

九點圓深究

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1 Introduction

本文主要介紹與九點圓所相關的位似、垂足圓、等角共軛以及九點圓例題

2 位似

2.1 定義和性質

Definition 1. 若有三點 O, P_1, P_2 皆在 l 上，則稱 P_1 以 O 為位似中心 r 倍位似於 P_2 ，其中 $r = \frac{OP_1}{OP_2}$

Property 2. 若 $(A_1, A_2), (B_1, B_2), (C_1, C_2)$ 皆對某點 O 有位似，則我們可以稱 $\triangle A_1 B_1 C_1, \triangle A_2 B_2 C_2$ 位似於 O ，而且 $\odot(A_1 B_1 C_1), \odot(A_2 B_2 C_2)$ 亦對 O 有位似關係

看完兩個基本定義，就先前進到今天的主題吧！

2.2 例題

Exaple 3.(九點圓) 證明三角形三邊中點、三邊垂足、三個頂點到垂心的中點共圓，而且圓心為垂心和外心中點

hint: 可以將垂心對中點，垂足依次反射找位似關係

Example 4.(USAMO 1993/2) Let $ABCD$ be a convex quadrilateral such that diagonals AC and BD intersect at right angles, and let E be their intersection. Prove that the reflections of E across AB, BC, CD, DA are concyclic.

hint:角度計算可以透過相似得到位似

Example 5.(小Lemma) 旁切圓圓心，高的中點，內切圓切於邊的點三點共線

hint:透過內切圓和旁切圓之間的位似關係慢慢把它化解

2.3 習題

習題 1.(Korea National Olympiad 2009 P1) Let I, O be the incenter and the circumcenter of triangle ABC , and D, E, F be the circumcenters of triangle BIC, CIA, AIB . Let P, Q, R be the midpoints of segments DI, EI, FI . Prove that the circumcenter of triangle PQR , M , is the midpoint of segment IO .

習題 2.(Unknow source) 一銳角 $\triangle ABC$ ，垂心為 H ，求證： $\triangle ABH, \triangle BCH, \triangle ACH$ 外心連成之三角形全等於 $\triangle ABC$

習題 3.(Brazil Mathematical Olympiad 2012 Day1 P2) ABC is a non-isosceles triangle.

T_A is the tangency point of incircle of ABC in the side BC (define T_B, T_C analogously).

I_A is the ex-center relative to the side BC (define I_B, I_C analogously).

X_A is the mid-point of $I_B I_C$ (define X_B, X_C analogously).

Show that $X_A T_A, X_B T_B, X_C T_C$ meet in a common point, colinear with the in-

center and circumcenter of ABC .

習題 4.(USAMO 2015 P2) Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment \overline{PQ} . Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that \overline{XT} is perpendicular to \overline{AX} . Let M denote the midpoint of chord \overline{ST} . As X varies on segment \overline{PQ} , show that M moves along a circle.

3 九點圓例題

3.1 例題們

Example1. 設 H 為 $\triangle ABC$ 之垂心， L 為 BC 中點， P 為 AH 中點，過 L 做 PL 垂線交 AB 於 G ，交 AC 延長線於 K

求證： G, B, K, C 四點共圓

hint:透過 P 為中點找出平行性質進行角度計算

Example2. (2018HKTST1 P6) A triangle ABC has its orthocentre H distinct from its vertices and from the circumcenter O of $\triangle ABC$. Denote by M, N and P respectively the circumcenters of triangles HBC, HCA and HAB . Show that the lines AM, BN, CP and OH are concurrent.

hint:有很多九點圓共圓心

3.2 習題們

習題 1.(裸性質題) 試證： $\triangle ABC$ 垂心與外接圓上的點的連線被其九點圓平分

習題 2.(中國全國高中聯賽題) $\triangle ABC$ 中， O 為外心，有三條高 AD, BE, CF 交於 H ，直線 ED, AB 交於 M ， FD, AC 交於 N

試證：(1) $OB \perp DF$ ， $OC \perp DE$ (2) $OH \perp MN$

習題 3.(IMO 1989 P2) ABC is a triangle, the bisector of angle A meets the circumcircle of triangle ABC in A_1 , points B_1 and C_1 are defined similarly. Let AA_1 meet the lines that bisect the two external angles at B and C in A_0 . Define B_0 and C_0 similarly. Prove that

(i) the area of triangle $A_0B_0C_0 = 2 \cdot$ area of hexagon

(ii) $AC_1BA_1CB_1 \geq 4 \cdot$ area of triangle ABC .

習題 4.(ISL 1997 P18) The altitudes through the vertices A, B, C of an acute-angled triangle ABC meet the opposite sides at D, E, F , respectively. The line through D parallel to EF meets the lines AC and AB at Q and R , respectively. The line EF meets BC at P . Prove that the circumcircle of the triangle PQR passes through the midpoint of BC .

4 等角共軛與垂足圓

4.1 定義和性質

Definition 1. 若兩線 l_1, l_2 滿足對另外兩線 m, n 有 $\angle(l_1, m) = \angle(l_2, n)$ ，則稱 l_1, l_2 為對 m, n 之等角共軛線組(也可以反過來說)

Definition 2.(等角共軛點) 對 $\triangle ABC$ 內一點 P ，若有一點 P^* 滿足對於任意兩對邊都有他們的頂點連 P, P^* 為等角線組，則 P, P^* 為一組等角共軛點(可以輕易由角原西瓦定理確認它的存在性)

Lemma 3.(Isogonal Lines Lemma)

Lemma :

Let AP, AS and AQ, AR be two pairs of isogonal lines with respect to $\angle BAC$.

Let $PR \cap QS = X$ and $PQ \cap RS = Y$.

Then AX, AY are isogonal line with respect to $\angle BAC$

Proof :

Without loss of generality let AY intersect PR, QS in K, L respectively. Considering the perspectivity that sends line $PR \rightarrow QS$ (Y is the centre of perspectivity).

$$\frac{PK \cdot XR}{XK \cdot PR} = \frac{QL \cdot SX}{XL \cdot SQ}$$

Now using the property of cross ratio w.r.t point A .

Let $\angle PAQ = \angle RAS = x, \angle QAL = y, \angle XAR = z$

Using the cross ratios,

$$\sin(x + y) \cdot \sin z = \sin y \cdot \sin(x + z)$$

$$\sin y = \sin z$$

$$y = z \text{ (configuration matters here)}$$

$$\angle QAY = \angle RAX$$

our lemma.

p.s.因為之前打過英文版本就先直接放了

Lemma 4.(佩多圓)給定 $\triangle ABC$ 內兩共軛點 P, P^* ，有他們倆個分別打到三邊的垂足六點共圓，且圓心為 PP^* 中點

hint:可以先想想看怎麼透過定義找四個點共圓，再把圓心標出來

Observation:其實外心垂心就是一組等角共軛點，所以九點圓就是他們的佩多圓

4.2 例題們

Example 1.(Tournament of Towns Spring Senior A 2006) In triangle ABC , let AA' be the bisector, and let X be any point on AA' and let BX, CX meet AC, AB again at B', C' . Let BX meet $A'C'$ at P and CX meet $A'B'$ at Q . Prove that $\angle PAC = \angle QAB$.

hint:Trivial by Lemma

Example 2.(Iran TST 2015) AH is the altitude of triangle ABC and H' is the reflection of H through the midpoint of BC . If the tangent lines to the circumcircle of ABC at B and C , intersect each other at X and the perpendicular line to XH' at H' , intersects AB and AC at Y and Z respectively, prove that $\angle ZXC = \angle YXB$.

hint:試著將 X 打攝影到 AB 上為 P

Example 2.(Iran TST 2015) AH is the altitude of triangle ABC and H' is the reflection of H through the midpoint of BC . If the tangent lines to the circumcircle of ABC at B and C , intersect each other at X and the perpendicular line to XH' at H' , intersects AB and AC at Y and Z respectively, prove that $\angle ZXC = \angle YXB$.

hint:試著將 X 打攝影到 AB 上為 P

Example 3.(常見性質) P 為 $\triangle ABC$ 內一點，則 P 的垂足三角形與三頂點打到 $\odot ABC$ 三點形成之三角形相似(稱這個三角形為圓西瓦三角形)

Observation:垂心其實就是這樣的特例(前面關於九點圓的討論)

4.3 習題

習題 1. (China TST 2002) Let E and F be the intersections of opposite sides of a convex quadrilateral $ABCD$. The two diagonals meet at P . Let O be the foot of the perpendicular from O to EF . Show that $\angle BOC = \angle AOD$.

習題 2. (2015 APMOC P5) 三角形 ABC 中，點 L, M, N 分別在 BC, AC, AB 邊上，已知 $\triangle ANM, \triangle BLN, \triangle CML$ 皆為銳角三角形，他們垂心分別為 H_A, H_B, H_C ，設 AH_A, BH_B, CH_C 三線共點。證明： LH_A, MH_B, NH_C 三線亦共點

習題 3. (India Postals 2015 Set 2) Let $ABCD$ be a convex quadrilateral. In the triangle ABC let I and J be the incenter and the excenter opposite the vertex A , respectively. In the triangle ACD let K and L be the incenter and the excenter opposite the vertex A , respectively. Show that the lines IL and JK , and the bisector of the angle BCD are concurrent.

習題 4. (2021 Taiwan TST 3J Mock6) Let $ABCD$ be a rhombus with center O . P is a point lying on the side AB . Let I, J , and L be the incenters of triangles PCD, PAD , and PBC , respectively. Let H and K be orthocenters of triangles PLB and PJA , respectively.

Prove that $OI \perp HK$.

5 後記

鯨魚覺得累了晚安，之後會慢慢補充我的講義的(?)