MATH 2574 (Calculus III)
Fall 2015

Name: SOLUTIONS

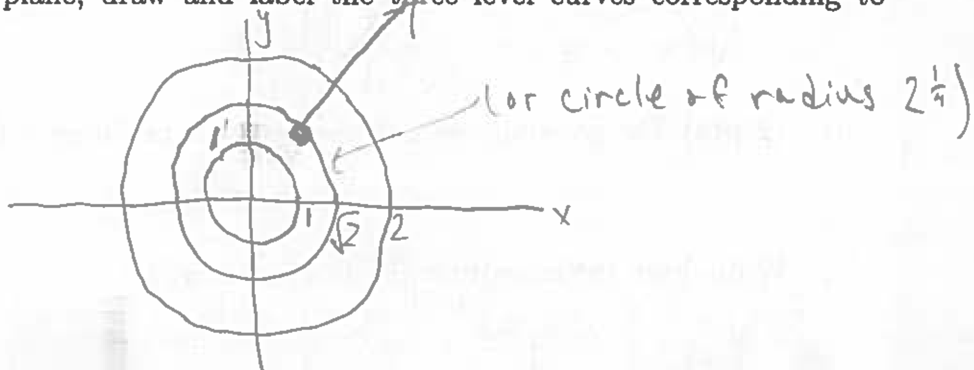
Thurs 8 Oct 2015

Quiz 6: Directional Derivatives (∮12.4-12.6)

Directions: You have 30 minutes to complete this quiz. You may collaborate.

1. The level curves of the surface $z = x^2 + y^2$ are circles in the xy-plane centered at the origin.

   (a) **(1 pt)** In the xy-plane, draw and label the three level curves corresponding to $z_0 = 1$, 2, and 4.



   (b) **(2 pts)** On the same picture, draw the gradient vector at the point $(1, 1)$.

   (c) **(2 pts)** Write down the gradient vector: $\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \rangle = \langle 2x, 2y \rangle$

2. **(2 pts)** Compute the directional derivative of $h(x, y)$, in the direction of the vector $\mathbf{v}$, at the point $(\ln 2, \ln 3)$, where

$$\mathbf{v} = \langle 1, 1 \rangle \quad \text{and} \quad h(x, y) = e^{-x-y}.$$

Hint: Make sure your answer has the correct magnitude!

Let $\hat{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1, 1 \rangle}{\sqrt{2}}$

$\nabla h = \langle -e^{-x-y}, -e^{-x-y} \rangle = \left\langle \frac{-1}{e^{x+y}}, \frac{-1}{e^{x+y}} \right\rangle$

$\nabla h(\ln 2, \ln 3)$
$= \left\langle \frac{-1}{e^{\ln 2} e^{\ln 3}}, \frac{-1}{e^{\ln 2} e^{\ln 3}} \right\rangle = \left\langle \frac{-1}{6}, \frac{-1}{6} \right\rangle$

$D_{\hat{u}} h(\ln 2, \ln 3) = \left\langle \frac{-1}{6}, \frac{-1}{6} \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

$= \frac{-2}{6\sqrt{2}} = \boxed{\frac{-1}{3\sqrt{2}}}$

3. Suppose our Sun is centered at the origin in $\mathbb{R}^3$ and some other star is a distance r away, at the coordinates (x, y, z). The **gravitational potential** between the two stars (or any two objects) is the function

$$V(r) = \frac{-GMm}{r},$$

where G is the gravitational constant, and in this case m denotes the mass of the Sun and M denotes the mass of the other star.

(a) **(1 pt)** Write down V as a function of $x, y,$ and z.

$$V(x,y,z) = \frac{-GMm}{\sqrt{x^2+y^2+z^2}}$$

(b) **(2 pts)** The gravitational force between the two stars is the vector-valued function

$$\mathbf{F} = -\nabla V(x, y, z).$$

Write down the magnitude $|\mathbf{F}|$ as a function of r.

$$-\nabla V = \Big(-\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)(-GMm),\ -\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2}(2y)(-GMm),\ -\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2}(2z)(-GMm) \Big)$$

$$|\vec{F}| = (x^2+y^2+z^2)^{-3/2}(+GMm)\sqrt{x^2+y^2+z^2}$$

$$= \frac{GMm(r)}{r^3} = \frac{GMm}{r^2} \leftarrow \text{Inverse Square Law!!}$$