Solutions: Mon 7 Mar Slides

Cal I Spring 2016

Slide 8. Determine the slope of the tangent line to the graph $f(x)=4^x$ at $x=0$.

• Solution.

$f'(x) = 4^x(\ln 4)$.

$f'(0) = 4^0(\ln 4) = \boxed{\ln 4} = 2\ln 2$

Slide 9: Story Problem Example

The energy (in Joules) released by an earthquake of magnitude M is given by the equation

$$E = 25000 \cdot 10^{1.5M}$$

(a) How much energy is released in a magnitude 3.0 earthquake?

• Solution. $E(3.0) = 25000 \cdot 10^{1.5(3.0)}$

$= 790{,}569{,}415 \text{ J}$

$\boxed{\approx 7.901 \times 10^8 \text{ J}}$

(b) What size earthquake releases 8 million Joules of energy?

$\log_{10}\left(E(M) = 25000 \cdot 10^{1.5M} = 8000000\right)$

$$\log_{10}\left(25000 \cdot 10^{1.5M}\right) = \log_{10}(8\,000\,000)$$

$$\log_{10}(25000) + \log_{10}\left(10^{1.5M}\right) = \log_{10}(8 \times 10^6)$$

$$1.5M = \log_{10}(8 \times 10^6) - \log_{10}(2.5 \times 10^4)$$

$$= \log_{10}\left(\frac{8 \times 10^6}{2.5 \times 10^4}\right) = \log_{10}\left(\frac{8}{2.5} \times 10^2\right)$$

$$\Rightarrow M = \frac{2 + \log_{10}(3.2)}{1.5}$$

$$\boxed{\approx 1.670}$$

(c) What is $\frac{dE}{dM}$ and what does it tell you?

• Solution.

$$\frac{dE}{dM} = 25000 \cdot 10^{1.5M} \cdot \ln(10) \cdot 1.5$$

$$= 25000 \cdot 10^{1.5M}\left(\ln(10^{1.5})\right)$$

measures how much the energy of an earthquake increases given an increase of magnitude.

Slide 18 · Evaluate the following limits:

- $\lim\limits_{x \to 3} \dfrac{x^2 - x - 6}{x^2 - 9}$

- **Solution**. Factor: $\lim\limits_{x \to 3} \dfrac{\cancel{(x-3)}(x+2)}{\cancel{(x-3)}(x+3)}$

$$= \frac{(3)+2}{(3)+3} = \boxed{\frac{5}{6}}$$

- $\lim\limits_{\theta \to 0} \dfrac{\sec\theta \tan\theta}{\theta}$

- **Solution**. Change to sines and cosines:

$$\lim_{\theta \to 0} \frac{\frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta}}{\theta} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta \cos^2\theta}$$

$$= \underbrace{\cancel{\lim_{\theta \to 0} \frac{\sin\theta}{\theta}}}_{1} \left(\underbrace{\cancel{\lim_{\theta \to 0} \frac{1}{\cos^2\theta}}}_{\frac{1}{\cos^2 0} = 1} \right) = \boxed{1}$$

Slide 21. Determine the horizontal asymptote(s) for the function

$$f(x) = \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$$

• Solution.

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}} \left(\frac{\frac{1}{x^3}}{\frac{1}{x^3}} \right)$$

$x > 0$ so $\sqrt{x^6} = x^3$

$$= \lim_{x \to \infty} \frac{10 - \cancel{\frac{3}{x}}^{\to 0} + \cancel{\frac{8}{x^3}}^{\to 0}}{\sqrt{25 + \underbrace{\cancel{\frac{1}{x^2}}}_{0} + \underbrace{\cancel{\frac{2}{x^6}}}_{0}}} = \frac{10}{\sqrt{25}} = \frac{10}{5} = 2$$

$\Rightarrow$ HA @ $\boxed{y = 2}$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}} \left(\frac{\frac{1}{-x^3}}{\frac{1}{-x^3}} \right)$$

$x < 0$ so $\sqrt{x^6} = -x^3$

$$= \lim_{x \to -\infty} \frac{-10 + \cancel{\frac{3}{x}}^{\to 0} - \cancel{\frac{8}{x^3}}^{\to 0}}{\sqrt{25 + \underbrace{\cancel{\frac{1}{x^2}}}_{0} + \underbrace{\cancel{\frac{2}{x^6}}}_{0}}} = \frac{-10}{5} = -2$$

$\Rightarrow$ HA @ $\boxed{y = -2}$

$\longrightarrow$

Slide 23. Determine the value of k so the function is continuous on $0 \le x \le 2$.

$$f(x) = \begin{cases} x^2 + k & 0 \le x \le 1 \\ -2kx + 4 & 1 < x \le 2 \end{cases}$$

- Solution Use the continuity checklist for $a = 1$:

① $f(1) = 1^2 + k = k + 1$

② $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + k) = k + 1$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (-2kx + 4) = -2k(1) + 4$
$= -2k + 4$

Want $k + 1 = -2k + 4$, which will also satisfy ③.

$3k = 3$

$\boxed{k = 1}$

Slide 27. Given that $y=3x+2$ is tangent to $f(x)$ at $x=1$ and that $y=-5x+6$ is tangent to $g(x)$ at $x=1$, write the equation of the tangent line to $h(x)=f(x)g(x)$ at $x=1$.

- Solution. Know: $f'(1)=3$, $f(1)=3(1)+2=5$
(b/c tangent at this point)

$$g'(1)=-5, \quad g(1)=-5(1)+6=1$$

$$h'(x)=f'(x)g(x)+f(x)g'(x)$$

$$h'(1)=f'(1)g(1)+f(1)g'(1)$$

$$=3(1)+5(-5)=3-25=-22$$

$$h(1)=f(1)g(1)=5(1)=5$$

line:

$$\boxed{y-5=-22(x-1)}$$

Slide 29. Suppose you have the following information about the functions f and g:

$$f(1)=6 \quad f'(1)=2 \quad g(1)=2 \quad g'(1)=3$$

- Let $F=2f+3g$. What is $F(1)$? What is $F'(1)$?
- Solution. $F(1)=2f(1)+3g(1)$

$$=2(6)+3(2)=12+6=\boxed{18}$$

- Let $G = fg$. What is $G(1)$? What is $G'(1)$?
- Solution. $G(1) = f(1)g(1) = 6(2) = \boxed{12}$

$$G'(1) = f'(1)\,g(1) + f(1)\,g'(1)$$
$$= 2(2) + 6(3) = 4 + 18 = \boxed{22}.$$

Slide 31. Calculate the derivative of the following functions:

- $f(x) = (1 + \sec x)\sin^3 x$
- Solution. $f'(x) = (0 + \sec x \tan x)(\sin^3 x)$

$$+ (1 + \sec x)(3\sin^2 x)(\cos x).$$

- $g(x) = \dfrac{\sin x + \cot x}{\cos x}$

$\dfrac{\cos x}{\sin x}$

- Solution. Rewrite: $g(x) = \dfrac{\sin x}{\cos x} + \dfrac{\cot x}{\cos x}$

$$= \tan x + \sec x$$

Then $\boxed{g'(x) = \sec^2 x + \sec x \tan x}$

Evaluate $\displaystyle\lim_{x \to -3} \frac{\sin(x+3)}{x^2 + 8x + 15}$.

- Solution. Factor: $\displaystyle\lim_{x \to -3} \frac{\sin(x+3)}{(x+5)(x+3)}$. Change variables:

Let $u = x+3$ and rewrite $\displaystyle\lim_{u \to 0} \frac{\sin u}{(u+2)(u)}$

$$= \underbrace{\lim_{u \to 0} \frac{\sin u}{u}}_{1} \overbrace{\left(\lim_{u \to 0} \frac{1}{u+2}\right)}^{\frac{1}{0+2}} = \boxed{\frac{1}{2}}$$

Slide 35. Suppose $f'(9) = 10$ and $g(x) = f(x^2)$.
What is $g'(3)$?

- Solution. $g'(x) = f'(x^2) \cdot 2x$

$$\Rightarrow g'(3) = f'(3^2) \cdot 2(3)$$
$$= 10(6) = \boxed{60}$$

Slide 37. Use implicit differentiation to calculate $\frac{dz}{dw}$ for $e^{2w} = \sin(wz)$.

- Solution. $\frac{d}{dw}\left(e^{2w} = \sin(wz)\right)$

$$2e^{2w} = \cos(wz)\left((1)z + w\frac{dz}{dw}\right)$$
$$2e^{2w} = z\cos(wz) + w\cos(wz)\frac{dz}{dw}$$
$$\rightarrow \boxed{\frac{dz}{dw} = \frac{2e^{2w} - z\cos(wz)}{w\cos(wz)}}$$

If $\sin x = \sin y$, then

- $\frac{dy}{dx} = ?$

- Solution. $\frac{d}{dx}\left(\sin x = \sin y\right)$

$$\cos x = \cos y \frac{dy}{dx}$$
$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{\cos x}{\cos y}}$$

$\longrightarrow$

• $\frac{d^2y}{dx^2} = ?$

• Solution. $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\cos x}{\cos y}\right)$

$$= \frac{\cos y(-\sin x) - \cos x(-\sin y)\frac{dy}{dx}}{\cos^2 y}$$

$$= \frac{-\cos y \sin x + \cos x \sin y\left(\frac{\cos x}{\cos y}\right)}{\cos^2 y}$$

$$= \frac{-\cos^2 y \sin x + \cos^2 x \sin y}{\cos^3 y}.$$