Math 116 Quiz 3: $\oint$ 7.5, 8.1-8.2 (sans arc length) **Tue 25 Sep 2012**
Name: SOLUTIONS

You have 30 minutes to complete this quiz. Eyes on your own paper and good luck!

1. **Definitions/Concepts.** (2 pts) Fill in the following inequalities using the symbols TRAP(n) or MID(n).

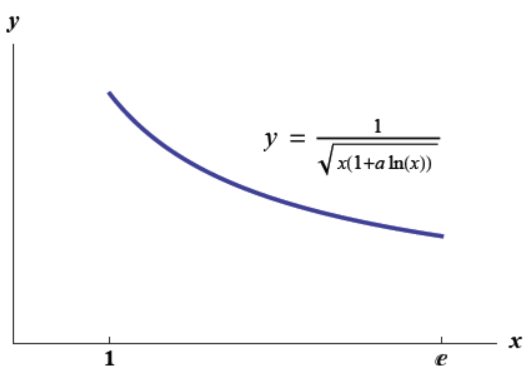
   If the graph of f is concave down on $[a, b]$, then

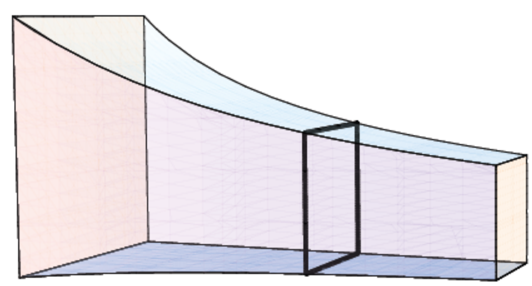
   $$\text{TRAP}(n) \leq \int_a^b f(x)dx \leq \text{MID}(n)$$

   If the graph of f is concave up on $[a, b]$, then

   $$\text{MID}(n) \leq \int_a^b f(x)dx \leq \text{TRAP}(n)$$

2. **Questions/Problems.** (8 pts) Let S be the solid whose base is the region bounded by the graph of the curve $y = \frac{1}{\sqrt{x(1+a\ln x)}}$ (for some positive constant $a > 0$), the x-axis, the lines $x = 1$ and $x = e$. The cross-sections of S perpendicular to the x-axis are squares. Find the exact volume of S.





The volume of one infinitessimal slice will be the area of the square multiplied by a width dx. The length of one side of the square is given by $\frac{1}{\sqrt{x(1+a\ln x)}}$. Now we can compute the total volume using an integral

$$V = \int_1^e \frac{dx}{x(1+a\ln x)}.$$

The most efficient way to evaluate this integral is to use the substitution $w = 1 + a\ln x$, $dw = \frac{a}{x}dx$. Now we evaluate the indefinite integral:

$$\frac{1}{a}\int \frac{dw}{w} = \frac{1}{a}\ln|w|$$

Resubstituting to get an expression in terms of x and applying the bounds of integration gives

$$\frac{1}{a}\ln|1+a\ln x|\Big|_1^e = \frac{1}{a}\ln|1+a|.$$

In fact, since we are told $a > 0$ we can drop the absolute value signs to get $\frac{1}{a}\ln(1+a)$.

3. **Computations/Algebra.**

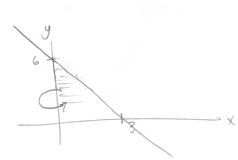
   (a) $\int_0^6 \pi(3-y/2)^2 dy$

   i. (1 pt) Which shape is being integrated? Choose one:
      A. triangle
      B. part of a circle
      C. hemisphere
      D. cone

   ii. (2 pts) If you chose triangle, write down the base and height, indicating which is which. If you chose part of a circle or hemisphere, write down the radius. If you chose cone, write down the radius and the height, indicating which is which.

      radius $= 3 - \frac{y}{2}$
      height $= 6$

   iii. (2 pts) Draw a picture to justify your answers to parts i. and ii.



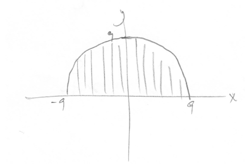
   (b) $\int_{-9}^9 \sqrt{81-x^2}dx$

   i. (1 pt) Which shape is being integrated? Choose one:
      A. triangle
      B. part of a circle
      C. hemisphere
      D. cone

   ii. (2 pts) If you chose triangle, write down the base and height, indicating which is which. If you chose part of a circle or hemisphere, write down the radius. If you chose cone, write down the radius and the height, indicating which is which.

      radius $= 9$

iii. (2 pts) Draw a picture to justify your answers to parts i. and ii.



ChAlLeNgE pRoBlEm: Rotate the bell curve $y = e^{-x^2/2}$ around the y-axis, forming a hill-shaped solid of revolution. Using horizontal slices, find the exact volume of this hill.

We will integrate discs of thickness dy. The radius of each disc is half of the length of the horizonal strip. We can get the radius exactly by solving for the x-coordinate:

$$\begin{aligned} y &= e^{-x^2/2} \\ \ln y &= \frac{-x^2}{2} \\ -2\ln y &= x^2 \\ \ln y^{-2} &= x^2 \\ \ln\frac{1}{y^2} &= x^2 \\ \sqrt{\ln\frac{1}{y^2}} &= x \end{aligned}$$

As $x \to \pm\infty$ the bell curve approaches 0, so we integrate y from γ to 1 (γ will be a "dummy" variable representing y) and take the limit as γ approaches zero.

$$\begin{aligned} \pi \int_\gamma^1 \left(\sqrt{\ln\frac{1}{y^2}}\right)^2 dy &= \pi \int_\gamma^1 \ln\frac{1}{y^2}\,dy \\ &= \pi \int_\gamma^1 -2\ln y\,dy \\ &= -2\pi \int_\gamma^1 \ln y\,dy \\ \text{(see p. 343 of the text on how to integrate natural log)} \quad &= -2\pi\left(y\ln y|_\gamma^1 - \int_\gamma^1 dy\right) \\ &= -2\pi\left(1\cdot\ln 1 - \gamma\cdot\ln\gamma - (1-\gamma)\right) \\ &= -2\pi(-\gamma\ln\gamma - 1 + \gamma) \end{aligned}$$

At this point the limit

$$\lim_{\gamma\to 0} \gamma\ln\gamma$$

is $0 \cdot (-\infty)$, which is not well-defined. However, we can apply L'Hôpital's rule by writing

$$\begin{aligned}
\lim_{\gamma \to 0} \gamma \ln \gamma &= \lim_{\gamma \to 0} \frac{\ln \gamma}{\frac{1}{\gamma}} \\
&= \lim_{\gamma \to 0} \frac{\frac{d}{d\gamma}(\ln \gamma)}{\frac{d}{d\gamma}\left(\frac{1}{\gamma}\right)} \\
&= \lim_{\gamma \to 0} \frac{\frac{1}{\gamma}}{-\frac{1}{\gamma^2}} \\
&= \lim_{\gamma \to 0} -\gamma \\
&= 0.
\end{aligned}$$

Then taking the limit of the integral, we get

$$\begin{aligned}
\lim_{\gamma \to 0} -2\pi(-\gamma \ln \gamma - 1 + \gamma) &= -2\pi(0 - 1 + 0) \\
&= 2\pi.
\end{aligned}$$