Math 2603 Exam 2
Wed 24 Sep 2014

Name: SOLUTIONS

Discrete Math
Exam 2 (Ch. 3-6 as we have covered)

Please provide the following data:

Drill Time: ____________________

Student ID: ____________________

Exam Instructions: You have 50 minutes to complete this exam. One 3×5 inch notecard is allowed. No graphing calculators. No programmable calculators. No phones, iDevices, computers, etc. If you finish early then you may leave, UNLESS there are less than 5 minutes of class left. To prevent disruption, if you finish with less than 5 minutes of class remaining then please stay seated and quiet.

Your signature below indicates that you have read this page and agree to follow the Academic Honesty Policies of the University of Arkansas.

Signature:

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Good luck!

12 pts 1. Determine which of the following are functions. If they are functions decide if they are one-to-one, onto, or both.

(a) $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = 4x - 5$.

f is a line, so is well-defined and bijective.

(b) $R = \{(a, y), (b, z), (c, x), (d, x)\} \subset \{a, b, c, d\} \times \{x, y, z\}$

R is well-defined since a, b, c, d each has exactly one image.

not one-to-one because $(c, x), (d, x) \in R$

Onto because range $= \{x, y, z\} =$ codomain

10 pts 2. Find an explicit formula for the nth term of each of the following sequences. You must state where your indices n start.

(a) $s = \left\{\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}, \ldots\right\}$

$$s_n = \frac{n}{(n+1)^2} \qquad n \geq 1$$

(b) $t = \left\{0, \frac{-1}{2}, \frac{2}{3}, \frac{-3}{4}, \frac{4}{5}, \frac{-5}{6}, \frac{6}{7}, \ldots\right\}$

$$t_n = \frac{(-1)^n n}{n+1} \qquad n \geq 0$$

18pts 3. Suppose R is a relation on a set X. Write down the following definitions:

(a) R is *reflexive* means

$\forall x \in X, (x,x) \in R$

(b) R is *symmetric* means

$\forall x,y \in X, (x,y) \in R \rightarrow (y,x) \in R$

(c) R is *antisymmetric* means

$\forall x,y \in X, (x,y),(y,x) \in R \rightarrow x = y$

(d) R is *transitive* means

$\forall x,y,z \in X, (x,y),(y,z) \in R \longrightarrow (x,z) \in R$

(e) R is an *equivalence relation* means

R is reflexive, symmetric, and transitive.

(f) R is a *partial order* means

R is reflexive, antisymmetric, and transitive.

30 pts 4. Define the relation R on $\mathbb{Z}$ by mRn if and only if $3 \mid (m^2 - n^2)$. Which of the properties from Problem 3 does R satisfy? Give justification (proof or counterexample) for each of the items (a)-(f) from Problem 3.

(yes) reflexive: $m^2 - m^2 = 0 = 3 \cdot 0$

so $3 \mid (m^2 - m^2)$, i.e., mRm for all $m \in \mathbb{Z}$.

(yes) Symmetric: Suppose $\exists\, m, n \in \mathbb{Z}$ with $3 \mid (m^2 - n^2)$.

Then $\exists\, q \in \mathbb{Z}$ such that

$$m^2 - n^2 = 3q, \text{ so}$$

$$n^2 - m^2 = -(m^2 - n^2) = -3q = 3(-q)$$

So $3 \mid (n^2 - m^2)$.

(no) antisymmetric: Put $m = 4$, $n = 5$.

Then $3 \mid \underbrace{(m^2 - n^2)}_{=\,9}$ and $3 \mid (n^2 - m^2)$ by symmetry.

(yes) transitive: Suppose for some $m, n, p \in \mathbb{Z}$ we have

$$m^2 - n^2 = 3q_1$$
$$n^2 - p^2 = 3q_2$$

so mRn, nRp.

Then $m^2 - p^2 = (m^2 - n^2) + (n^2 - p^2) = 3(q_1 + q_2)$. So mRp.

(yes) equiv. relation: refl, symm, trans.

(no) partial order: not antisymmetric.

15 pts 5. Prove that for the Fibonacci sequence ($f_1 = 1,\ f_2 = 1, \ldots$)

$$\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}.$$

Use induction on n.

<u>Base</u>. $n=1$.

Then $\sum_{i=1}^{n} f_i^2 = f_1^2 = 1^2 = 1$

and $f_1 f_2 = 1 \cdot 1 = 1$. ✓

<u>Induce</u>. Suppose for some $n \geq 1$

$\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$. We must show

$\sum_{i=1}^{n+1} f_i^2 = f_{n+1} f_{n+2}$. Write the left-hand side,

$$\sum_{i=1}^{n+1} f_i^2 = \sum_{i=1}^{n} f_i^2 + f_{n+1}^2$$

$= f_n f_{n+1} + f_{n+1}^2$ (by the induction hypothesis)

$= f_{n+1}(f_n + f_{n+1})$

$= f_{n+1} f_{n+2}$ (by recursive relation on Fibonacci #s).

$\boxtimes$

6. A student council consists of 16 students, 5 of whom are men and 11 of whom are women.

5pts (a) In how many ways can a committee of 5 be selected from the membership of the council?

$$\binom{16}{5}$$

5pts (b) How many committees of 5 contain at least one man?

Subtract from (a) the committees with all women:

$$\binom{16}{5} - \binom{11}{5}$$

5pts (c) Suppose the council has 2 freshmen, 4 sophomores, 6 juniors, and 4 seniors. How many committees of 8 contain exactly 2 representatives from each class?

Choose 2 per class:

$$\binom{2}{2} \cdot \binom{4}{2} \cdot \binom{6}{2} \cdot \binom{4}{2}$$

no pts 7. **cHaLlEnGe PrObLeM** The sequence of *Catalan numbers* is defined as

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

for each $n \in \mathbb{Z}_{\geq 1}$.

(a) Prove that $C_n = \binom{2n}{n} - \binom{2n}{n+1}$. Hint: Use the following Lemma (and prove it).

Lemma. *For all nonnegative integers n and r with $r+1 \leq n$,*

$$\binom{n}{r+1} = \frac{n-r}{r+1}\binom{n}{r}.$$

proof of Lemma.

The left-hand side is $\frac{n!}{(r+1)!(n-r-1)!}$.

The right-hand side is $\left(\frac{n-r}{r+1}\right)\frac{n!}{r!(n-r)!}$, which simplifies to the left-hand side because $(r+1)r! = (r+1)!$ and $\frac{n-r}{(n-r)!} = \frac{1}{(n-r-1)!}$.

By the Lemma,

$$\binom{2n}{n} - \binom{2n}{n+1} = \binom{2n}{n} - \frac{2n-n}{n+1}\binom{2n}{n} = \left(1 - \frac{n}{n+1}\right)\binom{2n}{n}$$

$$= \frac{1}{n+1}\binom{2n}{n}.$$

(b) Find the first 6 Catalan numbers.

$C_1 = \frac{1}{2}\binom{2}{1} = \boxed{1}$

$C_2 = \frac{1}{3}\binom{4}{2} = \boxed{2}$

$C_3 = \frac{1}{4}\binom{6}{3} = \frac{6\cdot 5}{3!} = \boxed{5}$

$C_4 = \frac{1}{5}\binom{8}{4} = \frac{\cancel{8}\cdot 7\cdot \cancel{6}^{2}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = \boxed{14}$

$C_5 = \frac{1}{6}\binom{10}{5} = \frac{10\cdot\cancel{9}^{3}\cdot\cancel{8}^{2}\cdot 7}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = \boxed{42}$

$C_6 = \frac{1}{7}\binom{12}{6} = \frac{\cancel{12}\cdot 11\cdot\cancel{10}^{2}\cdot\cancel{9}^{3}\cdot\cancel{8}^{2}}{\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = \boxed{132}$

6 pts **8. EXTRA CREDIT** Find integers s, t such that

$$\gcd(825, 315) = 825s + 315t.$$

Use the Euclidean Algorithm:

$825 \bmod 315 = 825 - 2(315) = 195$

$315 \bmod 195 = 315 - 1(195) = 120$

$195 \bmod 120 = 195 - 1(120) = 75$

$120 \bmod 75 = 120 - 1(75) = 45$

$75 \bmod 45 = 75 - 1(45) = 30$

$45 \bmod 30 = 45 - 1(30) = 15$

$30 \bmod 15 = 30 - 2(15) = 0.$

$\gcd(825, 315) = 15$

$= 45 - 1(30)$

$= (120 - 1(75)) - 1[75 - 1(120 - 1(75))]$

$= 2(120) + (-3)(75)$

$= 2(315 - 1(195)) + (-3)[195 - 1(315 - 1(195))]$

$= (2+3)(315) + (-2-3-3)(195)$

$= 5(315) + (-8)[825 - 2(315)]$

$= (5+16)(315) + (-8)(825).$

So $s = -8$

$t = 21$