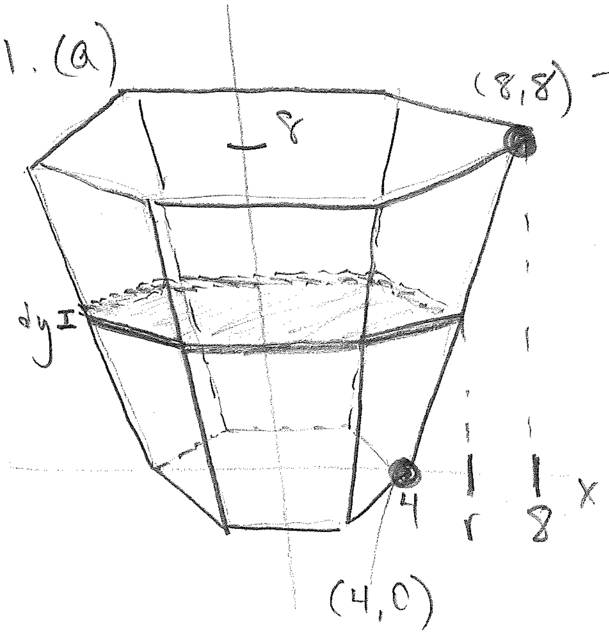
Take-Home Quiz No. 3

Math 236 (Calc II)
Fall 2017

SOLUTIONS

1. (a)



Total volume is given by summing the volumes of the thin hexagons of radius r. The radius is on the line determined by the two points (8,8) and (4,0):

$$y-0=\left(\frac{8-0}{8-4}\right)(x-4)$$

$$y=2(x-4)=2x-8$$

$$\Rightarrow r=x=\frac{y+8}{2}=\frac{y}{2}+4$$

The volume is:

$$\int_0^8 \frac{3\sqrt{3}}{2}\left(\frac{y}{2}+4\right)^2 dy$$

$$=\frac{3\sqrt{3}}{2}\int_0^8\left(\frac{y^2}{4}+4y+16\right)dy=\frac{3\sqrt{3}}{2}\left(\frac{1}{4}\left(\frac{y^3}{3}\right)+4\left(\frac{y^2}{2}\right)+16y\right)\Bigg|_0^8 \leftarrow \text{terms vanish}$$

$$=\frac{3\sqrt{3}}{2}\left(\frac{8^3}{12}+2(8^2)+16(8)\right)$$

$$=448\sqrt{3}\approx\boxed{776.0 \text{ in}^3}$$

$\longrightarrow$

(b) Finding the volume is similar to part (a), only this time r is given by the line with points (8,6) and (4,0):

$$y - 0 = \left(\frac{6-0}{8-4}\right)(x-4)$$

$$y = \frac{3}{2}(x-4) = \frac{3}{2}x - 6$$

$$\Rightarrow r = \frac{y+6}{\frac{3}{2}} = \frac{2}{3}y + 4$$

Total volume is: $\displaystyle\int_0^6 2\sqrt{2}\left(\frac{2}{3}y+4\right)^2 dy$

$$= 2\sqrt{2}\int_0^6 \left(\frac{4}{9}y^2 + \frac{16}{3}y + 16\right) dy$$

$$= 2\sqrt{2}\left(\frac{4}{9}\left(\frac{y^3}{3}\right) + \frac{16}{3}\left(\frac{y^2}{2}\right) + 16y\right)\Bigg|_0^6$$

terms vanish

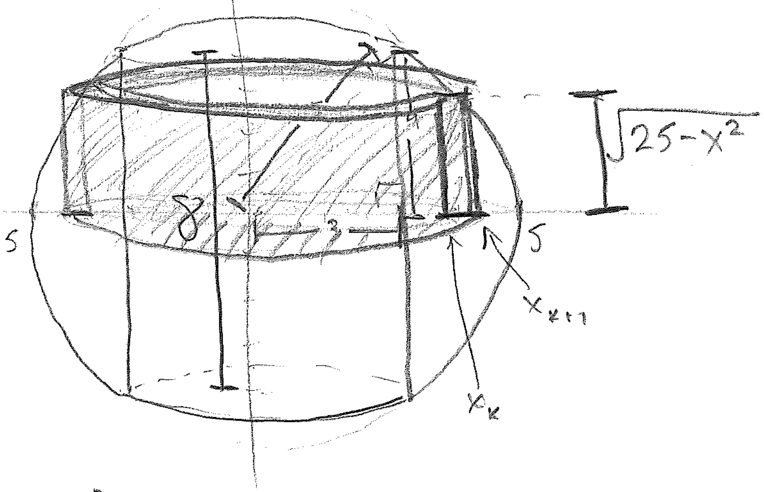
$$= 2\sqrt{2}\left(\frac{4(6^3)}{27} + \frac{16(6^2)}{6} + 16(6)\right)$$

$$= 448\sqrt{2} \approx \boxed{633.6 \text{ in}^3}$$

(c) Geek likes the <u>June</u> spouse more.

2. (a) For one shell, the volume is

$$\pi\left(x_{k+1}^2 - x_k^2\right)\sqrt{25-(x_k)^2}$$



⇒ Total volume is

$$2\int_3^5 \pi(2x\,dx)\sqrt{25-x^2}$$

$$= 4\pi\int_3^5 x\sqrt{25-x^2}\,dx$$

$u = 25-x^2$

$\Rightarrow du = -2x\,dx$ $u(5) = 25-5^2 = 0$

$x\,dx = -\frac{du}{2}$ $u(3) = 25-3^2 = 16$

$$= 4\pi\int_{16}^{0}\sqrt{u}\left(-\frac{du}{2}\right) = 2\pi\int_0^{16} u^{1/2}\,du$$

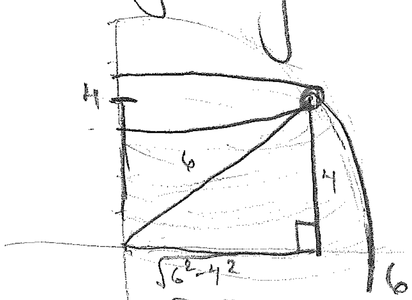
$$= 2\pi\left(\frac{2}{3}u^{3/2}\right)\Bigg|_0^{16} = 2\pi\left(\frac{2}{3}\left(16^{3/2}\right)\right)$$

0 ← terms vanish

$$= \frac{256}{3}\pi \approx \boxed{268.1\ cm^3}$$

(for Mark's napkin holder)

For Alına's napkin holder, the lower bound of the integral is found using the Pythagorean Theorem:



$\sqrt{6^2-4^2}$
$= \sqrt{36-16}$
$= \sqrt{20} = 2\sqrt{5}$

The integral becomes

$$4\pi\int_{2\sqrt{5}}^{6} x\sqrt{36-x^2}\,dx$$

Let $u = 36 - x^2 \rightarrow u(6) = 36 - 6^2 = 0$

$du = -2x\,dx$

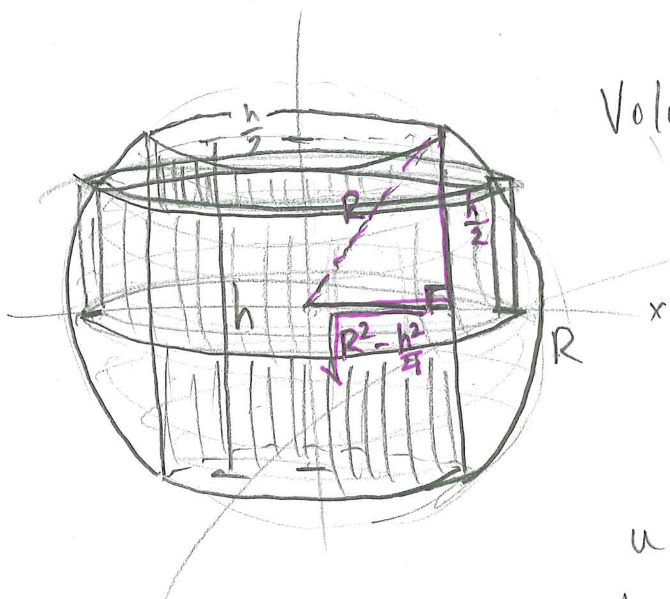
$x\,dx = -\frac{du}{2}$

$u(2\sqrt{5}) = 36 - (2\sqrt{5})^2$

$= 36 - 20 = 16$

$\Rightarrow \text{Volume} = 4\pi \int_{16}^{0} \sqrt{u}\left(-\frac{du}{2}\right) \leftarrow$ same integral as before

(b)



Volume =

$$4\pi \int_{\sqrt{R^2 - \frac{h^2}{4}}}^{R} x\sqrt{R^2 - x^2}\,dx$$

$u = R^2 - x^2 \quad u(R) = 0$

$du = -2x\,dx \quad u\left(\sqrt{R^2 - \frac{h^2}{4}}\right) = -\frac{h^2}{4}$

$$= 4\pi \int_{-\frac{h^2}{4}}^{0} -\frac{\sqrt{u}}{2}\,du$$

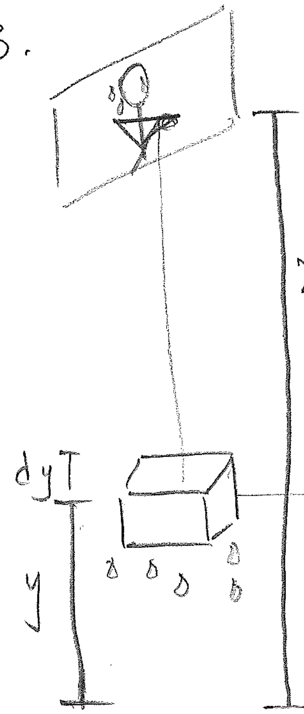
$$= 2\pi \int_{0}^{h^2/4} u^{1/2}\,du = 2\pi\left(\frac{2}{3}u^{3/2}\right)\Bigg|_{0}^{h^2/4}$$

$0 \leftarrow$ terms vanish

$$= \frac{4\pi}{3}\left(\frac{h^2}{4}\right)^{3/2}$$

$$= \frac{4\pi}{3}\left(\frac{h^3}{8_2}\right) = \boxed{\frac{\pi}{6}h^3 \leftarrow \text{Only depends on } h.}$$

3.



force = 100lb $- \left(2 \frac{lb}{ft}\right)(y\ ft)$

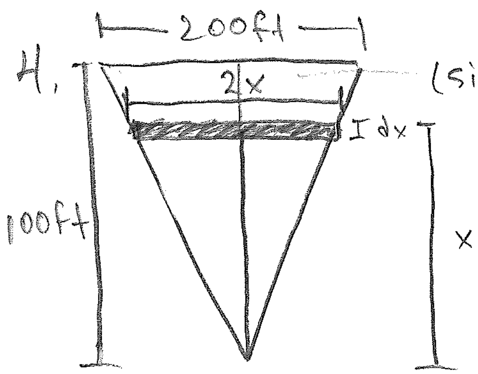
$$\text{Work} = \int_0^{22} \underbrace{(100-2y)}_{\text{force}}\ \underbrace{dy}_{\text{distance}}$$

$$= 100y - y^2 \Big|_0^{22} \leftarrow \text{terms vanish}$$

$$= 100(22) - (22)^{22}$$

$$\boxed{= 1716 \text{ ft-lb}}$$

4.



(similar triangles)

$$\text{Hydrostatic force} = \omega A d$$

weight-density (62.4 $\frac{lb}{ft^3}$ for water)

area

depth

area of shaded rectangle $= 2x\,dx$ ft^2

$$\text{h.s.f.} = \int_0^{100} \omega(2x\,dx)(100-x)$$

depth of water at the shaded rectangle $= 100 - x$ ft

$$= 2\omega \int_0^{100} (100x - x^2)\,dx = 2\omega\left(100\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^{100} \leftarrow \text{terms vanish}$$

$$= \cancel{2}\omega\left(\frac{100^3}{\cancel{2}}\left(\frac{3}{3}\right) - \frac{100^3}{3}\left(\frac{2}{\cancel{2}}\right)\right)$$

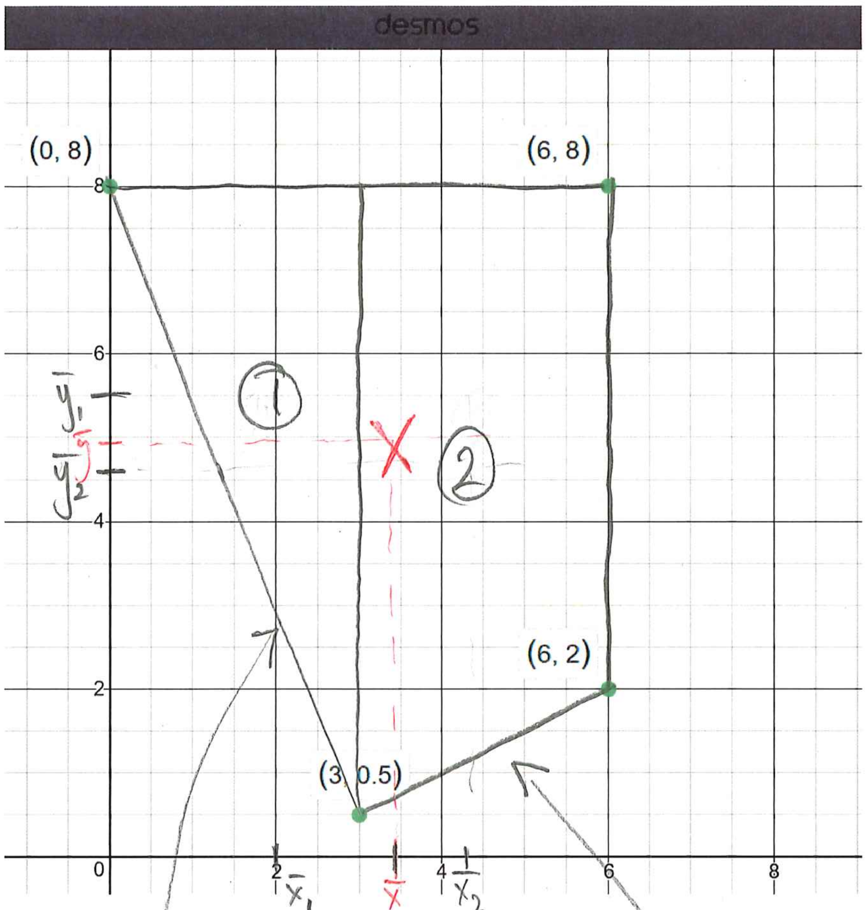
$$= \frac{1000000}{3}\omega = \frac{1000000}{3}(62.4) =$$

$$\boxed{20{,}800{,}000 \text{ lb}}$$

5. Ian must attach the cable to the centroid of the fluke, given by

$$\left(\frac{\bar{x}_1 A_1 + \bar{x}_2 A_2}{A}, \frac{\bar{y}_1 A_1 + \bar{y}_2 A_2}{A}\right)$$

where $(\bar{x}_i, \bar{y}_i)$ is the centroid of region i $(i = 1, 2)$, A_i is the area of region i, and A is the total area of the fluke.



$$y - 8 = \left(\frac{8 - 0.5}{0 - 3}\right)(x - 0)$$

$$\Rightarrow y = -\frac{5}{2}x + 8$$

$$y - 2 = \left(\frac{2 - 0.5}{6 - 3}\right)(x - 6)$$

$$\Rightarrow y = \frac{1}{2}x - 3 + 2 = \frac{1}{2}x - 1$$

$$\bar{x}_1 = \frac{\int_0^3 x\left|\cancel{8} - \left(-\frac{5}{2}x + \cancel{8}\right)\right|dx}{\int_0^3 \left|\cancel{8} - \left(-\frac{5}{2}x + \cancel{8}\right)\right|dx} = \frac{\frac{5}{2}\int_0^3 x^2\,dx}{\frac{5}{2}\int_0^3 x\,dx} = \frac{\left(\frac{5}{2}\right)\frac{x^3}{3}\Big|_0^3}{\left(\frac{5}{2}\right)\frac{x^2}{2}\Big|_0^3}$$

← terms vanish

$$= \frac{\left(\frac{5}{2}\right)\frac{\cancel{3^3}\,3^2}{\cancel{3}}}{\left(\frac{5}{2}\right)\frac{3^2}{2}} = \frac{\frac{45}{2}}{\frac{45}{4}}$$

$\bar{x}_1$

← A_1

⟶

$$\bar{y}_1 = \frac{\frac{1}{2}\int_0^3 \left| 8^2 - \overbrace{\left(-\frac{5}{2}x+8\right)^2}^{64-\left(\frac{25}{4}x^2-40x+64\right)} \right| dx}{\frac{45}{4}} = \frac{\frac{1}{2}\int_0^3 \left|\frac{25}{4}x^2 - 40x\right| dx}{\frac{45}{4}}$$

$\frac{25}{4}x^2 > 40x$

$\Rightarrow x > \frac{160}{25}$, but x is between 0 and 3

$$= \frac{\frac{1}{2}\int_0^3 \left(40x - \frac{25}{4}x^2\right) dx}{\frac{45}{4}}$$

$$= \frac{\frac{1}{2}\left(40\left(\frac{x^2}{2}\right) - \frac{25}{4}\left(\frac{x^3}{3}\right)\right)\Big|_0^3}{\frac{45}{4}} \leftarrow \text{t.u.}$$

$$= \frac{10(3^2) - \frac{25}{8}(3^2)}{\frac{45}{4}} = \frac{\frac{495}{8}}{\frac{45}{4}}$$

$\bar{y}_1$

$9-\frac{1}{2}x \leftarrow$ pos. for $x\in[3,6]$

$$\bar{x}_2 = \frac{\int_3^6 x\left|8-\left(\frac{1}{2}x-1\right)\right| dx}{\int_3^6 \left|8-\left(\frac{1}{2}x-1\right)\right| dx}$$

$$= \frac{\int_3^6 \left(9x - \frac{1}{2}x^2\right) dx}{\int_3^6 \left(9 - \frac{1}{2}x\right) dx}$$

$$= \frac{\left(9\left(\frac{x^2}{2}\right) - \frac{1}{2}\left(\frac{x^3}{3}\right)\right)\Big|_3^6}{\left(9x - \frac{1}{2}\left(\frac{x^2}{2}\right)\right)\Big|_3^6} = \frac{\frac{9}{2}\overbrace{(6^2-3^2)}^{36-9} - \frac{1}{6}\overbrace{(6^3-3^3)}^{216-9}}{9\underbrace{(6-3)}_{3} - \frac{1}{4}\underbrace{(6^2-3^2)}_{27}} = \frac{\frac{243}{2} - \frac{69}{2}}{\frac{81}{4}} = \frac{87}{\frac{81}{4}}$$

$\bar{x}_2$

A_2

$64-\left(\frac{1}{4}x^2-x+1\right) = 63-\frac{1}{4}x^2+x > 0$ when $x\in[3,6]$

$$\bar{y}_2 = \frac{\frac{1}{2}\int_3^6 \left|8^2 - \left(\frac{1}{2}x-1\right)^2\right| dx}{\frac{81}{4}} = \frac{\frac{1}{2}\int_3^6 \left(63 - \frac{1}{4}x^2 + x\right) dx}{\frac{81}{4}} = \frac{\frac{1}{2}\left(63x - \frac{1}{4}\left(\frac{x^3}{3}\right) + \frac{x^2}{2}\right)\Big|_3^6}{\frac{81}{4}}$$

$\longrightarrow$

$$= \frac{\frac{1}{2}\left(63(6-3) - \frac{1}{12}(6^3-3^3) + \frac{1}{2}(6^2-3^2)\right)}{\frac{81}{4}} = \frac{\frac{741}{8}}{\frac{81}{4}} \quad \bar{y}_2$$

The centroid of the fluke is

$$(\bar{x}, \bar{y}) = \left(\frac{\frac{45}{2} + 87}{\frac{45}{4} + \frac{81}{4}}, \frac{\frac{495}{8} + \frac{741}{8}}{\frac{45}{4} + \frac{81}{4}} \right)$$

$$= \left(\frac{\frac{219}{\cancel{2}}}{\frac{63}{\cancel{2}}}, \frac{\frac{309}{\cancel{2}}}{\frac{63}{\cancel{2}}} \right) \approx \boxed{(3.48, 4.90) \text{ inches}}$$