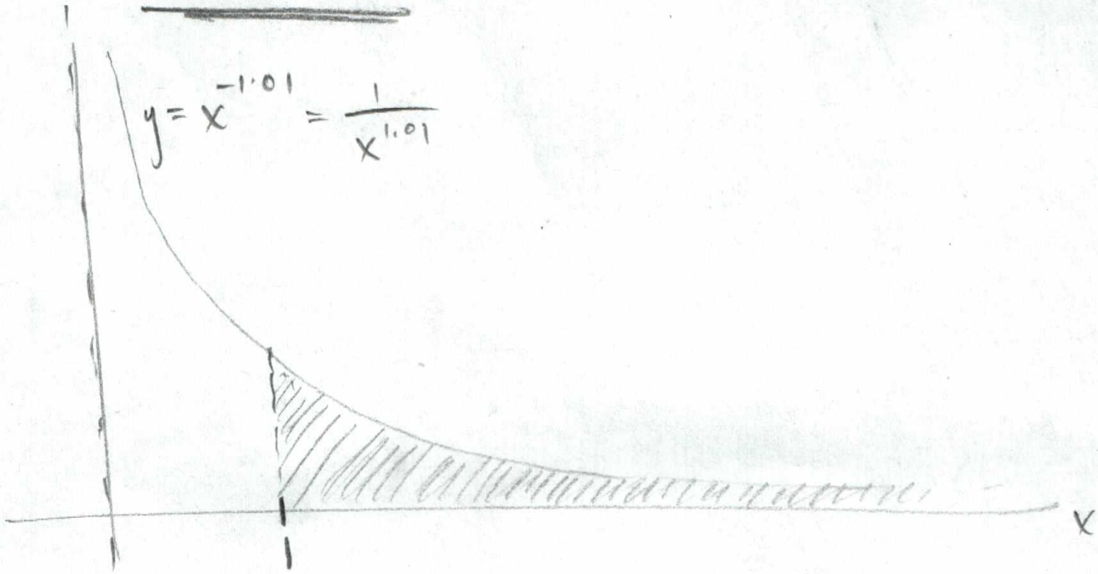
Take-Home Quiz #5

Math 236 (Calc II)

Fall 2017

SOLUTIONS

1.



- $\int_0^1 \frac{1}{x^{1.01}}\,dx = \lim_{A\to 0^+} \left. \frac{x^{-1.01+1}}{-1.01+1} \right|_A^1 = \lim_{A\to 0^+}\left(\frac{1^{-0.01}}{-0.01} - \frac{A^{-0.01}}{-0.01}\right)$

↑ vertical asymptote at $x=0$

$\frac{A^{-0.01}}{-0.01} = \cancel{\frac{-100}{A^{0.01}}} \to -\infty$

$\boxed{= \infty}$

- $\int_1^\infty \frac{1}{x^{1.01}}\,dx = \lim_{B\to\infty} \left. \frac{x^{-0.01}}{-0.01} \right|_1^B = \lim_{B\to\infty}\left(\frac{B^{-0.01}}{-0.01} + \frac{1^{-0.01}}{+0.01}\right)$

$\frac{B^{-0.01}}{-0.01} = \cancel{\frac{-1}{0.01B^{0.01}}}$ 0

$= \frac{1}{0.01} = \boxed{100}$

2. $Q(T) = \int_T^{\infty} k e^{-0.023t}\,dt = k \lim_{B\to\infty} \left.\frac{e^{-0.023t}}{-0.023}\right|_T^B$

$$= k \lim_{B\to\infty} \left(\cancel{\frac{1}{-0.023e^{0.023B}}} \to 0 - \frac{1}{-0.023e^{0.023T}} \right)$$

$$\Rightarrow Q(T) = \frac{k}{0.023e^{0.023T}}$$

(a) "Now" means $T = 12$ years.

$$Q(12) = \frac{k}{0.023e^{0.023(12)}} = 0.95$$

$$\Rightarrow k = 0.95(0.023)e^{0.023(12)}$$

When the tank was first filled, it contained

$$Q(0) = \frac{k}{0.023e^{0.023(0)}}$$

$$= \frac{0.95(\cancel{0.023})e^{0.023(12)}}{\cancel{0.023e^{0.023(0)}}} = 0.95e^{0.023(12)}$$

$\boxed{\approx 1.25 \text{ lb of cesium}}$

(b) For this tank,

$$Q(0) = \frac{k}{0.023e^{0.023(0)}} = 1.5$$

$$\Rightarrow k = 1.5(0.023)$$

Right now, $Q(T) = \dfrac{k}{0.023e^{0.023T}} = 0.85$

$$\frac{k}{0.023(0.85)} = e^{0.023T}$$

$$\ln\left(\frac{k}{0.023(0.85)}\right) = 0.023T$$

$$\Rightarrow T = \frac{\ln\left(\frac{k}{0.023(0.85)}\right)}{0.023}$$

$$= \frac{\ln\left(\frac{1.5(\cancel{0.023})}{\cancel{0.023}(0.85)}\right)}{0.023} = \frac{\ln\left(\frac{1.5}{0.85}\right)}{0.023} \boxed{\approx 25 \text{ years ago}}$$

3. $3.454545\ldots =$

- $3 + 45(0.01) + 45(0.0001) + 45(0.000001) + \ldots$

$= \boxed{3 + 45\sum_{k=1}^{\infty}(0.01)^k}$

- $3 + \frac{45}{99} = \boxed{\frac{342}{99}}$

4. Since r is a percentage write $\rho = \frac{r}{100}$. The area of the largest square is 1. Subtract the area of the largest white square, which is ρ^2. Add back the area of the second largest black square, which has side length $r\%$ of $r\%$, or ρ^2, so its area is ρ^4. The second largest white square has side length equal to $r\%$ of $r\%$ of $r\%$, which is ρ^3, so its area is $(\rho^3)^2 = \rho^6$. Continuing in this manner, the area of the entire shaded region is

$$1 - \rho^2 + \rho^4 - \rho^6 + \rho^8 - \ldots$$

$$= (1 + \rho^4 + \rho^8 + \rho^{12} + \rho^{16} + \ldots) - (\rho^2 + \rho^6 + \rho^{10} + \rho^{14} + \ldots)$$

$$= \sum_{k=0}^{\infty}(\rho^4)^k - \sum_{k=0}^{\infty}\rho^2(\rho^4)^k$$

$\longrightarrow$

$$=\left(\sum_{k=0}^{\infty}\rho^{4k}\right)-\rho^2\left(\sum_{k=0}^{\infty}\rho^{4k}\right)$$

Factor out $\sum_{k=0}^{\infty}\rho^{4k}$!

$$=(1-\rho^2)\underbrace{\sum_{k=0}^{\infty}\rho^{4k}}$$

geometric series — converges since $0<\rho^4<1$

$$=(1-\rho^2)\cdot\frac{1}{1-\rho^4}$$

$$=\frac{1-\rho^2}{1-\rho^4}=\frac{\cancel{1-\rho^2}}{(\cancel{1-\rho^2})(1+\rho^2)}=\frac{1}{1+\rho^2}=\boxed{\frac{1}{1+\left(\frac{r}{100}\right)^2}}$$

Alternatively,

$$1-\rho^2+\rho^4-\rho^6+\rho^8=\underbrace{\sum_{k=0}^{\infty}(-\rho^2)^k}$$

geometric series — converges since $0<|-\rho^2|<1$

$$=\frac{1}{1-(-\rho^2)}=\frac{1}{1+\rho^2}=\boxed{\frac{1}{1+\left(\frac{r}{100}\right)^2}}$$