§3.2 Graphing the Derivative

Recall: The graph of the derivative is essentially the graph of the collection of slopes of the tangent lines of a graph. If you just have a graph (without an equation for the graph), the best you can do is approximate the graph of the derivative.

Simple Checklist:

1. Note where $f'(x) = 0$.
2. Note where $f'(x) > 0$.

Question

What does this look like?

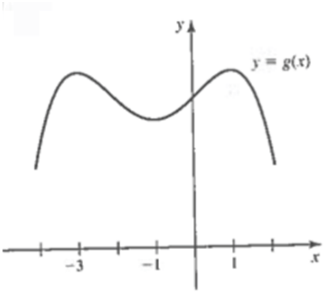
3. Note where $f'(x) < 0$.

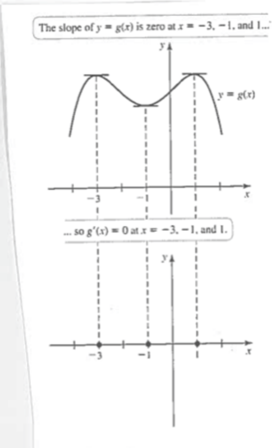
Question

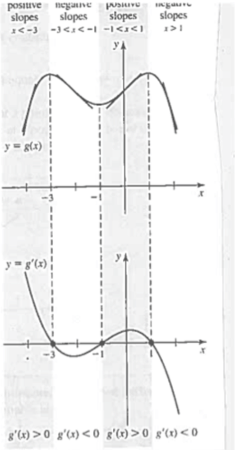
What does this look like?

Example

Given the graph of $g(x)$, sketch the graph of $g'(x)$.

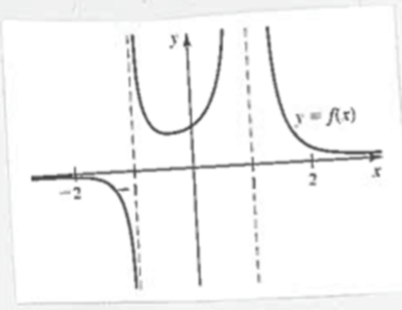


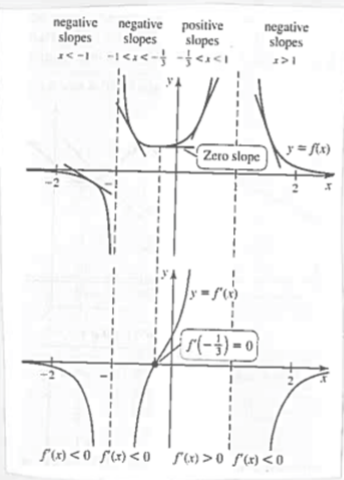

The slope of y = g(x) is zero at x = −3, −1, and 1...
y = g(x)
... so g′(x) = 0 at x = −3, −1, and 1.


positive slopes x < −3
negative slopes −3 < x < −1
positive slopes −1 < x < 1
negative slopes x > 1
y = g(x)
y = g′(x)
g′(x) > 0
g′(x) < 0
g′(x) > 0
g′(x) < 0

Example (With Asymptotes)

Given the graph of $f(x)$, sketch the graph of $f'(x)$.





negative slopes
$x < -1$
negative slopes
$-1 < x < -\frac{1}{3}$
positive slopes
$-\frac{1}{3} < x < 1$
negative slopes
$x > 1$
Zero slope
$y = f(x)$
$y = f'(x)$
$f'\left(-\frac{1}{3}\right) = 0$
$f'(x) < 0$
$f'(x) < 0$
$f'(x) > 0$
$f'(x) < 0$

Recall the relationship between differentiability and continuity.

Exercise

If a function g is not continuous at $x = a$, then g

A. must be undefined at $x = a$.

B. is not differentiable at $x = a$.

C. has an asymptote at $x = a$.

D. all of the above.

E. A. and B. only.

3.2 Book Problems

5-14

§3.3 Rules of Differentiation

Recall the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(as a function of x, i.e., a formula).
And, for any particular point a, we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Constant Functions

The constant function $f(x) = c$ is a horizontal line with a slope of 0 at every point. This is consistent with the definition of the derivative:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{c - c}{h} \\ &= \lim_{h \to 0} 0 = 0. \end{aligned}$$

Therefore, for constant functions, $\frac{d}{dx} c = 0$.

Fact: For any positive integer n, we can factor

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

For example, when $n = 2$, we get

$$x^2 - a^2 = (x-a)(x+a),$$

which is the difference of squares formula.

Power Rule, cont.

Suppose $f(x) = x^n$ where n is a positive integer. Then at a point a,

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a} \\ &= \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{x - a} \\ &= (a^{n-1} + a^{n-2} \cdot a + \cdots + a \cdot a^{n-2} + a^{n-1}) = na^{n-1}. \end{aligned}$$

Using the formula for the derivative as a function of x, one can show $\frac{d}{dx}(x^n) = nx^{n-1}$.

Consider a function of the form $cf(x)$, where c is a constant. Just like with limits, we can factor out the constant:

$$\begin{aligned}
\frac{d}{dx}[cf(x)] &= \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} \\
&= \lim_{h\to 0} \frac{c[f(x+h) - f(x)]}{h} = c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= cf'(x)
\end{aligned}$$

Therefore, $\frac{d}{dx}[cf(x)] = cf'(x)$.

Sum Rule

Sums of functions also behave under the same limit laws when we differentiate:

$$
\begin{aligned}
\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0} \left[\frac{[f(x+h)-f(x)]}{h} + \frac{[g(x+h)-g(x)]}{h} \right] \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= f'(x)+g'(x)
\end{aligned}
$$

So if f and g are differentiable at x,

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x).$$

The Sum Rule can be generalized for more than two functions to include n functions.

Note: Using the Sum Rule and the Constant Multiple Rule produces the Difference Rule:

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x).$$

Exercise

Using the differentiation rules we have discussed, calculate the derivatives of the following functions. Note which rule(s) you are using.

1. $y = x^5$
2. $y = 4x^3 - 2x^2$
3. $y = -1500$
4. $y = 3x^3 - 2x + 4$

Let $f(x) = b^x$, where $b > 0$, $b \neq 1$. To differentiate at 0, we write

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{b^x - b^0}{x} = \lim_{x \to 0} \frac{b^x - 1}{x}.$$

It is not obvious what this limit should be. However, consider the cases $b = 2$ and $b = 3$. By constructing a table of values, we can see that

$$\lim_{x \to 0} \frac{2^x - 1}{x} \approx 0.693 \quad \text{and} \quad \lim_{x \to 0} \frac{3^x - 1}{x} \approx 1.099.$$

So, $f'(0) < 1$ when $b = 2$ and $f'(0) > 1$ when $b = 3$. As it turns out, there is a particular number b, with $2 < b < 3$, whose graph has a tangent line with slope 1 at $x = 0$. In other words, such a number b has the property that

$$\lim_{x \to 0} \frac{b^x - 1}{x} = 1.$$

Question

What number is it?

Answer: This number is $e = 2.718281828459\ldots$ (known as the Euler number). The function $f(x) = e^x$ is called the **natural exponential function**.

Now, using $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$, we can find the formula for $\frac{d}{dx}(e^x)$:

$$\begin{aligned}
\frac{d}{dx}(e^x) &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} \\
&= e^x \left(\lim_{h \to 0} \frac{e^h - 1}{h} \right) \\
&= e^x \cdot 1 = e^x
\end{aligned}$$

Exercise

(a) Find the slope of the line tangent to the curve $f(x) = x^3 - 4x - 4$ at the point $(2, -4)$.

(b) Where does this curve have a horizontal tangent?

If we can write the derivative of f as a function of x, then we can take *its* derivative, too. The derivative of the derivative is called the **second derivative** of f, and is denoted f''.

In general, we can differentiate f as often as needed. If we do it n times, the nth derivative of f is

$$f^{(n)}(x) = \frac{d^n f}{dx^n} = \frac{d}{dx}[f^{(n-1)}(x)].$$

3.3 Book Problems

9-48 (every 3rd problem), 51-53, 58-60

- For these problems, use only the rules we have derived so far.

§3.4 The Product and Quotient Rules

Issue: Derivatives of products and quotients do NOT behave like they do for limits.

As an example, consider $f(x) = x^2$ and $g(x) = x^3$. We can try to differentiate their product in two ways:

- $$\begin{aligned} \frac{d}{dx}[f(x)g(x)] &= \frac{d}{dx}\left(x^5\right) \\ &= 5x^4 \end{aligned}$$
- $$\begin{aligned} f'(x)g'(x) &= (2x)(3x^2) \\ &= 6x^3 \end{aligned}$$

Question

Which answer is the correct one?

Product Rule

If f and g are any two functions that are differentiable at x, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x).$$

In the example from the previous slide, we have

$$\begin{aligned}
\frac{d}{dx}[x^2 \cdot x^3] &= \frac{d}{dx}(x^2) \cdot (x^3) + x^2 \cdot \frac{d}{dx}(x^3) \\
&= (2x) \cdot (x^3) + x^2 \cdot (3x^2) \\
&= 2x^4 + 3x^4 \\
&= 5x^4
\end{aligned}$$

Derivation of the Product Rule

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left(\frac{f(x+h)g(x+h) + [-f(x)g(x+h) + f(x)g(x+h)] - f(x)g(x)}{h} \right)$$

$$= \lim_{h\to 0} \left(\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} \right) + \left(\lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h} \right)$$

Derivation of the Product Rule (cont.)

$$= \lim_{h\to 0}\left(g(x+h)\frac{f(x+h)-f(x)}{h}\right) + \left(\lim_{h\to 0} f(x)\frac{g(x+h)-g(x)}{h}\right)$$

$$= g(x)f'(x) + f(x)g'(x)$$

Exercise

Use the product rule to find the derivative of the function $(x^2 + 3x)(2x - 1)$.

A. $2(2x + 3)$

B. $6x^2 + 10x - 3$

C. $2x^3 + 5x^2 - 3x$

D. $2x(x + 3) + x(2x - 1)$

Derivation of Quotient Rule

Question

Let $q(x) = \dfrac{f(x)}{g(x)}$. What is $\dfrac{d}{dx}q(x)$?

Answer: We can write $f(x) = q(x)g(x)$ and then use the Product Rule:

$$f'(x) = q'(x)g(x) + g'(x)q(x)$$

and now solve for $q'(x)$:

$$q'(x) = \frac{f'(x) - q(x)g'(x)}{g(x)}.$$

Then, to get rid of $q(x)$, plug in $\dfrac{f(x)}{g(x)}$:

$$q'(x) = \frac{f'(x) - g'(x)\frac{f(x)}{g(x)}}{g(x)}$$

$$= \frac{g(x)\left(f'(x) - g'(x)\frac{f(x)}{g(x)}\right)}{g(x)\cdot g(x)}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

"LO-D-HI minus HI-D-LO over LO squared"

Just as with the product rule, the derivative of a quotient is not a quotient of derivatives, i.e.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \neq \frac{f'(x)}{g'(x)}.$$

Here is the correct rule, the Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}.$$

Exercise

Use the Quotient Rule to find the derivative of

$$\frac{4x^3 + 2x - 3}{x + 1}.$$

Exercise

Find the slope of the tangent line to the curve

$$f(x) = \frac{2x - 3}{x + 1} \text{ at the point } (4, 1).$$

The Quotient Rule also allows us to extend the Power Rule to negative numbers – if n is any integer, then

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}.$$

Question

How?

Exercise

If $f(x) = \dfrac{x(3-x)}{2x^2}$, find $f'(x)$.

Derivative of e^{kx}

For any real number k,

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}.$$

Exercise

What is the derivative of $x^2 e^{3x}$?

The derivative provides information about the instantaneous rate of change of the function being differentiated (compare to the limit of the slopes of the secant lines from §2.1).

For example, suppose that the population of a culture can be modeled by the function $p(t)$. We can find the instantaneous growth rate of the population at any time $t \geq 0$ by computing $p'(t)$ as well as the **steady-state population** (also called the **carrying capacity** of the population). The steady-state population equals

$$\lim_{t \to \infty} p(t).$$

3.4 Book Problems

9-49 (every 3rd problem), 57, 59, 63, 75-79 (odds)