數論—常用技巧

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1 Introduction

本文主要介紹估計，直接構造，數學歸納法在數論解題的應用

2 估計

2.1 簡介

主要就是利用次方級的不同，或者整除本身帶來的大小關係以及整數的離散性質來做估計並求出有限的答案

2.2 常用觀念

1.若 $a|b$ ，則 $|a| \leq |b|\ or\ b = 0$

2.若$a \leq b$ 則 $a < b + 1$

3.n到$n + 1$之間沒有任何整數

沒錯，掌握了這些看似平常的東西很多問題就能迎刃而解了

2.3 例題

Example 1.

解所有正整數對滿足 $ab^2+b+7|a^2b+a+b$

觀察：事實上，或許可以經過乘以某個係數，將整除左側的數字變成一個次方級小於右側（ab^2+b+7）

Solution 1.

由題，有 $ab^2+b+7|b(a^2b+a+b)-a(ab^2+b+7)$

$\Rightarrow ab^2+b+7|b^2-7a$

$\Rightarrow |b^2-7a| \geq |ab^2+b+7|\ or\ b^2-7a=0$

若 $b^2-7a=0$，$(a,b)=(7n^2,7n)$

若$|b^2-7a| \geq ab^2+b+7$，有 $b^2-7a>0$

則 $(b^2-7a)-(ab^2+b+7)\geq 0$，但 $a\in N(a>0)$ ，故不合理

剩下考慮若 $|b^2-7a|\leq ab^2+b+7$

$\Rightarrow a(7-b^2)-b(b+1)\geq 0$

$\Rightarrow b=1or2$

若 $b=1$, $a=11or49$

若 $b=2$, a 無解

綜上所述，有 $(a,b)=(7n^2,7n),(11,1),(49,1)$

Example 2.(APMO 2011 P1)

試證： $a^2+b+c, b^2+a+c, c^2+a+b$ 當 $a,b,c\in N$時不全為完全平方數

觀察：整數的平方差具有離散姓(就是中間會有一段距離)

Solution 2.

利用反證法

不失一般性，假設$a \geq b \geq c$，則令 $a^2+b+c=n^2 (n \in N)$

有$n^2 = a^2+b+c \geq (a+1)^2 \Rightarrow b+c \geq 2a+1$ 矛盾

由反證法得證

2.4 習題

Problem 1.(2020 Japan MO Final P1)

Find all pairs of positive integers (m, n) such that $\frac{n^2+1}{2m}$ and $\sqrt{2^{n-1}+m+4}$ are both integers.

Problem 2.(2019 Japan MO Final P1)

Find all triples $(a,\ b,\ c)$ of positive integers such that

$$a^2+b+3=(b^2-c^2)^2.$$

Problem 3.(2022 Canadian Mathematical Olympiad Qualification P1)

Determine all pairs of integers (m, n) such that m^2+n and n^2+m are both perfect squares.

Problem 4.(2017 Singapore MO Open P3)

Find the smallest positive integer n so that $\sqrt{\frac{1^2+2^2+...+n^2}{n}}$ is an integer.

Problem 5.(2019 ISL N2)

Find all triples (a, b, c) of positive integers such that $a^3+b^3+c^3=(abc)^2$.

Problem 6.(2022 APMO P1)

Find all pairs (a, b) of positive integers such that a^3 is multiple of b^2 and $b-1$ is multiple of $a-1$.

Problem 7.(2013 APMO P2)

Determine all positive integers n for which $n^2+1[\sqrt{n}]^2+2$ is an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

Problem 8.(2021 ISL N1)

Find all positive integers $n \geq 1$ such that there exists a pair (a, b) of positive integers, such that a^2+b+3 is not divisible by the cube of any prime, and

$$n = \frac{ab+3b+8}{a^2+b+3}.$$

Problem 9.(2015 ISL N2)

Let a and b be positive integers such that $a!+b!$ divides $a!b!$. Prove that $3a \geq 2b+2$.

Problem 10.(2002 APMO P2)

Find all positive integers a and b such that

$$\frac{a^2+b}{b^2-a} \quad \text{and} \quad \frac{b^2+a}{a^2-b}$$

are both integers.

Problem 11.(2012 Bulgaria MO P1)

The sequence $a_1, a_2, a_3 \ldots$, consisting of natural numbers, is defined by the rule:

$$a_{n+1} = a_n + 2t(n)$$

for every natural number n, where $t(n)$ is the number of the different divisors of n (including 1 and n). Is it possible that two consecutive members of the sequence are squares of natural numbers?

3 直接構造

3.1 簡介

利用題目給定的條件，將小答案構造出來後進行歸納直接舉例

Example 1.(2010 ISL N1)

找出最小的正整數 n 使的存在一個集合 $\{s_1, s_2, \ldots, s_n\}$ 包含 n 個不同的正整數使得

$$\left(1-\frac{1}{s_1}\right)\left(1-\frac{1}{s_2}\right)\cdots\left(1-\frac{1}{s_n}\right)=\frac{51}{2010}.$$

觀察：可以先估計n至少要是多少，最後再進行構造

Solution 1.

$$(1-\frac{1}{s_1})(1-\frac{1}{s_2})\cdots(1-\frac{1}{s_n})\le(1-\frac{1}{2})(1-\frac{1}{3})\cdots(1-\frac{1}{n+1})=\frac{1}{n}$$

, 所以$n\ge 39$. 現在

$$(1-\frac{1}{2})(1-\frac{1}{3})\cdots(1-\frac{1}{33})\times(1-\frac{1}{35})(1-\frac{1}{36})\cdots(1-\frac{1}{40})\times(1-\frac{1}{67})=\frac{1}{33}\times\frac{34}{40}\times\frac{66}{67}=\frac{51}{2010}$$

. 於是$n=39$.

Example 2.(2017 ISL N3)解出所有 $n \geq 2$ 滿足以下條件: 對任意整數 $a_1, a_2, \ldots, a_n$ 他們的和不是n的倍數，則存在$1 \leq i \leq n$使得下列數組

$$a_i, a_i + a_{i+1}, \ldots, a_i + a_{i+1} + \ldots + a_{i+n-1}$$

接不被n整除. 在這裡我們令 $a_i = a_{i-n}$ 當 $i > n$.

觀察：可以先找出當n非質數時的構造反例，再考慮n是質數時的證明

Solution 2.

答案：只有 n 是質數時. 若 n 是合數，令$p \mid n$ 是質數,

構造 $(0, n/p, n/p, n/p, \ldots, n/p)$，即得到矛盾。

利用反證法，令 $n = p$. 假設存在 k 使 $a_k + a_{k+1} + \ldots + a_{\ell(k)} \equiv 0 \pmod p$ 對於某些 $\ell(k)$ 取決於 k. 考慮有向圖

$$k \rightarrow \ell(k) + 1$$

則可以找到一個循環. 於是 $0 \equiv (a_1 + \ldots + a_{\ell_1}) + (a_{\ell_1+1} + \ldots + a_{\ell_2}) + \ldots + (a_{\ell_r+1} + \ldots + a_p)$

$= c\,(a_1 + \ldots + a_p)$這裡 c 是k到達p的距離，於是$c < p$. 所以 $a_1 + \ldots + a_p \equiv 0 \pmod p$，矛盾

3.2 習題

Problem 1.(2016 IMO P4)

A set of positive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible positive integer value of b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

is fragrant?

Problem 2.(2021 ISL N2)

Let $n100$ be an integer. Ivan writes the numbers $n, n+1, \ldots, 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square

Problem 3.(2020 ISL N2)

For each prime p, construct a graph G_p on $\{1, 2, \ldots p\}$, where $m \neq n$ are adjacent if and only if p divides $(m^2 + 1 - n)(n^2 + 1 - m)$. Prove that G_p is disconnected for infinitely many p

Problem 4.(2014 ISL N1)

Let $n \geq 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in Z,\ 0 \leq k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n .

4 數學歸納法

4.1 介紹

數學歸納法使用時，一定要記得在一開始宣告使用數學歸納法，做出base case以及推導過程，最後再宣告由數學歸納法成立

4.2 例題

Example 1.(2017 ISL N1)

對於所有正整數 $a_0 > 1$, 定義數列 $a_0, a_1, a_2, \ldots$ 對於所有 $n \geq 0$ 有如下規則

$a_{n+1} = \sqrt{a_n}$ 如果 $\sqrt{a_n}$是整數

$a_{n+1} = a_n + 3$ 在其他情況下

找出所有 a_0 的值使得存在 A 滿足 $a_n = A$ 有無限多項 n.

觀察：什麼東西會回到已經出現過的情況

Solution 1.

首先，若 $a_0 \equiv 2(mod3)$，則因 $x^2 \equiv 0 or 1(mod3)$有數列發散 再來，若 $a_0 \equiv 1(mod3)$，利用數學歸納法，當$a_0 = 1$，有$a_1 = 4$, $a_2 = 2$，又當$a_k \equiv 2(mod3)$，數列開始發散，故$a_0 = 1$不滿足題意

令$a_0 = 1$ $3k+1$時只要非3的倍數皆不滿足，則當$a_0 = 3k+4$，往下推導至a_i是完全平方數時，有$a_{i+1} \leq 3k+1$且$a_{i+1} \equiv 1 or 2(mod3)$，於是由歸納假設即得數列發散

最後，若a_0是三的倍數，$a_2 = 9, a_3 = 3$滿足了一個循環

利用數學歸納法，假設對於$1 \leq i \leq k$，$a_0 = 3i$ 皆滿足，往下推導至a_i是完全

平方數時，有$a_{i+1} \leq 3k$且$a_{i+1} \equiv 0(mod 3)$，於是由歸納假設即得數列呈現循環，循環內所有數字皆出現無限多次

4.3 習題

Problem 1.(2013 ISL N1)

Let $Z_{>0}$ be the set of positive integers. Find all functions $f : Z_{>0} \to Z_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

Problem 2.(2019 ISL N3)

We say that a set S of integers is rootiful if, for any positive integer n and any $a_0, a_1, \cdots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1x + \cdots + a_nx^n$ are also in S. Find all rootiful sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers a and b.

Problem 3.(2018 ISL N3)

Define the sequence $a_0, a_1, a_2,$ by $a_n = 2^n + 2^{\lfloor n/2 \rfloor}$. Prove that there are infinitely many terms of the sequence which can be expressed as a sum of (two or more) distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.