Take-Home Quiz #7

Math 236 (Calc II)
Fall 2017

$\boxed{\text{SOLUTIONS}}$

1. This problem had a typo! The differential equation should read

$$x^2y''+xy'+(x^2-p^2)y=0.$$

(a) Differentiate the Bessel function term-by-term:

$$J'_p(x)=\sum_{k=0}^{\infty}\underbrace{\frac{(-1)^k}{k!(k+p)!2^{2k+p}}}_{\text{coefficient}}\underbrace{(2k+p)x^{(2k+p)-1}}_{\text{power Rule}}$$

$$J''_p(x)=\sum_{k=0}^{\infty}\frac{(-1)^k(2k+p)(2k+p-1)}{k!(k+p)!2^{2k+p}}x^{2k+p-2}$$

Substitute these expressions into the differential equation:

$$x^2\left(\sum_{k=0}^{\infty}\frac{(-1)^k(2k+p)(2k+p-1)}{k!(k+p)!2^{2k+p}}x^{2k+p-2}\right)+x\left(\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)}{k!(k+p)!2^{2k+p}}x^{2k+p-1}\right)$$

$$+(x^2-p^2)\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+p)!2^{2k+p}}x^{2k+p}\right)$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k(2k+p)(2k+p-1)}{k!(k+p)!2^{2k+p}}x^{2k+p}+\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)}{k!(k+p)!2^{2k+p}}x^{2k+p}$$

$$+\sum_{k=0}^{\infty}\frac{(-1)^k(x^2-p^2)}{k!(k+p)!2^{2k+p}}x^{2k+p}$$

$\longrightarrow$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+p)!2^{2k+p}}\Big(\underbrace{(2k+p)(2k+p-1)+(2k+p)}_{\substack{(2k+p)(2k+p-\cancel{1}+\cancel{1}) \\ 4k^2+4kp+p^2}}+(x^2-p^2)\Big)x^{2k+p}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k(4k^2+4kp+x^2)}{k!(k+p)!2^{2k+p}}x^{2k+p}$$

Claim: This is a telescoping series. Look at the partial sums:

$$S_1 = (-1)^0\left(\frac{4(0)^2+4(0)p+x^2}{0!(0+p)!2^{2(0)+p}}\right)x^{2(0)+p} + (-1)^1\left(\frac{4(1)^2+4(1)p+x^2}{1!(1+p)!2^{2(1)+p}}\right)x^{2(1)+p}$$

$$= \frac{1}{p!2^p}x^{2+p} - \left(\frac{4+4p+x^2}{(1+p)!2^{2+p}}\right)x^{2+p}$$

$$= \frac{\cancel{(1+p)\cdot 2^2}}{(1+p)!2^{2+p}}x^{2+p} - \frac{\cancel{4}+\cancel{4p}+x^2}{(1+p)!2^{2+p}}x^{2+p} = -\frac{1}{(1+p)!2^{2+p}}x^{4+p}$$

$$S_2 = S_1 + \frac{(-1)^2(4(2)^2+4(2)p+x^2)}{2!(2+p)!2^{2(2)+p}}x^{2(2)+p}$$

$$= \frac{-1}{(1+p)!2^{2+p}}x^{4+p} + \frac{16+8p+x^2}{2!(2+p)!2^{4+p}}x^{4+p}$$

$$= -\frac{\cancel{2!(2+p)\cdot 2^2}}{2!(2+p)!2^{4+p}}x^{4+p} + \frac{\cancel{16}+\cancel{8p}+x^2}{2!(2+p)!2^{4+p}}x^{4+p} = \frac{1}{2!(2+p)!2^{4+p}}x^{6+p}$$

$\vdots$

$$S_n = \frac{(-1)^n}{n!(n+p)!2^{2n+p}}x^{2(n+1)+p}$$

The series is $\lim_{n\to\infty} \frac{(-1)^n}{n!(n+p)!2^{2n+p}} x^{2(n+1)+p} = 0$, as desired, since factorial functions dominate exponential functions.

(b). $\lim_{k\to\infty} \dfrac{\dfrac{|x^{2(k+1)+p}|}{(k+1)!(k+1+p)!2^{2(k+1)+p}}}{\dfrac{|x^{2k+p}|}{k!(k+p)!2^{2k+p}}} = \lim_{k\to\infty} \frac{|x^2|}{(k+1)(k+1+p)\cdot 4}$

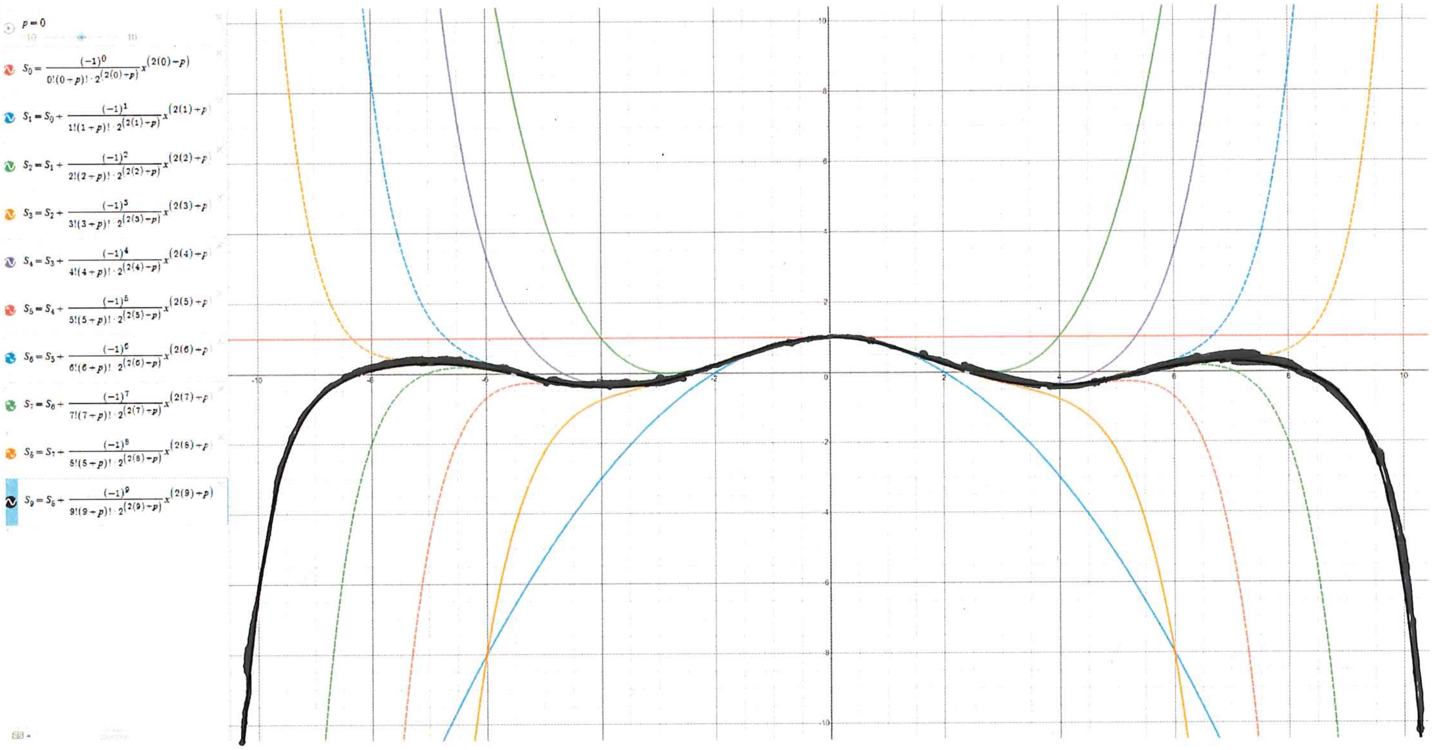
$= \frac{|x^2|}{4} \lim_{k\to\infty} \frac{1}{(k+1)(k+1+p)}$

$= 0$ for all x

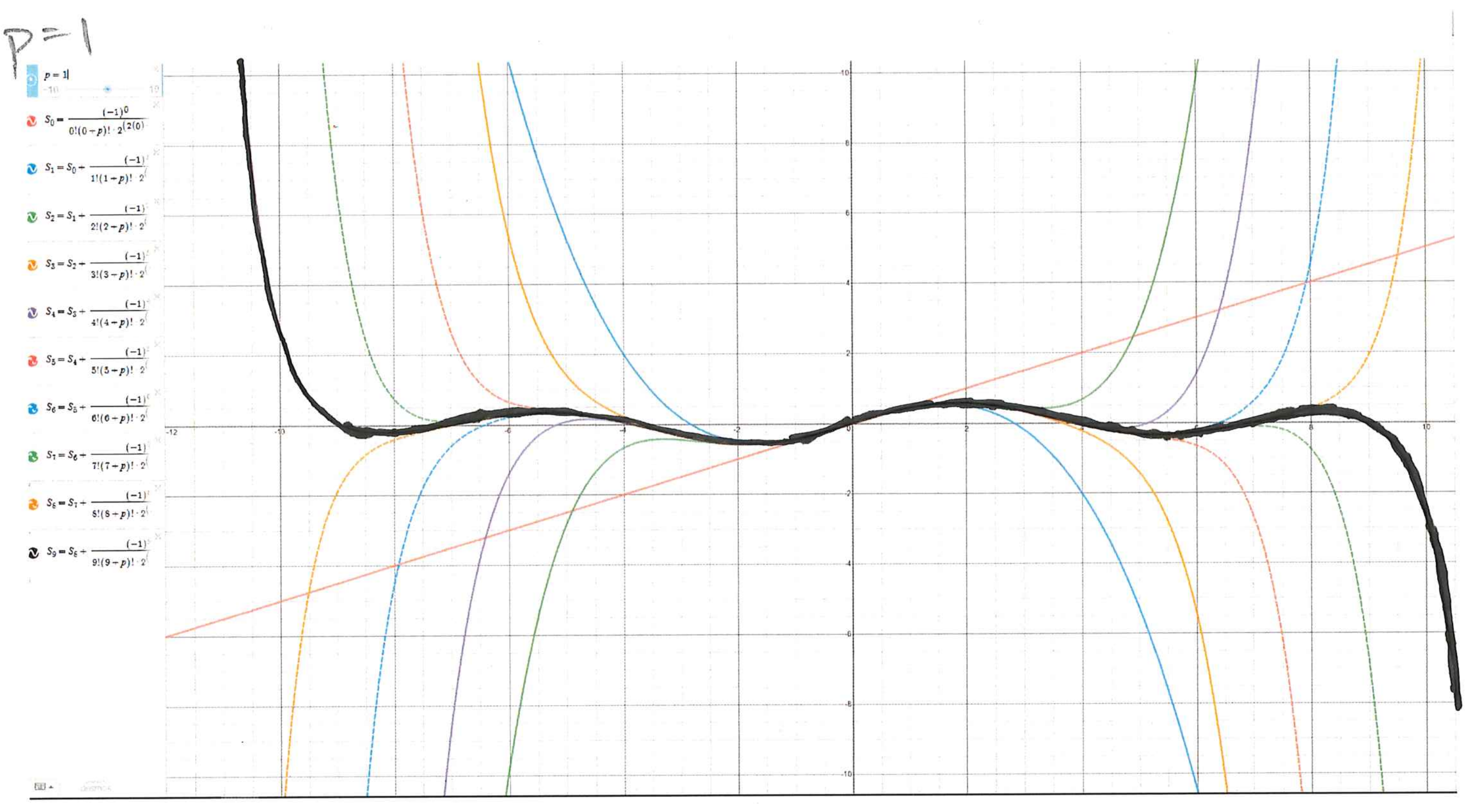
$\Rightarrow J_p(x)$ converges for all $x \in \mathbb{R}$.

(c) – see attached –

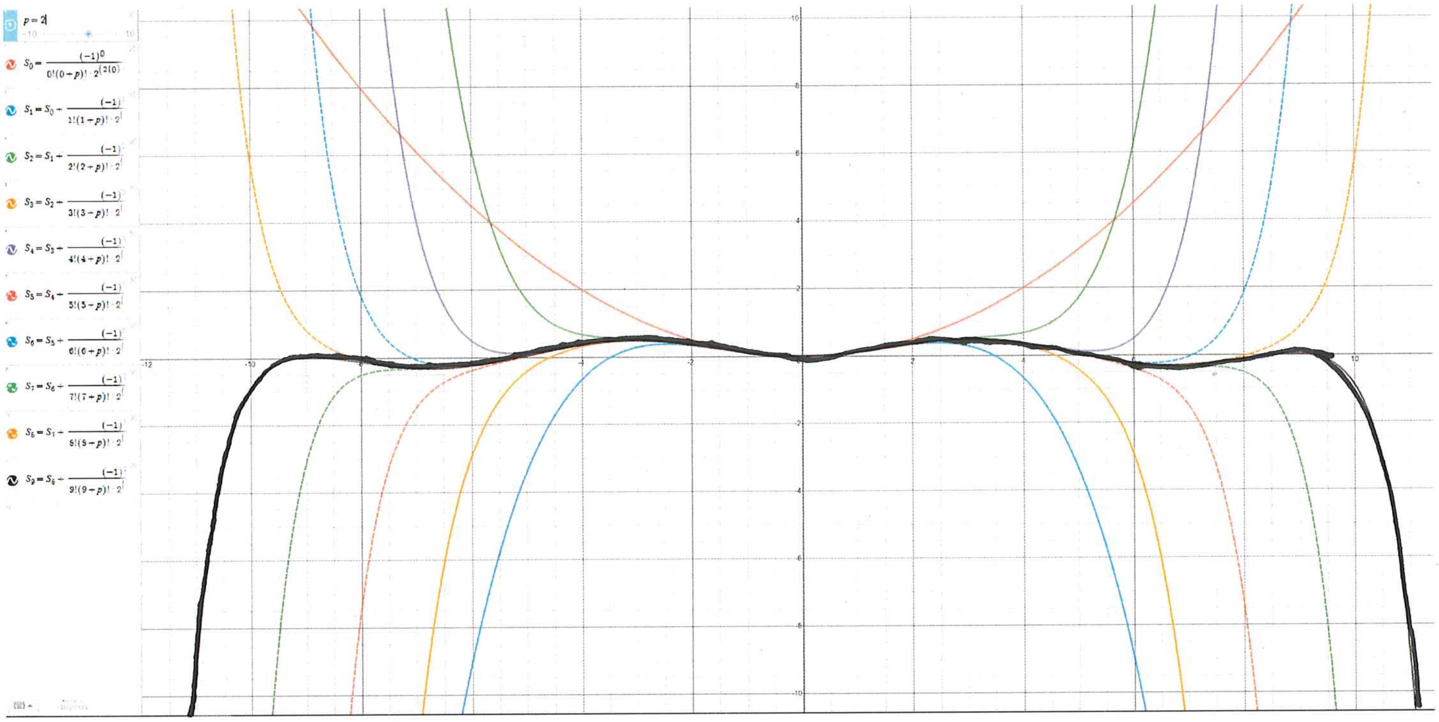
p=0


p = 0
S0 = (−1)^0 / 0!(0+p)!·2^(2(0)−p) x^(2(0)−p)
S1 = S0 + (−1)^1 / 1!(1+p)!·2^(2(1)−p) x^(2(1)+p)
S2 = S1 + (−1)^2 / 2!(2+p)!·2^(2(2)−p) x^(2(2)+p)
S3 = S2 + (−1)^3 / 3!(3+p)!·2^(2(3)−p) x^(2(3)+p)
S4 = S3 + (−1)^4 / 4!(4+p)!·2^(2(4)−p) x^(2(4)+p)
S5 = S4 + (−1)^5 / 5!(5+p)!·2^(2(5)−p) x^(2(5)+p)
S6 = S5 + (−1)^6 / 6!(6+p)!·2^(2(6)−p) x^(2(6)+p)
S7 = S6 + (−1)^7 / 7!(7+p)!·2^(2(7)−p) x^(2(7)+p)
S8 = S7 + (−1)^8 / 8!(8+p)!·2^(2(8)−p) x^(2(8)+p)
S9 = S8 + (−1)^9 / 9!(9+p)!·2^(2(9)−p) x^(2(9)+p)

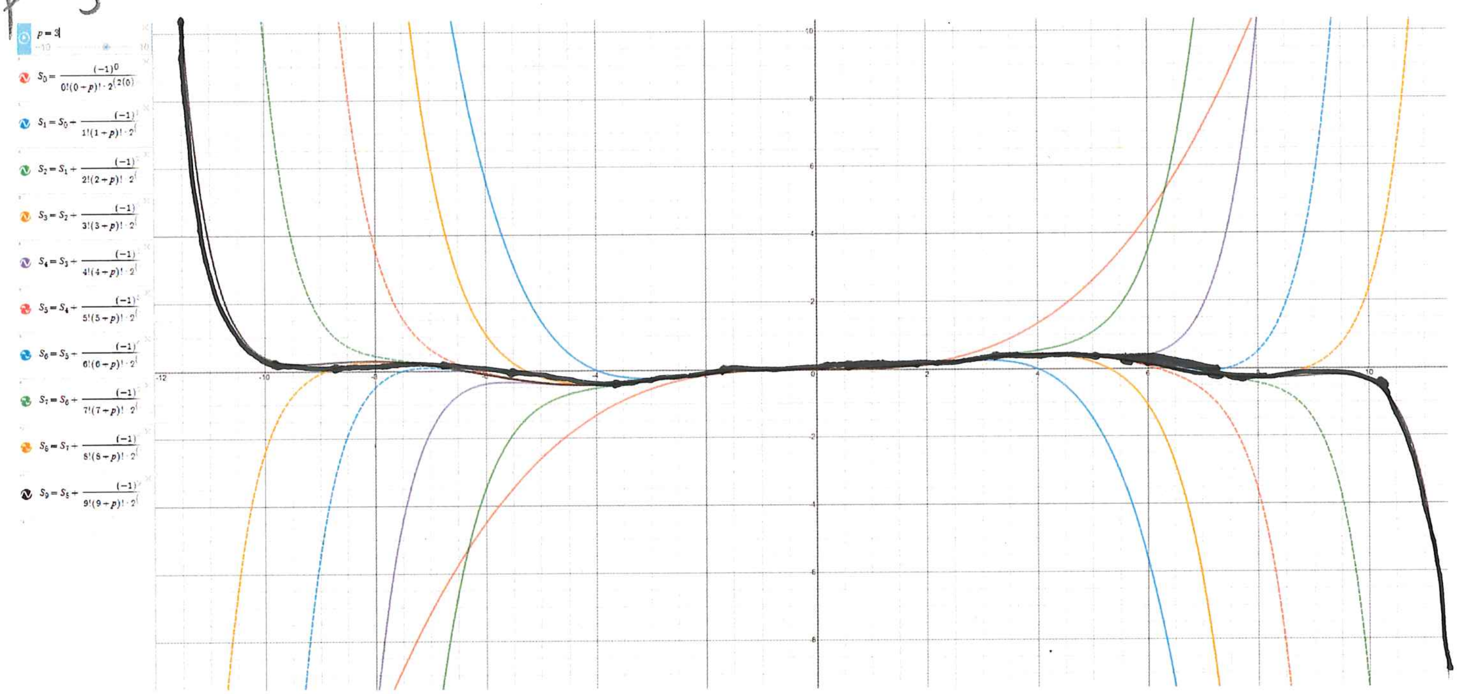
p=1


p = 1
S0 = (−1)^0 / 0!(0−p)!·2^(2(0)
S1 = S0 + (−1) / 1!(1−p)!·2
S2 = S1 + (−1) / 2!(2−p)!·2
S3 = S2 + (−1) / 3!(3−p)!·2
S4 = S3 + (−1) / 4!(4−p)!·2
S5 = S4 + (−1) / 5!(5−p)!·2
S6 = S5 + (−1) / 6!(6−p)!·2
S7 = S6 + (−1) / 7!(7−p)!·2
S8 = S7 + (−1) / 8!(8−p)!·2
S9 = S8 + (−1) / 9!(9−p)!·2

p=2



p=3



2. $f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2}(x)^{-1/2}$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2}$$

$$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2}$$

$$f''''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-7/2}$$

(a) $P_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2} + \frac{f'''(1)(x-1)^3}{3!} + \frac{f^{(4)}(1)(x-1)^4}{4!}$

$$= \boxed{1 + \frac{1}{2}(x-1) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}(x-1)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(x-1)^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}(x-1)^4}$$

(b) $P(x) = 1 + \sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\cdot\ldots\cdot\left(\frac{1}{2}-k+1\right)}{k!}(x-1)^k$

or $\boxed{1 + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k}(x-1)^k}$

$\longrightarrow$

(c) Write $R_n(x) = \dfrac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\cdot\cdots\cdot\left(\frac{1}{2}-n\right)}{(n+1)!}\, c^{\frac{1}{2}-n-1}(x-1)^{n+1}$

$$= \frac{(-1)^n\,(1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1))\sqrt{c}}{(n+1)!\,(2c)^{n+1}}\,(x-1)^{n+1}$$

$$\Rightarrow \lim_{n\to\infty} R_n(x) = \sqrt{c}\left(\lim_{n\to\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(n+1)!}\right)\left(\lim_{n\to\infty}\frac{(-1)^n}{(2c)^{n+1}}\right)\left(\lim_{n\to\infty}(x-1)^{n+1}\right)$$

Note that $x \in \left(\frac{1}{2}, \frac{3}{2}\right)$ and $c \in (1, x)$. In particular,

$2c > 1 \Rightarrow (2c)^{n+1} \longrightarrow \infty$ as $n \longrightarrow \infty$ and

$|x-1| < 1 \Rightarrow (x-1)^{n+1} \longrightarrow 0$ as $n \longrightarrow \infty$.

The limit becomes

$$\sqrt{c}\left(\lim_{n\to\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(n+1)!}\right)(0)(0) = 0$$

2 by the Ratio Test:

$$\frac{\dfrac{\cancel{1}\cdot\cancel{3}\cdot\cancel{5}\cdot\cdots\cdot\overset{2n+2-1}{(2(n+1)-1)}}{((n+1)+1)\cancel{!}}}{\dfrac{\cancel{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}}{\cancel{(n+1)!}}} = \frac{2n+1}{n+2} \longrightarrow 2 \text{ as } n\longrightarrow\infty$$

Since the limit equals 0 when $x \in \left(\frac{1}{2}, \frac{3}{2}\right)$, $f(x)$ must equal its Taylor series on that interval.

3. (a) $P(x) = 9x \sum_{k=0}^{\infty} \left(\frac{x^2}{9}\right)^k = \sum_{k=0}^{\infty} 9x \frac{x^{2k}}{9^k}$

$$= \boxed{\sum_{k=0}^{\infty} \frac{x^{2k+1}}{9^{k-1}}}$$

(b) Since the interval of convergence for $\frac{1}{1-x}$ is $(-1,1)$, we must have $\frac{x^2}{9} \in (-1,1)$

$$\Rightarrow x^2 \in (-9,9)$$

$$\Rightarrow \boxed{x \in (-3,3)}$$

4. (a) $P(x) = \boxed{\ln\left(\frac{1}{4}\right) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{x^2}{4}\right)^k}$

Since $\frac{x^2}{4} \in (-1,1]$, the interval of convergence

$\Rightarrow x^2 \in (-4,4]$ is $\boxed{x \in [-2,2]}$

(b) $\int_{0.5}^{1} \left(\ln\frac{1}{4} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{x^2}{4}\right)^k\right) dx$

$$= \ln\left(\frac{1}{4}\right)x \Big|_{0.5}^{1} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k \cdot 4^k} \int_{0.5}^{1} x^{2k}\, dx$$

$$= \left(\ln\frac{1}{4} - \frac{1}{2}\ln\frac{1}{4}\right) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4^k k} \frac{x^{2k+1}}{2k+1}\Bigg|_{0.5}^{1}$$

$$= \frac{1}{2}\ln\frac{1}{4} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4^k k(2k+1)}\left(1^{2k+1} - \left(\frac{1}{2}\right)^{2k+1}\right)$$

$$= \ln\left(\frac{1}{2}\right) + \sum_{k=1}^{\infty} \frac{\overset{3k+2}{(-1)^{k+1}(-1)^{2k+1}}}{2^{2k} k(2k+1)2^{2k+1}}$$

$$\boxed{= \ln\left(\frac{1}{2}\right) + \sum_{k=1}^{\infty} \frac{(-1)^{3k+2}}{k(2k+1)2^{4k+1}}}$$