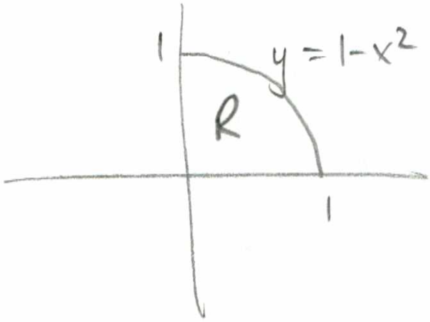
THQ6 SOLUTIONS

Math 2574 (Cal III)
Spring 2017

1. $\iint_R (2-y-1)\,dA$



$$=\int_0^1\int_0^{1-x^2} \underbrace{(2-y-1)}_{1-y}\,dy\,dx$$

$$=\int_0^1 \left(y-\frac{y^2}{2}\right)\Bigg|_0^{1-x^2} dx = \int_0^1 \underbrace{\left(1-x^2-\frac{\overbrace{(1-x^2)^2}^{1-2x^2+x^4}}{2}\right)}_{1-\cancel{x^2}-\frac{1}{2}-\frac{x^4}{2}+\cancel{x^2}} dx$$

terms vanish

$$=\int_0^1 \frac{1}{2}(1-x^4)\,dx$$

$$=\frac{1}{2}\left(x-\frac{x^5}{5}\Bigg|_0^1\right)=\frac{1}{2}\underbrace{\left(1-\frac{1}{5}\right)}^{\frac{4}{5}}=\boxed{\frac{2}{5}}$$

2. $$\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin\sqrt{yz}}{x^{3/2}}\, dy\, dx\, dz$$

$$= \int_0^{\pi^2} \int_1^4 \sin\sqrt{yz} \int_z^{4z} x^{-3/2}\, dx\, dz\, dy$$

$$= \int_0^{\pi^2} \int_1^4 \sin\sqrt{yz} \left(\frac{-2}{\sqrt{x}}\right)\Bigg|_z^{4z} dz\, dy = \int_0^{\pi^2} \int_1^4 \sin\sqrt{yz} \underbrace{\left(\frac{-2}{\sqrt{4z}} - \left(\frac{-2}{\sqrt{z}}\right)\right)}_{\frac{1}{\sqrt{z}}} dz\, dy$$

Put $u = \sqrt{yz}$

$du = \sqrt{y}\,\frac{1}{2} z^{-1/2}\, dz$

$= \frac{1}{2}\sqrt{\frac{y}{z}}\, dz$

$\Rightarrow \frac{2}{\sqrt{y}}\, du = \frac{1}{\sqrt{z}}\, dz$

$$= \int_0^{\pi^2} \int_1^4 \frac{\sin\sqrt{yz}}{\sqrt{z}}\, dz\, dy$$

$$= \int_0^{\pi^2} \frac{2}{\sqrt{y}} \int_{\sqrt{y}}^{\sqrt{4y}} \sin u\, du\, dy$$

$$= \int_0^{\pi^2} \frac{2}{\sqrt{y}} \left(-\cos\sqrt{4y} + \cos\sqrt{y}\right) dy$$

$$= 4\int_{\sqrt{0}}^{\sqrt{\pi^2}} (-\cos 2w + \cos w)\, dw$$

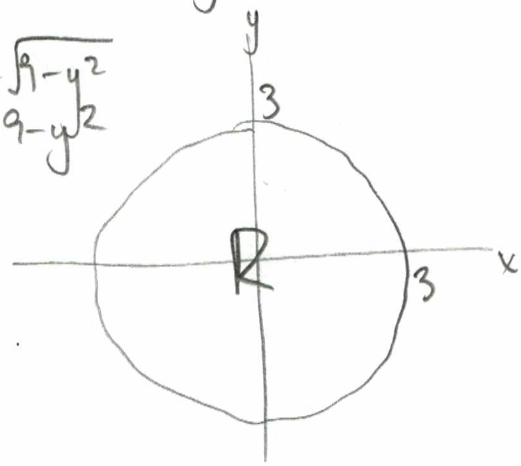
$$\left(\begin{aligned} &\text{Put } w = \sqrt{y} \\ &\Rightarrow dw = \frac{1}{2\sqrt{y}}\, dy \\ &2\, dw = \frac{dy}{\sqrt{y}} \end{aligned}\right)$$

$$= 4\left(-\frac{\sin 2w}{2} + \sin w\right)\Bigg|_0^{\pi} \boxed{= 0}$$

3. $$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^{9-3\sqrt{x^2+y^2}} dz\,dx\,dy = \iint_R \left(\int_0^{9-3r} dz \right) dA$$

$x = \pm\sqrt{9-y^2}$

$x^2 = 9-y^2$



$$= \iint_R (9-3r)\,dA$$

$$= \int_0^{2\pi}\int_0^3 \underbrace{(9-3r)r}_{9r-3r^2}\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{9r^2}{2} - r^3 \right)\Big|_0^3 d\theta$$

$$= \int_0^{2\pi} \underbrace{\left(\frac{9(9)}{2} - 27 \right)}_{\frac{81-54}{2} = \frac{27}{2}} d\theta$$

$$= \frac{27}{2}(2\pi) = \boxed{27\pi}$$

4. $$\int_0^{2\pi}\int_0^{\pi}\int_0^{1+\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\varphi \left(\frac{\rho^3}{3} \Bigg|_0^{1+\cos\varphi} \right) d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\sin\varphi}{3} \underbrace{(1+\cos\varphi)^3}_{1+3\cos\varphi+3\cos^2\varphi+\cos^3\varphi} d\varphi \, d\theta$$

Put $u = \cos\varphi$

$du = -\sin\varphi \, d\varphi$

$-du = \sin\varphi \, d\varphi$

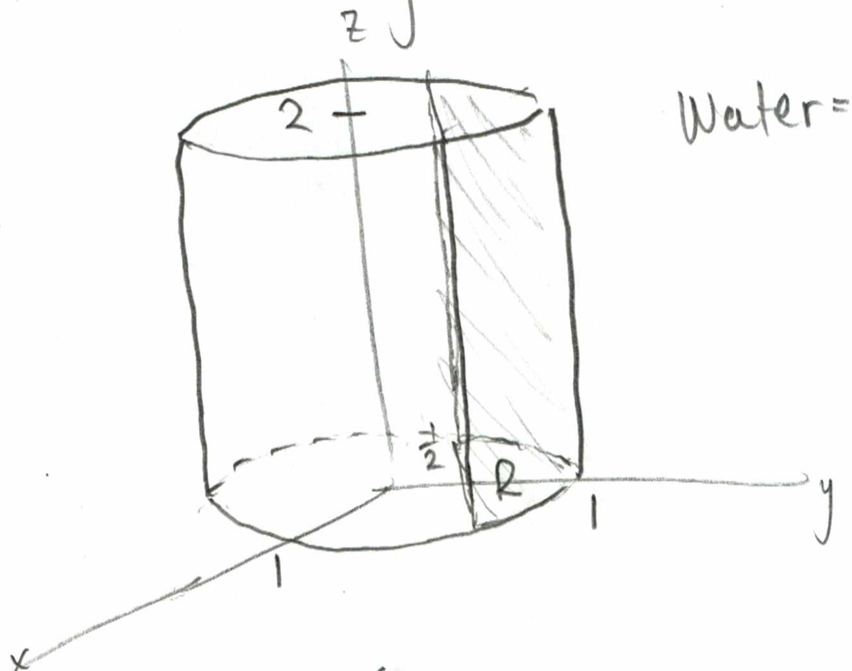
$$= -\frac{1}{3}\int_0^{2\pi}\int_{\cos 0}^{\cos\pi} \underbrace{(1+u)^3}_{1+3u+3u^2+u^3} du \, d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi} \left(u + \frac{3}{2}u^2 + u^3 + \frac{u^4}{4}\right)\Bigg|_1^{-1} d\theta$$

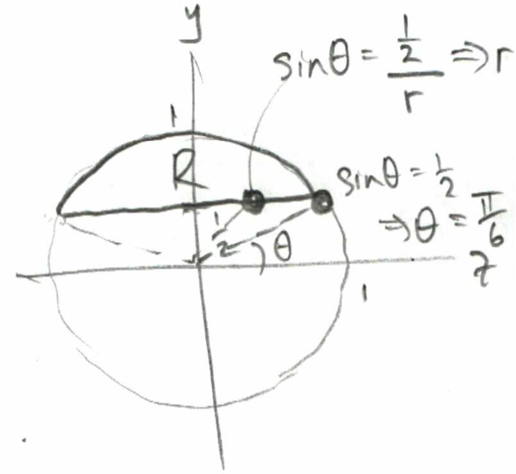
$$= -\frac{1}{3}\left[\left(\underbrace{-1}_{-1} + \cancel{\frac{3}{2}(-1)^2} + \underbrace{(-1)^3}_{-1} + \cancel{\frac{(-1)^4}{4}}\right) - \underbrace{\left(1 + \cancel{\frac{3}{2}} + 1 + \cancel{\frac{1}{4}}\right)}_{-(1+1)}\right] \cdot 2\pi$$

$$= -\frac{1}{3}(-4)(2\pi) = \boxed{\frac{8}{3}\pi}$$

5. Use strategic coordinates:



$$\text{Water} = \iint_R \left(\int_0^2 dz \right) dA$$



$\sin\theta = \frac{1/2}{r} \Rightarrow r = \frac{\csc\theta}{2}$

$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

$$\Rightarrow \text{Water} = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{\frac{\csc\theta}{2}}^{1} 2r\,dr\,d\theta$$

$$= \cancel{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{r^2}{\cancel{2}} \Bigg|_{\frac{\csc\theta}{2}}^{1} d\theta = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(1^2 - \frac{\csc^2\theta}{4} \right) d\theta$$

$$= \theta + \frac{1}{4}\cot\theta \Bigg|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$= \frac{5\pi}{6} + \frac{1}{4}(-\sqrt{3}) - \left(\frac{\pi}{6} + \frac{1}{4}\sqrt{3} \right)$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

$$\boxed{\approx 1.228\ ft^3}$$