MATH 2554 (Calculus I)
Summer 2015

Name: SOLUTIONS

Thurs 2 July 2015

Exam 3: Using Derivatives (∮3.10-4.6)

Version A

Exam Instructions: You have 50 minutes to complete this exam. Follow the directions and answer the question, using boss notation where appropriate. Justification is required for all problems.

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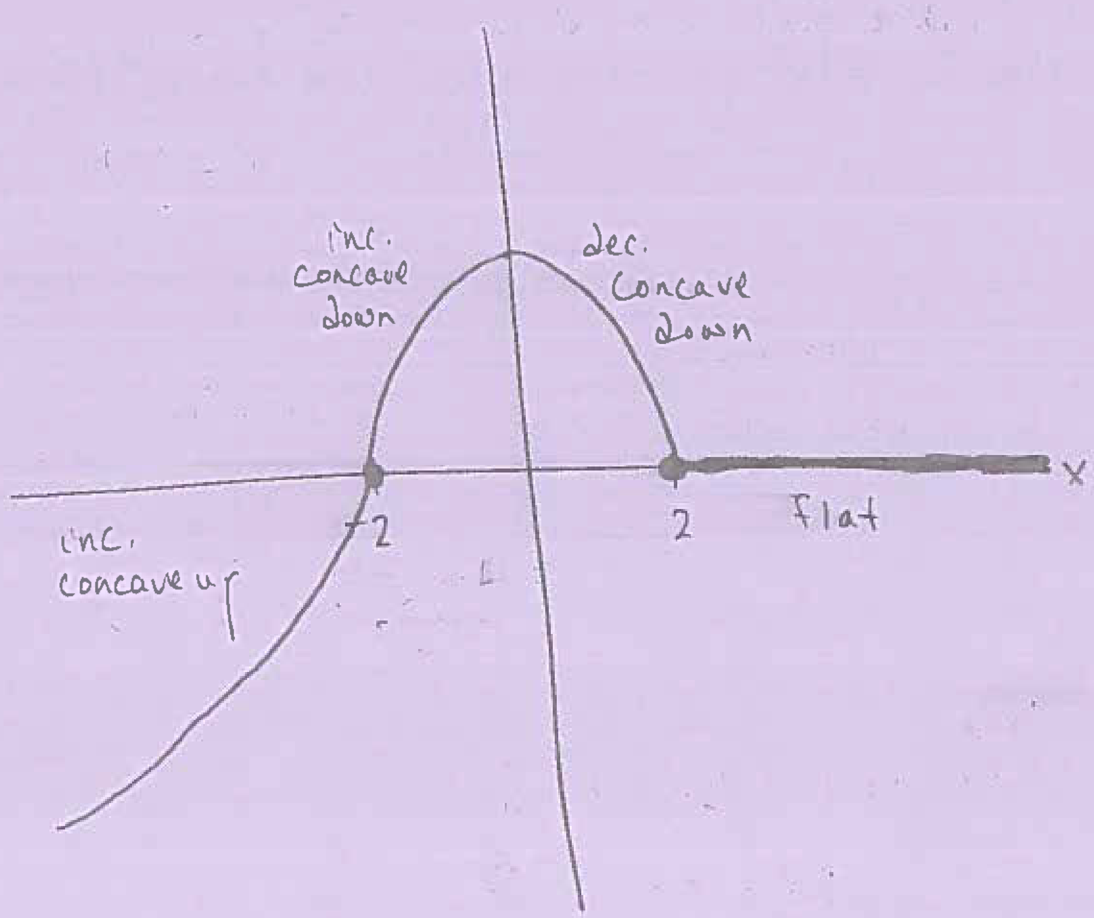
Signature: (1 pt) ______________________________

Good luck!

1. **(20 pts)** Sketch a graph of a function $f(x)$, continuous on $(-\infty, \infty)$, that satisfies all of the following criteria:

- $f(-2) = f(2) = 0$
- $f'(x) > 0$ and $f''(x) > 0$ on $(-\infty, -2)$
- $f'(x) > 0$ and $f''(x) < 0$ on $(-2, 0)$
- $f'(x) < 0$ and $f''(x) < 0$ on $(0, 2)$
- $f'(x) = 0$ on $(2, \infty)$



2. (a) **(9 pts)** What are the three hypotheses for Rolle's Theorem?

1. f is continuous on $[a,b]$
2. f is smooth (differentiable) on (a,b)
3. $f(a)=f(b)$

(b) **(7 pts)** Given the three hypotheses, what is the conclusion of Rolle's Theorem?

There exists c, between a and b, where $f'(c)=0$.

(c) **(7 pts)** The *Mean Value Theorem* applies to $f(x) = x(x^2+x+2)$ $= x^3+x^2+2x$ on $[-1,1]$. (You don't have to prove that.) Find the point(s) guaranteed to exist by the Mean Value Theorem.

slope of secant line: $\frac{f(1)-f(-1)}{1-(-1)} = \frac{(1^3+1^2+2(1)) - ((-1)^3+(-1)^2+2(-1))}{2}$

$= \frac{4-(-2)}{2} = \frac{6}{2} = 3$

solve for c:

$f'(c) = 3c^2+2c+2 = 3$

$3c^2+2c-1=0$

$(3c-1)(c+1)=0$

$\Rightarrow \boxed{c=\frac{1}{3}}, -1$

(in the interval)

3. **(7 pts ea)** Let $f(x) = \ln x + \sin(2 - x)$.

(a) Write the equation for the linear approximation to $f(x)$ at $x = 2$.

$$f(2) = \ln 2 + \sin(2-2) = \ln 2 + \sin 0 = \ln 2$$

$$f'(x) = \frac{1}{x} - \cos(2-x)$$

$$f'(2) = \frac{1}{2} - \cos(2-2) = \frac{1}{2} - \cos 0 = \frac{1}{2} - 1 = -\frac{1}{2}$$

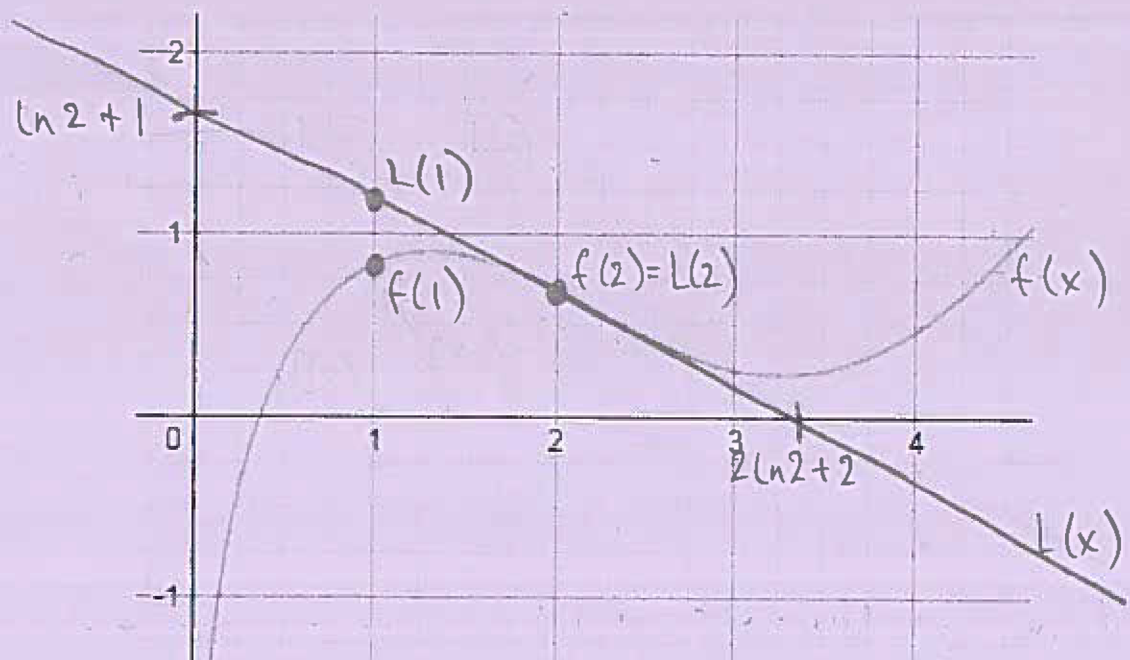
$$L(x) = f(2) + f'(2)(x-2)$$

$$\boxed{= \ln 2 - \frac{1}{2}(x-2) = -\frac{1}{2}x + (\ln 2 + 1)}$$

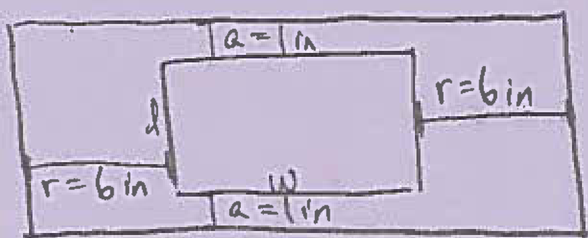
(b) Use your answer to (a) to approximate $f(1)$.

$$f(1) \approx L(1) = -\frac{1}{2}(1) + \ln 2 + 1 = \boxed{\ln 2 + \frac{1}{2}}$$

(c) Below is the graph of $f(x)$, drawn at the website desmos.com/calculator. On the same axis, draw your tangent line. Label both $f(1)$ and your approximation from part (b).



4. **(20 pts)** A landscaper wants to make a rectangular flower garden with an area of 24 in^2, surrounded by 6 in of rocks on either side and 1 in of astroturf above and below. What dimensions of the garden will minimize the combined area of the garden with its rocks and astroturf borders? Use the 2nd Derivative Test to justify your answer.



Objective: Minimize area

$A = (2a+l)(2r+w)$

$= 4ar + 2rl + 2aw + lw$

Constraint: $lw = 24 \text{ in}^2 \Rightarrow l = \frac{24}{w}$

Rewrite:

$A(w) = 4ar + 2r\left(\frac{24}{w}\right) + 2aw + 24$

constants

$A'(w) = -\frac{2r(24)}{w^2} + 2a$

$= 0 \Rightarrow 2aw^2 - 2(24)r = 0$

$w^2 = \frac{24r}{a}$

Dimensions:

$w = \sqrt{\frac{24r}{a}} = \sqrt{\frac{(24 \text{ in}^2)(6 \text{ in})}{(1 \text{ in})}} = \boxed{12 \text{ in}}$

$\Rightarrow l = \frac{24 \text{ in}^2}{12 \text{ in}} = \boxed{2 \text{ in}}$

2nd Derivative Test:

$A''(w) = \frac{2(2)(24)r}{w^3} > 0 \rightarrow$ $A(w)$ is always concave up and so $w = 12$ in does give the minimum.

5. **(10 pts ea)** Let $f(x)$ be a function, continuous on $(-\infty, \infty)$, such that

$$f'(x) = \frac{2-2x^2}{(1+x^2)^2} \quad \text{and} \quad f''(x) = \frac{6x^3-10x}{(1+x^2)^3}.$$

(a) Determine the intervals on which $f(x)$ is increasing and decreasing.

$f'(x) = \frac{2-2x^2}{(1+x^2)^2} > 0 \Rightarrow 2-2x^2 > 0$

always positive $\rightarrow (1+x^2)^2$

$2 > 2x^2$

$1 > x^2$

$1 > |x|$

$\boxed{f \text{ is increasing on } (-1,1)}$

$f'(x) < 0 \Rightarrow 2-2x^2 < 0$

$\Rightarrow 1 < |x|$ $\boxed{f \text{ is decreasing on } (-\infty,-1), (1,\infty)}$

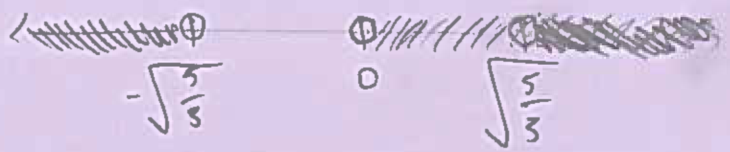
(b) Determine the intervals on which $f(x)$ is concave up and concave down.

$f''(x) = \frac{6x^3-10x}{(1+x^2)^3} > 0 \Rightarrow 6x^3 - 10x > 0$

always positive $\rightarrow (1+x^2)^3$

$2x(3x^2-5) > 0$

$\Rightarrow$ either $x > 0$ and $3x^2-5>0$



or $x<0$ and $3x^2-5<0$

$\Rightarrow$ $\boxed{f \text{ is concave up on } \left(\sqrt{\frac{5}{3}}, \infty\right) \text{ and } \left(\sqrt{\frac{5}{3}}, 0\right)}$

$f''(x) < 0$

$\Rightarrow 2x(3x^2-5) < 0$

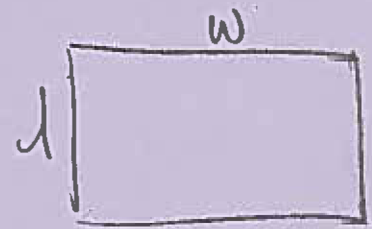
$\Rightarrow$ either $x>0$ and $3x^2-5<0$

or

$x<0$ and $3x^2-5>0$

$\Rightarrow$ $\boxed{f \text{ is concave down on } \left(-\infty, -\sqrt{\frac{5}{3}}\right) \text{ and } \left(0, \sqrt{\frac{5}{3}}\right)}$

6. **(20 pts)** A rectangle initially has dimensions 2 cm by 4 cm. All sides begin increasing in length at a rate of 1 cm/sec. At what rate is the area of the rectangle increasing after 20 sec?



Know:

$$\frac{dl}{dt} = \frac{dw}{dt} = 1 \text{ cm}/\text{sec}$$

$l(0) = 2$ cm $\quad \Rightarrow \quad l(t) = 2 + t$

$w(0) = 4$ cm $\quad \quad \quad w(t) = 4 + t$

WTF: area

$$\left.\frac{dA}{dt}\right|_{t=20\text{sec}}$$

$$A = lw$$
$$= (2+t)(4+t)$$
$$= 8 + 4t + 2t + t^2$$
$$= 8 + 6t + t^2$$

$$\frac{dA}{dt} = 6 + 2t \quad - - - \quad \left.\frac{dA}{dt}\right|_{t=20\text{sec}} = 6 + 2(20) = \boxed{46 \text{ cm}^2/\text{sec}}$$