The base for these slides was done by Dr. Shannon Dingman, later encoded into LaTeX by Dr. Brad Lutes and modified/formatted by Dr. Ashley K. Wheeler.

§5.4 Working with Integrals

- Integrating Even and Odd Functions
- Average Value of a Function
- Mean Value Theorem for Integrals
- Book Problems

§5.4 Working with Integrals

Now that we have methods to use in integrating functions, we can examine applications of integration. These applications include:

- Integration of even and odd functions;
- Finding the average value of a functions;
- Developing the Mean Value Theorem for Integrals.

Integrating Even and Odd Functions

Recall the definition of an even function,

$$f(-x) = f(x),$$

and of an odd function,

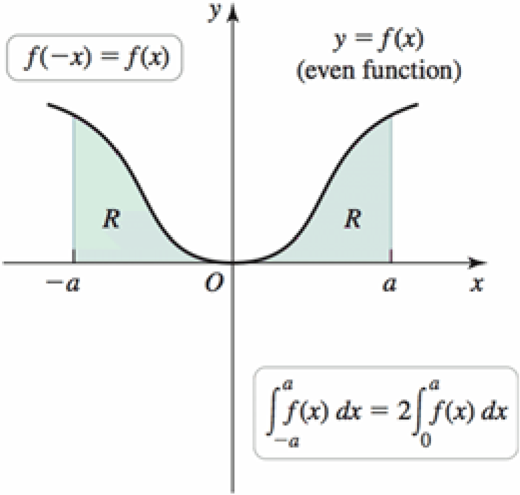
$$f(-x) = -f(x).$$

These properties simplify integrals when the interval in question is centered at the origin.

Even functions are symmetric about the y-axis. So

$$\int_{-a}^{0} f(x)\ dx = \int_{0}^{a} f(x)\ dx$$

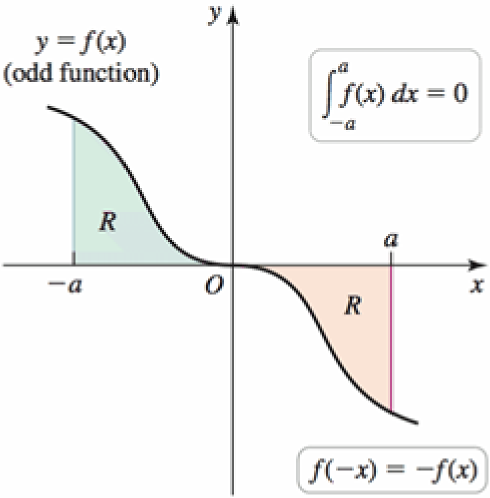
i.e., the area under the curve to the left of the y-axis is equal to the area under the curve to the right.


y
$f(-x) = f(x)$
$y = f(x)$
(even function)
R
R
$-a$
O
a
x
$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

On the other hand, odd functions have 180° rotation symmetry about the origin. So

$$\int_{-a}^{0} f(x)\ dx = -\int_{0}^{a} f(x)\ dx$$

i.e., the area under the curve to the left of the origin is the negative of the area under the curve to the right of the origin.



y
y = f(x)
(odd function)
$\int_{-a}^{a} f(x)\,dx = 0$
R
R
a
−a
O
x
f(−x) = −f(x)

Exercise

Evaluate the following integrals using the properties of even and odd functions:

(1) $\int_{-4}^{4} (3x^2 - x)\ dx$

(2) $\int_{-1}^{1} (1 - |x|)\ dx$

(3) $\int_{-\pi}^{\pi} \sin x\ dx$

Average Value of a Function

Finding the average value of a function is similar to finding the average of a set of numbers. We can estimate the average of $f(x)$ between points a and b by partitioning the interval $[a, b]$ into n equally sized sections and choosing y-values $f(\overline{x}_k)$ for each $[x_{k-1}, x_k]$. The average is approximately

$$\begin{aligned}
\frac{f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)}{n} &= \frac{f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)}{\left(\frac{b-a}{\Delta x}\right)} \\
&= \frac{1}{b-a}\left(f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)\right)\Delta x \\
&= \frac{1}{b-a}\sum_{k=1}^{n} f(\overline{x}_k)\Delta x
\end{aligned}$$

The estimate gets more accurate, the more y-values we take. Thus the average value of an integrable function f on the interval $[a, b]$ is

$$\begin{aligned}\overline{f} &= \lim_{n\to\infty}\left(\frac{1}{b-a}\sum_{k=1}^{n} f(\overline{x}_k)\Delta x\right)\\ &= \frac{1}{b-a}\left(\lim_{n\to\infty}\sum_{k=1}^{n} f(\overline{x}_k)\Delta x\right)\\ &= \frac{1}{b-a}\int_a^b f(x)\,dx.\end{aligned}$$

Example

The elevation of a path is given by $f(x) = x^3 - 5x^2 + 10$, where x measures horizontal distances. Draw a graph of the elevation function and find its average value for $0 \leq x \leq 4$.

Exercise

Find the average value of the function $f(x) = x(1 - x)$ on the interval $[0, 1]$.

Mean Value Theorem for Integrals

The average value of a function leads to the Mean Value Theorem for Integrals. Similar to the Mean Value Theorem from §4.6, the MVT for integrals says we can find a point c between a and b so that $f(c)$ is the average value of the function.

Theorem (Mean Value Theorem for Integrals)

If f is continuous on $[a, b]$, then there is at least one point c in $[a, b]$ such that

$$f(c) = \overline{f} = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

In other words, the horizontal line $y = \overline{f} = f(c)$ intersects the graph of f for some point c in $[a, b]$.

Exercise

Find or approximate the point(s) at which $f(x) = x^2 - 2x + 1$ equals its average value on $[0, 2]$.

5.4 Book Problems

7-27 (odds), 31-39 (odds)