Exam 3: Transformations and line integrals (§13.7-14.5)

Exam Instructions: You have 50 minutes to complete this exam. Justification is required for all problems. No electronic devices (phones, iDevices, computers, etc) except for a **basic scientific calculator**. On story problems, round to one decimal place. If you finish early then you may leave, UNLESS there are less than 5 minutes of class left. To prevent disruption, if you finish with less than 5 minutes of class remaining then please stay seated and quiet.

In addition, please provide the following data:

Drill Instructor: _____

Drill Time: _____

Your signature below indicates that you have read this page and agree to follow the Academic Honesty Policies of the University of Arkansas.

Signature: **(1 pt)** _____

Good luck!

Formulas you may need:

Jacobian Determinant of a Transformation of Two Variables

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Jacobian Determinant of a Transformation of Three Variables

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Recall that by expanding about the first row,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$$
$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Scalar Line Integrals in $\mathbb{R}^2$

$$\int_C f\,ds = \int_a^b f(x(t),y(t))|\mathbf{r}'(t)|dt = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

Scalar Line Integrals in $\mathbb{R}^3$

$$\int_C f\,ds = \int_a^b f(x(t),y(t),z(t))|\mathbf{r}'(t)|dt = \int_a^b f(x(t),y(t),z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt.$$

Green's Theorem—Circulation Form

The circulation form of Green's Theorem is also called the *tangential*, or *curl*, form.

$$\underbrace{\oint_C \mathbf{F}\cdot d\mathbf{r}}_{\text{circulation}} = \underbrace{\oint_C f\,dx + g\,dy}_{\text{circulation}} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dA.$$

Area of a Plane Region by Line Integrals

Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

$$\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx).$$

Green's Theorem, Flux Form

The flux form of Green's Theorem is also called the *normal*, or *divergence*, form.

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds}_{\text{outward flux}} = \underbrace{\oint_C f\,dy - g\,dx}_{\text{outward flux}} = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)dA$$

Curl of a Vector Field

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \begin{matrix} \leftarrow \text{Unit vectors} \\ \leftarrow \text{Components of } \nabla \\ \leftarrow \text{Components of } \mathbf{F} \end{matrix}$$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}.$$

Divergence of a Vector Field

$$\text{div }\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

Different Forms of Line Integrals of Vector Fields

The line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ may be expressed in the following forms,

$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_a^b \left(f(t)x'(t) + g(t)y'(t) + h(t)z'(t)\right)dt$$
$$= \int_C f\,dx + g\,dy + h\,dz$$
$$= \int_C \mathbf{F}\cdot d\mathbf{r}.$$

For line integrals in the plane,

$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_a^b \left(f(t)x'(t) + g(t)y'(t)\right)dt = \int_C f\,dx + g\,dy = \int_C \mathbf{F}\cdot d\mathbf{r}.$$

Flux

The **flux** of the vector field $\mathbf{F}$ across C is

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b \left(f(t)y'(t) - g(t)x'(t)\right)dt,$$

where $\mathbf{n} = \mathbf{T}\times\mathbf{k}$ is the unit normal vector and $\mathbf{T}$ is the unit tangent vector consistent with the orientation.

| Question	Points	Score
1	16	
2	16	
3	21	
4	18	
5	16	
6	12	
Total:	99	

1. **(16 pts)** Compute the Jacobian, $J(\rho, \varphi, \theta)$, of the following transformation taking Cartesian to spherical coordinates:

$$x = \rho \sin \varphi \cos \theta \qquad y = \rho \sin \varphi \sin \theta \qquad z = \rho \cos \varphi$$

You must show your work and simplify.

2. **(16 pts)** Use Green's Theorem to find the area inside an ellipse with major and minor axes of length 12 and 8, respectively. In case you need them, the half-angle formulas are $\cos^2 x = \dfrac{1 + \cos 2x}{2}$ and $\sin^2 x = \dfrac{1 - \cos 2x}{2}$.

3. The vector field $\mathbf{F} = \langle x, y^2 \rangle$, the circle C of radius 2 centered at the origin, and two points $P = (-1, 1)$ and $Q = (-1, -1)$, are given in the figure below.
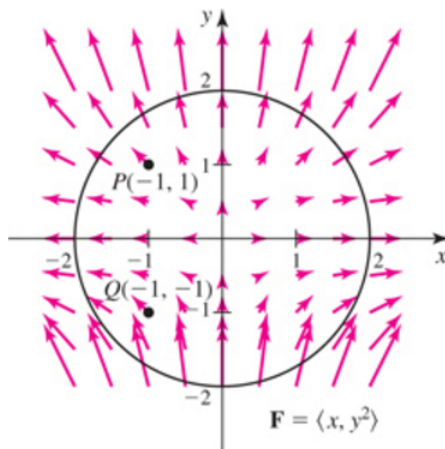
(a) **(5 pts)** Without computing the divergence, does the graph suggest that the divergence is positive or negative at P and Q? Justify your answer.

(b) **(5 pts)** Compute the divergence of $\mathbf{F}$ at P and Q to confirm your answer to part (a).

(c) **(3 pts)** Label on the graph where the flux across C is outward.

(d) **(8 pts)** Is the **net** outward flux across C positive or negative? You must justify your answer.

4. (**pts**) Let $\mathbf{F} = \langle 2xyz, x^2z, x^2y \rangle$.

    (a) (**8 pts**) What is the curl of $\mathbf{F}$?

    (b) (**10 pts**) What is the circulation of $\mathbf{F}$ along C, where C is the closed curve formed by the square whose corners are the points $(1, 1)$, $(-1, 1)$, $(-1, -1)$, and $(1, -1)$?

5. **(16 pts)** Evaluate the scalar line integral $\displaystyle\int_C (x^2 + y^3)\, ds$, where C is the line segment from $(0,0)$ to $(5,5)$.

6. **(12 pts)** Match vector fields (a)-(d) with graphs (A)-(D).

   (a) $\mathbf{F} = \langle 0, x^2 \rangle$

   (b) $\mathbf{F} = \langle x - y, x \rangle$

   (c) $\mathbf{F} = \langle 2x, -y \rangle$

   (d) $\mathbf{F} = \langle y, x \rangle$