

§3.9 Derivatives of Logarithmic and Exponential Functions

The natural exponential function $f(x) = e^x$ has an inverse function, namely $f^{-1}(x) = \ln x$. This relationship has the following properties:

1. $e^{\ln x} = x$ for $x > 0$ and $\ln(e^x) = x$ for all x .
2. $y = \ln x \iff x = e^y$
3. For real numbers x and $b > 0$,

$$b^x = e^{\ln(b^x)} = e^{x \ln b}.$$

Derivative of $y = \ln x$

Using 2. from the last slide, plus implicit differentiation, we can find $\frac{d}{dx}(\ln x)$. Write $y = \ln x$. We wish to find $\frac{dy}{dx}$. From 2.,

$$\frac{d}{dx}(x = e^y) \Rightarrow \frac{d}{dx}x = \frac{d}{dx}(e^y)$$

$$1 = e^y \left(\frac{dy}{dx} \right)$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\text{So } \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Derivative of $y = \ln |x|$

Recall, we can only take “ln” of a positive number. However:

- For $x > 0$, $\ln |x| = \ln x$, so

$$\frac{d}{dx}(\ln |x|) = \frac{1}{x}.$$

- For $x < 0$, $\ln |x| = \ln(-x)$, so

$$\frac{d}{dx}(\ln |x|) = \frac{d}{dx}(\ln(-x)) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}.$$

In other words, the absolute values do not change the derivative of natural log.

Exercise

Find the derivative of each of the following functions:

- $f(x) = \ln(15x)$
- $g(x) = x \ln x$
- $h(x) = \ln(\sin x)$

Derivative of $y = b^x$

What about other logs? Say $b > 0$. Since $b^x = e^{\ln b^x} = e^{x \ln b}$ (by 3. on the earlier slide),

$$\begin{aligned}\frac{d}{dx}(b^x) &= \frac{d}{dx}(e^{x \ln b}) \\ &= e^{x \ln b} \cdot \ln b \\ &= b^x \ln b.\end{aligned}$$

Exercise

Find the derivative of each of the following functions:

- $f(x) = 14^x$
- $g(x) = 45(3^{2x})$

Exercise

Determine the slope of the tangent line to the graph

$$f(x) = 4^x \text{ at } x = 0.$$

Story Problem Example

Example

The energy (in Joules) released by an earthquake of magnitude M is given by the equation

$$E = 25000 \cdot 10^{1.5M}.$$

- (a) How much energy is released in a magnitude 3.0 earthquake?
- (b) What size earthquake releases 8 million Joules of energy?
- (c) What is $\frac{dE}{dM}$ and what does it tell you?

Derivatives of General Logarithmic Functions

The relationship $y = \ln x \iff x = e^y$ applies to logarithms of other bases:

$$y = \log_b x \iff x = b^y.$$

Now taking $\frac{d}{dx}(x = b^y)$ we obtain

$$1 = b^y \ln b \left(\frac{dy}{dx} \right)$$

$$\frac{dy}{dx} = \frac{1}{b^y \ln b}$$

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

Example

The derivative of $f(x) = \log_2(10x)$ is

- A. $\frac{1}{10x}$
- B. $\frac{1}{x \ln 2}$
- C. $\frac{1}{x}$
- D. $\frac{10}{x \ln 2}$

Neat Trick: Logarithmic Differentiation

Example

Compute the derivative of $f(x) = \frac{x^2(x-1)^3}{(3+5x)^4}$.

Solution: We can use logarithmic differentiation – first take the natural log of both sides and then use properties of logarithms.

$$\begin{aligned}\ln(f(x)) &= \ln\left(\frac{x^2(x-1)^3}{(3+5x)^4}\right) \\ &= \ln x^2 + \ln(x-1)^3 - \ln(3+5x)^4 \\ &= 2 \ln x + 3 \ln(x-1) - 4 \ln(3+5x)\end{aligned}$$

Now we take $\frac{d}{dx}$ on both sides:

$$\frac{1}{f(x)} \left(\frac{df}{dx} \right) = 2 \left(\frac{1}{x} \right) + 3 \left(\frac{1}{x-1} \right) - 4 \left(\frac{1}{3+5x} \right) \quad (5)$$

$$\frac{f'(x)}{f(x)} = \frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}$$

Finally, solve for $f'(x)$:

$$\begin{aligned} f'(x) &= f(x) \left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x} \right] \\ &= \frac{x^2(x-1)^3}{(3+5x)^4} \left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x} \right] \end{aligned}$$

Exercise

Use logarithmic differentiation to calculate the derivative of

$$f(x) = \frac{(x+1)^{\frac{3}{2}}(x-4)^{\frac{5}{2}}}{(5x+3)^{\frac{2}{3}}}.$$

3.9 Book Problems

9-29 (odds), 55-67 (odds)

Exam #2 Review

- §3.2 Working with Derivatives
 - Be able to use the graph of a function to sketch the graph of its derivative, without computing derivatives
 - Know the 3 conditions for when a function is not differentiable at a point, and why these three conditions make a function not differentiable at the given point
 - Be able to determine where a function is not differentiable

Exam #2 Review (cont.)

- §3.3 Rules for Differentiation
 - Be able to use the various rules for differentiation (e.g., constant rule, power rule, constant multiple rule, sum and difference rule) to calculate the derivative of a function.
 - Know the derivative of e^x .
 - Be able to find slopes and/or equations of tangent lines.
 - Be able to calculate higher-order derivatives of functions.

Exam #2 Review (cont.)

Exercise

Given that $y = 3x + 2$ is tangent to $f(x)$ at $x = 1$ and that $y = -5x + 6$ is tangent to $g(x)$ at $x = 1$, write the equation of the tangent line to $h(x) = f(x)g(x)$ at $x = 1$.

Exam #2 Review (cont.)

- §3.4 The Product and Quotient Rules

- Be able to use the product and/or quotient rules to calculate the derivative of a given function.
- Be able to use the product and/or quotient rules to find tangent lines and/or slopes at a given point.
- Know the derivative of e^{kx} .
- Be able to combine derivative rules to calculate the derivative of a function.

Note: Functions are not always given by a formula.

When faced with a problem where you don't know where to start, go through the rules first.

Exam #2 Review (cont.)

Exercise

Suppose you have the following information about the functions f and g :

$$f(1) = 6 \quad f'(1) = 2 \quad g(1) = 2 \quad g'(1) = 3$$

- Let $F = 2f + 3g$. What is $F(1)$? What is $F'(1)$?
- Let $G = fg$. What is $G(1)$? What is $G'(1)$?

Exam #2 Review (cont.)

- §3.5 Derivatives of Trigonometric Functions

- Know the two special trigonometric limits

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$$

and be able to use them to solve other similar limits.

- Know the derivatives of $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$, and be able to use the quotient rule to derive the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$.
 - Be able to calculate derivatives (including higher order) involving trig functions using the rules for differentiation.

Exam #2 Review (cont.)

Exercise

Calculate the derivative of the following functions:

- $f(x) = (1 + \sec x) \sin^3 x$

- $g(x) = \frac{\sin x + \cot x}{\cos x}$

Exercise

Evaluate $\lim_{x \rightarrow -3} \frac{\sin(x+3)}{x^2 + 8x + 15}$.

Exam #2 Review (cont.)

- §3.6 Derivatives as Rates of Change
 - Be able to use the derivative to answer questions about rates of change involving:
 - Position and velocity
 - Speed and acceleration
 - Growth rates
 - Business applications

Exam #2 Review (cont.)

- Be able to use a position function to answer questions involving velocity, speed, acceleration, height/distance at a particular time t , maximum height, and time at which a given height/distance is achieved.
- Be able to use growth models to answer questions involving growth rate and average growth rate, and cost functions to answer questions involving average and marginal costs.

Exam #2 Review (cont.)

- §3.7 The Chain Rule

- Be able to use both versions of the Chain Rule to find the derivative of a composition function.
- Be able to use the Chain Rule more than once in a calculation involving more than two composed functions.
- Know and be able to use the Chain Rule for Powers:

$$\frac{d}{dx} (f(x))^n = n (f(x))^{n-1} f'(x)$$

Exam #2 Review (cont.)

Exercise

Suppose $f(9) = 10$ and $g(x) = f(x^2)$. What is $g'(3)$?

Exam #2 Review (cont.)

- §3.7 Implicit Differentiation

- Be able to use implicit differentiation to calculate $\frac{dy}{dx}$.
- Be able to use the derivative found from implicit differentiation to find the slope at a given point and/or a line tangent to the curve at the given point.
- Be able to calculate higher-order derivatives of implicitly defined functions.
- Be able to calculate $\frac{dy}{dx}$ when working with functions containing rational functions.

Exercise

Use implicit differentiation to calculate $\frac{dz}{dw}$ for

$$e^{2w} = \sin(wz)$$

Exam #2 Review (cont.)

Exercise

If $\sin x = \sin y$, then

- $\frac{dy}{dx} = ?$
- $\frac{d^2y}{dx^2} = ?$

Running Out of Time on the Exam Plus other Study Tips

- Do practice problems completely, from beginning to end (as if it were a quiz). You might think you understand something but when it's time to write down the details things are not so clear.
- Find a buddy who understands concepts a little better than you and work on problems for 2-3 hours. Then find a buddy who is struggling and work with them 2-3 hours.
- Don't count on cookie cutter problems. If you are doing a practice problem where you've memorized all the steps, make sure you understand why each step is needed. The exam problems may have a small variation from homeworks and quizzes. If you're not prepared, it'll come as a "twist" on the exam...

Running Out of Time on the Exam Plus other Study Tips (cont.)

- If you encounter an unfamiliar type of problem on the exam, relax, because it's most likely not a trick! The solutions will always rely on the information from the required reading/assignments. Take your time and do each baby step carefully.
- During the exam, do the problems you are most confident with first!
- During the exam, budget your time. Count the problems and divide by 50 minutes. The easier questions will take less time so doing them first leaves extra time for the harder ones. When studying, aim for 10 problems per hour (i.e., 6 minutes per problem).

Running Out of Time on the Exam Plus other Study Tips (cont.)

- Always make sure you **answer the question**. This is also a good strategy if you're not sure how to start a problem, figure out what the question wants first.
- The exam is not a race. If you finish early take advantage of the time to check your work. You don't want to leave feeling smug about how quickly you finished only to find out next week you lost a letter grade's worth of points from silly mistakes.

Other Study Tips

- Brush up on algebra, especially radicals, logs, common denominators, etc. Many times knowing the right algebra will simplify the problem!
- When in doubt, show steps.
- You will be punished for wrong notation. The slides for §3.1 show different notations for the derivative. Make sure whichever one you use in your work, that you are using it correctly.
- Read the question!
- Do the book problems.
- Look at the pictures in the book and the interactive applets on MLP.