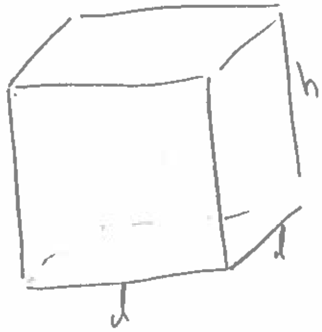
Math 2554 Quiz 10: Optimization (§4.4)

SOLUTIONS

1.



Objective: Maximize Volume
$V = l^2 h$

Constraint: $2l + h = 108$ in.

$\Rightarrow 0 \le l \le 54$
$0 \le h \le 108$ / Intervals of Interest

2 ways:

Solve for l:

$l = \frac{108-h}{2}$

$\Rightarrow V = \left(\frac{108-h}{2}\right)^2 h$

$\frac{dV}{dh} = 2\left(\frac{108-h}{2}\right)\left(\frac{-1}{2}\right)h + \left(\frac{108-h}{2}\right)^2$

$= \left(\frac{108-h}{2}\right)\left(\frac{-2h}{2} + \frac{108-h}{2}\right)$

$= 0 = (108-h)(108-3h)$

$h = 36, 108$

Solve for h:

$h = 108 - 2l$

$\Rightarrow V = l^2(108-2l)$
$= 108l^2 - 2l^3$

$\frac{dV}{dl} = 2(108)l - 6l^2$
$= l\,(2(108) - 6l) = 0$

$l = 0, 36$

Check for a max:

$V(0) = 0^2(108-2(0)) = 0$ in^3

$V(36) = 36^2(108-2(36)) = 36^3$ in^3

$V(54) = 54^2(108-2(54)) = 0$ in^3

Check for a max:

$V(0) = \left(\frac{108-0}{2}\right)^2(0) = 0$

$V(36) = \left(\frac{108-36}{2}\right)^2(36) = 36^3$

$V(108) = \left(\frac{108-108}{2}\right)^2(108) = 0.$

The maximum occurs when $h = 36$ in

$\Rightarrow l = \frac{108-36}{2}$

$= 36$ in

The dimensions are $\boxed{36 \text{ in} \times 36 \text{ in} \times 36 \text{ in}}$

to give a maximum volume of $\boxed{36 \text{ in}^3}$.

The maximum occurs when $l = 36$ in

$\Rightarrow h = 108 - 2(36)$

$= 36$ in.

The dimensions are $\boxed{36 \text{ in} \times 36 \text{ in} \times 36 \text{ in}}$

to give a maximum volume of $\boxed{36 \text{ in}^3}$.

2.(a) $g(0) = \dfrac{0(85-0)}{60} = 0$ miles/gal

$g(40) = \dfrac{40(85-40)}{60} = 30$ miles/gal

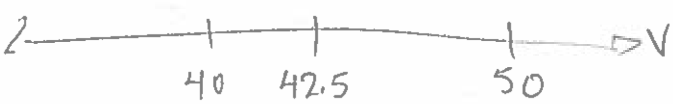
$g(60) = \dfrac{60(85-60)}{60} = 25$ miles/gal

It makes sense for there to be a "sweet spot" where gas mileage is the best. Fuel efficiency decreases for speeds that are too fast.

(b) $g'(v) = \dfrac{85-2v}{60} = 0$

$v = 42.5$ mph

No endpoints given, but can check it gives a maximum in two ways:

1st Derivative Test	2nd Derivative Test
	$g''(v) = -2 < 0$
$v'(40) = \dfrac{85-2(40)}{60} = \dfrac{5}{60} > 0$	The gas mileage function is always concave down,
$v'(50) = \dfrac{85-2(50)}{60} = \dfrac{-15}{60} < 0$	so the critical point $\longrightarrow$

The function goes from increasing to decreasing, so $v = 42.5$ mph will give the maximum gas mileage.

$v = 42.5$ mph maximizes the gas mileage function.

(c) Look at units to explain the cost function:

$$c(v) = \frac{(L \text{ miles})\left(\frac{\$ \, p}{\text{gal}}\right)}{g \, \frac{\text{miles}}{\text{gal}}} + \frac{(L \text{ miles})\left(\frac{\$ \, w}{\text{hour}}\right)}{v \, \frac{\text{miles}}{\text{hour}}}$$

cost of fuel

cost of driver

(d) $$C'(v) = \frac{-Lp}{g(v)^2} \cdot g'(v) - \frac{Lw}{v^2} = 0$$

$$\Rightarrow p \, g'(v) v^2 + w \, g(v)^2 = 0$$

$$p\left(\frac{85-2v}{60}\right)\cancel{v^2} + w \, \frac{\cancel{v^2}(85-v)^2}{60^2} = 0$$

$$60p(85-2v) + w(85-v)^2 = 0$$

Rewrite, then use the quadratic formula:

$$wv^2 + \left(-2(60p) - 2(85w)\right)v + 60p(85) + 85^2 w$$

$$v = \frac{2(60p + 85w) \pm \sqrt{\left(-2(60p + 85w)\right)^2 - 4w\left(85(60p + 85w)\right)}}{2w}$$

Plug in $L = 40$ miles

$p = \$4/\text{gal}$

$w = \$20/\text{hour}$

$\Rightarrow v = 97 \pm 2\sqrt{291}$ mph

$\approx 62.9,\ 131.1$ mph

Endpoints are not explicitly given, and computing the 2^{nd} derivative is masochistic, so use the 1^{st} Derivative Test to check which gives a minimum:

$$50 \qquad 97 - 2\sqrt{291} \qquad 100 \qquad 97 + 2\sqrt{291} \qquad 150 \qquad v$$

$$C'(50) = \frac{-Lp}{\frac{50^2(85-50)^2}{60^2}} \cdot \frac{85 - 2(50)}{60} - \frac{Lw}{50^2}$$

$\longrightarrow$

$$= -\frac{(40)(4)}{50^2(35^2)} \cdot 60 \cdot (-15) - \frac{40(20)}{50^2} < 0$$

$$C'(100) = -\frac{(40)(4)}{100^2(-15)^2} \cdot 60 \cdot (-115) - \frac{40(20)}{100^2} > 0$$

$$C'(150) = -\frac{(40)(4)}{150^2(-65)^2} \cdot 60(-215) - \frac{40(20)}{150^2} < 0$$

* values were punched into a calculator

The cost function only goes from decreasing to increasing when $v = 97 - 2\sqrt{291}$

$\boxed{\approx 62.9 \text{ mph}}$, so this is the speed to minimize cost.

(e) L is a constant factor in the cost function, so the critical points are not affected by it. Therefore the optimal speed is the same.

⟶

(f) In the quadratic formula that gives the critical points, plug in $p=4.2$:

$$V = 97.6 \pm 2\sqrt{307.44} \text{ mph}$$

$$\approx 62.5,\ 132.7 \text{ mph}$$

Since changing p simply scales part of the cost function by a positive factor, the first critical point,

$$V \approx 62.5 \text{ mph},$$

still gives the minimum cost, so the speed should <u>decrease</u>.

(g) As in (f), plug in $w=15$:

$$V = 101 \pm 2\sqrt{404} \text{ mph}$$

$$\approx 60.8,\ 141.2 \text{ mph}$$

and 60.8 mph gives the minimum cost, so the speed should <u>decrease</u>.

→

3.

Objective: Minimize surface area

$$A = 2\pi r h + 2\pi r^2$$

Constraint: Volume $= \pi r^2 h = 354 \text{ cm}^3$

$$\Rightarrow 0 \le r \le \sqrt{\frac{354}{\pi}}$$

$$0 \le h \le \frac{354}{\pi}$$

Intervals of Interest

2 ways!

Solve for r:

$$r = \sqrt{\frac{354}{\pi h}}$$

$$\Rightarrow A = 2\pi \sqrt{\frac{354}{\pi h}}\, h + 2\pi \left(\frac{354}{\pi h}\right)$$

$$= 2\sqrt{354\pi h} + \frac{2(354)}{h}$$

$$\frac{dA}{dh} = \frac{\frac{1}{2}(2)(\sqrt{354\pi})}{\sqrt{h}} - \frac{2(354)}{h^2} = 0$$

$$\Rightarrow \sqrt{354\pi}\, h^{3/2} - 2(354) = 0$$

$$h = \left(\frac{2(354)}{\sqrt{354\pi}}\right)^{2/3}$$

$$\approx 7.67 \text{ cm}$$

Solve for h:

$$h = \frac{354}{\pi r^2}$$

$$\Rightarrow A = 2\pi r\left(\frac{354}{\pi r^2}\right) + 2\pi r^2$$

$$= \frac{2(354)}{r} + 2\pi r^2$$

$$\frac{dA}{dr} = \frac{-2(354)}{r^2} + 4\pi r = 0$$

$$\Rightarrow -2(354) + 4\pi r^3 = 0$$

$$r = \sqrt[3]{\frac{2(354)}{4\pi}}$$

$$\approx 3.83 \text{ cm}$$

$\longrightarrow$

Check endpoints for a min!

$$A(0) = 2\sqrt{354\pi}(0) + \frac{2(354)}{0}$$

UNDEFINED

$$A\left(\left(\frac{2(354)}{\sqrt{354\pi}}\right)^{2/3}\right)$$

$$= 2\sqrt{354\pi}\left(\frac{2(354)}{\sqrt{354\pi}}\right)^{2/3} + 2(354)\left(\frac{\sqrt{354\pi}}{2(354)}\right)^{2/3}$$

$$\approx 603.7 \text{ cm}^2$$

$$A\left(\frac{354}{\pi}\right) = 2\sqrt{354\pi}\left(\frac{354}{\pi}\right) + \frac{2(\cancel{354})\pi}{\cancel{354}}$$

$$\approx 7521.8 \text{ cm}^2$$

So $\boxed{h = \left(\frac{2(354)}{\sqrt{354\pi}}\right)^{2/3} \approx 7.67 \text{ cm}}$ gives the minumum

$$\Rightarrow \boxed{r = \left(\sqrt{\frac{354}{\pi}}\right)\left(\left(\frac{2(354)}{\sqrt{354\pi}}\right)^{2/3}\right)^{-1/2} \approx 3.83 \text{ cm}}$$

Check endpoints for a min! 5

$$A(0) = \frac{2(354)}{0} + 2\pi(0^2)$$

UNDEFINED

$$A\left(\sqrt[3]{\frac{2(354)}{4\pi}}\right) = \frac{2(354)}{\left(\frac{2(354)}{4\pi}\right)^{1/3}} + 2\pi\left(\frac{2(354)}{4\pi}\right)^{2/3}$$

$$\approx 277.0 \text{ cm}^2$$

$$A\left(\sqrt{\frac{354}{\pi}}\right) = \frac{2(354)}{\sqrt{\frac{354}{\pi}}} + 2\pi\left(\sqrt{\frac{354}{\pi}}\right)^2$$

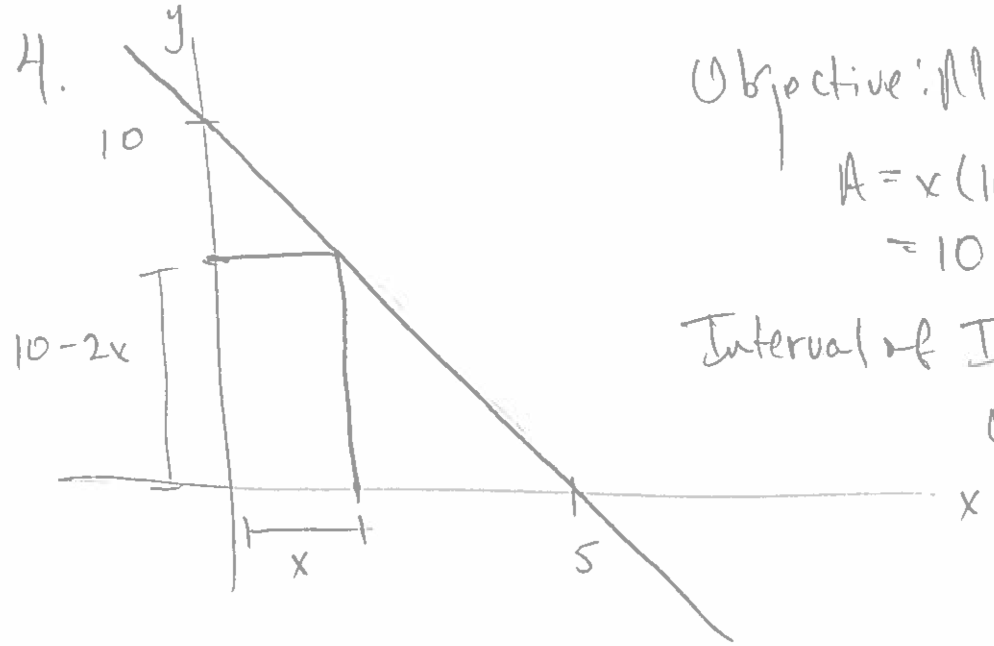
$$\approx 3560.5 \text{ cm}^2$$

So $\boxed{r = \sqrt[3]{\frac{177}{\pi}} \approx 3.83 \text{ cm}}$ (simplified)

$$\Rightarrow \boxed{h = \frac{354}{\pi\left(\frac{177}{\pi}\right)^{2/3}} \approx 7.67 \text{ cm}}$$

give the minimum surface area

4.



Objective: Maximize Area

$A = x(10-2x)$

$= 10x - 2x^2$

Interval of Interest:

$0 \le x \le 5$

Constraints are given in the picture and the objective function is already in one variable:

$\frac{dA}{dx} = 10 - 4x = 0$

$x = \frac{5}{2}.$

Check for the maximum:

$A(0) = 0(10-2(0)) = 0$

$A(\frac{5}{2}) = \frac{5}{2}(10 - 2(\frac{5}{2})) = \frac{25}{2}$

$A(5) = 5(10-2(5)) = 0.$

The maximum area is $\frac{25}{2}$ and is at $x = \frac{5}{2}$.