1 Week 13: 18-22 April

§5.2 Definite Integrals

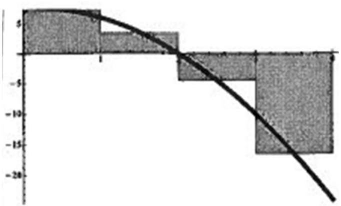
In §5.1, we saw how we can use Riemann sums to approximate the area under a curve. However, the curves we worked with were all non-negative.

Question

What happens when the curve is negative?

Example

Let $f(x) = 8 - 2x^2$ over the interval $[0, 4]$. Use a left, right, and midpoint Riemann sum with $n = 4$ to approximate the area under the curve.



Net Area

In the previous example, the areas where f was positive provided positive contributions to the area, while areas where f was negative provided negative contributions. The difference between positive and negative contributions is called the **net area**.

Definition

Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis minus the sum of the areas of the parts of R that lie below the x-axis on $[a, b]$.

The Riemann sums give approximations for the area under the curve. To make these approximations more and more accurate, we divide the region into more and more subintervals. To make these approximations exact, we allow the number of subintervals $n \to \infty$, thereby allowing the length of the subintervals $\Delta x \to 0$. In terms of limits:

$$\text{Net Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(\overline{x}_k)\Delta x.$$

General Riemann Sums

Suppose $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ are subintervals of $[a, b]$ with $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. Let Δx_k be the length of the subinterval $[x_{k-1}, x_k]$ and let $\overline{x}_k$ be any point in $[x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$. If f is defined on $[a, b]$, then the sum

$$\sum_{k=1}^{n} f(\overline{x}_k)\Delta x_k = f(\overline{x}_1)\Delta x_1 + f(\overline{x}_2)\Delta x_2 + \cdots + f(\overline{x}_n)\Delta x_n$$

is called a **general Riemann sum for f on $[a, b]$**.

Note: In this definition, the lengths of the subintervals do not have to be equal.

As $n \to \infty$, all of the $\Delta x_k \to 0$, even the largest of these. Let Δ be the largest of the Δx_k's.

Definition

The **definite integral of f from a to b** is

$$\int_a^b f(x)\ dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(\overline{x}_k)\Delta x_k,$$

where f is a function defined on $[a, b]$. When this limit exists – over *all* partitions of $[a, b]$ and *all* choices of $\overline{x}_k$ on a partition – f is called **integrable**.

Theorem

If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$.

See Figure 5.23, p. 325, for an example of a noncontinuous function that is integrable.

Knowing the limit of a Riemann sum, we can now translate that to a definite integral.

Example

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} (4\overline{x}_k - 3)\Delta x_k \text{ on } [-1, 4] \quad \equiv \quad \int_{-1}^{4} (4x - 3)\, dx$$

Without formally examining methods to evaluate definite integrals, we can use geometry.

Exercise

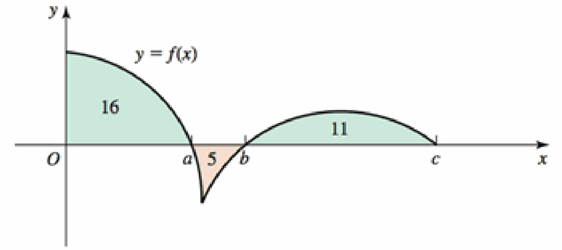
Using geometry, evaluate $\int_1^2 (4x-3)\ dx$.

(*Hint:* The area of a trapezoid is $A = \frac{h(l_1+l_2)}{2}$, where h is the height of the trapezoid and l_1 and l_2 are the lengths of the two parallel bases.)

Exercise

Using the picture below, evaluate the following definite integrals:

1. $\int_0^a f(x)\,dx$ 2. $\int_0^b f(x)\,dx$ 3. $\int_0^c f(x)\,dx$ 4. $\int_a^c f(x)\,dx$



Properties of Integrals

1. (Reversing Limits) $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$
2. (Identical Limits) $\int_a^a f(x)\ dx = 0$
3. (Integral of a Sum)
   $\int_a^b (f(x) + g(x))\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$
4. (Constants in Integrals) $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$

Properties of Integrals, cont.

5. (Integrals over Subintervals) If c lies between a and b, then

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

6. (Integrals of Absolute Values) The function $|f|$ is integrable on $[a, b]$ and $\int_a^b |f(x)|\ dx$ is the sum of the areas of regions bounded by the graph of f and the x-axis on $[a, b]$. (See Figure 5.31 on p. 329)

   (This is the total area, no negative signs.)

Exercise

If $\int_2^4 f(x)\ dx = 3$ and $\int_4^6 f(x)\ dx = -2$, then compute $\int_2^6 f(x)\ dx$.

5.2 Book Problems

11-45 (odds), 67-74

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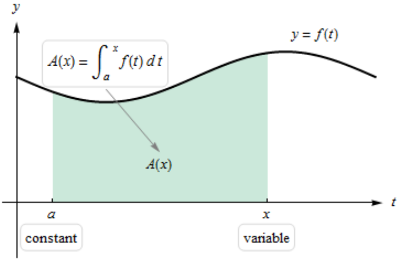
§5.3 Fundamental Theorem of Calculus

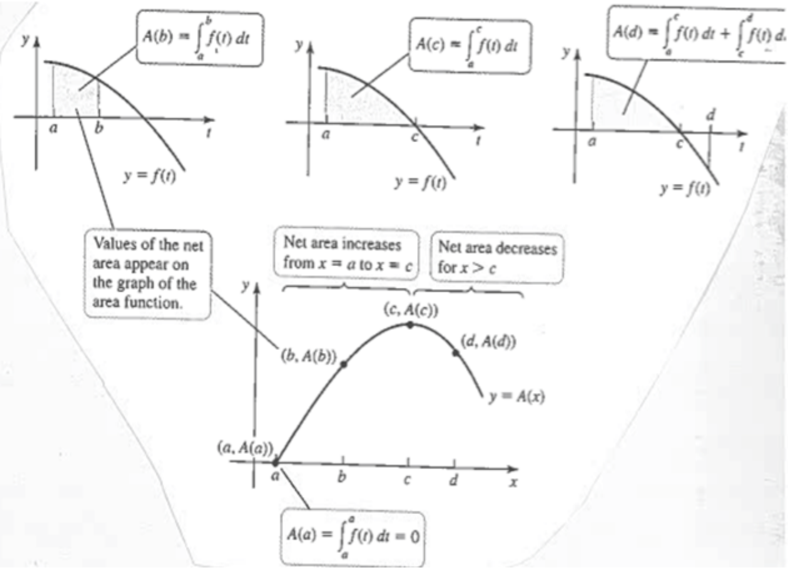
Using Riemann sums to evaluate definite integrals is usually neither efficient nor practical. We will develop methods to evaluate integrals and also tie together the concepts of differentiation and integration.

To connect the concepts of differention and integration, we first must define the concept of an area function.

Let $y = f(t)$ be a continuous function which is defined for all $t \geq a$, where a is a fixed number. The area function for f with left endpoint at a is given by $A(x) = \int_a^x f(t)\, dt$.



This gives the net area of the region between the graph of f and the t-axis between the points $t = a$ and $t = x$.



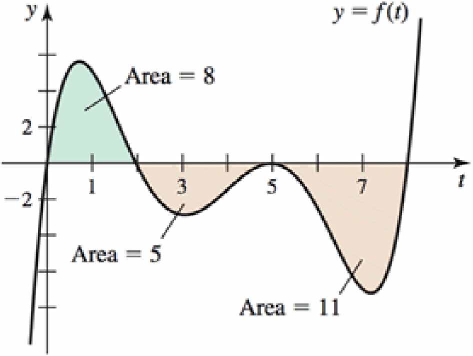
$A(b) = \int_a^b f(t)\,dt$
$A(c) = \int_a^c f(t)\,dt$
$A(d) = \int_a^c f(t)\,dt + \int_c^d f(t)\,dt$
y
a
b
c
d
t
$y = f(t)$
Values of the net area appear on the graph of the area function.
Net area increases from $x = a$ to $x = c$
Net area decreases for $x > c$
$(a, A(a))$
$(b, A(b))$
$(c, A(c))$
$(d, A(d))$
$y = A(x)$
x
$A(a) = \int_a^a f(t)\,dt = 0$

Example

The graph of f is shown below. Let

$$A(x) = \int_0^x f(t)\,dt \quad \text{and} \quad F(x) = \int_2^x f(t)\,dt$$

be two area functions for f. Compute $A(2), F(5), A(5), F(8)$.



Linear functions help to build the rationale behind the Fundamental Theorem of Calculus.

Example

Let $f(t) = 4t + 3$ and define $A(x) = \int_1^x f(t)\, dt$. What is $A(2)$? $A(4)$? $A(x)$? $A'(x)$?

In general, the property illustrated with this linear function works for all continuous functions and is one part of the FTOC (Fundamental Theorem of Calculus).

Theorem (FTOC I)

If f is continuous on $[a, b]$, then the area function $A(x) = \int_a^x f(t)\ dt$ for $a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$; or equivalently,

$$A'(x) = \frac{d}{dx} \int_a^x f(t)\ dt = f(x)$$

which means that the area function of f is an antiderivative of f.

The Fundamental Theorem of Calculus (Part 2)

Since A is an antiderivative of f, we now have a way to evaluate definite integrals and find areas under curves.

Theorem (FTOC II)

If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

We use the notation $F(x)|_a^b = F(b) - F(a)$.

In essence, to evaluate an integral, we

- Find any antiderivative of f, and call if F.
- Compute $F(b) - F(a)$, the difference in the values of F between the upper and lower limits of integration.

The two parts of the FTOC illustrate the inverse relationship between differentiation and integration – the integral "undoes" the derivative.

Example

(1) Use Part 1 of the FTOC to simplify $\dfrac{d}{dx}\displaystyle\int_x^{10} \frac{dz}{z^2+1}$.

(2) Use Part 2 of the FTOC to evaluate $\displaystyle\int_0^{\pi} (1 - \sin x)\, dx$.

(3) Compute $\displaystyle\int_1^{y} h'(p)\, dp$.

Exercise

(1) Simplify $\dfrac{d}{dx}\displaystyle\int_{3x^4}^{4} \frac{t-5}{t^2+1}\,dt$.

(2) Evaluate $\displaystyle\int_{1}^{5} (x^2-4)\,dx$.

5.3 Book Problems

11-17, 19-57 (odds), 61-67 (odds)

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§5.4 Working with Integrals

Now that we have methods to use in integrating functions, we can examine applications of integration. These applications include:

- Integration of even and odd functions;
- Finding the average value of a functions;
- Developing the Mean Value Theorem for Integrals.

Recall the definition of an even function,

$$f(-x) = f(x),$$

and of an odd function,

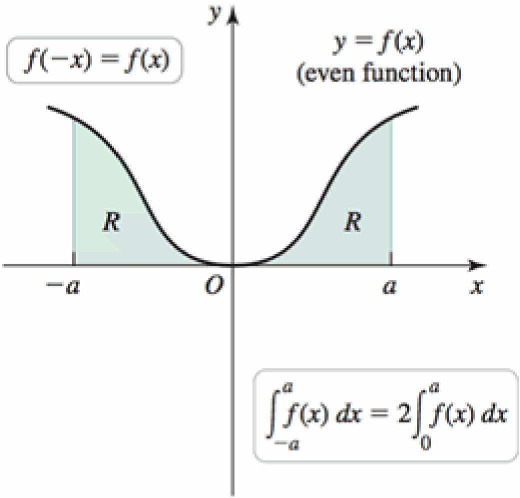
$$f(-x) = -f(x).$$

These properties simplify integrals when the interval in question is centered at the origin.

Even functions are symmetric about the y-axis. So

$$\int_{-a}^{0} f(x)\ dx = \int_{0}^{a} f(x)\ dx$$

i.e., the area under the curve to the left of the y-axis is equal to the area under the curve to the right.

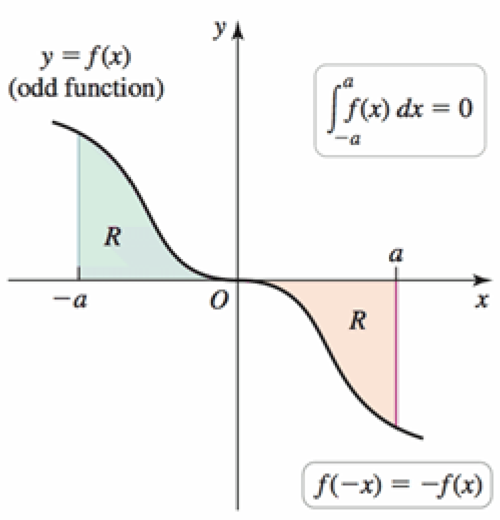


$f(-x) = f(x)$
$y = f(x)$
(even function)
y
R
R
$-a$
O
a
x
$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

On the other hand, odd functions have 180° rotation symmetry about the origin. So

$$\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx$$

i.e., the area under the curve to the left of the origin is the negative of the area under the curve to the right of the origin.



y
$y = f(x)$
(odd function)
$\int_{-a}^{a} f(x)\,dx = 0$
R
R
$-a$
O
a
x
$f(-x) = -f(x)$

Exercise

Evaluate the following integrals using the properties of even and odd functions:

(1) $\int_{-4}^{4}(3x^2 - x)\ dx$

(2) $\int_{-1}^{1}(1 - |x|)\ dx$

(3) $\int_{-\pi}^{\pi}\sin x\ dx$

Finding the average value of a function is similar to finding the average of a set of numbers. We can estimate the average of $f(x)$ between points a and b by partitioning the interval $[a, b]$ into n equally sized sections and choosing y-values $f(\overline{x}_k)$ for each $[x_{k-1}, x_k]$. The average is approximately

$$\begin{aligned}
\frac{f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)}{n} &= \frac{f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)}{\left(\frac{b-a}{\Delta x}\right)} \\
&= \frac{1}{b-a}\left(f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)\right)\Delta x \\
&= \frac{1}{b-a}\sum_{k=1}^{n} f(\overline{x}_k)\Delta x
\end{aligned}$$

The estimate gets more accurate, the more y-values we take. Thus the average value of an integrable function f on the interval $[a, b]$ is

$$\begin{aligned}
\overline{f} &= \lim_{n\to\infty}\left(\frac{1}{b-a}\sum_{k=1}^{n} f(\overline{x}_k)\Delta x\right) \\
&= \frac{1}{b-a}\left(\lim_{n\to\infty}\sum_{k=1}^{n} f(\overline{x}_k)\Delta x\right) \\
&= \frac{1}{b-a}\int_a^b f(x)\,dx.
\end{aligned}$$

Example

The elevation of a path is given by $f(x) = x^3 - 5x^2 + 10$, where x measures horizontal distances. Draw a graph of the elevation function and find its average value for $0 \leq x \leq 4$.

Exercise

Find the average value of the function $f(x) = x(1 - x)$ on the interval $[0, 1]$.

Mean Value Theorem for Integrals

The average value of a function leads to the Mean Value Theorem for Integrals. Similar to the Mean Value Theorem from §4.6, the MVT for integrals says we can find a point c between a and b so that $f(c)$ is the average value of the function.

Theorem (Mean Value Theorem for Integrals)

If f is continuous on $[a, b]$, then there is at least one point c in $[a, b]$ such that

$$f(c) = \overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

In other words, the horizontal line $y = \overline{f} = f(c)$ intersects the graph of f for some point c in $[a, b]$.

Exercise

Find or approximate the point(s) at which $f(x) = x^2 - 2x + 1$ equals its average value on $[0, 2]$.

5.4 Book Problems

7-27 (odds), 31-39 (odds)