1 Week 8: 13-16 July

§5.5 Substitution Rule

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We have seen a few methods to find antiderivatives (e.g., power rule, knowledge of derivatives, etc.). However, for many functions, it is more challenging to find the antiderivative.

Today we examine the substitution rule as a method to integrate.

Integration by Trial and Error

One somewhat inefficient method to find an antiderivative is by trial and error (with a natural check – find the derivative).

Example

$\int \cos(2x+5)\ dx$

Guess: Is it $\sin(2x+5)+C$?

Check: $\frac{d}{dx}\sin(2x+5) = 2\cdot\cos(2x+5)$

Question

How can you use your first attempt to refine your guess?

So we try $\frac{1}{2}\sin(2x+5)+C$.

Check:

$$\frac{d}{dx}\left(\frac{1}{2}\sin(2x+5)+C\right)=\frac{1}{2}(2\cdot\cos(2x+5))=\cos(2x+5)$$

So $\int\cos(2x+5)=\frac{1}{2}\sin(2x+5)+C$.

Trial and error can work in particular settings, but it is not an effiient strategy and doesn't work with some functions.

However, just as the Chain Rule helped us differentiate complex functions, the substitution rule (based on the Chain Rule) allows us to integrate complex functions.

Idea: Suppose we have $F(g(x))$, where F is an antiderivative of f. Then

$$\frac{d}{dx}\Big[F(g(x))\Big] = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$$

and $\displaystyle\int f(g(x)) \cdot g'(x)\ dx = F(g(x)) + C.$

If we let $u = g(x)$, then $du = g'(x)\ dx$. The integral becomes

$$\int f(g(x)) \cdot g'(x)\ dx = \int f(u)\ du.$$

Let $u = g(x)$, where g' is continuous on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

u-Substitution is the Chain Rule, backwards.

Example

Evaluate $\int 8x\cos(4x^2+3)\ dx$.

Solution: Look for a function whose derivative also appears.

$$u(x) = 4x^2 + 3$$

$$\text{and } u'(x) = \frac{du}{dx} = 8x$$

$$\implies du = 8x\ dx.$$

Now rewrite the integral and evaluate. Replace u at the end with its expression in terms of x.

$$\begin{aligned}\int 8x\cos(4x^2+3)\ dx &= \int \cos(\underbrace{4x^2+3}_{u})\,\underbrace{8x\ dx}_{du}\\ &= \int \cos u\ du\\ &= \sin u + C\\ &= \sin(4x^2+3) + C\end{aligned}$$

We can check the answer – by the Chain Rule,

$$\frac{d}{dx}\left(\sin\left(4x^2+3\right)+C\right) = 8x\cos\left(4x^2+3\right).$$

Procedure for Substitution Rule (Change of Variables)

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\ dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

Warning: Not all integrals yield to the Substitution Rule.

Example

Evaluating the integral $\int \frac{x}{x^2+1}\, dx$ yields the result

A. $x \arctan x + C$

B. $\frac{\frac{x^2}{2}}{\frac{x^3}{3+x}} + C$

C. $\frac{1}{2} \ln\left(x^2+1\right) + C$

D. $\ln|x| + C$

Exercise

Evaluate the following integrals. Check your work by differentiating each of your answers.

1. $\int \sin^{10} x \cos x \ dx$
2. $-\int \frac{\csc x \cot x}{1+\csc x} \ dx$
3. $\int \frac{1}{(10x-3)^2} \ dx$
4. $\int (3x^2 + 8x + 5)^8 (3x + 4) \ dx$

There are times when the u-substitution is not obvious or that more work must be done.

Example

Evaluate $\int \frac{x^2}{(x+1)^4}\ dx$.

Solution: Let $u = x + 1$. Then $x = u - 1$ and $du = dx$. Hence,

$$\int \frac{x^2}{(x+1)^4}\ dx = \int \frac{(u-1)^2}{u^4}\ du$$
$$= \int \frac{u^2 - 2u + 1}{u^4}\ du$$

$$= \int \left(u^{-2} - 2u^{-3} + u^{-4}\right)\ du$$
$$= \frac{-1}{u} + \frac{1}{u^2} + \frac{-1}{3u^3} + C$$
$$= \frac{-1}{x+1} + \frac{1}{(x+1)^2} - \frac{1}{3(x+1)^3} + C$$

Exercise

Check it.

This type of strategy works, usually, on problems where u can be written as a linear function of x.

Exercise

$$\int \frac{x}{\sqrt{x+1}}\ dx$$

Substitution Rule for Definite Integrals

We can use the Substitution Rule for Definite Integrals in two different ways:

1. Use the Substitution Rule to find an antiderivative F, and then use the Fundamental Theorem of Calculus to evaluate $F(b) - F(a)$.
2. Alternatively, once you have changed variables from x to u, you may also change the limits of integration and complete the integration with respect to u. Specifically, if $u = g(x)$, the lower limit $x = a$ is replaced by $u = g(a)$ and the upper limit $x = b$ is replaced by $u = g(b)$.

The second option is typically more efficient and should be used whenever possible.

Example

Evaluate $\int_0^4 \frac{x}{\sqrt{9+x^2}}\, dx$.

Solution: Let $u = 9 + x^2$. Then $du = 2x\, dx$. Because we have changed the variable of integration from x to u, the limits of integration must also be expressed in terms of u. Recall, u is a function of x (the $g(x)$ in the Chain Rule). For this example,

$$x = 0 \implies u(0) = 9 + 0^2 = 9$$

$$x = 4 \implies u(4) = 9 + 4^2 = 25$$

We had $u = 9 + x^2$ and $du = 2x\ dx \implies \frac{1}{2}du = x\ dx$. So:

$$\begin{aligned}\int_0^4 \frac{x}{\sqrt{9+x^2}}\,dx &= \frac{1}{2}\int_9^{25} \frac{du}{\sqrt{u}} \\ &= \frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right)\Bigg|_9^{25} \\ &= \sqrt{25} - \sqrt{9} \\ &= 5 - 3 = 2.\end{aligned}$$

Exercise

Evaluate $\int_0^2 \frac{2x}{(x^2+1)^2}\ dx$.

5.5 Book Problems

13-51 (odds), 63-77 (odds)