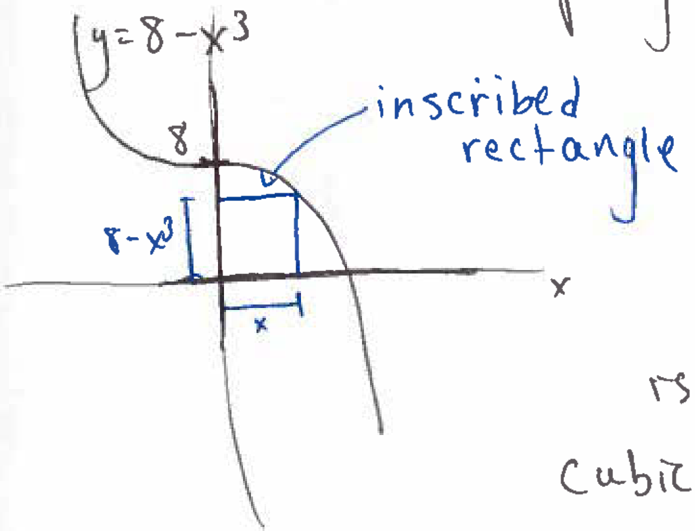
Wed 30 Mar 2016

Exercise. Find the dimensions of the rectangle of largest area which can be inscribed in the closed region bounded by the x-axis, y-axis, and the graph of $y = 8 - x^3$.

Solution:

The graph $y = 8 - x^3$ is an upside-down cubic, shifted up by 8:



The width of the inscribed rectangle is just the x-coordinate, so call it "x". The height is the y-coordinate of the cubic at that given x, so it's given by the formula $8 - x^3$.

The objective function is the area of the rectangle, $A = x(8 - x^3)$, and is already given in terms of one variable. The constraints also give the domain of A: The smallest x can be is 0, since the y-axis is one of →

the sides of the rectangle. The largest x can be is the x-intercept of the function $y = 8 - x^3$:

$$0 = 8 - x^3 \Rightarrow x^3 = 8 \Rightarrow x = 2.$$

So the domain is $[0, 2]$.

Differentiate:

$$A'(x) = 8 - 4x^3 = 0$$

$$\Rightarrow x^3 = 2 \Rightarrow x = \sqrt[3]{2}$$. is a critical point.

• Verify the critical point gives a max by plugging it and the endpoints of the domain into the original area function:

$$A(0) = (0)(8 - 0^3) = 0$$

$$A(\sqrt[3]{2}) = \sqrt[3]{2}\left(8 - (\sqrt[3]{2})^3\right) = \sqrt[3]{2}(8 - 2) = 6\sqrt[3]{2}$$. ← max.

$$A(2) = (2)(8 - 2^3) = 0.$$

Now answer the question: The dimensions are $\boxed{\sqrt[3]{2} \times 6}$.

There are alternative ways to verify $x = \sqrt[3]{2}$ gives a max:

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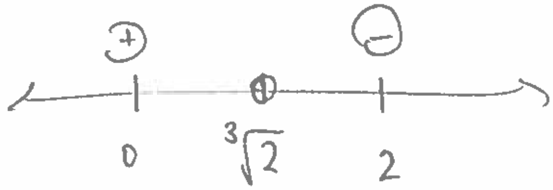
- 1^{st} Derivative Test:



$A'(0) = 8 - 4(0) > 0$

$A'(2) = 8 - 4(2^3) < 0$

A' changes from + to − so $x = \sqrt[3]{2}$ gives a local max; Since this is the only local extremum on the interval $[0,2]$ it must be absolute.

- 2^{nd} Derivative Test:

$A''(x) = -12x^2 < 0$

for all x means the area function is always concave down, and so the critical point must be a max; since it is the only local extremum on the interval $[0,2]$ it must be absolute.