

Some exercises on convergence of series:

§ 7.6

#48
$$\sum_{k=1}^{\infty} \left(\frac{1}{k^2} \right)^{\sqrt{k}}$$

① Divergence Test:

$$\lim_{k \rightarrow \infty} \left(\frac{1}{k^2} \right)^{\sqrt{k}} = \left(\lim_{k \rightarrow \infty} \underbrace{\frac{1}{k^2}}_0 \right)^{\sqrt{k}} = 0 \Rightarrow \underline{\text{inconclusive}}.$$

② Integral Test:

Let $a(x) = \left(\frac{1}{x^2} \right)^{\sqrt{x}}$

• continuous on $[1, \infty)$? ✓

• positive on $[1, \infty)$? ✓

• decreasing on $[1, \infty)$? ✓

$$\int_1^{\infty} a(x) dx = \int_1^{\infty} \left(\frac{1}{x^2} \right)^{\sqrt{x}} dx \approx 3.19313394028126 \dots$$

(via Wolfram Alpha & evaluated numerically)

$\Rightarrow \sum_{k=1}^{\infty} \left(\frac{1}{k^2} \right)^{\sqrt{k}}$ follows the same behavior, so converges.

③ Comparison Test:

$$\left(\frac{1}{k^2}\right)^{\sqrt{k}} = \frac{1}{k^{2\sqrt{k}}} \leq \frac{1}{k^2}$$

$$\Rightarrow \sum_{k=1}^{\infty} \left(\frac{1}{k^2}\right)^{\sqrt{k}} \leq \underbrace{\sum_{k=1}^{\infty} \frac{1}{k^2}}_{p\text{-series, } p>1} = \frac{\pi^2}{6}$$

$$\Rightarrow \sum_{k=1}^{\infty} \left(\frac{1}{k^2}\right)^{\sqrt{k}} \text{ converges.}$$

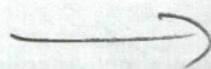
④ Limit Comparison Test:

$$\lim_{k \rightarrow \infty} \frac{\left(\frac{1}{k^2}\right)^{\sqrt{k}}}{\frac{1}{k^2}} = \lim_{k \rightarrow \infty} \underbrace{\left(\frac{1}{k^2}\right)^{\sqrt{k}-1}}_0 = 0$$

$$\Rightarrow \text{Since } \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges, so does } \sum_{k=1}^{\infty} \left(\frac{1}{k^2}\right)^{\sqrt{k}}.$$

⑤ Ratio Test:

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\left(\frac{1}{(k+1)^2}\right)^{\sqrt{k+1}}}{\left(\frac{1}{k^2}\right)^{\sqrt{k}}} &= \lim_{k \rightarrow \infty} \frac{(k^2)^{\sqrt{k}}}{((k+1)^2)^{\sqrt{k+1}}} \\ &= \frac{\lim_{k \rightarrow \infty} k^{2\sqrt{k}}}{\lim_{k \rightarrow \infty} (k+1)^{2\sqrt{k+1}}} \end{aligned}$$



Use logarithms to eliminate the exponents:

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$$\begin{aligned} \ln \left(\frac{\lim_{k \rightarrow \infty} k^{2\sqrt{k}}}{\lim_{k \rightarrow \infty} (k+1)^{2\sqrt{k+1}}} \right) &= \ln \left(\lim_{k \rightarrow \infty} k^{2\sqrt{k}} \right) - \ln \left(\lim_{k \rightarrow \infty} (k+1)^{2\sqrt{k+1}} \right) \\ &= \lim_{k \rightarrow \infty} \ln(k^{2\sqrt{k}}) - \lim_{k \rightarrow \infty} \ln((k+1)^{2\sqrt{k+1}}) \\ &= \lim_{k \rightarrow \infty} 2\sqrt{k} \ln k \rightarrow \infty - \lim_{k \rightarrow \infty} 2\sqrt{k+1} \ln(k+1) \rightarrow \infty \\ &\quad \nearrow \text{indeterminate form} \end{aligned}$$

Apply the Root Test:

⑥ Root Test:

$$\lim_{k \rightarrow \infty} \left(\left(\frac{1}{k^2} \right)^k \right)^{1/k} = \lim_{k \rightarrow \infty} \frac{1}{k^2} = 0 \Rightarrow \sum_{k=1}^{\infty} \left(\frac{1}{k^2} \right)^k \text{ converges.}$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{k^2} \right)^{\sqrt{k}} \leq \sum_{k=1}^{\infty} \left(\frac{1}{k^2} \right)^k, \text{ and so also} \\ \underline{\text{converges.}}$$

#50 $\sum_{k=0}^{\infty} \frac{2^k}{k!}$

① Divergence Test:

$$\lim_{k \rightarrow \infty} \frac{2^k}{k!} \xrightarrow[\text{eliminate the exponent}]{\text{use logs to}} \lim_{k \rightarrow \infty} \frac{k! \ln(2)}{\ln(k!)} = \infty,$$

since $k!$ dominates $\ln(k!)$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{2^k}{k!} \text{ diverges .}$$

② Integral Test:

$$\text{let } a(x) = \frac{2^x}{x!}.$$

• continuous on $[1, \infty)$? \checkmark (look up how to extend the factorial function's domain)

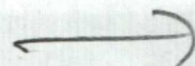
• positive on $[1, \infty)$? \checkmark

• decreasing on $[1, \infty)$? \times

③ Comparison Test:

Suspiciously divergent, so try the p -series $\sum_{k=1}^{\infty} \frac{1}{k}$.

$$\frac{1}{k} \frac{(k-1)!}{(k-1)!} = \frac{(k-1)!}{k!} < \frac{2^{(k-1)!}}{k!} \quad \text{b/c for any } x > 1,$$



$x < 2^x$, and so

$$\frac{1}{k} < \frac{2^{(k-1)!}}{k!} < \frac{2^{k!}}{k!}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{2^{k!}}{k!} \text{ diverges} \rightarrow \sum_{k=0}^{\infty} \frac{2^{k!}}{k!} \text{ diverges.}$$

④ Limit Comparison Test:

$$\lim_{k \rightarrow \infty} \frac{\frac{2^{k!}}{k!}}{\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{2^{k!}}{(k-1)!} = \infty, \text{ b/c this limit is larger than}$$

$$\lim_{k \rightarrow \infty} \frac{2^{k!}}{k!}, \text{ which we saw}$$

in the Divergence Test approach
infinity

$$\Rightarrow \text{Since } \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges, so does } \sum_{k=1}^{\infty} \frac{2^{k!}}{k!}, \text{ and}$$

$$\text{hence so does } \sum_{k=0}^{\infty} \frac{2^{k!}}{k!}.$$

⑤ Ratio Test:

$$\lim_{k \rightarrow \infty} \frac{\frac{2^{(k+1)!}}{(k+1)!}}{\frac{2^{k!}}{k!}} = \lim_{k \rightarrow \infty} \frac{2^{(k+1)!} - k!}{k+1}$$

Factor out $k!$:



$$= \lim_{k \rightarrow \infty} \frac{2^{k!(k+1-1)}}{k+1} = \infty,$$

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Since factorial exponential functions dominate linear functions.

$$\Rightarrow \sum_{k=0}^{\infty} \frac{2^k}{k!} \text{ diverges .}$$

⑥ Root Test:

$$\lim_{k \rightarrow \infty} \left(\frac{2^k}{k!} \right)^{1/k} = \lim_{k \rightarrow \infty} \frac{2^{(k-1)!}}{k^{1/k} (k-1)!} = \infty$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{2^k}{k!} \text{ diverges .}$$

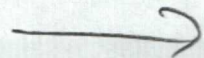
#58 $\sum_{k=0}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdots (3k+1)}{100^k}$

① Divergence Test

$$\lim_{k \rightarrow \infty} \frac{1 \cdot 4 \cdot 7 \cdots (3k+1)}{100^k} \left(\frac{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3k-1) \cdot 3k}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3k-1) \cdot 3k} \right)$$

$$= \lim_{k \rightarrow \infty} \frac{(3k+1)!}{100^k} \left(\frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3k-1) \cdot 3k} \right) \rightarrow 0$$

∞ b/c factorials dominate exponentials



But $\infty \cdot 0$ is an indeterminate form.

X

② Integral Test:

$$\text{let } a(x) = \frac{1 \cdot 4 \cdot 7 \cdots (3x+1)}{100^x}$$

• continuous on $[0, \infty)$?

Is it even well-defined for all x ??

X

③ Comparison Test:

... compare to ???

X

④ Limit Comparison Test:

$$\lim_{k \rightarrow \infty} \frac{1 \cdot 4 \cdot 7 \cdots (3k+1)}{100^k}$$

$$\frac{(3k+1)!}{100^k}$$

$$= \lim_{k \rightarrow \infty} \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3k-1) \cdot 3k} = 0$$

But $\sum_{k=0}^{\infty} \frac{(3k+1)!}{100^k}$ diverges \Rightarrow inconclusive.

—————>

⑤ Ratio Test:

$$\lim_{k \rightarrow \infty} \frac{1 \cdot 4 \cdot 7 \cdot \dots \cdot (3k+3+1)}{100^{k+1}}$$

$$\frac{1 \cdot 4 \cdot 7 \cdot \dots \cdot (3k+1)}{100^k}$$

$$= \lim_{k \rightarrow \infty} \frac{(3k+2)(3k+3)(3k+4)}{100} = \infty$$

\Rightarrow diverges.

⑥ Root Test:

$$\lim_{k \rightarrow \infty} \left(\frac{1 \cdot 4 \cdot 7 \cdot \dots \cdot (3k+1)}{100^k} \right)^{1/k}$$

$$= \lim_{k \rightarrow \infty} \frac{1 \cdot 4 \cdot 7 \cdot \dots \cdot (3k+1)}{100} \rightarrow \infty^0$$

Indeterminate Form

§ 7.5

#24 $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{1+k^2}$

Since this series only features powers

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of k , try to compare to a p -series!

$$\frac{\sqrt{k}}{1+k^2} < \frac{\sqrt{k}}{k^2} = \frac{1}{k^{3/2}}$$

$\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges, since $p=3/2 > 1$.

$\Rightarrow \sum_{k=1}^{\infty} \frac{\sqrt{k}}{1+k^2}$ converges.