

§5.3 Fundamental Theorem of Calculus

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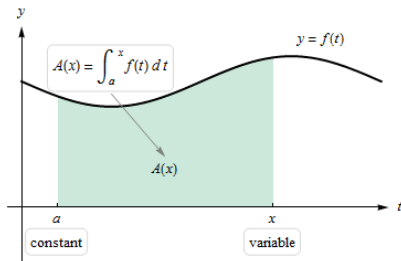
§5.3 Fundamental Theorem of Calculus

Using Riemann sums to evaluate definite integrals is usually neither efficient nor practical. We will develop methods to evaluate integrals and also tie together the concepts of differentiation and integration.

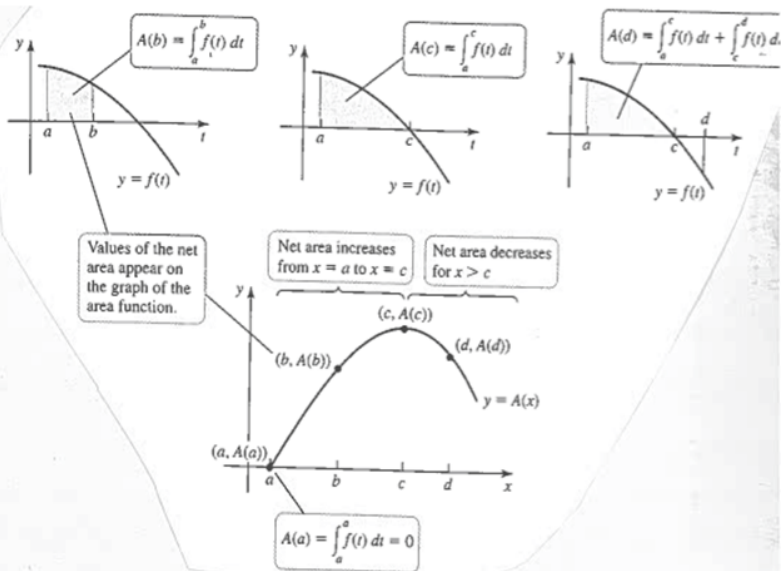
To connect the concepts of differentiation and integration, we first must define the concept of an area function.

Area Functions

Let $y = f(t)$ be a continuous function which is defined for all $t \geq a$, where a is a fixed number. The area function for f with left endpoint at a is given by $A(x) = \int_a^x f(t) dt$.



This gives the net area of the region between the graph of f and the t -axis between the points $t = a$ and $t = x$.

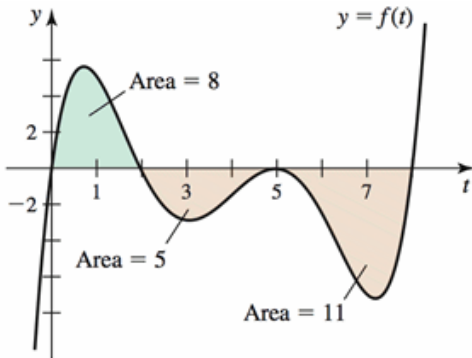


Example

The graph of f is shown below. Let

$$A(x) = \int_0^x f(t) dt \quad \text{and} \quad F(x) = \int_2^x f(t) dt$$

be two area functions for f . Compute $A(2)$, $F(5)$, $A(5)$, $F(8)$.



The Fundamental Theorem of Calculus (Part 1)

Linear functions help to build the rationale behind the Fundamental Theorem of Calculus.

Example

Let $f(t) = 4t + 3$ and define $A(x) = \int_1^x f(t) dt$. What is $A(2)$? $A(4)$? $A(x)$? $A'(x)$?

In general, the property illustrated with this linear function works for all continuous functions and is one part of the FTOC (Fundamental Theorem of Calculus).

Theorem (FTOC I)

If f is continuous on $[a, b]$, then the area function $A(x) = \int_a^x f(t) dt$ for $a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on (a, b) . The area function satisfies $A'(x) = f(x)$; or equivalently,

$$A'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

which means that the area function of f is an antiderivative of f .

The Fundamental Theorem of Calculus (Part 2)

Since A is an antiderivative of f , we now have a way to evaluate definite integrals and find areas under curves.

Theorem (FTOC II)

If f is continuous on $[a, b]$ and F is any antiderivative of f , then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

We use the notation $F(x)|_a^b = F(b) - F(a)$.

Overview of FTC

In essence, to evaluate an integral, we

- Find any antiderivative of f , and call it F .
- Compute $F(b) - F(a)$, the difference in the values of F between the upper and lower limits of integration.

The two parts of the FTC illustrate the inverse relationship between differentiation and integration – the integral “undoes” the derivative.

Example

- (1) Use Part 1 of the FTC to simplify $\frac{d}{dx} \int_x^{10} \frac{dz}{z^2 + 1}$.
- (2) Use Part 2 of the FTC to evaluate $\int_0^\pi (1 - \sin x) \, dx$.
- (3) Compute $\int_1^y h'(p) \, dp$.

Exercise

(1) Simplify $\frac{d}{dx} \int_{3x^4}^4 \frac{t-5}{t^2+1} dt$.

(2) Evaluate $\int_1^5 (x^2 - 4) dx$.

5.3 Book Problems

11-17, 19-57 (odds), 61-67 (odds)