Take-Home Quiz 6

Math 236 (Calc II)

Fall 2017

SOLUTIONS:

1. $S = \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$

(a) $\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\infty} \frac{du}{u^2} = \lim_{B\to\infty} -u^{-1}\Big|_{\ln 2}^{B}$

$u = \ln x \quad u(2) = \ln 2$

$du = \frac{dx}{x} \quad \lim_{x\to\infty} u = \infty$

$= \lim_{B\to\infty}\left(-\underbrace{\frac{1}{B}}_{\to 0} - \left(-\frac{1}{\ln 2}\right)\right)$

$= \frac{1}{\ln 2}$

$\Rightarrow \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges

(b) For any $n \geq 2$, $S_n \leq \underbrace{\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}}_{S} \leq S_n + \int_n^{\infty} \frac{dx}{x(\ln x)^2}$

$S_{10} = \frac{1}{2(\ln 2)^2} + \frac{1}{3(\ln 3)^2} + \cdots + \frac{1}{10(\ln 10)^2} \boxed{\approx 1.685}$

$\Rightarrow S_{10} \leq S \leq S_{10} + \frac{1}{\ln 10} \rightarrow \boxed{S \in (1.685, 2.119)}$

(c) $0 \leq R_{10} \leq \int_{10}^{\infty} \frac{dx}{x(\ln x)^2} \rightarrow \boxed{R_{10} \leq \frac{1}{\ln(10)}}$

$\longrightarrow$

(d) — see part (b) —

(e) $R_n = S - S_n \leq 10^{-6}$

From Theorem 7.31, $R_n \leq \frac{1}{\ln(n)} \leq 10^{-6}$

$$\Rightarrow 10^6 \leq \ln(n)$$

$$e^{10^6} \leq n$$

$\Rightarrow$ smallest $\boxed{n = \lceil e^{10^6} \rceil}$

This notation is the ceiling function.

$\lceil x \rceil$ = smallest integer larger than or equal to x.

2. (a) $\lim_{x\to\infty} \frac{g(x+1)}{g(x)}$

$$= \lim_{x\to\infty} \frac{\dfrac{a_0 + a_1(x+1) + a_2(x+1)^2 + \ldots + a_n(x+1)^n}{b_0 + b_1(x+1) + b_2(x+1)^2 + \ldots + b_m(x+1)^m}}{\dfrac{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \ldots + b_m x^m}}$$

Since this is a rational function, look at the highest order terms to compute the limit!

$$= \lim_{x\to\infty} \frac{\frac{a_n(x+1)^n}{b_m(x+1)^m}}{\frac{a_n x^n}{b_m x^m}} = \lim_{x\to\infty} \frac{\frac{a_n x^n}{b_m x^m}}{\frac{a_n x^n}{b_m x^m}} \boxed{=1}$$

(b) The ratio test for a rational function will always give the limit in part (a). Since that limit equals 1, the test is inconclusive every time.

3. (a) $\sum_{k=1}^{\infty} \frac{\ln k}{k^p}$. Use the limit Comparison Test with p-series:

$$\lim_{k\to\infty} \frac{\frac{\ln k}{k^p}}{\frac{1}{k^p}} = \lim_{k\to\infty} \ln k = \infty \Rightarrow \sum_{k=1} \frac{\ln k}{k^p} \text{ dominates.}$$

When $p \le 1$, $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges and hence so does $\sum_{k=1}^{\infty} \frac{\ln k}{k^p}$.

On the other hand,

$$\lim_{k\to\infty} \frac{\frac{\ln k}{k^p}}{\frac{1}{k^{p-1}}} = \lim_{k\to\infty} \frac{\ln k}{k} = 0 \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^{p-1}} \text{ dominates.}$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^{p-1}}$ converges when $p-1>1$, or $p>2$, it follows that $\sum_{k=1}^{\infty} \frac{\ln k}{k^p}$ converges when $p>2$.

What about $1<p\leq 2$? Use the integral test:

$$\int_1^\infty \frac{\ln x}{x^p}\,dx = \lim_{B\to\infty}\left((\ln x)\frac{x^{-p+1}}{-p+1}\Big|_1^B - \int_1^B \frac{x^{-p+1}}{-p+1}\cdot\frac{dx}{x}\right)$$

$u=\ln x \Rightarrow du=\frac{dx}{x}$

$dv=\frac{dx}{x^p} \rightarrow v=\frac{x^{-p+1}}{-p+1}$

$$= \lim_{B\to\infty} \frac{\ln x}{(1-p)x^{p-1}}\Big|_1^B - \frac{1}{1-p}\int_1^B \frac{dx}{x^p}$$

$$= \lim_{B\to\infty}\left(\cancel{\frac{\ln B}{(1-p)B^{p-1}}} - \cancel{\frac{\ln(1)}{(1-p)(1)^{p-1}}}\right)$$

0, since $1<p\leq 2$

$$-\frac{1}{1-p}\left(\frac{x^{-p+1}}{-p+1}\right)\Big|_1^B$$

$$= \lim_{B\to\infty} \frac{-1}{(1-p)^2}\left(\underbrace{\cancel{\frac{1}{B^{p-1}}}}_{0} - \frac{1}{1^{p-1}}\right) = \frac{1}{(1-p)^2}$$

$$\Rightarrow \boxed{\sum_{k=1}^{\infty}\frac{\ln k}{k^p} \text{ converges for } p>1, \text{ diverges else.}}$$

Since all terms of the series are positive, there is no conditional vs. absolute convergence.

(b) Use the integral test:

$$\int_1^\infty \frac{1}{(c+x)^p}\,dx = \int_{c+1}^\infty \frac{1}{u^p}\,du \le \int_1^\infty \frac{1}{u^p}\,du$$

$u = c+x \quad u(1) = c+1$

$du = dx \quad \lim_{x\to\infty} u = \infty$

$\boxed{\text{converges for } p>1}$

(diverges else)

No issue of absolute or conditional convergence since all terms in the series are positive.

* In general, the integral test works because we can define a function $a(x)$, consistent with the terms a_k of the series, such that $a(x)$ is
  - continuous on $[1,\infty)$
  - positive on $[1,\infty)$
  - decreasing on $[1,\infty)$.

(c) Again, all terms of the series are positive so there is no issue of absolute vs. conditional convergence.

First, if $p \le 1$ then $\frac{(\ln k)^p}{k^2} \le \frac{\ln k}{k^2}$.

$\longrightarrow$

From part (a), $\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ converges and hence

so does $\sum_{k=1}^{\infty} \frac{(\ln k)^p}{k^2}$.

For $p > 1$, note that

$$\frac{(\ln k)^p}{k^2} \le \frac{\left(k^{\frac{1}{2p}}\right)^p}{k^2}$$ for $k \gg 0$, because power functions dominate log functions.

The right-hand side equals $\frac{k^{1/2}}{k^2} = \frac{1}{k^{3/2}}$. Since $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges,

so does $\sum_{k=1}^{\infty} \frac{(\ln k)^p}{k^2}$. Therefore $\sum_{k=1}^{\infty} \frac{(\ln k)^p}{k^2}$

$\boxed{\text{converges for all } p.}$

(d) $\sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k^p \ln k}$ is an alternating series:

- $$\frac{1}{(k+1)^p \ln(k+1)} \overset{?}{<} \frac{1}{k^p \ln k}$$

Yes, if $p \ge 0$ since $(k+1)^p > k^p$ and $\ln(k+1) > \ln k$.

- $\lim_{k\to\infty} \frac{1}{k^p \ln k} = 0$ if $p \geq 0$

$= \infty$ if $p < 0$

$\Rightarrow$ diverges if $p<0$.

Absoluteness of convergence:

Does $\sum_{k=2}^{\infty} \left| \frac{(-1)^{k+1}}{k^p \ln k} \right| = \sum_{k=2}^{\infty} \frac{1}{k^p \ln k}$ converge for $p \geq 0$?

Use limit comparison to a p-series:

$$\lim_{k\to\infty} \frac{\frac{1}{k^p \ln k}}{\frac{1}{k^p}} = \lim_{k\to\infty} \frac{1}{\ln k} = 0$$

$\Rightarrow \sum_{k=2}^{\infty} \frac{1}{k^p \ln k}$ is dominated by

$\sum_{k=2}^{\infty} \frac{1}{k^p} \Rightarrow$ converges for $p>1$.

$$\lim_{k\to\infty} \frac{\frac{1}{k^p \ln k}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^{1-p}}{\ln k} = \infty \text{ if } 0 \leq p < 1$$

$\Rightarrow \sum_{k=2}^{\infty} \frac{1}{k^p \ln k}$ dominates $\sum_{k=2}^{\infty} \frac{1}{k}$, so diverges

$\longrightarrow$

Finally, if $p=1$ then

$$\int_2^\infty \frac{dx}{x\ln x} = \int_{\ln 2}^{\infty} \frac{du}{u} = \lim_{B\to\infty} \ln|u|\Big|_{\ln 2}^{B}$$

$$= \lim_{B\to\infty} \ln B - \ln(\ln 2) = \infty$$

Conclusion: $\sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k^p \ln k}$ diverges for $p<0$

converges conditionally for $0 \le p \le 1$

converges absolutely for $p>1$.

(e) $\sum_{k=0}^{\infty} (-1)^k k^p e^{-k^2} = \sum_{k=0}^{\infty} \frac{(-1)^k k^p}{e^{k^2}}$

is an alternating series

- $\frac{(k+1)^p}{e^{(k+1)^2}} \overset{?}{<} \frac{k^p}{e^{k^2}}\left(\frac{e^{2k+1}}{e^{2k+1}}\right) = \frac{k^p e^{2k+1}}{e^{(k+1)^2}}$

Look at numerators:

$(k+1)^p \overset{?}{<} k^p e^{2k+1}$, i.e., $\left(\frac{k+1}{k}\right)^p \overset{?}{<} e^{2k+1}$

Yes, because $\left(\frac{k+1}{k}\right)^p < e$.

⟶

- $\lim_{k\to\infty} \frac{k^p}{e^{k^2}} = 0$, since exponential functions dominate power functions.

$\Rightarrow$ converges for all p.

Absolute convergence?

Look at $\sum_{k=0}^{\infty} \frac{k^p}{e^{k^2}}$.

Compare: For any p, $\frac{k^p}{e^{k^2}} < \frac{k^p}{k^{p+2}} = \frac{1}{k^2}$.

Since $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, $\sum_{k=1}^{\infty} \frac{k^p}{e^{k^2}}$ must also converge.

Adding the $k=0$ term, $\Rightarrow \sum_{k=0}^{\infty} \frac{k^p}{e^{k^2}}$ converges.

$\boxed{\text{Conclusion: Converges absolutely for all } p.}$

4. (a) The formula for the number of returning fish is recursive:

$$P_{k+1} = 0.2(p_k + h)$$

$$= 0.2(0.2(p_{k-1} + h) + h) = 0.2^2 p_{k-1} + 0.2^2 h + 0.2h$$

$$= 0.2(0.2(0.2(p_{k-2} + h) + h) + h)$$

$$= 0.2^3 p_{k-2} + 0.2^3 h + 0.2^2 h + 0.2h$$

$$\vdots$$

$$= 0.2^{k+1} p_{k-k} + 0.2^{k+1} h + 0.2^k h + \cdots + 0.2h$$

$$= 0.2^{k+1} p_0 + \sum_{n=1}^{k+1} 0.2^n h$$

The sustained number of fish approaches

$$P_\infty = \lim_{k\to\infty} P_{k+1}$$

$$= \lim_{k\to\infty} \overbrace{0.2^{k+1}}^{\to 0} p_0 + \lim_{k\to\infty} \sum_{n=1}^{k+1} 0.2^n h$$

$$= \sum_{n=1}^{\infty} 0.2^n h = h \sum_{k=1}^{\infty} 0.2^k$$ (factor out an h and then rename the indices).

→

(b) P_∞ is a geometric series:

$$h\sum_{k=1}^{\infty}(0.2)^k = h\left(\frac{1}{1-0.2} - 0.2^0\right)$$

$$= h\left(\frac{1}{0.8} - \frac{0.8}{0.8}\right) = \frac{0.2}{0.8}h = \boxed{\frac{1}{4}h}$$

(c) In order for $\frac{1}{4}h = P$, Leila must add the spawn from $h = 4P$ fish.

5. (a) $$f_\infty = s\left(1 + \sum_{l=0}^{\infty}\left(\prod_{j=0}^{l} r_j\right)\right)$$

$$= s + s\sum_{l=0}^{\infty}\prod_{j=0}^{l} r_j$$

Ratio Test: $$\lim_{l\to\infty} \frac{\prod_{j=0}^{l+1} r_j}{\prod_{j=0}^{l} r_j} = \lim_{l\to\infty} \frac{\cancel{r_0}\cancel{r_1}\cancel{r_2}\cdots r_{l+1}}{\cancel{r_0}\cancel{r_1}\cancel{r_2}\cdots \cancel{r_l}} = r_{l+1}.$$

Since $r_{l+1} \in [0.65, 0.95]$, the series must converge.

(b) No, since r_j is different each year.