MATH 2554 (Calculus I)

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Table of Contents

1. Week 6: 16-20 February
   - $\oint$ 3.5 Derivatives as Rates of Change
   - $\oint$ 3.6 The Chain Rule
   - $\oint$ 3.7 Implicit Differentiation

Monday 16 February (Week 6)

- SNOW DAY (no class)
- Read $\oint 3.5$. We begin $\oint 3.6$ on Wednesday.

∮ 3.5 Derivatives as Rates of Change

Position and Velocity Suppose an object moves along a straight line and its location at time t is given by the position function $s = f(t)$.

The **displacement** of the object between $t = a$ and $t = a + \Delta t$ is
$$\Delta s = f(a + \Delta t) - f(a).$$

Here Δt represents how much time has elapsed.

We now define average velocity as

$$\frac{\Delta s}{\Delta t} = \frac{f(a+\Delta t) - f(a)}{\Delta t}.$$

Recall that the limit of the average velocities as the time interval approaches 0 was the instantaneous velocity (which we denote here by v). Therefore, the instantaneous velocity at a is

$$v(a) = \lim_{\Delta t \to 0} \frac{f(a+\Delta t) - f(a)}{\Delta t} = f'(a).$$

In mathematics, speed and velocity are related but not the same:

If the *velocity* of an object at any time t is given by $v(t)$, then the *speed* of the object at any time t is given by

$$|v(t)| = |f'(t)|.$$

By definition, acceleration (denoted by a) is the instantaneous rate of change of the velocity of an object at time t.

Therefore,

$$a(t) = v'(t)$$

and since velocity was the derivative of the position function $s = f(t)$, then

$$a(t) = v'(t) = f''(t).$$

Summary: Given the position function $s = f(t)$, the velocity at time t is the first derivative, the speed at time t is the absolute value of the first derivative, and the acceleration at time t is the second derivative.

Question: Given the position function $s = f(t)$ of an object launched into the air, how would you know:

1. The highest point the object reaches?
2. How long it takes to hit the ground?
3. The speed at which the object hits the ground?

Growth Models

Suppose $p = f(t)$ is a function of the growth of some quantity of interest. The average growth rate of p between times $t = a$ and a later time $t = a + \Delta t$ is the change in p divided by the elapsed time Δt:

$$\frac{\Delta p}{\Delta t} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

As Δt approaches 0, the average growth rate approaches the derivative $\dfrac{dp}{dt}$, which is the instantaneous growth rate (or just simply the growth rate). Therefore,

$$\frac{dp}{dt} = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t}.$$

Exercise

The population of the state of Georgia (in thousands) from 1995 ($t = 0$) to 2005 ($t = 10$) is modeled by the polynomial

$$p(t) = -0.27t^2 + 101t + 7055.$$

1. What was the average growth rate from 1995 to 2005?
2. What was the growth rate for Georgia in 1997?
3. What can you say about the population growth rate in Georgia between 1995 and 2005?

Average and Marginal Cost

Suppose a company produces a large amount of a particular quantity. Associated with manufacturing the quantity is a **cost function** $C(x)$ that gives the cost of manufacturing x items. This cost may include a **fixed cost** to get started as well as a **unit cost** (or **variable cost**) in producing one item.

If a company produces x items at a cost of $C(x)$, then the average cost is $\dfrac{C(x)}{x}$.

This average cost indicates the cost of items already produced. Having produced x items, the cost of producing another Δx items is $C(x + \Delta x) - C(x)$. So the average cost of producing these extra Δx items is

$$\frac{\Delta C}{\Delta x} = \frac{C(x + \Delta x) - C(x)}{\Delta x}.$$

If we let Δx approach 0, we have

$$\lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = C'(x)$$

which is called the **marginal cost**.

The marginal cost is the approximate cost to produce one additional item after producing x items.

Note: In reality, we can't let Δx approach 0 because Δx represents whole numbers of items.

Exercise

If the cost of producing x items is given by

$$C(x) = -0.04x^2 + 100x + 800$$

for $0 \leq x \leq 1000$, find the average cost and marginal cost functions. Also, determine the average and marginal cost when $x = 500$.

HW from Section 3.5

Do problems 9–12, 17–18, 22–23, 27–37 odd (pp. 171–175 in textbook).

Wednesday 18 February (Week 6)

- Quiz 5 due tomorrow in drill.
- Exam 2 Friday 27 Feb (next week!)
- Midterm the following Tues.

$\oint$ 3.6 The Chain Rule

Suppose that Yvonne (y) can run twice as fast as Uma (u). Therefore

$$\frac{dy}{du} = 2.$$

Suppose that Uma can run four times as fast as Xavier (x). So

$$\frac{du}{dx} = 4.$$

How much faster can Yvonne run than Xavier?

In this case, we would take both our rates and multiply them together:

$$\frac{dy}{du} \cdot \frac{du}{dx} = 2 \cdot 4 = 8.$$

Version 1 of the Chain Rule

If g is differentiable at x, and $y = f(u)$ is differentiable at $u = g(x)$, then the composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Guidelines for Using the Chain Rule

Assume the differentiable function $y = f(g(x))$ is given.

1. Identify the outer function f, the inner function g, and let $u = g(x)$.
2. Replace $g(x)$ by u to express y in terms of u:
$$y = f(g(x)) \implies y = f(u)$$
3. Calculate the product $\dfrac{dy}{du} \cdot \dfrac{du}{dx}$
4. Replace u by $g(x)$ in $\dfrac{dy}{du}$ to obtain $\dfrac{dy}{dx}$.

Example: Use Version 1 of the Chain Rule to calculate $\dfrac{dy}{dx}$ for $y = (5x^2 + 11x)^{20}$.

- inner function: $u = 5x^2 + 11x$
- outer function: $y = u^{20}$

We have $y = f(g(x)) = (5x^2 + 11x)^{20}$. Differentiate:

$$\begin{aligned}\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} &= 20u^{19} \cdot (10x + 11) \\ &= 20(5x^2 + 11x)^{19} \cdot (10x + 11)\end{aligned}$$

Use the first version of the Chain Rule to calculate $\dfrac{dy}{dx}$ for

$$y = \left(\frac{3x}{4x+2}\right)^5.$$

Version 2 of the Chain Rule

Notice if $y = f(u)$ and $u = g(x)$, then $y = f(u) = f(g(x))$, so we can also write:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= f'(u) \cdot g'(x) \\ &= f'(g(x)) \cdot g'(x). \end{aligned}$$

Use Version 2 of the Chain Rule to calculate $\dfrac{dy}{dx}$ for $y = (7x^4 + 2x + 5)^9$.

- inner function: $g(x) = 7x^4 + 2x + 5$
- outer function: $f(u) = u^9$

Then

$$f'(u) = 9u^8 \implies f'(g(x)) = 9(7x^4 + 2x + 5)^8$$
$$g'(x) = 28x^3 + 2.$$

Putting it together,

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x) = 9(7x^4 + 2x + 5)^8 \cdot (28x^3 + 2)$$

Chain Rule for Powers

If g is differentiable for all x in the domain and n is an integer, then

$$\frac{d}{dx}\left[(g(x))^n\right] = n(g(x))^{n-1} \cdot g'(x).$$

Example:

$$\begin{aligned}\frac{d}{dx}\left[(1-e^x)^4\right] &= 4(1-e^x)^3 \cdot (-e^x)\\ &= -4e^x(1-e^x)^3\end{aligned}$$

Composition of 3 or more functions

Compute $\dfrac{d}{dx}\left[\sqrt{(3x-4)^2+3x}\right]$.

$$\begin{aligned}
\frac{d}{dx}\left[\sqrt{(3x-4)^2+3x}\right] &= \frac{1}{2}\left((3x-4)^2+3x\right)^{-\frac{1}{2}}\cdot\frac{d}{dx}\left[(3x-4)^2+3x\right]\\
&= \frac{1}{2\sqrt{\left((3x-4)^2+3x\right)}}\cdot\left[2(3x-4)\frac{d}{dx}(3x-4)+3\right]\\
&= \frac{1}{2\sqrt{\left((3x-4)^2+3x\right)}}\cdot\left[2(3x-4)\cdot 3+3\right]\\
&= \frac{18x-21}{2\sqrt{\left((3x-4)^2+3x\right)}}
\end{aligned}$$

HW from Section 3.6

Do problems 7–29 odd, 30, 33–43 odd, 49 (pp. 180–181 in textbook)

Friday 20 February (Week 6)

- Exam 2 Friday 27 Feb (next week!)
- Midterm the following Tues.

∮ 3.7 Implicit Differentiation

Up to now, we have calculated derivatives of functions of the form $y = f(x)$, where y is defined **explicitly** in terms of x.

In this section, we examine relationships between variables that are **implicit** in nature, meaning that y either is not defined explicitly in terms of x or cannot be easily manipulated to solve for y in terms of x.

Examples of functions implicitly defined

$$x^2 + y^2 = 9$$

$$x + y^3 - xy = 4$$

$$\cos(x - y) + \sin y = \sqrt{2}$$

The goal of **implicit differentiation** is to find a single expression for the derivative directly from an equation of the form $F(x, y) = 0$ without first solving for y.

Calculate $\dfrac{dy}{dx}$ directly from the equation for the circle

$$x^2 + y^2 = 9.$$

Solution: To note that x is our independent variable and that we are differentiating with respect to x, we replace y with $y(x)$:

$$x^2 + (y(x))^2 = 9.$$

Now differentiate each term with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}((y(x))^2) = \frac{d}{dx}(9).$$

By the Chain Rule, $\dfrac{d}{dx}((y(x))^2) = 2y(x)y'(x)$, or $\dfrac{d}{dx}(y^2) = 2y\dfrac{dy}{dx}$.

So

$$2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}.$$

Now $\frac{dy}{dx} = -\frac{x}{y}$, so we can find slopes of tangent lines at various points along the circle.

The slope of the tangent line at (0,3) is

$$\left.\frac{dy}{dx}\right|_{(x,y)=(0,3)} = -\frac{0}{3} = 0.$$

The slope of the tangent line at $(1, 2\sqrt{2})$ is

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,2\sqrt{2})} = -\frac{1}{2\sqrt{2}}.$$

Example

Find $\dfrac{dy}{dx}$ for $xy + y^3 = 1$.

Finding tangent lines

Find an equation of the line tangent to the curve $x^4 - x^2y + y^4 = 1$ at the point $(-1, 1)$.

Higher Order Derivatives

Find $\dfrac{d^2y}{dx^2}$ if $xy + y^3 = 1$.

Power Rule for Rational Exponents

Implicit differentiation also allows us to extend the power rule to rational exponents:

Assume p and q are integers with $q \neq 0$. Then

$$\frac{d}{dx}(x^{p/q}) = \frac{p}{q}x^{p/q-1}$$

provided $x \geq 0$ when q is even.

HW from Section 3.7

Do problems 5–21 odd, 27–45 odd (pp. 188–189 in textbook)