Take Home Quiz 3

Math 2574 (Cal III)
Spring 2017

Solutions

1. $u(x,t) = 5\cos(2(x+ct)) + 3\sin(x-ct)$

$$u_x = 5(-\sin(2(x+ct)))(2) + 3\cos(x-ct)$$
$$= -10\sin(2x+2ct) + 3\cos(x-ct)$$
$$\Rightarrow u_{xx} = -10\cos(2x+2ct)(2) - 3\sin(x-ct)$$
$$= -20\cos(2x+2ct) - 3\sin(x-ct)$$
$$u_t = 5(-\sin(2(x+ct)))(2c) + 3\cos(x-ct)(-c)$$
$$= -10c\sin(2x+2ct) - 3c\cos(x-ct)$$
$$\Rightarrow u_{tt} = -10c\cos(2x+2ct)(2c) - (-3c)\sin(x-ct)(-c)$$
$$= -20c^2\cos(2x+2ct) - 3c^2\sin(x-ct)$$
$$= c^2\underbrace{(-20\cos(2x+2ct) - 3\sin(x-ct))}_{u_{xx}} \quad \checkmark$$

2. (a) $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$

$= f(\Delta x, \Delta y) - f(0,0)$

$= 2\Delta x + 3(\Delta y)^2 - \cancel{(2(0)+3(0)^2)}$

$\boxed{= 2\Delta x + 3(\Delta y)^2}$

(b) $f_x(0,0) = 2$

$f_y(0,0) = 6(0)^2 = 0$

(c) $\varepsilon_1 = \dfrac{f(\Delta x, 0) - f(0,0)}{\Delta x} - f_x(0,0)$

$= \dfrac{2\cancel{\Delta x} - 0}{\cancel{\Delta x}} - 2 = 2 - 2 = \boxed{0}$

$\varepsilon_2 = \dfrac{f(0,\Delta y) - f(0,0)}{\Delta y} - f_y(0,0)$

$= \dfrac{3(\Delta y)^{\cancel{2}} - 0}{\cancel{\Delta y}} - 0 = \boxed{3\Delta y}$

In the definition, check:

- Does $\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$?

ans:

RHS (right-hand side of the equation) is:

$2\Delta x + 0\Delta y + 0\Delta x + 3\Delta y(\Delta y)$

$= 2\Delta x + 3(\Delta y)^2 = \Delta z$ ✓

- Does $\varepsilon_1 \to 0$ as $\Delta x \to 0$?

ans.

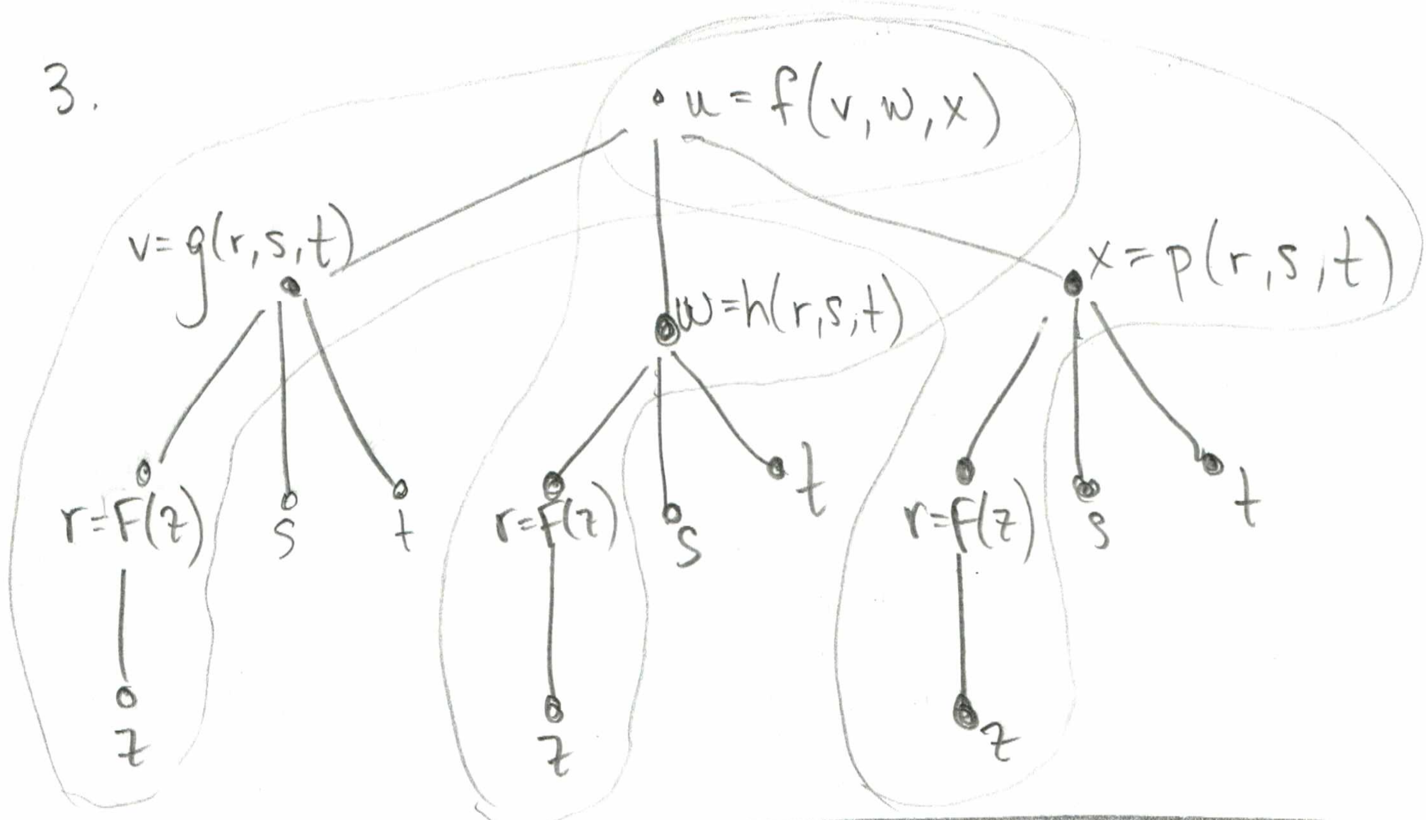
Yes, b/c $\varepsilon_1 = 0$.

- Does $\varepsilon_2 \to 0$ as $\Delta y \to 0$?

ans.

$\lim_{\Delta y \to 0} \varepsilon_2 = \lim_{\Delta y \to 0} 3\Delta y = 0$ ✓

3.



$$\boxed{\frac{du}{dz} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial r}\frac{dr}{dz} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial r}\frac{dr}{dz} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial r}\frac{dr}{dz}}$$

4. (a) $x_r = \cos\theta$

$y_r = \sin\theta$

$x_\theta = -r\sin\theta$

$y_\theta = r\cos\theta$

(b) $r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2+y^2}$ since $r \geq 0$.

$$\Rightarrow r_x = \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2}(\cancel{2}x)$$

$$\boxed{\begin{aligned} r_x &= \frac{x}{\sqrt{x^2+y^2}} \\ r_y &= \frac{y}{\sqrt{x^2+y^2}} \end{aligned}}$$

$$\tan\theta = \frac{y}{x} \Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$$

$$\theta_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}(-yx^{-2}) = \boxed{\frac{-y}{x^2+y^2} = \theta_x}$$

$$\theta_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{1}{x+\frac{y^2}{x}} = \boxed{\frac{x}{x^2+y^2} = \theta_y}$$

$\longrightarrow$

(c) Chain Rule:

$$z_r = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \boxed{f_x \cos\theta + f_y \sin\theta = z_r}$$

$$z_\theta = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = \boxed{-f_x r\sin\theta + f_y r\cos\theta = z_\theta}$$

(d)

$$w_x = \frac{\partial g}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial x} = \boxed{g_r\left(\frac{x}{\sqrt{x^2+y^2}}\right) - g_\theta\left(\frac{y}{x^2+y^2}\right) = w_x}$$

$$w_y = \frac{\partial g}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial y} = \boxed{g_r\left(\frac{y}{\sqrt{x^2+y^2}}\right) + g_\theta\left(\frac{x}{x^2+y^2}\right) = w_y}$$

5. Level curve: $f(-1,-2) = -4 + 6(-1)^2 + 3(-2)^2 = 14$

$\quad\quad 6 \quad\quad 12$

is an ellipse given by

$$14 = -4 + 6x^2 + 3y^2$$

$$\Rightarrow 6x^2 + 3y^2 = 18$$

$$\frac{x^2}{3} + \frac{y^2}{6} = 1$$

OR $\vec{r}(t) = \langle \sqrt{3}\cos t, \sqrt{6}\sin t \rangle$

$\longrightarrow$

Max increase: $\nabla f = \langle 12x, 6y \rangle$

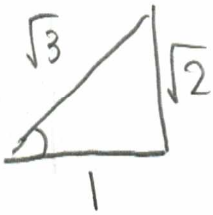
$$\nabla f(-1,-2) = \langle 12(-1), 6(-2) \rangle$$
$$= \langle -12, -12 \rangle$$

Max decrease $= -\nabla f(-1,-2) = \langle 12, 12 \rangle$

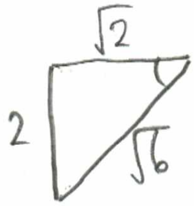
no change: $\vec{r}\,'(t) = \langle -\sqrt{3}\sin t, \sqrt{6}\cos t \rangle$

When $\vec{r}(t) = \langle -1, -2 \rangle \Rightarrow -1 = \sqrt{3}\cos t$

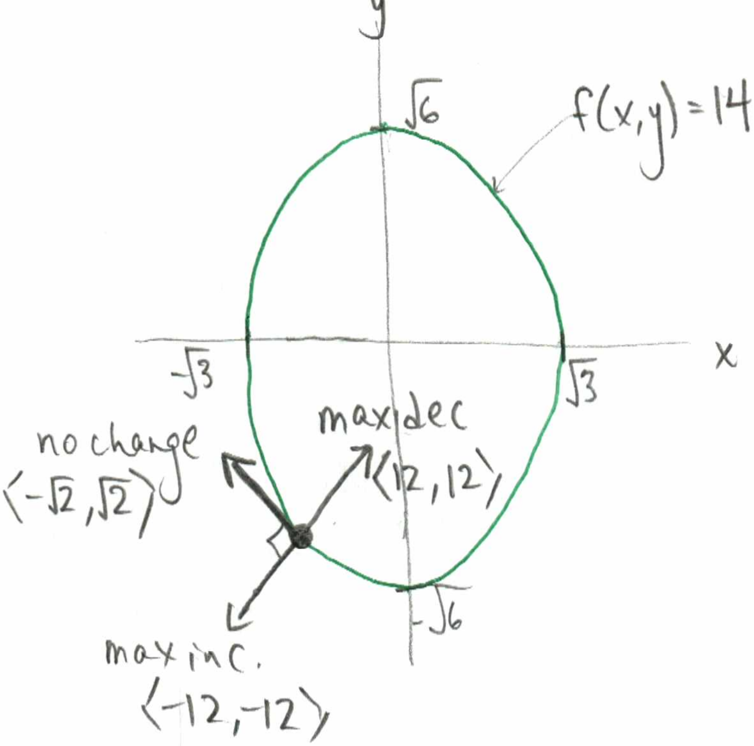
$$-2 = \sqrt{6}\sin t$$



$$-\sqrt{3}\sin\left(\arccos\left(\frac{-1}{\sqrt{3}}\right)\right) = -\sqrt{2}$$



$$\sqrt{6}\cos\left(\arcsin\left(\frac{-2}{\sqrt{6}}\right)\right) = \sqrt{2}$$



6. (a) $\nabla f = \langle y+z, x-z, x-y \rangle$

$\nabla f(1,1,1) = \langle 1+1, 1-1, 1-1 \rangle$

$\boxed{= \langle 2,0,0 \rangle}$

(b) Tail at P (and head at (x,y,z)):

$\langle x-1, y-1, z-1 \rangle$

Orthogonal to gradient!

$\nabla f \cdot \langle x-1, y-1, z-1 \rangle = 0$

$\langle 2,0,0 \rangle \cdot \langle x-1, y-1, z-1 \rangle = 0$

$\boxed{\text{Tangent plane: } 2(x-1)=0}$

or $x=1$

7. $\Delta x = \Delta y = dx = dy = 10^{-16}$

(a) $dz = f_x\,dx + f_y\,dy = y(10^{-16}) + x(10^{-16})$

$\Rightarrow \boxed{dz = 10^{-16}(x+y)}$

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$
$$= (x+10^{-16})(y+10^{-16}) - xy$$
$$= \cancel{xy} + 10^{-16}(x+y) + 10^{-32} - \cancel{xy}$$

$$\left|\frac{\Delta z - dz}{\Delta z}\right| = \left|\frac{\cancel{10^{-16}(x+y)} + 10^{-32} - \cancel{10^{-16}(x+y)}}{10^{-16}(x+y) + 10^{-32}}\right|$$
$$= \boxed{\left|\frac{10^{-16}}{x+y+10^{-16}}\right|}$$

(b) $dz = \frac{1}{y}(10^{-16}) + (-xy^{-2})(10^{-16})$

$$\rightarrow \boxed{dz = 10^{-16}\left(\frac{1}{y} - \frac{x}{y^2}\right)} \quad \text{or} \quad \frac{10^{-16}(y-x)}{y^2}$$

$$\Delta z = \frac{x+10^{-16}}{y+10^{-16}} - \frac{x}{y} = \frac{y(x+10^{-16}) - x(y+10^{-16})}{y(y+10^{-16})}$$
$$= \frac{\cancel{xy} + 10^{-16}y - \cancel{xy} - 10^{-16}x}{y(y+10^{-16})}$$
$$= \frac{10^{-16}(y-x)}{y(y+10^{-16})}$$

$\longrightarrow$

$$\left|\frac{\Delta z - dz}{\Delta z}\right| = \left| \frac{\frac{10^{-16}(y-x)}{y(y+10^{-16})} - \frac{10^{-16}(y-x)}{y^2}}{\frac{10^{-16}(y-x)}{y(y+10^{-16})}} \right|$$

$$= \left| \frac{\frac{\cancel{10^{-16}(y-x)}(\cancel{y}-\cancel{y}-10^{-16})}{y^{\cancel{2}}\cancel{(y+10^{-16})}}}{\frac{\cancel{10^{-16}(y-x)}}{\cancel{y(y+10^{-16})}}} \right|$$

$$= \left|\frac{-10^{-16}}{y}\right| = \boxed{\left|\frac{10^{-16}}{y}\right|}$$

(c) $dz = \Delta z = 10^{-16}$

$dw = f_x\,dx + f_y\,dy + f_z\,dz$

$\Rightarrow \boxed{dw = 10^{-16}(yz + xz + xy)}$

$\Delta w = f(x+10^{-16}, y+10^{-16}, z+10^{-16}) - f(x,y,z)$

$= (x+10^{-16})(y+10^{-16})(z+10^{-16}) - xyz$

$\longrightarrow$

$$= \cancel{xyz} + 10^{-16}(xy + yz + xz) + 10^{-32}(x+y+z)$$
$$+ 10^{-48} - \cancel{xyz}$$

$$\left|\frac{\Delta w - dw}{\Delta w}\right| = \frac{\left|10^{-16}\cancel{(xy+yz+xz)} + 10^{\overset{16}{-\cancel{32}}}(x+y+z) + 10^{\overset{32}{-\cancel{48}}} - \cancel{10^{-16}(xy+yz+xz)}\right|}{\left|\cancel{10^{-16}}(xy+yz+xz) + 10^{\overset{16}{-\cancel{32}}}(x+y+z) + 10^{\overset{32}{-\cancel{48}}}\right|}$$

$$\boxed{= \left|\frac{10^{-16}(x+y+z) + 10^{-32}}{xy + yz + xz + 10^{-16}(x+y+z) + 10^{-32}}\right|}$$

(d) $dw = \frac{1}{yz}\,dx + \frac{-x}{y^2 z}\,dy + \frac{-x}{y z^2}\,dz$

$$\boxed{\left| = \frac{10^{-16}(yz - xz - xy)}{y^2 z^2}\right|}$$

$$\Delta w = \frac{x + 10^{-16}}{(y + 10^{-16})(z + 10^{-16})} - \frac{x}{yz}$$

$$\cancel{xyz} + 10^{-16}yz \qquad x(\cancel{yz} + 10^{-16}z + 10^{-16}y + 10^{-32})$$

$$= \frac{(x+10^{-16})yz - x(y+10^{-16})(z+10^{-16})}{yz(y+10^{-16})(z+10^{-16})}$$

$$= \frac{10^{-16}(yz - xz - xy) + 10^{-32}x}{yz(y+10^{-16})(z+10^{-16})}$$

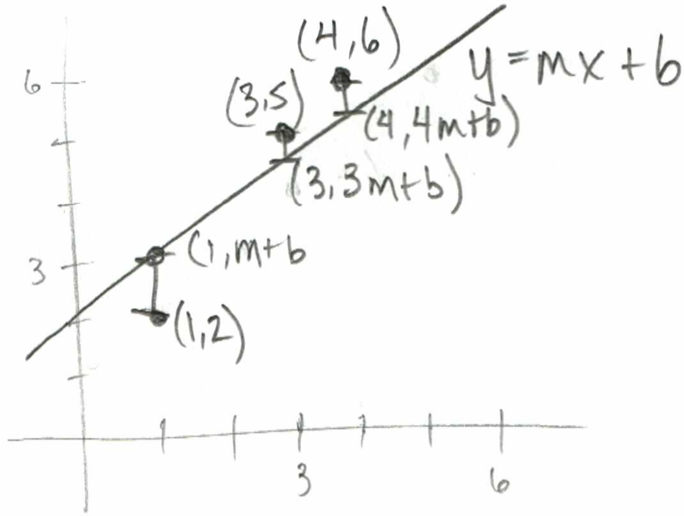
$$\left|\frac{\Delta w - dw}{\Delta w}\right| = \left| \frac{\dfrac{10^{-16}(yz-xz-xy)+10^{-32}x}{\cancel{yz}(y+10^{-16})(z+10^{-16})} - \dfrac{10^{-16}(yz-xz-xy)}{y^{\cancel{2}}z^{\cancel{2}}}}{\dfrac{10^{-16}(yz-xz-xy)+10^{-32}x}{\cancel{yz}(y+10^{-16})(z+10^{-16})}} \right|$$

$$(\cancel{yz} + 10^{-16}y + 10^{-16}z + 10^{-32})$$

$$= \left| \frac{\dfrac{\cancel{10^{-16}}(yz-xz-xy)(\cancel{yz} - (y+10^{-16})(z+10^{-16})) + 10^{\cancel{-32}\,16}xyz}{yz\cancel{(y+10^{-16})(z+10^{-16})}}}{\dfrac{\cancel{10^{-16}}(yz-xz-xy)+10^{\cancel{-32}\,16}x}{\cancel{(y+10^{-16})(z+10^{-16})}}} \right|$$

$$\boxed{= \left| \frac{(yz-xz-xy)(-y-z-10^{-16}) + 10^{-16}xyz}{(yz-xz-xy) + 10^{-16}xyz} \right|}$$

8. (a)



At the point (1,2), the vertical distance to the regression line is measured from the point (1, m+b). Summing the squares of the distances gives

$$F(m,b) = (m+b-2)^2 + (3m+b-5)^2 + (4m+b-6)^2.$$

(b) $F_m = 2(m+b-2) + 2(3m+b-5)(3) + 2(4m+b-6)(4)$

$= 2m+2b-4+18m+6b-30+32m+8b-48$

$= 52m+16b-82 = 0 \Rightarrow b = \dfrac{82-52m}{16} = \dfrac{41-26m}{8}$

$F_b = 2(m+b-2) + 2(3m+b-5) + 2(4m+b-6)$

$= 2m+2b-4+6m+2b-10+8m+2b-12$

$= 16m+6b-26 = 0 \Rightarrow m = \dfrac{26-6b}{16} = \dfrac{13-3b}{8}$

$\Rightarrow m = \dfrac{13 - \dfrac{3(41-26m)}{8}}{8}$

$\longrightarrow$

$$= \frac{13(8) - 3(41) - 3(26m)}{64}$$

$$\Rightarrow \underbrace{\left(64 - 3(26)\right)}_{-14} m = \underbrace{13(8) - 3(41)}_{-19}$$

$$m = \frac{19}{14} \Rightarrow b = \frac{41 - 26\left(\frac{19}{14}\right)}{8} = \frac{41(14) - 26(19)}{8(14)}$$

$$= \frac{5}{7}$$

$$D(m,b) = \begin{vmatrix} F_{mm} & F_{mb} \\ F_{bm} & F_{bb} \end{vmatrix} = \begin{vmatrix} 52 & 16 \\ 16 & 6 \end{vmatrix} = 52(6) - 16^2 = 56 > 0$$

$\Rightarrow \left(\frac{19}{14}, \frac{5}{7}\right)$ gives a min for $F(m,b)$.

(c) $y = \frac{19}{14}x + \frac{5}{7}$

$0 = \frac{19}{14}x + \frac{5}{7}$

$-\frac{5}{7}\left(\frac{14}{19}\right) = x$