Some exercises on convergence of series:

§7.6

#48 $\sum_{k=1}^{\infty} \left(\frac{1}{k^2}\right)^{\sqrt{k}}$

① Divergence Test:

$\lim_{k\to\infty} \left(\frac{1}{k^2}\right)^{\sqrt{k}} = \Big(\lim_{k\to\infty} \underbrace{\frac{1}{k^2}}_{0}\Big)^{\sqrt{k}} = 0 \Rightarrow$ inconclusive.

② Integral Test:

Let $a(x) = \left(\frac{1}{x^2}\right)^{\sqrt{x}}$

- continuous on $[1,\infty)$? ✓
- positive on $[1,\infty)$? ✓
- decreasing on $[1,\infty)$? ✓

$\int_1^{\infty} a(x)\,dx = \int_1^{\infty} \left(\frac{1}{x^2}\right)^{\sqrt{x}} dx \approx 3.19313394028126\ldots$
(via Wolfrom Alpha & evaluated numerically)

$\Rightarrow \sum_{k=1}^{\infty} \left(\frac{1}{k^2}\right)^{\sqrt{k}}$ follows the same behavior, so converges.

⟶

③ Comparison Test:

$$\left(\frac{1}{k^2}\right)^{\sqrt{k}} = \frac{1}{k^{2\sqrt{k}}} \leq \frac{1}{k^2}$$

$$\Rightarrow \sum_{k=1}^{\infty}\left(\frac{1}{k^2}\right)^{\sqrt{k}} \leq \underbrace{\sum_{k=1}^{\infty}\frac{1}{k^2}}_{\text{p-series, } p>1} = \frac{\pi^2}{6}$$

$$\Rightarrow \sum_{k=1}^{\infty}\left(\frac{1}{k^2}\right)^{\sqrt{k}} \text{ \underline{converges}.}$$

④ Limit Comparison Test:

$$\lim_{k\to\infty} \frac{\left(\frac{1}{k^2}\right)^{\sqrt{k}}}{\frac{1}{k^2}} = \lim_{k\to\infty} \left(\underset{\downarrow\, 0}{\frac{1}{k^2}}\right)^{\sqrt{k}-1} = 0$$

$\Rightarrow$ Since $\sum_{k=1}^{\infty}\frac{1}{k^2}$ <u>converges</u>, so does $\sum_{k=1}^{\infty}\left(\frac{1}{k^2}\right)^{\sqrt{k}}$.

⑤ Ratio Test:

$$\lim_{k\to\infty} \frac{\left(\frac{1}{(k+1)^2}\right)^{\sqrt{k+1}}}{\left(\frac{1}{k^2}\right)^{\sqrt{k}}} = \lim_{k\to\infty} \frac{(k^2)^{\sqrt{k}}}{((k+1)^2)^{\sqrt{k+1}}}$$

$$= \frac{\lim_{k\to\infty} k^{2\sqrt{k}}}{\lim_{k\to\infty} (k+1)^{2\sqrt{k+1}}}$$

$\longrightarrow$

Use logarithms to eliminate the exponents:

$$\ln\left(\frac{\lim\limits_{k\to\infty} k^{2\sqrt{k}}}{\lim\limits_{k\to\infty}(k+1)^{2\sqrt{k+1}}}\right) = \ln\left(\lim_{k\to\infty} k^{2\sqrt{k}}\right) - \ln\left(\lim_{k\to\infty}(k+1)^{2\sqrt{k+1}}\right)$$

$$= \lim_{k\to\infty} \ln\left(k^{2\sqrt{k}}\right) - \lim_{k\to\infty}\left(\ln (k+1)^{2\sqrt{k+1}}\right)$$

$$= \lim_{k\to\infty} 2\sqrt{k}\ln k \to \infty - \lim_{k\to\infty} 2\sqrt{k+1}\ln(k+1) \to \infty$$

indeterminate form

. . .

⑥ Root Test:

$$\lim_{k\to\infty}\left(\left(\frac{1}{k^2}\right)^k\right)^{1/k} = \lim_{k\to\infty}\frac{1}{k^2} = 0 \Rightarrow \sum_{k=1}^{\infty}\left(\frac{1}{k^2}\right)^k \text{ converges.}$$

$$\sum_{k=1}^{\infty}\left(\frac{1}{k^2}\right)^{\sqrt{k}} \leq \sum_{k=1}^{\infty}\left(\frac{1}{k^2}\right)^k,$$ and so also

converges.

#50 $\sum_{k=0}^{\infty} \frac{2^{k!}}{k!}$

① Divergence Test:

$\lim_{k\to\infty} \frac{2^{k!}}{k!} \xrightarrow[\text{eliminate the exponent}]{\text{use logs to}} \lim_{k\to\infty} \frac{k!\ln(2)}{\ln(k!)} = \infty$, since $k!$ dominates $\ln(k!)$

$\Rightarrow \sum_{k=0}^{\infty} \frac{2^{k!}}{k!}$ diverges.

② Integral Test:

Let $a(x) = \frac{2^{x!}}{x!}$.

- continuous on $[1,\infty)$? ✓ (* look up how to extend the factorial function's domain)
- positive on $[1,\infty)$? ✓
- decreasing on $[1,\infty)$? X

③ Comparison Test:

Suspiciously divergent, so try the p-series $\sum_{k=1}^{\infty} \frac{1}{k}$.

$\frac{1}{k}\frac{(k-1)!}{(k-1)!} = \frac{(k-1)!}{k!} < \frac{2^{(k-1)!}}{k!}$ b/c for any $x > 1$,

→

$x < 2^x$, and so

$$\frac{1}{k} < \frac{2^{(k-1)!}}{k!} < \frac{2^{k!}}{k!}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{2^{k!}}{k!} \text{ diverges} \rightarrow \sum_{k=0}^{\infty} \frac{2^{k!}}{k!} \text{ \underline{diverges}.}$$

④ Limit Comparison Test:

$$\lim_{k\to\infty} \frac{\frac{2^{k!}}{k!}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{2^{k!}}{(k-1)!} = \infty,$$ b/c this limit is larger than $\lim_{k\to\infty} \frac{2^{k!}}{k!}$, which we saw in the Divergence Test approach infinity.

$\Rightarrow$ Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, so does $\sum_{k=1}^{\infty} \frac{2^{k!}}{k!}$, and hence so does $\sum_{k=0}^{\infty} \frac{2^{k!}}{k!}$.

⑤ Ratio Test:

$$\lim_{k\to\infty} \frac{\frac{2^{(k+1)!}}{(k+1)!}}{\frac{2^{k!}}{k!}} = \lim_{k\to\infty} \frac{2^{(k+1)!-k!}}{k+1}$$

Factor out $k!$:

$\longrightarrow$

$$= \lim_{k \to \infty} \frac{2^{k!(k+1-1)}}{k+1} = \infty,$$

since factorial exponential functions dominate linear functions.

$$\Rightarrow \sum_{k=0}^{\infty} \frac{2^{k!}}{k!} \ \underline{\text{diverges}}.$$

⑥ Root Test:

$$\lim_{k \to \infty} \left(\frac{2^{k!}}{k!} \right)^{1/k} = \lim_{k \to \infty} \frac{2^{(k-1)!}}{\underbrace{k^{1/k}}_{1}(k-1)!} = \infty$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{2^{k!}}{k!} \ \underline{\text{diverges}}.$$

#58 $\displaystyle\sum_{k=0}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdots (3k+1)}{100^k}$

① Divergence Test:

$$\lim_{k \to \infty} \frac{1 \cdot 4 \cdot 7 \cdots (3k+1)}{100^k} \left(\frac{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3k-1) \cdot 3k}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3k-1) \cdot 3k} \right)$$

$$= \lim_{k \to \infty} \underbrace{\frac{(3k+1)!}{100^k}}_{\infty} \underbrace{\left(\frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3k-1) \cdot 3k} \right)}_{\to 0}$$

∞ b/c factorials dominate exponentials

$\longrightarrow$

But $\infty \cdot 0$ is an indeterminate form.

X

2) Integral Test:

let $a(x) = \dfrac{1 \cdot 4 \cdot 7 \cdots (3x+1)}{100^x}$

- continuous on $[0, \infty)$?

  Is it even well-defined for all x??

X

3) Comparison Test:

... Compare to ???

X

4) Limit Comparison Test:

$$\lim_{k \to \infty} \frac{\dfrac{1 \cdot 4 \cdot 7 \cdots (3k+1)}{\cancel{100^k}}}{\dfrac{(3k+1)!}{\cancel{100^k}}} = \lim_{k \to \infty} \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3k-1) \cdot 3k} = 0$$

But $\displaystyle\sum_{k=0}^{\infty} \frac{(3k+1)!}{100^k}$ diverges $\Rightarrow$ inconclusive.

$\longrightarrow$

⑤ Ratio Test:

$$\lim_{k\to\infty} \frac{\dfrac{1\cdot 4\cdot 7\cdot \ldots \cdot (\overset{3k+3+1}{3(k+1)+1})}{100^{k+1}}}{\dfrac{1\cdot 4\cdot 7\cdot \ldots \cdot (3k+1)}{100^k}}$$

$$= \lim_{k\to\infty} \frac{(3k+2)(3k+3)(3k+4)}{100} = \infty$$

$\Rightarrow$ diverges.

⑥ Root Test:

$$\lim_{k\to\infty} \left(\frac{1\cdot 4\cdot 7\cdot \ldots \cdot (3k+1)}{100^k} \right)^{1/k}$$

$$= \lim_{k\to\infty} \frac{(1\cdot 4\cdot 7\cdot \ldots \cdot (3k+1))^{1/k}}{100} \to \infty^0$$

Indeterminate Form

§7.5

$$\#24 \quad \sum_{k=1}^{\infty} \frac{\sqrt{k}}{1+k^2}$$

Since this series only features powers

of k, try to compare to a p-series!

$$\frac{\sqrt{k}}{1+k^2} < \frac{\sqrt{k}}{k^2} = \frac{1}{k^{3/2}}$$

$\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges, since $p = 3/2 > 1$.

$\Rightarrow \sum_{k=1}^{\infty} \frac{\sqrt{k}}{1+k^2}$ converges.