Name: SOLUTIONS

Drill: ———

MATH 2574 (Calculus III)
Fall 2015

Tues 3 Nov 2015

Quiz 9: The Jacobian (§13.4-13.5, 13.7)

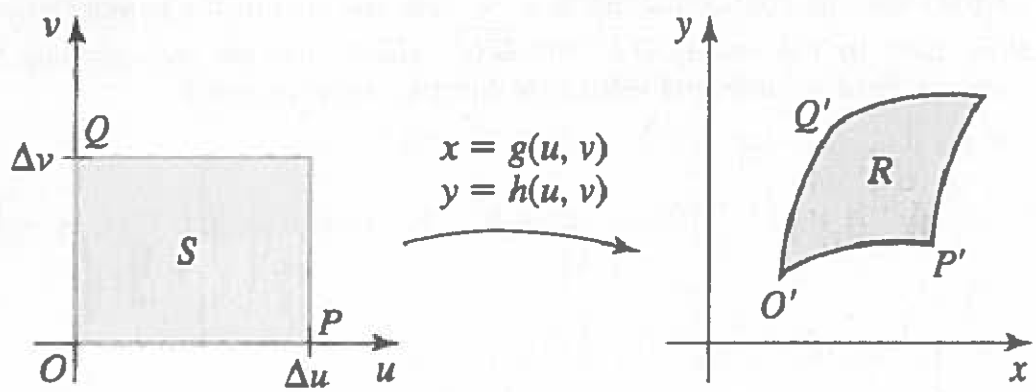
The Jacobian is a magnification (or reduction) factor that relates the area of a small region, or neighborhood, near a point (u, v) in $\mathbb{R}^2$ (uv-plane), to the area of the *preimage** of that region near the point (x, y) in $\mathbb{R}^2$ (xy-plane), where

$$T : x = g(u, v) \quad \text{and} \quad y = h(u, v)$$

is a one-to-one transformation.

Suppose S is a little rectangle in the uv-plane with vertices $(0, 0), (\Delta u, 0), (\Delta u, \Delta v), (0, \Delta v)$. The preimage of S under the transformation given above is a small region R in the xy-plane. The arrows ($\mapsto$) below, and the picture, indicate the respective preimages of each of the following points:

$$(0, 0) = O \mapsto O'$$
$$(\Delta u, 0) = P \mapsto P'$$
$$(0, \Delta v) = Q \mapsto Q'$$



(a) **(3 pts)** Write down the coordinates for each of the points O', P', Q'.

$$O' = \big(g(0,0), h(0,0)\big)$$
$$P' = \big(g(\Delta u,0), h(\Delta u,0)\big)$$
$$Q' = \big(g(0,\Delta v), h(0,\Delta v)\big)$$

*The text instead uses the term *image* and writes $T(S) = R$. The reason for the discrepancy is delicate and relevant to the field of Algebraic Geometry. The transformation T can be described in two ways:

T : { unknowns in the xs and ys } $\to$ { unknowns in the us and vs } *(Algebraic Paradigm)*

or

T : { known values in the us and vs } $\to$ { corresponding x- and y-values } *(Geometric Paradigm)*

(b) The linear approximation of $g(u,v)$ near the point $O=(0,0)$ is:

$$g(u,v) \approx g(0,0) + g_u(0,0)\cdot u + g_v(0,0)\cdot v$$

i. **(1 pt)** Write down the linear approximation of $h(u,v)$ near O.

$h(u,v) \approx h(0,0) + h_u(0,0)\cdot u + h_v(0,0)\cdot v$

ii. **(2 pts)** The points P and Q are close to the point O; use the linear approximations for g and h to compute

$g(\Delta u, 0) \approx g(0,0) + g_u(0,0)\Delta u$ $\quad$ $h(\Delta u, 0) \approx h(0,0) + h_u(0,0)\Delta u$

$g(0, \Delta v) \approx g(0,0) + g_v(0,0)\Delta v$ $\quad$ $h(0, \Delta v) \approx h(0,0) + h_v(0,0)\Delta v$

iii. **(2 pts)** Use the approximations in ii. to find the area of the parallelogram with sides given by the vectors $\overrightarrow{O'P'}$ and $\overrightarrow{O'Q'}$. *(Hint: Use the cross product by first adding a third variable and setting its direction equal to zero.)*

$\overrightarrow{O'P'} = \langle g(\Delta u,0) - g(0,0), h(\Delta u,0) - h(0,0)\rangle$

$\approx \langle \cancel{g(0,0)} + g_u(0,0)\Delta u - \cancel{g(0,0)}, \cancel{h(0,0)} + h_u(0,0)\Delta u - \cancel{h(0,0)}\rangle$

Similarly,

$\overrightarrow{O'Q'} \approx \langle g_v(0,0)\Delta v, h_v(0,0)\Delta v\rangle$

$$\begin{pmatrix}\text{area of}\\ R\end{pmatrix} \approx \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ g_u(0,0)\Delta u & h_u(0,0)\Delta u & 0 \\ g_v(0,0)\Delta v & h_v(0,0)\Delta v & 0 \end{vmatrix} \right\| = \left| [g_u(0,0)\Delta u\, h_v(0,0)\Delta v - h_u(0,0)\Delta u\, g_v(0,0)\Delta v]\hat{k} \right|$$

$$= |g_u(0,0)h_v(0,0) - h_u(0,0)g_v(0,0)|\,\Delta u \Delta v \approx \begin{pmatrix}\text{area of}\\ S\end{pmatrix}$$

iv. **(2 pts)** What is the approximate ratio of the area of R to the area of S?

$|J(0,0)|$, where $J(u,v)$ is the Jacobian for the transformation T