

Math 2554 Exam 4: § 4.6 – 5.4  
Friday 24 April 2015

Name: SOLUTIONS

## Calculus I Exam 4

Please provide the following data:

Drill Instructor: \_\_\_\_\_

Drill Time: \_\_\_\_\_

Student ID or clicker #: \_\_\_\_\_

**Exam Instructions:** You have 50 minutes to complete this exam. This exam is open resources. Write on your test which resources you used, and where in the problem you used them. Justification is required for all problems. If your answer is wrong, showing more steps can earn you partial credit. The grading will be picky on notation.

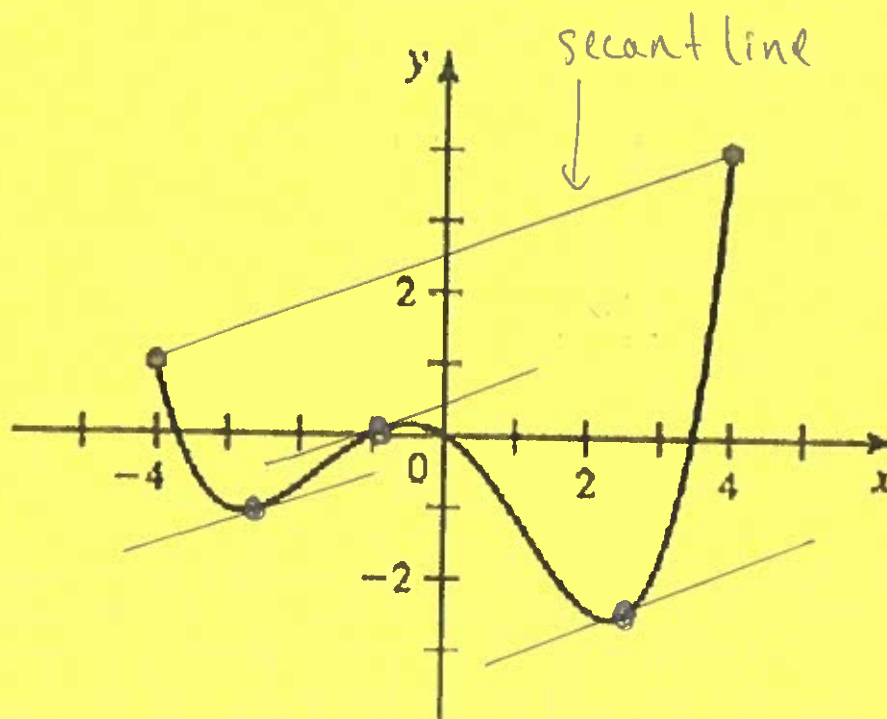
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Good luck!

1. (5 points) By visual inspection, locate all points on the graph at which the slope of the tangent line equals the average rate of change of the function over the interval  $[-4, 4]$ .



from Briggs, et al.

Since the function is continuous on  $[-4, 4]$  and differentiable on  $(-4, 4)$ , the Mean Value Theorem applies.

2. (3 pts ea) Suppose

$$\int_1^4 f(x) dx = 8 \quad \text{and} \quad \int_1^6 f(x) dx = 5.$$

Evaluate the following integrals:

$$\begin{aligned} \text{(a)} \quad \int_1^4 (-3f(x)) dx &= -3 \int_1^4 f(x) dx \\ &= -3(8) \\ &= -24 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_6^4 12f(x) dx &= -12 \int_4^6 f(x) dx \\ &= -12 \left[ \int_1^6 f(x) dx - \int_1^4 f(x) dx \right] \\ &= -12(5-8) \\ &= -12(-3) = 36 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \int_4^6 (f(x) + 3x) dx &= \int_4^6 f(x) dx + 3 \int_4^6 x dx \\ &= -3 + 3 \left. \frac{x^2}{2} \right|_4^6 \\ &= -3 + 3 \left( \frac{36}{2} - \frac{16}{2} \right) = -3 + 3(10) = 27 \end{aligned}$$

3. (6 pts ea) Evaluate the following limits:

$$(a) \lim_{x \rightarrow 1^-} (1-x) \tan\left(\frac{\pi x}{2}\right) \quad (0 \cdot \infty)$$

$$= \lim_{x \rightarrow 1^-} \frac{1-x}{\cot \frac{\pi x}{2}} \quad \left(\frac{0}{0}\right)$$

L'Hôpital:

$$= \lim_{x \rightarrow 1^-} \frac{-1}{-\csc^2\left(\frac{\pi x}{2}\right) \cdot \frac{\pi}{2}} = \frac{2 \sin^2\left(\frac{\pi}{2}\right)}{\pi}$$

$$= \frac{2}{\pi}$$

$$(b) \lim_{\theta \rightarrow \frac{\pi}{2}^-} (\tan \theta - \sec \theta) = \lim_{\theta \rightarrow \frac{\pi}{2}^-} \frac{\sin \theta - 1}{\cos \theta} \quad \left(\frac{0}{0}\right)$$

L'Hôpital:

$$= \lim_{\theta \rightarrow \frac{\pi}{2}^-} \frac{\cos \theta}{-\sin \theta} = 0$$

$$(c) \lim_{z \rightarrow \infty} \left(1 + \frac{10}{z^2}\right)^{z^2}$$

$$\text{Put } L = \ln \left( \lim_{z \rightarrow \infty} \left(1 + \frac{10}{z^2}\right)^{z^2} \right)$$

$$= \lim_{z \rightarrow \infty} z^2 \ln \left(1 + \frac{10}{z^2}\right) \quad (\infty \cdot 0)$$

$$= \lim_{z \rightarrow \infty} \frac{\ln \left(1 + \frac{10}{z^2}\right)}{\frac{1}{z^2}} \quad \left(\frac{0}{0}\right)$$

$$\text{Let } t = \frac{1}{z^2}.$$

$$= \lim_{t \rightarrow 0^+} \frac{\ln(1 + 10t)}{t}$$

L'Hôpital:

$$= \lim_{t \rightarrow 0^+} \frac{\frac{1}{1+10t} \cdot 10}{1} = 10$$

$$\text{Then } \lim_{z \rightarrow \infty} \left(1 + \frac{10}{z^2}\right)^{z^2} = e^L = e^{10}.$$

4. (7 points) Find the point(s) at which the function

$$f(x) = 1 - |x|$$

equals its average value on the interval  $[-1, 1]$ . Then draw the picture of  $f(x)$ , labelling the points and the average value you computed.

$$\bar{f} = \frac{1}{1 - (-1)} \int_{-1}^1 (1 - |x|) dx$$

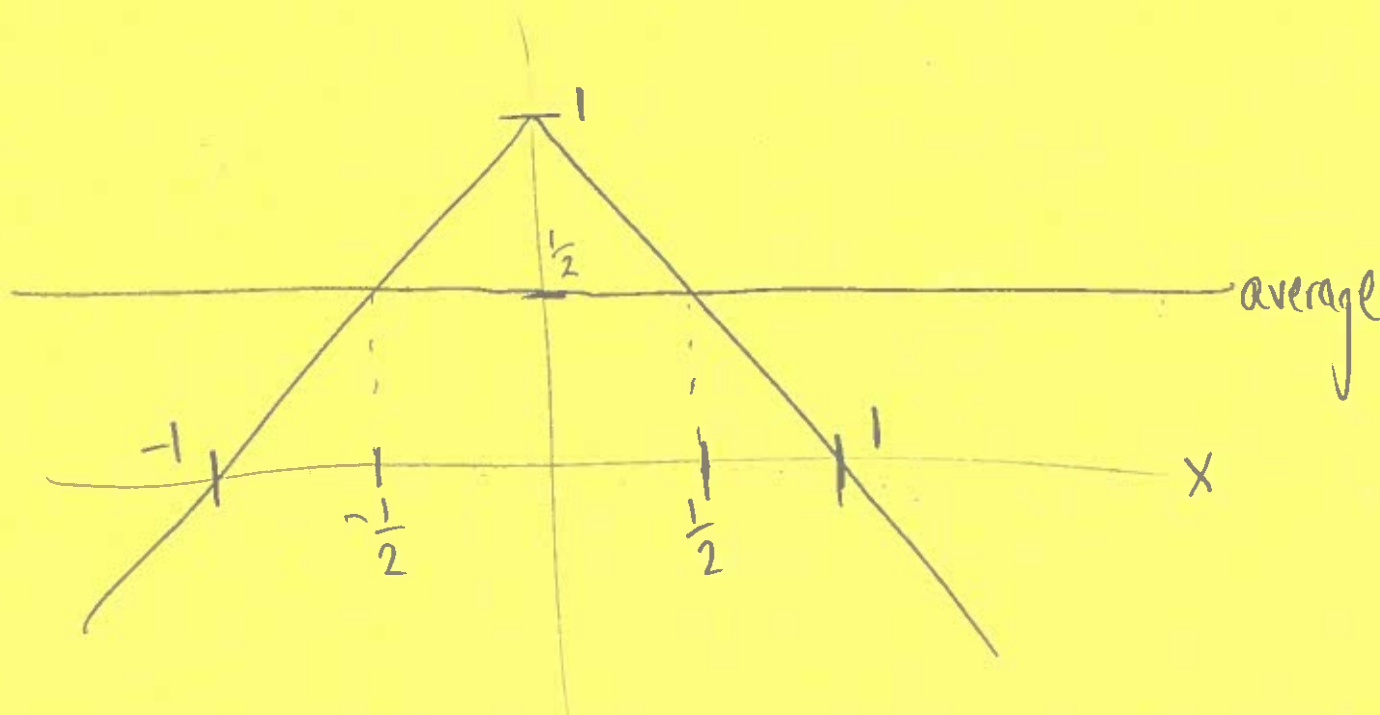
← even function

$$= \frac{1}{2} \left( 2 \int_0^1 (1 - x) dx \right)$$

$$= x - \frac{x^2}{2} \Big|_0^1 = \left( 1 - \frac{1}{2} \right) - \left( 0 - \frac{0}{2} \right) = \frac{1}{2}$$

$$f(x) = 1 - |x| = \frac{1}{2}$$

$$1 - \frac{1}{2} = |x| \Rightarrow x = \pm \frac{1}{2}$$



5. (3 pts ea) Fill in the blanks.

(a)  $\sum_{k=1}^{10} f(1+2k) \cdot 2$  is a right Riemann sum for  $f$  on the interval  $[\underline{1}, \underline{21}]$

with  $n = \underline{10}$ .

$$\Delta x = 2$$

$$\bar{x}_k = a + k\Delta x = 1 + 2k$$

$$a + \cancel{2k} = 1 + \cancel{2k}$$

$$a = 1$$

$$\frac{b-a}{\Delta x} = n$$

$$\frac{b-1}{2} = 10 \Rightarrow b = 21$$

(b)  $\sum_{k=1}^4 f\left(\frac{3}{2} + \frac{k}{2}\right) \cdot \frac{1}{2}$  is a midpoint Riemann sum for  $f$  on the interval  $[\underline{\frac{7}{4}}, \underline{\frac{15}{4}}]$

with  $n = \underline{4}$ .

$$\Delta x = \frac{1}{2}$$

$$\bar{x}_k = a + \left(k - \frac{1}{2}\right)\Delta x = \frac{3}{2} + \frac{k}{2}$$

$$a + \frac{\cancel{k}}{2} - \frac{1}{4} = \frac{3}{2} + \frac{\cancel{k}}{2}$$

$$\Rightarrow a = \frac{3}{2} + \frac{1}{4} = \frac{7}{4}$$

$$\frac{b-a}{\Delta x} = n$$

$$\frac{b - \frac{7}{4}}{\frac{1}{2}} = 4$$

$$b = 2 + \frac{7}{4} = \frac{15}{4}$$

(c)  $\sum_{k=1}^5 f(2+k) \cdot 1$  is a left Riemann sum for  $f$  on the interval  $[\underline{3}, \underline{8}]$

with  $n = \underline{5}$ .

$$\Delta x = 1$$

$$\bar{x}_k = a + (k-1)\Delta x = 2 + k$$

$$a + \cancel{k} - 1 = 2 + \cancel{k}$$

$$a = 3$$

$$\frac{b-a}{\Delta x} = n$$

$$\frac{b-3}{1} = 5$$

$$\Rightarrow b = 8$$



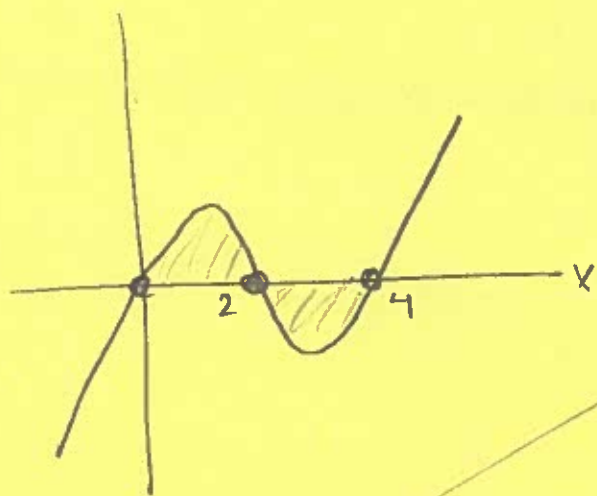
6. (4 pts ea) Evaluate each:

$$\begin{aligned} \text{(a)} \quad \int_0^{\ln 8} e^x dx &= e^x \Big|_0^{\ln 8} \\ &= e^{\ln 8} - e^0 \\ &= 8 - 1 \\ &= 7 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{d}{dx} \int_x^0 \frac{dp}{p^2 + 1} &= - \frac{d}{dx} \int_0^x \frac{dp}{p^2 + 1} \\ &= \frac{-1}{x^2 + 1} \end{aligned}$$



- (c) the net area of the region bounded between the  $x$ -axis and the function  
 $f(x) = x(x-2)(x-4) = x(x^2 - 6x + 8)$



$$\int_0^4 (x^3 - 6x^2 + 8x) dx$$

$$= \left. \frac{x^4}{4} - \frac{6x^3}{3} + \frac{8x^2}{2} \right|_0^4$$

$$= \left( \frac{4^4}{4} - 2(4^3) + 4(4^2) \right) - 0$$

$$= 4^3 - 2(4^3) + 4^3 = 0$$

(d)  $\frac{d}{dy} \int_2^{y^3} (t^2 + t + 1) dt$

$$= 3y^2 \left( (y^3)^2 + y^3 + 1 \right)$$

$$= 3y^2 (y^6 + y^3 + 1)$$

$$= 3y^8 + 3y^5 + 3y^2$$

7. (10 points) A mass oscillates up and down on the end of a spring. Find its position  $s$  relative to the equilibrium position if its acceleration is

$$a(t) = \sin(\pi t),$$

its initial velocity is  $v(0) = 3$ , and its initial position is  $s(0) = 0$ .

$$v(t) = \int a(t) dt$$

$$= \int \sin(\pi t) dt = -\frac{1}{\pi} \cos(\pi t) + C$$

$$v(0) = -\frac{1}{\pi} \cos(0) + C = 3$$

$$\Rightarrow C = 3 + \frac{1}{\pi}$$

$$s(t) = \int \left( -\frac{1}{\pi} \cos(\pi t) + 3 + \frac{1}{\pi} \right) dt$$

$$= -\frac{1}{\pi^2} \sin(\pi t) + 3t + \frac{1}{\pi} t + C$$

$$s(0) = -\frac{1}{\pi^2} \sin(0) + 3(0) + \frac{1}{\pi}(0) + C = 0$$

$$\Rightarrow C = 0$$

$$s(t) = -\frac{1}{\pi^2} \sin(\pi t) + \left( 3 + \frac{1}{\pi} \right) t$$