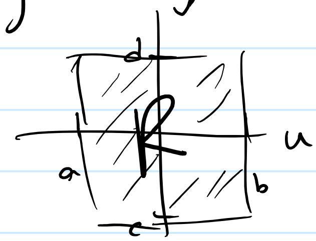
§14.7: Stokes' Theorem

Friday, December 4, 2015 10:07 AM

recall: Surface Integral of a Scalar-Valued Function

$f(x,y,z)$ is a continuous function on a (piecewise) smooth surface S parametrized by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$,

$a \le u \le b$
$c \le v \le d$



Then

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_R f(x(u,v), y(u,v), z(u,v)) \left| \vec{r}_u \times \vec{r}_v \right| dA}$$

usual double integral from Chapter 13

Surface Integral of a Vector Field:

flux integral! $\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_R \vec{F} \cdot \vec{n} \left| \vec{r}_u \times \vec{r}_v \right| dA$

with u's and v's plugged in

But just as in §14.2 we had

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F} \cdot \underbrace{\frac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}}}_{\vec{T}} \underbrace{\cancel{|\vec{r}\,'(t)|}\, dt}_{ds} = \int_a^b \vec{F} \cdot d\vec{r}$$

for the flux surface integral we have

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_R \vec{F}\cdot\underbrace{\frac{(\vec{r}_u\times\vec{r}_v)}{\cancel{|\vec{r}_u\times\vec{r}_v|}}}_{\hat{n}}\underbrace{\cancel{|\vec{r}_u\times\vec{r}_v|}\,dA}_{dS}$$

in particular,

$$= \iint_R \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA$$

Q: What about circulation?

Ans:

§14.7 : Stokes' Theorem

(aka circulation form of Green's Theorem, but one dimension higher)

recall: $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \underbrace{(g_x - f_y)}_{\text{curl}\,\vec{F}}\,dA$

In $\mathbb{R}^3$, $\vec{F} = \langle f, g, h\rangle$ and

$\text{curl}\,\vec{F} = \nabla\times\vec{F} = (h_y - g_z)\hat{i} + (f_z - h_x)\hat{j} + (g_x - f_y)\hat{k}$

dot with $\hat{k} = \langle 0, 0, 1\rangle$: get $g_x - f_y$. So Green's Theorem really says

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (\nabla \times \vec{F})\cdot \hat{k}\, dA$$

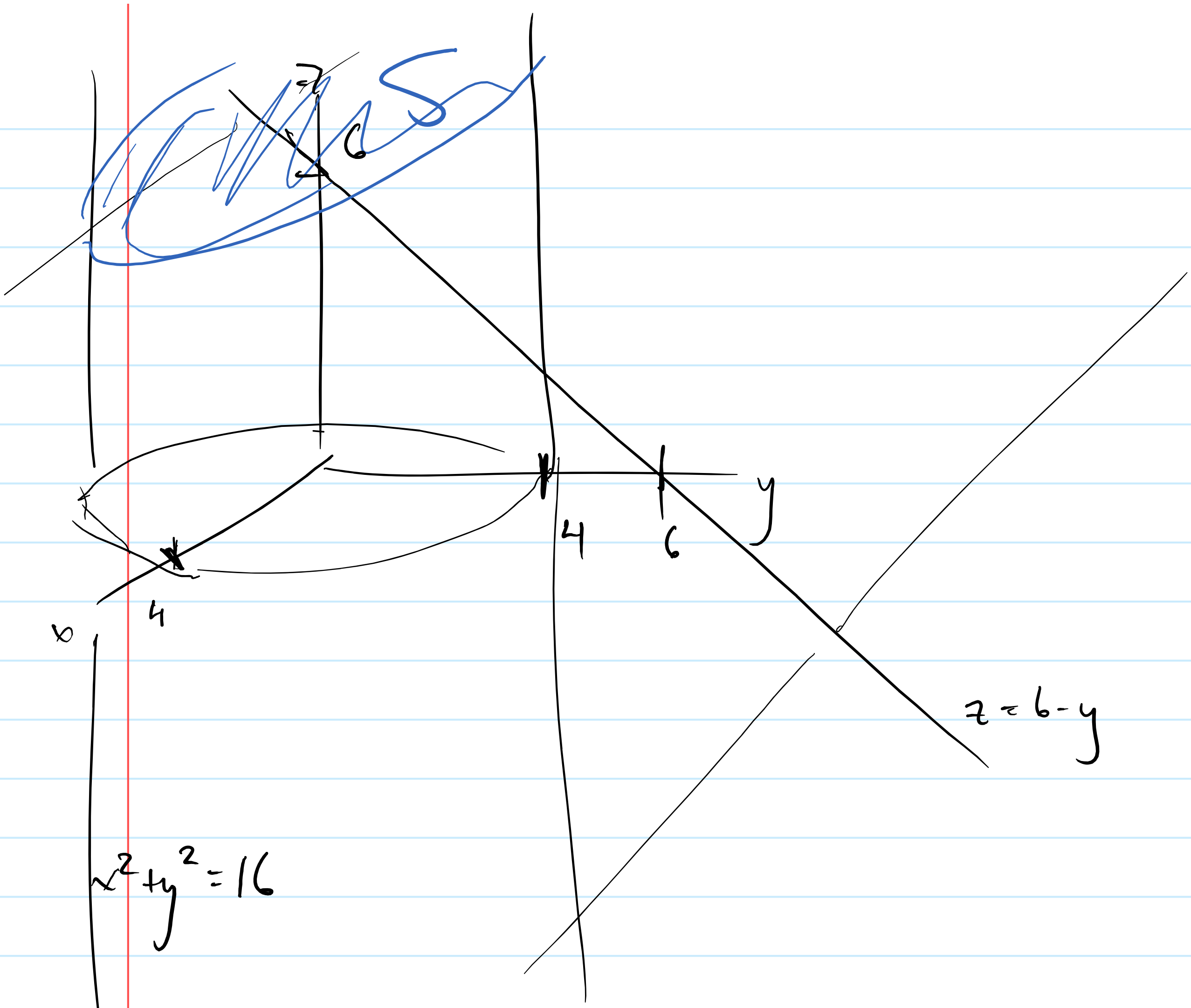
Stokes' Theorem: The 2D region R becomes a surface, S. In $\mathbb{R}^2$, $\hat{k}$ was normal to R.
For S, we need the normal vector $\vec{n}$:

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot \vec{n}\, dS}$$ ← proof is in the text

Caution: Orientations have to be consistent with the RHR, and C oriented CCW when viewed from above.

ex: #10
$\vec{F} = \langle -y, -x-z, y-x \rangle$
and
C is the boundary of S, which is the part of the plane $z = 6-y$ that lies in the cylinder $x^2+y^2=16$.

Verify Stokes' Theorem



RHS: Parametrize C :

The boundary is where these two surfaces intersect.

$$\vec{r}(t) = \langle \underbrace{4\cos t, 4\sin t}_{x^2+y^2=16}, \underbrace{6-4\sin t}_{z=6-y} \rangle, \quad 0 \le t \le 2\pi$$

$$\vec{r}\,'(t) = \langle -4\sin t, 4\cos t, -4\cos t \rangle,$$

$$\vec{F}(t) = \langle -4\sin t, -4\cos t - \underset{-6\ +4\sin t}{(6-4\sin t)}, 4\sin t - 4\cos t \rangle,$$

So
$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \overset{16\sin^2 t}{(-4\sin t)(-4\sin t\, dt)} +$$

$-16\cancel{\cos^2 t} - 24\cos t + 16\cancel{\cos t\sin t} +$

$0 \qquad 4\cos t(-4\cos t - 6 + 4\sin t)\,dt$

$+ (-4\cos t)(4\sin t - 4\cos t)\,dt$

$\cancel{-16\cos t\sin t} + \cancel{16\cos^2 t}$

$$= \int_0^{2\pi} (16\sin^2 t - 24\cos t)\, dt$$

$$= 8 \int_0^{2\pi} (\underbrace{2\sin^2 t}_{\left(\frac{-\cos(2t)}{\cancel{2}}\right)} - 3\cos t)\, dt$$

(Half-Angle)

$$= 8\left(t - \frac{1}{2}\sin 2t - 3\sin t\right)\Big|_0^{2\pi}$$

$2\pi \leftarrow \sin 2\pi = 0$

$0 \leftarrow \sin 0 = 0$ so terms vanish;

$$= 8(2\pi) = \boxed{16\pi}$$

RHS: $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS = ?$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & - & y-x \end{vmatrix} = \langle 1-(-1), -(-1-0), -1-(-1) \rangle$$

$$1+1 \qquad 1 \qquad -1+1$$

$$= \langle 2, 1, 0 \rangle$$

The plane $z = 6 - y$ is parallel to the x-axis, so an outward normal is $\langle 0, 1, 1 \rangle$ $(\hat{j} + \hat{k})$

$$\iint_R 1\, dA$$

parametrize S:

$\vec{r}(u,v) = \langle 4\cos u,$

$4\sin u,$

*Q: How come we don't have to normalize?