is always positive.
Case 2: $f''(x) < 0$ and $f'''(x) > 0$.
Since $f''(x) < 0$, the graph of $f(x)$ is concave down and hence the graph of $f(x)$ lies below its tangent line at $x = 0$. Thus,

$$f(x) \leq f(0) + f'(0)x$$

and this implies

$$\begin{aligned} f\left(\frac{-f(0)}{f'(0)}\right) &\leq f(0) + f'(0) \cdot \left(\frac{-f(0)}{f'(0)}\right) \\ &= 0 \end{aligned}$$

contradicting $f(x)$ is always positive.

$\square$