

# Midterm Review

- §2.1-2.2
  - **Material may not be explicitly tested, but the topics here are foundational to later sections.**
- §2.3 Techniques for Computing Limits
  - **Be able to do questions similar to 1-48.**
  - Know and be able to compute limits using analytical methods (e.g., limit laws, additional techniques).
  - Be able to evaluate one-sided and two-sided limits of functions.
  - Know the Squeeze Theorem and be able to use this theorem to determine limits.

# Midterm Review (cont.)

Exercise (problems from past midterm)

Evaluate the following limits:

- $\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x^2 - 9}$

- $\lim_{\theta \rightarrow 0} \frac{\sec \theta \tan \theta}{\theta}$

# Midterm Review (cont.)

- §2.4 Infinite Limits
  - **Be able to do questions similar to 17-30.**
  - Be able to use a graph, a table, or analytical methods to determine infinite limits.
  - Be able to use analytical methods to evaluate one-sided limits.
  - Know the **definition** of a vertical asymptote and be able to determine whether a function has vertical asymptotes.

# Midterm Review (cont.)

- §2.5 Limits at Infinity
  - **Be able to do questions similar to 9-30 and 38-46.**
  - Be able to find limits at infinity and horizontal asymptotes.
  - Know how to compute the limits at infinity of rational functions and algebraic functions.
  - Be able to list horizontal and/or vertical asymptotes of a function.

# Midterm Review (cont.)

## Exercise

Determine the horizontal asymptote(s) for the function

$$f(x) = \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$$

A.  $y = 2$

B.  $y = 0$

C.  $y = -2$

D.  $y = \pm 2$

# Midterm Review (cont.)

- §2.6 Continuity
  - **Be able to do questions similar to 9-44.**
  - Know the definition of continuity and be able to apply the continuity checklist.
  - Be able to determine the continuity of a function (including those with roots) on an interval.
  - Be able to apply the Intermediate Value Theorem to a function.

# Midterm Review (cont.)

Exercise (problem from past midterm)

Determine the value of  $k$  so the function is continuous on  $0 \leq x \leq 2$ .

$$f(x) = \begin{cases} x^2 + k & 0 \leq x \leq 1 \\ -2kx + 4 & 1 < x \leq 2 \end{cases}$$

# Midterm Review (cont.)

- §3.1 Introducing the Derivative
  - **Be able to do questions similar to 11-32.**
  - Know the definition of a derivative and be able to use this definition to calculate the derivative of a given function.
  - Be able to determine the equation of a line tangent to the graph of a function at a given point.
  - Know the 3 conditions for when a function is not differentiable at a point, and why these three conditions make a function not differentiable at the given point.



# Midterm Review (cont.)

- §3.2 Working with Derivatives
  - Be able to use the graph of a function to sketch the graph of its derivative, without computing derivatives
  - Know the 3 conditions for when a function is not differentiable at a point, and why these three conditions make a function not differentiable at the given point
  - Be able to determine where a function is not differentiable

# Midterm Review (cont.)

- §3.3 Rules for Differentiation
  - **Be able to do questions similar to 7-41.**
  - Be able to use the various rules for differentiation (e.g., constant rule, power rule, constant multiple rule, sum and difference rule) to calculate the derivative of a function.
  - Know the derivative of  $e^x$ .
  - Be able to find slopes and/or equations of tangent lines.
  - Be able to calculate higher-order derivatives of functions.

# Midterm Review (cont.)

## Exercise

Given that  $y = 3x + 2$  is tangent to  $f(x)$  at  $x = 1$  and that  $y = -5x + 6$  is tangent to  $g(x)$  at  $x = 1$ , write the equation of the tangent line to  $h(x) = f(x)g(x)$  at  $x = 1$ .

# Midterm Review (cont.)

- §3.4 The Product and Quotient Rules
  - **Be able to do questions similar to 7-42 and 47-52.**
  - Be able to use the product and/or quotient rules to calculate the derivative of a given function.
  - Be able to use the product and/or quotient rules to find tangent lines and/or slopes at a given point.
  - Know the derivative of  $e^{kx}$ .
  - Be able to combine derivative rules to calculate the derivative of a function.

**Note:** Functions are not always given by a formula. When faced with a problem where you don't know where to start, go through the rules first.

# Midterm Review (cont.)

## Exercise

Suppose you have the following information about the functions  $f$  and  $g$ :

$$f(1) = 6 \quad f'(1) = 2 \quad g(1) = 2 \quad g'(1) = 3$$

- Let  $F = 2f + 3g$ . What is  $F(1)$ ? What is  $F'(1)$ ?
- Let  $G = fg$ . What is  $G(1)$ ? What is  $G'(1)$ ?

# Midterm Review (cont.)

- §3.5 Derivatives of Trigonometric Functions

- **Be able to do questions similar to 1-55.**
- Know the two special trigonometric limits

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$$

and be able to use them to solve other similar limits.

- Know the derivatives of  $\sin x$ ,  $\cos x$ ,  $\tan x$ ,  $\cot x$ ,  $\sec x$ ,  $\csc x$ , and be able to use the quotient rule to derive the derivatives of  $\tan x$ ,  $\cot x$ ,  $\sec x$ , and  $\csc x$ .
- Be able to calculate derivatives (including higher order) involving trig functions using the rules for differentiation.

# Midterm Review (cont.)

## Exercise

Calculate the derivative of the following functions:

- $f(x) = (1 + \sec x) \sin^3 x$
- $g(x) = \frac{\sin x + \cot x}{\cos x}$

## Exercise

Evaluate  $\lim_{x \rightarrow -3} \frac{\sin(x+3)}{x^2 + 8x + 15}$ .

# Midterm Review (cont.)

- §3.6 Derivatives as Rates of Change
  - **Be able to do questions similar to 11-18.**
  - Be able to use the derivative to answer questions about rates of change involving:
    - Position and velocity
    - Speed and acceleration
    - Growth rates
    - Business applications



# Midterm Review (cont.)

- Be able to use a position function to answer questions involving velocity, speed, acceleration, height/distance at a particular time  $t$ , maximum height, and time at which a given height/distance is achieved.
- Be able to use growth models to answer questions involving growth rate and average growth rate, and cost functions to answer questions involving average and marginal costs.

# Midterm Review (cont.)

- §3.7 The Chain Rule
  - **Be able to do questions similar to 7-43.**
  - Be able to use both versions of the Chain Rule to find the derivative of a composition function.
  - Be able to use the Chain Rule more than once in a calculation involving more than two composed functions.
  - Know and be able to use the Chain Rule for Powers:

$$\frac{d}{dx} (f(x))^n = n (f(x))^{n-1} f'(x)$$

# Midterm Review (cont.)

## Exercise

Suppose  $f'(9) = 10$  and  $g(x) = f(x^2)$ . What is  $g'(3)$ ?

# Midterm Review (cont.)

- §3.8 Implicit Differentiation

- **Be able to do questions similar to 5-26 and 33-46.**
- Be able to use implicit differentiation to calculate  $\frac{dy}{dx}$ .
- Be able to use the derivative found from implicit differentiation to find the slope at a given point and/or a line tangent to the curve at the given point.
- Be able to calculate higher-order derivatives of implicitly defined functions.
- Be able to calculate  $\frac{dy}{dx}$  when working with functions containing rational functions.

# Midterm Review (cont.)

## Exercise

Use implicit differentiation to calculate  $\frac{dz}{dw}$  for

$$e^{2w} = \sin(wz)$$

## Exercise

If  $\sin x = \sin y$ , then

- $\frac{dy}{dx} = ?$
- $\frac{d^2y}{dx^2} = ?$

# Midterm Review (cont.)

- §3.9 Derivatives of Logarithmic and Exponential Functions
  - Be able to compute derivatives involving  $\ln x$  and  $\log_b x$
  - Be able to compute derivatives of exponential functions of the form  $b^x$
  - Be able to use logarithmic differentiation to determine  $f'(x)$

## Running Out of Time on the Exam Plus other Study Tips

- Do practice problems completely, from beginning to end (as if it were a quiz). You might think you understand something but when it's time to write down the details things are not so clear.
- Find a buddy who understands concepts a little better than you and work on problems for 2-3 hours. Then find a buddy who is struggling and work with them 2-3 hours.
- Don't count on cookie cutter problems. If you are doing a practice problem where you've memorized all the steps, make sure you understand why each step is needed. The exam problems may have a small variation from homeworks and quizzes. If you're not prepared, it'll come as a "twist" on the exam...

## Running Out of Time on the Exam Plus other Study Tips (cont.)

- If you encounter an unfamiliar type of problem on the exam, relax, because it's most likely not a trick! The solutions will always rely on the information from the required reading/assignments. Take your time and do each baby step carefully.
- During the exam, do the problems you are most confident with first!
- During the exam, budget your time. Count the problems and divide by 50 minutes. The easier questions will take less time so doing them first leaves extra time for the harder ones. When studying, aim for 10 problems per hour (i.e., 6 minutes per problem).



## Running Out of Time on the Exam Plus other Study Tips (cont.)

- Always make sure you **answer the question**. This is also a good strategy if you're not sure how to start a problem, figure out what the question wants first.
- The exam is not a race. If you finish early take advantage of the time to check your work. You don't want to leave feeling smug about how quickly you finished only to find out next week you lost a letter grade's worth of points from silly mistakes.

## Other Study Tips

- Brush up on algebra, especially radicals, logs, common denominators, etc. Many times knowing the right algebra will simplify the problem!
- When in doubt, show steps.
- You will be punished for wrong notation. The slides for §3.1 show different notations for the derivative. Make sure whichever one you use in your work, that you are using it correctly.
- Read the question!
- Do the book problems.
- Look at the pictures in the book and the interactive applets on MLP.

## §3.10 Derivatives of Inverse Trigonometric Functions

**Recall:** If  $y = f(x)$ , then  $f^{-1}(x)$  is the value of  $y$  such that  $x = f(y)$ .

### Example

If  $f(x) = 3x + 2$ , then what is  $f^{-1}(x)$ ?

**NOTE:**  $f^{-1}(x) \neq f(x)^{-1} \left( = \frac{1}{f(x)} \right)$

To avoid this confusion, we use  $\arcsin x$ ,  $\arccos x$ ,  $\arctan x$ , ... to denote inverse trig functions.

## Derivative of Inverse Sine

Trig functions are functions, too. Just like with “ $f$ ”, there has to be something to “plug in”. It makes no sense to just say  $\sin$ , without having  $\sin(\text{something})$ .

$$y = \sin^{-1} x \iff x = \sin y$$

The derivative of  $y = \sin^{-1} x$  can be found using implicit differentiation:

$$x = \sin y$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$$

$$1 = (\cos y) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

We still need to replace  $\cos y$  with an expression in terms of  $x$ . We use the trig identity  $\sin^2 y + \cos^2 y = 1$  (careful with notation: in this case we mean  $(\sin y)^2 + (\cos y)^2 = 1$ ). Then

$$\cos y = \pm \sqrt{1 - \sin^2 y} = \pm \sqrt{1 - x^2}.$$

The range of  $y = \sin^{-1} x$  is  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ . In this range, cosine is never negative, so we can just take the positive portion of the square root. Therefore,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}} \implies \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.$$

## Exercise

Compute the following:

1.  $\frac{d}{dx} (\sin^{-1}(4x^2 - 3))$
2.  $\frac{d}{dx} (\cos(\sin^{-1} x))$

## Derivative of Inverse Tangent

Similarly to inverse sine, we can let  $y = \tan^{-1} x$  and use implicit differentiation:

$$x = \tan y$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan y)$$

$$1 = (\sec^2 y) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$



Use the trig identity  $\sec^2 y - \tan^2 y = 1$  to replace  $\sec^2 y$  with  $1 + x^2$ :

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$$

## Derivative of Inverse Secant

Again, use the same method as with inverse sine:

$$y = \sec^{-1} x$$

$$x = \sec y$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sec y)$$

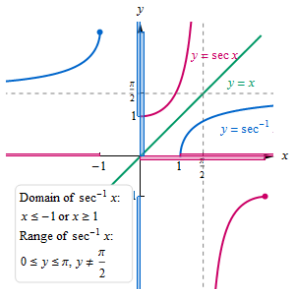
$$1 = \sec y \tan y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

Use the trig identity  $\sec^2 y - \tan^2 y = 1$  again to get

$$\tan y = \pm \sqrt{\sec^2 y - 1} = \pm \sqrt{x^2 - 1}.$$

This time, the  $\pm$  matters:



- If  $x \geq 1$ , then  $0 \leq y < \frac{\pi}{2}$  and so  $\tan y > 0$ .
- If  $x \leq -1$ , then  $\frac{\pi}{2} < y \leq \pi$  and so  $\tan y < 0$ .

Therefore,

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

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Using other trig identities (which you do not need to prove)

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2} \quad \cot^{-1} x + \tan^{-1} x = \frac{\pi}{2} \quad \csc^{-1} x + \sec^{-1} x = \frac{\pi}{2}$$

we can get the rest of the inverse trig derivatives.

## All Other Inverse Trig Derivatives

To summarize:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \\ (-1 < x < 1)$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2} \\ (-\infty < x < \infty)$$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}} \\ (|x| > 1)$$

## Example

Compute the derivatives of  $f(x) = \tan^{-1}\left(\frac{1}{x}\right)$  and  $g(x) = \sin\left(\sec^{-1}(2x)\right)$ .

## Derivatives of Inverse Functions in General

Let  $f$  be differentiable and have an inverse on an interval  $I$ . Let  $x_0$  be a point in  $I$  at which  $f'(x_0) \neq 0$ . Then  $f^{-1}$  is differentiable at  $y_0 = f(x_0)$  and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

where  $y_0 = f(x_0)$ .

### Example

Let  $f(x) = 3x + 4$ . Find  $f^{-1}(x)$  and  $(f^{-1})'(\frac{1}{3})$ .

## 3.10 Book Problems

7-33 (odds), 37-41 (odds)



## §3.11 Related Rates

In this section, we use our knowledge of derivatives to examine how variables change with respect to time.

The prime feature of these problems is that two or more variables, which are related in a known way, are themselves changing in time.

The goal of these types of problems is to determine the rate of change (i.e., the derivative) of one or more variables at a specific moment in time.

## Problem

The edges of a cube increase at a rate of 2 cm/sec. How fast is the volume changing when the length of each edge is 50 cm?

- **Variables:**  $V$  (Volume of the cube) and  $x$  (length of edge)
- **How Variables are related:**  $V = x^3$
- **Rates Known:**  $\frac{dx}{dt} = 2$  cm/sec
- **Rate We Seek:**  $\frac{dV}{dt}$  when  $x = 50$  cm

Note that both  $V$  and  $x$  are functions of  $t$  (their respective sizes are dependent upon how much time has passed).

So we can write  $V(t) = x(t)^3$  and then differentiate this with respect to  $t$ :

$$V'(t) = 3x(t)^2 \cdot x'(t).$$

**Note that  $x(t)$  is the length of the cube's edges at time  $t$ , and  $x'(t)$  is the rate at which the edges are changing at time  $t$ .**

We can rewrite the previous equation as

$$\frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt}.$$

So the rate of change of the volume when  $x = 50$  cm is

$$\left. \frac{dV}{dt} \right|_{x=50} = 3 \cdot 50^2 \cdot 2 = 15000 \text{ cm}^3/\text{sec}.$$

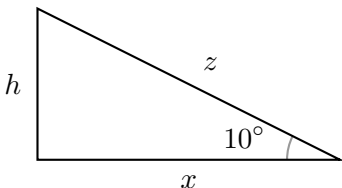
## Steps for Solving Related Rates Problems

1. Read the problem carefully, making a sketch to organize the given information. Identify the rates that are given and the rate that is to be determined.
2. Write one or more equations that express the basic relationships among the variables.
3. Introduce rates of change by differentiating the appropriate equation(s) with respect to time  $t$ .
4. Substitute known values and solve for the desired quantity.
5. Check that the units are consistent and the answer is reasonable.

## The Jet Problem

A jet ascends at a  $10^\circ$  angle from the horizontal with an airspeed of 550 miles/hr (its speed along its line of flight is 550 miles/hr). How fast is the altitude of the jet increasing? If the sun is directly overhead, how fast is the shadow of the jet moving on the ground?

**Step 1:** There are three variables: the distance the shadow has traveled ( $x$ ), the altitude of the jet ( $h$ ), and the distance the jet has actually traveled on its line of flight ( $z$ ). We know that  $\frac{dz}{dt} = 550$  miles/hr and we want to find  $\frac{dx}{dt}$  and  $\frac{dh}{dt}$ . We also see that these variables are related through a right triangle:



**Step 2:** To answer how fast the altitude is increasing, we need an equation involving only  $h$  and  $z$ . Using trigonometry,

$$\sin(10^\circ) = \frac{h}{z} \implies h = \sin(10^\circ) \cdot z.$$

To answer how fast the shadow is moving, we need an equation involving only  $x$  and  $z$ . Using trigonometry,

$$\cos(10^\circ) = \frac{x}{z} \implies x = \cos(10^\circ) \cdot z.$$



**Step 3:** We can now differentiate each equation to answer each question:

$$h = \sin(10^\circ) \cdot z \implies \frac{dh}{dt} = \sin(10^\circ) \frac{dz}{dt}$$
$$x = \cos(10^\circ) \cdot z \implies \frac{dx}{dt} = \cos(10^\circ) \frac{dz}{dt}$$

**Step 4:** We know that  $\frac{dz}{dt} = 550$  miles/hr. So

$$\frac{dh}{dt} = \sin(10^\circ) \cdot 550 \approx 95.5 \text{ miles/hr}$$
$$\frac{dx}{dt} = \cos(10^\circ) \cdot 550 \approx 541.6 \text{ miles/hr}$$

**Step 5:** Because both answers are in terms of miles/hr and both answers seem reasonable within the context of the problem, we conclude that the jet is gaining altitude at a rate of 95.5 miles/hr, while the shadow on the ground is moving at about 541.6 miles/hr.

## Example

The sides of a cube increase at a rate of  $R$  cm/sec. When the sides have a length of 2 cm, what is the rate of change of the volume?

## Example

Two boats leave a dock at the same time. One boat travels south at 30 miles/hr and the other travels east at 40 miles/hr. After half an hour, how fast is the distance between the boats increasing?

## Exercise

A 13 foot ladder is leaning against a vertical wall when Jack begins pulling the foot of the ladder away from the wall at a rate of 0.5 ft/sec. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 5 ft from the wall?

## 3.11 Book Problems

5-14, 16-19, 21-24, 37-38