Fri 22 Jan

- comp.uark.edu/~ashleykw/Cal1Spring2016/cal1spr16.html
  Course website. All information is here, including a link to MLP, lecture slides, administrative information, etc. You should have already seen the **syllabus** by now.
- MyLabsPlus (MLP) has the graded homework. Solutions to Quizzes and Drill exercises will be posted there, under "Menu $\rightarrow$ Course Tools $\rightarrow$ Document Sharing".

Fri 22 Jan (cont.)

- Lecture slides are available on the course website. I'll try to have the week's slides posted in advance but the individual lectures might not be posted until right before class. **Don't try to take notes from the slides.** Instead, print out the slides beforehand or else follow along on your tablet/phone/laptop. You should, however, take notes when we do exercises during lecture.
- For old Calculus materials, see the parent page `comp.uark.edu/~ashleykw` and look for links under "Previous Semesters".
- Next week: Attendance using clickers.

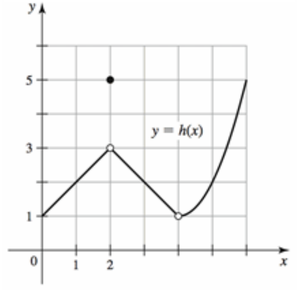
Definition (limit)

Suppose the function f is defined for all x near a, except possibly at a. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = L$$

and say **the limit of** $f(x)$ **as** x **approaches** a **equals** L.

Exercise



Determine the following:

(a) $h(1)$

(b) $h(2)$

(c) $h(4)$

(d) $\lim_{x \to 2} h(x)$

(e) $\lim_{x \to 4} h(x)$

(f) $\lim_{x \to 1} h(x)$

Question

Does $\lim_{x \to a} f(x)$ always equal $f(a)$?

(Hint: Look at the example from the previous slide!)

Exercise

Suppose $f(x) = \dfrac{x^2 + x - 20}{x - 4}$.

(a) Create a table of values of $f(x)$ when

$$x = 3.9, 3.99, 3.999, \text{ and}$$
$$x = 4.1, 4.01, 4.001$$

(b) What can you conjecture about $\lim_{x \to 4} f(x)$?

Up to this point we have been working with two-sided limits; however, for some functions it makes sense to examine one-sided limits.

Notice how in the previous example we could approach $f(x)$ from both sides as x approaches a, i.e., when $x > a$ and when $x < a$.

Definition (right-hand limit)

Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

$$\lim_{x \to a^+} f(x) = L$$

and say **the limit of** $f(x)$ **as** x **approaches** a **from the right equals** L.

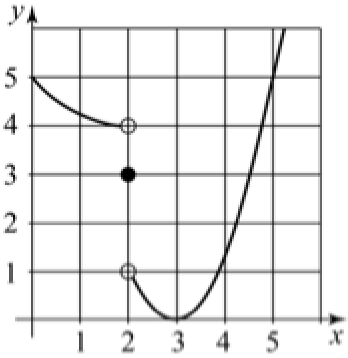
Definition (left-hand limit)

Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

$$\lim_{x \to a^-} f(x) = L$$

and say **the limit of** $f(x)$ **as** x **approaches** a **from the left equals** L.

Exercise



Determine the following:

(a) $g(2)$

(b) $\lim_{x\to 2^+} g(x)$

(c) $\lim_{x\to 2^-} g(x)$

(d) $\lim_{x\to 2} g(x)$

Relationship Between One- and Two-Sided Limits

Theorem

If f is defined for all x near a except possibly at a, then $\lim_{x \to a} f(x) = L$ if and only if both $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$.

In other words, the only way for a two-sided limit to exist is if the one-sided limits equal the same number (L).

2.2 Book Problems

1-4, 7, 9, 11, 13, 19, 23, 29, 31

§2.3 Techniques for Computing Limits

Exercise

Given the function $f(x) = 4x + 7$, find $\lim_{x \to -2} f(x)$

(a) graphically;

(b) numerically (i.e., using a table of values near -2)

(c) via a direct computation method of your choosing.

Compare and contrast the methods in this exercise – which is the most favorable?

This section provides various laws and techniques for determining limits. These constitute **analytical** methods of finding limits. The following is an example of a very useful limit law:

Limits of Linear Functions: Let a, b, and m be real numbers. For linear functions $f(x) = mx + b$,

$$\lim_{x \to a} f(x) = f(a) = ma + b.$$

This rule says we if $f(x)$ is a linear function, then in taking the limit as $x \to a$, we can just plug in the a for x.

IMPORTANT! Using a table or a graph to compute limits, as in the previous sections, can be helpful. However, "analytical" does not include those techniques.

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, c is a real number, and m, n are positive integers.

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1. **Sum:** $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. **Difference:** $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

In other words, if we are taking a limit of two things added together or subtracted, then we can first compute each of their individual limits one at a time.

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, c is a real number, and m, n are positive integers.

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3. **Constant Multiple:** $\lim_{x\to a}(cf(x)) = c\left(\lim_{x\to a} f(x)\right)$

4. **Product:** $\lim_{x\to a}(f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$

The same is true for products. If one of the factors is a constant, we can just bring it outside the limit. In fact, a constant is its own limit.

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, c is a real number, and m, n are positive integers.

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5. **Quotient:** $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

   (provided $\lim_{x\to a} g(x) \neq 0$)

Question

Why the caveat?

Limit Laws, cont.

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, c is a real number, and m, n are positive integers.

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6. **Power:** $\lim_{x\to a} (f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$

7. **Fractional Power:** $\lim_{x\to a} (f(x))^{\frac{n}{m}} = \left(\lim_{x\to a} f(x)\right)^{\frac{n}{m}}$

   (provided $f(x) \geq 0$ for x near a if m is even and $\frac{n}{m}$ is in lowest terms)

Question

Explain the caveat in **7.**

Laws **1.-6.** hold for one-sided limits as well. But **7.** must be modified:

7. Fractional Power (one-sided limits):

- $$\lim_{x \to a^+} (f(x))^{\frac{n}{m}} = \left(\lim_{x \to a^+} f(x) \right)^{\frac{n}{m}}$$
  (provided $f(x) \geq 0$ for x near a with $x > a$, if m is even and $\frac{n}{m}$ is in lowest terms)

- $$\lim_{x \to a^-} (f(x))^{\frac{n}{m}} = \left(\lim_{x \to a^-} f(x) \right)^{\frac{n}{m}}$$
  (provided $f(x) \geq 0$ for x near a with $x < a$, if m is even and $\frac{n}{m}$ is in lowest terms)

Limits of Polynomials and Rational Functions

Assume that $p(x)$ and $q(x)$ are polynomials and a is a real number.

- **Polynomials:** $\lim_{x \to a} p(x) = p(a)$
- **Rational functions:** $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$

  (provided $q(a) \neq 0$)

For polynomials and rational functions we can plug in a to compute the limit, as long as we don't get zero in the denominator. Linear functions count as polynomials. A rational function is a "fraction" made of polynomials.

Exercise

Evaluate the following limits analytically.

1. $\displaystyle\lim_{x\to 1} \frac{4f(x)g(x)}{h(x)}$, given that

$$\lim_{x\to 1} f(x) = 5, \ \lim_{x\to 1} g(x) = -2, \text{ and } \lim_{x\to 1} h(x) = -4.$$

2. $\displaystyle\lim_{x\to 3} \frac{4x^2+3x-6}{2x-3}$

3. $\displaystyle\lim_{x\to 1^-} g(x)$ and $\displaystyle\lim_{x\to 1^+} g(x)$, given that

$$g(x) = \begin{cases} x^2 & \text{if } x \leq 1; \\ x+2 & \text{if } x > 1. \end{cases}$$

Additional (Algebra) Techniques

When direct substitution (a.k.a. plugging in a) fails try using algebra:

- Factor and see if the denominator cancels out.

Example

$$\lim_{t\to 2} \frac{3t^2 - 7t + 2}{2 - t}$$

- Look for a common denominator.

Example

$$\lim_{h\to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h}$$

Exercise

Evaluate $\lim_{s \to 3} \dfrac{\sqrt{3s+16} - 5}{s-3}$.

This method for evaluating limits uses the relationship of functions with each other.

Theorem (Squeeze Theorem)

Assume $f(x) \leq g(x) \leq h(x)$ *for all values of* x *near* a*, except possibly at* a*, and suppose*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.$$

Then since g *is always between* f *and* h *for* x*-values close enough to* a*, we must have*

$$\lim_{x \to a} g(x) = L.$$

Example

(a) Draw a graph of the inequality

$$-|x| \leq x^2 \ln\left(x^2\right) \leq |x|.$$

(b) Compute $\lim_{x \to 0} x^2 \ln\left(x^2\right)$.

2.3 Book Problems

12-30 (every 3rd problem), 33, 39-51 (odds), 55, 57, 61-67 (odds)

In general, review your algebra techniques, since they can save you some headache.