1 Week 11: 4-8 April

§4.6 Mean Value Theorem

- Consequences of MVT
- Book Problems

Exam #3 Review

§4.6 Mean Value Theorem

In this section, we examine the Mean Value Theorem, one of the "big ideas" that provides the basis for much of calculus.

Before we get to the mean Value Theorem, we examine Rolle's Theorem, where the property $f(a) = f(b)$ holds, for some function $f(x)$ defined on an interval $[a, b]$.

Question

If you have two points $(a, f(a))$ and $(b, f(b))$, with the property that $f(a) = f(b)$, what might this look like?

Theorem (Rolle's Theorem)

Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. Then there is at least one point c in (a, b) such that $f'(c) = 0$.

Essentially what Rolle's Theorem concludes is that at some point(s) between a and b, f has a horizontal tangent.

Question

Note the hypotheses in this theorem: f is continuous on $[a, b]$ and differentiable on (a, b). Why are these important?

Exercise

Determine whether Rolle's Theorem applies to the function $f(x) = x^3 - 2x^2 - 8x$ on the interval $[-2, 0]$.

- If it doesn't, find an interval for which Rolle's Thm could apply to that function.
- If it does, what is the "c" value so that $f'(c) = 0$?

Theorem (Mean Value Theorem (MVT))

If f is continuous on a closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

See Figure 4.68 on p. 276 for a visual justification of MVT.

The slope of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$ is

$$\frac{f(b) - f(a)}{b - a}.$$

MVT says that there is a point c on f where the tangent line at c (whose slope is $f'(c)$) is parallel to this secant line.

Question

Suppose you leave Fayetteville for a location in Fort Smith that is 60 miles away. If it takes you 1 hour to get there, what can we say about your speed? If it takes you 45 minutes to get there, what can we say about your speed?

Example

Let $f(x) = x^2 - 4x + 3$.

1. Determine whether the MVT applies to $f(x)$ on the interval $[-2, 3]$.
2. If so, find the point(s) that are guaranteed to exist by the MVT.

Example

How many points c satisfy the conclusion of the MVT for $f(x) = x^3$ on the interval $[-1, 1]$? Justify your answer.

Consequences of MVT

Theorem (Zero Derivative Implies Constant Function)

If f is differentiable and $f'(x) = 0$ at all points of an interval I, then f is a constant function on I.

Theorem (Functions with Equal Derivatives Differ by a Constant)

If two functions have the property that $f'(x) = g'(x)$ for all x of an interval I, then $f(x) - g(x) = C$ on I, where C is a constant.

Theorem (Intervals of Increase and Decrease)

Suppose f is continuous on an interval I and differentiable at all interior points of I.

- *If $f'(x) > 0$ at all interior points of I, then f is increasing on I.*
- *If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.*

4.6 Book Problems

7-14, 17-24

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Exam #3 Review

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- §3.10 Derivatives of Inverse Trig Functions
    - Know the derivatives of the six inverse trig functions.
    - Also: You are responsible for every derivative rule and every derivative formula we have covered this semester.
- §3.11 Related Rates
    - Know the steps to solving related rates problems, and be able to use them to solve problems given variables and rates of change.
    - Be able to solve related rates problems. If, while doing the HW (paper or computer), you were provided a formula in order to solve the problem, then I will do the same. If you were not provided a formula while doing the HW (paper or computer), then I also will not provide the formula.

Exam # 3 Review (cont.)

Exercise

An inverted conical water tank with a height of 12 ft and a radius of 6 ft is drained through a hole in the vertex at a rate of 2 ft^3/sec. What is the rate of change of the water depth when the water depth is 3 ft?

- §4.1 Maxima and Minima
  - Know the definitions of maxima, minima, and what makes these points local or absolute extrema (both analytically and graphically).
  - Know how to find critical points for a function.
  - Given a function on a given interval, be able to find local and/or absolute extrema.
  - Given specified properties of a function, be able to sketch a graph of that function.

Exam # 3 Review (cont.)

- §4.2 What Derivatives Tell Us
  - Be able to use the first derivative to determine where a function is increasing or decreasing.
  - Be able to use the First Derivative Test to identify local maxima and minima. Be able to explain in words how you arrived at your conclusion.
  - Be able to find critical points, absolute extrema, and inflection points for a function.
  - Be able to use the second derivative to determine the concavity of a function.
  - Be able to use the Second Derivative Test to determine whether a given point is a local max or min. Be able to explain in words how you arrived at your conclusion.
  - Know your Derivative Properties!!! (see Figure 4.36 on p. 256)

Exam # 3 Review (cont.)

- §4.3 Graphing Functions
  - Be able to find specific characteristics of a function that are spelled out in the Graphing Guidelines on p. 261 (e.g., know how to find x- and y-intercepts, vertical/horizontal asymptotes, critical points, inflection points, intervals of concavity and increasing/decreasing, etc.).
  - Be able to use these specific characteristics of a function to sketch a graph of the function.
- §4.4 Optimization Problems
  - Be able to solve optimization problems that maximize or minimize a given quantity.
  - Be able to identify and express the constraints and objective function in an optimization problem.

Exam # 3 Review (cont.)

- Be able to determine your interval of interest in an optimization problem (e.g., what range of x-values are you searching for your extreme points?)
- **As to formulas, the same comment made above with respect to formulas for related rates problems applies here as well.**

Exercise

What two nonnegative real numbers a and b whose sum is 23 will

(a) minimize $a^2 + b^2$?

(b) maximize $a^2 + b^2$?

- §4.5 Linear Approximation and Differentials
  - Be able to find a linear approximation for a given function.

Exam # 3 Review (cont.)

- Be able to use a linear approximation to estimate the value of a function at a given point.
- Be able to use differentials to express how the change in x (dx) impacts the change in y (dy).

- §4.6 Mean Value Theorem (for Derivatives)
  - Know and be able to state Rolle's Thm and the Mean Value Thm, including knowing the hypotheses and conclusions for both.
  - Be able to apply Rolle's Thm to find a point in a given interval.
  - Be able to apply the MVT to find a point in a given interval.
  - Be able to use the MVT to find equations of secant and tangent lines.

Exam # 3 Review (cont.)

Exercise (s)

Determine whether the Mean Value Theorem (or Rolle's Theorem) applies to the following functions. If it does, then find the point(s) guaranteed by the theorem to exist.

(1) $f(x) = \sin(2x)$ on $\left[0, \frac{\pi}{2}\right]$

(2) $g(x) = \ln(2x)$ on $[1, e]$

(3) $h(x) = 1 - |x|$ on $[-1, 1]$

Exam # 3 Review (cont.)

Exercise (s)

(4) $j(x) = x + \frac{1}{x}$ on $[1, 3]$

(5) $k(x) = \frac{x}{x+2}$ on $[-1, 2]$