Mon 18 Apr

- April 22: Last day to drop with a "W".
- Exam 4 next week, probably Friday. Covers §4.7-5.4

Sigma Notation

Riemann sums become more accurate when we make n (the number of rectangles) bigger, but obviously writing it all down is no fun! Sigma notation gives a shorthand. Here is how sigma notation works, through an example:

Example

$\sum_{n=1}^{5} n^2$ is the sum all integer values from the lowest limit $(n = 1)$ to the highest limit $(n = 5)$ in the summand n^2 (in this case n is the indexing variable).

$$\sum_{n=1}^{5} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

Example

Evaluate $\sum_{k=0}^{3}(2k-1)$.

Solution: In this example, k is the indexing variable. It starts at 0 and goes to 3, which means we write down the expression in the parentheses for each of the integers from 0 to 3, then add the results together:

$$\begin{aligned}\sum_{k=0}^{3}(2k-1) &= (2(0)-1)+(2(1)-1)+(2(2)-1)+(2(3)-1)\\ &= -1+1+3+5=8.\end{aligned}$$

Σ-Shortcuts

(n is always a positive integer)

$$\sum_{k=1}^{n} c = cn \text{ (where } c \text{ is a constant)}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Question

What is the indexing variable in these formulas?

Riemann Sums Using Sigma Notation

Suppose f is defined on a closed interval $[a, b]$ which is divided into n subintervals of equal length Δx. As before, $\overline{x}_k$ denotes a point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$. Recall that $x_0 = a$ and $x_n = b$.

Here is how we can write the Riemann sum in a much more compact form:

$$R = f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x = \sum_{k=1}^{n} f(\overline{x}_k)\Delta x.$$

With sigma-notation we can even derive explicit formulas for the basic Riemann sums (the expression in red is $\overline{x}_k$ for each case:

1. $\displaystyle\sum_{k=1}^{n} f(a+(k-1)\Delta x)\Delta x =$ left Riemann sum
2. $\displaystyle\sum_{k=1}^{n} f(a+k\Delta x)\Delta x =$ right Riemann sum
3. $\displaystyle\sum_{k=1}^{n} f\left(a+\left(k-\tfrac{1}{2}\right)\Delta x\right)\Delta x =$ midpoint Riemann sum

Exercise

(a) Use sigma notation to write the left, right, and midpoint Riemann sums for the function $f(x) = x^2$ on the interval $[1, 5]$ given that $n = 4$.

(b) Based on these approximations, estimate the area bounded by the graph of $f(x)$ over $[1, 5]$.

Suggestion: As n gets very big, Riemann sums, along with the Σ-shortcuts plus algebra, often make the problem way more manageable.

5.1 Book Problems

9-37

§5.2 Definite Integrals

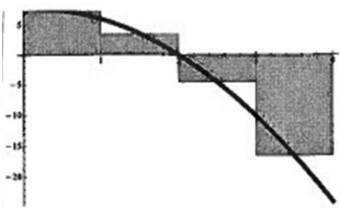
In §5.1, we saw how we can use Riemann sums to approximate the area under a curve. However, the curves we worked with were all non-negative.

Question

What happens when the curve is negative?

Example

Let $f(x) = 8 - 2x^2$ over the interval $[0, 4]$. Use a left, right, and midpoint Riemann sum with $n = 4$ to approximate the area under the curve.



Net Area

In the previous example, the areas where f was positive provided positive contributions to the area, while areas where f was negative provided negative contributions. The difference between positive and negative contributions is called the **net area**.

Definition

Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis minus the sum of the areas of the parts of R that lie below the x-axis on $[a, b]$.

The Riemann sums give approximations for the area under the curve. To make these approximations more and more accurate, we divide the region into more and more subintervals. To make these approximations exact, we allow the number of subintervals $n \to \infty$, thereby allowing the length of the subintervals $\Delta x \to 0$. In terms of limits:

$$\text{Net Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(\overline{x}_k)\Delta x.$$

Suppose $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ are subintervals of $[a, b]$ with $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. Let Δx_k be the length of the subinterval $[x_{k-1}, x_k]$ and let $\overline{x}_k$ be any point in $[x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$. If f is defined on $[a, b]$, then the sum

$$\sum_{k=1}^{n} f(\overline{x}_k)\Delta x_k = f(\overline{x}_1)\Delta x_1 + f(\overline{x}_2)\Delta x_2 + \cdots + f(\overline{x}_n)\Delta x_n$$

is called a **general Riemann sum for f on $[a, b]$**.

Note: In this definition, the lengths of the subintervals do not have to be equal.

As $n \to \infty$, all of the $\Delta x_k \to 0$, even the largest of these. Let Δ be the largest of the Δx_k's.

Definition

The **definite integral of f from a to b** is

$$\int_a^b f(x)\ dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(\overline{x}_k) \Delta x_k,$$

where f is a function defined on $[a, b]$. When this limit exists – over *all* partitions of $[a, b]$ and *all* choices of $\overline{x}_k$ on a partition – f is called **integrable**.

Evaluating Definite Integrals

Theorem

If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$.

See Figure 5.23, p. 325, for an example of a noncontinuous function that is integrable.

Knowing the limit of a Riemann sum, we can now translate that to a definite integral.

Example

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} (4\overline{x}_k - 3)\Delta x_k \text{ on } [-1, 4] \quad \equiv \quad \int_{-1}^{4} (4x - 3)\, dx$$

Without formally examining methods to evaluate definite integrals, we can use geometry.

Exercise

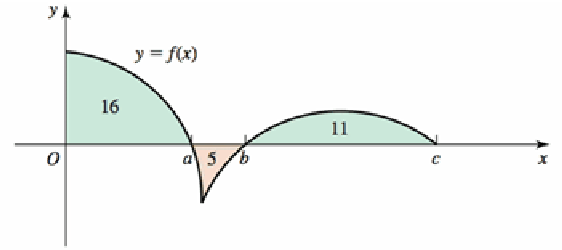
Using geometry, evaluate $\int_1^2 (4x-3)\ dx$.

(*Hint:* The area of a trapezoid is $A = \frac{h(l_1+l_2)}{2}$, where h is the height of the trapezoid and l_1 and l_2 are the lengths of the two parallel bases.)

Exercise

Using the picture below, evaluate the following definite integrals:

1. $\displaystyle\int_0^a f(x)\,dx$ 2. $\displaystyle\int_0^b f(x)\,dx$ 3. $\displaystyle\int_0^c f(x)\,dx$ 4. $\displaystyle\int_a^c f(x)\,dx$



Properties of Integrals

1. (Reversing Limits) $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$
2. (Identical Limits) $\int_a^a f(x)\ dx = 0$
3. (Integral of a Sum)
   $\int_a^b (f(x) + g(x))\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$
4. (Constants in Integrals) $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$

Properties of Integrals, cont.

5. (Integrals over Subintervals) If c lies between a and b, then

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

6. (Integrals of Absolute Values) The function $|f|$ is integrable on $[a, b]$ and $\int_a^b |f(x)|\ dx$ is the sum of the areas of regions bounded by the graph of f and the x-axis on $[a, b]$. (See Figure 5.31 on p. 329)

   (This is the total area, no negative signs.)

Exercise

If $\int_2^4 f(x)\ dx = 3$ and $\int_4^6 f(x)\ dx = -2$, then compute $\int_2^6 f(x)\ dx$.

5.2 Book Problems

11-45 (odds), 67-74