Exam 3: Taylor series (Ch. 8)

Math 236 (Calc II)
Fall 2017

SOLUTIONS

1. The Maclaurin series formula is

$$T(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

Compute the derivatives:

$f'(x) = 2f(x)$

$f''(x) = 2f'(x) = 4f(x)$

$f'''(x) = 2f''(x) = 8f(x)$

$f^{(4)}(x) = 2f'''(x) = 2^4 f(x)$

$\vdots$

$f^{(n)}(x) = 2f^{(n-1)}(x) = 2^n f(x) \implies f^{(n)}(0) = 2^n \cdot (-3)$

$$\implies T(x) = -3 - 6x - \frac{12}{2}x^2 - \frac{24}{3!}x^3 - \frac{48}{4!}x^4 - \cdots$$

or $= -3 - 6x - 6x^2 - 4x^3 - 2x^4 - \cdots$

or $= \boxed{\sum_{k=0}^{\infty} (-3)\frac{2^k}{k!}x^k}$

2. According to the formula sheet, the Maclaurin series for $\cos x$ is

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$\text{or} \quad = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \quad \text{for } x \in \mathbb{R}.$$

Substitute x^3 for x, then multiply by x, to get

$$F'(x) = x \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (x^3)^{2k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{6k+1} \quad \text{for } x \in \mathbb{R}$$

$$\text{or} \quad = x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + \frac{x^{25}}{8!} - \cdots$$

$$\Rightarrow F(x) = \int F'(x)\,dx = \int \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{6k+1}\,dx$$

$$= \sum_{k=0}^{\infty} \left(\int \frac{(-1)^k}{(2k)!} x^{6k+1}\,dx \right)$$

$\longrightarrow$

$$= \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{(2k)!} \int x^{6k+1} \, dx \right)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{x^{6k+2}}{6k+2} + C_k \right)$$

may be a different constant for each k

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{x^{6k+2}}{6k+2} + \sum_{k=0}^{\infty} C_k$$

ends up being a constant, C, if it converges.

Use the initial value:

$$F(0) = -1 = \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{0^{6k+2}}{6k+2}}_{\text{equals } 0} + C \Rightarrow C = -1$$

$$\Rightarrow F(x) = \boxed{-1 + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{x^{6k+2}}{6k+2}} \quad \text{for } x \in \mathbb{R}$$

or $= -1 + \frac{x^2}{2} - \frac{x^8}{2!(8)} + \frac{x^{14}}{3!(14)} - \frac{x^{20}}{4!(20)} + \dots$

3. Use the Maclaurin series for e^x and $\cos x$:

$$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)$$

e^x for $x \in \mathbb{R}$

$$\cdot \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right)$$

$\cos x$ for $x \in \mathbb{R}$

0 1 2

$$= (1)(1) + (x)(1) + \left(\frac{x^2}{2!}\right)(1) + (1)\cancel{\left(-\frac{x^2}{2!}\right)}$$

3 4

$$+ (x)\left(-\frac{x^2}{2!}\right) + \left(\frac{x^3}{3!}\right)(1) + (1)\left(\frac{x^4}{4!}\right) + \left(\frac{x^2}{2!}\right)\left(-\frac{x^2}{2!}\right) + \left(\frac{x^4}{4!}\right)(1)$$

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$$+ (x)\left(\frac{x^4}{4!}\right) + \left(\frac{x^3}{3!}\right)\left(-\frac{x^2}{2!}\right) + \left(\frac{x^5}{5!}\right)(1) + \cdots$$

$$= 1 + x + \left(\frac{-1}{2!} + \frac{1}{3!}\right)x^3 + \left(\frac{2}{4!} + \frac{1}{2!2!}\right)x^4$$

$$+ \left(\frac{1}{4!} - \frac{1}{3!2!} + \frac{1}{5!}\right)x^5 + \cdots$$

for $x \in \mathbb{R}$

4. Replace x with x-1 in the Maclaurin series for e^x:

$$e^{x-1} = 1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \cdots$$

for $x \in \mathbb{R}$

Multiply by e:

$$e \cdot e^{x-1} = e^x = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \cdots$$

$$\boxed{= \sum_{k=0}^{\infty} \frac{e}{k!}(x-1)^k \quad \text{for } x \in \mathbb{R}}$$

5. Compare to the Maclaurin series

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = (-1) \sum_{k=1}^{\infty} \frac{(-1)^k}{k} x^k$$

for $x \in (-1, 1]$

Since $\frac{1}{\pi} \in (-1, 1]$,

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\frac{1}{\pi}\right)^k = \frac{\ln\left(1 + \frac{1}{\pi}\right)}{(-1)} = -\ln\left(\frac{\pi+1}{\pi}\right)$$

$$\boxed{= \ln\left(\frac{\pi}{\pi+1}\right) \approx -0.276}$$

6. Compare to the Maclaurin series

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \text{for } x \in \mathbb{R}$$

Then

$$\sin(3x+7) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (3x+7)^{2k+1}$$

$$\text{for } 3x+7 \in \mathbb{R} \Rightarrow \boxed{x \in \mathbb{R}}$$