MATH 2554 (Calculus I)

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$\oint 4.6$ Mean Value Theorem

Theorem (Rolle's Theorem)

Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. Then there is at least one point c in (a, b) such that $f'(c) = 0$.

Theorem (Mean Value Theorem (MVT))

If f is continuous on a closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

See Figure 4.68 on p. 276 for a visual justification of MVT.

The slope of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$ is

$$\frac{f(b) - f(a)}{b - a}.$$

MVT says that there is a point c on f where the tangent line at c (whose slope is $f'(c)$) is parallel to this secant line.

Example

Let $f(x) = x^2 - 4x + 3$.

1. Determine whether the MVT applies to $f(x)$ on the interval $[-2, 3]$.
2. If so, find the point(s) that are guaranteed to exist by the MVT.

Example

How many points c satisfy the conclusion of the MVT for $f(x) = x^3$ on the interval $[-1, 1]$? Justify your answer.

Consequences of MVT

Theorem (Zero Derivative Implies Constant Function)

If f is differentiable and $f'(x) = 0$ at all points of an interval I, then f is a constant function on I.

Theorem (Functions with Equal Derivatives Differ by a Constant)

If two functions have the property that $f'(x) = g'(x)$ for all x of an interval I, then $f(x) - g(x) = C$ on I, where C is a constant.

Theorem (Intervals of Increase and Decrease)

Suppose f is continuous on an interval I and differentiable at all interior points of I.

- *If $f'(x) > 0$ at all interior points of I, then f is increasing on I.*
- *If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.*

HW from Section 4.6

Do problems 7, 10, 11, 13, 15, 17, 20–22, 24–26, 29 (pp. 279–280 in textbook)

$\oint$ 4.7 L'Hôpital's Rule

In Ch. 2, we examined limits that were computed using analytical techniques. Some of these limits, in particular those that were indeterminate, could not be computed with simple analytical methods.

For example,

$$\lim_{x\to 0} \frac{\sin x}{x} \quad \text{and} \quad \lim_{x\to 0} \frac{1-\cos x}{x}$$

are both limits that can't be computed by substitution, because plugging in 0 for x gives $\frac{0}{0}$.

Theorem (L'Hôpital's Rule $\left(\frac{0}{0}\right)$)

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is $\pm\infty$).

(The rule also applies if $x \to a$ is replaced by $x \to \pm\infty$, $x \to a^+$ or $x \to a^-$.)

Example

Evaluate the following limit:

$$\lim_{x\to -1} \frac{x^4 + x^3 + 2x + 2}{x+1}.$$

Solution: By direct substitution, we obtain 0/0. So we must apply l'Hôpital's Rule (LR) to evaluate the limit:

$$\lim_{x\to -1} \frac{x^4 + x^3 + 2x + 2}{x+1} \overset{\text{LR}}{=} \lim_{x\to -1} \frac{\frac{d}{dx}\left(x^4 + x^3 + 2x + 2\right)}{\frac{d}{dx}(x+1)}$$

$$= \lim_{x\to -1} \frac{4x^3 + 3x^2 + 2}{1}$$

$$= -4 + 3 + 2 = 1$$

Theorem (L'Hôpital's Rule ($\frac{\infty}{\infty}$))

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is $\pm\infty$).

(The rule also applies if $x \to a$ is replaced by $x \to \pm\infty$, $x \to a^+$ or $x \to a^-$.)

Exercise

Evaluate the following limits using l'Hôpital's Rule:

- $\lim\limits_{x \to \infty} \dfrac{4x^3 - 2x^2 + 6}{\pi x^3 + 4}$
- $\lim\limits_{x \to 0} \dfrac{\tan 4x}{\tan 7x}$

Wednesday 8 April (Week 12)

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L'Hôpital's Rule in disguise

Other indeterminate limits in the form $0 \cdot \infty$ or $\infty - \infty$ cannot be evaluated directly using l'Hôpital's Rule.

For $0 \cdot \infty$ cases, we must rewrite the limit in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. A common technique is to divide by the reciprocal:

$$\lim_{x\to\infty} x^2 \sin\left(\frac{1}{5x^2}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{5x^2}\right)}{\frac{1}{x^2}}$$

Exercise

Compute $\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right)$.

For $\infty - \infty$, we can divide by the reciprocal as well as use a change of variables:

Example

Find $\lim\limits_{x\to\infty} x - \sqrt{x^2 + 2x}$.

Solution:

$$\begin{aligned}
\lim_{x\to\infty} x - \sqrt{x^2 + 2x} &= \lim_{x\to\infty} x - \sqrt{x^2(1 + \frac{2}{x})} \\
&= \lim_{x\to\infty} x - x\sqrt{1 + \frac{2}{x}} \\
&= \lim_{x\to\infty} x\left(1 - \sqrt{1 + \frac{2}{x}}\right) \\
&= \lim_{x\to\infty} \frac{1 - \sqrt{1 + \frac{2}{x}}}{\frac{1}{x}}
\end{aligned}$$

This is now in the form $\frac{0}{0}$, so we can apply l'Hôpital's Rule and evaluate the limit.

In this case, it may even help to change variables. Let $t = \frac{1}{x}$:

$$\lim_{x\to\infty} \frac{1-\sqrt{1+\frac{2}{x}}}{\dfrac{1}{x}} = \lim_{t\to 0^+} \frac{1-\sqrt{1+2t}}{t}.$$

Other Indeterminate Forms

Limits in the form 1^∞, 0^0, and ∞^0 are also considered indeterminate forms, and to use l'Hôpital's Rule, we must rewrite them in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Here's how:

Assume $\lim\limits_{x\to a} f(x)^{g(x)}$ has the indeterminate form 1^∞, 0^0, or ∞^0.

1. Evaluate $L = \lim\limits_{x\to a} g(x)\ln f(x)$. This limit can often be put in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, which can be handled by l'Hôpital's Rule.
2. Then $\lim\limits_{x\to a} f(x)^{g(x)} = e^L$. Don't forget this step!

Example

Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Solution: This is in the form 1^∞, so we need to examine

$$\begin{aligned} L &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \\ &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \\ &\overset{\text{LR}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1. \end{aligned}$$

NOT DONE! Write

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^L = e^1 = e.$$

Examining Growth Rates

We can use l'Hôpital's Rule to examine the rate at which functions grow in comparison to one another.

Definition

Suppose f and g are functions with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Then **$f$ grows faster than g** as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \text{ or } \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty.$$

$g \ll f$ means that f grows faster than g as $x \to \infty$.

Definition

The functions f and g have **comparable growth rates** if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M, \text{ where } 0 < M < \infty.$$

Pitfalls in Using l'Hôpital's Rule

1. L'Hôpital's Rule says that $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$. NOT

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \left[\frac{f(x)}{g(x)}\right]' \text{ or } \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \left[\frac{1}{g(x)}\right]' f'(x)$$

   (i.e., don't confuse this rule with the Quotient Rule).

2. Be sure that the limit with which you are working is in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
3. When using l'Hôpital's Rule more than once, simplify as much as possible before repeating the rule.
4. If you continue to use l'Hôpital's Rule in an unending cycle, another method must be used.

HW from Section 4.7

Do problems 13–39 odd, 43–51 odd, 63–69 odd (pp. 290–291 in textbook)

$\oint$ 4.8 Antiderivatives

With differentiation, the goal of problems was to find the function f' given the function f.

With antidifferentiation, the goal is the opposite. Here, given a function f, we wish to find a function F such that the derivative of F is the given function f (i.e., $F' = f$).

Definition

A function F is called an **antiderivative** of a function f on an interval I provided $F'(x) = f(x)$ for all x in I.

Example

Given $f(x) = 4$, an antiderivative of $f(x)$ is $F(x) = 4x$.

NOTE: Antiderivatives are not unique!

They differ by a constant (C):

Theorem

Let F be any antiderivative of f. Then **all** *the antiderivatives of f have the form $F + C$, where C is an arbitrary constant.*

Recall: $\dfrac{d}{dx} f(x) = f'(x)$ is the derivative of $f(x)$.

Now: $\displaystyle\int f(x)\,dx = F + C$ *is* the antiderivative of $f(x)$. It doesn't matter which F you choose, since writing the C will show you are talking about all the antiderivatives at once. The C is also why we call it the *indefinite* integral.

Indefinite Integrals

Example

$\int 4x^3 \, dx = x^4 + C$, where C is the **constant of integration**.

The dx is called the **differential** and it is the same dx from Section 4.5. Like the $\frac{d}{dx}$, it shows which variable you are talking about. The function written between the $\int$ and the dx is called the **integrand**.

Friday 10 April (Week 12)

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Rules for Indefinite Integrals

Power Rule: $\displaystyle\int x^p \, dx = \frac{x^{p+1}}{p+1} + C$

(p is any real number except -1)

Constant Multiple Rule: $\displaystyle\int cf(x) \, dx = c \int f(x) \, dx$

Sum Rule: $\displaystyle\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx$

Exercise

Evaluate the following indefinite integrals:

- $$\int \left(3x^{-2} - 4x^2 + 1\right)\, dx$$
- $$\int 6\sqrt[3]{x}\, dx$$

Indefinite Integrals of Trig Functions

Table 4.5 (p. 296) provides us with rules for finding indefinite integrals of trig functions.

1. $\frac{d}{dx}(\sin ax) = a\cos ax \longrightarrow \int \cos ax \; dx = \frac{1}{a}\sin ax + C$
2. $\frac{d}{dx}(\cos ax) = -a\sin ax \longrightarrow \int \sin ax \; dx = -\frac{1}{a}\cos ax + C$
3. $\frac{d}{dx}(\tan ax) = a\sec^2 ax \longrightarrow \int \sec^2 ax \; dx = \frac{1}{a}\tan ax + C$
4. $\frac{d}{dx}(\cot ax) = -a\csc^2 ax \longrightarrow \int \csc^2 ax \; dx = -\frac{1}{a}\cot ax + C$
5. $\frac{d}{dx}(\sec ax) = a\sec ax\tan ax \longrightarrow \int \sec ax\tan ax \; dx = \frac{1}{a}\sec ax + C$
6. $\frac{d}{dx}(\csc ax) = -a\csc ax\cot ax \longrightarrow \int \csc ax\cot ax \; dx = -\frac{1}{a}\csc ax + C$

Other Indefinite Integrals

Table 4.6 (p. 297) provides us with rules for finding other indefinite integrals.

$$7.\ \frac{d}{dx}(e^{ax}) = ae^{ax} \longrightarrow \int e^{ax}\ dx = \frac{1}{a}e^{ax} + C$$

$$8.\ \frac{d}{dx}(\ln|x|) = \frac{1}{x} \longrightarrow \int \frac{dx}{x} = \ln|x| + C$$

$$9.\ \frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2 - x^2}} \longrightarrow \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$10.\ \frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{a^2 + x^2} \longrightarrow \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$11.\ \frac{d}{dx}\left(\sec^{-1}\left|\frac{x}{a}\right|\right) = \frac{a}{x\sqrt{x^2 - a^2}} \longrightarrow \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$

Example

Evaluate the following indefinite integral: $\displaystyle\int 2\sec^2 2x\ dx$.

Solution: Using rule 3, with $a = 2$, we have

$$\int 2\sec^2 2x\ dx = 2\int \sec^2 2x\ dx = 2\left[\frac{1}{2}\tan 2x\right] + C = \tan 2x + C.$$

Exercise

Evaluate $\displaystyle\int 2\cos(2x)\ dx$.

Initial Value Problems

In some instances, you have enough information to determine the value of C in the antiderivative. These are often called **initial value problems**.

Example

If $f'(x) = 7x^6 - 4x^3 + 12$ and $f(1) = 24$, find $f(x)$.

Solution: $f(x) = \displaystyle\int (7x^6 - 4x^3 + 12)\, dx = x^7 - x^4 + 12x + C$. Now find out which C gives $f(1) = 24$:

$$24 = f(1) = 1 - 1 + 12 + C,$$

so $C = 12$. Hence, $f(x) = x^7 - x^4 + 12x + 12$.

Exercise

Find the function f that satisfies $f''(t) = 6t$ with $f'(0) = 1$ and $f(0) = 2$.

HW from Section 4.8

Do problems 11–45 odd, 55–59 odd, 63, 65 (pp. 301–302 in textbook)

To solve 55–59 odd, 63, and 65, read pp. 299–300, focusing in on Example 7.

$\oint 5.1$ Approximating Area Under Curves

Example

Suppose you ride your bike at a constant velocity of 8 miles per hour for 1.5 hours.

- What is the velocity function that models this scenario?
- What does the graph of the velocity function look like?
- What is the position function for this scenario?
- Where is the displacement (i.e., the distance you've traveled) represented when looking at the graph of the velocity function?

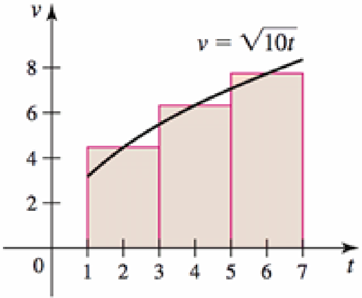
In the previous example, the velocity was constant. In most cases, this is not accurate (or possible). How could we find displacement when the velocity is changing over an interval?

One strategy is to divide the time interval into a particular number of subintervals and approximate the velocity on each subinterval with a constant velocity. Then for each subinterval, the displacement can be evaluate and summed.

Note: This provides us with only an approximation, but with a larger number of subintervals, the approximation becomes more accurate.

Example

Suppose the velocity of an object moving along a line is given by $v(t) = \sqrt{10t}$ on the interval $1 \leq t \leq 7$. Divide the time interval into $n = 3$ subintervals, assuming the object moves at a constant velocity equal to the value of v evaluated at the midpoint of the subinterval. Estimate the displacement of the object on $[1, 7]$. Repeat for $n = 6$ subintervals.



Riemann Sums

Consider a function f over the interval $[a, b]$. Divide $[a, b]$ into n subintervals of equal length:

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$$

with $x_0 = a$ and $x_n = b$. The length of each subinterval is denoted

$$\Delta x = \frac{b-a}{n}.$$

In each subinterval $[x_{k-1}, x_k]$, we can choose any point $\overline{x}_k$ and create a rectangle with a height of $f(\overline{x}_k)$. The area of the rectangle is the product of its height and base, or $f(\overline{x}_k)\Delta x$.

Doing this for each subinterval, and then summing each rectangle's area, produces an approximation of the overall area. This approximation is called a **Riemann sum**

$$R = f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x.$$

Note: We let k vary from 1 to n, and we always have

$$x_{k-1} \leq \overline{x}_k \leq x_k.$$

We usually choose $\overline{x}_k$ so that it is consistent across all the subintervals. The most common ways to do this are with **left Riemann sums**, **right Riemann sums**, and **midpoint Riemann sums**. (See Figures 5.9–5.11 on pp. 310–312 for pictures of these sums.)

Let $R = f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x$.

1. R is a **left Riemann sum** when we choose $\overline{x}_k = x_{k-1}$ for each k.
2. R is a **right Riemann sum** when we choose $\overline{x}_k = x_k$ for each k.
3. R is a **midpoint Riemann sum** when we take $\overline{x}_k$ to be the midpoint between x_{k-1} and x_k, for each k.

Sigma Notation

Riemann sums become more accurate when we make n (the number of rectangles) bigger. However, writing it down becomes a pain. Sigma notation gives a shorthand.

Example

$$\sum_{n=1}^{5} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

Exercise

Evaluate $\displaystyle\sum_{k=0}^{3} (2k-1)$.

Σ-Shortcuts

(n is always a positive integer)

$$\sum_{k=1}^{n} c = cn \text{ (where } c \text{ is a constant)}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Riemann Sums Using Sigma Notation

Now using sigma notation, we can write the Riemann sum in a much more compact form:

$$R = f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x$$

$$= \sum_{k=1}^{n} f(\overline{x}_k)\Delta x.$$

Left, Right, and Midpoint Riemann Sums in Sigma Notation

Suppose f is defined on a closed interval $[a, b]$ which is divided into n subintervals of equal length Δx. As before, let $\overline{x}_k$ denote a point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$. Also, notice that $x_0 = a$ and $x_n = b$.

1. $\displaystyle\sum_{k=1}^{n} f(a + (k-1)\Delta x)\Delta x$ gives a left Riemann sum.
2. $\displaystyle\sum_{k=1}^{n} f(a + k\Delta x)\Delta x$ gives a right Riemann sum.
3. $\displaystyle\sum_{k=1}^{n} f\left(a + \left(k - \frac{1}{2}\right)\Delta x\right)\Delta x$ gives a midpoint Riemann sum.

Exercise

Use sigma notation to write the left, right, and midpoint Riemann sums for the function $f(x) = x^2$ on the interval $[1, 5]$ given that $n = 4$.

Based on these approximations, estimate the area bounded by the graph of $f(x)$ over $[1, 5]$.

HW from Section 5.1

Do problems 9, 11, 15–23 odd, 31, 33, 53–57 all (pp. 315–320 in textbook)