MATH 2554 (Calculus I)
Spring 2016

Name: SOLUTIONS

Fri 12 Feb 2016

Exam 1: Limits (§2.1-3.1)

Exam Instructions: You have 50 minutes to complete this exam. Justification is required for all problems.

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Signature: (1 pt) ______________________________

Good luck!

1. **(10 pts)** Let $f(x) = \dfrac{x+7}{x^4-49x^2}$. Identify all vertical asymptotes for f (or if there are none, say so and why). Then, for each vertical asymptote a, find $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$.

$$f(x) = \frac{x+7}{x^4-49x^2} = \frac{\cancel{x+7}}{x^2\cancel{(x+7)}(x-7)}$$

Check $a=0$, $a=7$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x^2(x-7)} = -\infty$$

0, pos $\quad$ -7

$$\lim_{x\to 0^+} \frac{1}{x^2(x-7)} = -\infty$$

0, pos $\quad$ -7

$$\boxed{\text{VA's @ } x = 0, 7}$$

$$\lim_{x\to 7^-} \frac{1}{x^2(x-7)} = -\infty$$

49 $\quad$ 0, neg.

$$\lim_{x\to 7^+} \frac{1}{x^2(x-7)} = \infty$$

49 $\quad$ 0, pos

2. (10 pts) Determine the end behavior of $f(x) = \dfrac{x+1}{\sqrt{9x^2+x}}$. If there are any horizontal asymptotes then identify them.

$$\lim_{x\to\infty} \frac{\cancel{\frac{x}{x}}^{\,1} + \cancel{\frac{1}{x}}^{\,0}}{\sqrt{\underbrace{\cancel{\frac{9x^2}{x^2}}}_{9} + \underbrace{\cancel{\frac{x}{x^2}}}_{0}}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$$

$$\lim_{x\to-\infty} \frac{\cancel{\frac{x}{x}}^{\,-1} + \cancel{\frac{1}{x}}^{\,0}}{\sqrt{\underbrace{\cancel{\frac{9x^2}{x^2}}}_{9} + \underbrace{\cancel{\frac{x}{x^2}}}_{0}}} = \frac{-1}{\sqrt{9}} = -\frac{1}{3}$$

$$\boxed{\text{HA's @ } y = \tfrac{1}{3},\ -\tfrac{1}{3}}$$

3. (3 pts ea) Evaluate the following limits analytically:

(a) $\lim\limits_{y\to 3} \dfrac{\sqrt{3y+16}-5}{y-3}\left(\dfrac{\sqrt{3y+16}+5}{\sqrt{3y+16}+5}\right)$

$$= \lim_{y\to 3} \frac{3y+16-25}{(y-3)(\sqrt{3y+16}+5)} = \lim_{y\to 3} \frac{3\cancel{(y-3)}}{\cancel{(y-3)}\sqrt{3y+16}+5}$$

$$= \frac{3}{\sqrt{3(3)+16}+5} = \boxed{\frac{3}{10}}$$

(b) $\lim\limits_{x\to 0}(2x^{-8}+4x^3) = \lim\limits_{x\to 0} \dfrac{2}{x^8} + \lim\limits_{x\to 0} 4x^3$

0, pos

$$\boxed{= \infty}$$

(c) $\lim_{x\to\infty} \pi e^{-x}$ = $\pi \lim_{x\to\infty} e^{-x} = \boxed{0}$

(d) $\lim_{t\to -1} f(t)g(t)$, given that $\lim_{t\to -1} f(t) = 2$ and $\lim_{t\to -1} g(t) = 8$.

$= \left(\lim_{t\to -1} f(t)\right)\left(\lim_{t\to -1} g(t)\right) = 2(8) = \boxed{16}$

4. **(10 pts)** Use the Intermediate Value Theorem to show $f(x) = 4x^3 - 6x^2 + 3x - 2$ must cross the line $y = 10$ in the interval $(1, 2)$.

Since f is a polynomial, it is continuous on the interval $[1,2]$.

$$f(1) = 4(1)^3 - 6(1)^2 + 3(1) - 2$$

$$= 4 - 6 + 3 - 2 = -1$$

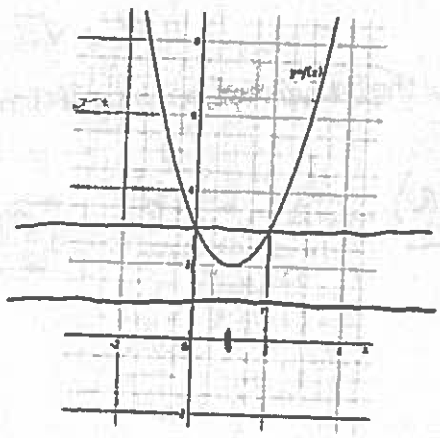
$$f(2) = 4(2)^3 - 6(2)^2 + 3(2) - 2$$

$$= 32 - 24 + 6 - 2 = 12$$

So $f(1) < 10 < f(2)$

$\therefore$ By IVT, there exists c between 1 and 2 where $f(c) = 10$.

5. (3 pts ea) Let $f(x) = x^2 - 2x + 3$. Below is a graph of $f(x)$, drawn at desmos.com.



(a) Use the graph to find a number $\delta > 0$ such that if $|x-1| < \delta$ then $|f(x) - 2| < 1$. If no such number exists, then say so.

$\delta = 1$

(b) If, when we use smaller and smaller values $\epsilon < 1$, we can always find a corresponding value $\delta > 0$, as in (a), then we will have proved that

$\lim_{x\to ?} f(x) = ?$ (rewrite the limit, with the ?s filled in).

$\lim_{x\to 1} f(x) = 2$

(c) For *any* $\epsilon > 0$, find $\delta > 0$ so that $|f(x) - 2| < \epsilon$ whenever $0 < |x-1| < \delta$. *Hint: Your answer will be an expression with ϵs in it.*

Want $|f(x) - 2| < \varepsilon$

$|x^2 - 2x + 3 - 2| < \varepsilon$

$|x^2 - 2x + 1| = |(x-1)^2| < \varepsilon$

so $(x-1)^2 < \varepsilon$

$\Rightarrow |x-1| < \sqrt{\varepsilon}$

$\Rightarrow \boxed{\delta = \sqrt{\varepsilon}}$

6. When computing derivatives in this problem you must use the limit definitions. Given the function,

$$s(t) = \sqrt{5t}$$

(a) **(5 pts)** write the formula for the slope of the secant line joining the points $(a, s(a))$ and $(b, s(b))$;

$$\frac{s(b)-s(a)}{b-a} = \frac{\sqrt{5b}-\sqrt{5a}}{b-a} = \frac{\sqrt{5}\left(\sqrt{b}-\sqrt{a}\right)}{b-a}$$

(b) **(5 pts)** find $s'(1)$;

$$s'(1) = \lim_{t\to 1} \frac{s(t)-s(1)}{t-1} = \lim_{t\to 1} \frac{\sqrt{5}\left(\sqrt{t}-1\right)}{t-1}\left(\frac{\sqrt{t}+1}{\sqrt{t}+1}\right)$$

$b = t$

$a = 1$

$$= \sqrt{5}\lim_{t\to 1} \frac{\cancel{t-1}}{(\cancel{t-1})(\sqrt{t}+1)} = \sqrt{5}\left(\frac{1}{\sqrt{1}+1}\right) = \boxed{\frac{\sqrt{5}}{2}}$$

(c) **(3 pts)** write the equation of the line tangent to $s(t)$ at $t = 1$.

$$y - s(1) = \frac{\sqrt{5}}{2}(t-1)$$

$$\boxed{y - \sqrt{5} = \frac{\sqrt{5}}{2}(t-1)}$$

7. **(5 pts)** Find constants b and c in the polynomial $p(x) = x^2 + bx + c$ so that

$$\lim_{x \to 2} \frac{p(x)}{x-2} = 6.$$

For the limit to exist, the denominator should cancel. So $p(x) = (x-2) \times$ (something else).
$p(x)$ should also be quadratic (degree 2) and monic (no coefficient on x^2). So let (something else) $= x - a$.
Solve $2 - a = 6 \Rightarrow a = -4$. Then

$$p(x) = (x-2)(x+4)$$
$$= x^2 - 2x + 4x - 8 = x^2 + 2x - 8$$

$$\Rightarrow \boxed{b = 2,\ c = -8}$$

8. **(5 pts)** Determine the interval(s) of continuity for

$$f(x) = \frac{x+2}{x^2-4}.$$

$$= \frac{\cancel{x+2}}{\cancel{(x+2)}(x-2)} = \frac{1}{x-2} \text{ when } x \neq -2$$

Continuous on

$$\boxed{(-\infty,-2),(-2,2),(2,\infty)}$$