Ch. 13 Review

- **13.7**: Know how to choose u, v in 2 variables;

Keep your work organized and answer the question.

ex: #33

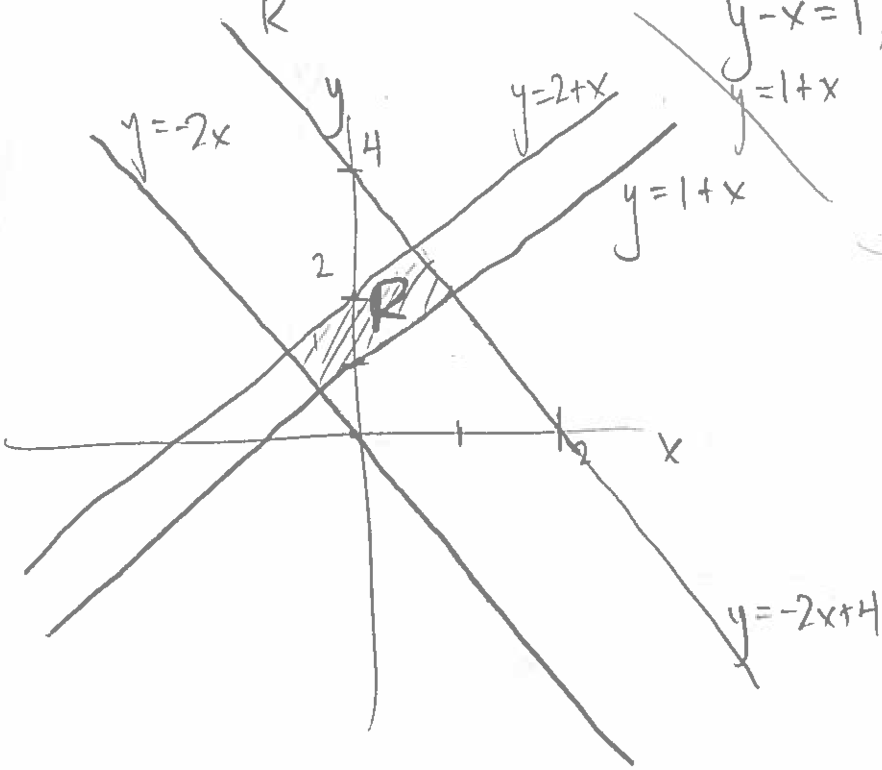
Evaluate using a change of variables. Sketch both regions of integration.

$$\iint_R \left(\frac{y-x}{y+2x+1}\right)^4 dA;$$

R = parallelogram bounded by

$y-x=1$, $y-x=2$, $y+2x=0$, $y+2x=4$

$y=1+x$, $y=2+x$, $y=-2x$, $y=-2x+4$



Let $\boxed{\begin{array}{l} u = y-x \\ v = y+2x \end{array}}$

Then $u = y-x$

$u + x = y$

$v = (u+x)+2x$

$= u + 3x$

$\Rightarrow \boxed{x = \frac{v-u}{3} = g(u,v)}$

$\Rightarrow y = u + \left(\frac{v-u}{3}\right)$

$= \frac{3u+v-u}{3}$

$= \boxed{\frac{2u+v}{3} = h(u,v)}$

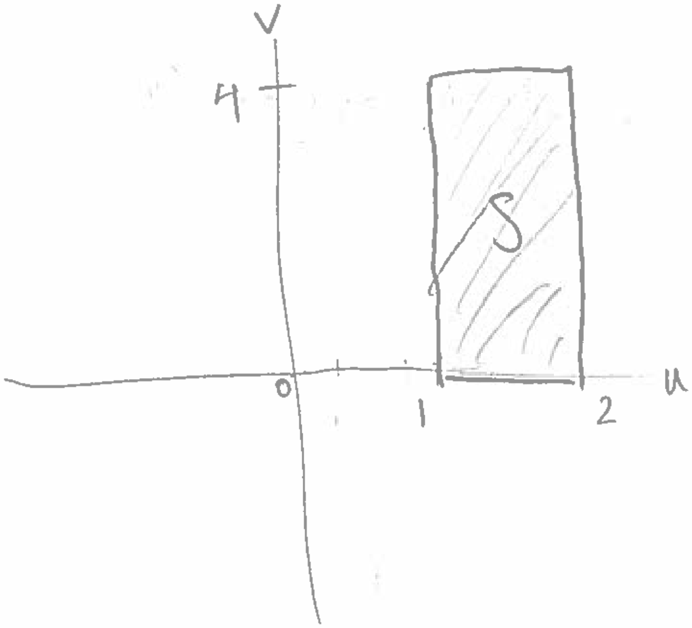
Jacobian:

$$J(u,v) = \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix} = \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \left(\frac{-1}{3}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$$

$$= -\frac{1}{3}$$

The parallelogram equations imply

$$1 \le u \le 2$$
$$0 \le v \le 4$$



and the integral becomes

$$\frac{1}{3}\int_0^4\int_1^2 \left(\frac{u}{v+1}\right)^4 du\,dv$$

$$= \frac{1}{3}\int_0^4 \left(\frac{1}{v+1}\right)^4\left(\frac{u^5}{5}\Big|_1^2\right)dv$$

$$= \frac{1}{3}\left(\frac{2^5}{5}-\frac{1^5}{5}\right)\int_0^4\left(\frac{1}{v+1}\right)^4 dv$$

$w = v+1$
$dw = dv$
$w(0) = 1$
$w(4) = 5$

$$= \left(\frac{2^5-1^5}{5}\right)\frac{1}{3}\int_1^5 w^{-4}\,dw = \frac{31}{9}\,\frac{w^{-3}}{-3}\Big|_1^5$$

$$\boxed{= \frac{31}{15}\cdot\frac{1}{3}\cdot\left(\frac{1}{5^3}-1\right)}$$

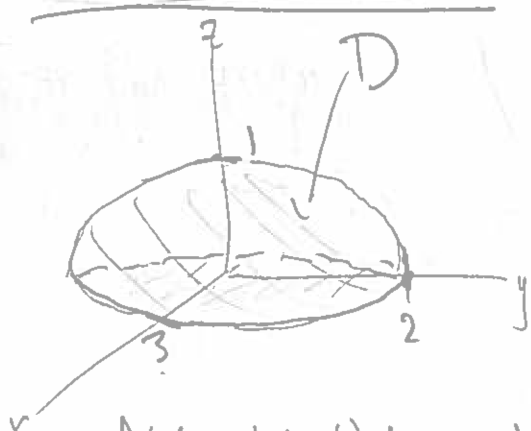
ex: #44

Evaluate $\iiint_D dV$; D is bounded by the upper half of the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$ and the xy-plane.

Use the change of variables
$x = 3u = g(u,v,w)$
$y = 2v = h(u,v,w)$
$z = w = p(u,v,w)$

Jacobian:

$$J(u,v,w) = \begin{vmatrix} g_u & g_v & g_w \\ h_u & h_v & h_w \\ p_u & p_v & p_w \end{vmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$



$\overset{(*)}{=} 6$ (determinant of a diagonal matrix is always the product of the entries on the main diagonal)

Note: We'd love to compute the volume using spherical coordinates — this change-of-variables will set it up.

From the equations, $0 \le z \le \sqrt{1 - \left(\frac{x}{3}\right)^2 - \left(\frac{y}{2}\right)^2}$ and

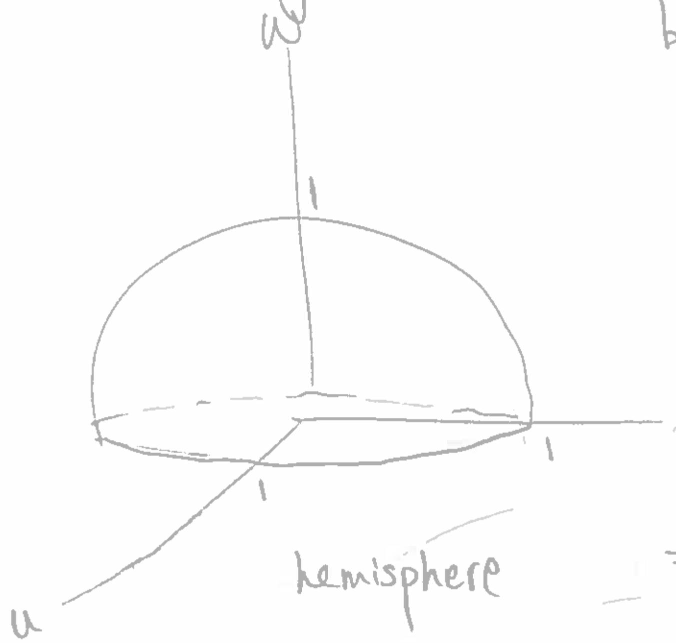
$$0 \le \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 \le 1.$$

Substitute the new variables:

$$0 \le w \le \sqrt{1-\left(\frac{\cancel{3}u}{\cancel{3}}\right)^2-\left(\frac{\cancel{2}v}{\cancel{2}}\right)^2}$$

$$0 \le \left(\frac{\cancel{3}u}{\cancel{3}}\right)^2+\left(\frac{\cancel{2}v}{\cancel{2}}\right)^2 \le 1$$

The new region, E, is a hemisphere; the integral becomes



$$\int_0^1\int_0^{\sqrt{1-u^2}}\int_0^{\sqrt{1-u^2-v^2}} 6\,dw\,dv\,du$$

Jacobian.

Using spherical coordinates,

$$=6\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$=6\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin\varphi\left(\underbrace{\frac{\rho^3}{3}\Big|_0^1}_{\frac{1^3}{3}-\frac{0^3}{3}}\right)d\varphi\,d\theta = \frac{1}{3}(6)\int_0^{2\pi}\left(\underbrace{-\cos\varphi\Big|_0^{\frac{\pi}{2}}}_{\substack{\cos\frac{\pi}{2}-\cos 0\\ 0-1}}\right)d\theta$$

$$=-(-1)(2)\int_0^{2\pi}d\theta = \boxed{4\pi}$$

Ch 13 Review

(or class notes Fri. 30 Oct)

- 13.5: Look at the pictures on p. 918-919, 925-926, and get comfortable switching between coordinate systems;

  Always use the picture and the equations to set up the integral — making sure they are consistent with each other will minimize mistakes;

  Don't forget $r\,dr\,d\theta$, $\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$;
  Common mistake!

  In the pictures, recognize and understand the difference between r and ρ, θ and φ.

- 13.4: Triple integrals — do them neatly. When studying these problems focus on speed and accuracy.

- 13.3: Cartesian $\longleftrightarrow$ polar;
  drawing the pictures; don't worry about lemniscate but you should know how to draw cardiods, annuli, and roses.

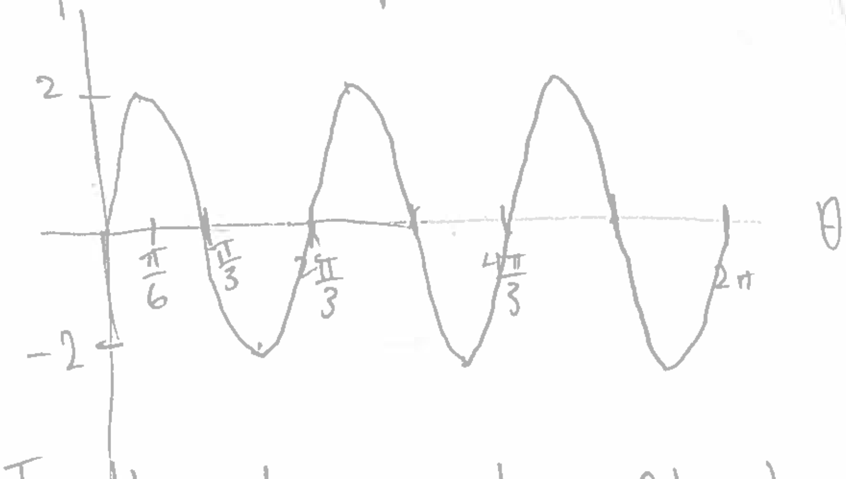
$\longrightarrow$

ex: (MLP)

Sketch the region R, given outside the circle $r=1$ and inside the rose $r=2\sin 3\theta$ in the first quadrant. Then express $\iint_R f(r,\theta)\,dA$ as an iterated integral over R.

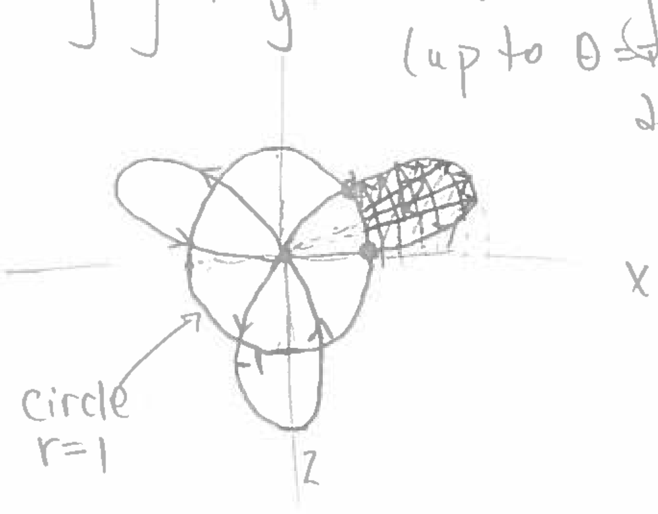
means there will be 3 sine curves; the amplitude is 2

With polar graphing, it helps to draw the "flat" graph first, up to $\theta = 2\pi$



In the polar graph, pretend you are trying to draw a circle, starting at $\theta=0$, but some force is pushing your hand in a way that keeps changing the radius as you draw.

(up to $\theta = \pi$, in this example $\pi \le \theta \le 2\pi$ draws over the same picture)



Since formulas for r are given, this will be the inner integral: $1 \le r \le 2\sin 3\theta$

Ch. 13 Review

To find the points of intersection, set the boundaries equal:

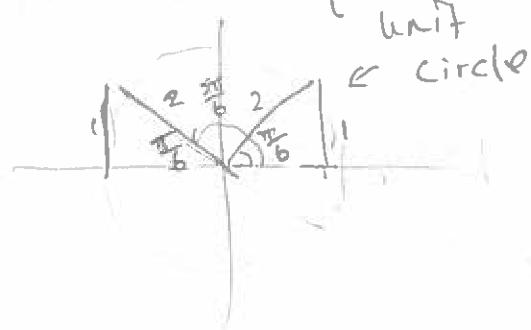
When is sine equal to $\frac{1}{2}$?

$$1 = 2\sin 3\theta$$

$$\frac{1}{2} = \sin 3\theta$$

$$\Rightarrow 3\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\Rightarrow \theta = \frac{\pi}{18}, \frac{5\pi}{18}$$



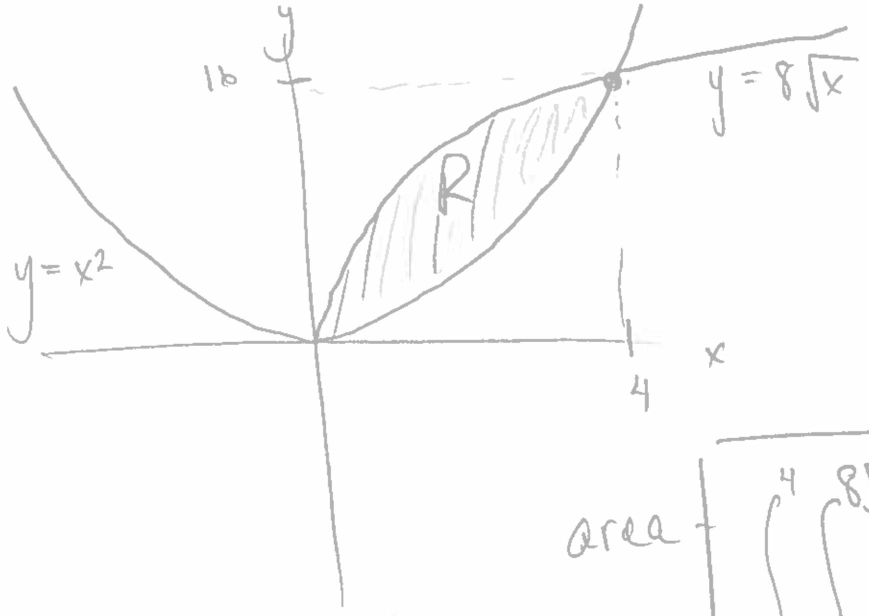
The integral ~~becomes~~

$$\boxed{\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} \int_{1}^{2\sin 3\theta} f(r,\theta)\, r\, dr\, d\theta}$$

- 13.2 Double Integrals; Change-of-Perspective; Drawing R

ex: #12 Sketch R and write an iterated integral for its area.

$$R = \{(x,y) \mid 0 \le x \le 4,\ x^2 \le y \le 8\sqrt{x}\}$$



area $= \boxed{\int_0^4 \int_{x^2}^{8\sqrt{x}} dy\,dx}$

- Know how to find the "average value" of a function (13.1?)