九點圓深究

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1 Introduction

本文主要介紹與九點圓所相關的位似、垂足圓、等角共軛以及九點圓例題

2 位似

2.1 定義和性質

Definition 1. 若有三點 O, P_1, P_2皆在 l上，則稱P_1以O為位似中心r倍位似於P_2，其中$r = \frac{OP_1}{OP_2}$

Property 2. 若$(A_1, A_2), (B_1, B_2), (C_1, C_2)$皆對某點$O$有位似，則我們可以稱$\Delta A_1B_1C_1, \Delta A_2B_2C_2$位似於$O$，而且 $\odot(A_1B_1C_1), \odot(A_2B_2C_2)$亦對$O$有位似關係

看完兩個基本定義，就先前進到今天的主題吧！

2.2 例題

Exaple 3.(九點圓) 證明三角形三邊中點、三邊垂足、三個頂點到垂心的中點共圓，而且圓心為垂心和外心中點

hint:可以將垂心對中點，垂足依次反射找位似關係

Example 4.(USAMO 1993/2) Let $ABCD$ be a convex quadrilateral such that diagonals AC and BD intersect at right angles, and let E be their intersection. Prove that the reflections of E across AB, BC, CD, DA are concyclic.

hint:角度計算可以透過相似得到位似

Example 5.(小Lemma) 旁切圓圓心，高的中點，內切圓切於邊的點三點共線

hint:透過內切圓和旁切圓之間的位似關係慢慢把它化解

2.3 習題

習題 1.(Korea National Olympiad 2009 P1) Let I, O be the incenter and the circumcenter of triangle ABC, and D, E, F be the circumcenters of triangle BIC, CIA, AIB. Let P, Q, R be the midpoints of segments DI, EI, FI. Prove that the circumcenter of triangle PQR, M, is the midpoint of segment IO.

習題 2.(Unknow source) 一銳角ΔABC，垂心為H，求證：$\Delta ABH, \Delta BCH$, ΔACH 外心連成之三角形全等於ΔABC

習題 3.(Brazil Mathematical Olympiad 2012 Day1 P2) ABC is a non-isosceles triangle.

T_A is the tangency point of incircle of ABC in the side BC (define T_B,T_C analogously).

I_A is the ex-center relative to the side BC (define I_B,I_C analogously).

X_A is the mid-point of I_BI_C (define X_B,X_C analogously).

Show that X_AT_A,X_BT_B,X_CT_C meet in a common point, colinear with the incenter and circumcenter of ABC.

習題 4.(USAMO 2015 P2) Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle.

3 九點圓例題

3.1 例題們

Example1. 設H為ΔABC之垂心，L為BC中點，P為AH中點，過L做PL垂線交AB於G，交 AC延長線於K

求證：G, B, K, C四點共圓

hint:透過P為中點找出平行性質進行角度計算

Example2. (2018HKTST1 P6)A triangle ABC has its orthocentre H distinct from its vertices and from the circumcenter O of $\triangle ABC$. Denote by M, N and P respectively the circumcenters of triangles HBC, HCA and HAB. Show that the lines AM, BN, CP and OH are concurrent.

hint:有很多九點圓共圓心

3.2 習題們

習題 1.(裸性質題) 試證：ΔABC垂心與外接圓上的點的連線被其九點圓平分

習題 2.(中國全國高中聯賽題) ΔABC中，O為外心，有三條高AD, BE, CF交於H，直線ED, AB交於M，FD, AC交於N

試證：(1)$OB \perp DF$，$OC \perp DE$(2)$OH \perp MN$

習題 3.(IMO 1989 P2)ABC is a triangle, the bisector of angle A meets the circumcircle of triangle ABC in A_1, points B_1 and C_1 are defined similarly. Let AA_1 meet the lines that bisect the two external angles at B and C in A_0. Define B_0 and C_0 similarly. Prove that

(i)the area of triangle $A_0B_0C_0 = 2\cdot$ area of hexagon

(ii)$AC_1BA_1CB_1 \geq 4\cdot$ area of triangle ABC.

習題 4.(ISL 1997 P18) The altitudes through the vertices A, B, C of an acute-angled triangle ABC meet the opposite sides at D, E, F, respectively. The line through D parallel to EF meets the lines AC and AB at Q and R, respectively. The line EF meets BC at P. Prove that the circumcircle of the triangle PQR passes through the midpoint of BC.

4 等角共軛與垂足圓

4.1 定義和性質

Definition 1. 若兩線l_1, l_2滿足對另外兩線m, k有$\angle(l_1, m) = \angle(l_2, n)$，則稱$l_1, l_2$為對$m, n$之等角共軛線組(也可以反過來說)

Definition 2.(等角共軛點) 對ΔABC內一點P，若有一點P^*滿足對於任意兩對邊都有他們的頂點連P, P^*為等角線組，則$P, P*$為一組等角共軛點(可以輕易由角原西瓦定理確認它的存在性)

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Lemma 3.(Isogonal Lines Lemma)

Lemma :

Let AP, AS and AQ, AR be two pairs of isogonal lines with respect to $\angle BAC$.

Let $PR \cap QS = X$ and $PQ \cap RS = Y$.

Then AX, AY are isogonal line with respect to $\angle BAC$

Proof：

Without loss of generality let AY intersect PR, QS in K, L respectively. Considering the perspectivity that sends line $PR \rightarrow QS$ (Y is the centre of perspectivity).

$\frac{PK \cdot XR}{XK \cdot PR} = \frac{QL \cdot SX}{XL \cdot SQ}$

Now using the property of cross ratio w.r.t point A.

Let $\angle PAQ = \angle RAS = x, \angle QAL = y, \angle XAR = z$

Using the cross ratios,

$sin(x+y) \cdot sinz = siny \cdot sin(x+z)$

$siny = sinz$

$y = z$ (configuration matters here)

$\angle QAY = \angle RAX$

our lemma.

p.s.因為之前打過英文版本就先直接放了

Lemma 4.(佩多圓)給定ΔABC內兩共軛點P, P^*，有他們倆個分別打到三邊的垂足六點共圓，且圓心為PP^*中點

hint:可以先想想看怎麼透過定義找四個點共圓，再把圓心標出來

Observation:其實外心垂心就是一組等角共軛點，所以九點圓就是他們的佩多圓

4.2 例題們

Example 1.(Tournament of Towns Spring Senior A 2006) In triangle ABC, let AA' be the bisector, and let X be any point on AA' and let BX, CX meet AC, AB again at B', C'. Let BX meet $A'C'$ at P and CX meet $A'B'$ at Q. Prove that $\angle PAC = \angle QAB$.

hint:Trivial by Lemma

Example 2.(Iran TST 2015)AH is the altitude of triangle ABC and H' is the reflection of H trough the midpoint of BC. If the tangent lines to the circumcircle of ABC at B and C, intersect each other at X and the perpendicular line to XH' at H', intersects AB and AC at Y and Z respectively, prove that $\angle ZXC = \angle YXB$.

hint:試著將X打攝影到AB上為P

Example 2.(Iran TST 2015)AH is the altitude of triangle ABC and H' is the reflection of H trough the midpoint of BC. If the tangent lines to the circumcircle of ABC at B and C, intersect each other at X and the perpendicular line to XH' at H', intersects AB and AC at Y and Z respectively, prove that $\angle ZXC = \angle YXB$.

hint:試著將X打攝影到AB上為P

Example 3.(常見性質)P為ΔABC內一點，則P的垂足三角形與三頂點打到$\odot ABC$三點形成之三角形相似(稱這個三角形為圓西瓦三角形)

Observation:垂心其實就是這樣的特例(前面關於九點圓的討論)

4.3 習題

習題 1.(China TST 2002) Let E and F be the intersections of opposite sides of a convex quadrilateral $ABCD$. The two diagonals meet at P. Let O be the foot of the perpendicular from O to EF. Show that $\angle BOC = \angle AOD$.

習題 2.(2015 APMOC P5)三角形ABC中，點L, M, N分別在BC, AC, AB邊上，已知$\Delta ANM, \Delta BLN, \Delta CML$皆為銳角三角形，他們垂心分別為$H_A, H_B, H_C$，設$AH_A, BH_B, CH_C$三線共點。證明：$LH_A, MH_B, NH_C$三線亦共點

習題 3.(India Postals 2015 Set 2) Let $ABCD$ be a convex quadrilateral. In the triangle ABC let I and J be the incenter and the excenter opposite the vertex A, respectively. In the triangle ACD let K and L be the incenter and the excenter opposite the vertex A, respectively. Show that the lines IL and JK, and the bisector of the angle BCD are concurrent.

習題 4.(2021 Taiwan TST 3J Mock6) Let $ABCD$ be a rhombus with center O. P is a point lying on the side AB. Let I, J, and L be the incenters of triangles PCD, PAD, and PBC, respectively. Let H and K be orthocenters of triangles PLB and PJA, respectively.
Prove that $OI \perp HK$.

5 後記

鯨魚覺得累了晚安，之後會慢慢補充我的講義的(?)