1 Week 12: 11-15 April

§4.7 L'Hôpital's Rule

In Ch. 2, we examined limits that were computed using analytical techniques. Some of these limits, in particular those that were indeterminate, could not be computed with simple analytical methods.

For example,

$$\lim_{x\to 0}\frac{\sin x}{x} \quad \text{and} \quad \lim_{x\to 0}\frac{1-\cos x}{x}$$

are both limits that can't be computed by substitution, because plugging in 0 for x gives $\frac{0}{0}$.

Theorem (L'Hôpital's Rule $(\frac{0}{0})$)

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is $\pm\infty$).

(The rule also applies if $x \to a$ is replaced by $x \to \pm\infty$, $x \to a^+$ or $x \to a^-$.)

Example

Evaluate the following limit:

$$\lim_{x\to -1} \frac{x^4 + x^3 + 2x + 2}{x+1}.$$

Solution: By direct substitution, we obtain 0/0. So we must apply l'Hôpital's Rule (LR) to evaluate the limit:

$$\begin{aligned}
\lim_{x\to -1} \frac{x^4 + x^3 + 2x + 2}{x+1} &\overset{\text{LR}}{=} \lim_{x\to -1} \frac{\dfrac{d}{dx}\left(x^4 + x^3 + 2x + 2\right)}{\dfrac{d}{dx}(x+1)} \\
&= \lim_{x\to -1} \frac{4x^3 + 3x^2 + 2}{1} \\
&= -4 + 3 + 2 = 1
\end{aligned}$$

Theorem (L'Hôpital's Rule ($\frac{\infty}{\infty}$))

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is $\pm\infty$).

(The rule also applies if $x \to a$ is replaced by $x \to \pm\infty$, $x \to a^+$ or $x \to a^-$.)

Exercise

Evaluate the following limits using l'Hôpital's Rule:

- $\lim_{x\to\infty} \dfrac{4x^3 - 2x^2 + 6}{\pi x^3 + 4}$
- $\lim_{x\to 0} \dfrac{\tan 4x}{\tan 7x}$

L'Hôpital's Rule in disguise

Other indeterminate limits in the form $0 \cdot \infty$ or $\infty - \infty$ cannot be evaluated directly using l'Hôpital's Rule.

For $0 \cdot \infty$ cases, we must rewrite the limit in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. A common technique is to divide by the reciprocal:

$$\lim_{x\to\infty} x^2 \sin\left(\frac{1}{5x^2}\right) = \lim_{x\to\infty} \frac{\sin\left(\dfrac{1}{5x^2}\right)}{\dfrac{1}{x^2}}$$

Exercise

Compute $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$.

For $\infty - \infty$, we can divide by the reciprocal as well as use a change of variables:

Example

Find $\lim_{x\to\infty} x - \sqrt{x^2+2x}$.

Solution:

$$\begin{aligned}
\lim_{x\to\infty} x - \sqrt{x^2+2x} &= \lim_{x\to\infty} x - \sqrt{x^2(1+\frac{2}{x})} \\
&= \lim_{x\to\infty} x - x\sqrt{1+\frac{2}{x}} \\
&= \lim_{x\to\infty} x\left(1 - \sqrt{1+\frac{2}{x}}\right) \\
&= \lim_{x\to\infty} \frac{1-\sqrt{1+\frac{2}{x}}}{\frac{1}{x}}
\end{aligned}$$

This is now in the form $\frac{0}{0}$, so we can apply l'Hôpital's Rule and evaluate the limit.

In this case, it may even help to change variables. Let $t = \frac{1}{x}$:

$$\lim_{x\to\infty} \frac{1-\sqrt{1+\frac{2}{x}}}{\frac{1}{x}} = \lim_{t\to 0^+} \frac{1-\sqrt{1+2t}}{t}.$$

Limits in the form 1^∞, 0^0, and ∞^0 are also considered indeterminate forms, and to use l'Hôpital's Rule, we must rewrite them in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Here's how:

Assume $\lim_{x \to a} f(x)^{g(x)}$ has the indeterminate form 1^∞, 0^0, or ∞^0.

1. Evaluate $L = \lim_{x \to a} g(x) \ln f(x)$. This limit can often be put in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, which can be handled by l'Hôpital's Rule.
2. Then $\lim_{x \to a} f(x)^{g(x)} = e^L$. Don't forget this step!

Example

Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Solution: This is in the form 1^∞, so we need to examine

$$\begin{aligned} L &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \\ &= \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \\ &\overset{\text{LR}}{=} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1. \end{aligned}$$

NOT DONE! Write

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^L = e^1 = e.$$

Examining Growth Rates

We can use l'Hôpital's Rule to examine the rate at which functions grow in comparison to one another.

Definition

Suppose f and g are functions with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Then **$f$ grows faster than g** as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \text{ or } \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty.$$

$g \ll f$ means that f grows faster than g as $x \to \infty$.

Definition

The functions f and g have **comparable growth rates** if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M, \text{ where } 0 < M < \infty.$$

Pitfalls in Using l'Hôpital's Rule

1. L'Hôpital's Rule says that $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$. NOT

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \left[\frac{f(x)}{g(x)}\right]' \text{ or } \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \left[\frac{1}{g(x)}\right]' f'(x)$$

   (i.e., don't confuse this rule with the Quotient Rule).

2. Be sure that the limit with which you are working is in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
3. When using l'Hôpital's Rule more than once, simplify as much as possible before repeating the rule.
4. If you continue to use l'Hôpital's Rule in an unending cycle, another method must be used.

4.7 Book Problems

13-59 (odds), 69-79 (odds)

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§4.9 Antiderivatives

With differentiation, the goal of problems was to find the function f' given the function f.

With antidifferentiation, the goal is the opposite. Here, given a function f, we wish to find a function F such that the derivative of F is the given function f (i.e., $F' = f$).

Definition

A function F is called an **antiderivative** of a function f on an interval I provided $F'(x) = f(x)$ for all x in I.

Example

Given $f(x) = 4$, an antiderivative of $f(x)$ is $F(x) = 4x$.

NOTE: Antiderivatives are not unique!

They differ by a constant (C):

Theorem

Let F be any antiderivative of f. Then **all** *the antiderivatives of f have the form $F + C$, where C is an arbitrary constant.*

Recall: $\dfrac{d}{dx}f(x) = f'(x)$ is the derivative of $f(x)$.

Now: $\int f(x)\, dx = F + C$ *is* the antiderivative of $f(x)$. It doesn't matter which F you choose, since writing the C will show you are talking about all the antiderivatives at once. The C is also why we call it the *indefinite* integral.

Example

Find the antiderivatives of the following functions:

(1) $f(x) = -6x^{-7}$

(2) $g(x) = -4\cos 4x$

(3) $h(x) = \csc^2 x$

Example

$\int 4x^3 \, dx = x^4 + C$, where C is the **constant of integration**.

The dx is called the **differential** and it is the same dx from Section 4.5. Like the $\frac{d}{dx}$, it shows which variable you are talking about. The function written between the $\int$ and the dx is called the **integrand**.

Rules for Indefinite Integrals

Power Rule: $\int x^p \, dx = \dfrac{x^{p+1}}{p+1} + C$

(p is any real number except -1)

Constant Multiple Rule: $\int cf(x) \, dx = c \int f(x) \, dx$

Sum Rule: $\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx$

Exercise

$$\int (5x^4 + 2x + 1)\, dx =$$

A. $20x^3 + 2 + C$

B. $x^5 + x^2 - x + C$

C. $x^5 + x^2 + C$

D. $x^5 + 2x^2 - x + C$

Exercise

Evaluate the following indefinite integrals:

(1) $\int (3x^{-2} - 4x^2 + 1)\ dx$

(2) $\int 6\sqrt[3]{x}\ dx$

(3) $\int 2\cos(2x)\ dx$

Indefinite Integrals of Trig Functions

Table 4.9 (p. 322) provides us with rules for finding indefinite integrals of trig functions.

1. $\dfrac{d}{dx}(\sin ax) = a\cos ax \longrightarrow \displaystyle\int \cos ax\ dx = \frac{1}{a}\sin ax + C$
2. $\dfrac{d}{dx}(\cos ax) = -a\sin ax \longrightarrow \displaystyle\int \sin ax\ dx = -\frac{1}{a}\cos ax + C$
3. $\dfrac{d}{dx}(\tan ax) = a\sec^2 ax \longrightarrow \displaystyle\int \sec^2 ax\ dx = \frac{1}{a}\tan ax + C$
4. $\dfrac{d}{dx}(\cot ax) = -a\csc^2 ax \longrightarrow \displaystyle\int \csc^2 ax\ dx = -\frac{1}{a}\cot ax + C$
5. $\dfrac{d}{dx}(\sec ax) = a\sec ax\tan ax \longrightarrow \displaystyle\int \sec ax\tan ax\ dx = \frac{1}{a}\sec ax + C$
6. $\dfrac{d}{dx}(\csc ax) = -a\csc ax\cot ax \longrightarrow \displaystyle\int \csc ax\cot ax\ dx = -\frac{1}{a}\csc ax + C$

Example

Evaluate the following indefinite integral: $\int 2\sec^2 2x \; dx$.

Solution: Using rule 3, with $a = 2$, we have

$$\int 2\sec^2 2x \; dx = 2\int \sec^2 2x \; dx = 2\left[\frac{1}{2}\tan 2x\right] + C = \tan 2x + C.$$

Exercise

Evaluate $\int 2\cos(2x) \; dx$.

Other Indefinite Integrals

7. $\dfrac{d}{dx}(e^{ax}) = ae^{ax} \longrightarrow \displaystyle\int e^{ax}\ dx = \frac{1}{a}e^{ax} + C$

8. $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x} \longrightarrow \displaystyle\int \frac{dx}{x} = \ln|x| + C$

9. $\dfrac{d}{dx}\left(\sin^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{1}{\sqrt{a^2-x^2}} \longrightarrow \displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

10. $\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{a}{a^2+x^2} \longrightarrow \displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

11. $\dfrac{d}{dx}\left(\sec^{-1}\left|\dfrac{x}{a}\right|\right) = \dfrac{a}{x\sqrt{x^2-a^2}} \longrightarrow \displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

Initial Value Problems

In some instances, you have enough information to determine the value of C in the antiderivative. These are often called **initial value problems**. Finding $f(x)$ is often called **finding the solution**.

Example

If $f'(x) = 7x^6 - 4x^3 + 12$ and $f(1) = 24$, find $f(x)$.

Solution: $f(x) = \int (7x^6 - 4x^3 + 12)\, dx = x^7 - x^4 + 12x + C$. Now find out which C gives $f(1) = 24$:

$$24 = f(1) = 1 - 1 + 12 + C,$$

so $C = 12$. Hence, $f(x) = x^7 - x^4 + 12x + 12$.

Exercise

Find the function f that satisfies $f''(t) = 6t$ with $f'(0) = 1$ and $f(0) = 2$.

4.9 Book Problems

11-45 (odds), 59-73 (odds), 83-93 (odds)

Advice: To solve 83-93 (odds), read pages 325-326, focusing on Example 8.

§5.1 Approximating Area Under Curves

In the previous two chapters, we have come to see the derivative of a function associated with the rate of change of a function as well as the slope of the tangent line to the curve.

In the first section of Chapter 5, we now examine the meaning of the integral.

Question

If we know the velocity function of a particular object, what does that tell us about its position function?

Example

Suppose you ride your bike at a constant velocity of 8 miles per hour for 1.5 hours.

(a) What is the velocity function that models this scenario?

(b) What does the graph of the velocity function look like?

(c) What is the position function for this scenario?

(d) Where is the displacement (i.e., the distance you've traveled) represented when looking at the graph of the velocity function?

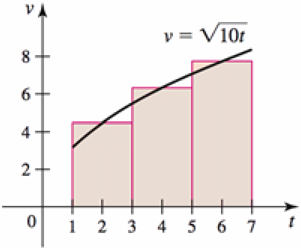
In the previous example, the velocity was constant. In most cases, this is not accurate (or possible). How could we find displacement when the velocity is changing over an interval?

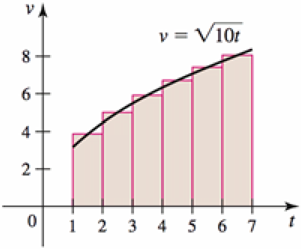
One strategy is to divide the time interval into a particular number of subintervals and approximate the velocity on each subinterval with a constant velocity. Then for each subinterval, the displacement can be evaluate and summed.

Note: This provides us with only an approximation, but with a larger number of subintervals, the approximation becomes more accurate.

Example

Suppose the velocity of an object moving along a line is given by $v(t) = \sqrt{10t}$ on the interval $1 \le t \le 7$. Divide the time interval into $n = 3$ subintervals, assuming the object moves at a constant velocity equal to the value of v evaluated at the midpoint of the subinterval. Estimate the displacement of the object on $[1, 7]$. Repeat for $n = 6$ subintervals.





The more subintervals you divide your time interval into, the more accurate your approximation of displacement will be.We now examine a method for approximating areas under curves.

Consider a function f over the interval $[a, b]$. Divide $[a, b]$ into n subintervals of equal length:

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

with $x_0 = a$ and $x_n = b$. The length of each subinterval is denoted

$$\Delta x = \frac{b-a}{n}.$$

In each subinterval $[x_{k-1}, x_k]$ (where k is any number from 1 to n), we can choose any point $\overline{x}_k$ (note $\overline{x}_k$ might be different, depending on which k), and create a rectangle with a height of $f(\overline{x}_k)$.

The area of the rectangle is "base times height", written $f(\overline{x}_k)\Delta x$, since the base is the length of the subinterval.

Doing this for each subinterval, and then summing each rectangle's area, produces an approximation of the overall area. This approximation is called a **Riemann sum**

$$R = f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x.$$

The symbol "k" is what's known as an **indexing variable**. We let k vary from 1 to n, and we always have $x_{k-1} \leq \overline{x}_k \leq x_k$.

Note: We usually choose $\overline{x}_k$ so that it is consistent across all the subintervals.

Definition

Suppose

$$R = f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x$$

is a Riemann sum. Then:

1. R is a **left Riemann sum** when we choose $\overline{x}_k = x_{k-1}$ for each k (so $\overline{x}_k$ is the left endpoint of the subinterval).
2. R is a **right Riemann sum** when we choose $\overline{x}_k = x_k$ for each k (so $\overline{x}_k$ is the right endpoint of the subinterval).
3. R is a **midpoint Riemann sum** when we take $\overline{x}_k$ to be the midpoint between x_{k-1} and x_k, for each k.

(See pages 337-339 for picture of these.)

Example

Calculate the left Riemann sum for the function $f(x) = x^2 - 1$ on the interval $[2, 4]$ when $n = 4$.

A. 13.75

B. 19.75

C. 27.5

D. 55

Exercise

Compute the left, right, and midpoint Riemann sums for the function $f(x) = 2x^3$ on the interval $[0, 8]$ with $n = 4$.

Sigma Notation

Riemann sums become more accurate when we make n (the number of rectangles) bigger, but obviously writing it all down is no fun! Sigma notation gives a shorthand. Here is how sigma notation works, through an example:

Example

$\sum_{n=1}^{5} n^2$ is the sum all integer values from the lowest limit $(n = 1)$ to the highest limit $(n = 5)$ in the summand n^2 (in this case n is the indexing variable).

$$\sum_{n=1}^{5} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

Example

Evaluate $\sum_{k=0}^{3}(2k-1)$.

Solution: In this example, k is the indexing variable. It starts at 0 and goes to 3, which means we write down the expression in the parentheses for each of the integers from 0 to 3, then add the results together:

$$\begin{aligned}\sum_{k=0}^{3}(2k-1) &= (2(0)-1)+(2(1)-1)+(2(2)-1)+(2(3)-1)\\ &= -1+1+3+5=8.\end{aligned}$$

Σ-Shortcuts

(n is always a positive integer)

$$\sum_{k=1}^{n} c = cn \text{ (where } c \text{ is a constant)}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Question

What is the indexing variable in these formulas?

Suppose f is defined on a closed interval $[a, b]$ which is divided into n subintervals of equal length Δx. As before, $\overline{x}_k$ denotes a point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$. Recall that $x_0 = a$ and $x_n = b$.

Here is how we can write the Riemann sum in a much more compact form:

$$R = f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x = \sum_{k=1}^{n} f(\overline{x}_k)\Delta x.$$

With sigma-notation we can even derive explicit formulas for the basic Riemann sums (the expression in red is $\overline{x}_k$ for each case:

1. $\displaystyle\sum_{k=1}^{n} f(a+(k-1)\Delta x)\Delta x =$ left Riemann sum
2. $\displaystyle\sum_{k=1}^{n} f(a+k\Delta x)\Delta x =$ right Riemann sum
3. $\displaystyle\sum_{k=1}^{n} f\left(a+\left(k-\tfrac{1}{2}\right)\Delta x\right)\Delta x =$ midpoint Riemann sum

Exercise

(a) Use sigma notation to write the left, right, and midpoint Riemann sums for the function $f(x) = x^2$ on the interval $[1, 5]$ given that $n = 4$.

(b) Based on these approximations, estimate the area bounded by the graph of $f(x)$ over $[1, 5]$.

Suggestion: As n gets very big, Riemann sums, along with the Σ-shortcuts plus algebra, often make the problem way more manageable.

5.1 Book Problems

9-37