1. $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{f(i-j)}{N}(n_i-1)(n_j+1)\,p(n_0,n_1,\dots,n_i-1,\dots,n_j+1,\dots;t)$:

(Have n_i-1, n_j+1 particles of i,j states. A particle of state j copies the state of particle in state i.)

$+\sum_{i=0}^{\infty}(n_i+1)\lambda_1\, p(n_0,n_1,\dots,n_i+1,n_{i+1}-1,\dots;t)$:

(Have n_i+1, $n_{i+1}-1$ particles of i, $i+1$ states. A particle of state i updates its information spontaneously to be in state $i+1$.)

$+\sum_{i=0}^{\infty}(n_i+1)\lambda_2\, p(n_0,n_1,\dots,n_{i-1}-1,n_i+1,\dots;t)$:

(Have $n_{i-1}-1$, n_i+1 particles of $i-1$, i states. A particle of state i updates its information spontaneously to be in state $i-1$.)

$-\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{f(i-j)}{N}n_in_j+\lambda_1\sum_{n=0}^{\infty}n_i+\lambda_2\sum_{i=1}^{\infty}n_i\right)p(n_0,n_1\dots;t)$: (All possible ways of leaving the $n_0,n_1\dots$ state by either pairwise copying or spontaneous information update) Require $p=0$ if any $n_i<0$.

[5]

2. Multiply $\frac{d}{dt}p(n_0,n_1,\dots;t)$ by n_k, sum over all states, reindex sums:

$$\frac{d\langle n_k\rangle}{dt}=\lambda_1(\langle n_{k-1}\rangle-\langle n_k\rangle)+\lambda_2(\langle n_{k+1}\rangle-\langle n_k\rangle)+\sum_{i=0}^{\infty}\left[\frac{f(k-i)-f(i-k)}{N}\right]\langle n_kn_i\rangle \qquad \text{for } k=1,2,\dots$$

where:

$$\frac{d\langle n_0\rangle}{dt}=-\lambda_1\langle n_0\rangle+\lambda_2\langle n_1\rangle+\sum_{i=0}^{\infty}\frac{f(-i)-f(i)}{N}\langle n_0n_i\rangle$$

Closed equations using mean field equations:

$$\frac{d\langle n_k\rangle}{dt}=\lambda_1(\langle n_{k-1}\rangle-\langle n_k\rangle)+\lambda_2(\langle n_{k+1}\rangle-\langle n_k\rangle)+\langle n_k\rangle\sum_{i=0}^{\infty}\frac{f(k-i)-f(i-k)}{N}\langle n_i\rangle \qquad k=1,2,\dots$$

$$\frac{d\langle n_0\rangle}{dt}=-\lambda_1\langle n_0\rangle+\lambda_2\langle n_1\rangle+\langle n_0\rangle\sum_{i=0}^{\infty}\frac{f(-i)-f(i)}{N}\langle n_i\rangle.$$

[5]

3. If f is even, $f(k-i) = f(-(k-i)) = f(i-k)$

so the sum over $\sum_{i=0}^{\infty} \frac{f(k-i)-f(i-k)}{N}$ vanishes in both the stochastic system and the mean field model.

In both cases we are left with:

$$\frac{d\langle n_0\rangle}{dt} = -\lambda_1\langle n_0\rangle + \lambda_2\langle n_1\rangle$$

$$\frac{d\langle n_k\rangle}{dt} = \lambda_1(\langle n_{k-1}\rangle - \langle n_k\rangle) + \lambda_2(\langle n_{k+1}\rangle - \langle n_k\rangle)$$

for $k = 1, 2, \ldots$

[3]

4. $K+1$ states. Considering $\frac{d\langle n_{K+1}\rangle}{dt}$ separately:

$$\frac{d\langle n_j\rangle}{dt} = \lambda_1(\langle n_{j-1}\rangle - \langle n_j\rangle) + \lambda_2(\langle n_{j+1}\rangle - \langle n_j\rangle) + \sum_{i=0}^{K+1} \frac{f(j-i)-f(i-j)}{N}\langle n_j n_i\rangle$$

index changed to j for clarity.

for $j = 1, 2, \ldots, K$

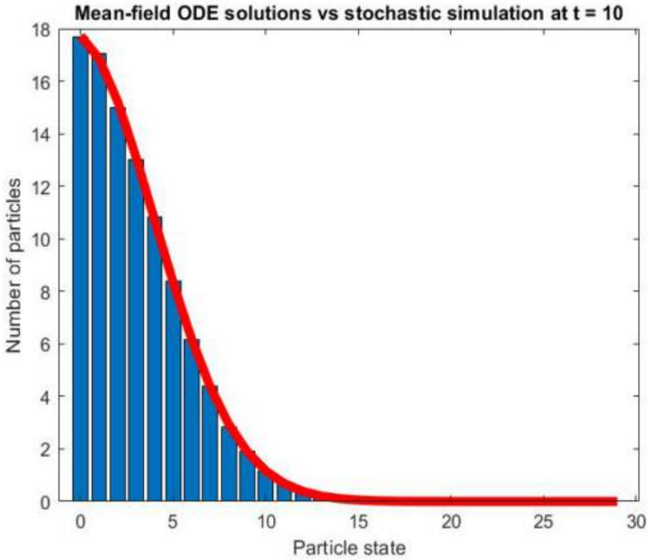
$$\frac{d\langle n_0\rangle}{dt} = -\lambda_1\langle n_0\rangle + \lambda_2\langle n_1\rangle + \sum_{i=0}^{K+1} \frac{f(-i)-f(i)}{N}\langle n_0 n_i\rangle$$

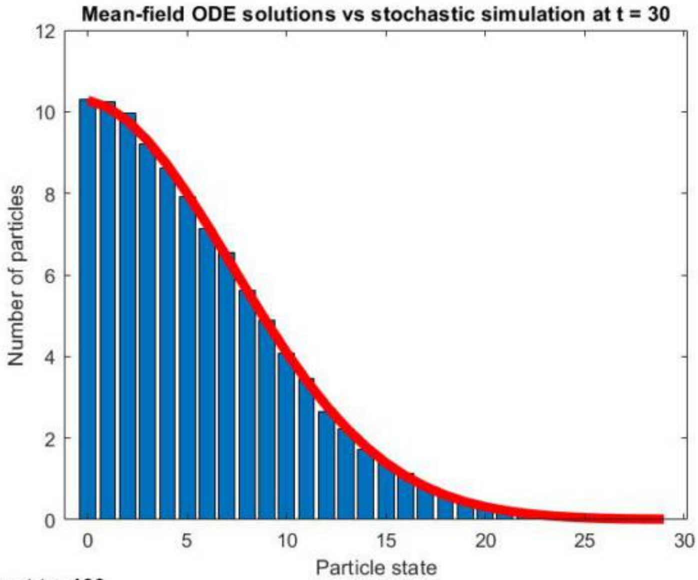
$$\frac{d\langle n_{K+1}\rangle}{dt} = \lambda_1\langle n_K\rangle - \lambda_2\langle n_{K+1}\rangle + \sum_{i=0}^{K+1} \frac{f((K+1)-i)-f(i-(K+1))}{N}\langle n_{K+1} n_i\rangle$$

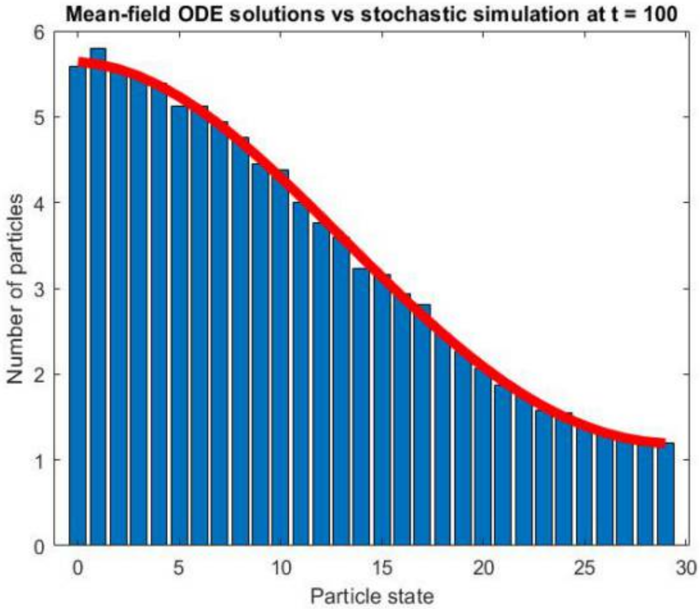
[3]

5.

f is even so can use eq'ns in Q3).







6. (Note $\lambda_2 = 0$).

Subbing in $f(z) = -\alpha z H(-z)$ in eq'ns derived in Q2:

$$\frac{d\langle n_0\rangle}{dt} = \frac{\langle n_0\rangle}{N}\sum_{i=0}^{\infty} -\alpha(-i)H(i) - (-\alpha i\, H(-i))\langle n_i\rangle \quad - \lambda_1\langle n_0\rangle$$

$$= \alpha\frac{\langle n_0\rangle}{N}\sum_{i=0}^{\infty}\Big[i\underbrace{H(i)}_{=1 \text{ for } i=0,1,\dots} + i\underbrace{H(-i)}_{=0 \text{ for } i=0,1,\dots}\Big]\langle n_i\rangle \quad - \lambda_1\langle n_0\rangle$$

$$\frac{d\langle n_0\rangle}{dt} = \alpha\langle n_0\rangle\cdot\frac{1}{N}\sum_{i=0}^{\infty} i\langle n_i\rangle \quad - \lambda_1\langle n_0\rangle$$

$$\frac{d\langle n_k\rangle}{dt} = \frac{\langle n_k\rangle}{N}\sum_{i=0}^{\infty}\Big[-\alpha(k-i)H(i-k) + \alpha(k-i)H(k-i)\Big]\langle n_i\rangle \quad + \lambda_1(\langle n_{k-1}\rangle - \langle n_k\rangle)$$

$$= \frac{\alpha\langle n_k\rangle}{N}\sum_{i=0}^{\infty}(i-k)\Big[\underbrace{H(i-k)}_{\substack{=1 \text{ for}\\ i\geq k+1}} + \underbrace{H(k-i)}_{\substack{=1 \text{ for}\\ i\leq k-1}}\Big]\langle n_i\rangle \quad + \lambda_1(\langle n_{k-1}\rangle - \langle n_k\rangle)$$

$\longrightarrow$ so $H(i-k) + H(k-i) = \begin{cases} 1 & i \neq k \\ 0 & i = k \end{cases}$

$$= \frac{\alpha\langle n_k\rangle}{N}\sum_{i=0}^{\infty}(i-k)\langle n_i\rangle + \lambda_1(\langle n_{k-1}\rangle - \langle n_k\rangle)$$

$$= \alpha\langle n_k\rangle\Big[\frac{1}{N}\sum_{i=0}^{\infty} i\langle n_i\rangle - k\cdot\underbrace{\frac{1}{N}\sum_{i=0}^{\infty}\langle n_i\rangle}_{=1}\Big] + \lambda_1(\langle n_{k-1}\rangle - \langle n_k\rangle)$$

so:

$$\frac{d\langle n_k\rangle}{dt} = \alpha\langle n_k\rangle\Big(\frac{1}{N}\sum_{i=0}^{\infty} i\langle n_i\rangle \quad - k\Big) + \lambda_1\langle n_{k-1}\rangle - \lambda_1\langle n_k\rangle$$

for $k = 1, 2, \dots$ [6]

7.

Substituting $\bar{k} = \frac{1}{N}\sum_{i=0}^{\infty} i\langle n_i\rangle$ in the mean-field equations derived above gives:

$$\frac{d\langle n_0\rangle}{dt} = \alpha\langle n_0\rangle\bar{k} - \lambda_1\langle n_0\rangle \qquad \text{for } k = 0$$

$$\frac{d\langle n_k\rangle}{dt} = \alpha\langle n_k\rangle(\bar{k} - k) + \lambda_1\langle n_{k-1}\rangle - \lambda_1\langle n_k\rangle$$

for $k > 0$

[3]

8. Solve ODEs in 7) iteratively using initial condition:

Proceed by strong induction.

Base case $k=0$:

$$\frac{d\langle n_0\rangle}{dt} = (\alpha\bar{k} - \lambda_1)\langle n_0\rangle \quad \text{(by separation of variables)}$$

$$\int \frac{1}{\langle n_0\rangle} d\langle n_0\rangle = \int (\alpha\bar{k} - \lambda_1) dt \implies \langle n_0\rangle = Ae^{(\alpha\bar{k}-\lambda_1)t}$$

$$\langle n_0\rangle(0) = 0 \implies A = 0 \implies \langle n_0\rangle = 0 \quad \forall t.$$

Assume true for all $\langle n_i\rangle \leq i < k_0 - 1$

i.e. $\langle n_0\rangle = \langle n_1\rangle = \dots = \langle n_{i-1}\rangle = 0$, we also know $\langle n_i\rangle(0) = 0$

For $i+1$:

$$\frac{d\langle n_i\rangle}{dt} = (\alpha\bar{k} - \lambda_1)\langle n_i\rangle - \lambda_1 \underbrace{\langle n_{i-1}\rangle}_{=0 \text{ by assumption}}$$

$$\int \frac{1}{\langle n_i\rangle} d\langle n_i\rangle = \int (\alpha\bar{k} - \lambda_1) dt \implies \langle n_i\rangle = Ae^{(\alpha\bar{k}-\lambda_1)t}$$

But $\langle n_i\rangle(0) = 0 \implies A = 0 \implies \langle n_i\rangle = 0 \quad \forall t.$

So by strong induction, $\langle n_k\rangle = 0 \ \forall t$ for $k < k_0$

(Note when we reach k_0 there is no longer any initial condition, and so $\langle n_{k_0}\rangle$ and above cannot be found without unknown constants).

[4]

9. Solve iteratively $\frac{d\langle n_k \rangle}{dt} = 0$ starting at k_0

$$\frac{d\langle n_{k_0}\rangle}{dt} = (\alpha\bar{k} - \alpha k_0 - \lambda_1)\langle n_{k_0}\rangle - \lambda_1 \underbrace{\langle n_{k_0-1}\rangle}_{=0 \text{ by assumption.}}$$

$$0 = (\alpha\bar{k} - \alpha k_0 - \lambda_1)\langle n^{st}_{k_0}\rangle$$

assuming $\langle n^{st}_{k_0}\rangle \neq 0$ as $=0$ is not an interesting case (will see later this forces all other st. sts. $=0$ also).

$\Rightarrow 0 = \alpha\bar{k} - \alpha k_0 - \lambda_1 \Rightarrow \alpha\bar{k} - \lambda_1 = \alpha k_0$.

Prove by induction that: $\langle n_{k_0+i}\rangle = \left(\frac{\lambda_1}{\alpha}\right)^i \frac{1}{i!}\langle n_{k_0}\rangle$ for $i = 0, 1, \ldots$

Base case true by plugging in $i=0$.

Assume true for $\langle n_{k_0+k}\rangle$, for $\langle n_{k_0+k+1}\rangle$:

$$\frac{d\langle n_{k_0+k+1}\rangle}{dt} = \langle n_{k_0+k+1}\rangle(\alpha\bar{k} - \lambda_1 - \alpha(k_0+k+1)) + \lambda_1\langle n_{k_0+k}\rangle$$

$$0 = \langle n^{st}_{k_0+k+1}\rangle(\alpha k_0 - \alpha k_0 - \alpha k - \alpha) + \lambda_1\langle n_{k_0+k}\rangle$$

$$0 = -\langle n^{st}_{k_0+k+1}\rangle(\alpha(k+1)) + \lambda_1\langle n_{k_0+k}\rangle$$

so

$$\langle n^{st}_{k_0+k+1}\rangle = \frac{\lambda_1}{\alpha}\cdot\frac{1}{k+1}\langle n^{st}_{k_0+k}\rangle = \frac{\lambda_1}{\alpha}\cdot\left(\frac{\lambda_1}{\alpha}\right)^k\left(\frac{1}{k+1}\right)\left(\frac{1}{k!}\right)\langle n^{st}_{k_0+k}\rangle$$

$$= \left(\frac{\lambda_1}{\alpha}\right)^{k+1}\left(\frac{1}{(k+1)!}\right)\langle n^{st}_{k_0+k}\rangle$$

Now use normalisation condition to find $\langle n^{st}_{k_0}\rangle$:

$$N = \sum_{i=0}^{\infty}\langle n^{st}_{k_0+i}\rangle = \underbrace{\sum_{i=0}^{\infty}\left(\frac{\lambda_1}{\alpha}\right)^i\left(\frac{1}{i!}\right)}_{\text{series for exp}}\langle n^{st}_{k_0}\rangle = e\left(\frac{\lambda_1}{\alpha}\right)\langle n^{st}_{k_0}\rangle$$

$$\Rightarrow \langle n^{st}_{k_0}\rangle = N\exp\left(-\frac{\lambda_1}{\alpha}\right)$$

Hence, $\langle n^{st}_k\rangle = N\exp\left(-\frac{\lambda_1}{\alpha}\right)\left(\frac{\lambda_1}{\alpha}\right)^{k-k_0}\frac{1}{(k-k_0)!}$ $\square$

[8]

10.

Define $\hat{K} = \frac{1}{N}\sum_{i=0}^{K+1} i\langle n_i\rangle$

$$\frac{d\langle n_0\rangle}{dt} = \alpha\langle n_0\rangle\hat{k} - \lambda_1\langle n_0\rangle$$

$$\frac{d\langle n_j\rangle}{dt} = \alpha\langle n_j\rangle(\hat{k} - j) + \lambda_1\langle n_{j-1}\rangle - \lambda_1\langle n_j\rangle \qquad 0<j<K+1$$

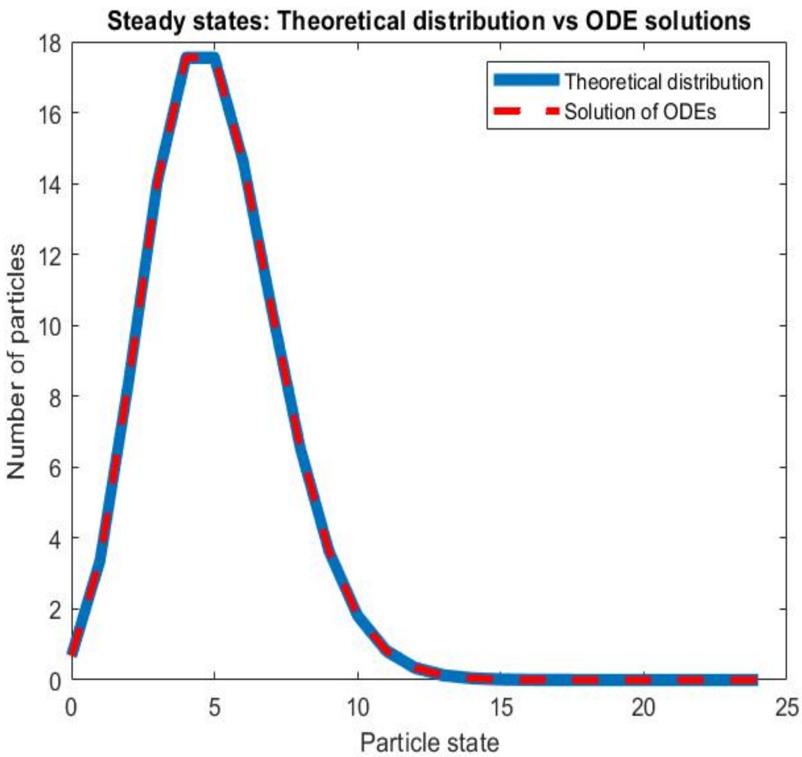
$$\frac{d\langle n_{K+1}\rangle}{dt} = \alpha\langle n_{K+1}\rangle(\hat{k} - (K+1)) + \lambda_1\langle n_K\rangle$$

[3]

11.

For example, if the first 4 reactions of the stochastic simulation were particles in box 0 jumping to box 1, then this is effectively the same as starting a different simulation with 0 particles in box 0 and 8 in box 1. Q9 would predict a different steady state in this case.

So we see the random nature of the stochastic system could move it away from the steady state as certain reactions could occur in sequence.



12.

$$\frac{d\langle n_0\rangle}{dt} = \alpha\langle n_0\rangle\hat{k} - \lambda_1\langle n_0\rangle + \lambda_2\langle n_1\rangle$$

$$\frac{d\langle n_j\rangle}{dt} = \alpha\langle n_j\rangle(\hat{k}-j) + \lambda_1\langle n_{j-1}\rangle - \lambda_1\langle n_j\rangle + \lambda_2\langle n_{j+1}\rangle - \lambda_2\langle n_j\rangle$$

for $j=1,\dots,K$

$$\frac{d\langle n_{K+1}\rangle}{dt} = \alpha\langle n_{K+1}\rangle(\hat{k}-(K+1)) + \lambda_1\langle n_K\rangle - \lambda_2\langle n_{K+1}\rangle$$

where $\hat{k} = \sum_{i=0}^{K+1} i\langle n_i\rangle$ as before.