Math 115 Quiz 5: $\oint$ 3.1-4 Basic Shortcuts **Mon 25 October 2010**

Name: ______________________________

You have 30 minutes to complete this quiz. Make your variables clear and consistent (so if you want to say, for example, $\frac{dy}{dx}$, you should also mention $y = f(x)$, or "y is a function of x"). Calculators are OK.

1. **Definitions/Concepts.** (1 pt each) State the following:

   (a) Product Rule:

   If $u = f(x)$ and $v = g(x)$ are differentiable, then

   $$(fg)' = f'g + g'f.$$

   The product rule can also be written

   $$\frac{d(uv)}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}.$$

   In words:
   The derivative of a product is the derivative of the first times the second plus the first times the derivative of the second.

   (b) Quotient Rule:

   If $u = f(x)$ and $v = g(x)$ are differentiable, then

   $$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2},$$

   or equivalently,

   $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx} \cdot v - u \cdot \frac{dv}{dx}}{v^2}.$$

   In words:
   The derivative of a quotient is the derivative of the numerator times the denominator minus the numerator times the derivative of the denominator, all over the denominator squared.

   (c) Chain Rule:

   If f and g are differentiable functions, then

   $$\frac{d}{dx} f\left(g(x)\right) = f'\left(g(x)\right) \cdot g'(x).$$

   In words:
   The derivative of a composite function is the product of the derivatives of the outside and inside functions. The derivative of the outside function must be evaluated at the inside function.

2. **Questions/Problems.** The acceleration due to gravity, g, at a distance r from the center of the earth is given by

$$g = \frac{GM}{r^2},$$

where M is the mass of the earth and G is a constant.

(a) (1 pt) Find $\frac{dg}{dr}$.

Use the power rule applied to r^{-2} and multiply by the constant GM.

$$\begin{aligned} \frac{dg}{dr} &= -2 \cdot r^{-3} \cdot GM \\ &= \frac{-2GM}{r^3} \end{aligned}$$

(b) (2 pts) What is the practical interpretation (in terms of acceleration) of $\frac{dg}{dr}$? Why would you expect it to be negative?

The derivative $\frac{dg}{dr}$ is the rate of change of acceleration due to gravity, as the distance, r, from the center of the earth increases. Gravitational pull toward the earth decreases at distances far from the earth, so g decreasing implies $\frac{dg}{dr} < 0$.

(c) (1 pt) You are told that $M = 6 \cdot 10^{24}$ and $G = 6.67 \cdot 10^{-20}$ where M is in kilograms and r in kilometers. What is the value of $\frac{dg}{dr}$ at the surface of the earth ($r = 6400$ km)?

The typo in this problem is fixed.
Replace the constants with the given values and evaluate the derivative at $r = 6400$.

$$\begin{aligned} g'(6400) &= -2\frac{(6.67 \cdot 10^{-20})(6 \cdot 10^{24})}{6400^3} \\ &\approx -3.053 \cdot 10^{-6} \text{ kg/km}^3 \end{aligned}$$

(d) (1pt) What does this tell you about whether or not it is reasonable to assume g is constant near the surface of the earth?

It is totally reasonable. According to part (c), near the surface of the earth the acceleration due to gravity decreases very slightly if r increases by 1 km. To ease computations, assuming g is constant near the surface of the earth will only give an error on the order of 10^{-6} kg/km^2.

3. **Computations/Algebra.** (1 pt each) Differentiate with respect to x. You must show work to get credit.

(a) $f(x) = \frac{x^2+3x+2}{x+1}$

$$\begin{aligned}
f'(x) &= \frac{(x+1)\frac{d}{dx}(x^2+3x+2) - (x^2+3x+2)\frac{d}{dx}(x+1)}{(x+1)^2} \\
&= \frac{(x+1)(2x+3) - (x^2+3x+2)(1)}{(x+1)^2} \\
&= \frac{x^2+2x+1}{(x+1)^2} \\
&= 1
\end{aligned}$$

Alternate answer: Notice

$$\begin{aligned}
\frac{x^2+3x+2}{x+1} &= \frac{(x+2)(x+1)}{x+1} \\
&= x+2.
\end{aligned}$$

Then

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(x+2) \\
&= 1.
\end{aligned}$$

(b) $g(x) = x^k + k^x$

$$g'(x) = kx^{k-1} + (\ln k)k^x$$

ChAlLeNgE PrObLeM: Use the identity

$$\ln(a^x) = x \ln a$$

and the chain rule to write an alternate justification of the formula

$$\frac{d}{dx}a^x = (\ln a)a^x.$$

The point of this exercise was to segue into ∮3.6. The answer is part of the text; see p. 147.
Differentiate both sides of the identity with respect to x:

$$\begin{aligned}
\frac{d}{dx}(\ln(a^x) &= x \ln a) \\
\frac{d}{d(a^x)}\ln(a^x) \cdot \frac{d}{dx}a^x &= \frac{d}{dx}x \ln a \\
\frac{d}{dx}a^x &= \frac{\ln a}{\frac{d}{d(a^x)}\ln(a^x)}
\end{aligned}$$

The last step is to verify

$$\frac{d}{d(a^x)} \ln(a^x) = \frac{1}{a^x}.$$

To simplify notation, put $y = a^x$. We want to find

$$\frac{d}{dy} \ln y.$$

Recall the identity $e^{\ln y} = y$. Now differentiate both sides with respect to y, using the chain rule on the lefthand side.

$$\begin{aligned} \frac{d}{dy}\left(e^{\ln y}\right. &= \left.y\right) \\ \frac{d}{dy} e^{\ln y} &= \frac{d}{dy} y \\ e^{\ln y} \cdot \frac{d}{dy}(\ln y) &= 1. \end{aligned}$$

Use the identity $e^{\ln y} = y$ again and solve for $\frac{d}{dy} \ln y$.

$$\begin{aligned} y \cdot \frac{d}{dy} \ln y &= 1 \\ \frac{d}{dy} \ln y &= \frac{1}{y}. \end{aligned}$$

Since $y = a^x$, indeed,

$$\frac{d}{d(a^x)} \ln(a^x) = \frac{1}{a^x}.$$

Therefore

$$\begin{aligned} \frac{d}{dx} a^x &= \frac{\ln a}{\frac{d}{d(a^x)} \ln(a^x)} \\ &= \frac{\ln a}{\frac{1}{a^x}} \\ &= (\ln a) a^x. \end{aligned}$$