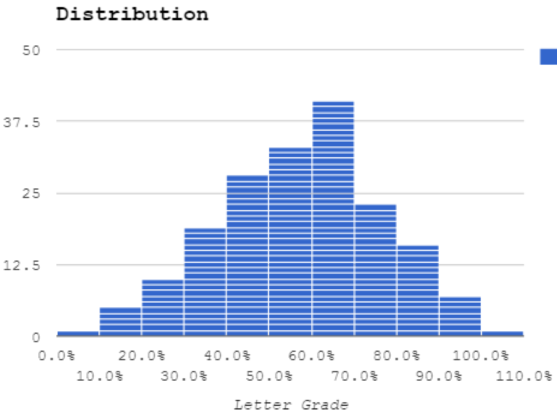
# Fri 11 Mar

- Exam 2: Curve, etc. is posted.



# Fri 11 Mar (cont.)

- Midterm: expect it back next week in drill. Don't expect a curve. :(
- "Fast Track Calculus": Dr. Kathleen Morris will be teaching a second 8 weeks Calculus One class. "If you have a student who is maybe doing poorly because of illness or a tragic event in their life during the beginning of the semester, this might be an opportunity for a new start for them." The class requires departmental consent so the student will need to contact Kathleen to get permission to enroll.

# §3.10 Derivatives of Inverse Trigonometric Functions

**Recall:** If $y = f(x)$, then $f^{-1}(x)$ is the value of $y$ such that $x = f(y)$.

Example

If $f(x) = 3x + 2$, then what is $f^{-1}(x)$?

**NOTE:** $f^{-1}(x) \neq f(x)^{-1} \left(= \frac{1}{f(x)}\right)$

To avoid this confusion, we use $\arcsin x$, $\arccos x$ $\arctan x, \ldots$ to denote inverse trig functions.

Trig functions are functions, too. Just like with "$f$", there has to be something to "plug in". It makes no sense to just say $\sin$, without having $\sin($*something*$)$.

$$y = \sin^{-1} x \iff x = \sin y$$

The derivative of $y = \sin^{-1} x$ can be found using implicit differentiation:

$$\begin{aligned} x &= \sin y \\ \frac{d}{dx}(x) &= \frac{d}{dx}(\sin y) \\ 1 &= (\cos y)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\cos y} \end{aligned}$$

We still need to replace $\cos y$ with an expression in terms of $x$. We use the trig identity $\sin^2 y + \cos^2 y = 1$ (careful with notation: in this case we mean $(\sin y)^2 + (\cos y)^2 = 1$). Then

$$\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - x^2}.$$

The range of $y = \sin^{-1} x$ is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. In this range, cosine is never negative, so we can just take the positive portion of the square root. Therefore,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}} \implies \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.$$

## Exercise

Compute the following:

1. $\dfrac{d}{dx}\left(\sin^{-1}(4x^2-3)\right)$
2. $\dfrac{d}{dx}\left(\cos(\sin^{-1} x)\right)$

# Derivative of Inverse Tangent

Similarly to inverse sine, we can let $y = \tan^{-1} x$ and use implicit differentiation:

$$\begin{aligned}
x &= \tan y \\
\frac{d}{dx}(x) &= \frac{d}{dx}(\tan y) \\
1 &= (\sec^2 y)\frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{1}{\sec^2 y}
\end{aligned}$$

Use the trig identity $\sec^2 y - \tan^2 y = 1$ to replace $\sec^2 y$ with $1 + x^2$:

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$$

# Derivative of Inverse Secant

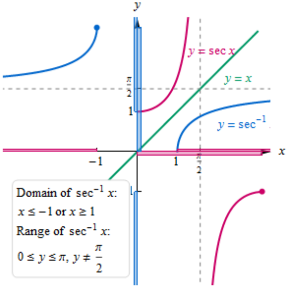
Again, use the same method as with inverse sine:

$$
\begin{aligned}
y &= \sec^{-1} x \\
x &= \sec y \\
\frac{d}{dx}(x) &= \frac{d}{dx}(\sec y) \\
1 &= \sec y \tan y \frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{1}{\sec y \tan y}
\end{aligned}
$$

Use the trig identity $\sec^2 y - \tan^2 y = 1$ again to get

$$\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}.$$

This time, the $\pm$ matters:




- If $x \geq 1$, then $0 \leq y < \frac{\pi}{2}$ and so $\tan y > 0$.
- If $x \leq -1$, then $\frac{\pi}{2} < y \leq \pi$ and so $\tan y < 0$.

Therefore,

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}.$$

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Using other trig identities (which you do not need to prove)

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2} \quad \cot^{-1} x + \tan^{-1} x = \frac{\pi}{2} \quad \csc^{-1} x + \sec^{-1} x = \frac{\pi}{2}$$

we can get the rest of the inverse trig derivatives.

## All Other Inverse Trig Derivatives

To summarize:

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$$
$$(-1 < x < 1)$$
$$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$
$$(-\infty < x < \infty)$$
$$\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{|x|\sqrt{x^2-1}}$$
$$(|x| > 1)$$

## Example

Compute the derivatives of $f(x) = \tan^{-1}\left(\frac{1}{x}\right)$ and $g(x) = \sin\left(\sec^{-1}(2x)\right)$.

# Derivatives of Inverse Functions in General

Let $f$ be differentiable and have an inverse on an interval $I$. Let $x_0$ be a point in $I$ at which $f'(x_0) \neq 0$. Then $f^{-1}$ is differentiable at $y_0 = f(x_0)$ and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}$$

where $y_0 = f(x_0)$.

**Example**

Let $f(x) = 3x + 4$. Find $f^{-1}(x)$ and $\left(f^{-1}\right)'\left(\frac{1}{3}\right)$.

## 3.10 Book Problems

7-33 (odds), 37-41 (odds)

# §3.11 Related Rates

In this section, we use our knowledge of derivatives to examine how variables change with respect to time.

The prime feature of these problems is that two or more variables, which are related in a known way, are themselves changing in time.

The goal of these types of problems is to determine the rate of change (i.e., the derivative) of one or more variables at a specific moment in time.

## Problem

The edges of a cube increase at a rate of 2 cm/sec. How fast is the volume changing when the length of each edge is 50 cm?

- **Variables:** $V$ (Volume of the cube) and $x$ (length of edge)
- **How Variables are related:** $V = x^3$
- **Rates Known:** $\dfrac{dx}{dt} = 2$ cm/sec
- **Rate We Seek:** $\dfrac{dV}{dt}$ when $x = 50$ cm

Note that both $V$ and $x$ are functions of $t$ (their respective sizes are dependent upon how much time has passed).

So we can write $V(t) = x(t)^3$ and then differentiate this with respect to $t$:

$$V'(t) = 3x(t)^2 \cdot x'(t).$$

**Note that $x(t)$ is the length of the cube's edges at time $t$, and $x'(t)$ is the rate at which the edges are changing at time $t$.**

We can rewrite the previous equation as

$$\frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt}.$$

So the rate of change of the volume when $x = 50$ cm is

$$\left.\frac{dV}{dt}\right|_{x=50} = 3 \cdot 50^2 \cdot 2 = 15000 \text{ cm}^3/\text{sec}.$$

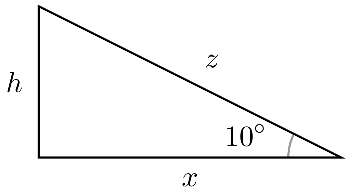
## Steps for Solving Related Rates Problems

1. Read the problem carefully, making a sketch to organize the given information. Identify the rates that are given and the rate that is to be determined.
2. Write one or more equations that express the basic relationships among the variables.
3. Introduce rates of change by differentiating the appropriate equation(s) with respect to time $t$.
4. Substitute known values and solve for the desired quantity.
5. Check that the units are consistent and the answer is reasonable.

## The Jet Problem

A jet ascends at a $10^\circ$ angle from the horizontal with an airspeed of 550 miles/hr (its speed along its line of flight is 550 miles/hr). How fast is the altitude of the jet increasing? If the sun is directly overhead, how fast is the shadow of the jet moving on the ground?

**Step 1:** There are three variables: the distance the shadow has traveled ($x$), the altitude of the jet ($h$), and the distance the jet has actually traveled on its line of flight ($z$). We know that $\frac{dz}{dt} = 550$ miles/hr and we want to find $\frac{dx}{dt}$ and $\frac{dh}{dt}$. We also see that these variables are related through a right triangle:



**Step 2:** To answer how fast the altitude is increasing, we need an equation involving only $h$ and $z$. Using trigonometry,

$$\sin(10^\circ) = \frac{h}{z} \implies h = \sin(10^\circ) \cdot z.$$

To answer how fast the shadow is moving, we need an equation involving only $x$ and $z$. Using trigonometry,

$$\cos(10^\circ) = \frac{x}{z} \implies x = \cos(10^\circ) \cdot z.$$

**Step 3:** We can now differentiate each equation to answer each question:

$$h = \sin(10^\circ) \cdot z \implies \frac{dh}{dt} = \sin(10^\circ)\frac{dz}{dt}$$
$$x = \cos(10^\circ) \cdot z \implies \frac{dx}{dt} = \cos(10^\circ)\frac{dz}{dt}$$

**Step 4:** We know that $\frac{dz}{dt} = 550$ miles/hr. So

$$\frac{dh}{dt} = \sin(10^\circ) \cdot 550 \approx 95.5 \text{ miles/hr}$$
$$\frac{dx}{dt} = \cos(10^\circ) \cdot 550 \approx 541.6 \text{ miles/hr}$$

**Step 5:** Because both answers are in terms of miles/hr and both answers seem reasonable within the context of the problem, we conclude that the jet is gaining altitude at a rate of 95.5 miles/hr, while the shadow on the ground is moving at about 541.6 miles/hr.

### Example

The sides of a cube increase at a rate of $R$ cm/sec. When the sides have a length of 2 cm, what is the rate of change of the volume?

### Example

Two boats leave a dock at the same time. One boat travels south at 30 miles/hr and the other travels east at 40 miles/hr. After half an hour, how fast is the distance between the boats increasing?

### Exercise

A 13 foot ladder is leaning against a vertical wall when Jack begins pulling the foot of the ladder away from the wall at a rate of 0.5 ft/sec. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 5 ft from the wall?

## 3.11 Book Problems

5-14, 16-19, 21-24, 37-38