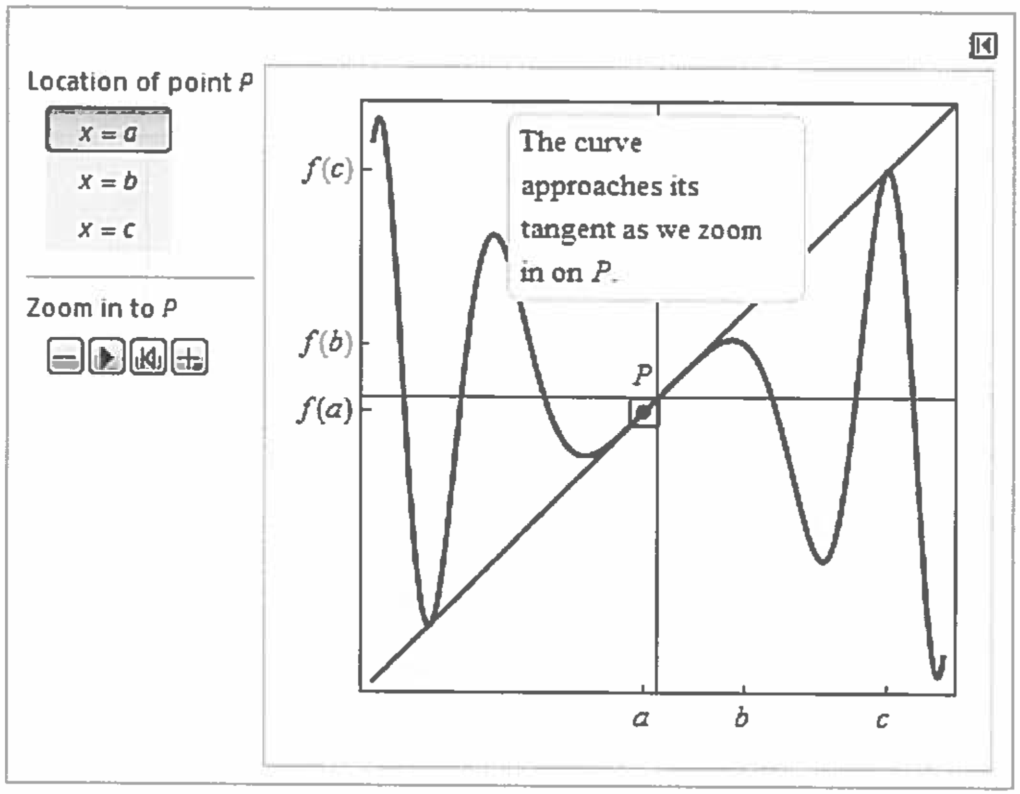
§4.5 Linear Approximation and Differentials

Suppose f is a function such that f' exists at some point P. If you zoom in on the graph, the curve appears more and more like the tangent line to f at P.



Linear Approximation

This idea – that smooth curves (i.e., curves without corners) appear straighter on smaller scales – is the basis of linear approximations.

One of the properties of a function that is differentiable at a point P is that it is locally linear near P (i.e., the curve approaches the tangent line at P.)

Therefore, it makes sense to approximate a function with its tangent line, which matches the value and slope of the function at P.

This is why you've had to do so many "find the equation for the tangent line to the given point" problems!

Definition

Suppose f is differentiable on an interval I containing the point a. The **linear approximation** to f at a is the linear function

$$L(x) = f(a) + f'(a)(x - a) \quad \text{for } x \text{ in } I.$$

Remarks: Compare this definition to the following: At a given point $P = (a, f(a))$, the slope of the line tangent to the curve at P is $f'(a)$. So the equation of the tangent line is

$$y - \underset{y_0}{f(a)} = f'(a)(x - \underset{x_0}{a}).$$

(Yes, it is the same thing!)

Exercise

Write the equation of the line that represents the linear approximation to $L(x) = f(a) + f'(a)(x-a)$.

$$f(x) = \frac{x}{x+1} \qquad \text{at } a = 1.$$

Then *use* the linear approximation to estimate $\boxed{f(1.1)}$. $\approx L(1.1)$

Solution: First compute

$$f'(x) = \frac{1}{(x+1)^2}, \quad f(a) = \frac{1}{2}, \quad f'(a) = \frac{1}{4}.$$

$$L(x) = \frac{1}{2} + \frac{1}{4}(x-1) = \frac{1}{4}x + \frac{1}{4}.$$ ← same as the equation of the tangent line to $f(x)$ at $x=1$.

Solution (continued):

Because $x = 1.1$ is near $a = 1$, we can estimate $f(1.1)$ using $L(1.1)$:

$$f(1.1) \approx L(1.1) = 0.525$$

Note that $f(1.1) = 0.5238$, so the error in this estimation is

$$\frac{0.525 - 0.5238}{0.5238} \times 100 = 0.23\%.$$

Exercise

(a) The linear approximation to $f(x) = \sqrt{1+x}$ at the point $x = 0$ is (choose one):

$L(x) = f(a) + f'(a)(x-a)$

$f'(x) = \frac{1}{2\sqrt{1+x}}$

$f'(0) = \frac{1}{2}$

$f(0) = 1$

$L(x) \approx 1 + \frac{1}{2}(x)$

A. $L(x) = 1$
B. $L(x) = 1 + \frac{x}{2}$
C. $L(x) = x$
D. $L(x) = 1 - \frac{x}{2}$

(b) What is an approximation for $f(0.1)$?

$f(0.1) \approx L(0.1) = 1 + \frac{1}{2}(0.1)$

$= 1 + 0.05 = \boxed{1.05}$

$f(0.1) \approx 1.005$

Intro to Differentials

Our linear approximation $L(x)$ is used to approximate $f(x)$ when a is fixed and x is a nearby point:

$$f(x) \approx \underbrace{f(a) + f'(a)(x-a)}_{L(x)}$$

When rewritten,

$$\underbrace{f(x) - f(a)}\approx \underbrace{f'(a)(x-a)}$$

$$\implies \Delta y \approx f'(a)\Delta x.$$

A change in y can be approximated by the corresponding change in x, magnified or diminished by a factor of $f'(a)$.

This is another way to say that $f'(a)$ is the rate of change of y with respect to x!

$$\Delta y \approx f'(a)\Delta x$$

$$\frac{\Delta y}{\Delta x} \approx f'(a)$$

So if f is differentiable on an interval I containing the point a, then the change in the value of f (the Δy), between two points a and $\underbrace{a + \Delta x}_{x \text{ (close to } a)}$ in I, is approximately $f'(x)\Delta x$.

We now have two different, but related quantities:

- The change in the function $y = f(x)$ as x changes from a to $a + \Delta x$ (which we call Δy).
- The change in the linear approximation $y = L(x)$ as x changes from a to $a + \Delta x$ (called the differential, dy).

$$\Delta y \approx dy$$

When the x-coordinate changes from a to $a + \Delta x$:

- The function change is **exactly** $\Delta y = f(\underbrace{a + \Delta x}) - f(a)$.
- The linear approximation change is

$$\Delta L = L(a + \Delta x) - L(a)$$

$$= (f(a) + f'(a)(a + \Delta x - a)) - (f(a) + f'(a)(a - a))$$

$$= f'(a)\Delta x$$

and this is $\boxed{dy.} = \Delta L$

$\Delta y = f(a + \Delta x) - f(a)$

$dy \approx f(a + \Delta x) - f(a)$

$dy = L(a + \Delta x) - L(a)$

We define the differentials dx and dy to distinguish between the change in the function (Δy) and the change in the linear approximation (ΔL):

- dx is simply the change in x, i.e. Δx.
- dy is the change in the linear approximation, which is $\Delta L = f'(a)\Delta x$.

SO:

$$\Delta L = f'(a)\Delta x$$

$$dy = f'(a)dx$$

$$\frac{dy}{dx} = f'(a) \quad \text{(at } x = a\text{)}$$

Definition

Let f be differentiable on an interval containing x. $\Delta x = dx$ but $\Delta y \neq dy$

- A small change in x is denoted by the **differential** dx.
- The corresponding change in $y = f(x)$ is approximated by the **differential** $dy = f'(x)dx$; that is,

$$\Delta y = f(x + \Delta x) - f(x)$$

$$\approx dy = f'(x)dx.$$

$\Delta L = L(x + \Delta x) - L(x)$

The use of differentials is critical as we approach integration.

Example

Use the notation of differentials $[dy = f'(x)dx]$ to approximate the change in $f(x) = x - x^3$ given a small change dx.

Solution: $f'(x) = 1 - 3x^2$, so $\boxed{dy = (1 - 3x^2)dx.}$
A small change dx in the variable x produces an approximate change of $dy = (1 - 3x^2)dx$ in y.

$dx = \Delta x$

"a"

For example, if x increases from 2 to 2.1, then $\overset{\Delta x}{dx} = 0.1$ and

$$dy = \underbrace{\boxed{\left(1 - 3(2)^2\right)}}_{f'(2)} \underbrace{(0.1)}_{\cdot\, \Delta x} = -1.1.$$

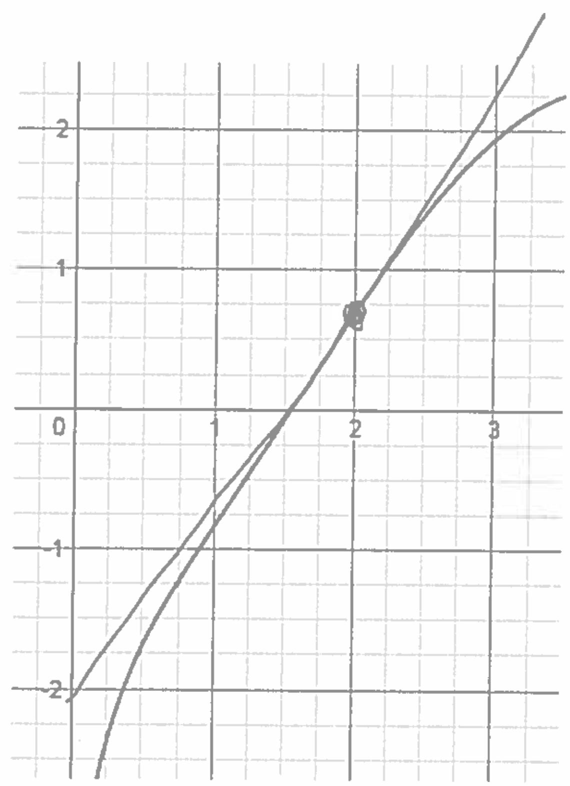
This means as x increases by 0.1, y decreases by ≈ 1.1.

3. (7 pts ea) Let $f(x) = \ln x - \sin(2 - x)$.

(a) Write the equation for the linear approximation to $f(x)$ at $x = 2$.

(b) Use your answer to (a) to approximate $f(1)$.

(c) Below is the graph of $f(x)$, drawn at the website `desmos.com/calculator`. On the same axis, draw your tangent line. Label both $f(1)$ and your approximation from part (b).



$$\Delta y = f(\overbrace{a+\Delta x}^{1}) - f(a) = f(1) - f(2)$$

3. **(7 pts ea)** Let $f(x) = \ln x + \sin(2-x)$.

(a) Write the equation for the linear approximation to $f(x)$ at $x = 2$. a

$$L(x) = f(a) + f'(a)\underbrace{(x-a)}_{\Delta x}$$

tangent line $\boxed{L(x) = \ln 2 - \frac{1}{2}(x-2)}$

$$f'(x) = \frac{1}{x} + \cos(2-x)(-1)$$

$$f'(2) = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$f(2) = \ln 2 + 0$$

$$dy = L(a+dx) - L(a)$$

$$\Delta x = -1 = 1 - 2$$

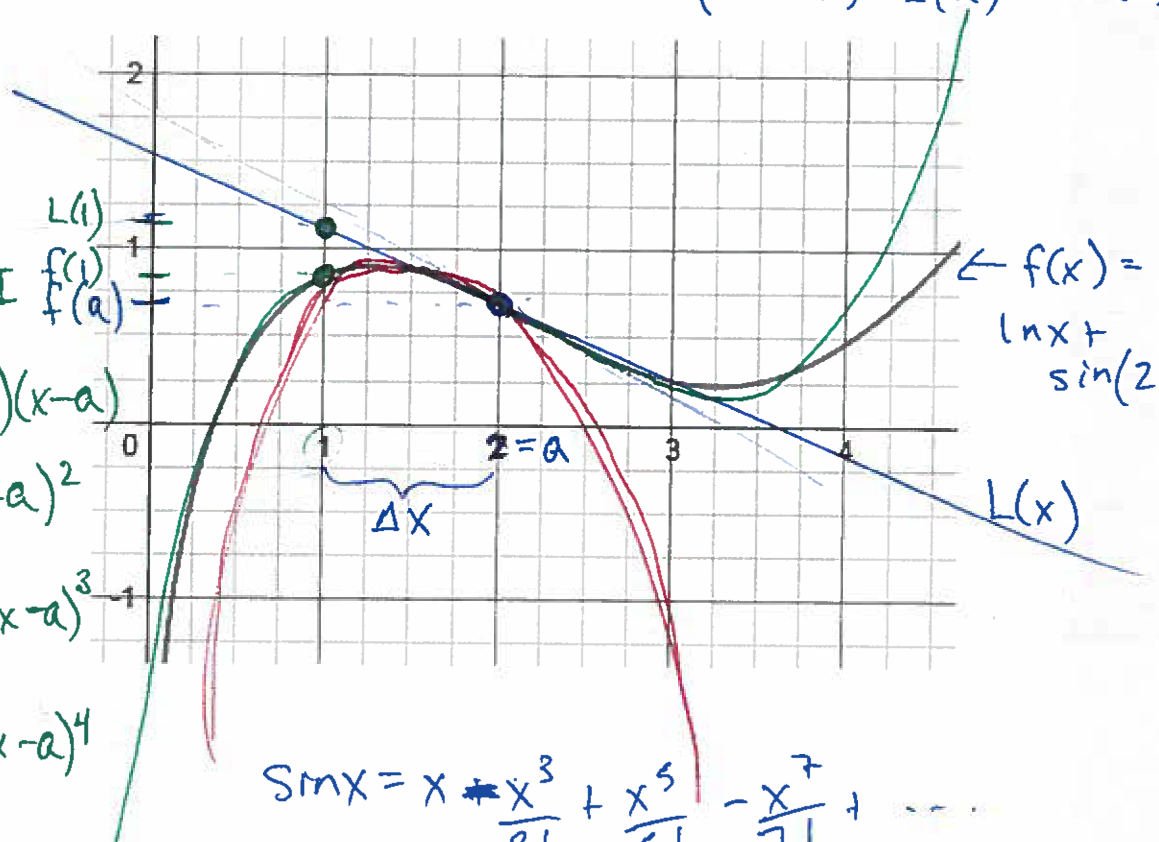
$$\Delta x = x - a = dx$$

(b) Use your answer to (a) to approximate $f(1)$. "x"

$$f(1) \approx L(1) = \overset{f(a)}{\ln 2} - \overset{f'(a)}{\frac{1}{2}}\overset{\Delta x}{(1-2)} = \ln 2 + \frac{1}{2} \Rightarrow dy = L(1) - \ln 2 = \frac{1}{2}$$

$L(a+\Delta x) - L(a)$ $\quad$ $f'(a)dx$

(c) Below is the graph of $f(x)$, drawn at the website desmos.com/calculator. On the same axis, draw your tangent line. Label both $f(1)$ and your approximation from part (b).



$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \frac{1}{4!}f''''(a)(x-a)^4 + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

1. **(3 pts ea)** Let $g(x) = \ln(1 + x)$.

   (a) Write the equation for the linear approximation to $g(x)$ at $x = 0$.

   (b) Use your answer to (a) to approximate $g(0.9)$.

   (c) Below is the graph of $g(x)$, drawn at the website `desmos.com/calculator`. On the same axis, draw your tangent line. Label both $g(0.9)$ and your approximation from part (b).