

Data X

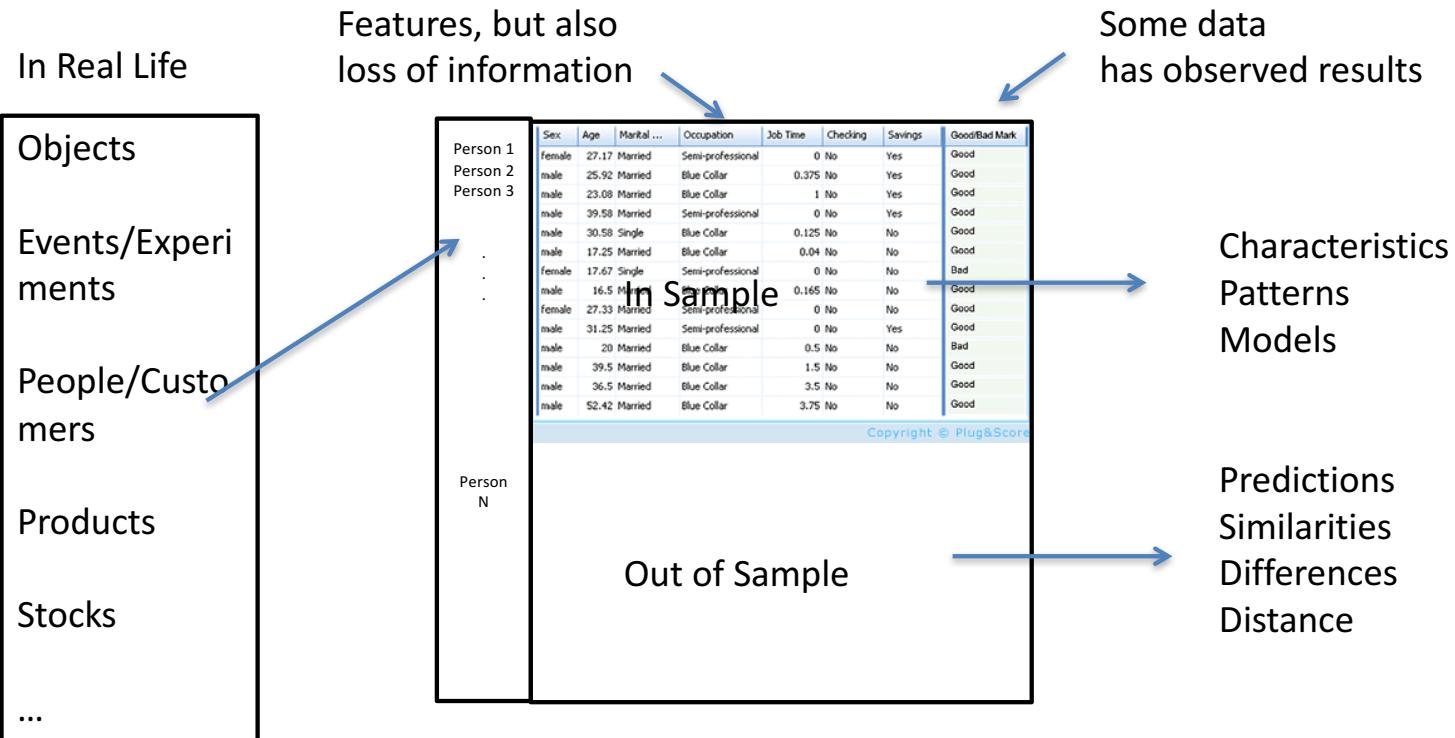
Data as a Signal: Convolution
Data X: A Course on Data, Signals, and Systems

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Sutardja Center for Entrepreneurship & Technology
IEOR Emerging Area Professor Award, UC Berkeley

Before We Discuss Convolutional Neural Networks
Lets Discuss Convolution and LTI Systems First



A High Level Framework

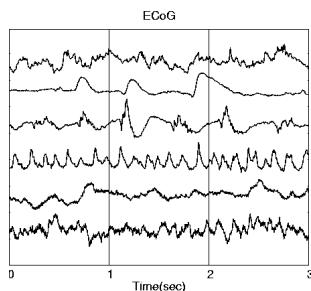


Data X

Converting From Time Sequence Data to Features

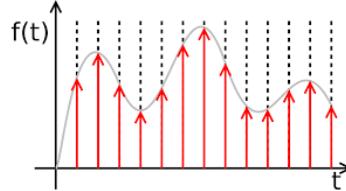
Many Types of data
are signals in time

- Stock market
- Temperature
- Instrument readings



Continuous signals
 $x(t)$

Sometimes we
sample them,
record at intervals
of T



Sampled signals (data)
 $x(nT)$

We get a
list in a table,
array, or vector

Rec	Observed
1	60.323
2	61.122
3	60.171
4	61.187
5	63.221
6	63.639
7	64.999
8	63.761
9	66.019
10	67.857
11	68.169
12	66.513
13	68.655
14	69.564
15	69.331
16	70.551

Discrete data
 $x_n = x_1, x_2, x_3, \dots$

(might lose time reference)

What we want
(for now):
features and
characteristics

For example:

- Means
- Variances
- Pattern matches
- Changes
- accumulation
- Frequency

- But from when, and over what time period



Simple things we can do with a sequence: Mean

Discrete data

$$x_n = x_1, x_2, x_3, \dots$$

Rec	Observed
1	60.323
2	61.122
3	60.171
4	61.187
5	63.221
6	63.639
7	64.989
8	63.761
9	66.019
10	67.857
11	68.169
12	66.513
13	68.655
14	69.564
15	69.331
16	70.551

- Mean
- Variance
- Standard Deviation
- Many other estimators or features
- But from when, and over what time period

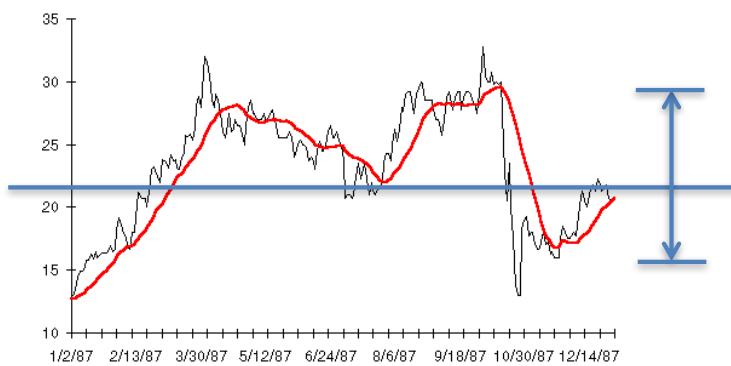


Simple things we can do with a sequence: Mean, Variance

Discrete data

$x_n = x_1, x_2, x_3, \dots$

Rec	Observed
1	60.323
2	61.122
3	60.171
4	61.187
5	63.221
6	63.639
7	64.989
8	63.761
9	66.019
10	67.857
11	68.169
12	66.513
13	68.655
14	69.564
15	69.331
16	70.551



variance

mean

Sample Mean	Population Mean
$\bar{x} = \frac{\sum x}{n}$	$\mu = \frac{\sum x}{N}$

where $\sum X$ is sum of all data values
N is number of data items in population
n is number of data items in sample

$$\sigma^2 = \frac{\sum (X - \mu)^2}{N}$$

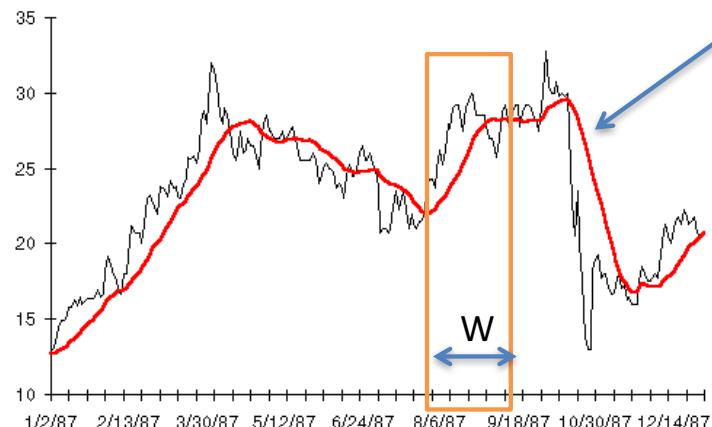
$$s^2 = \frac{\sum_{i=1}^n (X_i - \bar{X}_{avg})^2}{n-1}$$



Data X

Moving Average of Sequence

- Discrete data $x_n = x_1, x_2, x_3, \dots$
- Moving Average



(Simple) Moving Average

W = size of window
MA(n) is red line, at the right edge of the window

Note:

- * It's a function of n (and maybe W)
- Its smoother than the original (low pass filter)
- W = window size, can be in the summation

$$MA(n) = \frac{1}{W} \sum_{i=0}^{W-1} x_{n-i}$$

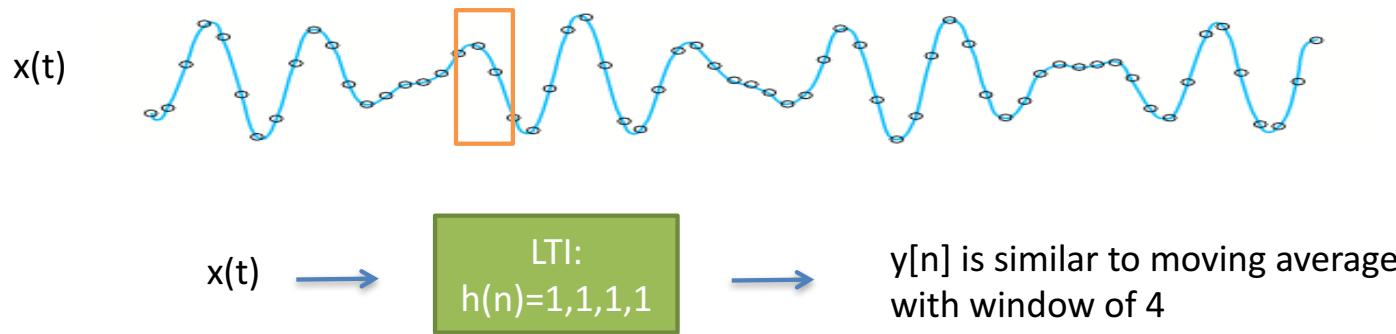
$$MA(n) = \sum_{i=0}^{W-1} \frac{x_{n-i}}{W}$$



Linear Time Invariant System Approaches

0 0 0 1 0 1 0 0 1 1 1 1 1 1 0 0 0 0 0 1 0 0 1 0 1 1 1 1 1 1 0
1 0 1 1 0 0 0 1 0 0 1 0 0 0 1 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 0
1 Data X 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 1 1 1 1 1 1 1 1 1 0

Another Way to Think about Moving Average

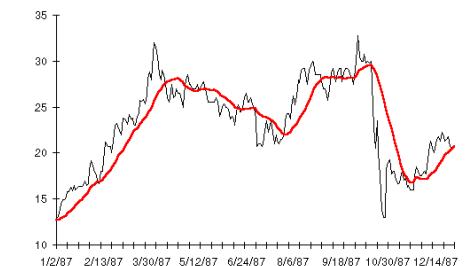


$x(n)$ = array with sequence of numbers = [10, 3, 6, 12, 43, 12, 1, 4]

$h(n)$ = impulse response function from linear systems
= [1, 1, 1, 1]

MA(n, W) = $y(n) = x(t) * h(t)$ (convolution)

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



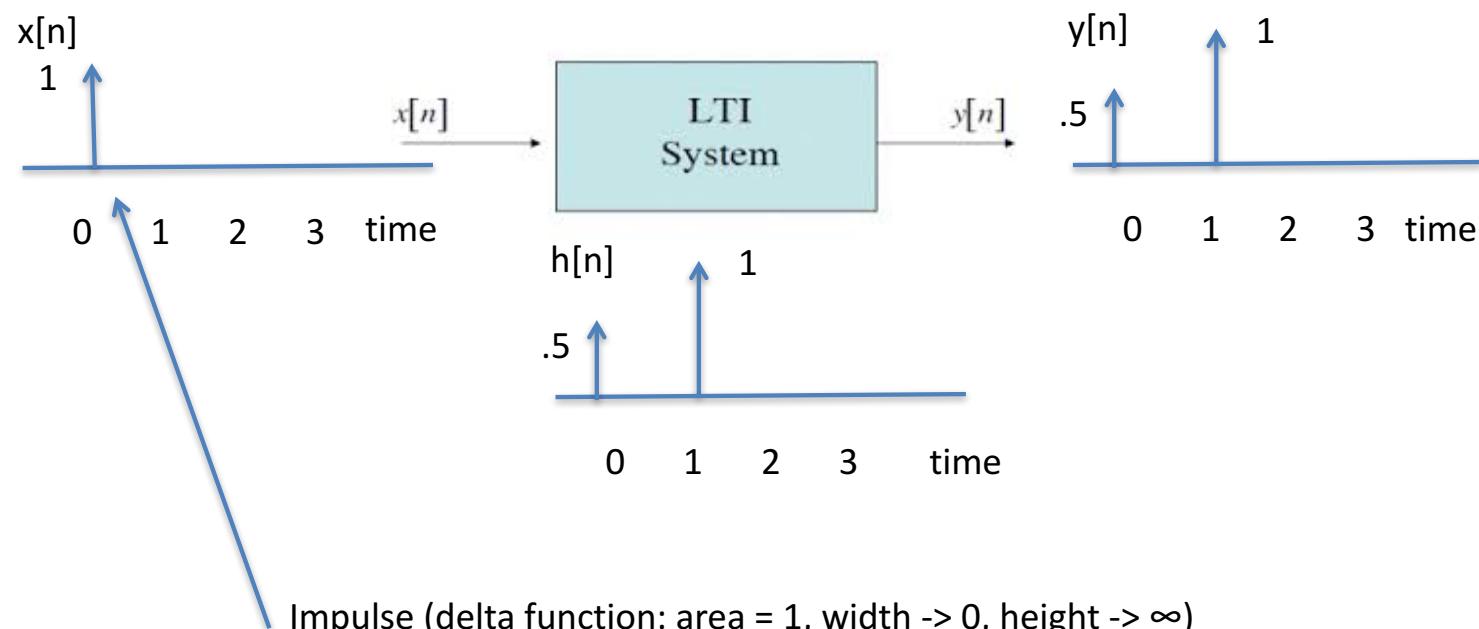
Continuous $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t - \tau)d\tau = x(t) * h(t)$

Circular

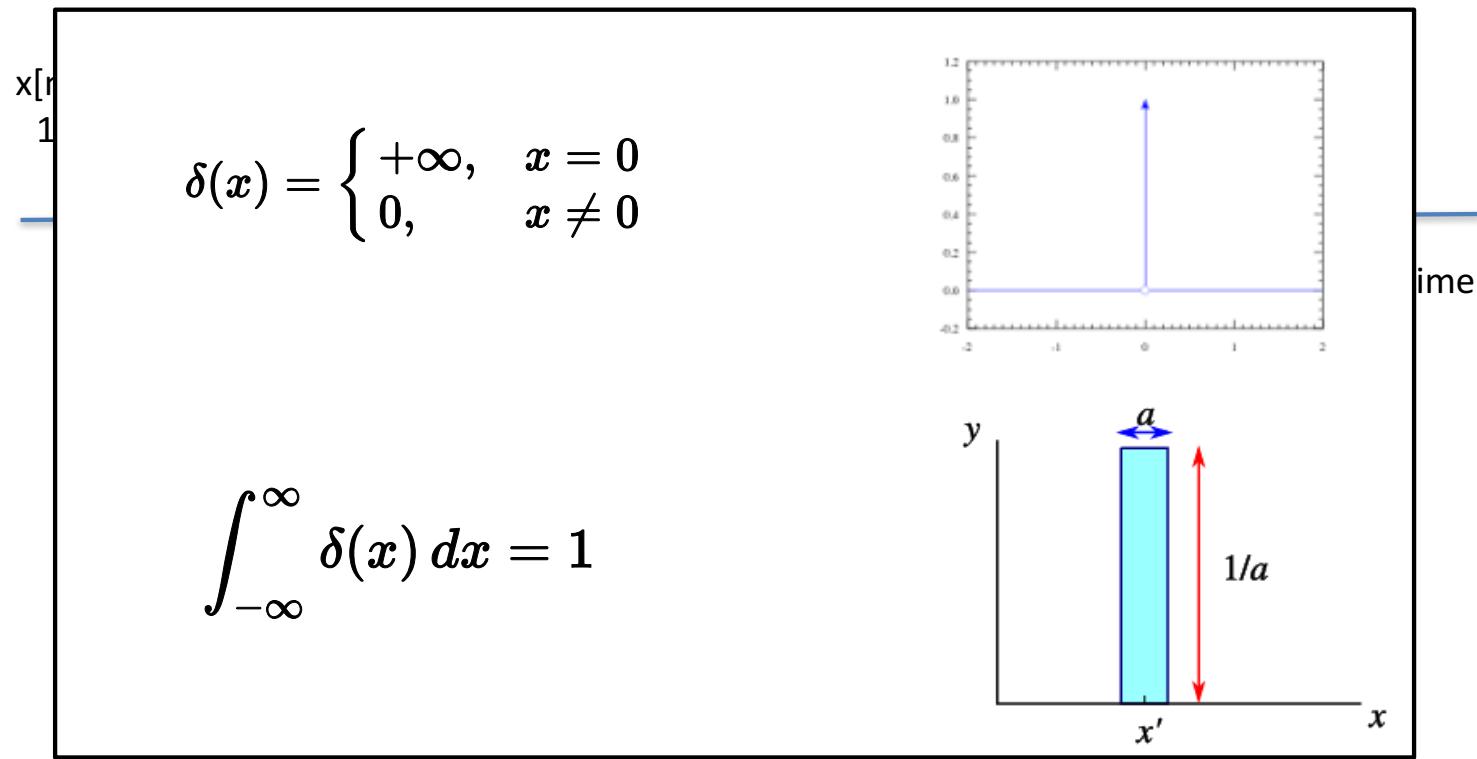
$$(x \circledast y)_n \triangleq \sum_{m=0}^{N-1} x(m)y(n-m)$$



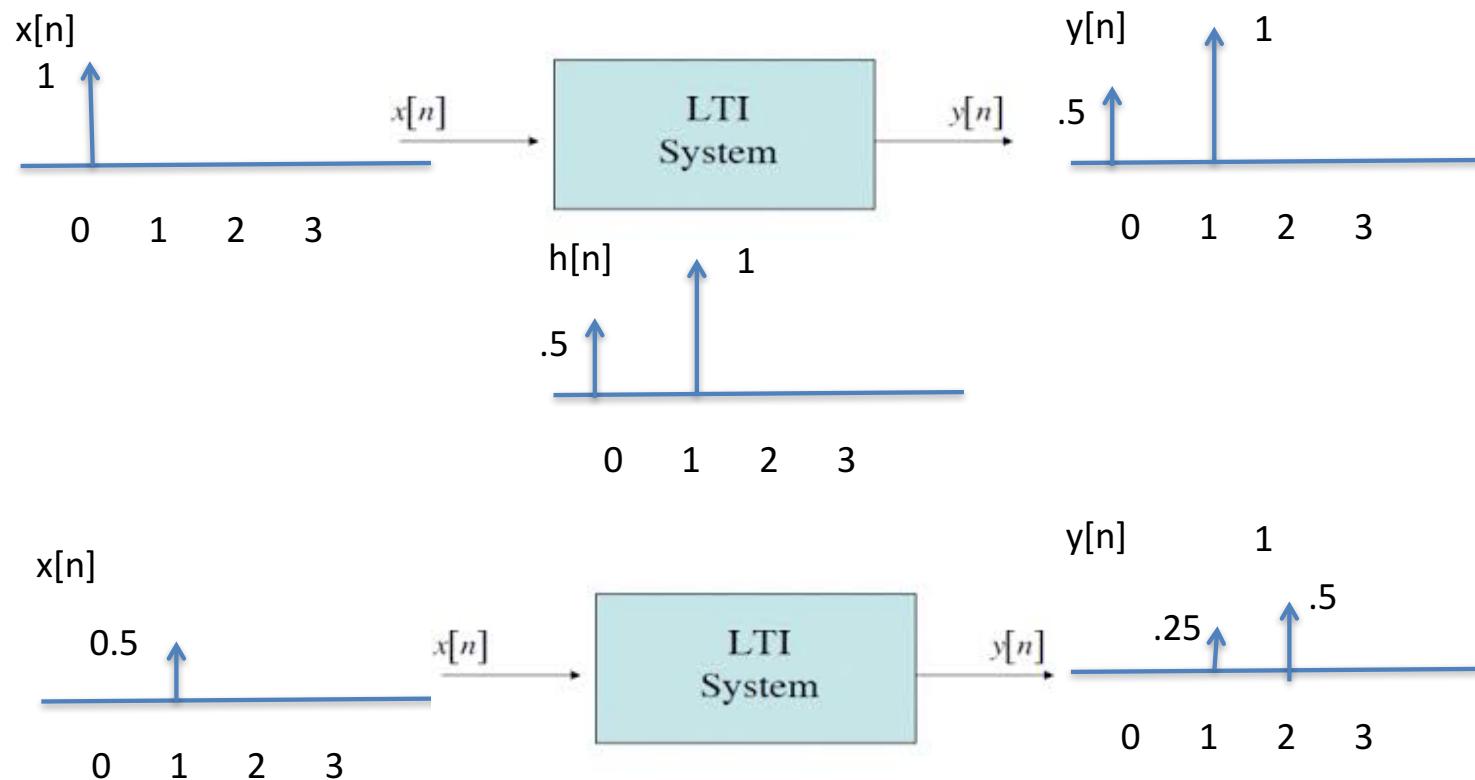
Linear Time Invariant System



Aside on Delta Functions



Linear Time Invariant System



Convolution is actually really simple, but powerful

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

- $x[n] = 1 \ 2 \ 3 \ 2 \ 4 \ 0 \ 0 \dots$

- $h[n] = 1 \ 1 \ 1$

- $y[n] = \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \\ 4 & 4 & 4 \end{matrix}$

←
←
←
←
←

$h[n-0] x[0]$
 $h[n-1] x[1]$
 $h[n-2] x[2]$
 $h[n-3] x[3]$
 $h[n-4] x[4]$

$$y[n] = 1 \ 3 \ 6 \ 7 \ 9 \ 6 \ 4 \dots$$

Note: if $h[n] = 1/3, 1/3, 1/3$, then $y[n] = 1/3, 1, 2, 7/3, 3, 2, 4/3$

Note, MA with Window = 3
(if divided by 3)



Another Convolution, different impulse response

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

- $x[n] = 1 \ 2 \ 3 \ 2 \ 4 \ 0 \ 0 \dots$

- $h[n] = 3 \ 2 \ 1$

- $y[n] = 3 \ 2 \ 1$ $\leftarrow h[n-0] x[0]$
 6 4 1 $\leftarrow h[n-1] x[1]$
 9 6 1 $\leftarrow h[n-2] x[2]$
 6 4 1 $\leftarrow h[n-3] x[1]$
 12 8 4 $\leftarrow h[n-1] x[1]$

$$y[n] = 3 \ 8 \ 14 \ 13 \ 17 \ 9 \ 4 \dots$$

This is called filtering the data with an FIR filter with impulse response $h[n] = 3 \ 2 \ 1$



Code Example of Convolution

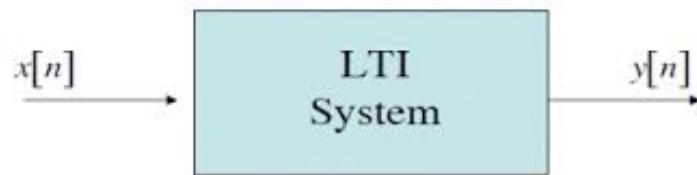
```
>>> np.convolve([1, 2, 3], [0, 1, 0.5])
array([ 0.,  1.,  2.5,  4.,  1.5])
```

`numpy.convolve(a, v, mode='full')[source]`

- Returns the discrete, linear convolution of two one-dimensional sequences.
- The convolution operator is often seen in signal processing, where it models the effect of a linear time-invariant system on a signal [R17]. In probability theory, the sum of two independent random variables is distributed according to the convolution of their individual distributions.



Convolution: Why We Care



FIR: Finite Impulse Response Filter

Take Any Input Sequence $x[n]$
Filter for what you want with $h[n]$
Get result $y[n]$

$h[n]$	Result
$1/W, 1/W, 1/W, 1/W$ (a_1, a_2, \dots, a_{10})	Low pass filter or moving average, with $1/W$ cutoff Any linear weighted sum
1, -1	Differential
Any sequence	Cross-correlation, allows you to look for similar pattern



Properties of Convolution

EQUATION 7-6

The commutative property of convolution.
This states that the order in which signals
are convolved can be exchanged.

$$a[n] * b[n] = b[n] * a[n]$$

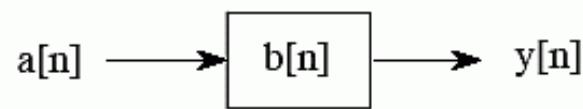
EQUATION 7-8

The distributive property of con-
volution describes how parallel
systems are analyzed.

$$a[n]*b[n] + a[n]*c[n] = a[n] * (b[n] + c[n])$$



IF



THEN

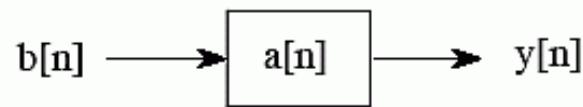
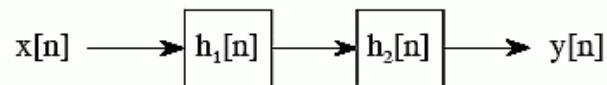


FIGURE 7-8

The commutative property in system theory. The commutative property of convolution allows the input signal and the impulse response of a system to be exchanged without changing the output. While interesting, this usually has no physical significance. (A signal appearing inside of a box, such as $b[n]$ and $a[n]$ in this figure, represent the *impulse response* of the system).



IF



THEN



ALSO

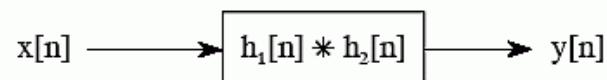
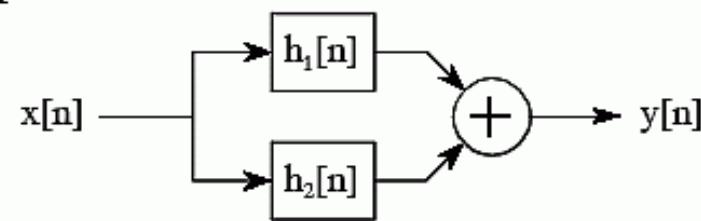


FIGURE 7-9

The associative property in system theory. The associative property provides two important characteristics of *cascaded* linear systems. First, the order of the systems can be rearranged without changing the overall operation of the cascade. Second, two or more systems in a cascade can be replaced by a single system. The impulse response of the replacement system is found by convolving the impulse responses of the stages being replaced.



IF



THEN

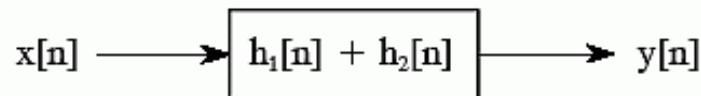


FIGURE 7-10

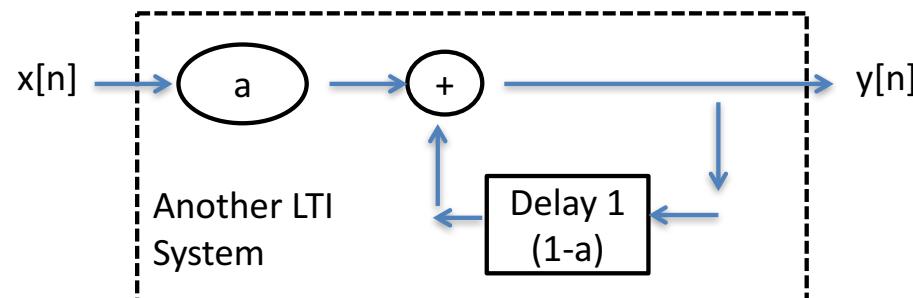
The distributive property in system theory. The distributive property shows that parallel systems with added outputs can be replaced with a single system. The impulse response of the replacement system is equal to the sum of the impulse responses of all the original systems.



Another LTI System: IIR: Infinite Impulse Response Filter

- Discrete data $x_n = x_1, x_2, x_3, \dots$

$$y(n) = \alpha x(n) + (1-\alpha)y(n-1)$$



$$y(n) = \alpha x(n) + \alpha^2 x(n-1) + \alpha^3 x(n-2) \dots$$

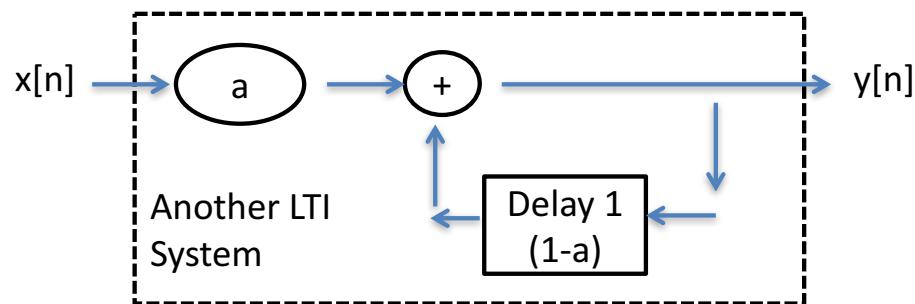
Infinitely back in $x[n]$



Another LTI System: IIR: Infinite Impulse Response Filter

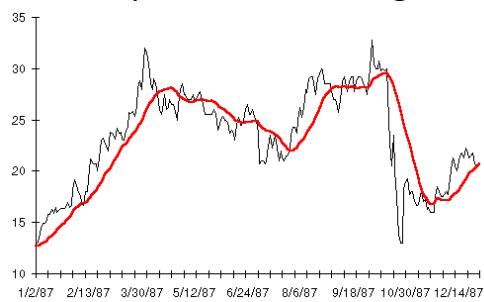
- Discrete data $x_n = x_1, x_2, x_3, \dots$

$$y(n) = \alpha x(n) + (1-\alpha)y(n-1)$$



$$y(n) = \alpha x(n) + \alpha^2 x(n-1) + \alpha^3 x(n-2) \dots$$

Exponential Moving Average



α = smoothing factor of $2/(W+1)$

W is number of time periods

Infinitely back in $x[n]$

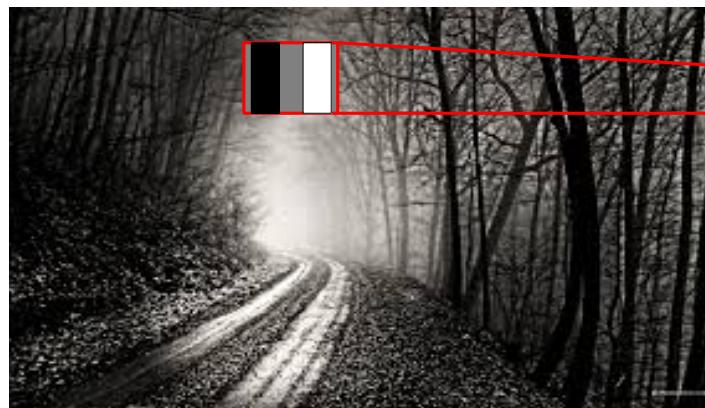


With Images
Convolutions are 2D

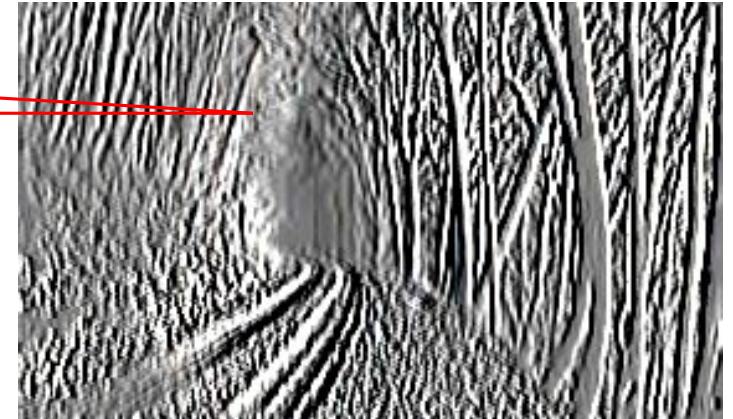
Examples in this section are from
Alyosha Efros, Slides in Guest Lecture



Convolutional of Two Signals



$$* \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} =$$



$$f[x,y] * g[x,y] = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} f[n_1, n_2] \cdot g[x-n_1, y-n_2]$$

elementwise multiplication and sum
of a filter and the signal (image)



Image filtering

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	90	90	90	90	90	0
0	0	0	0	90	90	90	90	90	0
0	0	0	0	90	90	90	90	90	0
0	0	0	0	90	0	90	90	90	0
0	0	0	0	90	90	90	90	90	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[.,.]$



Image filtering

$f[.,.]$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0	0
0	0	0	90	90	90	90	90	0	0	0
0	0	0	90	90	90	90	90	0	0	0
0	0	0	90	0	90	90	90	0	0	0
0	0	0	90	90	90	90	90	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$g[.,.]$

	0	10								



Image filtering

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[.,.]$

	0	10	20						



Image filtering

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[.,.]$

	0	10	20	30					



Image filtering

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[.,.]$

	0	10	20	30	30				



Image filtering

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[.,.]$

	0	10	20	30	30				

$$h[\cdot, \cdot] \frac{1}{9} \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$



Image filtering

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[.,.]$

	0	10	20	30	30				



$$h[\cdot, \cdot] \frac{1}{9} \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

Image filtering

$$h[\cdot, \cdot] \quad \frac{1}{9} \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

$$f[\cdot, \cdot]$$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$$g[\cdot, \cdot]$$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

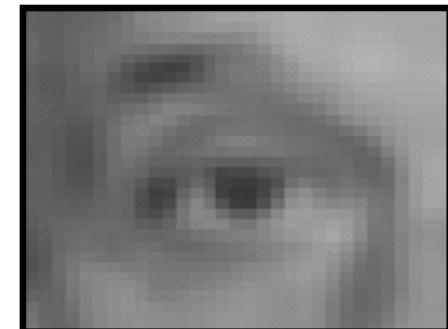


Linear filters: examples



Original

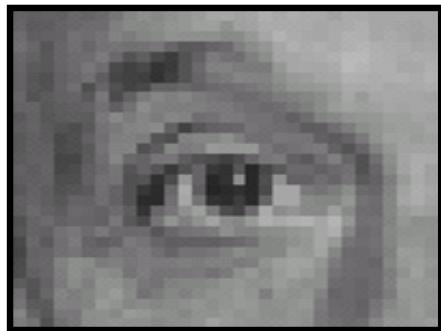
$$\frac{1}{9} \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array} =$$



Blur (with a mean filter)



Practice with linear filters



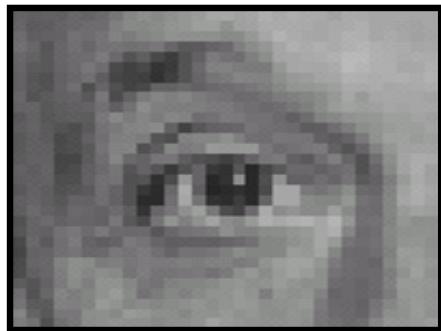
0	0	0
0	1	0
0	0	0

?

Original

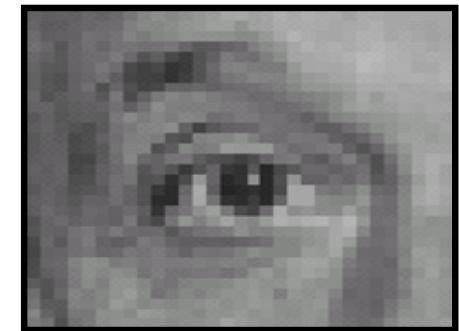
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 0 1 1 X 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
Data 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

Practice with linear filters



Original

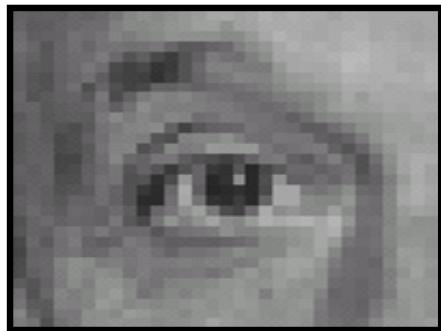
0	0	0
0	1	0
0	0	0



Filtered (no change)

Data X

Practice with linear filters



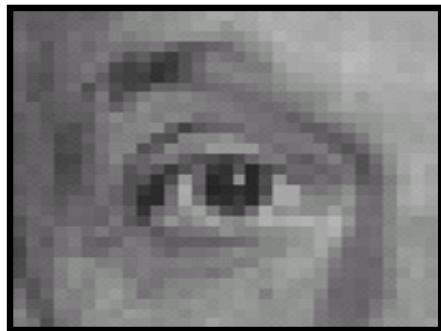
0	0	0
0	0	1
0	0	0

?

Original

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 0 1 1 X 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
Data 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

Practice with linear filters



Original

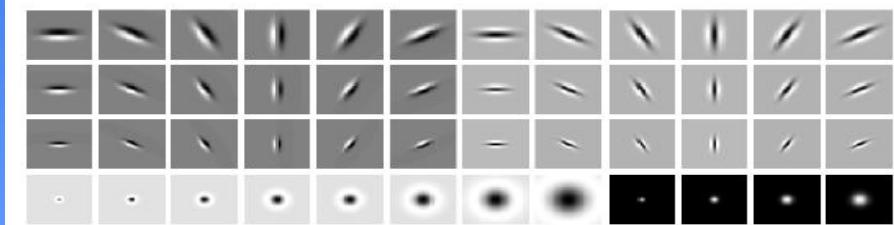
0	0	0
0	0	1
0	0	0



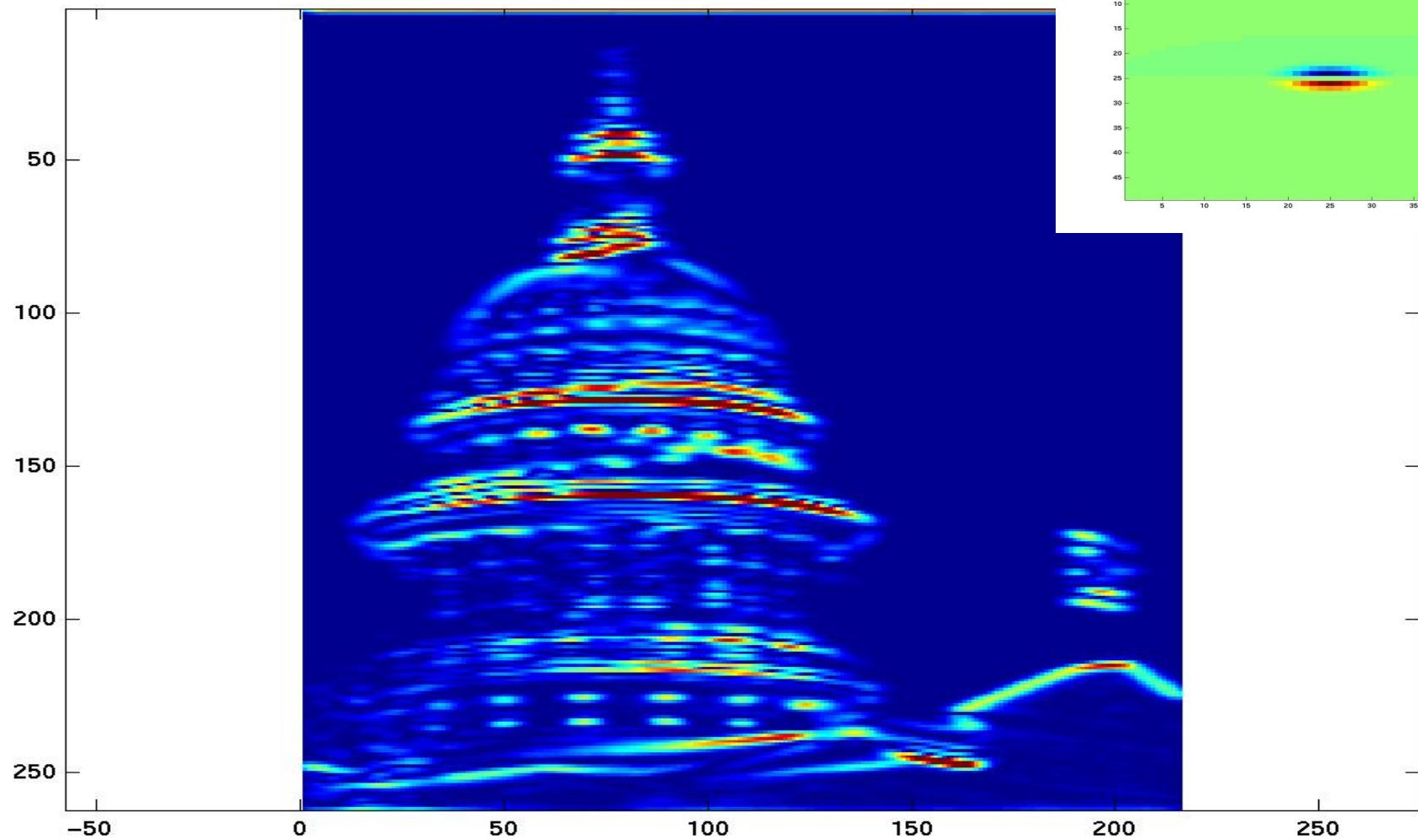
Shifted left
By 1 pixel

A horizontal strip of binary data (0s and 1s) with an 'X' marking the center pixel where the filter will be applied. The data is labeled "Data" at the bottom left.

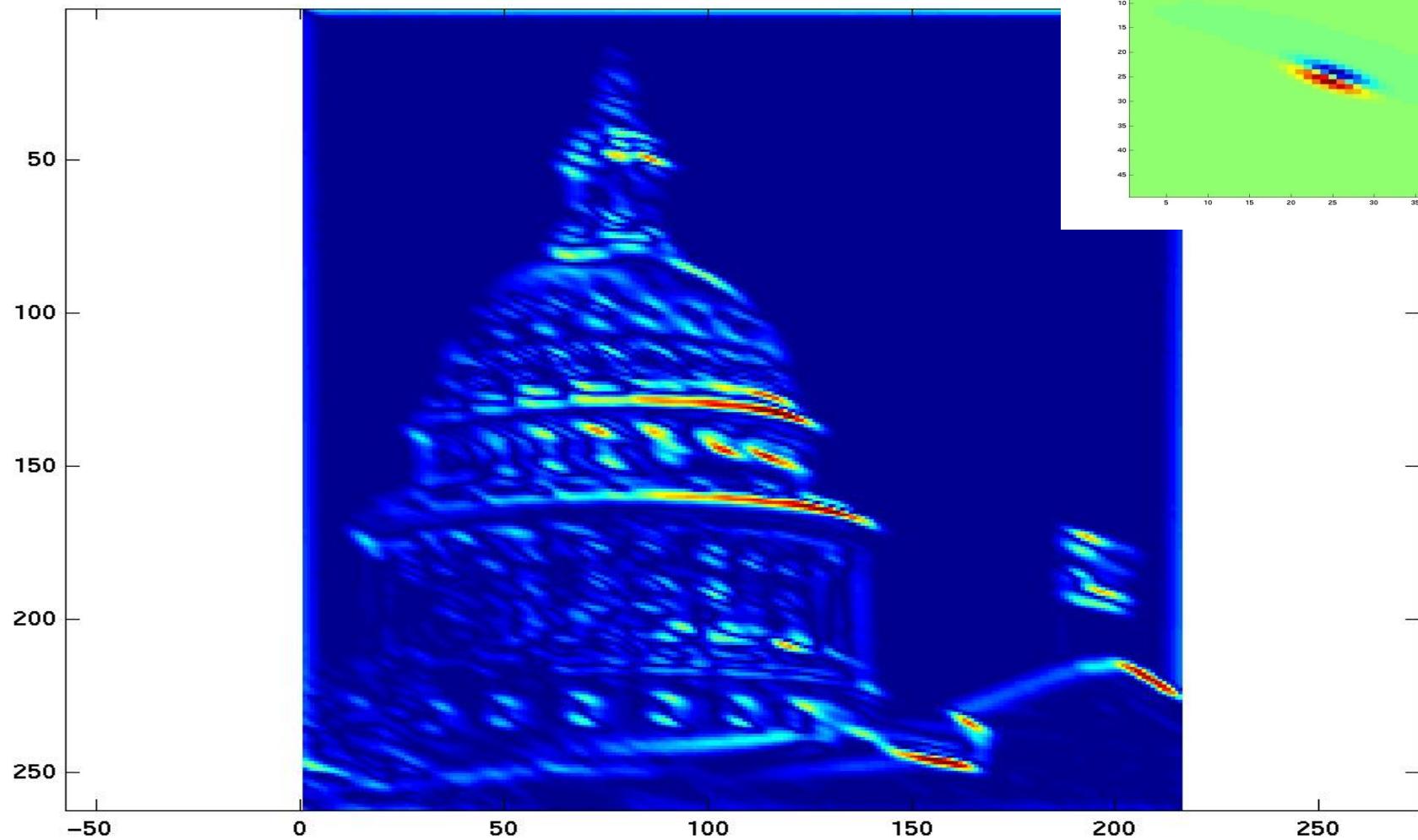
from <http://www.texaseexplorer.com/austincap2.jpg>



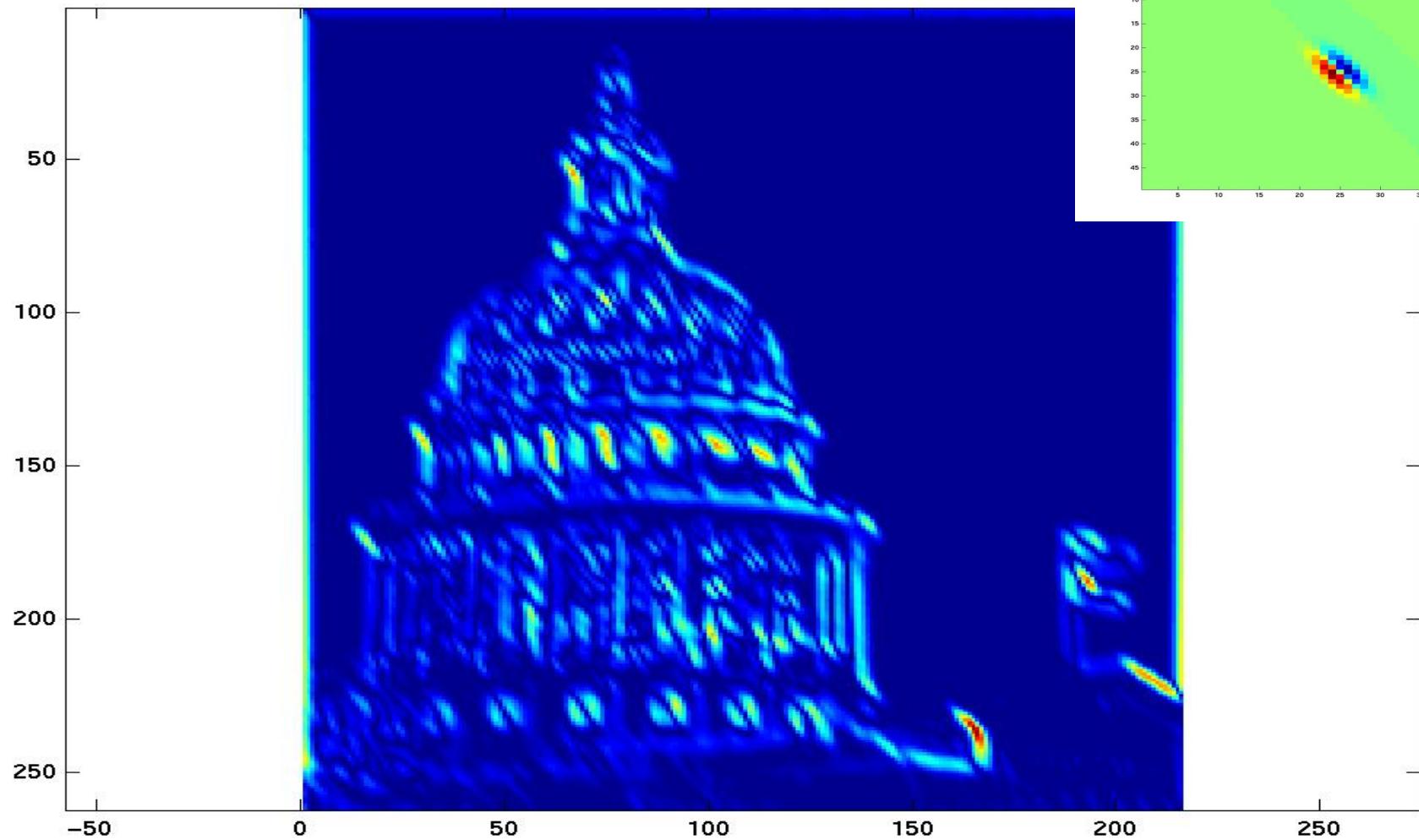
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1 1 0 1 1 0 1 1 0 1
Data 0 1 0 1 0 1 0 1 0 1
0 1 0 1 0 1 0 1 0 1



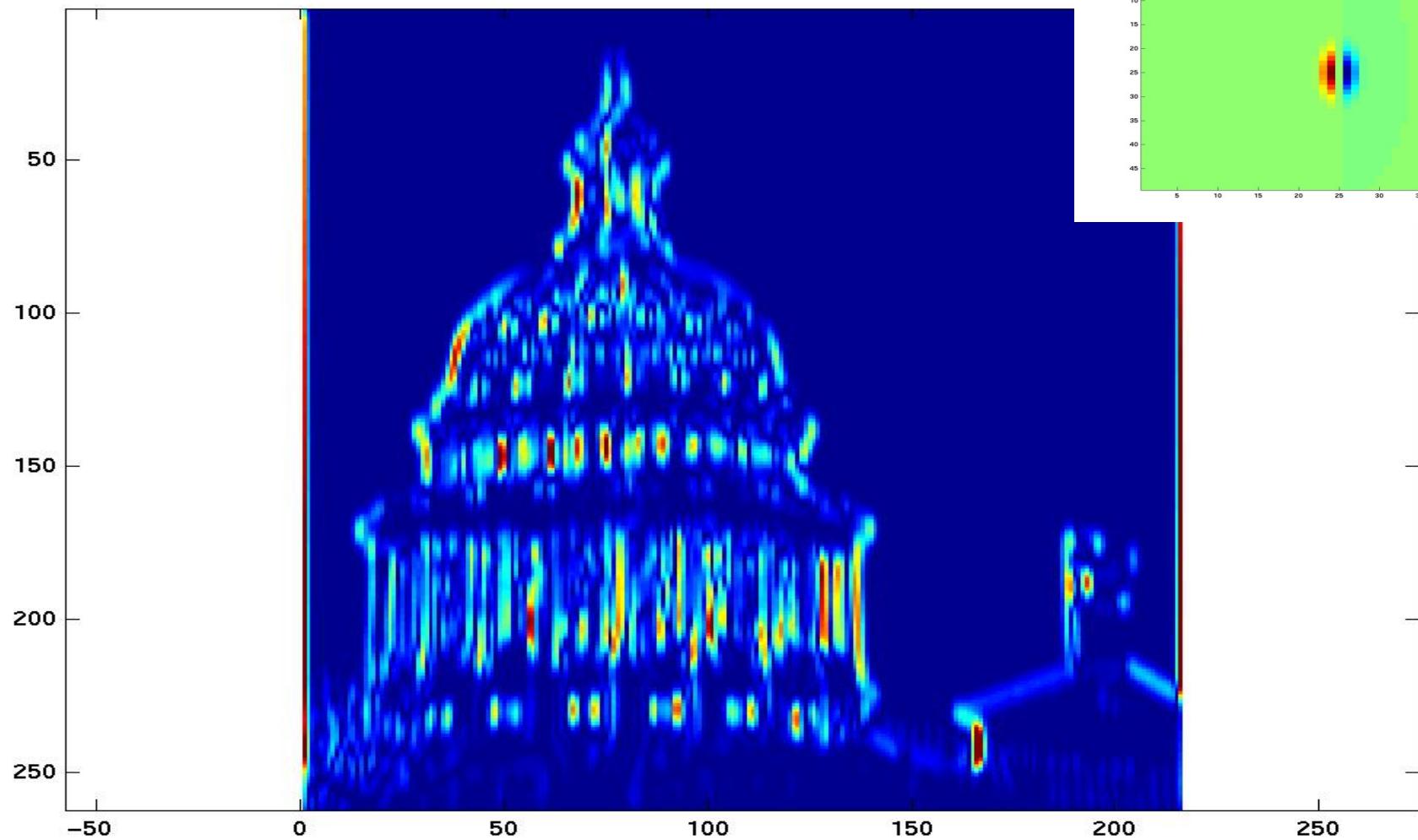
Kristen Grauman



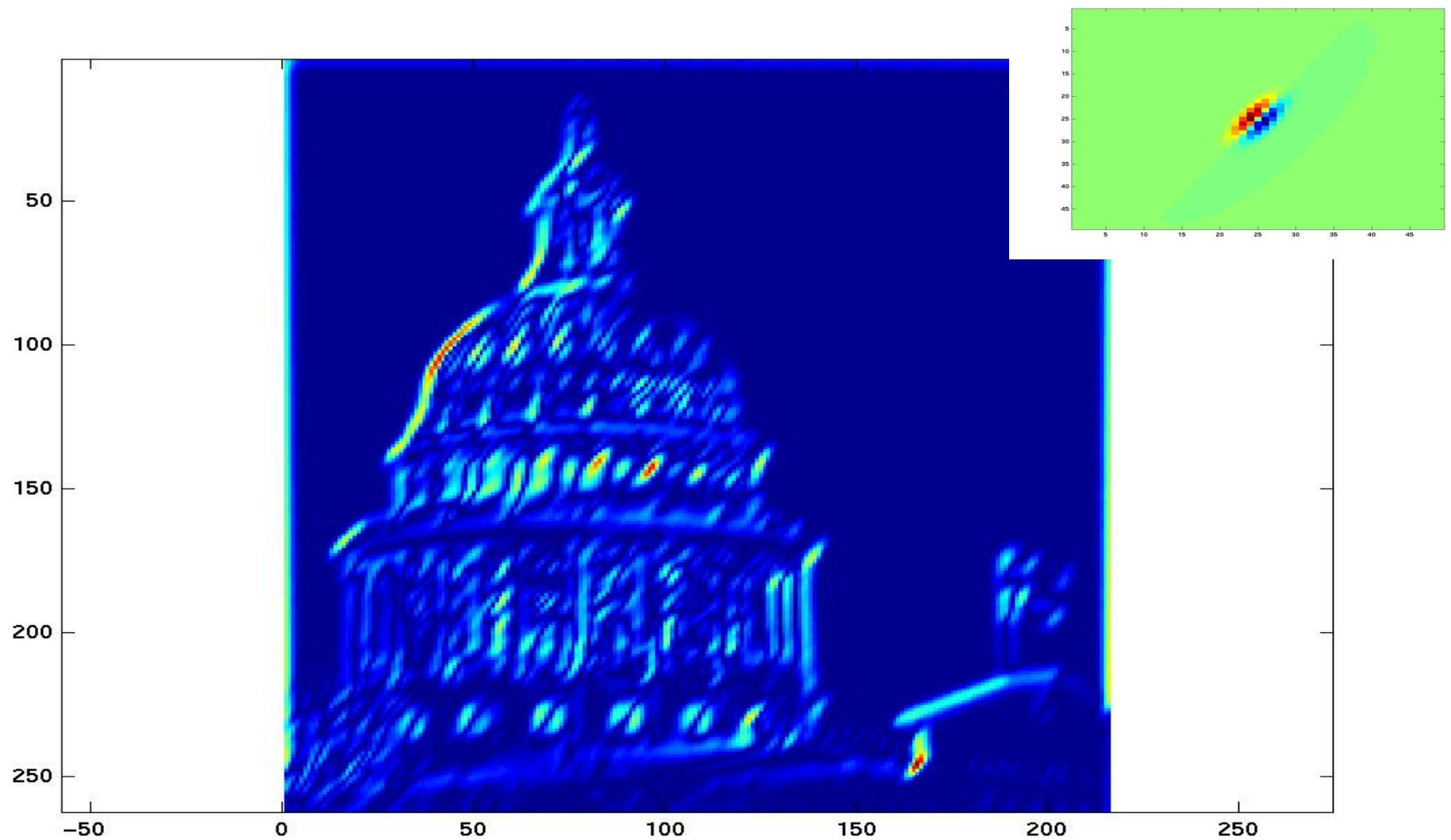
Kristen Grauman



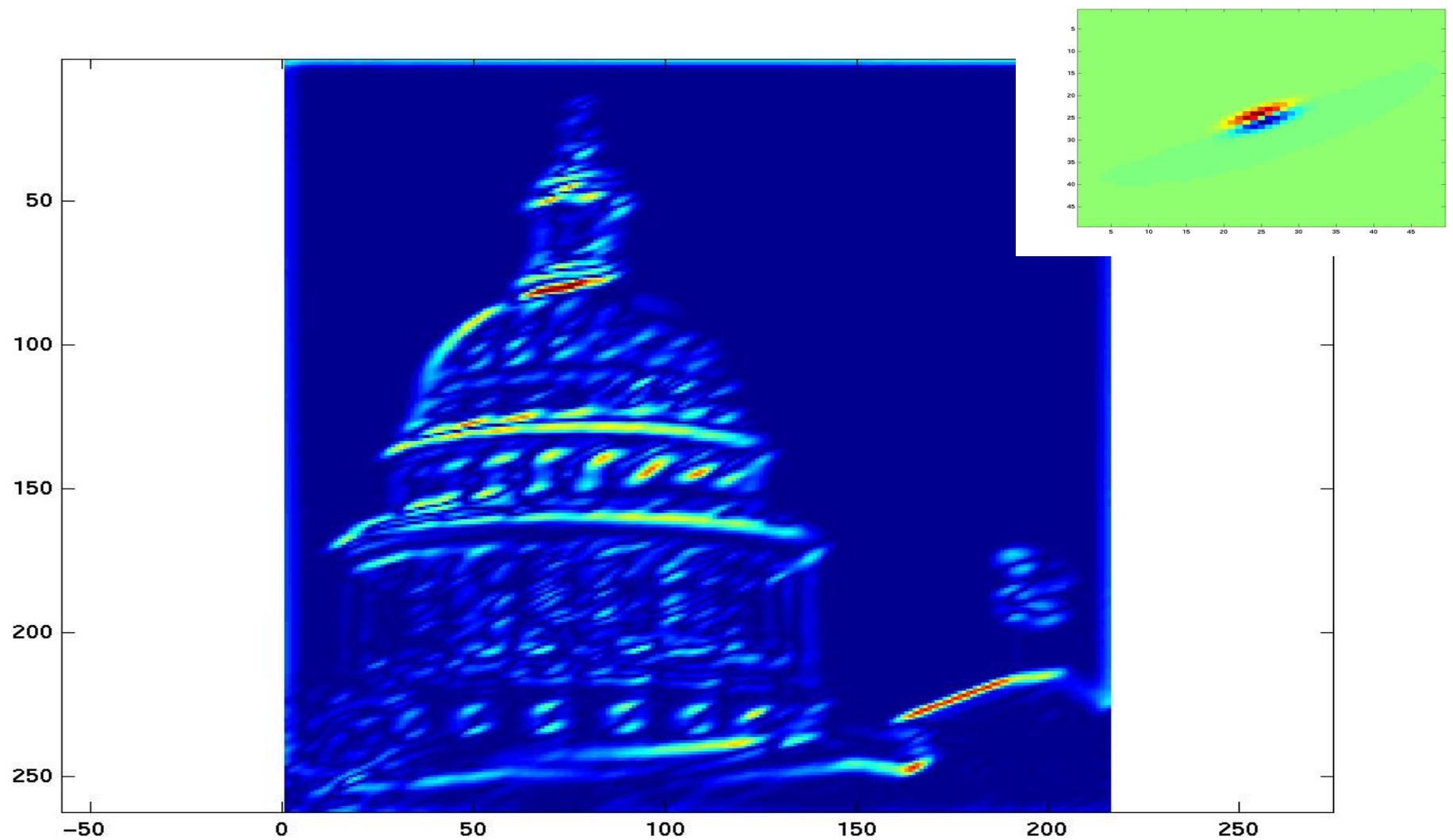
Kristen Grauman



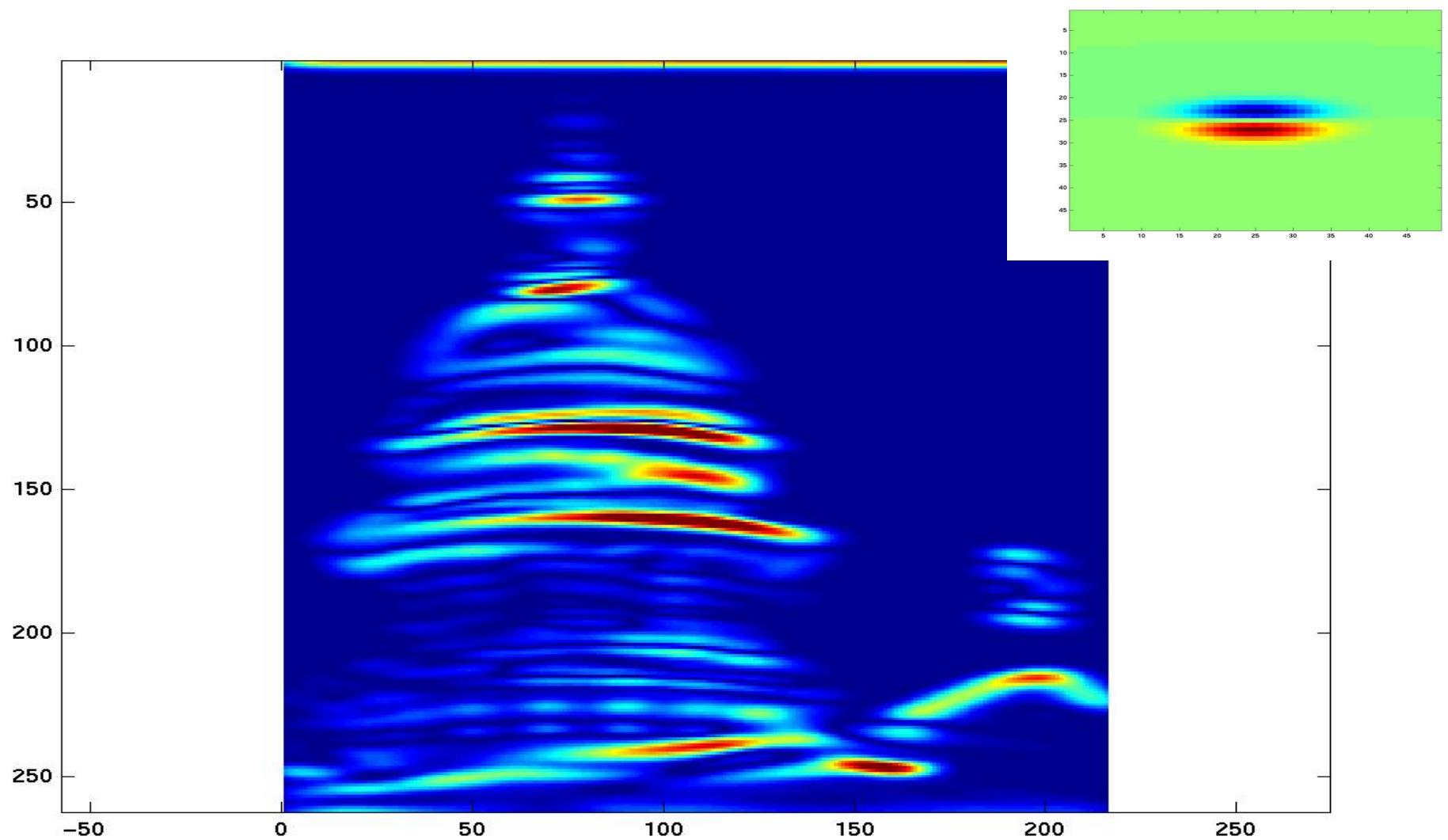
Kristen Grauman



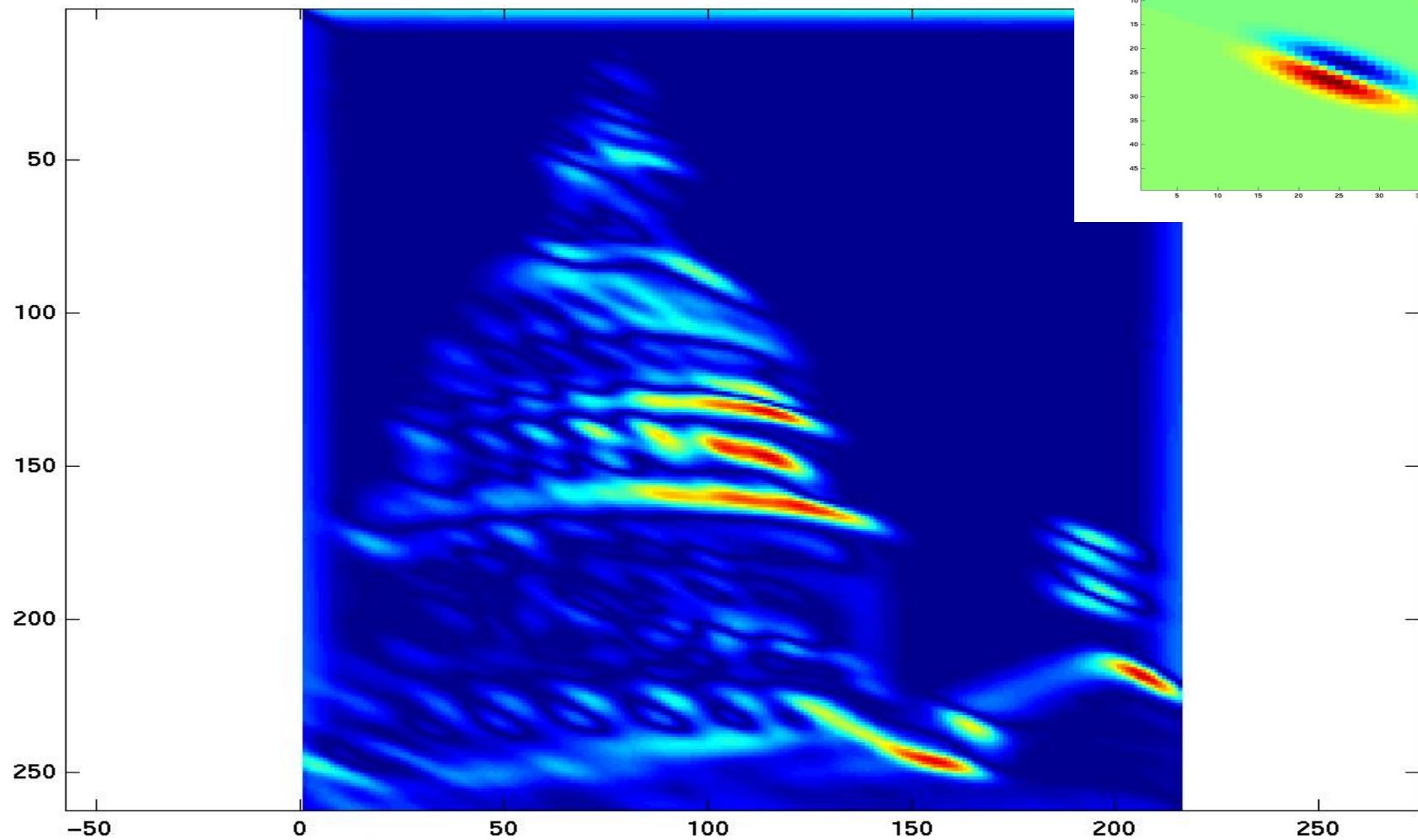
Kristen Grauman



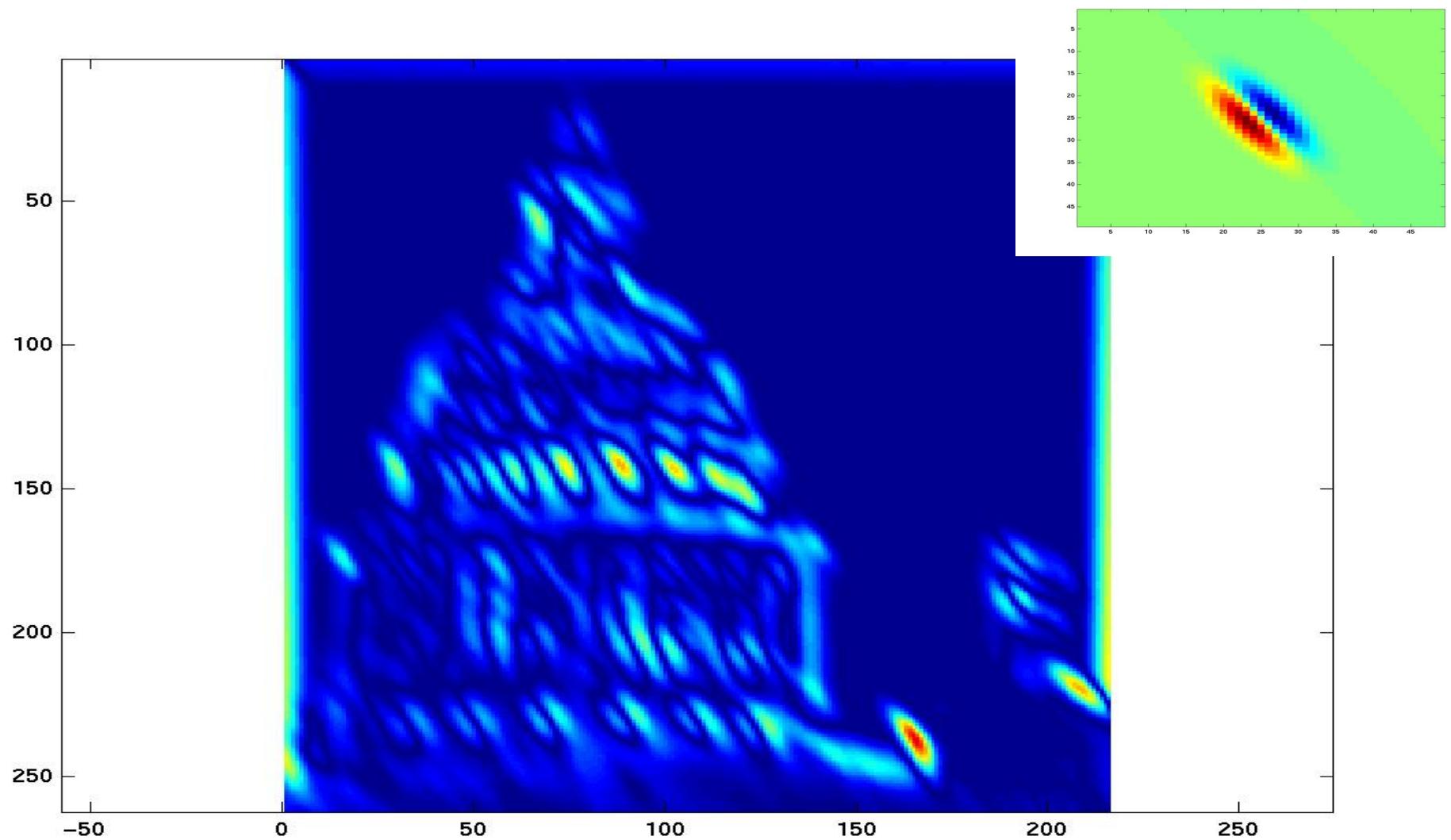
Kristen Grauman



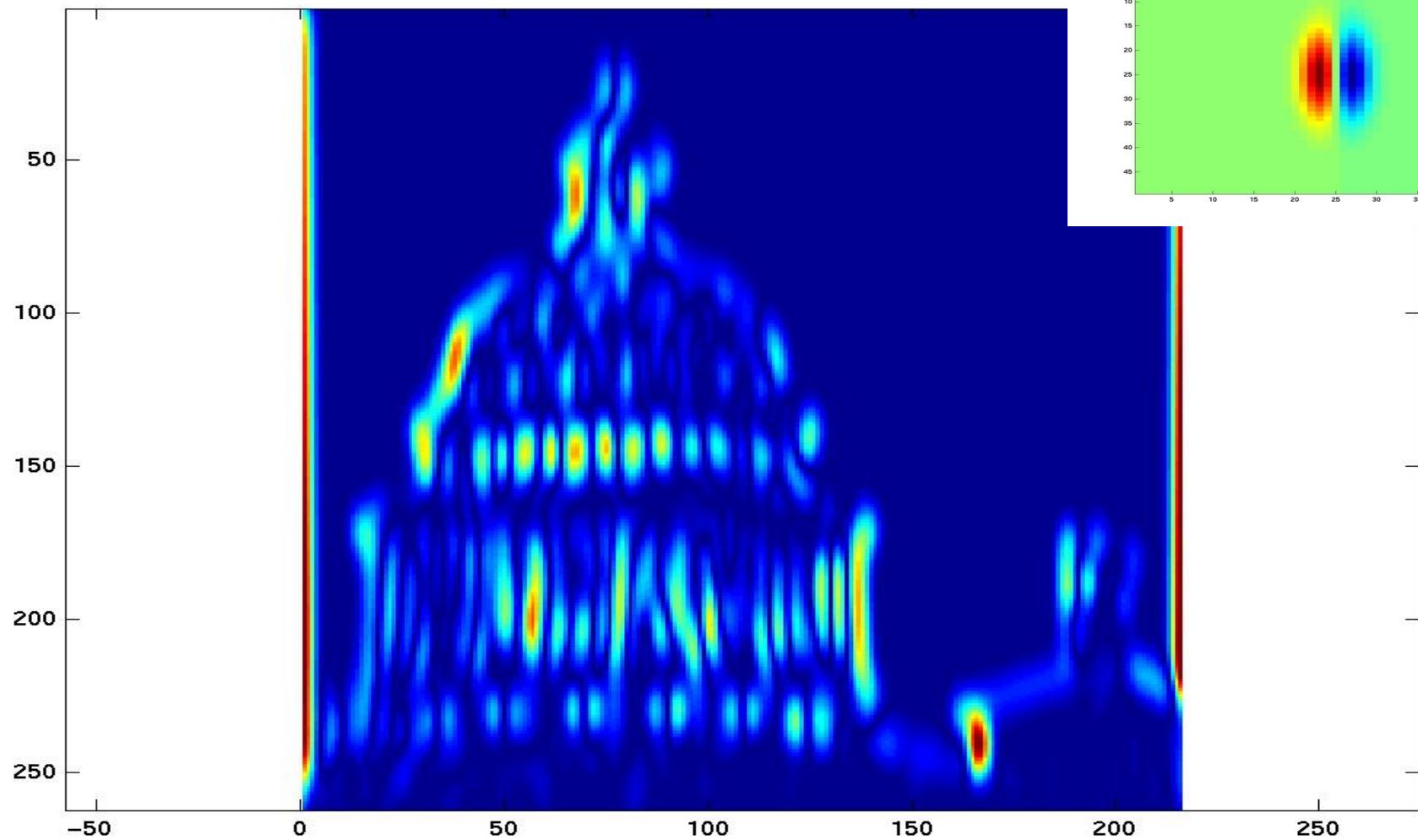
Kristen Grauman



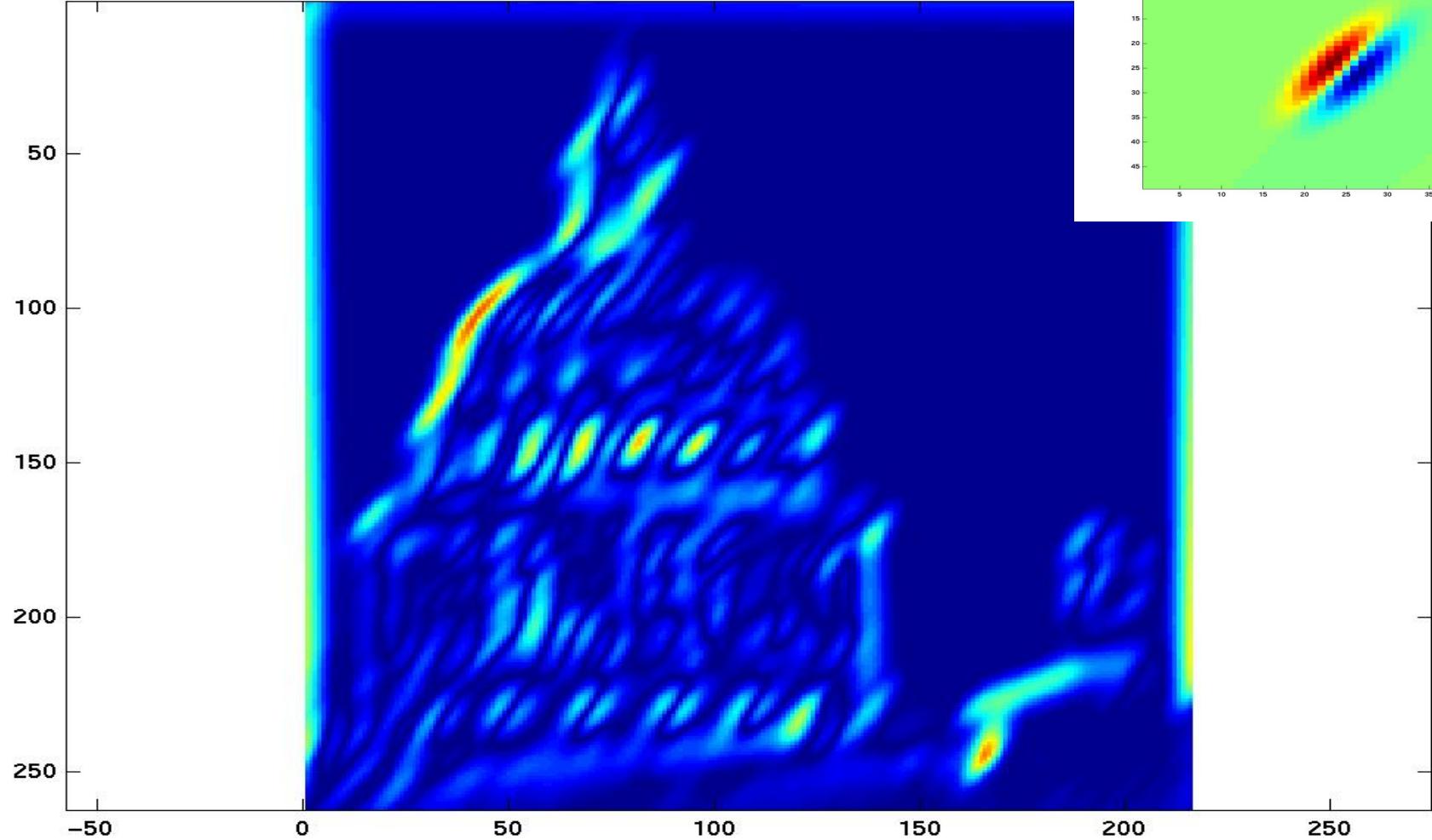
Kristen Grauman



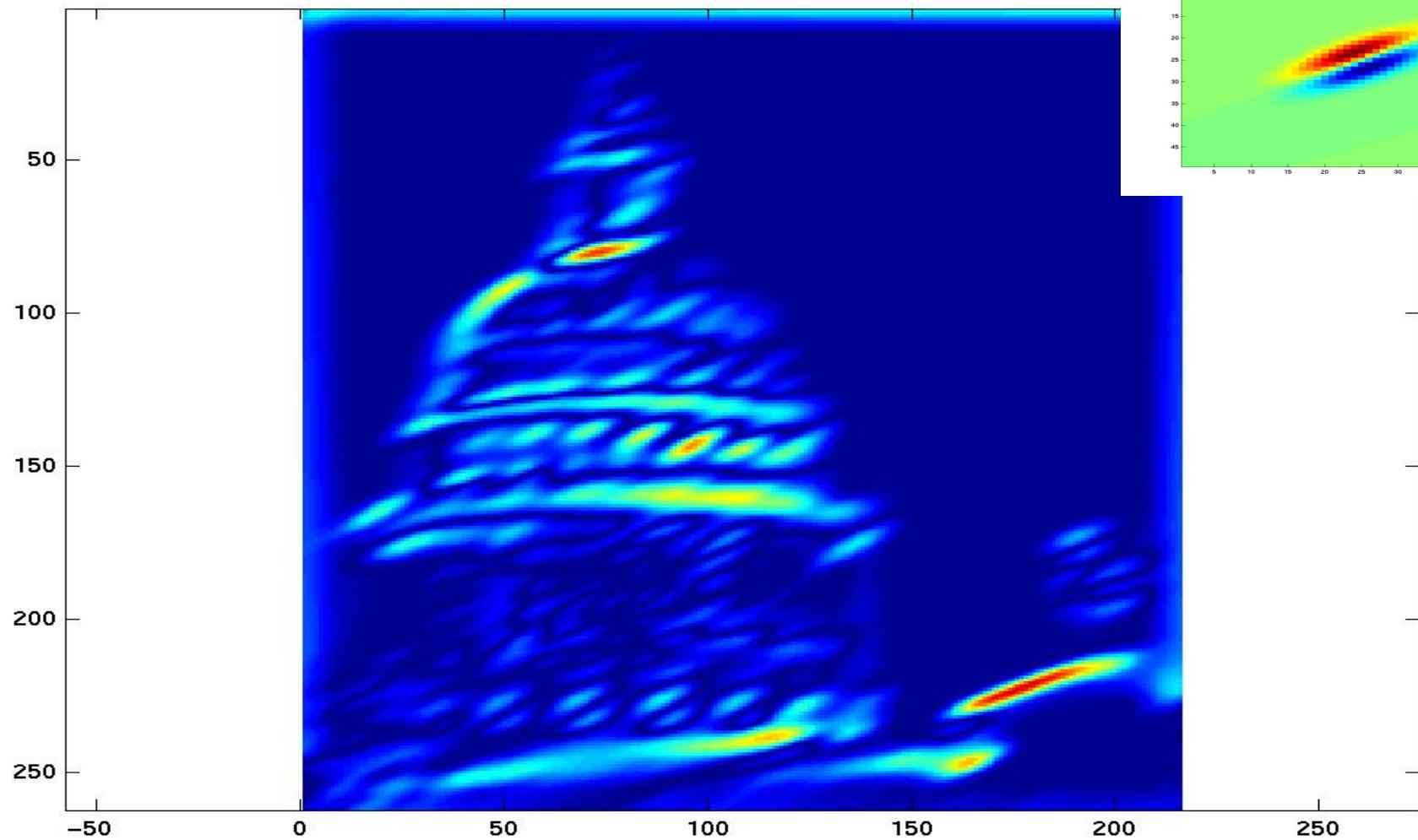
Kristen Grauman



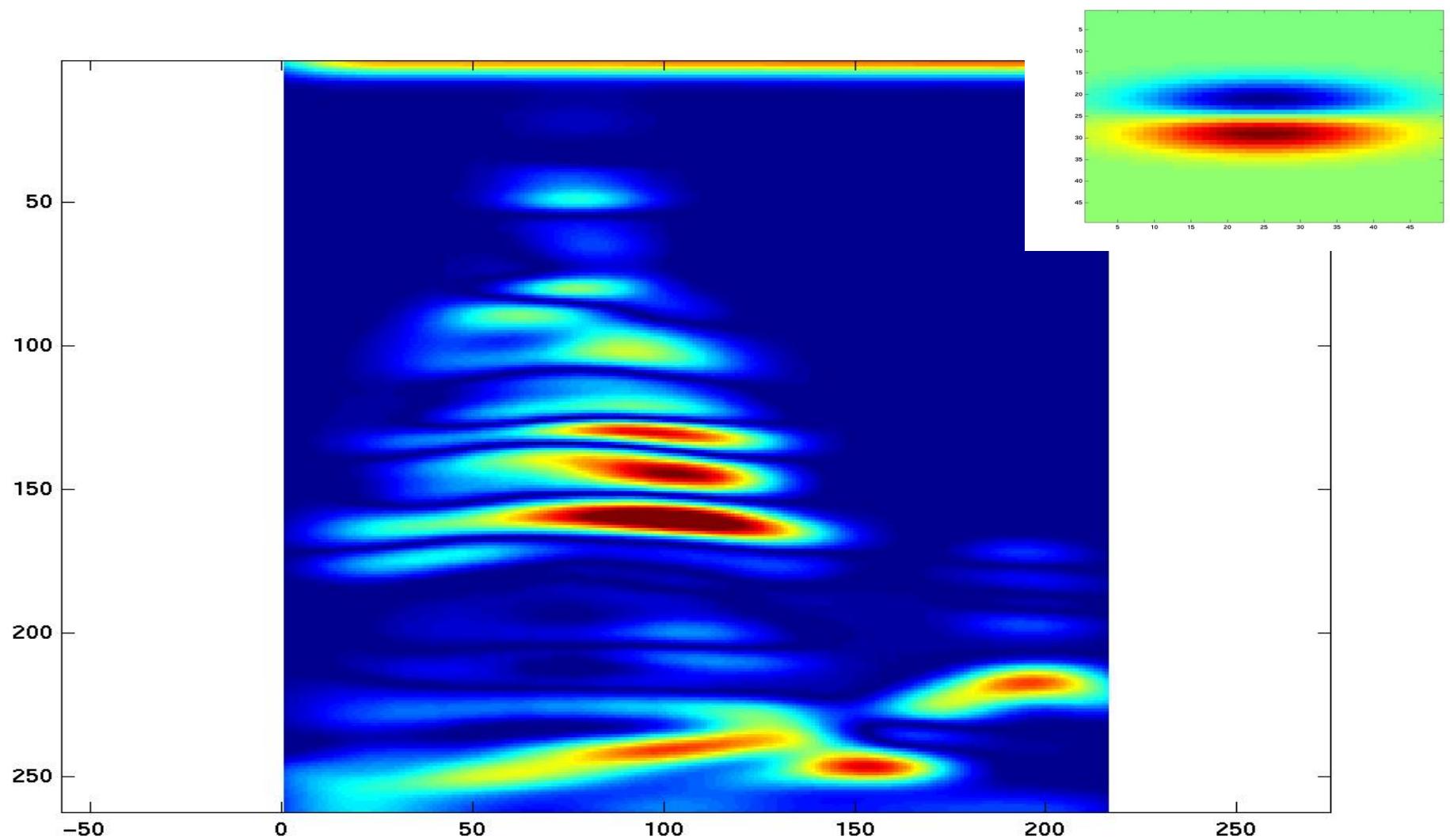
Kristen Grauman



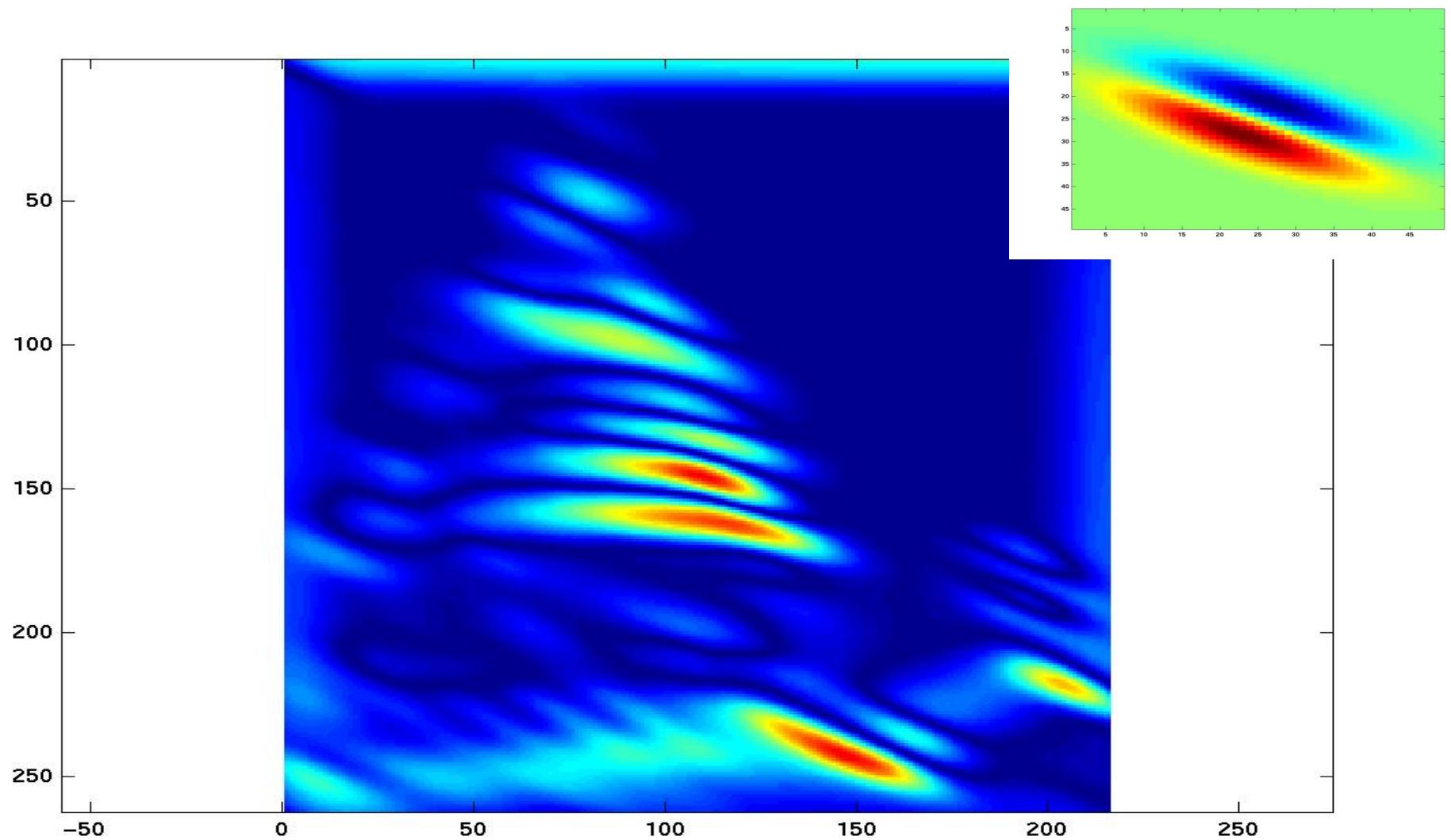
Kristen Grauman



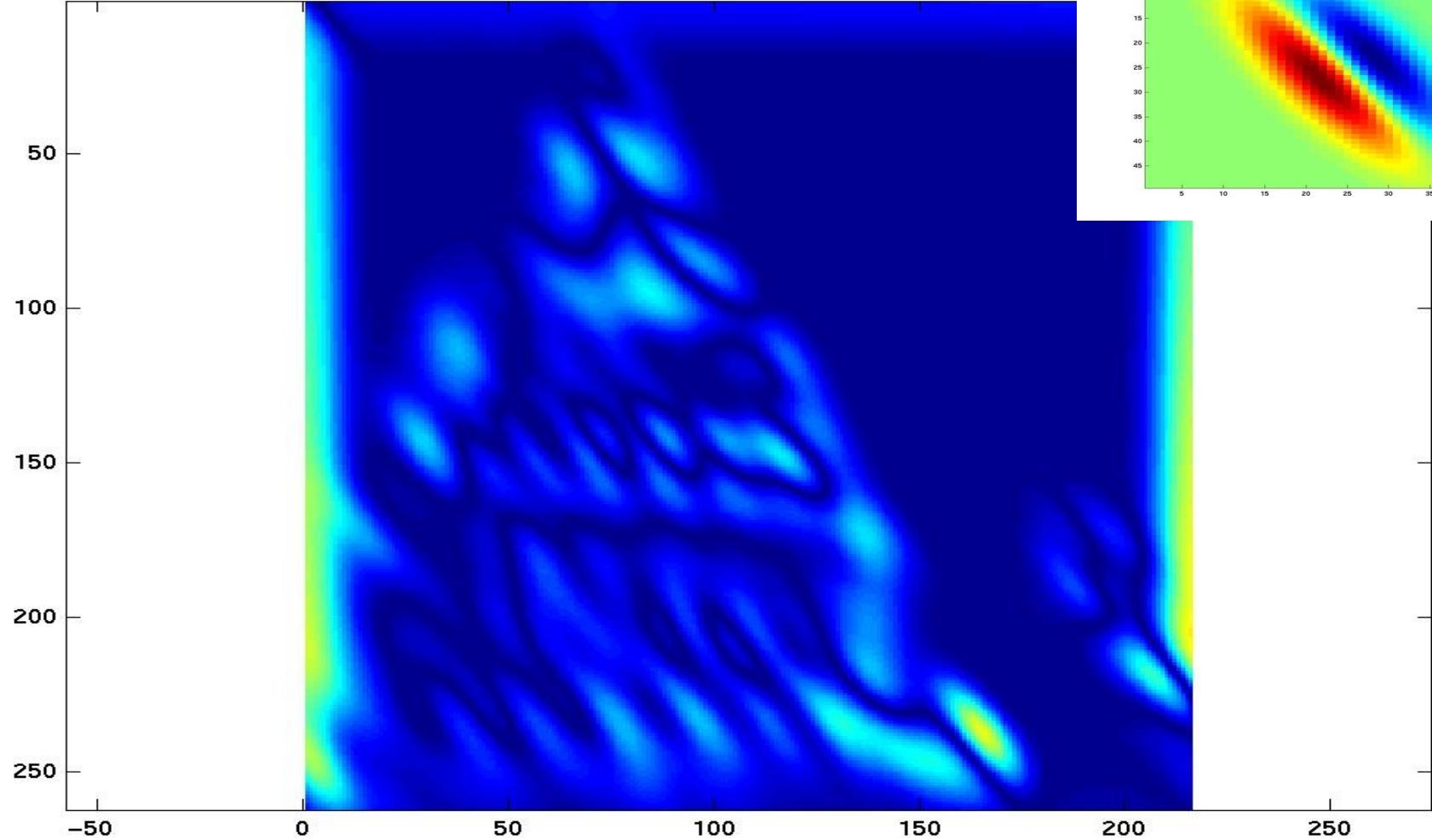
Kristen Grauman



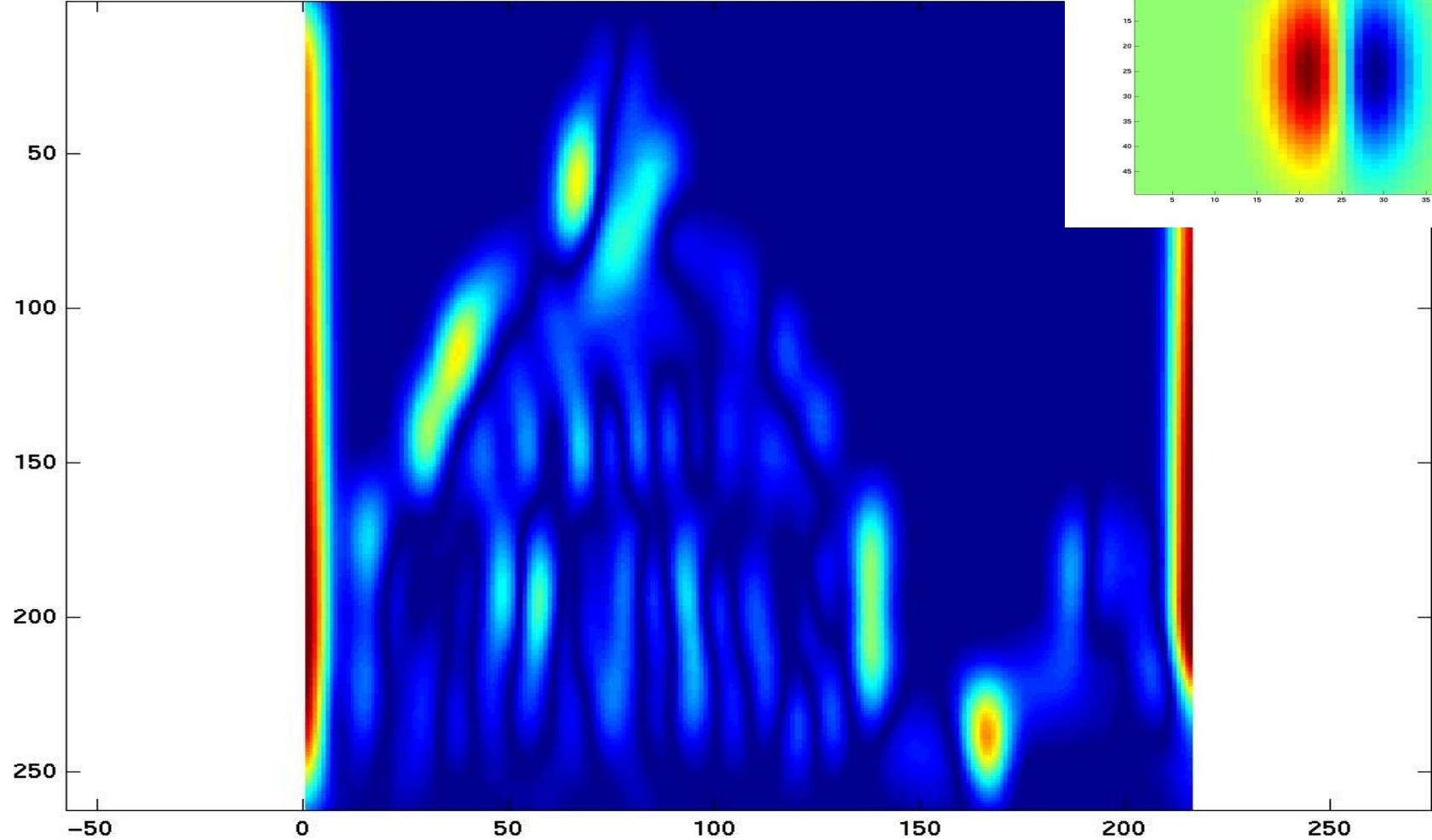
Kristen Grauman



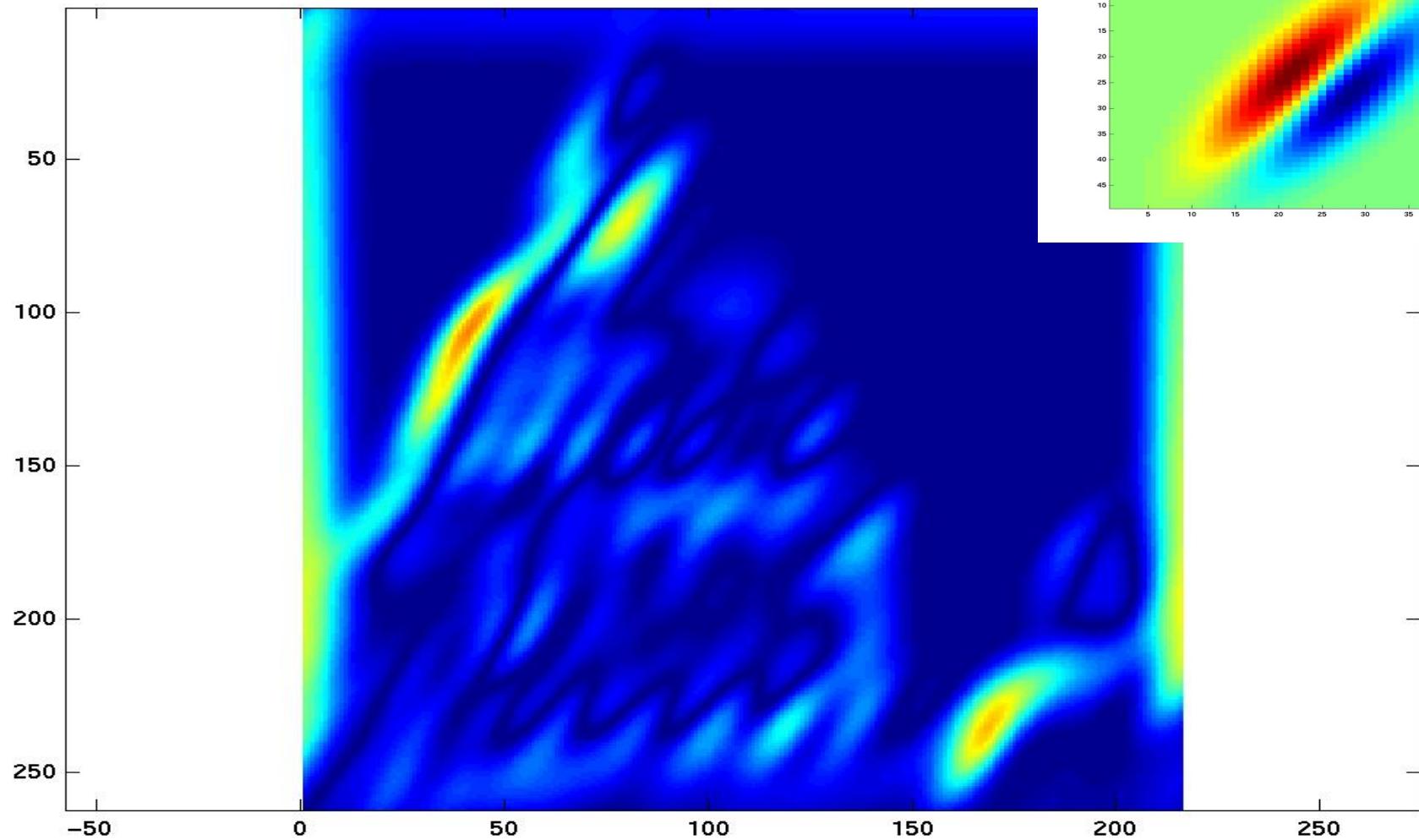
Kristen Grauman



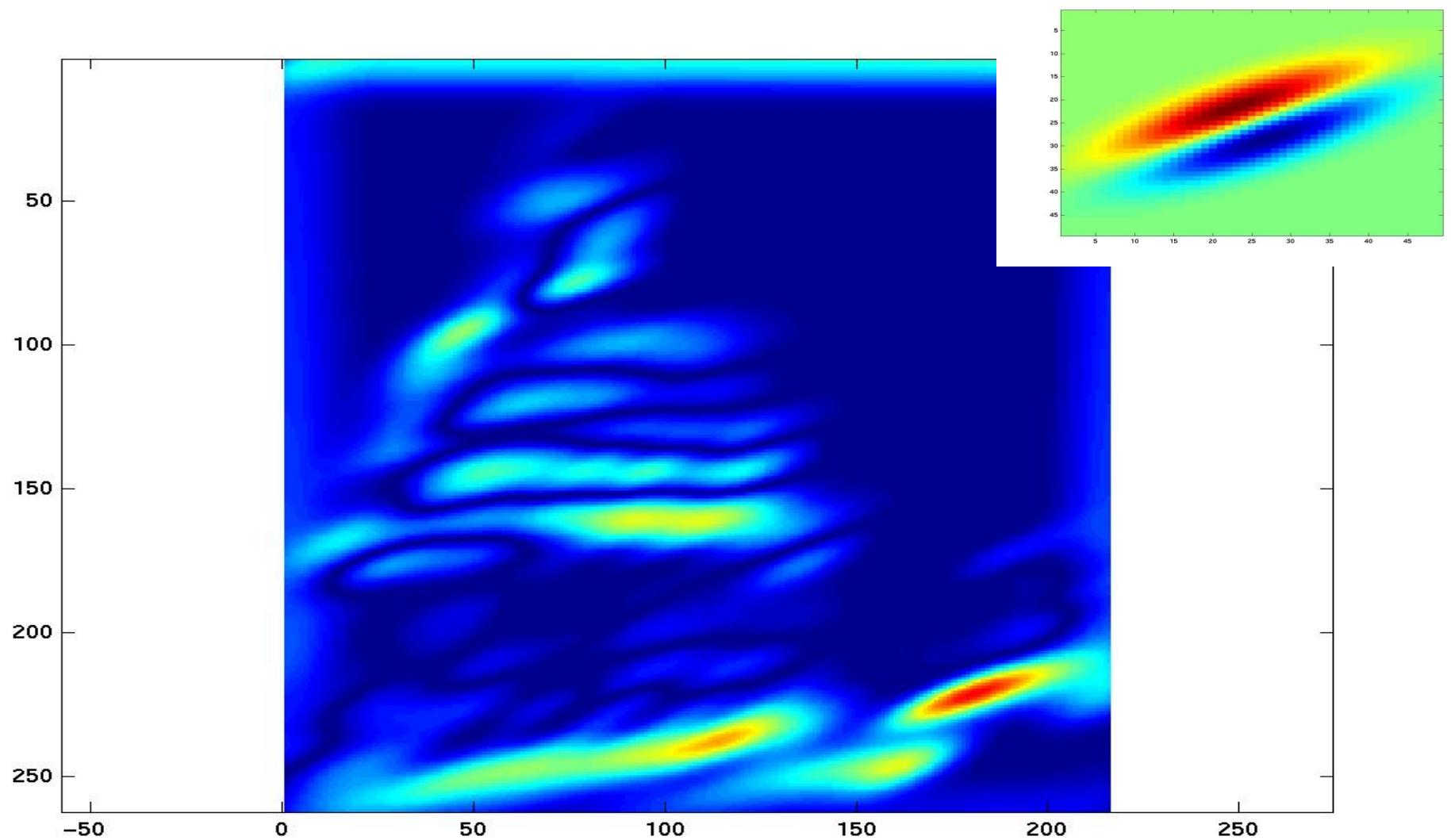
Kristen Grauman



Kristen Grauman



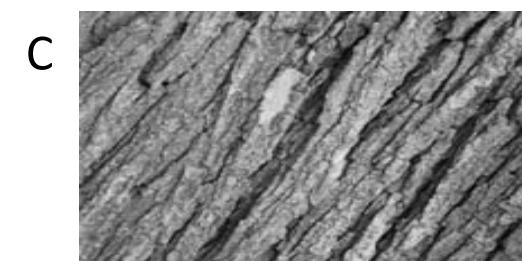
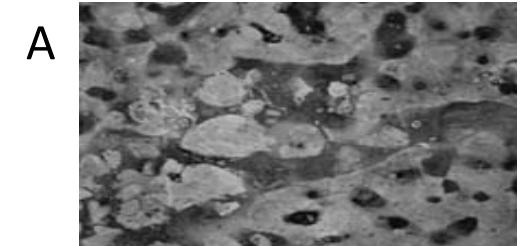
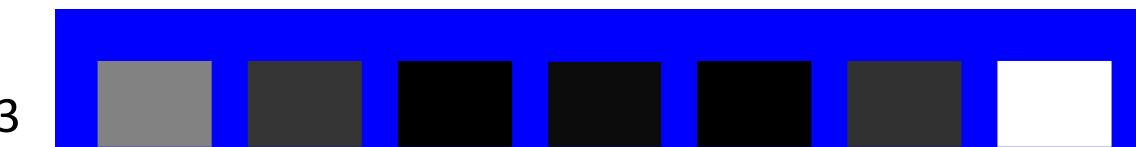
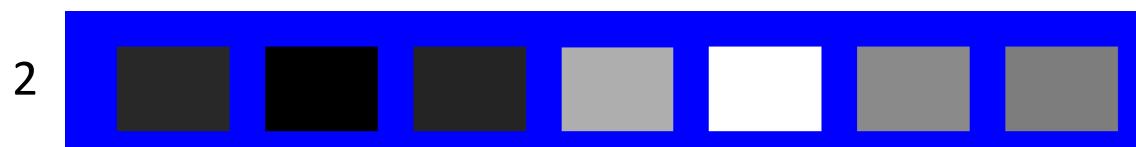
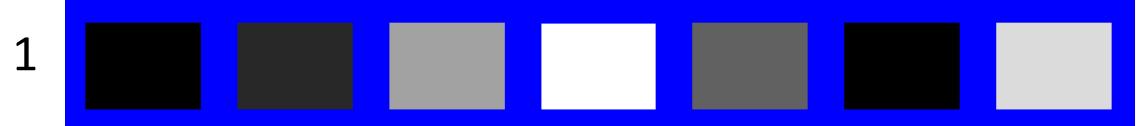
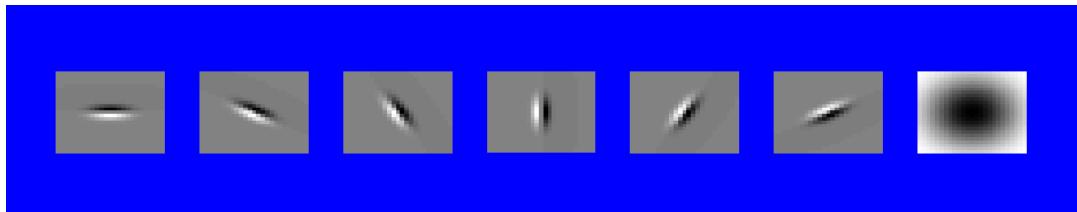
Kristen Grauman



Kristen Grauman

Can you match the texture to the response?

Filters



End of Section

0 0 0 1 0 1 0 1 0 1 1 1 0 0 0 0 0 0 1 0 0 1 0 1 0 1 1 1 0 0
1 0 1 1 X 1 1 0 0 1 0 1 0 0 1 0 1 0 1 0 1 1 1 1 1 0 1 0 1 1 1 0 0
1 Data 0 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 1 1 1 1 1 0 1 0 1 1 1 0 0