# Math 2554 Quiz 15: Substitution Rules (§5.5) L1

SOLUTIONS

1. (a) $\displaystyle\int \frac{(\sqrt{x}+1)^4}{2\sqrt{x}}\,dx \longrightarrow u=\sqrt{x}+1$

$du = \frac{1}{2}x^{-1/2}\,dx$

$= \frac{1}{2\sqrt{x}}\,dx$

$= \displaystyle\int u^4\,du$

$= \frac{u^5}{5} + C = \boxed{\frac{(\sqrt{x}+1)^5}{5} + C}$

Check: $\frac{d}{dx}\left(\frac{(\sqrt{x}+1)^5}{5} + C\right) = \frac{1}{5}\,5(\sqrt{x}+1)^4\cdot\frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{(\sqrt{x}+1)^4}{2\sqrt{x}}$ ✓

(b) $\displaystyle\int x^9\sin(x^{10})\,dx \longrightarrow u = x^{10}$

$du = 10x^9\,dx$

$\Rightarrow \frac{1}{10}\,du = x^9\,dx$

$= \frac{1}{10}\displaystyle\int \sin(u)\,du$

$= \frac{1}{10}(-\cos(u)) + C = \boxed{\frac{-\cos(x^{10})}{10} + C}$

Check: $\frac{d}{dx}\left(-\frac{\cos(x^{10})}{10} + C\right) = \frac{-1}{10}(-\sin(x^{10}))\cdot 10x^9 = x^9\sin(x^{10})$ ✓

$\longrightarrow$

(c) $\displaystyle\int \frac{y}{\sqrt{y-4}}\,dy \longrightarrow u = y-4$

$\Rightarrow y = u+4$

$du = dy$

$$= \int \frac{u+4}{\sqrt{u}}\,du$$

$$= \int \sqrt{u}\,du + 4\int \frac{1}{\sqrt{u}}\,du$$

$$= \frac{u^{3/2}}{\frac{3}{2}} + \frac{4u^{1/2}}{\frac{1}{2}} + C = \frac{2}{3}u^{3/2} + 8\sqrt{u} + C$$

$$\boxed{= \frac{2}{3}(y-4)^{3/2} + 8\sqrt{y-4} + C}$$

Check: $\displaystyle\frac{d}{dy}\left(\frac{2}{3}(y-4)^{3/2} + 8\sqrt{y-4} + C\right)$

$$= \frac{2}{3}\left(\frac{3}{2}\right)(y-4)^{1/2} + 8\cdot\frac{1}{2}(y-4)^{-1/2}$$

$$= \sqrt{y-4} + \frac{4}{\sqrt{y-4}} = \frac{y-4+4}{\sqrt{y-4}} = \frac{y}{\sqrt{y-4}} \checkmark$$

$\longrightarrow$

(d) $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx \longrightarrow \begin{array}{l} u = e^x + e^{-x} \\ du = (e^x - e^{-x})\,dx \end{array}$

$$= \int \frac{1}{u}\,du = \ln|u| + C$$

$$= \ln\left|e^x + e^{-x}\right| + C$$

always $> 0$

$$\boxed{= \ln(e^x + e^{-x}) + C}$$

Check: $\displaystyle\frac{d}{dx}\left(\ln(e^x + e^{-x}) + C\right) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ ✓

(e) $\displaystyle\int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta$

Half-Angle Formula

$$= \int \frac{1 - \cos\left(2\left(\theta + \frac{\pi}{6}\right)\right)}{2}\,d\theta$$

$$= \int \frac{1 - \cos\left(2\theta + \frac{\pi}{3}\right)}{2}\,d\theta \longrightarrow \begin{array}{l} u = 2\theta + \frac{\pi}{3} \\ du = 2\,d\theta \\ \Rightarrow \frac{1}{2}\,du = d\theta \end{array}$$

$$= \frac{1}{4}\int (1 - \cos u)\,du$$

$$= \frac{1}{4}\left(u - \sin u + C\right)$$

$\longrightarrow$

$$\boxed{= \frac{1}{4}\left(2\theta + \frac{\pi}{3} - \sin\left(2\theta + \frac{\pi}{3}\right)\right) + C}$$

Check: $\frac{d}{d\theta}\left(\frac{1}{4}\left(2\theta + \frac{\pi}{3} - \sin\left(2\theta + \frac{\pi}{3}\right)\right) + C\right)$

$$= \frac{1}{4}\left(2 - \cos\left(2\theta + \frac{\pi}{3}\right) \cdot 2\right)$$

$$= \frac{1 - \cos\left(2\theta + \frac{\pi}{3}\right)}{2} \checkmark$$

2. Recall, $u = g(x)$ is a function of $x$.

(a) $\int_0^1 2x(4-x^2)\,dx \longrightarrow u = g(x) = 4 - x^2$

$du = -2x\,dx$

$\Rightarrow -du = 2x\,dx$

$g(1) = 4 - 1^2 = 3$

$g(0) = 4 - 0^2 = 4$

$$= \int_{g(0)}^{g(1)} u(-du) = -\int_4^3 u\,du$$

$$= \int_3^4 u\,du$$

$\longrightarrow$

$$= \frac{u^2}{2}\Big|_3^4$$

$$= \frac{4^2}{2} - \frac{3^2}{2} = \frac{16}{2} - \frac{9}{2} = \boxed{\frac{7}{2}}$$

Alternate Way (no subs. required):

$$\int_0^1 2x(4-x^2)\,dx = \int_0^1 (8x-2x^3)\,dx$$

$$= 8\frac{x^2}{2} - 2\frac{x^4}{4}\Big|_0^1$$

$$= \left(4(1) - \frac{1}{2}\right) - 0$$

$$= \boxed{\frac{7}{2}}$$

(b) $\int_{-1}^{2} x^2 e^{x^3+1}\,dx \longrightarrow u = g(x) = x^3+1$

$du = 3x^2\,dx$

$\Rightarrow \frac{1}{3}\,du = x^2\,dx$

$g(2) = 2^3 + 1 = 9$

$g(-1) = (-1)^3 + 1 = 0$

$$= \frac{1}{3}\int_0^9 e^u\,du = \frac{1}{3}e^u\Big|_0^9 = \frac{1}{3}\left(e^9 - e^0\right) = \boxed{\frac{1}{3}\left(e^9 - 1\right)}$$

$\longrightarrow$

(c) $\displaystyle\int_0^{\pi/4} \frac{\sin x}{\cos^2 x}\,dx \longrightarrow u = g(x) = \cos x$

$du = -\sin x\,dx$

$\Rightarrow -du = \sin x\,dx$

$g\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$g(0) = \cos(0) = 1$

$$= -\int_1^{\frac{\sqrt{2}}{2}} \frac{1}{u^2}\,du = -\left(\frac{-1}{u}\right)\Bigg|_1^{\frac{\sqrt{2}}{2}}$$

$$= \frac{1}{\frac{\sqrt{2}}{2}} - 1 = \frac{2}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) - 1$$

$$\boxed{= \sqrt{2} - 1}$$

(d) $\displaystyle\int_1^{e^2} \frac{\ln x}{x}\,dx \longrightarrow u = g(x) = \ln x$

$du = \frac{1}{x}\,dx$

$g(e^2) = \ln(e^2) = 2$

$g(1) = \ln(1) = 0$

$$= \int_0^2 u\,du = \frac{u^2}{2}\Bigg|_0^2 = \frac{2^2}{2} - 0 = \boxed{2}$$

$\longrightarrow$

(e) Compare to the derivation of $\frac{d}{dx}(\arctan x)$:

$$\int_0^6 \frac{dz}{z^2+36} \longrightarrow z = 6\tan u \Rightarrow u = \arctan\left(\frac{z}{6}\right)$$

$$dz = 6\sec^2 u\,du$$

$$= \int_{\arctan(0)}^{\arctan(1)} \frac{6\sec^2 u}{36\tan^2 u + 36}\,du$$

Note, $\tan\theta = \frac{\sin\theta}{\cos\theta} = 1$

when $\sin\theta = \cos\theta$

$\Rightarrow \theta = \frac{\pi}{4}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = 0$

when $\sin\theta = 0$

$\Rightarrow \theta = 0$

$$= \int_0^{\frac{\pi}{4}} \frac{6\sec^2 u}{36(\tan^2 u + 1)}\,du$$

Recall, the trig identity $\sec^2 u - \tan^2 u = 1$

$$= \int_0^{\pi/4} \frac{1}{6} \frac{\cancel{\sec^2 u}}{\cancel{\sec^2 u}} \, du$$

$$= \frac{1}{6} \int_0^{\pi/4} du = \frac{1}{6} u \Big|_0^{\pi/4}$$

$$= \frac{1}{6}\left(\frac{\pi}{4} - 0\right) \boxed{= \frac{\pi}{24}}$$

Alternate Way (no subs required):

$$\int_0^6 \frac{dz}{z^2+36} = \frac{1}{6} \arctan\left(\frac{z}{6}\right)\Big|_0^6$$

$$= \frac{1}{6}\left(\arctan(1) - \arctan(0)\right)$$

$$= \frac{1}{6}\left(\frac{\pi}{4} - 0\right) \boxed{= \frac{\pi}{24}}$$