Quiz 6: Inverses and Related Rates

SOLUTIONS
Tues. 15 Mar 2016

1. $$f(x) = \frac{\sec x \sin^2(\tan^{-1}(\ln x))}{\log_6\left(e^{x^2\csc^{-1}(\pi x)}\right)} = \frac{\sec x \sin^2(\tan^{-1}(\ln x))}{x^2\csc^{-1}(\pi x)\log_6(e)}$$

via log algebra: $\log(A^B) = B\log A$

$$= \frac{\ln 6 \sec x \sin^2(\tan^{-1}(\ln x))}{x^2\csc^{-1}(\pi x)}$$

via log base change: $\log_6(e) = \frac{\ln(e)}{\ln(6)}$

In other words, $\log_6\left(e^{x^2\csc^{-1}(\pi x)}\right) = \frac{x^2\csc^{-1}(\pi x)}{\ln 6}$.

Now use log differentiation:

$$\frac{d}{dx}\left(\ln f(x) = \ln(\ln 6) + \ln(\sec x) + 2\ln(\sin(\tan^{-1}(\ln x))) - 2\ln x - \ln(\csc^{-1}(\pi x))\right)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = 0 + \frac{1}{\cancel{\sec x}}\cdot \cancel{\sec x}\tan x + 2\frac{1}{\sin(\tan^{-1}(\ln x))}\cdot\cos(\tan^{-1}(\ln x))\cdot\frac{1}{1+(\ln x)^2}\cdot\frac{1}{x}$$
$$-2\left(\frac{1}{x}\right) - \frac{1}{\csc^{-1}(\pi x)}\cdot\frac{-1}{|\cancel{\pi} x|\sqrt{(\pi x)^2-1}}\cdot\cancel{\pi}$$

$$= \tan x + \frac{2\underbrace{\cot(\tan^{-1}(\ln x))}_{=\frac{1}{\ln x}}}{x(1+\ln^2 x)} - \frac{2}{x} + \frac{1}{|x|\csc^{-1}(\pi x)\sqrt{(\pi x)^2-1}}$$

$$\boxed{\Rightarrow f'(x) = \frac{\ln 6\sec x\sin^2(\tan^{-1}(\ln x))}{x^2\csc^{-1}(\pi x)}\left[\tan x + \frac{2}{(\ln x)(x)(1+\ln^2 x)} - \frac{2}{x} + \frac{1}{|x|\csc^{-1}(\pi x)\sqrt{(\pi x)^2-1}}\right]}$$

OR differentiate directly: $\longrightarrow$

$$\frac{d}{dx}\left(\frac{\sec x\sin^2(\tan^{-1}(\ln x))}{\log_6(e^{x^2\csc^{-1}(\pi x)})}\right) = \frac{\log_6(e^{x^2\csc^{-1}(\pi x)})\left[\sec x\tan x\sin^2(\tan^{-1}(\ln x)) + \sec x\left(2\sin(\tan^{-1}(\ln x))\right)\cos(\tan^{-1}(\ln x))\cdot\frac{1}{1+(\ln x)^2}\cdot\frac{1}{x}\right]}{\log_6^2(e^{x^2\csc^{-1}(\pi x)})}$$

$$-\frac{\sec x\sin^2(\tan^{-1}(\ln x))\cdot\frac{1}{(\ln 6)(e^{x^2\csc^{-1}(\pi x)})}\cdot e^{x^2\csc^{-1}(\pi x)}\cdot\left(2x\csc^{-1}(\pi x)+x^2\cdot\frac{(-1)}{|\pi x|\sqrt{(\pi x)^2-1}}\cdot\pi\right)}{\log_6^2(e^{x^2\csc^{-1}(\pi x)})}$$

To check, factor out $f(x)$:

$$= \frac{\cancel{\log_6(e^{x^2\csc^{-1}(\pi x)})}\left[\sec x\tan x\sin^2(\tan^{-1}(\ln x)) + \sec x\left(2\sin(\tan^{-1}(\ln x))\right)\cos(\tan^{-1}(\ln x))\cdot\frac{1}{1+(\ln x)^2}\cdot\frac{1}{x}\right]}{\log_6^{\cancel{2}}(e^{x^2\csc^{-1}(\pi x)})}$$

$$-\frac{\sec x\sin^2(\tan^{-1}(\ln x))\cdot\frac{1}{(\ln 6)\cancel{(e^{x^2\csc^{-1}(\pi x)})}}\cdot \cancel{e^{x^2\csc^{-1}(\pi x)}}\cdot\left(2x\csc^{-1}(\pi x)+x^2\cdot\frac{(-1)}{|\cancel{\pi} x|\sqrt{(\pi x)^2-1}}\cdot\cancel{\pi}\right)}{\log_6^{\cancel{2}}(e^{x^2\csc^{-1}(\pi x)})}$$

$$= \frac{\sec x\sin^2(\tan^{-1}(\ln x))}{\log_6(e^{x^2\csc^{-1}\pi x})}\left[\tan x + 2\,\frac{\cos(\tan^{-1}(\ln x))}{\sin(\tan^{-1}(\ln x))}\cdot\frac{1}{1+\ln^2 x}\cdot\frac{1}{x} - \underbrace{\left(2x\csc^{-1}(\pi x) - \frac{x^2}{|x|\sqrt{(\pi x)^2+1}}\right)\cdot\frac{1}{\ln 6\,\log_6(e^{x^2\csc^{-1}\pi x})}}_{-\left(\frac{2x\csc^{-1}\pi x}{x^2\csc^{-1}\pi x} - \frac{x^2}{|x|\sqrt{(\pi x)^2+1}}\left(\frac{1}{x^2\csc^{-1}(\pi x)}\right)\right)}\right]$$

$$\frac{\cos(\tan^{-1}(\ln x))}{\sin(\tan^{-1}(\ln x))} = \frac{1}{\ln x}$$

2.(a) $y = \dfrac{x}{x+5}$

$$y(x+5) = x$$
$$xy + 5y = x$$
$$5y = x - xy$$
$$= x(1-y)$$
$$\frac{5y}{1-y} = x$$
$$\Rightarrow g^{-1}(x) = \frac{5x}{1-x}$$

$\longrightarrow$

(b) $$(g^{-1})'(x) = \frac{(1-\cancel{x})(\cancel{5}) - \cancel{5x(-1)}}{(1-x)^2}$$

$$\boxed{= \frac{5}{(1-x)^2}}$$

Check: $$\frac{1}{g'(u)} = \frac{(u+5)^2}{(\cancel{u}+5)(1) - \cancel{u}(1)}$$

$$= \frac{1}{5}(u+5)^2$$

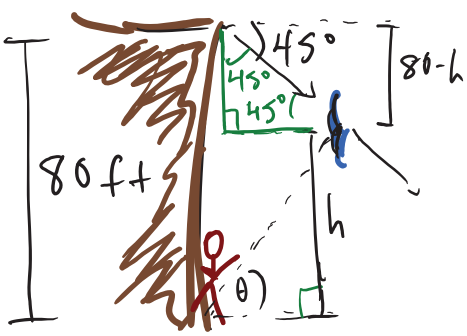
$$= \frac{1}{5}\left(\frac{5x}{1-x}+5\right)^2$$

$$= \frac{1}{5}\left(\frac{\cancel{5x}+5(1-\cancel{x})}{1-x}\right)^2$$

$$= \frac{1}{5}\left(\frac{5}{1-x}\right)^2$$

$$= \frac{5}{(1-x)^2} \quad \checkmark$$

3.



(a) The two green sides have equal length, $80-h$ ft.

Then $\tan\theta = \frac{h}{80-h}$

$$\Rightarrow \boxed{\theta = \arctan\left(\frac{h}{80-h}\right)}$$

(b) $$\frac{d\theta}{dh} = \frac{1}{1+\left(\frac{h}{80-h}\right)^2} \cdot \left(\frac{(80-\cancel{h})(1) - \cancel{h(-1)}}{(80-h)^2}\right)$$

$$= \frac{\cancel{(80-h)^2}}{(80-h)^2+h^2}\left(\frac{80}{\cancel{(80-h)^2}}\right)$$

$80^2 - 2(80)h + 2h^2$

$$\Rightarrow \left.\frac{d\theta}{dh}\right|_{h=60\text{ft}} = \frac{8\cancel{0}}{20^2+60^2} \boxed{= 0.02 \text{ rad/ft}}$$

$40\cancel{0}$ $\quad 360\cancel{0}$

4. (a)

- $\frac{da}{dt} = 1 \text{ cm/min}$
- $\frac{dA}{dt} = 2 \text{ cm}^2/\text{min}$
- $\frac{db}{dt}$
- $a = 10 \text{ cm}$
- $A = 100 \text{ cm}^2$

(b) $A = \frac{1}{2}ab \Longrightarrow \boxed{b = 2\frac{A}{a}}$

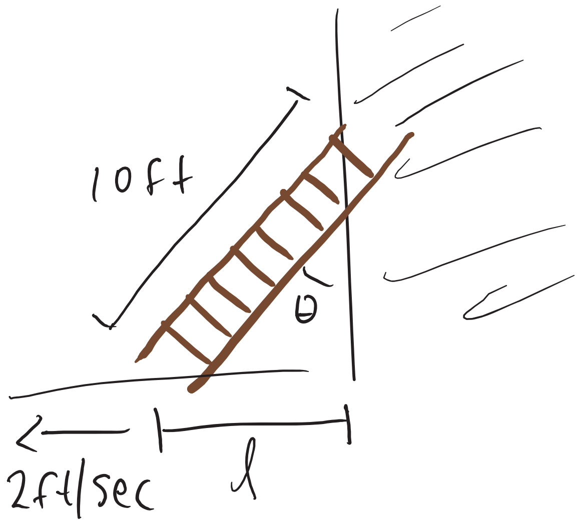
$$\boxed{\frac{db}{dt} = 2\frac{\left(a\frac{dA}{dt} - A\frac{da}{dt}\right)}{a^2}}$$

(c) $$\left.\frac{db}{dt}\right|_{\substack{a=10\text{cm} \\ A=100\text{cm}^2}} = 2\left[\frac{(10\text{cm})(2\text{cm}^2/\text{min}) - 100\text{cm}^2\left(\frac{1\text{cm}}{\text{min}}\right)}{(10\text{ cm})^2}\right]$$

$$= 2\left(\frac{20-100}{100}\right) = 2\left(\frac{-4}{5}\right) = \boxed{-1.6 \text{ cm/min}}$$

$\longrightarrow$

5.



- $\dfrac{dl}{dt} = 2$ ft/sec
- WTF: $\left.\dfrac{d\theta}{dt}\right|_{l=6ft}$

$$\sin\theta = \frac{l}{10} \Rightarrow \theta = \sin^{-1}\left(\frac{l}{10}\right)$$

$$\frac{d\theta}{dt} = \frac{1}{\sqrt{1-\left(\frac{l}{10}\right)^2}} \cdot \frac{1}{10} \cdot \frac{dl}{dt}$$

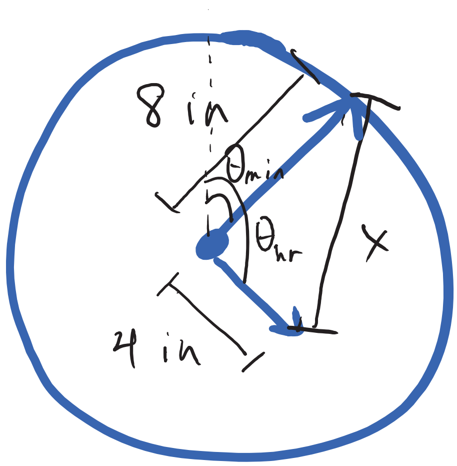
$$\Rightarrow \left.\frac{d\theta}{dt}\right|_{l=6ft} = \frac{1}{\sqrt{1-\left(\frac{6\,ft}{10}\right)^2}} \cdot \frac{1}{10}(2\text{ ft/sec})$$

$$= \frac{2}{\sqrt{\frac{100-36}{100}}} \cdot \frac{1}{10} = \frac{2}{\cancel{\frac{1}{10}}\sqrt{64}} \cdot \cancel{\frac{1}{10}}$$

$$= \frac{2}{8} = \boxed{\frac{1}{4}\text{ rad/sec}}$$

$\longrightarrow$

6.



- $\dfrac{d\theta_{min}}{dt} = \dfrac{2\pi}{\cancel{hr}}\left(\dfrac{\cancel{hr}}{60\,min}\right) = \dfrac{\pi}{30}$ rad/min
- $\dfrac{d\theta_{hr}}{dt} = \dfrac{1}{12}(2\pi)\dfrac{radians}{\cancel{hr}}\left(\dfrac{\cancel{hr}}{60\,min}\right) = \dfrac{\pi}{360}$ rad/min.
- 1 o'clock $\Longleftrightarrow$ $\theta_{min} = 0$ and $\theta_{hr} = \dfrac{2\pi}{12} = \dfrac{\pi}{6}$
- WTF: $\left.\dfrac{dx}{dt}\right|_{\substack{\theta_{min}=0 \\ \theta_{hr}=\frac{\pi}{6}}}$

Let $\theta = |\theta_{min} - \theta_{hr}|$

$$\Rightarrow \frac{d\theta}{dt} = \left|\frac{d\theta_{min}}{dt} - \frac{d\theta_{hr}}{dt}\right|$$

Law of Cosines:

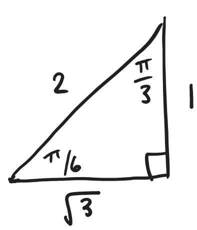
$$x^2 = 8^2 + 4^2 - 2(8)(4)\cos\theta \Longrightarrow x = \sqrt{80 - 64\cos\theta}$$

$$= 4\sqrt{5 - 4\cos\theta} \text{ inches}$$

$$2x\frac{dx}{dt} = -64(-\sin\theta)\frac{d\theta}{dt}$$

$$\left.\frac{dx}{dt}\right|_{\substack{\theta_{min}=0 \\ \theta_{hr}=\frac{\pi}{6}}} = \frac{64\sin\left|0-\frac{\pi}{6}\right| \cdot \left|\frac{\pi}{360} - \frac{\pi}{30}\right|}{2\left(4\sqrt{5-4\cos\left|0-\frac{\pi}{6}\right|}\right)} = \frac{\cancel{32}^{2}\left(\frac{1}{\cancel{2}}\right)\left|\frac{\pi - 12\pi}{360}\right|}{\cancel{8}\sqrt{5-\cancel{4}^{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right)}}$$

$$= \frac{22\pi}{360\left(\sqrt{5-2\sqrt{3}}\right)} \approx \boxed{0.155 \text{ in/min}}$$



OR ≈ 9.295 in/hr