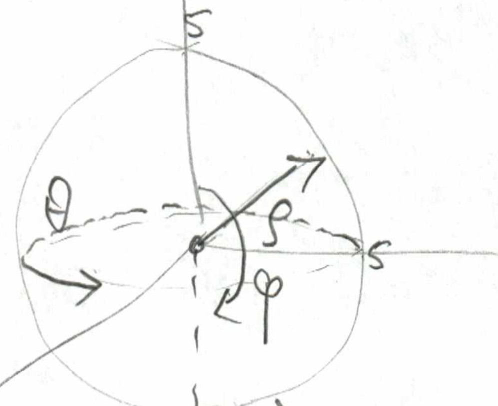
§14.8 #20 | $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$

Spherical coords: $0 \le \rho \le 5$

$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$



$\operatorname{div}\vec{F} = 2x + 2y + 2z$

$= 2(5\cos\theta\sin\varphi + 5\sin\theta\sin\varphi + 5\cos\varphi)$

$$10\int_0^{2\pi}\int_0^{\pi}\int_0^{5} \rho^2\sin\varphi(\underbrace{\cos\theta\sin\varphi + \sin\theta\sin\varphi}_{\text{group } \theta\text{s together}} + \cos\varphi)\,d\rho\,d\varphi\,d\theta$$

$$= 10\left(\frac{5^3}{3}\right)\int_0^{2\pi}\int_0^{\pi}\Big[(\cos\theta+\sin\theta)\underbrace{\sin^2\varphi}_{\text{half angle formula}} + \underbrace{\cancel{\cos\varphi\sin\varphi}}_{u\text{-sub}}\Big]\,d\varphi\,d\theta$$

u-sub: $u = \sin\varphi$

$du = \cos\varphi\,d\varphi$

$u(\pi) = 0$

$u(0) = 0$

$$= \frac{\overset{5}{\cancel{10}}}{3}(5^3)\int_0^{2\pi}\cancel{\tfrac{1}{2}}(\cos\theta+\sin\theta)\int_0^{\pi}(1-\cos2\varphi)\,d\varphi\,d\theta$$

$$= \frac{5^4}{3}\int_0^{2\pi}(\cos\theta+\sin\theta)\left(\varphi - \cancel{\tfrac{1}{2}\sin2\varphi}\right)\Big|_0^{\pi}\,d\theta$$

$\longrightarrow$

$$= \frac{5^4}{3}\pi\left(\cancel{\sin\theta} - \cos\theta\right)\Big|_0^{2\pi}$$

$$= -\frac{5^4}{3}\pi\left(\cos(2\pi) - \cos(0)\right) = \boxed{0}$$

Just for fun

$$\iint_S \vec{F}\cdot\hat{n}\, dS = \iint_S \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\, dv\, du$$

Put $\vec{r}(u,v) = \langle 5\cos u\sin v, 5\sin u\sin v, 5\cos v\rangle$

$0 \le u \le 2\pi,\ 0 \le v, \pi$

$$\vec{r}_u = 5\langle -\sin u\sin v, \cos u\sin v, 0\rangle$$

$$\vec{r}_v = 5\langle \cos u\cos v, \sin u\cos v, -\sin v\rangle$$

$$\vec{r}_u \times \vec{r}_v = 25\langle -\cos u\sin^2 v, -(\sin u\sin^2 v), \underbrace{-\sin^2 u\cos v\sin v - \cos^2 u\cos v\sin v}_{-(\cancel{\sin^2 u + \cos^2 u})\cos v\sin v}\rangle$$

$\longrightarrow$

$$\vec{F} = \langle 25\cos^2 u\sin^2 v, 25\sin^2 u\sin^2 v, 25\cos^2 v\rangle$$

$$-25^2\int_0^{2\pi}\int_0^{\pi}\left(\cos^3 u\sin^4 v + \sin^3 u\sin^4 v + \underbrace{\cancel{\cos^3 v\sin v}}_{}\right) dv\, du$$

w-sub: $w = \cos v$

$dw = -\sin v\, dv$

$w(\pi) = -1$

$w(0) = 1$

$$-\int_1^{-1} w^3\, dw = -\left(\frac{w^4}{4}\Big|_1^{-1}\right)$$

$$= -\cancel{\left(\frac{(-1)^4}{4} - \frac{1}{4}\right)}$$

$$= -25^2\int_0^{2\pi}(\cos^3 u + \sin^3 u)\int_0^{\pi}\sin^4 v\, dv$$

$$= -25^2\int_0^{2\pi}(\cos^3 u + \sin^3 u)\underbrace{\int_0^{\pi}\left(\frac{1-\cos 2v}{2}\right)^2 dv}_{}$$

$$\frac{1}{4}\int_0^{\pi}(1 - 2\cos 2v + \cos^2 2v)\, dv$$

$$= \frac{1}{4}(v - \cancel{\sin 2v})\Big|_0^{\pi} + \frac{1}{4}\int_0^{\pi}\left(\frac{1+\cos 4v}{2}\right) dv$$

$$= \frac{1}{4}\pi + \frac{1}{4}\left(\frac{1}{2}\right)\left(v + \cancel{\frac{1}{4}\sin 4v}\right)\Big|_0^{\pi}$$

$$= \frac{1}{4}\pi\left(\underbrace{1 + \frac{1}{2}}_{\frac{3}{2}}\right) = \frac{3}{8}\pi$$

$\longrightarrow$

$$= -25^2\left(\frac{3}{8}\pi\right)\int_0^{2\pi}\left(\cos^3 u + \sin^3 u\right)du$$

via trig integral formulas at the end of the book

$$= -25^2\left(\frac{3}{8}\pi\right)\left[\left(-\frac{1}{3}\sin^3 u + \sin u\right) + \left(\frac{1}{3}\cos^3 u - \cos u\right)\right]_0^{2\pi}$$

$$\cancel{\frac{1}{3}(1^3) - 1 - \left(\frac{1}{3}(1^3) - 1\right)}$$

$$\boxed{= 0}$$