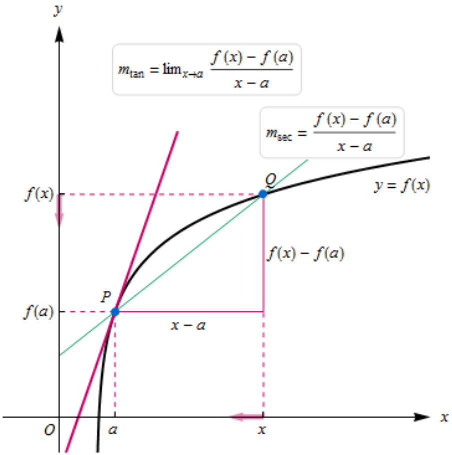
§3.1 Introducing the Derivative

Recall from Ch 2: We said that the slope of the tangent line at a point is the limit of the slopes of the secant lines as the points get closer and closer.

- slope of secant line: $\dfrac{f(x) - f(a)}{x - a}$ (average rate of change)
- slope of tangent line: $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ (instantaneous rate of change)


y
$m_{\tan} = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$
$m_{\sec} = \dfrac{f(x) - f(a)}{x - a}$
Q
$y = f(x)$
$f(x)$
$f(x) - f(a)$
P
$f(a)$
$x - a$
O
a
x
x

Exercise

Use the relationship between secant lines and tangent lines, specifically the slope of the tangent line, to find the equation of a line tangent to the curve $f(x) = x^2 + 2x + 2$ at the point $P = (1, 5)$.

In the preceding exercise, we considered two points

$$P = (a, f(a)) \quad \text{and} \quad Q = (x, f(x))$$

that were getting closer and closer together.

Instead of looking at the points approaching one another, we can also view this as the distance h between the points approaching 0. For

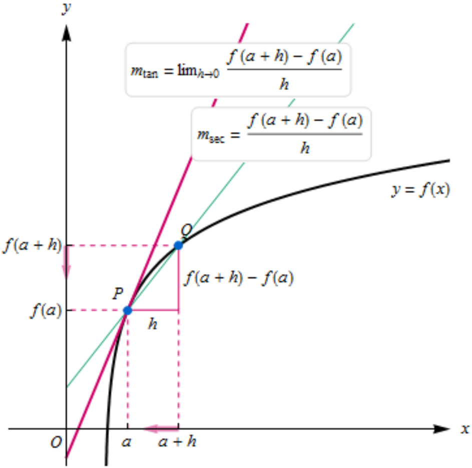
$$P = (a, f(a)) \quad \text{and} \quad Q = (a + h, f(a + h)),$$

- slope of secant line:

$$\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

- slope of tangent line:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$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$y = f(x)$
Q
$f(a+h)$
$f(a+h) - f(a)$
P
$f(a)$
h
O
a
$a+h$
x

Exercise

Find the equation of a line tangent to the curve $f(x) = x^2 + 2x + 2$ at the point $P = (2, 10)$.

Derivative Defined as a Function

The slope of the tangent line for the function f is itself a function of x (in other words, there is an expression where we can plug in any value $x = a$ and get the derivative at that point), called the derivative of f.

Definition

The **derivative** of f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. If $f'(x)$ exists, we say f is **differentiable** at x. If f is differentiable at every point of an open interval I, we say that f is differentiable on I.

Exercise

Use the definition of the derivative to find the derivative of the function $f(x) = x^2 + 2x + 2$.

A standard notation for change involves the Greek letter Δ.

$$\frac{f(x+h)-f(x)}{h} = \frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}.$$

Apply the limit:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

The following are alternative ways of writing $f'(x)$ (i.e., the derivative as a function of x):

$$\frac{dy}{dx} \qquad \frac{df}{dx} \qquad \frac{d}{dx}\left(f(x)\right) \qquad D_x(f(x)) \qquad y'(x)$$

The following are ways to notate the derivative of f evaluated at $x = a$:

$$f'(a) \qquad y'(a) \qquad \left.\frac{df}{dx}\right|_{x=a} \qquad \left.\frac{dy}{dx}\right|_{x=a}$$

Question

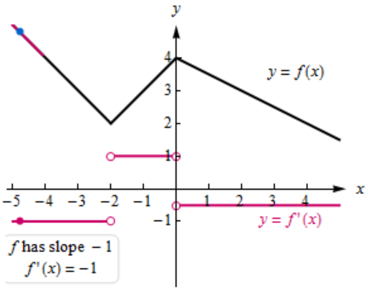
Do the words “derive” and “differentiate” mean the same thing?

The graph of the derivative is the graph of the collection of slopes of tangent lines of a graph. If you just have a graph (without an equation for the graph), the best you can do is approximate the graph of the derivative.

Example

Simple checklist:

1. Note where $f'(x) = 0$.
2. Note where $f'(x) > 0$.
   (What does this look like?)
3. Note where $f'(x) < 0$.
   (What does this look like?)



Key points about the relationship between differentiability and continuity:

- If f is differentiable at a, then f is continuous at a.
- If f is not continuous at a, then f is not differentiable at a.
- f can be continuous at a, but not differentiable at a.

A function f is not differentiable at a if at least one of the following conditions holds:

1. f is not continuous at a.
2. f has a corner at a.

Question

Why does this make f not differentiable?

3. f has a vertical tangent at a.

Question

Why does this make f not differentiable?

3.1 Book Problems

9-45 (odds), 49-53 (odds)

- **NOTE:** You do not know any rules for differentiation yet (e.g., Power Rule, Chain Rule, etc.) In this section, you are strictly using the definition of the derivative and the definition of slope of tangent lines we have derived.

Exam #1 Review

- $\oint$2.1 The Idea of Limits
  - Understand the relationship between average velocity & instantaneous velocity, and secant and tangent lines
  - Be able to compute average velocities and use the idea of a limit to approximate instantaneous velocities
  - Be able to compute slopes of secant lines and use the idea of a limit to approximate the slope of the tangent line

Exam #1 Review (cont.)

- $\oint$2.2 Definitions of Limits
  - Know the definition of a limit
  - Be able to use a graph of a table to determine a limit
  - Know the relationship between one- and two-sided limits
- $\oint$2.3 Techniques for Computing Limits
  - Know and be able to compute limits using analytical methods (e.g., limit laws, additional techniques)
  - Know the Squeeze Theorem and be able to use it to determine limits

Exam #1 Review (cont.)

Example

Evaluate $\lim\limits_{x \to 0} x \sin \dfrac{1}{x}$.

- $\oint$2.4 Infinite Limits
  - Be able to use a graph, a table, or analytical methods to determine infinite limits
  - Know the definition of a vertical asymptote and be able to determine whether a function has vertical asymptotes

Exam #1 Review (cont.)

- $\oint$2.5 Limits at Infinity
  - Be able to find limits at infinity and horizontal asymptotes
  - Know how to compute the limits at infinity of rational functions

Exam #1 Review (cont.)

Example

Determine the end behavior of $f(x)$. If there is a horizontal asymptote, then say so. Next, identify any vertical asymptotes. If $x = a$ is a vertical asymptote, then evaluate $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$.

$$f(x) = \frac{2x^3 + 10x^2 + 12x}{x^3 + 2x^2}$$

Exam #1 Review (cont.)

- $\oint$2.6 Continuity
  - Know the definition of continuity and be able to apply the continuity checklist
  - Be able to determine the continuity of a function (including those with roots) on an interval
  - Be able to apply the Intermediate Value Theorem to a function

Exam #1 Review (cont.)

Example

Determine the value for a that will make $f(x)$ continuous.

$$f(x) = \begin{cases} \dfrac{x^2 + 3x + 2}{x + 1} & x \neq -1 \\ a & x = -1 \end{cases}$$

Example

Show that $f(x) = 2$ has a solution on the interval $(-1, 1)$, with

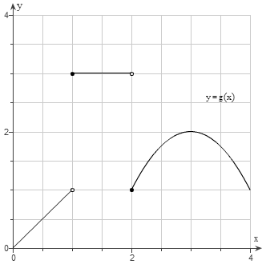
$$f(x) = 2x^3 + x.$$

Exam #1 Review (cont.)

- $\oint$ 2.7 Precise Definition of Limits
  - Understand the δ, ϵ relationship for limits
  - Be able to use a graph or analytical methods to find a value for $\delta > 0$ given an $\epsilon > 0$ (including finding symmetric intervals)

Exam #1 Review (cont.)

Example



Use the graph to find the appropriate δ.

(a) $|g(x) - 2| < \frac{1}{2}$ whenever

$$0 < |x - 3| < \delta$$

(b) $|g(x) - 1| < \frac{3}{2}$ whenever

$$0 < |x - 2| < \delta$$

In this example, the two-sided limits at $x = 1$ and $x = 2$ do not exist.

Exam #1 Review (cont.)

- $\oint$3.1 Introducing the Derivative
  - Know the definition of a derivative and be able to use this definition to calculate the derivative of a given function
  - Be able to determine the equation of a line tangent to the graph of a function at a given point
  - Know the 3 conditions for when a function is not differentiable at a point, and why these three conditions make a function not differentiable at the given point

Exam #1 Review (cont.)

Example

(a) Use the limit definition of the derivative to find an equation for the line tangent to $f(x)$ at a, where

$$f(x) = \frac{1}{x}; \quad a = -5.$$

(b) Using the same $f(x)$ from part (a), find a formula for $f'(x)$ (using the limit definition).

(c) Plug -5 into your answer for (b) and make sure it matches your answer for (a).

Other Study Tips

- Brush up on algebra, especially radicals.
- When in doubt, show steps. Defer to class notes and old exams to get an idea of what's expected.
- You will be punished for wrong notation; e.g., the limit symbol.
- Read the question! Several students always lose points because they didn't answer the question or they didn't follow directions.
- Do the book problems.
- Budget your time. You don't have to do the problems in order. Do the easier ones first.

∮ 3.2 Rules of Differentiation

Recall the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(as a function of x, i.e., a formula).
And, for any particular point a, we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Constant Functions

The constant function $f(x) = c$ is a horizontal line with a slope of 0 at every point. This is consistent with the definition of the derivative:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{c - c}{h} \\ &= \lim_{h \to 0} 0 = 0. \end{aligned}$$

Therefore, for constant functions, $\frac{d}{dx} c = 0$.

Fact: For any positive integer n, we can factor

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

For example, when $n = 2$, we get

$$x^2 - a^2 = (x-a)(x+a),$$

which is the difference of squares formula.

Power Rule, cont.

Suppose $f(x) = x^n$ where n is a positive integer. Then at a point a,

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{x^n - a^n}{x - a} \\ &= \lim_{x\to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{x - a} \\ &= (a^{n-1} + a^{n-2}\cdot a + \cdots + a \cdot a^{n-2} + a^{n-1}) = na^{n-1}. \end{aligned}$$

Using the formula for the derivative as a function of x, one can show $\frac{d}{dx}(x^n) = nx^{n-1}$.

Constant Multiple Rule

Consider a function of the form $cf(x)$, where c is a constant. Just like with limits, we can factor out the constant:

$$\begin{aligned} \frac{d}{dx}[cf(x)] &= \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} \\ &= \lim_{h\to 0} \frac{c[f(x+h) - f(x)]}{h} = c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= cf'(x) \end{aligned}$$

Therefore, $\frac{d}{dx}[cf(x)] = cf'(x)$.

Sums of functions also behave under the same limit laws when we differentiate:

$$\begin{aligned}
\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0}\left[\frac{[f(x+h)-f(x)]}{h}+\frac{[g(x+h)-g(x)]}{h}\right] \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= f'(x)+g'(x)
\end{aligned}$$

So if f and g are differentiable at x,

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x).$$

The Sum Rule can be generalized for more than two functions to include n functions.

Note: Using the Sum Rule and the Constant Multiple Rule produces the Difference Rule:

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x).$$

Exercise

Using the differentiation rules we have discussed, calculate the derivatives of the following functions. Note which rule(s) you are using.

1. $y = x^5$
2. $y = 4x^3 - 2x^2$
3. $y = -1500$
4. $y = 3x^3 - 2x + 4$

Let $f(x) = b^x$, where $b > 0$, $b \neq 1$. To differentiate at 0, we write

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{b^x - b^0}{x} = \lim_{x \to 0} \frac{b^x - 1}{x}.$$

It is not obvious what this limit should be. However, consider the cases $b = 2$ and $b = 3$. By constructing a table of values, we can see that

$$\lim_{x \to 0} \frac{2^x - 1}{x} \approx 0.693 \quad \text{and} \quad \lim_{x \to 0} \frac{3^x - 1}{x} \approx 1.099.$$

So, $f'(0) < 1$ when $b = 2$ and $f'(0) > 1$ when $b = 3$. As it turns out, there is a particular number b, with $2 < b < 3$, whose graph has a tangent line with slope 1 at $x = 0$. In other words, such a number b has the property that

$$\lim_{x \to 0} \frac{b^x - 1}{x} = 1.$$

Question

What number is it?

Answer: This number is $e = 2.718281828459\ldots$ (known as the Euler number). The function $f(x) = e^x$ is called the **natural exponential function**.

Now, using $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$, we can find the formula for $\frac{d}{dx}(e^x)$:

$$\begin{aligned}
\frac{d}{dx}(e^x) &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} \\
&= e^x \left(\lim_{h \to 0} \frac{e^h - 1}{h} \right) \\
&= e^x \cdot 1 = e^x
\end{aligned}$$

Exercise

(a) Find the slope of the line tangent to the curve $f(x) = x^3 - 4x - 4$ at the point $(2, -4)$.

(b) Where does this curve have a horizontal tangent?

If we can write the derivative of f as a function of x, then we can take *its* derivative, too. The derivative of the derivative is called the **second derivative** of f, and is denoted f''.

In general, we can differentiate f as often as needed. If we do it n times, the nth derivative of f is

$$f^{(n)}(x) = \frac{d^n f}{dx^n} = \frac{d}{dx}[f^{(n-1)}(x)].$$

3.2 Book Problems

3-45 (x3)

- For these problems, use only the rules we have derived so far.

$\oint$ 3.3 The Product and Quotient Rules

Issue: Derivatives of products and quotients do NOT behave like they do for limits.

As an example, consider $f(x) = x^2$ and $g(x) = x^3$. We can try to differentiate their product in two ways:

- $$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}\left(x^5\right)$$
  $$= 5x^4$$
- $$f'(x)g'(x) = (2x)(3x^2)$$
  $$= 6x^3$$

Question

Which answer is the correct one?

Product Rule

If f and g are any two functions that are differentiable at x, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x).$$

In the example from the previous slide, we have

$$\begin{aligned}
\frac{d}{dx}[x^2 \cdot x^3] &= \frac{d}{dx}(x^2) \cdot (x^3) + x^2 \cdot \frac{d}{dx}(x^3) \\
&= (2x) \cdot (x^3) + x^2 \cdot (3x^2) \\
&= 2x^4 + 3x^4 \\
&= 5x^4
\end{aligned}$$

Derivation of the Product Rule

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left(\frac{f(x+h)g(x+h) + [-f(x)g(x+h) + f(x)g(x+h)] - f(x)g(x)}{h} \right)$$

$$= \lim_{h\to 0} \left(\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} \right) + \left(\lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h} \right)$$

$$= \lim_{h\to 0}\left(g(x+h)\frac{f(x+h)-f(x)}{h}\right) + \left(\lim_{h\to 0} f(x)\frac{g(x+h)-g(x)}{h}\right)$$

$$= g(x)f'(x) + f(x)g'(x)$$

Question

Let $q(x) = \dfrac{f(x)}{g(x)}$. What is $\dfrac{d}{dx}q(x)$?

Answer: We can write $f(x) = q(x)g(x)$ and then use the Product Rule:

$$f'(x) = q'(x)g(x) + g'(x)q(x)$$

and now solve for $q'(x)$:

$$q'(x) = \frac{f'(x) - q(x)g'(x)}{g(x)}.$$

Then, to get rid of $q(x)$, plug in $\dfrac{f(x)}{g(x)}$:

$$q'(x) = \frac{f'(x) - g'(x)\frac{f(x)}{g(x)}}{g(x)}$$

$$= \frac{g(x)\left(f'(x) - g'(x)\frac{f(x)}{g(x)}\right)}{g(x) \cdot g(x)}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

"LO-D-HI minus HI-D-LO over LO squared"

Quotient Rule

Just as with the product rule, the derivative of a quotient is not a quotient of derivatives, i.e.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \neq \frac{f'(x)}{g'(x)}.$$

Here is the correct rule, the Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}.$$

Exercise

Use the Quotient Rule to find the derivative of

$$\frac{4x^3 + 2x - 3}{x + 1}.$$

Exercise

Find the slope of the tangent line to the curve

$$f(x) = \frac{2x - 3}{x + 1} \text{ at the point } (4, 1).$$

The Quotient Rule also allows us to extend the Power Rule to negative numbers – if n is any integer, then

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}.$$

Question

How?

Exercise

If $f(x) = \dfrac{x(3-x)}{2x^2}$, find $f'(x)$.

Derivative of e^{kx}

For any real number k,

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}.$$

Exercise

What is the derivative of $x^2 e^{3x}$?

The derivative provides information about the instantaneous rate of change of the function being differentiated (compare to the limit of the slopes of the secant lines from $\oint$2.1).

For example, suppose that the population of a culture can be modeled by the function $p(t)$. We can find the instantaneous growth rate of the population at any time $t \geq 0$ by computing $p'(t)$ as well as the **steady-state population** (also called the **carrying capacity** of the population). The steady-state population equals

$$\lim_{t\to\infty} p(t).$$

3.3 Book Problems

6-51 (x3)

$\oint$3.4 Derivatives of Trigonometric Functions

Derivative formulas for sine and cosine can be derived using the following limits:

- $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$

(We will prove these limits in Chapter 4.)

Exercise

Evaluate $\lim_{x\to 0} \frac{\sin 9x}{x}$ and $\lim_{x\to 0} \frac{\sin 9x}{\sin 5x}$.

Using the previous limits and the definition of the derivative, we obtain

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$

Trig Identities You Should Know

- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = 1 - 2 \sin^2 x$
- $\cos^2 x = \dfrac{1 + \cos 2x}{2}$
- $\sin^2 x = \dfrac{1 - \cos 2x}{2}$
- $\tan x = \dfrac{\sin x}{\cos x}$
- $\cot x = \dfrac{\cos x}{\sin x}$
- $\cot x = \dfrac{1}{\tan x}$
- $\sec x = \dfrac{1}{\cos x}$
- $\csc x = \dfrac{1}{\sin x}$

Derivatives of Other Trig functions

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{\cos x \cos x - (-\sin x)\sin x}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

So $\frac{d}{dx}(\tan x) = \sec^2 x$.

By using trig identities and the Quotient Rule, we obtain

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\csc x \cot x$$

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\tan x}\right) = -\csc^2 x$$

Exercise

Compute the derivative of the following functions:

$$f(x) = \frac{\tan x}{1 + \tan x} \qquad g(x) = \sin x \cos x$$

Higher-Order Trig Derivatives

There is a cyclic relationship between the higher order derivatives of $\sin x$ and $\cos x$:

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f^{(3)}(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \end{aligned}$$

$$\begin{aligned} g(x) &= \cos x \\ g'(x) &= -\sin x \\ g''(x) &= -\cos x \\ g^{(3)}(x) &= \sin x \\ g^{(4)}(x) &= \cos x \end{aligned}$$

3.4 Book Problems

7, 13, 17, 21-27, 33, 35, 44-46, 53-55

Suppose an object moves along a straight line and its location at time t is given by the position function $s = f(t)$. The **displacement** of the object between $t = a$ and $t = a + \Delta t$ is

$$\Delta s = f(a + \Delta t) - f(a).$$

Here Δt represents how much time has elapsed.

We now define average velocity as

$$\frac{\Delta s}{\Delta t} = \frac{f(a+\Delta t) - f(a)}{\Delta t}.$$

Recall that the limit of the average velocities as the time interval approaches 0 was the instantaneous velocity (which we denote here by v). Therefore, the instantaneous velocity at a is

$$v(a) = \lim_{\Delta t \to 0} \frac{f(a+\Delta t) - f(a)}{\Delta t} = f'(a).$$

In mathematics, speed and velocity are related but not the same – if the velocity of an object at any time t is given by $v(t)$, then the speed of the object at any time t is given by

$$|v(t)| = |f'(t)|.$$

By definition, acceleration (denoted by a) is the instantaneous rate of change of the velocity of an object at time t. Therefore,

$$a(t) = v'(t)$$

and since velocity was the derivative of the position function $s = f(t)$, then

$$a(t) = v'(t) = f''(t).$$

Summary: Given the position function $s = f(t)$, the velocity at time t is the first derivative, the speed at time t is the absolute value of the first derivative, and the acceleration at time t is the second derivative.

Question

Given the position function $s = f(t)$ of an object launched into the air, how would you know:

- The highest point the object reaches?
- How long it takes to hit the ground?
- The speed at which the object hits the ground?

Suppose $p = f(t)$ is a function of the growth of some quantity of interest. The average growth rate of p between times $t = a$ and a later time $t = a + \Delta t$ is the change in p divided by the elapsed time Δt:

$$\frac{\Delta p}{\Delta t} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

As Δt approaches 0, the average growth rate approaches the derivative $\dfrac{dp}{dt}$, which is the instantaneous growth rate (or just simply the growth rate). Therefore,

$$\frac{dp}{dt} = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t}.$$

Exercise

The population of the state of Georgia (in thousands) from 1995 ($t = 0$) to 2005 ($t = 10$) is modeled by the polynomial

$$p(t) = -0.27t^2 + 101t + 7055.$$

(a) What was the average growth rate from 1995 to 2005?

(b) What was the growth rate for Georgia in 1997?

(c) What can you say about the population growth rate in Georgia between 1995 and 2005?

Suppose a company produces a large amount of a particular quantity. Associated with manufacturing the quantity is a **cost function** $C(x)$ that gives the cost of manufacturing x items. This cost may include a **fixed cost** to get started as well as a **unit cost** (or **variable cost**) in producing one item.

If a company produces x items at a cost of $C(x)$, then the average cost is $\frac{C(x)}{x}$. This average cost indicates the cost of items already produced. Having produced x items, the cost of producing another Δx items is $C(x+\Delta x) - C(x)$. So the average cost of producing these extra Δx items is

$$\frac{\Delta C}{\Delta x} = \frac{C(x+\Delta x) - C(x)}{\Delta x}.$$

If we let Δx approach 0, we have

$$\lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = C'(x)$$

which is called the **marginal cost**. The marginal cost is the approximate cost to produce one additional item after producing x items.

Note: In reality, we can't let Δx approach 0 because Δx represents whole numbers of items.

Exercise

If the cost of producing x items is given by

$$C(x) = -0.04x^2 + 100x + 800$$

for $0 \leq x \leq 1000$, find the average cost and marginal cost functions. Also, determine the average and marginal cost when $x = 500$.

3.5 Book Problems

9-12, 17-18, 22-23, 27-37 (odds)

$\oint$ 3.6 The Chain Rule

Suppose that Yvonne (y) can run twice as fast as Uma (u). Therefore $\frac{dy}{du} = 2$.

Suppose that Uma can run four times as fast as Xavier (x). So $\frac{du}{dx} = 4$.

How much faster can Yvonne run than Xavier? In this case, we would take both our rates and multiply them together:

$$\frac{dy}{du} \cdot \frac{du}{dx} = 2 \cdot 4 = 8.$$

Version 1 of the Chain Rule

If g is differentiable at x, and $y = f(u)$ is differentiable at $u = g(x)$, then the composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Guidelines for Using the Chain Rule

Assume the differentiable function $y = f(g(x))$ is given.

1. Identify the outer function f, the inner function g, and let $u = g(x)$.
2. Replace $g(x)$ by u to express y in terms of u:

$$y = f(g(x)) \implies y = f(u)$$

3. Calculate the product $\dfrac{dy}{du} \cdot \dfrac{du}{dx}$
4. Replace u by $g(x)$ in $\dfrac{dy}{du}$ to obtain $\dfrac{dy}{dx}$.

Example

Use Version 1 of the Chain Rule to calculate $\frac{dy}{dx}$ for $y = (5x^2 + 11x)^{20}$.

- inner function: $u = 5x^2 + 11x$
- outer function: $y = u^{20}$

We have $y = f(g(x)) = (5x^2 + 11x)^{20}$. Differentiate:

$$\begin{aligned} \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} &= 20u^{19} \cdot (10x + 11) \\ &= 20(5x^2 + 11x)^{19} \cdot (10x + 11) \end{aligned}$$

Exercise

Use the first version of the Chain Rule to calculate $\frac{dy}{dx}$ for

$$y = \left(\frac{3x}{4x+2}\right)^5.$$

Version 2 of the Chain Rule

Notice if $y = f(u)$ and $u = g(x)$, then $y = f(u) = f(g(x))$, so we can also write:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= f'(u) \cdot g'(x) \\ &= f'(g(x)) \cdot g'(x).\end{aligned}$$

Example

Use Version 2 of the Chain Rule to calculate $\frac{dy}{dx}$ for $y = (7x^4 + 2x + 5)^9$.

- inner function: $g(x) = 7x^4 + 2x + 5$
- outer function: $f(u) = u^9$

Then

$$f'(u) = 9u^8 \implies f'(g(x)) = 9(7x^4 + 2x + 5)^8$$
$$g'(x) = 28x^3 + 2.$$

Putting it together,

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x) = 9(7x^4 + 2x + 5)^8 \cdot (28x^3 + 2)$$

Chain Rule for Powers

If g is differentiable for all x in the domain and n is an integer, then

$$\frac{d}{dx}\left[\,(g(x))^n\,\right] = n(g(x))^{n-1} \cdot g'(x).$$

Chain Rule for Powers (cont.)

Example

$$\frac{d}{dx}\left[(1-e^x)^4\right] = \text{?}$$

Answer:

$$\begin{aligned}\frac{d}{dx}\left[(1-e^x)^4\right] &= 4(1-e^x)^3 \cdot (-e^x) \\ &= -4e^x(1-e^x)^3\end{aligned}$$

Composition of 3 or More Functions

Example

Compute $\dfrac{d}{dx}\left[\sqrt{(3x-4)^2+3x}\right]$.

Composition of 3 or More Functions (cont.)

Answer:

$$\begin{aligned}
\frac{d}{dx}\left[\sqrt{(3x-4)^2+3x}\right] &= \frac{1}{2}\left((3x-4)^2+3x\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left[(3x-4)^2+3x\right] \\
&= \frac{1}{2\sqrt{\left((3x-4)^2+3x\right)}} \cdot \left[2(3x-4)\frac{d}{dx}(3x-4)+3\right] \\
&= \frac{1}{2\sqrt{\left((3x-4)^2+3x\right)}} \cdot \left[2(3x-4)\cdot 3+3\right] \\
&= \frac{18x-21}{2\sqrt{\left((3x-4)^2+3x\right)}}
\end{aligned}$$

3.6 Book Problems

7-29 (odds), 30, 33-43 (odds), 49