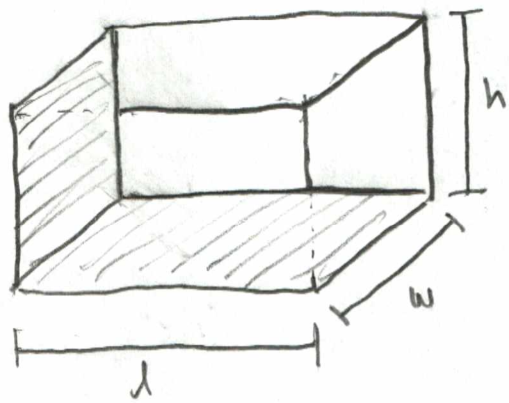
Take-Home Quiz no. 4

SOLUTIONS

Math 2574 (Cal III)

Spring 2017

1.



Objective: Maximize

$V(l,w,h) = lwh$

Constraint:

$lw + 2wh + 2lh = 2$

(a) Use the constraint to write V as a function of 2 variables:

$$lw + 2wh + 2lh = 2$$

$$h(2w+2l) = 2 - lw$$

$$\Rightarrow h = \frac{2-lw}{2w+2l}$$

$$\Rightarrow V(l,w) = \frac{lw(2-lw)}{2w+2l} = \frac{2lw - l^2w^2}{2w+2l}$$

Find critical points:

$$V_l = \frac{(2w+2l)(2w-2lw^2) - (2lw - l^2w^2)(2)}{(2w+2l)^2}$$

$\longrightarrow$

$$= \frac{\cancel{4}^{2}w^2 + \cancel{4lw} - \cancel{4}^{2}lw^3 - \cancel{4}^{2}l^2w^2 - \cancel{4lw} + \cancel{2}l^2w^2}{2(\cancel{2}w+\cancel{2}l)^2}$$

never 0

$$= \frac{2\cancel{w^2} - 2lw^{\cancel{3}} - l^2\cancel{w^2}}{\cancel{2(w+l)^2}} = 0$$ $\leftarrow$ never 0

$$\Rightarrow w = \frac{l^2-2}{-2l} = \frac{2-l^2}{2l}$$

$$V_w = \frac{(2w+2l)(2l-2l^2w) - (2lw - l^2w^2)(2)}{(2w+2l)^2}$$

$$= \frac{\cancel{4lw} + 4l^2 - 4l^2w^2 - 4l^3w - 4lw + 2l^2w^2}{(2w+2l)^2}$$

$$= \frac{2l^2 - 2l^3w - l^2w^2}{2(w+l)^2} = 0$$ (same simplification as above)

$$\Rightarrow l = \frac{2-w^2}{2w} = \frac{2-\left(\frac{2-l^2}{2l}\right)^2}{2\left(\frac{2-l^2}{2l}\right)}$$

$\longrightarrow$

$$\frac{\cancel{2l}(2-l^2)}{\cancel{2l}} = 2 - \left(\frac{2-l^2}{2l}\right)^2$$

$$(2l)^2(2-l^2) = 2(2l)^2 - (2-l^2)^2$$

$$\cancel{8l^2} - 4l^4 = \cancel{8l^2} - 4 + 4l^2 - l^4$$

$$0 = 3l^4 + 4l^2 - 4$$

Q-formula: $l^2 = \frac{2}{3} \Rightarrow l = \sqrt{\frac{2}{3}} \approx 0.8$

and $w = \frac{2 - \left(\sqrt{\frac{2}{3}}\right)^2}{2\sqrt{\frac{2}{3}}} = \sqrt{\frac{2}{3}}$

Max or min?

$$D(l,w) = V_{ll}V_{ww} - V_{lw}V_{wl}$$

$$V_{ll} = \frac{2(w+l)^{\cancel{2}}(-2w^3 - 2lw^2) - (2w^2 - 2lw^3 - l^2w^2)(4\cancel{(w+l)})}{4(w+l)^{\cancel{4}\,3}}$$

$$= \frac{-\cancel{4}w^4 - \cancel{4}lw^3 - \cancel{4}lw^2 - \cancel{4l^2w^2} - \cancel{8}^{2}w^2 + \cancel{8}^{2}lw^3 + \cancel{4l^2w^2}}{\cancel{4}(w+l)^3}$$

$$= \frac{-w^4 + lw^3 - lw^2 - 2w^2}{(w+l)^3} = \frac{-w^2(w^2 - lw + l + 2)}{(w+l)^3}$$

$\longrightarrow$

$$V_{ww} = \frac{-l^4 + l^3 w - l^2 w - 2l^2}{(w+l)^3} \text{ (by symmetry)}$$

$V_{lw} = V_{wl}$ (as long as $2w + 2l \neq 0$)

$$= \frac{2(w+l)^{\cancel{2}}(4l - 6l^2 w - 2lw^2) - (2l^2 - 2l^3 w - l^2 w^2)(4\cancel{(w+l)})}{4(w+l)^{\cancel{4}\,3}}$$

$$= \frac{\cancel{8lw} \cancel{+8l^2} - 12l^2w^2 - 12l^3 w - 4lw^3 - \cancel{4l^2w^2} - \cancel{8l^2} + 8l^3 w + \cancel{4l^2 w^2}}{4(w+l)^3}$$

$$= \frac{\cancel{8}^{2}lw - \cancel{12}^{3}l^2w^2 - \cancel{4}l^3 w - \cancel{4}lw^3}{\cancel{4}(w+l)^3} = \frac{2lw - 3l^2w^2 - l^3 w - lw^3}{(w+l)^3}$$

Check $D(l,w) =$

$$\left(\frac{-w^4 + lw^3 - lw^2 - 2w^2}{(w+l)^3}\right)\left(\frac{-l^4 + l^3 w - l^2 w - 2l^2}{(w+l)^3}\right) - \left(\frac{2lw - 3l^2w^2 - l^3 w - lw^3}{(w+l)^3}\right)^2$$

$$= \frac{(-w^2)(-l^2)(w^2 - lw + l + 2)(l^2 - lw + w + 2) - l^2w^2(2 - 3lw - l^2 - w^2)^2}{(w+l)^3}$$

$$= \frac{w^2 l^2}{(w+l)^3}\Big[l^2w^2 - lw^3 + w^3 + 2w^2 - l^3w + l^2w^2 - lw^2 - 2lw$$

$$+ l^3 - l^2w + lw + 2l + 2l^2 - 2lw + 2w + 4$$

$$- \big(4 - 6lw - 2l^2 - 2w^2 - 6lw + 9l^2w^2 + 3l^3w + 3lw^3$$

$$- 2l^2 + 3l^3w + l^4 + l^2w^2 - 2w^2 + 3lw^3 + l^2w^2 + w^4\big)$$

$$= \frac{w^2 l^2}{(w+l)^3}\big(-7lw^3 + w^3 + 6w^2 - 7l^3w - lw^2 + 9lw + l^3 - l^2w + 2l$$

$$+ 6l^2 + 2w - 9l^2w^2 - l^4 - w^4 \big)$$

always positive

$$D\left(\sqrt{\tfrac{2}{3}}, \sqrt{\tfrac{2}{3}}\right) = \frac{\frac{2}{3}\left(\frac{2}{3}\right)}{\left(2\sqrt{\frac{2}{3}}\right)^3}\Big[-7\left(\sqrt{\tfrac{2}{3}}\right)^4 + \left(\sqrt{\tfrac{2}{3}}\right)^3 + 6\left(\sqrt{\tfrac{2}{3}}\right)^2 - 7\left(\sqrt{\tfrac{2}{3}}\right)^4 - \left(\sqrt{\tfrac{2}{3}}\right)^3 + 9\left(\sqrt{\tfrac{2}{3}}\right)^2$$

$$+ \left(\sqrt{\tfrac{2}{3}}\right)^3 - \left(\sqrt{\tfrac{2}{3}}\right)^3 + 2\sqrt{\tfrac{2}{3}} + 6\left(\sqrt{\tfrac{2}{3}}\right)^2 + 2\sqrt{\tfrac{2}{3}}$$

$$- 9\left(\sqrt{\tfrac{2}{3}}\right)^4 - \left(\sqrt{\tfrac{2}{3}}\right)^4 - \left(\sqrt{\tfrac{2}{3}}\right)^4 \Big]$$

$$= -25\left(\sqrt{\tfrac{2}{3}}\right)^4 + 15\left(\sqrt{\tfrac{2}{3}}\right)^2 + 4\sqrt{\tfrac{2}{3}} = -25(4) + 15(2)(3) + 4(3)(2)\sqrt{6} > 0$$

$$\text{and } V_{ll}\left(\sqrt{\tfrac{2}{3}}, \sqrt{\tfrac{2}{3}}\right) = \frac{-\left(\sqrt{\frac{2}{3}}\right)^2\left[\left(\sqrt{\frac{2}{3}}\right)^2 - \left(\sqrt{\frac{2}{3}}\right)^2 + \sqrt{\frac{2}{3}} + 2\right]}{\left(2\sqrt{\frac{2}{3}}\right)^3} < 0$$

max

$$h = \frac{2 - \left(\sqrt{\frac{2}{3}}\right)^2}{2\cancel{4}\sqrt{\frac{2}{3}}} = \frac{1 - \frac{1}{3}}{2\sqrt{\frac{2}{3}}} = \frac{\frac{2}{3}}{2\sqrt{\frac{2}{3}}} = \frac{\sqrt{3}}{3\sqrt{2}} \approx 0.4$$

$$\boxed{\text{dimensions: } 0.8\,\text{m} \times 0.8\,\text{m} \times 0.4\,\text{m}}$$

(b) Let $g = lw + 2wh + 2lh - 2$

Set $\nabla V = \lambda \nabla g$

$\langle wh, lh, lw \rangle = \lambda \langle w+2h, l+2h, 2w+2l \rangle$

① $wh = \lambda(w+2h)$

$\lambda = \dfrac{wh}{w+2h}$

② $lh = \left(\dfrac{wh}{w+2h}\right)(l+2h)$

$lh(w+2h) = wh(l+2h)$

$\cancel{lwh} + 2lh^2 = \cancel{lwh} + 2wh^2$

$\cancel{2}l\cancel{h^2} = \cancel{2}w\cancel{h^2}$

$\Rightarrow l = w$ (or $h=0$)

③ $l(l) = \left(\dfrac{lh}{l+2h}\right)(2l+2l)$

$l^2(l+2h) = lh(4l) = 4l^2h$

$\Rightarrow l^2(l-2h) = 0 \Rightarrow l = 2h$ (or $l=0$)

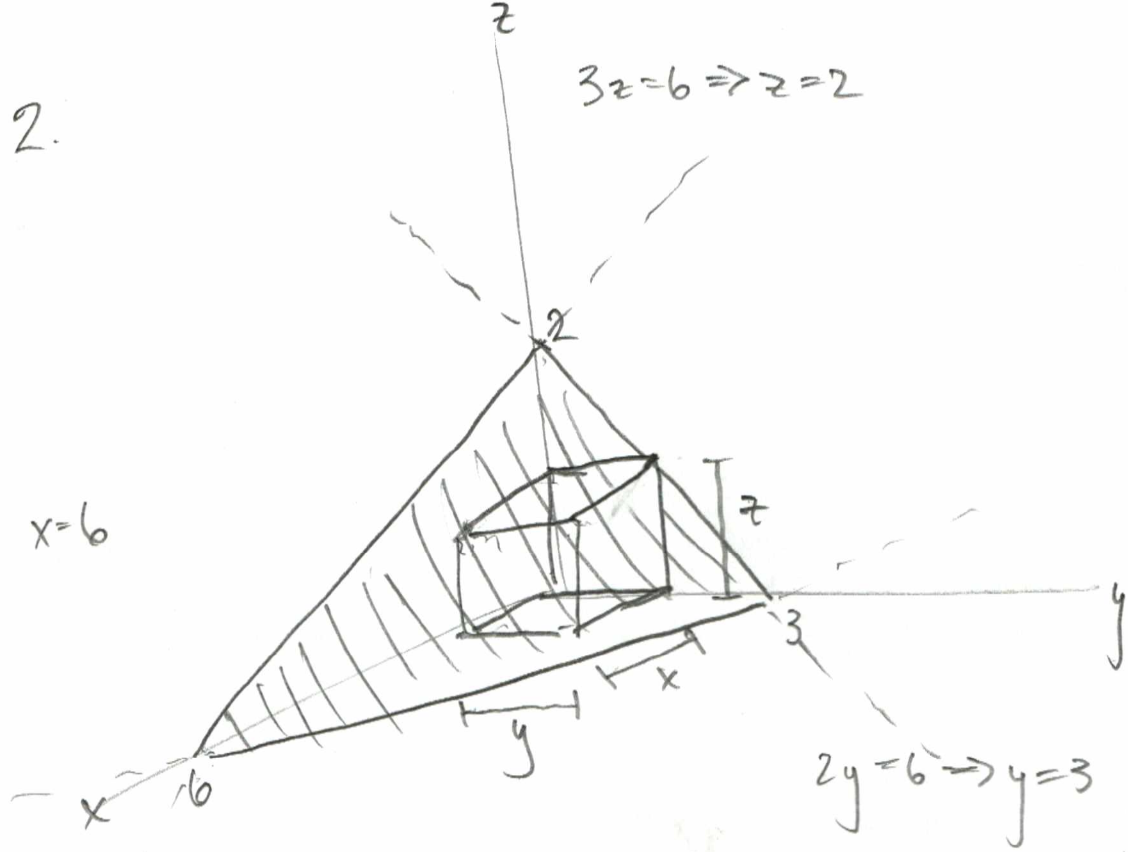
④ $g = 2h(2h) + 2(2h)h + 2(2h)h - 2 = 0$

$6h^2 - 1 = 0$

$\Rightarrow h = \sqrt{\dfrac{1}{6}}\left(\dfrac{\sqrt{3}}{\sqrt{3}}\right) = \dfrac{\sqrt{3}}{3\sqrt{2}} \approx 0.4 \Rightarrow l = w = 2\left(\dfrac{\sqrt{3}}{3\sqrt{2}}\right) = \sqrt{\dfrac{2}{3}} \approx 0.8$

$\boxed{\text{dimensions: } 0.8\,m \times 0.8\,m \times 0.4\,m}$

2.



<u>Objective</u>: Maximize $V(x,y,z) = xyz$

<u>Constraint</u>: $x+2y+3z=6$

(a) Write $x = 6-2y-3z$

$\Rightarrow V(y,z) = (6-2y-3z)yz = 6yz - 2y^2z - 3yz^2$

Critical points:

$V_y = 6z - 4yz - 3z^2 = 0 \Rightarrow y = \frac{6\cancel{z}-3z^{\cancel{2}}}{4\cancel{z}} = \frac{6-3z}{4}$

$V_z = 6y - 2y^2 - 6yz = 0$

$\longrightarrow$

$$\Rightarrow 6\left(\frac{6-3z}{4}\right) - 2\left(\frac{6-3z}{4}\right)^2 - 6\left(\frac{6-3z}{4}\right)z = 0$$

$$4(6)(6-3z) - 2(36-36z+9z^2) - 4(6)(6-3z)z = 0$$

$$\cancel{144} - \cancel{72z} - 72 + \cancel{72z} - 18z^2 - 144z + 72z^2 = 0$$

$$72 - 144z + 54z^2 = 0$$

$$\cancel{18}(4-8z+3z^2) = 0$$

$$\text{Q-formula: } z = \frac{8 \pm \sqrt{\overset{64 = 16(4) \;-\; 16(3)}{(-8)^2 - 4(3)(4)}}}{2(3)} = \frac{8 \pm 4}{2(3)}$$

$$= \frac{12}{6}, \frac{4}{6} = \cancel{2}, \frac{2}{3}$$

or else $x = y = 0$

$$\Rightarrow y = \frac{6 - 3\left(\frac{2}{\cancel{3}}\right)}{4} = 1$$

Classify:

$$D(y,z) = (-4z)(-6z) - (6-4y-6z)^2$$

$$= \cancel{24z^2} - 36 + 24y + 36z + 24y - 16y^2 - 24yz$$
$$+ 36z - 24yz - \cancel{36z^2}$$

$$= -16y^2 - 12z^2 + 48y + 72z - 48yz - 36$$

$$D\left(1, \frac{2}{3}\right) = -\cancel{16(1)^2} - 12\left(\frac{2}{3}\right)^2 + \cancel{48(1)} + \overset{24}{\cancel{72}}\left(\frac{2}{\cancel{3}}\right) - \overset{16}{\cancel{48}}(1)\left(\frac{2}{\cancel{3}}\right) - 36$$

$$= -4 + 36 - 36 < 0 \Rightarrow \underline{\text{max}} \checkmark$$

$x = 6 - 2(1) - 3\left(\frac{2}{3}\right) = 2$

$$\boxed{\text{dimensions: } 2 \times 1 \times \frac{2}{3}}$$

(b) Let $g = x + 2y + 3z - 6$

Set $\nabla V = \lambda \nabla g$

$\langle yz, xz, xy \rangle = \lambda \langle 1, 2, 3 \rangle$

① $yz = \lambda$

② $x\cancel{z} = \lambda(2) = 2y\cancel{z}$

$\Rightarrow x = 2y$ (or $z = 0$)

③ $xy = (yz)(3)$

$2y = 3z \Rightarrow y = \frac{3}{2}z \Rightarrow x = 3z$

④ $3z + 2\left(\frac{3}{2}z\right) + 3z - 6 = 0$

$\Rightarrow 9z - 6 = 0$

$z = \frac{2}{3} \Rightarrow y = \frac{3}{2}\left(\frac{2}{3}\right) = 1, \; x = 3\left(\frac{2}{3}\right) = 2$

$$\boxed{\text{dimensions: } 2 \times 1 \times \frac{2}{3}}$$