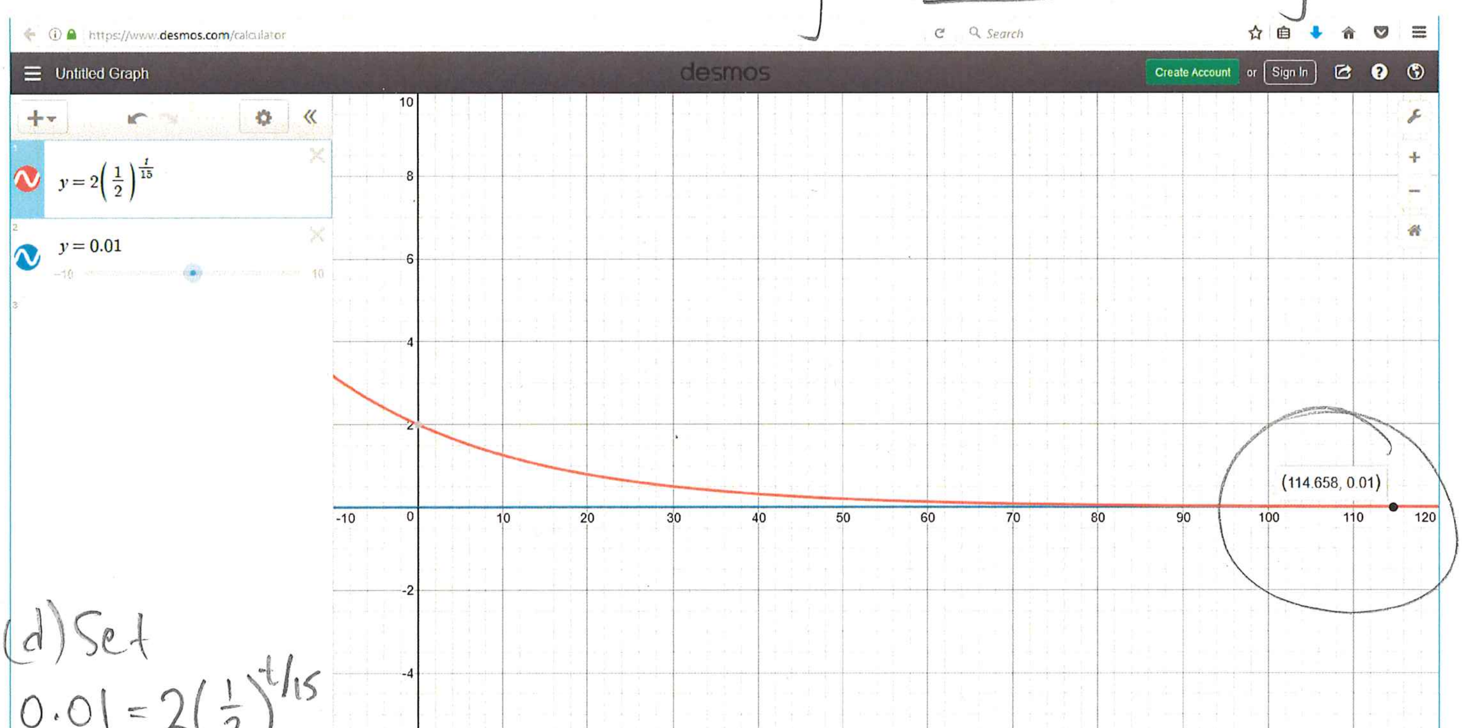
Take-Home Quiz #2 SOLUTIONS

Math 235 (Calc I)
Fall 2017

1.(a) After 15 hours $\frac{1}{2}$ of the sodium remains. After $60 = 4 \cdot (15 \text{ hours})$, $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$ of the sodium remains: $\frac{1}{16}(2\text{ g}) = \boxed{\frac{1}{8}\text{ g}}$

(b) The amount of ^{24}Na after t hours is $S(t) = S_0 b^t$. The initial amount is $S_0 = 2\text{ g}$.
To find b, use the half-life:
$\frac{1}{2}(2\text{ g}) = (2\text{ g})\,b^{15} \Rightarrow b = \left(\frac{1}{2}\right)^{1/15}$, so $\boxed{S(t) = 2\left(\frac{1}{2}\right)^{t/15}}$

(c) $4\text{ days}\left(\frac{24\text{ hours}}{\text{day}}\right) = 96\text{ hours}$, so the amount remaining is $\boxed{S(96) \approx 0.024\text{ g}}$

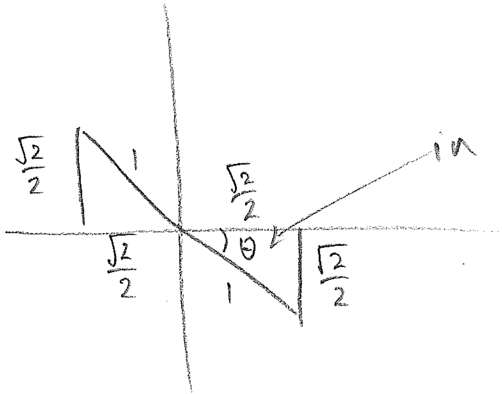


(d) Set
$0.01 = 2\left(\frac{1}{2}\right)^{t/15}$

$\Rightarrow t \approx 114.658\text{ hours}\left(\frac{\text{day}}{24\text{ hours}}\right) \boxed{\approx 4.8\text{ days}}$

2. (a) $\arctan(-1) = $ angle θ where $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = -1$

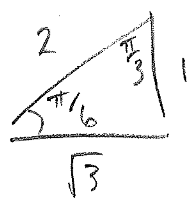
in the domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$\Rightarrow \boxed{\theta = -\frac{\pi}{4}}$

(b) $\arccos\left(\frac{\sqrt{3}}{2}\right) = $ angle θ where $\cos\theta = \frac{\sqrt{3}}{2}$

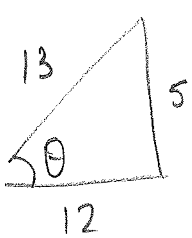
(in the interval $[0, \pi]$)



$= \boxed{\frac{\pi}{6}}$

(c) $\cos\big(2\underbrace{\arcsin\left(\tfrac{5}{13}\right)}_{\theta}\big) = 2\cos^2\theta - 1$ (trig identity)

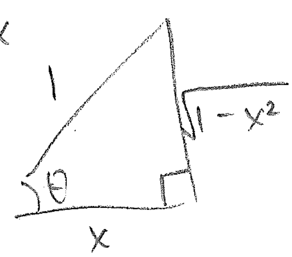
$= 2\left(\frac{12}{13}\right)^2 - 1 = \frac{288}{169} - \frac{169}{169}$



$= \boxed{\frac{119}{169}}$

(d) $\sin\big(2\underbrace{\arccos x}_{\theta}\big) = 2\sin\theta\cos\theta = 2\left(\sqrt{1-x^2}\right)x = \boxed{2x\sqrt{1-x^2}}$

$\Rightarrow \cos\theta = x$



$\leftarrow$ Pythagorean Theorem

3. (a) $m = f(v) = \dfrac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$

$\Rightarrow \sqrt{1 - \frac{v^2}{c^2}} = \dfrac{m_0}{m}$

$\Rightarrow 1 - \dfrac{v^2}{c^2} = \left(\dfrac{m_0}{m}\right)^2$

$\Rightarrow 1 - \left(\dfrac{m_0}{m}\right)^2 = \dfrac{v^2}{c^2}$

$\Rightarrow c^2 - c^2\left(\dfrac{m_0}{m}\right)^2 = v^2$

$\Rightarrow \dfrac{c^2 m^2 - c^2 m_0^2}{m^2} = v^2$

$\Rightarrow \boxed{v = \dfrac{c}{m}\sqrt{m^2 - m_0^2}}$

Represents the velocity required for an object with rest mass m_0 to achieve a mass of m.

(b) $\displaystyle\lim_{v \to c^-} \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \infty$, since $1 - \frac{v^2}{c^2} \to 1^-$. In other words, the mass of the object →

approaches infinity.

(c) $\lim_{v \to c^-} L = \lim_{v \to c^-} L_0 \sqrt{1 - \frac{v^2}{c^2}}$

$= \left(\lim_{v \to c^-} L_0\right) \sqrt{\lim_{v \to c^-} \left(1 - \frac{v^2}{c^2}\right)}$

$= L_0 \sqrt{\lim_{v \to c^-} \left(\frac{c^2}{c^2} - \frac{v^2}{c^2}\right)}$

$= L_0 \cdot 0 = \boxed{0}$.

It means to an observer an object approaching the speed of light will appear shorter and shorter. The left-hand limit is necessary, since objects cannot move faster than the speed of light.

4. (a) The coordinates for P are $(0, r)$. To find the coordinates for Q, use the respective equations of the circles:

$$C_1 : (x-1)^2 + y^2 = 1$$

$$C_2 : x^2 + y^2 = r^2$$

Q lies on both circles, and on their upper halves:

$$\underbrace{\sqrt{1-(x-1)^2}}_{\text{upper half of } C_1} = \underbrace{\sqrt{r^2 - x^2}}_{\text{upper half of } C_2}$$

$$\Rightarrow 1-(x-1)^2 = r^2 - x^2$$

$$\cancel{1} - \cancel{x^2} + 2x - \cancel{1} = r^2 - \cancel{x^2}$$

$$2x = r^2 \Rightarrow x = \frac{r^2}{2},$$

$$y = \sqrt{r^2 - \left(\frac{r^2}{2}\right)^2} = r\sqrt{1 - \frac{r^2}{4}}$$

Verify: $y = \sqrt{1-(x-1)^2}$

$$= \sqrt{1 - \left(\frac{r^2}{2} - 1\right)^2}$$

$$= \sqrt{\cancel{1} - \frac{r^4}{4} + \cancel{2}\frac{r^2}{\cancel{2}} - \cancel{1}} = r\sqrt{1 - \frac{r^2}{4}} \checkmark$$

$\longrightarrow$

The line joining P and Q has slope

$$\frac{r - r\sqrt{1-\frac{r^2}{4}}}{\underbrace{0}_{P} - \underbrace{\frac{r^2}{2}}_{Q}} = \frac{-2\cancel{r} + 2\cancel{r}\sqrt{1-\frac{r^2}{4}}}{r^{\cancel{2}}} = \frac{-2 + 2\sqrt{1-\frac{r^2}{4}}}{r}$$

and y-intercept r. The x-intercept is the x-coordinate for R, so solve:

$$0 = \left(\frac{-2+2\sqrt{1-\frac{r^2}{4}}}{r}\right)x + r$$

$$-r^2 = \left(-2+2\sqrt{1-\frac{r^2}{4}}\right)x$$

$$\Rightarrow \boxed{x = \frac{r^2}{2-2\sqrt{1-\frac{r^2}{4}}}}$$

(b) As C_2 shrinks, the y-coordinate of R does not change. The x-coordinate is given by

$$\lim_{r\to 0^+} x = \lim_{r\to 0^+} \frac{r^2}{2-2\sqrt{1-\frac{r^2}{4}}}\left(\frac{2+2\sqrt{1-\frac{r^2}{4}}}{2+2\sqrt{1-\frac{r^2}{4}}}\right)$$

$$= \lim_{r\to 0^+} \frac{r^2\left(2+2\sqrt{1-\frac{r^2}{4}}\right)}{4-4\left(1-\frac{r^2}{4}\right)}$$

$$= \lim_{r\to 0^+} \frac{\cancel{r^2}\left(2+2\sqrt{1-\frac{r^2}{4}}\right)}{\cancel{4}-\cancel{4}+\cancel{r^2}}$$

$$= \lim_{r\to 0^+} \left(2+2\sqrt{1-\frac{r^2}{4}}\right)$$

$$= 2+2\sqrt{1-\frac{0^2}{4}} = 4$$

$$\Rightarrow \boxed{R \to (4,0).}$$

5. (a) $$\lim_{x\to 0} f(x) = \lim_{x\to 0} f(x)\left(\frac{x^2}{x^2}\right) = \left(\lim_{x\to 0}\frac{f(x)}{x^2}\right)\left(\lim_{x\to 0} x^2\right)$$

$$= 5\cdot 0 = \boxed{0}$$

(b) $$\lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{f(x)}{x}\left(\frac{x}{x}\right) = \left(\lim_{x\to 0}\frac{f(x)}{x^2}\right)\left(\lim_{x\to 0} x\right)$$

$$= 5\cdot 0 = \boxed{0}$$

6. (a) i. $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} x = \boxed{1}$

ii. $\lim_{x\to 1} g(x) = \lim_{x\to 1^-} g(x) = \lim_{x\to 1^+} g(x)$, if it exists.

$\lim_{x\to 1^-} g(x) = 1$

$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} (2-x^2) = 2-1^2 = 1$

$= \boxed{1}$

iii. $g(1) = \boxed{3}$

iv. $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} (2-x^2) = 2-(2)^2 = \boxed{-2}$

v. $\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} (x-3) = 2-3 = \boxed{-1}$

vi. $\lim_{x\to 2} g(x)$ $\boxed{\text{DNE}}$, by iv. and v.

(b)