MA30257/MA50257- Methods for stochastic systems coursework

Date Set: Friday 18th November, 2022.

Deadline for submission of answer document: Noon, Friday 16th December, 2022.

Marks: This coursework will account for 25% of the total marks of the unit for both MA30257 and MA50257. The marks available for each question are in square brackets at the end of each question except for the coding questions. A total of 40 marks are allocated across the three coding questions.

Feedback: A marked up copy of the completed coursework will be returned within three semester weeks of submission.

Time: It is intended that students spend around 16 hours on this coursework.

Objective: Answer the questions below including writing `MATLAB` code to reproduce the requested figures.

Submission: Please add your answers (including any figures) to the blank answer sheet provided: `Coursework_Answer_Sheet.pdf`. Scanned/electronic answers and code (3 `.m` files max submitted separately and not as a zip file) should be submitted via moodle by 12 noon on Friday 16th December 2022. Please title your answers document in the following format: `candidatenumber_2022` and ensure your candidate number is appended to the title of your code(s). For example if your candidate number is 123456, then your file should be titled 123456_2022.pdf.

Because people failing to name their files correctly is a massive time sink when marking, there will be an automatic 5 mark deduction for anyone failing to name their submissions according to this format. Similarly, failure to use the answer sheet provided will carry a penalty of 10 marks.

Please ensure your document is submitted as a PDF.

Consider N particles undergoing a series of first- and second- order interactions on a semi-infinite lattice. Subpopulations are characterised by their state, $i \in \mathbb{N}_0$, which we shall refer to as a particle's 'state'.

This can be thought of as a model for information spread between particles, where the state represent particles with different levels of information.

The governing dynamics of the particle interactions are copying (a pairwise interaction)

$$X_i + X_j \xrightarrow{f(i-j)/N} 2X_i \quad \text{for} \quad i = 0, 1, 2, \dots \quad \text{and} \quad j = 0, 1, 2, \dots .$$

Note that pairwise copying takes place at a rate which is a function of the difference between the two states, $f : \mathbb{Z} \to \mathbb{R}_0^+$. We will choose $f(0) = 0$ throughout.

Particles can also update their information spontaneously according to the following reactions.

$$X_i \underset{\lambda_2}{\overset{\lambda_1}{\rightleftarrows}} X_{i+1}, \quad \text{for} \quad i = 0, 1, 2, \dots .$$

Question 1 Explain the terms in the probability master equation for $p(n_0, n_1, \dots; t)$, the probability of there being n_0 particles in state 0, n_1 particles in state 1 and so on:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}p(n_0,n_1,\ldots;t) =& \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{f(i-j)}{N}(n_i-1)(n_j+1)p(n_0,n_1,\ldots,n_i-1,\ldots,n_j+1\ldots;t)\\ &+\sum_{i=0}^{\infty}(n_i+1)\lambda_1 p(n_0,n_1,\ldots,n_i+1,n_{i+1}-1,\ldots;t)\\ &+\sum_{i=1}^{\infty}(n_i+1)\lambda_2 p(n_0,n_1,\ldots,n_{i-1}-1,n_i+1,\ldots;t)\\ &-\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{f(i-j)}{N}n_i n_j+\lambda_1\sum_{i=0}^{\infty}n_i+\lambda_2\sum_{i=1}^{\infty}n_i\right)p(n_0,n_1,\ldots;t),\end{aligned}$$

being careful to give any conventions necessary to make the master equation valid.

[5]

Question 2 Write down ODEs for the evolution of the mean number of particles at each state, k, and then write down closed versions of the same equations using the mean-field moment closure. [5]

Question 3 Show that if the function f is even, then the mean-field model agrees with the mean of the stochastic model. [3]

Question 4 Adapt your equations from Question 2 to the situation in which there are a finite number of states, $K+1$.

[3]

Question 5 Consider the case in which all $N = 100$ particles start at information state 0 on a domain with 30 states (i.e. $0,\ldots,29$). Let $\lambda_1 = \lambda_2 = 1$ and let

$$f(z) = |z|.$$

Write codes which simulate the stochastic and mean-field versions of the model (from Question 4). Average the stochastic simulations over 1000 repeats and compare to the solutions of the ODEs of the mean-field model at time points, 10, 30 and 100.

Include plots of these three comparisons in your solutions.

Set $\lambda_2 = 0$ in what follows until told otherwise.

Question 6 Now consider the case for which

$$f(z) = -\alpha z H(-z),$$

where H is the Heaviside function

$$H(z) = \begin{cases}0 \text{ if } z \leq 0,\\ 1 \text{ if } z > 0,\end{cases}$$

and $\alpha \in \mathbb{R}^+$. Write down the resulting mean-field equations for the infinite-state version of this system. [6]

Question 7 Define $\bar{k} = \frac{1}{N}\sum_{i=0}^{\infty} i\langle n_i\rangle$ to be the average state of the population. Demonstrate that the mean-field equations are given by

$$\begin{aligned}\frac{\mathrm{d}\langle n_0\rangle}{\mathrm{d}t} &= \alpha\langle n_0\rangle\bar{k} - \lambda_1\langle n_0\rangle \quad \text{for} \quad k = 0,\\ \frac{\mathrm{d}\langle n_k\rangle}{\mathrm{d}t} &= \alpha\langle n_k\rangle(\bar{k}-k) + \lambda_1\langle n_{k-1}\rangle - \lambda_1\langle n_k\rangle, \quad \text{for} \quad k > 0.\end{aligned}$$

[3]

Question 8 Demonstrate that if $\langle n_k \rangle(0) = 0$ for $k < k_0$ then $n_k = 0$ $\forall t$ for $k < k_0$. [4]

Question 9 Let k_0 be the first state for which the initial condition is non-zero. Demonstrate that the steady state values of the mean-field model, $\langle n_k^{st} \rangle$, are given by the scaled Poisson distribution:

$$\langle n_k^{st} \rangle = N \exp\left(-\frac{\lambda_1}{\alpha}\right)\left(\frac{\lambda_1}{\alpha}\right)^{k-k_0} \frac{1}{(k-k_0)!}.$$

[8]

Question 10 Adapt your equations from Question 7 to the situation in which there are a finite number of states, $K+1$. [3]

Question 11 Solve numerically the system of ODEs in Question 10 to steady state. Use parameters $\lambda_1 = 5$ and $\alpha = 1$ with 25 states using the initial condition that $n_k(0) = 4$ for $k = 0, \ldots, 24$. Compare the steady state of your ODE system to the theoretical distribution given in Question 9 and include a plot of the steady state in your solutions.

Suggest why the stochastic version of the system may not converge to the same steady state.

Question 12 Write down the general form of closed mean-field ODEs for the system (equivalent to those in Question 10 but with $\lambda_2 \neq 0$). Solve these ODEs numerically using the initial condition $n_5(0) = 1000$ and parameters $\lambda_1 = 4$, $\lambda_2 = 1$ and $\alpha = 1$ with 60 states (note λ_2 is non-zero in this question).

Use the output of your ODE solutions at time 50 (suitably rounded to give integer values) as an initial condition for a stochastic simulation of the system with the same parameters.

Plot the output of your deterministic and stochastic (averaged over 100 repeats) systems at $t = 10$ and $t = 50$, and include the plots in your solution document.

Coding marks: A total of 40 marks are allocated across the three coding questions. These will be allocated holistically based on your performance on the coding tasks and my evaluation of your code.