Math 2554 Exam 1: Limits **Name:** ______________________________
date

You have 50 minutes to complete this exam. Eyes on your own paper and good luck!

1. **Definitions/Concepts.**
   (a) (2.1) Suppose $s(t)$ is the position of an object moving along a line at a time $t \geq 0$. What is the average velocity between the times $t = a$ and $t = b$? What is the instantaneous velocity at $t = a$? Which one is the slope of the secant line between the points $(a, s(a))$ and $(b, s(b))$ on the graph of s? Which one gives the slope of the line tangent to the graph of s at $(a, s(a))$?
   (b) When $\lim_{x \to a} f(x)$ exists, it always equals $f(a)$. (True or False?)
   (c) (2.5) The end behavior for e^x and e^{-x} (on the entire real line) and $\ln x$ (on the interval $(0, \infty)$) is given by
      i. $\lim_{x \to \infty} e^x =$
      ii. $\lim_{x \to -\infty} e^x =$
      iii. $\lim_{x \to \infty} e^{-x} =$
      iv. $\lim_{x \to -\infty} e^{-x} =$
      v. $\lim_{x \to \infty} \ln x =$
      vi. $\lim_{x \to 0+} \ln x =$

2. (2.4) The function $f(x)$ has a vertical asymptote at the line $x = a$ means at least one of the following conditions holds:
   (a)
   (b)
   (c)

3. (2.6) Suppose a function $f(x)$ is continuous at the point a. Why is it OK to plug in the value $x = a$ when computing $\lim_{x \to a} f(x)$ (Hint: What are the three conditions on the Continuity Checklist?)?

Questions/Problems.

1. (2.1/2.2) Sketch the graph of a function with the given properties. You do not need to find a formula for the function.
$$f(1) = 0, f(2) = 4, f(3) = 6, \lim_{x \to 2^-} f(x) = -3, \lim_{x \to 2^+} f(x) = 5$$

2. The value of $\lim_{x \to 3} \frac{x^2-9}{x-3}$ does not exist. (t/f)

3. The value of $\lim_{x \to a} f(x)$ does not exist if $f(a)$ is undefined. (t/f)

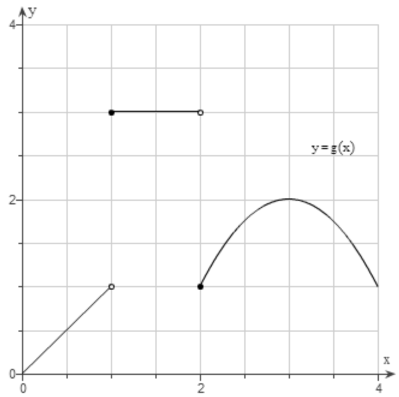

Figure 1: g (Briggs, W. and Cochran, L. *Calculus: Early ranscendentals*)

4. (2.1/2.2) Use the graph of g in the figure to find the following values, if they exist. If a limit does not exist, explain why.
   (a) $g(1)$
   (b) $\lim_{x\to 1} g(x)$
   (c) $\lim_{x\to 2^+} g(x)$
   (d) $\lim_{x\to 1^-} g(x)$
   (e) $g(2)$
   (f) $g(3)$
   (g) $\lim_{x\to 1^+} g(x)$
   (h) $\lim_{x\to 2^-} g(x)$
   (i) $\lim_{x\to 3} g(x)$

5. Using the figure as a guide, explain how the Squeeze Theorem can by used to compute $\lim_{x\to 0} x^2 \sin(\frac{1}{x})$, and then say what the limit is.
6. (CHALLENGE) Suppose a spaceship is traveling at velocity v, relative to an observer. Say the length of the spaceship is L_0. To the observer, the ship appears to have a smaller length, given by the Lorentz contraction formula:

$$\text{length to the observer} = L_0\sqrt{1-\frac{v^2}{c^2}},$$

where c is the speed of light.

   (a) If $v = 0.5c$, i.e., the ship is traveling at half the speed of light, then what is the length of the ship to the observer?
   (b) If the ship is traveling 75% of the speed of light then how long does the ship look to the observer?
   (c) Compute $\lim_{v\to c^-} L_0\sqrt{1-\frac{v^2}{c^2}}$. What is physically interesting about this limit?

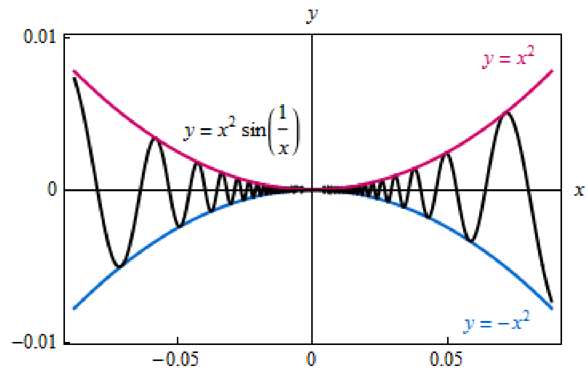

Figure 2: (Briggs, W. and Cochran, L. *Calculus: Early Transcendentals*)

7. (2.3) Suppose $\lim_{x\to 1} f(x) = 4$. What is $\lim_{x\to -1} f(x^2)$? (Hint: There is another limit law that says how to compute this limit.)

8. (2.3/2.4) Suppose $g(x) = \begin{cases} \frac{x-5}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Evaluate

   (a) $\lim_{x\to 0^+} g(x)$. Make sure your justification involves determining the sign of the numerator and denominator for x-values slightly larger than 0.

   (b) $\lim_{x\to 0^-} g(x)$. Make sure your justification involves determining the sign of the numerator and denominator for x-values slightly smaller than 0.

   (c) $\lim_{x\to 0} g(x)$.

   (d) $g(0)$.

9. (2.3/2.4) Suppose $h(x) = \begin{cases} \frac{x-5}{x} & x \lneq 0 \\ 0 & x \geq 0 \end{cases}$. Evaluate

   (a) $\lim_{x\to 0^+} h(x)$.

   (b) $\lim_{x\to 0^-} h(x)$. Make sure your justification involves determining the sign of the numerator and denominator for x-values slightly smaller than 0.

   (c) $\lim_{x\to 0} h(x)$.

   (d) $h(0)$.

   (e) Does h have a vertical asymptote at the line $x = 0$?

10. (2.4) The graph of f in the figure has vertical asymptotes at $x = 1$ and $x = 2$. Find the following limits, if possible. If not possible, then say so and why.

   (a) $\lim_{x\to 1^-} f(x)$

   (b) $\lim_{x\to 1^+} f(x)$

   (c) $\lim_{x\to 1} f(x)$

   (d) $\lim_{x\to 2^-} f(x)$

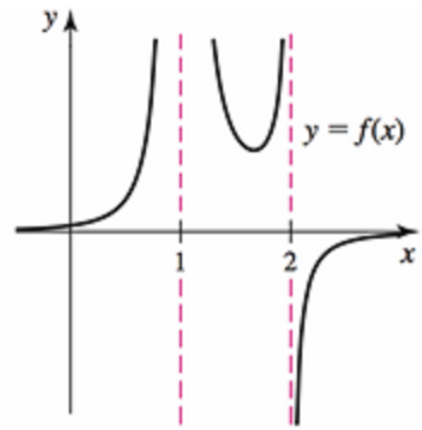

Figure 3: (Briggs, W. and Cochran, L. *Calculus: Early Transcendentals*)

(e) $\lim_{x\to 2^+} f(x)$

(f) $\lim_{x\to 2} f(x)$

11. (2.4) Sketch a possible graph of a function f, together with vertical asymptotes, satisfying all of the following conditions.

    (a) $f(1) = 0$

    (b) $f(3)$ is undefined

    (c) $\lim_{x\to 3} f(x) = 1$

    (d) $\lim_{x\to 0^+} = -\infty$

    (e) $\lim_{x\to 2} f(x) = \infty$

    (f) $\lim_{x\to 4^-} f(x) = \infty$

12. (2.1/3.1) Given the graph of f in the following figures, find the slope of the secant line that passes through $(0,0)$ and $(h, f(h))$, in terms of h, for $h > 0$ and $h < 0$. Then calculate the limit of that slope as $h \to 0^+$ and as $h \to 0^-$. What does this tell you about the tangent line to the curve at $(0,0)$? Why doesn't f itself have a vertical asymptote in either of these cases?

    (a) $f(x) = x^{\frac{1}{3}}$

    (b) $f(x) = x^{\frac{2}{3}}$

13. (up to 2.5) Sketch a possible graph of a function f that satisfies all of the given conditions. Be sure to identify all the vertical and horizontal asymptotes.

    (a) $f(-1) = -2$

    (b) $f(1) = 2$

    (c) $f(0) = 0$

    (d) $\lim_{x\to\infty} f(x) = 1$

    (e) $\lim_{x\to -\infty} f(x) = -1$

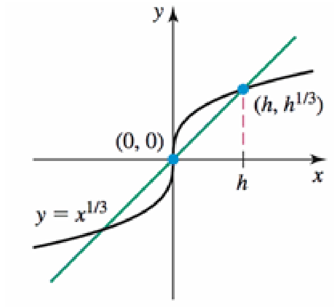

Figure 4: (Briggs, W. and Cochran, L. *Calculus: Early Transcendentals*)

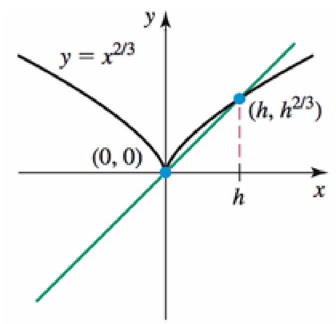

Figure 5: (Briggs, W. and Cochran, L. *Calculus: Early Transcendentals*)

14. (2.4/2.5) For each function $f(x)$, evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, and then identify any horizontal asymptotes. Next, find the vertical asymptotes. For each vertical asymptote $x = a$, evaluate $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$.

    (a) $f(x) = \frac{x^2-4x+3}{x-1}$

    (b) $f(x) = \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

    (c) $f(x) = \frac{x^2-4}{x(x-2)}$

15. (2.6) Does the function
$$f(x) = 2x^5 - 8x^3 + 5x^2 + 3x - 5$$
cross the horizontal line $y = -4$ for some x in the interval $[0, 1]$? (Yes, it does.) Justify your answer, and in particular, mention any important theorems you use and why they apply in this situation.

Computations/Algebra.

1. (2.3) $\lim_{x\to 4}(3x - 7)$

2. (2.3) $\lim_{x\to -5} \pi$

3. (2.3) $\lim_{x\to -5} 2$

4. (2.3) Suppose $\lim_{x\to 1} f(x) = 8, \lim_{x\to 1} g(x) = 3$, and $\lim_{x\to 1} h(x) = 2$. Compute the following limits and state the limit laws used (and why you are allowed to use them in that instance, if there is a caveat) to justify your computations. If the limit does not exist then say so.

   (a) $\lim_{x\to 1} 4f(x)$

   (b) $\lim_{x\to 1} \frac{f(x)g(x)}{h(x)}$

   (c) $\lim_{x\to 1} \sqrt[3]{f(x)g(x)+3}$

   (d) $\lim_{x\to 1}(2x^3 - 3x^2 + 4x + 5)$

   (e) $\lim_{x\to 1} \left(x^2 - x\right)^5$

   (f) $\lim_{x\to 3^-} \sqrt{\frac{x-3}{2-x}}$

   (g) $\lim_{x\to 3} \sqrt{\frac{x-3}{2-x}}$

   (h) $\lim_{x\to 3^+} \sqrt{\frac{x-3}{2-x}}$

   (i) $\lim_{x\to -b} \frac{(x+b)^7+(x+b)^{10}}{4(x+b)}$

   (j) $\lim_{t\to a} \frac{\sqrt{3t+1}-\sqrt{3a-1}}{t-a}$

   (k) $\lim_{x\to 0} \frac{a-\sqrt{a^2-x^2}}{x^2}$

   (l) $\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x}\right)$

   (m) $\lim_{x\to c} \frac{x^2-2cx+c^2}{x-c}$

   (n) $\lim_{x\to \infty} \frac{2x}{x+1}$

   (o) $\lim_{x\to -\infty} \left(5 + \frac{100}{x} + \frac{\sin^4 x^3}{x^2}\right)$

   (p) $\lim_{x\to -\infty} \left(3x^7 + x^2\right)$

   (q) $\lim_{x\to \infty} -12x^{-5}$