Math 116 Quiz 6: $\oint$ 4.8, 8.3, 7.7-7.8 **Name:** SOLUTIONS
Tue 6 Nov 2012

You have 45 minutes to complete this quiz. Eyes on your own paper and good luck!

1. **Definitions/Concepts.**

   (a) (3 pts) **Parametrizing a Line:** Given $\frac{dx}{dt} = a$ and $\frac{dy}{dt} = b$, a line passing through the point (x_0, y_0) has the following parametric equations:

   $$x(t) = x_0 + at$$
   $$y(t) = y_0 + bt$$

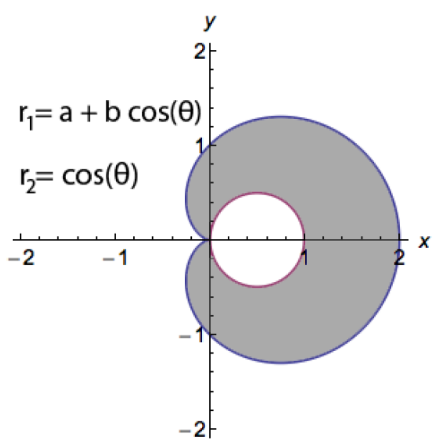
   The non-parametric equation for the same line is given by:

   $$y - y_0 = \frac{b}{a}(x - x_0)$$

   (b) (1 pt) Given polar coordinates (r, θ), the same point in Cartesian coordinates is

   $$x = r\cos\theta$$
   $$y = r\sin\theta$$

2. **Questions/Problems.** *(from Fall 2011 Exam 2)* Members of the recruitment committee for the Mars University (MU) chapter of the fraternity Epsilon Rho Rho (ERR) are designing a pledge pin to distribute during Rush Week. The pin takes the shape of a cardioid with a circular hole in it. The cardioid is given by a polar equation of the form $r_1 = a + b\cos\theta$, while the circular hole has the polar equation $r_2 = \cos\theta$. The pin is pictured below, where the x- and y-axes are measured in inches.



   (a) (5 pts) The committee plans on coating one side of the pin in gold plating, which costs 3 dollars per square inch. Give an expression representing the cost to plate one face of the pin in gold. Your answer may involve integrals and the constants a and b.

   -see the solution posted on the course website -

(b) (3 pts) Find a and b.

-see the solution posted on the course website -

3. **Computations/Algebra.** (2 pts) Determine if the following integrals converge or diverge. If an integral converges, compute the value to which it converges. If an integral diverges, you must explain why.

(a) $\int_{-2}^{2} \frac{dx}{x^2} =$

$$\int_{-2}^{0} \frac{dx}{x^2} + \int_{0}^{2} \frac{dx}{x^2},$$

since the integrand diverges when $x = 0$. We must take limits in order to evaluate, so we get

$$\begin{aligned}
\lim_{b\to 0^-} \int_{-2}^{b} \frac{dx}{x^2} + \lim_{a\to 0^+} \int_{a}^{2} \frac{dx}{x^2} &= \lim_{b\to 0^-} \left.\frac{-1}{x}\right|_{-2}^{b} + \lim_{a\to 0^+} \left.\frac{-1}{x}\right|_{a}^{2} \\
&= \lim_{b\to 0^-} \left(\frac{-1}{b} - \frac{-1}{-2}\right) + \lim_{a\to 0^+} \left(\frac{-1}{a} - \frac{-1}{2}\right) \\
&= \lim_{b\to 0^-} \left(\frac{-1}{b} - \frac{1}{2}\right) + \lim_{a\to 0^+} \left(\frac{-1}{a} + \frac{1}{2}\right) \\
&= \infty
\end{aligned}$$

and so the integral diverges.

(b) $\int_{-1}^{2} \frac{dx}{\sqrt{2-x}} =$

$$\begin{aligned}
\lim_{b\to 2^-} \int_{-1}^{b} \frac{dx}{\sqrt{2-x}} &= \lim_{b\to 2^-} -2\sqrt{2-x}\,\Big|_{-1}^{b} \\
&= -2\sqrt{2-2} - \left(-2\sqrt{2-(-1)}\right) \\
&= 2\sqrt{3}
\end{aligned}$$

(c) $\int_{10}^{\infty} \frac{5+2\sin 4\theta}{\theta} d\theta =$

Before integrating, notice the sine function is bounded between -1 and 1. So for $\theta >> 0$ (the $>>$ sign means "for θ sufficiently large"), we have the inequality

$$\frac{1}{\theta} \le \frac{3}{\theta} \le \frac{5+2\sin 4\theta}{\theta} \le \frac{7}{\theta},$$

but in particular,

$$\int_{10}^{\infty} \frac{1}{\theta}\, d\theta = \int_{1}^{\infty} \frac{1}{\theta} d\,\theta - \int_{1}^{10} \frac{1}{\theta} d\,\theta$$

and the right hand side diverges. Therefore the integral $\int_{10}^{\infty} \frac{5+2\sin 4\theta}{\theta} d\theta$ also diverges.

(d) $\int_{1}^{\infty} \frac{x}{1+x} dx =$

To conclude divergence, it is not enough to say the function $\frac{x}{1+x}$ behaves like the constant function 1 for $x >> 0$. However, we can set up an explicit inequality. Notice

$$\frac{1}{2} = \frac{x}{x+x} \leq \frac{x}{1+x}$$

for $x >> 0$. Integrating $\frac{1}{2}$ is the same as integrating the function 1 and then multiplying the result by $\frac{1}{2}$; i.e.,

$$\int_{1}^{\infty} \frac{1}{2}\, dx = \frac{1}{2} \int_{1}^{\infty} 1\, dx,$$

which diverges. Therefore the original integral $\int_{1}^{\infty} \frac{x}{1+x} dx$ diverges.