Wed 24 Feb

- Exam 1: see the course webpage for the curve
- MIDTERM
  - Tuesday 8 March 6-7:30p
  - If you have legitimate conflict, i.e., anything that is also scheduled in ISIS, I need to know now. If you are not sure if it conflicts with a course, please have that instructor contact me ASAP.
  - Cumulative. Covers up to §3.9
  - Morning Section: Walker rm 124
    Afternoon Section: Walker rm 218
- Sub on Friday 26 Feb and Monday 29 Feb.
- Possible sub on Wednesday 2 Mar.
- Exam 2: Friday 4 March. Covers up to §3.8.
- Quizzes: Only some of the quiz problems are graded now.

§3.5 Derivatives of Trigonometric Functions

Trig functions are commonly used to model cyclic or periodic behavior in everyday settings. Therefore it is important to know how these functions change across time.

Fact: Derivative formulas for sine and cosine can be derived using the following limits:

- $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$
- $\lim_{x\to 0} \dfrac{\cos x - 1}{x} = 0$

(We will prove these limits in Chapter 4.)

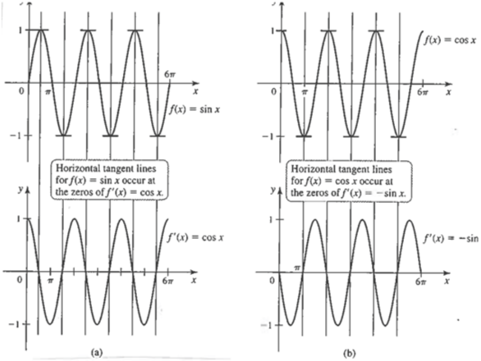
Exercise

Evaluate $\lim_{x \to 0} \frac{\sin 9x}{x}$ and $\lim_{x \to 0} \frac{\sin 9x}{\sin 5x}$.

Using the previous limits and the definition of the derivative, we obtain

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$

Examining the graphs of sine and cosine illustrate the relationship between the functions and their derivatives.



Trig Identities You Should Know

- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = 1 - 2 \sin^2 x$
- $\cos^2 x = \dfrac{1 + \cos 2x}{2}$
- $\sin^2 x = \dfrac{1 - \cos 2x}{2}$
- $\tan x = \dfrac{\sin x}{\cos x}$
- $\cot x = \dfrac{\cos x}{\sin x}$
- $\cot x = \dfrac{1}{\tan x}$
- $\sec x = \dfrac{1}{\cos x}$
- $\csc x = \dfrac{1}{\sin x}$

Derivatives of Other Trig functions

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \cos x - (-\sin x)\sin x}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}$$

So $\frac{d}{dx}(\tan x) = \sec^2 x$.

By using trig identities and the Quotient Rule, we obtain

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\csc x \cot x$$
$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \sec x \tan x$$
$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\tan x}\right) = -\csc^2 x$$

Exercise

Compute the derivative of the following functions:

$$f(x) = \frac{\tan x}{1 + \tan x} \qquad g(x) = \sin x \cos x$$

Exercise

Use the difference and product rules to find the derivative of the function $y = \cos x - x \sin x$.

A. $-\sin x + x \cos x$

B. $x \cos x$

C. $-2 \sin x - x \cos x$

D. $x \cos x - 2 \sin x$

Higher-Order Trig Derivatives

There is a cyclic relationship between the higher order derivatives of $\sin x$ and $\cos x$:

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f^{(3)}(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \end{aligned}$$

$$\begin{aligned} g(x) &= \cos x \\ g'(x) &= -\sin x \\ g''(x) &= -\cos x \\ g^{(3)}(x) &= \sin x \\ g^{(4)}(x) &= \cos x \end{aligned}$$

3.5 Book Problems

7-47 (odds), 57, 59, 61

§3.6 Derivatives as Rates of Change

Question

Why do we need derivatives in real life?

We look at four areas where the derivative assists us with determining the rate of change in various contexts.

Suppose an object moves along a straight line and its location at time t is given by the position function $s = f(t)$. The **displacement** of the object between $t = a$ and $t = a + \Delta t$ is

$$\Delta s = f(a + \Delta t) - f(a).$$

Here Δt represents how much time has elapsed.

We now define average velocity as

$$\frac{\Delta s}{\Delta t} = \frac{f(a+\Delta t) - f(a)}{\Delta t}.$$

Recall that the limit of the average velocities as the time interval approaches 0 was the instantaneous velocity (which we denote here by v). Therefore, the instantaneous velocity at a is

$$v(a) = \lim_{\Delta t \to 0} \frac{f(a+\Delta t) - f(a)}{\Delta t} = f'(a).$$

In mathematics, speed and velocity are related but not the same – if the velocity of an object at any time t is given by $v(t)$, then the speed of the object at any time t is given by

$$|v(t)| = |f'(t)|.$$

By definition, acceleration (denoted by a) is the instantaneous rate of change of the velocity of an object at time t. Therefore,

$$a(t) = v'(t)$$

and since velocity was the derivative of the position function $s = f(t)$, then

$$a(t) = v'(t) = f''(t).$$

Summary: Given the position function $s = f(t)$, the velocity at time t is the first derivative, the speed at time t is the absolute value of the first derivative, and the acceleration at time t is the second derivative.

Question

Given the position function $s = f(t)$ of an object launched into the air, how would you know:

- The highest point the object reaches?
- How long it takes to hit the ground?
- The speed at which the object hits the ground?

Exercise

A rock is dropped off a bridge and its distance s (in feet) from the bridge after t seconds is $s(t) = 16t^2 + 4t$. At $t = 2$ what are, respectively, the velocity of the rock and the acceleration of the rock?

A. 64 ft/s; 16 ft/s^2

B. 68 ft/s; 32 ft/s^2

C. 64 ft/s; 32 ft/s^2

D. 68 ft/s; 16 ft/s^2

Suppose $p = f(t)$ is a function of the growth of some quantity of interest. The average growth rate of p between times $t = a$ and a later time $t = a + \Delta t$ is the change in p divided by the elapsed time Δt:

$$\frac{\Delta p}{\Delta t} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

As Δt approaches 0, the average growth rate approaches the derivative $\frac{dp}{dt}$, which is the instantaneous growth rate (or just simply the growth rate). Therefore,

$$\frac{dp}{dt} = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t}.$$

Exercise

The population of the state of Georgia (in thousands) from 1995 ($t = 0$) to 2005 ($t = 10$) is modeled by the polynomial

$$p(t) = -0.27t^2 + 101t + 7055.$$

(a) What was the average growth rate from 1995 to 2005?

(b) What was the growth rate for Georgia in 1997?

(c) What can you say about the population growth rate in Georgia between 1995 and 2005?

Suppose a company produces a large amount of a particular quantity. Associated with manufacturing the quantity is a **cost function** $C(x)$ that gives the cost of manufacturing x items. This cost may include a **fixed cost** to get started as well as a **unit cost** (or **variable cost**) in producing one item.

If a company produces x items at a cost of $C(x)$, then the average cost is $\frac{C(x)}{x}$. This average cost indicates the cost of items already produced. Having produced x items, the cost of producing another Δx items is $C(x + \Delta x) - C(x)$. So the average cost of producing these extra Δx items is

$$\frac{\Delta C}{\Delta x} = \frac{C(x + \Delta x) - C(x)}{\Delta x}.$$

If we let Δx approach 0, we have

$$\lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = C'(x)$$

which is called the **marginal cost**. The marginal cost is the approximate cost to produce one additional item after producing x items.

Note: In reality, we can't let Δx approach 0 because Δx represents whole numbers of items.

Exercise

If the cost of producing x items is given by

$$C(x) = -0.04x^2 + 100x + 800$$

for $0 \leq x \leq 1000$, find the average cost and marginal cost functions. Also, determine the average and marginal cost when $x = 500$.

3.6 Book Problems

9-19, 21-24, 30-33 (odds)