Math 2603 Exam 1
Wed 24 Sep 2014

Name: KEY

Discrete Math
Exam 1 (Ch. 1-2: Set theory, logic, proofs)

Please provide the following data:

Drill Time: ______________________________

Student ID: ______________________________

Exam Instructions: You have 50 minutes to complete this exam. One 3×5 inch notecard is allowed. No graphing calculators. No programmable calculators. No phones, iDevices, computers, etc. If you finish early then you may leave, UNLESS there are less than 5 minutes of class left. To prevent disruption, if you finish with less than 5 minutes of class remaining then please stay seated and quiet.

Your signature below indicates that you have read this page and agree to follow the Academic Honesty Policies of the University of Arkansas.

Signature: ____

Good luck!

1. **Truth tables and valid arguments.**

(a) Fill in the truth table:

P	Q	R	$P \wedge Q$	$(P \wedge Q) \rightarrow (\neg R)$	$P \vee \neg Q$	$\neg Q \rightarrow P$
T	T	T	T	F	T	T
T	T	F	T	T	T	T
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	F	T	F	T
F	T	F	F	T	F	T
F	F	T	F	T	T	F
F	F	F	F	T	T	F

(b) Using table (a) explain if the argument below is valid.

$$\begin{array}{l} P \wedge Q \\ Q \\ \hline \therefore (P \wedge Q) \rightarrow (\neg R) \end{array}$$

The conclusion is false when P, Q, R are all true, but the hypotheses are true in this case. Therefore the argument is not valid

(c) Determine if the following argument is valid. You may extend the table in (a) if necessary, but either way, you must justify your answer.

$$\begin{array}{l} P \wedge Q \rightarrow \neg R \\ P \vee \neg Q \\ \neg Q \rightarrow P \\ \hline \therefore R \end{array}$$

There are two cases (according to the truth table) when the hypotheses are all true but the conclusion is false, namely, when P, Q are true and R is false and when P is true and R, Q are false. Therefore the argument is not valid.

2. Let $P =$ "If $\frac{6}{d} \in \mathbb{Z}$, where $d \in \mathbb{Z}$, then $d = 3$."

$\forall d \in \mathbb{Z} \left(\frac{6}{d} \in \mathbb{Z} \longrightarrow d = 3 \right)$

(a) Is P a proposition?

Yes.

(b) What is $\neg P$?

$\exists d \in \mathbb{Z} \left(\frac{6}{d} \in \mathbb{Z} \wedge d \neq 3 \right)$

(c) Give the contrapositive of P.

If $d \neq 3$, where $d \in \mathbb{Z}$, then $\frac{6}{d} \notin \mathbb{Z}$.

(d) Give the converse of P.

If $d = 3$, where $d \in \mathbb{Z}$, then $\frac{6}{d} \in \mathbb{Z}$

(e) Is P true or false? (Prove or give a counterexample.)

False. From (b), let $d = 6$. Then $\frac{6}{6} = 1 \in \mathbb{Z}$, but $d = 6 \neq 3$.

3. **For a set S let $\mathcal{P}(S)$ denote its power set.**

(a) **Prove $\mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y)$.**

($\subseteq$) Choose $S \in \mathcal{P}(X \cap Y)$. Then $S \subseteq X \cap Y$.
Therefore $S \subseteq X$ and $S \subseteq Y$. So
$S \in \mathcal{P}(X)$ and $S \in \mathcal{P}(Y)$

$$|||$$

$$S \in \mathcal{P}(X) \cap \mathcal{P}(Y).$$

($\supseteq$) Choose $S \in \mathcal{P}(X) \cap \mathcal{P}(Y)$. Then $S \subseteq X$ and $S \subseteq Y$.
So $S \subseteq X \cap Y$. Hence $S \in \mathcal{P}(X \cap Y)$.

$\boxtimes$

(b) **Disprove $\mathcal{P}(X \cup Y) \subseteq \mathcal{P}(X) \cup \mathcal{P}(Y)$.**

Counterexample: e.g., say $X \perp\!\!\!\perp Y$.
Let $X = \{1\}$
$Y = \{2\}$

$S = \{1,2\} \subseteq X \cup Y = \{1,2\}$. So $S \in \mathcal{P}(X \cup Y)$.

However, $\mathcal{P}(X) = \{\emptyset, \{1\}\}$
$\mathcal{P}(Y) = \{\emptyset, \{2\}\}$,
neither of which contains S.

4. **Prove by contrapositive: If $x^2 \in \mathbb{R} \setminus \mathbb{Q}$, then $x \in \mathbb{R} \setminus \mathbb{Q}$.**

Suppose $x \notin \mathbb{R} \setminus \mathbb{Q}$, i.e., $x \in \mathbb{Q}$. Then there exist integers a, b, with $b \neq 0$ and $x = \frac{a}{b}$. Therefore we may write

$$x^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}.$$

$a^2, b^2 \in \mathbb{Z}$ (axioms about integers)

and $b^2 \neq 0$ b/c $b \neq 0$.

We conclude $x^2 \in \mathbb{Q}$, equivalently, $x^2 \notin \mathbb{R} \setminus \mathbb{Q}$.

$\boxtimes$

5. **True/False. If true, use induction to prove it. If false, give a counterexample.**

(a) $1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n+1} n^2 = \dfrac{(-1)^{n+1} n(n+1)}{2}$, $n \in \mathbb{Z}_{>0}$.

We first check the base case, $n=1$:

$$(-1)^{n+1} 1^2 = 1^2 = \frac{(-1)^2 (1)(2)}{2}$$

$$1 = 1 \checkmark$$

Now we check the statement at the inductive step. Suppose there exist some $n \in \mathbb{Z}_{>0}$ (e.g., $n=1$, as in the base case) where the formula holds. Write the lefthand side (LHS):

$$1^2 - 2^2 + \cdots + (-1)^{n+1} n^2 + (-1)^{n+2} (n+1)^2$$

$$= \frac{(-1)^{n+1} n(n+1)}{2} + \frac{2(-1)^{n+2}(n+1)^2}{2}$$

$$= \frac{(-1)^{n+1} n(n+1) + 2(-1)^{n+1}(-1)(n+1)^2}{2} = \frac{(-1)^{n+1}\left[n(n+1) - 2(n+1)^2\right]}{2}$$

$$= (-1)^{n+1} \frac{n^2 + n - 2n^2 - 4n - 2}{2} = \frac{(-1)^{n+1}(-n^2 - 3n - 2)}{2} = \frac{(-1)^{n+2}(n+1)(n+2)}{2}$$

$\therefore$ True. $\blacksquare$

(b) For all $n \in \mathbb{Z}_{>0}$, $11^n - 6$ is divisible by 5.

Base. $n=1$.

$$11^1 - 6 = 11 - 6 = 5 \quad \checkmark$$

Induce. Suppose for some n,

$11^n - 6$ is divisible by 5.

Then we can write

$$11^{n+1} - 6 = (11^n - 6) + x \quad (\text{because } 11^{n+1} > 11^n).$$

If x is divisible by 5 then we are done.

$$x = 11^{n+1} - 6 - (11^n - 6)$$

$$= 11^{n+1} - 11^n = 11^n(11-1)$$

$$= 11^n(10) \leftarrow \text{divisible by } 5.$$

Since the induction hypothesis says $11^n - 6$ is divisible by 5, write

$$11^n - 6 = 5q \quad \text{for some } q \in \mathbb{Z}.$$

Then $11^{n+1} - 6 = 5q + 11^n(10)$

$$= 5\underbrace{(q + 2 \cdot 11^n)}_{\text{an integer}}$$

$\therefore 11^{n+1} - 6$ is divisible by 5. $\blacksquare$

6. **cHaLlEnGe PrObLeM Prove: Given any two rational numbers r and s with $r < s$, there is a rational number between r and s.**

We want to show there exists $x \in \mathbb{Q}$ such that $r \leq x \leq s$. Since $r, s \in \mathbb{Q}$, write

$$r = \frac{a}{b}, \quad s = \frac{c}{d}$$; without loss of generality we can assert r, s are in lowest terms, and $a, b, c, d \in \mathbb{Z}$, $b, d \neq 0$.

We claim $x = \dfrac{r+s}{2}$ does the job.

Write $x = \dfrac{\frac{a}{b} + \frac{c}{d}}{2} = \dfrac{ad+bc}{2bd} \in \mathbb{Q}$. It remains to show $r \leq x \leq s$. We work backwards to show $r \leq x$:

$$r = \frac{a}{b} \boxed{\overset{?}{\leq}} x = \frac{ad+bc}{2bd}$$

To verify the ? we will work to find an equivalent statement that is obviously true. Since $b, d \neq 0$ there are four cases:

Case 1. $b, d > 0$. Then

$2abd \leq b(ad+bc)$ iff

$2ad \leq ad + bc$ iff

$\longrightarrow$

$ad \le bc$ iff

$r = \frac{a}{b} \le \frac{c}{d} = s$ ← true, by hypothesis. ✓

<u>Case 2</u>. $b > 0$, $d < 0$. We cannot have $a, b < 0$, nor can we have $c, d < 0$, because we asserted r, s were in lowest terms. So we can equivalently suppose $c < 0$ and $d > 0$. Then we are in Case 1, which we already proved.

<u>Case 3</u>. $b < 0$, $d < 0$. Then again, we can suppose, equivalently, that $b, d > 0$ and $a, c < 0$, which again is Case 1.

<u>Case 4</u>. By arguments similar to those above, we can reduce to Case 1.

We conclude $r \le x$. We must now show $x \le s$. We can use the same strategy as before. Write

$$(*) \qquad x = \frac{ad + bc}{2bd} \overset{?}{\le} s = \frac{c}{d}$$

If we can find an obvious statement that is equivalent to $(*)$, then we may conclude $x \le s$ and the proof is complete.

⟶

Again, there are four cases to consider:

1. $b, d > 0$
2. $b > 0,\ d < 0$
3. $b < 0,\ d > 0$
4. $b, d < 0$

However, by the exact same arguments as before, it is enough to prove Case 1! Thus, suppose $b, d > 0$. Then

$$x \le s \quad \text{iff}$$

$$\frac{ad+bc}{2b\cancel{d}} \le \frac{c}{\cancel{d}} \quad \text{iff}$$

$$ad + bc \le 2bc \quad \text{iff}$$

$$ad \le bc \quad \text{iff}$$

$$r = \frac{a}{b} \le \frac{c}{d} = s. \leftarrow \text{True by hypothesis.}$$

Since we showed the claim "$x \le s$" is equivalent to $r \le s$, which is true by hypothesis, we are done. Therefore $x = \frac{ad+bc}{2bd} \in \mathbb{Q}$ and $r \le x \le s$, as desired.

7. **EXTRA CREDIT** Use the quotient-remainder theorem, with $d = 3$, to prove that the product of any three consecutive integers is divisible by 3.

Let $n_1 < n_2 < n_3$ denote 3 consecutive integers. By the QRT, there are 3 cases to consider:

① $n_1 = 3q$ or ② $n_1 = 3q+1$ or ③ $n_1 = 3q+2$

for some $q \in \mathbb{Z}$. We show in all 3 cases that the product is divisible by 3.

Case 1. $n_1 = 3q$. Then $n_2 = 3q+1$, $n_3 = 3q+2$, because the integers are consecutive. The product is

$$n_1 n_2 n_3 = \underset{\substack{\uparrow \\ \text{divisible by 3}}}{3}\underbrace{q(3q+1)(3q+2)}_{\in \mathbb{Z}}$$

Case 2. $n_1 = 3q+1$. Then

$n_2 = 3q+2$

$n_3 = 3q+3 = 3(q+1)$. The product is

$$n_1 n_2 n_3 = (3q+1)(3q+2)(3q+3)$$
$$= 3(3q+1)(3q+2)(q+1) \leftarrow \text{divisible by 3.}$$

Case 3. $n_1 = 3q+2$. Then

$n_2 = 3q+3 = 3(q+1)$

$n_3 = 3q+4$

divisible by 3.

and $n_1 n_2 n_3 = (3q+2)(3q+3)(3q+4) = 3(3q+2)(q+1)(3q+4)$

$\blacksquare$