More Final Practice Problems (Fall 2015 Cal III)

Ch 11 Review

#22 Find the angle θ between $\underset{\vec{u}}{\langle 2,0,-2\rangle}$ and $\underset{\vec{v}}{\langle 2,2,0\rangle}$ using

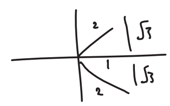
(a) the dot product

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

$$= \cos^{-1}\left(\frac{\overset{4}{2(2)} + \overset{0}{(0)(2)} + \overset{0}{(-2)(0)}}{\underbrace{\left(\sqrt{2^2+0^2+(-2)^2}\right)}_{\sqrt{4+4}}\underbrace{\left(\sqrt{2^2+2^2+0^2}\right)}_{\sqrt{4+4}}}\right) = \cos^{-1}\left(\frac{4}{8}\right) = \cos^{-1}\left(\frac{1}{2}\right)$$

When is cosine equal to $\frac{1}{2}$?



ans: $\pm\frac{\pi}{3}$

$$\boxed{=\frac{\pi}{3}}$$

(θ is always between 0 and π)

(b) the cross product

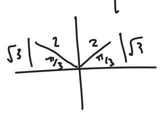
$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -2 \\ 2 & 2 & 0 \end{vmatrix} = \langle \underbrace{0-(-2)(2)}_{4}, \underbrace{-(0-(-2)(2))}_{-4}, \underbrace{4-0}_{4}\rangle$$

$$\theta = \sin^{-1}\left(\frac{\overset{4\sqrt{3}}{\sqrt{4^2+(-4)^2+4^2}}}{2\,\cancel{8}}\right)$$

from (a)

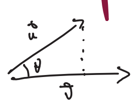
When is sine equal to $\frac{\sqrt{3}}{2}$?



ans: $\frac{\pi}{3}$, $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$$\boxed{=\frac{\pi}{3}}$$ (θ must be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$)

#18 (b) Compute $\text{proj}_{\vec{v}}\vec{u}$ and $\text{scal}_{\vec{v}}\vec{u}$, where $\vec{u} = -3\hat{j} + 4\hat{k}$ and $\vec{v} = -4\hat{i} + \hat{j} + 5\hat{k}$.



From the picture, $\cos\theta = \dfrac{|\text{proj}_{\vec{v}}\vec{u}|}{|\vec{u}|}$. ← $\text{scal}_{\vec{v}}\vec{u}$

The direction of $\text{proj}_{\vec{v}}\vec{u}$ is the same as $\vec{v}$, so normalize to get

$$\text{proj}_{\vec{v}}\vec{u} = \underbrace{|\vec{u}|\cos\theta}_{\text{scal}_{\vec{v}}\vec{u}}\,\frac{\vec{v}}{|\vec{v}|}.$$

We don't know θ, but $|\vec{u}|\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$,

so $$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\left(\frac{\vec{v}}{|\vec{v}|}\right)$$

$$= \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \frac{(0)\cancel{(-4)} + \overset{-3}{(-3)(1)} + \overset{20}{4(5)}}{\underset{16}{(-4)^2} + \underset{1}{1^2} + \underset{25}{5^2}}\langle -4,1,5\rangle = \boxed{\frac{17}{42}\langle -4,1,5\rangle}$$

and $$\text{scal}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \boxed{\frac{17}{\sqrt{42}}}$$

§11.7 #12

Find the velocity, speed, and acceleration of an object with position vector
$\vec{r}(t) = \langle 3\sin t, 5\cos t, 4\sin t \rangle \quad 0 \le t \le 2\pi$

velocity: $\boxed{\vec{r}\,'(t) = \langle 3\cos t, -5\sin t, 4\cos t \rangle}$

speed: $|\vec{r}\,'(t)| = \sqrt{(3\cos t)^2 + (-5\sin t)^2 + (4\cos t)^2}$

$= \sqrt{25\cos^2 t + 25\sin^2 t}$

$\boxed{= 5}$

acceleration: $\boxed{\vec{r}\,''(t) = \langle -3\sin t, -5\cos t, -4\sin t \rangle}$

§11.8 #30 Find the length of the spiral $r = 4\theta^2$ for $0 \le \theta \le 6$.

The arclength formula for a polar curve $r = f(\theta)$ is

$$\int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$

$$= \int_0^6 \sqrt{\underbrace{(4\theta^2)^2}_{16\theta^4} + \underbrace{(8\theta)^2}_{64\theta^2}}\, d\theta = 4\int_0^6 \theta\sqrt{\theta^2+4}\, d\theta$$

$u = \theta^2 + 4$

$du = 2\theta\, d\theta$

$$= 2\, \frac{(\theta^2+4)^{3/2}}{3/2}\Bigg|_0^6$$

$$= \frac{4}{3}\left[\underbrace{(6^2+4)^{3/2}}_{\substack{40^{3/2} \\ 8(10^{3/2})}} - \underbrace{(0^2+4)^{3/2}}_{8}\right]$$

$$\boxed{= \frac{32}{3}\left(10\sqrt{10} - 1\right)}$$

§12.3 #28 Evaluate the following limit or else justify why it doesn't exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^3+y^3}$$

Along the line $y=0$, $\displaystyle\lim_{x\to 0} \frac{x^3-0^3}{x^3+0^3} = 1$.

Along the line $x=0$, $\displaystyle\lim_{y\to 0} \frac{0^3-y^3}{0^3+y^3} = -1 \neq 1$

So, by the <u>Two Path Test</u>, the limit does not exist

§12.6 #16 Compute the directional derivative of $g(x,y)=\sin\pi(2x-y)$ at the point $P(-1,-1)$ in the direction of $\left\langle \frac{5}{13}, \frac{-12}{13}\right\rangle$.

Make sure the direction vector $\vec{u}$ is a unit vector:

$$\sqrt{\left(\frac{5}{13}\right)^2+\left(\frac{-12}{13}\right)^2} = \frac{1}{13}\sqrt{\underbrace{5^2+12^2}_{13^2}} = 1 \quad \checkmark$$

Then $D_{\vec{u}}\, g(-1,-1) = \underbrace{\langle g_x(-1,-1), g_y(-1,-1)\rangle}_{\text{gradient}} \cdot \left\langle \frac{5}{13}, \frac{-12}{13}\right\rangle$

gradient:

$$\nabla g = \langle \cos(\pi(2x-y))(2\pi), \cos(\pi(2x-y))(-\pi)\rangle$$

$$\nabla g(-1,-1) = \langle \underbrace{\cos(\pi(\underbrace{2(-1)-(-1)}_{-1}))}_{-1}(2\pi), -(-\pi)\rangle$$

$$= \langle -2\pi, \pi\rangle \cdot \left\langle \frac{5}{13}, \frac{-12}{13}\right\rangle = -\frac{10}{13}\pi - \frac{12}{13}\pi$$

$$\boxed{= \frac{-22}{13}\pi}$$

§12.7 #18 Find an equation of the plane tangent to the surface $z = \underbrace{\ln(1+xy)}_{g}$ at the point $(1, 2, \ln 3)$.

Equation is the same as the linear approximation formula, with z replacing $L(x,y)$.

$$z = g_x(1,2)(x-1) + g_y(1,2)(y-2) + \underbrace{g(1,2)}_{\ln 3}$$

$$\nabla g = \left\langle \frac{1}{1+xy}(y), \frac{1}{1+xy}(x) \right\rangle$$

$$= \frac{(2)}{1+(1)(2)}(x-1) + \frac{(1)}{1+(1)(2)}(y-2) + \ln 3$$

Move z to the other side to get

$$\boxed{\frac{2}{3}(x-1) + \frac{1}{3}(y-1) - (z - \ln 3) = 0}$$

* In an implicit function $F(x,y,z)$, the formula becomes

$$0 = F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c).$$

In this problem, $F_z(a,b,c) = -1$.

§12.8 #26 Find the critical points of the function $f(x,y) = \frac{xy(x-y)}{x^2+y^2}$ ($x^2y - xy^2$). Use the 2nd Derivative Test to determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or saddle point.

For a critical point, both partials must equal zero:

$$f_x = \frac{(x^2+y^2)(2xy-y^2) - (x^2y-xy^2)(2x)}{(x^2+y^2)^2} = 0 = \frac{\cancel{2x^3y} + 2xy^3 - x^2y^2 - y^4 - \cancel{2x^3y} + 2x^2y^2}{(x^2+y^2)^2} = \frac{y^2(2xy - x^2 - y^2 + 2x^2)}{(x^2+y^2)^2}$$

$$f_y = \frac{(x^2+y^2)(x^2-2xy) - (x^2y-xy^2)(2y)}{(x^2+y^2)^2} = 0$$

$$= \frac{x^4 + x^2y^2 - 2x^3y - \cancel{2xy^3} - 2x^2y^2 + \cancel{2xy^3}}{(x^2+y^2)^2} = \frac{x^2(x^2+y^2-2xy-2y^2)}{(x^2+y^2)^2}$$

If $y=0$ then

$$\frac{x^2(x^2 - 2x(0) - 0^2)}{(x^2+0^2)^2} = \frac{x^4}{x^4} = 0 \Rightarrow x = 0.$$

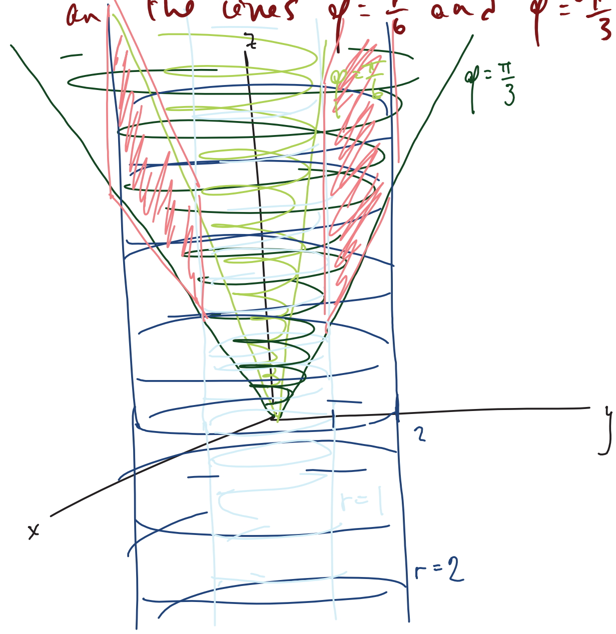
But then $\frac{0}{0}$ is undefined.

Similarly, we cannot have $x=0$.

So $2xy - \cancel{y^2} + \cancel{x^2} = \cancel{x^2} - \cancel{y^2} - 2xy$

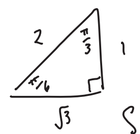
$\Rightarrow 4xy = 0$. But $x \neq 0$ and $y \neq 0$ so there are $\boxed{\text{no critical points.}}$

§13.5 #50 Find the volume of the region bounded by the cylinders $r=1$ and $r=2$ (cylindrical coords) and the cones $\varphi=\frac{\pi}{6}$ and $\varphi=\frac{\pi}{3}$ (spherical coords)!



Cylindrical coordinates:

The cone $\varphi=\frac{\pi}{6}$ satisfies $\frac{r}{z}=\tan\frac{\pi}{6}=\frac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}}=\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{1}{\sqrt{3}}$.



Similarly, for the cone $\varphi=\frac{\pi}{3}$,

$$\frac{r}{z}=\tan\frac{\pi}{3}=\frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}}=\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\sqrt{3}.$$

The volume is

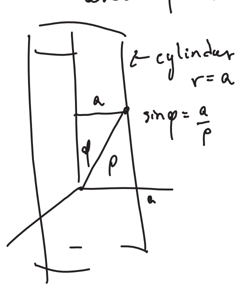
$$\int_0^{2\pi}\int_1^2\int_{\frac{r}{\sqrt{3}}}^{\sqrt{3}r} r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_1^2 r\left(\sqrt{3}r-\frac{r}{\sqrt{3}}\right)dr\,d\theta=\int_0^{2\pi}\left(\sqrt{3}-\frac{1}{\sqrt{3}}\right)\frac{r^3}{3}\Big|_1^2\,d\theta$$

$$=\frac{2}{\sqrt{3}}\left(\frac{8}{3}-\frac{1}{3}\right)(2\pi)=\boxed{\frac{28}{3\sqrt{3}}\pi}$$

Spherical Coordinates:

The cylinders are given by $\rho=\csc\varphi$ and $\rho=2\csc\varphi$ (Table 13.4)



The volume is

$$\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\pi/3}\int_{\csc\varphi}^{2\csc\varphi}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta=\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sin\varphi\,\frac{\rho^3}{3}\Big|_{\csc\varphi}^{2\csc\varphi}\,d\varphi\,d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\underbrace{\sin\varphi\Big(\underbrace{8\csc^3\varphi-\csc^3\varphi}_{7\csc^3\varphi}\Big)}_{7\csc^2\varphi}\,d\varphi\,d\theta$$

$$=\frac{7}{3}\int_0^{2\pi}-\cot\varphi\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}\,d\theta=-\frac{7}{3}\int_0^{2\pi}\left(\frac{\overset{\frac{1}{2}}{\cos\frac{\pi}{3}}}{\underset{\frac{\sqrt{3}}{2}}{\sin\frac{\pi}{3}}}-\frac{\overset{\frac{\sqrt{3}}{2}}{\cos\frac{\pi}{6}}}{\underset{\frac{1}{2}}{\sin\frac{\pi}{6}}}\right)d\theta$$

$$=-\frac{7}{3}\underbrace{\left(\frac{1}{\sqrt{3}}-\sqrt{3}\right)}_{-\frac{2}{\sqrt{3}}}(2\pi)$$

$$\boxed{=\frac{28}{3\sqrt{3}}\pi}$$

§14.2 #34 Calculate $\int_C \vec{F}\cdot\vec{T}\,ds$ for $\vec{F}=\langle -y,x\rangle$, where C is the semicircle given by $\vec{r}(t)=\langle 4\cos t,4\sin t\rangle$, $0\le t\le \pi$.

$\vec{r}\,'(t)=\langle -4\sin t,4\cos t\rangle$

$$\int_C \vec{F}\cdot\vec{T}\,ds=\int_C \vec{F}\cdot d\vec{r}=\int_0^{\pi}\left[-(4\sin t)(-4\sin t)+4\cos t(4\cos t)\right]dt$$

$$=\int_0 16\,dt=\boxed{16\pi}$$

#46 Given the force $\vec{F}=\dfrac{\langle x,y,z\rangle}{x^2+y^2+z^2}$, find the work required to move an object along the line segment from $\underset{P}{(1,1,1)}$ to $\underset{Q}{(8,4,2)}$.

$\vec{PQ}=\langle 8-1,4-1,2-1\rangle=\langle 7,3,1\rangle$

so $\vec{r}(t)=\langle 1+7t,1+3t,1+t\rangle$, $0\le t\le 1$

$\vec{r}\,'(t)=\langle 7,3,1\rangle$.

Work $=\int_0^1 \vec{F}\cdot d\vec{r}$

$$=\int_0^1 \frac{\overbrace{(1+7t)(7)+(1+3t)(3)+(1+t)(1)}^{7+49t+3+9t+1+t}}{\underbrace{(1+7t)^2+(1+3t)^2+(1+t)^2}_{1+14t+49t^2+1+6t+9t^2+1+2t+t^2}}\,dt=\int_0^1\frac{59t+11}{3+22t+59t^2}\,dt$$

$u=59t^2+22t+3$

$du=(2(59t)+22)\,dt$

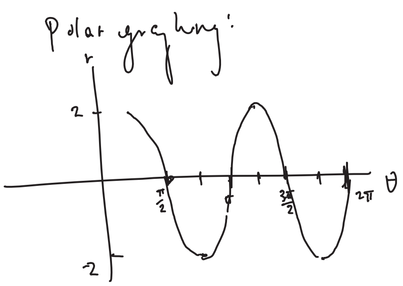
$$=\int_{t=0}^{t=1}\frac{1}{2}\left(\frac{1}{u}\right)du=\frac{1}{2}\ln(59t^2+22t+3)\Big|_0^1$$

$$=\frac{1}{2}\left(\ln(\underbrace{59+22+3}_{84})-\ln(3)\right)$$

$$=\frac{1}{2}\ln\left(\frac{84}{3}\right)=\boxed{\frac{1}{2}\ln(28)}$$

§13.3 #30 Sketch the region R and find its area; R is the region inside the leaf of the rose $r=2\sin 2\theta$ in the first quadrant. You do not need to evaluate the area integral, only set it up.

Polar graphing:



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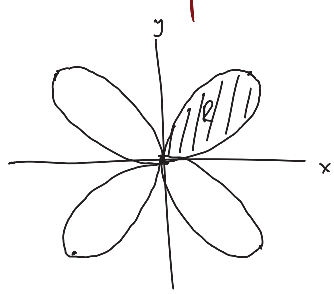
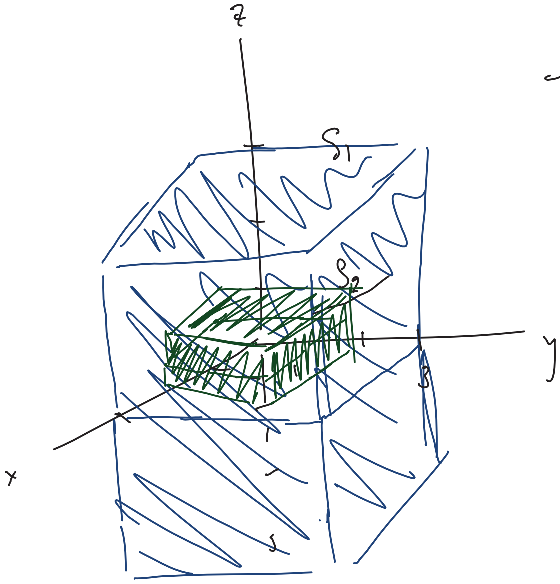


$$\text{area}=\int_0^{\frac{\pi}{2}}\int_0^{2\sin 2\theta} r\,dr\,d\theta$$

§14.8 #28 Compute the net outward flux of $\vec{F} = \langle z-y, x-z, 2y-x \rangle$ across the region

$$D = \{(x,y,z) \mid 1 \le |x| \le 3, 1 \le |y| \le 3, 1 \le |z| \le 3\}$$



The flux is the difference between the fluxes through each cube

$$\iint_{S_1} \vec{F} \cdot dS_1 - \iint_{S_2} \vec{F} \cdot dS_2 = \iiint_{D_1} \operatorname{div} \vec{F} \, dV - \iiint_{D_2} \operatorname{div} \vec{F} \, dV$$

(D)ivergence Theorem

$$= \int_1^3 \int_1^3 \int_1^3 (0+0+0) \, dx \, dy \, dz = \boxed{0}$$