Math 2603 Exam 3
Mon 24 Nov 2014

Name: SOLUTIONS

Discrete Math
Exam 3 (Ch. 6-8 as we have covered)

Please provide the following data:

Drill Time: ____________________

Student ID: ____________________

Exam Instructions: You have 50 minutes to complete this exam. One 3×5 inch notecard, two-sided, is allowed. No graphing calculators. No programmable calculators. No phones, iDevices, computers, etc. If you finish early then you may leave, UNLESS there are less than 5 minutes of class left. To prevent disruption, if you finish with less than 5 minutes of class remaining then please stay seated and quiet.

Your signature below indicates that you have read this page and agree to follow the Academic Honesty Policies of the University of Arkansas.

Signature: ____

Good luck!

1. (a) How many strings can be formed using all of the letters

$n=13$

ASSISTANTSHIP

$n_A=2$, $n_S=4$, $n_I=2$

$n_T=2$, $n_N=1$, $n_H=1$, $n_P=1$

$$\frac{13!}{2!4!2!2!1!1!1!}$$

(b) How many such strings have all the S's consecutive?

Treat the four S's as one letter:

$n=10$

$$\frac{10!}{2!1!2!2!1!1!1!}$$

(c) How many such strings from (a) have no consecutive S's?

There are 10 positions relative to the other 9 letters in which to place an S.

$$\frac{9!}{2!2!2!}\binom{10}{4}$$

2. (a) The Binomial Theorem (BT) states $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

(b) Using the BT, write down the coefficient for x^2yz in the expansion of $(2x+y-z)^4$.

Apply to $((2x+y)-z)^4$: $\binom{4}{1}(2x+y)^3(-z)$

Apply to $(2x+y)^3$: $\binom{3}{1}(2x)^2 y$

Coefficient is $-4\binom{4}{1}\binom{3}{1} = -48$

(c) Prove that

$$\left(\frac{m}{m+n}\right)^m \left(\frac{n}{m+n}\right)^n \cdot \binom{m+n}{m} < 1$$

for all $m, n \in \mathbb{Z}_{>0}$. *Hint:* Consider the term for $k = m$ in the BT expansion of $(x+y)^{m+n}$ for appropriate x and y.

Put $x = \frac{m}{m+n}$, $y = \frac{n}{m+n}$. Then

$$(x+y)^{m+n} = \left(\frac{m}{m+n} + \frac{n}{m+n}\right)^{m+n} = 1^{m+n} = 1.$$

But $\left(\frac{m}{m+n}\right)^m \left(\frac{n}{m+n}\right)^n \cdot \binom{m+n}{m}$ is the m^{th} term in the binomial expansion of 1^{m+n}. Since $m, n > 0$ all terms in the binomial expansion are positive, so must be less than 1.

EA

3. Professor Euclid is paid every other Friday. Using the Pigeonhole Principle, prove that in one year's time, there will be some month where she got paid three times. To get credit, you MUST state which version of the Principle Pigeonhole you used and how you applied it in solving the problem.

Pigeonhole Principle III:

$|X|$(pigeons) = 26 paydays per year

$|Y|$(pigeonholes) = 12 months per year

$k = \left\lceil \frac{26}{12} \right\rceil = 3$. So there are at least 3 paydays occuring in the same month. Months don't have enough days for 4 paydays, so there is a month where Euclid gets paid exactly 3 times. ☒

4. How many positive integer solutions are there of

$x_1, x_2, x_3 \in \mathbb{Z}_{>0}$. $x_1 + x_2 + x_3 = 20$?

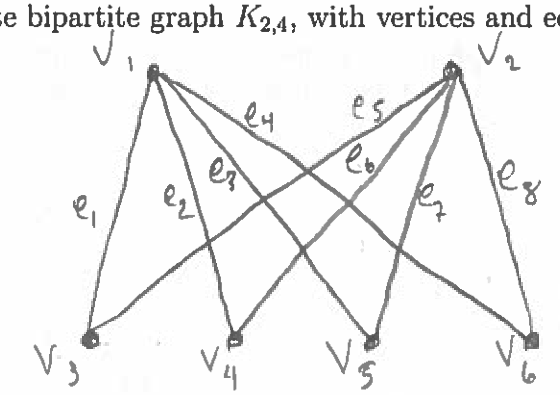
Let $y_1 = x_1 - 1$ The problem becomes:

$y_2 = x_2 - 1$ $y_1 + y_2 + y_3 = 20 - 3 = 17$.

$y_3 = x_3 - 1$ There are $t = 3$ variables that must add up to $k = 17$.

$$\binom{k+t-1}{t-1} = \binom{17+3-1}{3-1} = \boxed{\binom{19}{2} \text{ solutions}}$$

5. (a) Draw the complete bipartite graph $K_{2,4}$, with vertices and edges legibly labeled.



(b) Write the adjacency matrix for $K_{2,4}$.

$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

(c) Write the incidence matrix for $K_{2,4}$.

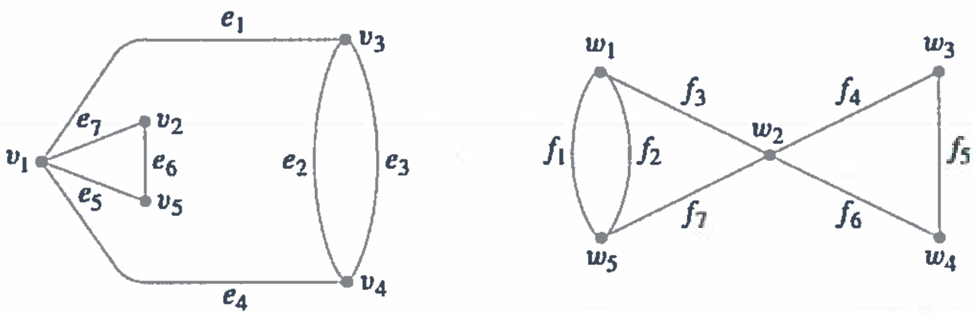
$$\begin{array}{c c} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

(d) Does $K_{2,4}$ have an Euler cycle? If yes, then list the ordering of edges that give one. If no, then prove why not.

Yes, because $K_{2,4}$ is connected and all of its vertices have even degree.

Cycle: $e_1 \to e_5 \to e_6 \to e_2 \to e_3 \to e_7 \to e_8 \to e_4$.

6. Are the following[1] graphs G, G' isomorphic? If so then, exhibit an isomorphism. If not, then state an invariant not shared by the two graphs. If the invariant you cite was not mentioned in class then you must prove it is actually an invariant.



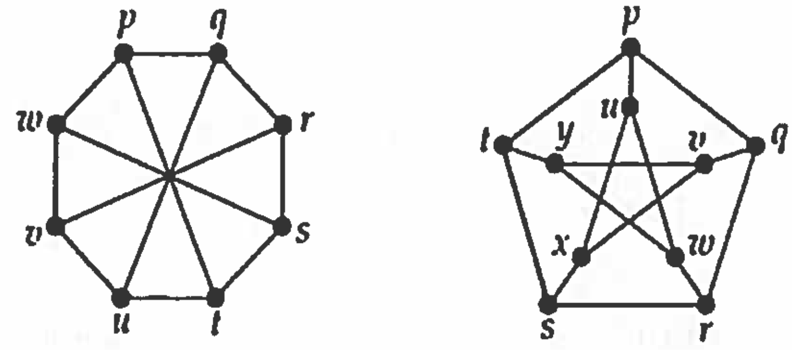
G $\qquad\qquad$ G'

Yes, via the adjacency matrices:

$$\begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 0 & 0 & 1 \\ v_3 & 1 & 0 & 0 & 2 & 0 \\ v_4 & 1 & 0 & 2 & 0 & 0 \\ v_5 & 1 & 1 & 0 & 0 & 0 \end{array} \;=\; \begin{array}{c|ccccc} & w_2 & w_3 & w_1 & w_5 & w_4 \\ \hline w_2 & 0 & 1 & 1 & 1 & 1 \\ w_3 & 1 & 0 & 0 & 0 & 1 \\ w_1 & 1 & 0 & 0 & 2 & 0 \\ w_5 & 1 & 0 & 2 & 0 & 0 \\ w_4 & 1 & 1 & 0 & 0 & 0 \end{array}$$

[1] Image Credit: Epp, Susanna. *Discrete Mathematics with Applications.* Cengage Learning, 2010. p. 677.

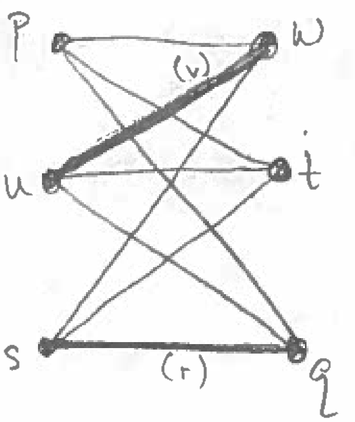
7. Choose one of the following[2] graphs, (a) or (b), to consider. NOTE: All vertices are labeled.



(a) (b)

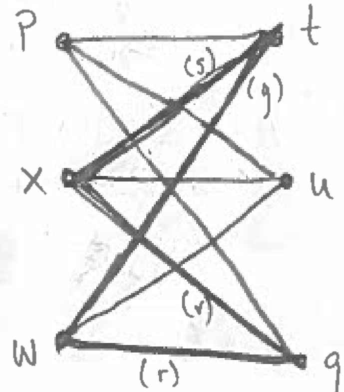
Is the graph you chose planar? Prove your answer: If it is planar then redraw it without any edges overlapping; if it is not planar then exhibit, by series reduction if necessary, a subgraph homeomorphic to $K_{3,3}$ or K_5.

(a) Look at the subgraph given by omitting (v,r). Do a series reduction at (w,v), (v,u) and (q,r), (r,s) to get a $K_{3,3}$ (so <u>no</u>).



(b) Look at the subgraph attained by removing (v,y) and (r,s). Do series reductions at $(t,s),(s,x)$; $(x,v),(v,q)$; $(w,y),(y,t)$; $(w,r),(r,q)$ to get a $K_{3,3}$.

(so <u>no</u>)



[2] Image Credit: Aldous, Joan M. and Wilson, Robin J. *Graphs and Applications: An Introductory Approach.* Springer-Verlag London, 2000. p. 262.

Strategy: In both graphs the vertices all have degree 3. After a little bit of experimenting with rearranging edges and vertices, when that doesn't work, look for $K_{3,3}$.

8. The Fibonacci sequence is defined by the recurrence relation

$$f_n = f_{n-1} + f_{n-2},$$

for $n \geq 3$.

(a) How many initial conditions should there be and what are they?

2 initial conditions: $f_1 = f_2 = 1$

(b) **EXTRA CREDIT** Solve the relation to get an explicit formula for f_n.

$$f_n - f_{n-1} - f_{n-2} = 0$$

is linear, homogeneous with constant coefficients

Solve $t^2 - t - 1 = 0$: $t = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$

Solve the system given by the initial conditions:

$$f_1 = x\left(\frac{1+\sqrt{5}}{2}\right)^1 + y\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1 \Rightarrow y = \frac{2}{1-\sqrt{5}}\left(1 - x\left(\frac{1+\sqrt{5}}{2}\right)\right)$$

$$f_2 = x\left(\frac{1+\sqrt{5}}{2}\right)^2 + y\left(\frac{1-\sqrt{5}}{2}\right)^2 = 1 = x\left(\frac{1+\sqrt{5}}{2}\right)^2 + \frac{1-\sqrt{5}}{2}\left(1 - x\left(\frac{1+\sqrt{5}}{2}\right)\right)$$

$$= x\left(\frac{1+\sqrt{5}}{2}\right)\left(\underbrace{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}_{2\sqrt{5}}\right) + \frac{1-\sqrt{5}}{2}$$

$$\Rightarrow x = \frac{2-(1-\sqrt{5})}{\sqrt{5}(1+\sqrt{5})} = \frac{1}{\sqrt{5}}$$

$$y = \frac{2}{1-\sqrt{5}}\left[1 - \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)\right] = \frac{\cancel{2}}{1-\sqrt{5}}\left(\frac{2\sqrt{5}-(1+\sqrt{5})}{\cancel{2}\sqrt{5}}\right)$$

$$= \frac{-1+\sqrt{5}}{(1-\sqrt{5})\sqrt{5}} = -\frac{1}{\sqrt{5}}$$

$$\Rightarrow \boxed{f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

9. **EXTRA CREDIT** Find a formula for the probability that out of n millennials, at least two will have the same birthday, where "same birthday" means same month, date, AND year. Millennials are those born in the years 1980-1995. Assume no leap years in that time period.

There are 16 years, 365 each, so the sample space is $(16 \cdot 365)^n$. There are $(16 \cdot 365)(16 \cdot 365 - 1) \cdot \ldots \cdot (16 \cdot 365 - n + 1)$ ways all n millenials will have different birthdays. The probability at least 2 have the same is

$$1 - \frac{(16 \cdot 365)(16 \cdot 365 - 1) \cdot \ldots \cdot (16 \cdot 365 - n + 1)}{(16 \cdot 365)^n}.$$

10. **cHaLlEnGe PrObLeM** Find such a formula, only this time, taking leap years into account.

Leap years are 1980, 1984, 1988, 1992, so add 4 to the sample space:

$$1 - \frac{(16 \cdot 365 + 4)(16 \cdot 365 + 4 - 1) \cdot \ldots \cdot (16 \cdot 365 + 4 + n - 1)}{(16 \cdot 365 + 4)^n}.$$