Take-Home Quiz #2

SOLUTIONS

Math 236 (Calc II)
Fall 2017

1. (a) $$\frac{x^3}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

$$?! \longrightarrow = \frac{Ax-2A+Bx-B}{(x-1)(x-2)}$$

no cubic terms on right-hand side of the equation.

For Fun!

Try $$\frac{Ax^2+Bx+C}{x-1}\left(\frac{x-2}{x-2}\right) + \frac{Dx^2+Ex+F}{x-2}\left(\frac{x-1}{x-1}\right)$$

$$\Rightarrow x^3 = Ax^3+Bx^2+Cx-2Ax^2-2Bx-2C+Dx^3+Ex^2+Fx-Dx^2-Ex-F$$

Collect terms:

$$1x^3 = (A+D)x^3$$

$$0x^2 = (B-2A+E-D)x^2$$

$$0x = (C-2B+F-E)x$$

$$0 = -2C-F$$

4 eqns, 6 unknowns $\Rightarrow$ We can choose "easy" numbers for 2 of the unknowns, then solve for the other 4.

Let $\underline{A=1}$:

$$\cancel{1} = \cancel{1} + D \Rightarrow D = 0$$

$$0 = B - 2(1) + E - \cancel{D} \Rightarrow 2 = B+E$$

$$0 = C - 2B + F - E$$

$$0 = -2C - F$$

$\longrightarrow$

Let $\underline{B=1}$:

$2 = 1 + E \Rightarrow \underline{E=1}$

$0 = C - 2(1) + F - E$ $\Rightarrow 3 = C + F \Rightarrow F = 3 - C$

$0 = -2C - F$ $\Rightarrow 0 = -2C - (3 - C)$

$= -2C - 3 + C$

$\Rightarrow \underline{C = -3}, \underline{F = 6}$

So $$\frac{x^3}{(x-1)(x-2)} = \frac{x^2+x-3}{x-1}\left(\frac{x-2}{x-2}\right) + \frac{x+6}{x-2}\left(\frac{x-1}{x-1}\right)$$

$$= \frac{x^3 + \cancel{x^2} - 3x - 2\cancel{x^2} - 2x + \cancel{6} + \cancel{x^2} + 6x - x - \cancel{6}}{(x-1)(x-2)} \checkmark$$

and $$\int \frac{x^3}{(x-1)(x-2)}\,dx = \int \frac{x^2+x-3}{x-1}\,dx + \int \frac{x+6}{x-2}\,dx$$

$u = x - 1$ $\quad$ $w = x - 2$

$\Rightarrow x = u + 1$ $\quad$ $\Rightarrow x = w + 2$

$$= \int \frac{(u+1)^2 + (u+1) - 3}{u}\,du + \int \frac{(w+2)+6}{w}\,dw$$

$$= \int \frac{u^2 + 2u + 1 + u + 1 - 3}{u}\,du + \int \frac{w+8}{w}\,dw$$

$$= \int \left(u + 3 - \frac{1}{u}\right) du + \int \left(1 + \frac{8}{w}\right) dw$$

$$= \frac{u^2}{2} + 3u - \ln|u| + w + 8\ln|w| + C$$

$$= \frac{(x-1)^2}{2} + 3(x-1) - \ln|x-1| + (x-2) + 8\ln|x-2| + C$$

$$= \frac{x^2}{2} - \cancel{\frac{2x}{2}} + \frac{1}{2} + 3x - 3 - \ln|x-1| + \cancel{x} - 2$$

$$+ 8\ln|x-2| + C$$

$$\boxed{= \frac{x^2}{2} + 3x - \ln|x-1| + 8\ln|x-2| + \frac{1}{2} - 3 - 2 + C}$$

$+C$

So you don't technically need long division for this integral.

However!

$$\begin{array}{r} x + 3 \\ (x-1)(x-2) \overline{\smash{)}\, x^3 + 0x^2 + 0x + 0} \\ x^2 - 3x + 2 \quad -(x^3 - 3x^2 + 2x) \\ \hline 3x^2 - 2x + 0 \\ -(3x^2 - 9x + 6) \\ \hline 7x - 6 \end{array}$$

$$\Rightarrow \frac{x^3}{(x-1)(x-2)} = x + 3 + \frac{7x-6}{(x-1)(x-2)} \longrightarrow = \frac{A}{x-1}\left(\frac{x-2}{x-2}\right) + \frac{B}{x-2}\left(\frac{x-1}{x-1}\right)$$

$$= \frac{Ax - 2A + Bx - B}{(x-1)(x-2)}$$

$$\Rightarrow A + B = 7 \Rightarrow A = 7 - B$$

$$-2A - B = -6$$

$$-2(7-B) - B = -6$$

$$-14 + 2B - B = -6$$

$$\Rightarrow \underline{B = 8}, \ \underline{A = -1}$$

$$\Rightarrow \int \frac{x^3}{(x-1)(x-2)}\,dx = \int (x+3)\,dx + \int \frac{-1}{x-1}\,dx + \int \frac{8}{x-2}\,dx$$

$$\boxed{= \frac{x^2}{2} + 3x - \ln|x-1| + 8\ln|x-2| + C}$$

Long-division made it faster.

(b) $\dfrac{1}{(x^2+1)^2} = \dfrac{A}{x^2+1} + \dfrac{B}{(x^2+1)^2}$

$$= \frac{A(x^2+1)+B}{(x^2+1)^2} = \frac{Ax^2+A+B}{(x^2+1)^2}$$

?! nothing happens

$\Rightarrow 0x^2 = A \Rightarrow A=0$

$1 = \cancel{A}+B$

$\Rightarrow B=1$

If you're curious: What about

$$\frac{1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \ ?$$

$$= \frac{(Ax+B)(x^2+1)+Cx+D}{(x^2+1)^2} = \frac{Ax^3+Bx^2+Ax+B+Cx+D}{(x^2+1)^2}$$

$\Rightarrow 0x^3 = Ax^3 \rightarrow \underline{A=0}$

$0x^2 = Bx^2 \rightarrow \underline{B=0}$

$0x = (\cancel{A}+C)x \Rightarrow \underline{C=0}$

$1 = \cancel{B}+D \Rightarrow \underline{D=1}$

In other words, $\dfrac{1}{(x^2+1)^2}$ is the best one can do with partial fractions.

2. (a) $\displaystyle\int \frac{e^x}{e^{3x}-2e^{2x}}\,dx$ Let $u=e^x \Rightarrow du=e^x\,dx$

$$= \int \frac{1}{u^3-2u^2}\,du = \int \frac{du}{u^2(u-2)}$$

$$\frac{1}{u^2(u-2)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-2}$$

$$= \frac{Au(u-2)+B(u-2)+Cu^2}{u^2(u-2)} = \frac{Au^2-2Au+Bu-2B+Cu^2}{u^2(u-2)}$$

Solve: $0 = A+C \Rightarrow A=-C$

$0 = -2A+B = -2(-C)+B \Longrightarrow B=-2C$

$1 = -2B = -2(-2C)$

$\Rightarrow C=\frac{1}{4}$, $B=\frac{-1}{2}$, $A=-\frac{1}{4}$

$$= \int \frac{-\frac{1}{4}}{u}\,du + \int \frac{-\frac{1}{2}}{u^2}\,du + \int \frac{\frac{1}{4}}{u-2}\,du$$

$$= -\frac{1}{4}\ln|u| - \frac{1}{2}(-u^{-1}) + \frac{1}{4}\ln|u-2| + C$$

$$= -\frac{1}{4}\ln|e^x| - \frac{1}{2}(-e^{-x}) + \frac{1}{4}\ln|e^x-2| + C$$

$$= \boxed{-\frac{1}{4}x + \frac{1}{2e^x} + \frac{1}{4}\ln|e^x-2| + C}$$

(b) $\displaystyle\int \frac{\ln x + 1}{x((\ln x)^2 - 4)}\,dx$ Let $u = \ln x \Rightarrow du = \dfrac{dx}{x}$

$$= \int \frac{u+1}{u^2-4}\,du = \int \frac{u+1}{(u-2)(u+2)}\,du$$

$$\frac{A}{u-2}\left(\frac{u+2}{u+2}\right) + \frac{B}{u+2}\left(\frac{u-2}{u-2}\right)$$

$$= \frac{Au + 2A + Bu - 2B}{(u-2)(u+2)}$$

$\Longrightarrow A + B = 1 \Longrightarrow B = 1 - A$

$2A - 2B = 1$

$2A - 2(1-A) = 1$

$2A - 2 + 2A = 1$

$4A = 3 \Rightarrow \underline{A = \frac{3}{4}}$, $\underline{B = \frac{1}{4}}$

$$= \frac{3}{4}\int \frac{du}{u-2} + \frac{1}{4}\int \frac{du}{u+2}$$

$$= \frac{3}{4}\ln|u-2| + \frac{1}{4}\ln|u+2| + C$$

$$\boxed{= \frac{3}{4}\ln\left|\ln x - 2\right| + \frac{1}{4}\ln\left|\ln x + 2\right| + C}$$

3. (a) First find the relevant Pythagorean identity:

$$\frac{\sin^2 x + \cos^2 x = 1}{\sin^2 x} \longrightarrow 1 + \cot^2 x = \csc^2 x$$

So $\cot^2 x = \csc^2 x - 1$. We also have $\frac{d}{dx}\cot x = -\csc^2 x$

and $\frac{d}{dx}\csc x = -\cot x \csc x$

$\longrightarrow$

There are no $\csc x$'s so we will never be able to factor out only one at a time. However, we can factor out two at a time and use a substitution $u = \cot x$.

$$\int \cot^k x\,dx = \int (\csc^2 x - 1)\cot^{k-2} x\,dx$$

$$= \underbrace{\int \csc^2 x \cot^{k-2} x\,dx}_{u=\cot x} - \int \cot^{k-2} x\,dx = \int \csc^2 x \cot^{k-4} x\,dx - \int \cot^{k-4} x\,dx \ \ldots$$

The last term will be

$$\pm \int \cot x\,dx = \pm \int \frac{\cos x}{\sin x}\,dx \quad \text{Put } w = \sin x, \ dw = \cos x\,dx$$

$$= \pm \ln|\sin x| + C$$

$$\Rightarrow \int \cot^k x\,dx = -\frac{\cot^{k-1} x}{k-1} + \frac{\cot^{k-3} x}{k-3} - \ldots \pm \frac{\cot^3 x}{3} \mp \ln|\sin x| + C$$

(k odd and $k > 1$)

~~Naive~~ Wheeler Way: $\int \cot^k x\,dx = \int \frac{\cos^k x}{\sin^k x}\,dx = \int \frac{(1-\sin^2 x)^{\frac{k-1}{2}} \cos x\,dx}{\sin^k x}$

Let $u = \sin x$, $du = \cos x\,dx$

$$= \int \frac{(1-u^2)^{\frac{k-1}{2}}}{u^k}\,du$$

Two Options: Multiply the numerator out $\longrightarrow$

or use a trig-sub.

MethodOne: $(A+B)^n = \sum_{m=0}^{n}\binom{n}{m}A^m B^{n-m}$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

from

$$\Rightarrow \int \frac{(1-u^2)^{\frac{k-1}{2}}}{u^k}\,du$$

$k-1-2m$

$$= \int \frac{\sum_{m=0}^{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{m}(-u^2)^{\frac{k-1}{2}-m}}{u^k}\,du = \sum_{m=0}^{\frac{k-1}{2}}\binom{\frac{k-1}{2}}{m}(-1)^{\frac{k-1}{2}-m}\int u^{-1-2m}\,du$$

$$= \pm\ln|\sin x| + \sum_{m=1}^{\frac{k-1}{2}}\binom{\frac{k-1}{2}}{m}(-1)^{\frac{k-1}{2}-m}\left(\frac{-u^{-2m}}{2m}\right)+C$$

$$= \pm\ln|\sin x| - \sum_{m=1}^{\frac{k-1}{2}}\binom{\frac{k-1}{2}}{m}\frac{(-1)^{\frac{k-1}{2}-m}}{2m}\csc^{2m}x + C$$

$$= -\sum_{m=1}^{\frac{k-1}{2}}\binom{\frac{k-1}{2}}{m}\frac{(-1)^{\frac{k-1}{2}-m}}{2m}\underbrace{(\cot^2 x+1)^m}_{\text{use Pascal's }\Delta\text{ again}} + C \pm \ln|\sin x|$$

$$\boxed{= -\sum_{m=1}^{\frac{k-1}{2}}\binom{\frac{k-1}{2}}{m}\frac{(-1)^{\frac{k-1}{2}-m}}{2m}\left(\sum_{l=0}^{m}\binom{m}{l}\cot^{2l}x\right) + C \pm \ln|\sin x|}$$

Which is "clearly" the same answer

~~Method Two: Let $w = \arcsin u$~~

$$\cancel{\Rightarrow u = \sin w \longrightarrow du = \cos w \, dw}$$

$$\cancel{\Rightarrow \int \frac{(1-u^2)^{\frac{k-1}{2}}}{u^k} du = \int \frac{(1-\sin^2 w)^{\frac{k-1}{2}}}{\sin^k w} \cos w \, dw}$$

~~... wait a minute, that's the same thing.~~

(b) $\int \csc^2 x \, dx = \boxed{-\cot x + C}$

(c) If m is odd, write

$$\int \sin^m x \cos^n x \, dx = \int \sin x \left(1 - \cos^2 x\right)^{\frac{m-1}{2}} \cos^n x \, dx,$$

then let $u = \cos x$.

If n is odd, write

$$\int \sin^m x \cos^n x \, dx = \int \sin^m x \left(1 - \sin^2 x\right)^{\frac{m-1}{2}} \cos x \, dx,$$

then let $u = \sin x$.

(d) Write $\int \sec^m x \tan^n x \, dx = \int \sec^2 x \left(1 + \tan^2 x\right)^{\frac{m-2}{2}} \tan^n x \, dx,$

then let $u = \tan x$.

(e) There is no way to get an expression with exactly one $\tan x$ nor exactly one $\sec^2 x$.

$\longrightarrow$

Instead, write

$$\int \sec^m x \tan^n x\,dx = \int \sec^m x \underbrace{\left(\sec^2 x - 1\right)^{\frac{n}{2}}}_{\sum_{k=0}^{\frac{n}{2}} \binom{\frac{n}{2}}{k} (\sec^{2k} x)(-1)^{\frac{n}{2}-2k}} dx$$

= an alternating sum of integrals of odd powers of $\sec x$ with binomial coefficients.

For each $k = 0, 1, 2, \ldots, \frac{n}{2}$:

$\int \sec^{m+2k} x\,dx$: Use integration by parts with

$$u = \sec^{m+2(k-1)} x \quad dv = \sec^2 x \Rightarrow v = \tan x$$

$$\Rightarrow du = (m+2k-2)\sec^{m+2(k-1)} x \tan x\,dx$$

$$= \sec^{m+2(k-1)} x \tan x - (m+2k-2)\int \sec^{m+2(k-1)} x \tan^2 x\,dx$$

$$= \int \sec^{m+2(k-1)} x \left(\sec^2 x - 1\right) dx$$

$$= \int \sec^{m+2k} x\,dx - \int \sec^{m+2(k-1)} x\,dx$$

Solve for $\int \sec^{m+2k} x\,dx$:

$$\left(1 + (m+2k-2)\right)\int \sec^{m+2k} x\,dx = \sec^{m+2(k-1)} x \tan x + (m+2k-2)\underbrace{\int \sec^{m+2(k-1)} x\,dx}_{\text{⟨repeat strategy⟩}}$$

4. (a) $\displaystyle\int_{-500}^{500} \sqrt{1 + \left(2(0.025x)(0.025)\right)^2}\, dx$

$\displaystyle = 2(0.025)^2 \int_{-500}^{500} \sqrt{\frac{1}{4(0.025)^4} + x^2}\, dx$

(b) Let $u = \arctan\left(2(0.025)^2 x\right)$

$\Rightarrow x = \dfrac{1}{2(0.025)^2} \tan u \;,\; dx = \dfrac{1}{2(0.025)^2} \sec^2 u \, du$

$\displaystyle = \cancel{2(0.025)^2} \int_{x=-500}^{x=500} \sqrt{\frac{\sec^2 u}{4(0.025)^4}} \left(\frac{1}{\cancel{2(0.025)^2}} \sec^2 u \, du \right)$

$\displaystyle = \frac{1}{2(0.025)^2} \int_{x=-500}^{x=500} \sec^3 u \, du \leftarrow$ Integration-by-Parts:

$w = \sec u \Rightarrow dw = \sec u \tan u \, du$

$dv = \sec^2 u \Rightarrow v = \tan u$

$\displaystyle \int \sec^3 u \, du = \sec u \tan u - \underbrace{\int \tan u (\sec u \tan u \, du)}$

$\displaystyle \int \tan^2 u \sec u \, du$

$\displaystyle = \int (\sec^2 u - 1) \sec u \, du$

$\longrightarrow$

Solve for $\int \sec^3 u\,du$:

$$\int \sec^3 u\,du = \sec u \tan u - \int \sec^3 u\,du + \int \sec u\,du$$

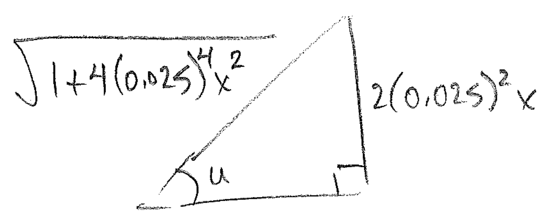
$$\Rightarrow \int \sec^3 u\,du = \frac{1}{2}\left(\sec u \tan u + \ln\left|\sec u + \tan u\right|\right) \quad (+c)$$

Cable Length is:

$$\frac{1}{2(0.025)^2} \cdot \frac{1}{2}\left(\sec u \tan u + \ln\left|\sec u + \tan u\right|\right)\Bigg|_{x=-500}^{x=500}$$

$$= \frac{1}{4(0.025)^2}\Bigg(\sec\left(\arctan\left(2(0.025)^2 x\right)\right)\tan\left(\arctan\left(2(0.025)^2 x\right)\right)$$

$$+ \ln\left|\sec\left(\arctan\left(2(0.025)^2 x\right)\right) + \tan\left(\arctan\left(2(0.025)^2 x\right)\right)\right|\Bigg)\Bigg|_{x=-500}^{x=500}$$



$$= \frac{1}{4(0.025)^2}\Bigg(\left(\sqrt{1+4(0.025)^4 x^2}\right)\left(2(0.025)^2 x\right) + \ln\left|\sqrt{1+4(0.025)^4 x^2} + 2(0.025)^2 x\right|\Bigg)\Bigg|_{-500}^{500}$$

400 $\quad \frac{1}{800^2}$ $\quad \frac{1}{800}$

$\longrightarrow$

$$= 400\left(\sqrt{1+\frac{500^2}{800^2}}\left(\frac{500}{800}\right) + \ln\left|\sqrt{1+\frac{500^2}{800^2}} + \frac{500}{800}\right|\right)$$

$$-400\left(\sqrt{1+\frac{(-500)^2}{800^2}}\left(\frac{-500}{800}\right) + \ln\left|\sqrt{1+\frac{(-500)^2}{800^2}} + \frac{-500}{800}\right|\right)$$

$$= 2(400)\sqrt{1+\left(\frac{5}{8}\right)^2}\left(\frac{5}{8}\right) + 400\ln\left(\frac{\sqrt{1+\left(\frac{5}{8}\right)^2}+\frac{5}{8}}{\sqrt{1+\left(\frac{5}{8}\right)^2}-\frac{5}{8}}\right)$$

$$\boxed{\approx 1061.7 \text{ ft}}$$

5. The domain of $\tan u$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$, and its range is $(-\infty, \infty)$, so $x = \tan u$ is a viable substitution. Write

$$\begin{aligned} x^2 + a^2 &= (a\tan u)^2 + a^2 \\ &= a^2\tan^2 u + a^2 \\ &= a^2(\tan^2 u + 1) \\ &= a^2\sec^2 u. \end{aligned}$$

(typo on the assignment page)

No absolute values are needed because no radical signs are involved.