Arc length and the Cirumfrence of a Circle.

Thurs. 17 Sep. 2015

Q: Ever wonder how we know the circumfrence formula is $2\pi R$ for a circle of radius R?

There are actually many ways to derive the formula, one is by directly using the arc length formula:

Using the peramatrization
$$\vec{r}(t) = \langle R\cos t, R\sin t \rangle,$$
the circle is drawn on the time $0 \le t \le 2\pi$. So the length of the circle is

$$\int_0^{2\pi} \sqrt{(-R\sin t)^2 + (R\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{R^2\sin^2 t + R^2\cos^2 t}\, dt$$

$$= R\int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= R\int_0^{2\pi} dt = R\left(t\Big|_0^{2\pi}\right) = R(2\pi - 0) = \boxed{2\pi R.}$$

☆ As an exercise, try to derive the same formula but using the parametrization

$$\vec{r}(t) = \langle R\cos at, R\sin at \rangle,$$

where a is some positive number. What do you have to do differently in order to get the correct answer, $2\pi R$?

Alternately you can derive the circumfrence formula from polar coordinates for a circle of radius R,

$$\rho(\theta) = R = \text{constant}.$$

The circle is drawn for $0 \le \theta \le 2\pi$:

$$\int_0^{2\pi} \sqrt{\rho'(\theta)^2 + \rho(\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{0^2 + R^2}\, d\theta = R\int_0^{2\pi} d\theta = R(2\pi - 0) \boxed{= 2\pi R}$$

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*You may have also noticed $0 \leq \theta \leq 2\pi$ is not the only possible domain for the circle. Derive the circumfrence formula using a different domain.