§4.9 Antiderivatives

With differentiation, the goal of problems was to find the function f' given the function f.

With antidifferentiation, the goal is the opposite. Here, given a function f, we wish to find a function F such that the derivative of F is the given function f (i.e., $F' = f$).

Definition

A function F is called an **antiderivative** of a function f on an interval I provided $F'(x) = f(x)$ for all x in I.

Example

Given $f(x) = 4$, an antiderivative of $f(x)$ is $F(x) = 4x$.

NOTE: Antiderivatives are not unique!

They differ by a constant (C):

Theorem

Let F be any antiderivative of f. Then **all** *the antiderivatives of f have the form $F + C$, where C is an arbitrary constant.*

Recall: $\frac{d}{dx}f(x) = f'(x)$ is the derivative of $f(x)$.

Now: $\int f(x)\, dx = F + C$ *is* the antiderivative of $f(x)$. It doesn't matter which F you choose, since writing the C will show you are talking about all the antiderivatives at once. The C is also why we call it the *indefinite* integral.

Example

Find the antiderivatives of the following functions:

(1) $f(x) = -6x^{-7}$

(2) $g(x) = -4\cos 4x$

(3) $h(x) = \csc^2 x$

Example

$\int 4x^3 \ dx = x^4 + C$, where C is the **constant of integration**.

The dx is called the **differential** and it is the same dx from Section 4.5. Like the $\frac{d}{dx}$, it shows which variable you are talking about. The function written between the $\int$ and the dx is called the **integrand**.

Rules for Indefinite Integrals

Power Rule: $\int x^p \ dx = \dfrac{x^{p+1}}{p+1} + C$

(p is any real number except -1)

Constant Multiple Rule: $\int cf(x) \ dx = c \int f(x) \ dx$

Sum Rule: $\int (f(x) + g(x)) \ dx = \int f(x) \ dx + \int g(x) \ dx$

Exercise

$$\int (5x^4 + 2x + 1)\, dx =$$

A. $20x^3 + 2 + C$

B. $x^5 + x^2 - x + C$

C. $x^5 + x^2 + C$

D. $x^5 + 2x^2 - x + C$

Exercise

Evaluate the following indefinite integrals:

(1) $\int \left(3x^{-2} - 4x^2 + 1\right)\ dx$

(2) $\int 6\sqrt[3]{x}\ dx$

(3) $\int 2\cos\left(2x\right)\ dx$

Table 4.9 (p. 322) provides us with rules for finding indefinite integrals of trig functions.

1. $\dfrac{d}{dx}(\sin ax) = a\cos ax \longrightarrow \displaystyle\int \cos ax\ dx = \frac{1}{a}\sin ax + C$
2. $\dfrac{d}{dx}(\cos ax) = -a\sin ax \longrightarrow \displaystyle\int \sin ax\ dx = -\frac{1}{a}\cos ax + C$
3. $\dfrac{d}{dx}(\tan ax) = a\sec^2 ax \longrightarrow \displaystyle\int \sec^2 ax\ dx = \frac{1}{a}\tan ax + C$
4. $\dfrac{d}{dx}(\cot ax) = -a\csc^2 ax \longrightarrow \displaystyle\int \csc^2 ax\ dx = -\frac{1}{a}\cot ax + C$
5. $\dfrac{d}{dx}(\sec ax) = a\sec ax\tan ax \longrightarrow \displaystyle\int \sec ax\tan ax\ dx = \frac{1}{a}\sec ax + C$
6. $\dfrac{d}{dx}(\csc ax) = -a\csc ax\cot ax \longrightarrow \displaystyle\int \csc ax\cot ax\ dx = -\frac{1}{a}\csc ax + C$

Example

Evaluate the following indefinite integral: $\int 2\sec^2 2x\ dx$.

Solution: Using rule 3, with $a = 2$, we have

$$\int 2\sec^2 2x\ dx = 2\int \sec^2 2x\ dx = 2\left[\frac{1}{2}\tan 2x\right] + C = \tan 2x + C.$$

Exercise

Evaluate $\int 2\cos(2x)\ dx$.

Other Indefinite Integrals

7. $\dfrac{d}{dx}(e^{ax}) = ae^{ax} \longrightarrow \displaystyle\int e^{ax}\, dx = \frac{1}{a}e^{ax} + C$

8. $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x} \longrightarrow \displaystyle\int \frac{dx}{x} = \ln|x| + C$

9. $\dfrac{d}{dx}\left(\sin^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{1}{\sqrt{a^2-x^2}} \longrightarrow \displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

10. $\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{a}{a^2+x^2} \longrightarrow \displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

11. $\dfrac{d}{dx}\left(\sec^{-1}\left|\dfrac{x}{a}\right|\right) = \dfrac{a}{x\sqrt{x^2-a^2}} \longrightarrow \displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

Initial Value Problems

In some instances, you have enough information to determine the value of C in the antiderivative. These are often called **initial value problems**. Finding $f(x)$ is often called **finding the solution**.

Example

If $f'(x) = 7x^6 - 4x^3 + 12$ and $f(1) = 24$, find $f(x)$.

Solution: $f(x) = \int (7x^6 - 4x^3 + 12)\, dx = x^7 - x^4 + 12x + C$. Now find out which C gives $f(1) = 24$:

$$24 = f(1) = 1 - 1 + 12 + C,$$

so $C = 12$. Hence, $f(x) = x^7 - x^4 + 12x + 12$.

Exercise

Find the function f that satisfies $f''(t) = 6t$ with $f'(0) = 1$ and $f(0) = 2$.

4.9 Book Problems

11-45 (odds), 59-73 (odds), 83-93 (odds)

Advice: To solve 83-93 (odds), read pages 325-326, focusing on Example 8.