Quiz 9: L'Hôpital's Rule and Antiderivatives

SOLUTIONS
Tues. 19 Apr. 2016

1. (a) $\lim_{\theta \to 0} \csc\theta - \cot\theta$

$= \lim_{\theta \to 0} \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}$

$\infty - \infty$

$= \lim_{\theta \to 0} \frac{1-\cos\theta}{\sin\theta} \quad \frac{0}{0}$

$\stackrel{LR}{=} \lim_{\theta \to 0} \frac{0-(-\sin\theta)}{\cos\theta} = \frac{\sin(0)}{\cos(0)} = \frac{0}{1} = \boxed{0}$

(b) $\lim_{u \to 1} \frac{u^{10}-1}{12u-12} \quad \frac{0}{0}$

$\stackrel{LR}{=} \lim_{u \to 1} \frac{10u^9 - 0}{12} = \boxed{\frac{10}{12}} = \frac{5}{6}$

OR $\lim_{u \to 1} \frac{u^{10}-1}{12u-12} = \lim_{u \to 1} \frac{\cancel{(u-1)}(u^9+u^8+\dots+u+1)}{\cancel{(u-1)} \cdot 12}$

$= \frac{1^9+1^8+\dots+1+1}{12} = \boxed{\frac{10}{12}} = \frac{5}{6}$

$\longrightarrow$

2. (a) $\lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt}$

Let $L = \ln\left(\lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt}\right)$

$$= \lim_{n\to\infty} \ln\left(A_0\left(1+\frac{r}{n}\right)^{nt}\right)$$

$$= \lim_{n\to\infty}\left(\ln A_0 + \ln\left(1+\frac{r}{n}\right)^{nt}\right)$$

$$= \lim_{n\to\infty}\left(\ln A_0 + nt\ln\left(1+\frac{r}{n}\right)\right)$$

$$= \lim_{n\to\infty} \ln A_0 + \lim_{n\to\infty} nt\ln\left(1+\frac{r}{n}\right)$$

$\infty \cdot 0$

$$= \ln A_0 + t\lim_{n\to\infty} \frac{\ln\left(1+\frac{r}{n}\right)}{\frac{1}{n}}$$

Let $m=\frac{1}{n}$. As $n\to\infty$, $m\to 0$

$$= \ln A_0 + t\lim_{m\to 0} \frac{\ln(1+rm)}{m} \quad \frac{0}{0}$$

$$\overset{LR}{=} \ln A_0 + t\lim_{m\to 0} \frac{\frac{1}{1+rm}(r)}{1}$$

$\longrightarrow$

$$= \ln A_0 + t \lim_{m \to 0} \frac{r}{1 + rm}$$

$$= \ln A_0 + t \frac{r}{1 + r(0)} = \ln A_0 + rt. \leftarrow L$$

Then $\displaystyle\lim_{n \to \infty} A_0 \left(1 + \frac{r}{n}\right)^{nt}$

$$= e^{L} = \boxed{e^{\ln A_0 + rt}}$$

$$= \left(e^{\ln A_0}\right)\left(e^{rt}\right)$$

$$= A_0 e^{rt}.$$

(b) 12% compounded monthly:

$$r_0 = 12 \ln\left(1 + \frac{0.12}{12}\right) \approx 11.940\%$$

13% compounded yearly:

$$r_0 = (1) \ln\left(1 + \frac{0.13}{1}\right) \approx 12.221\%$$

← higher continuously compounded interest rate.

3. $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$ $\frac{\infty}{\infty}$

$$\overset{LR}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{\cancel{2}}(x^2+1)^{-1/2}(\cancel{2}x)} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} \quad ?!$$

Use technique from Chapter 2:

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \cdot \frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}} = \lim_{x\to\infty} \frac{\frac{x}{\cancel{\sqrt{x^2}}} \to 1}{\sqrt{\underbrace{\frac{\cancel{x^2}}{\cancel{x^2}}}_{1} + \underbrace{\frac{1}{\cancel{x^2}}}_{0}}}$$

$$\boxed{= 1}$$

4. $\lim_{x\to\infty} \frac{b^x}{x^n}$ $\frac{\infty}{\infty}$ So as $x \to \infty$, $b^x \to \infty$ and $x^n \to \infty$

$$\overset{LR}{=} \lim_{x\to\infty} \frac{b^x \ln b}{n x^{n-1}} \quad \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{b^x (\ln b)^2}{n(n-1)x^{n-2}} \quad \frac{\infty}{\infty}$$

... keep applying L'Hôpital

$$\boxed{= \lim_{x\to\infty} \frac{b^x (\ln b)^n}{n!} = \infty}$$

means $b^x >> x^n$.

<u>Note</u>: $n! = n(n-1)(n-2)\cdots 1$

5. (a) $s'(t) = \int s''(t)\,dt = \int g\,dt = gt + C$

$$s'(0) = g(0) + C = v_0$$
$$\Rightarrow C = v_0.$$

$$s(t) = \int s'(t)\,dt = \int (gt + v_0)\,dt$$
$$= \frac{gt^2}{2} + v_0 t + C$$
$$s(0) = \cancel{\frac{g}{2}(0)^2} + v_0(0) + C = h$$
$$\Rightarrow C = h$$

$$\boxed{s(t) = \frac{g}{2}t^2 + v_0 t + h}$$

(b) $s(t) = \int v(t)\,dt$

$$= \int \left(\underbrace{-\frac{mg}{\beta}}_{\text{constant}} + \underbrace{\left(\frac{mg}{\beta} + v_0\right)}_{\text{constant}} e^{\overbrace{-\frac{\beta}{m}}^{\text{constant}}t} \right) dt$$

$$= -\frac{mg}{\beta}t + \left(\frac{mg}{\beta} + v_0\right)\left(-\frac{m}{\beta}\right)e^{-\frac{\beta}{m}t} + C$$

$$s(0) = \cancel{-\frac{mg}{\beta}(0)} + \left(\frac{mg}{\beta} + v_0\right)\left(-\frac{m}{\beta}\right)\cancel{e^{-\frac{\beta}{m}(0)}} + C = h$$

$$\Rightarrow C = h - \left(\frac{mg}{\beta} + v_0\right)\left(-\frac{m}{\beta}\right)$$

$$= h + \left(\frac{mg}{\beta} + v_0\right)\left(\frac{m}{\beta}\right)$$

$$\boxed{s(t) = -\frac{mg}{\beta}t - \left(\frac{mg}{\beta} + v_0\right)\left(\frac{m}{\beta}\right)e^{-\frac{\beta}{m}t} + h + \left(\frac{mg}{\beta} + V_0\right)\left(\frac{m}{\beta}\right)}$$

6. (a) $\int f(x)\,dx = \int 2x\,dx = \boxed{x^2 + C}$

(b) • $F(0) = 0^2 + C = 8 \Rightarrow C = 8$

• $F(1) = 1^2 + C = 1 \Rightarrow C = 0$