

# Signals and Systems (Lab)

## Lab 4: The Continuous-Time Fourier Transform

Dr. **Wu Guang**

**[wug@sustech.edu.cn](mailto:wug@sustech.edu.cn)**

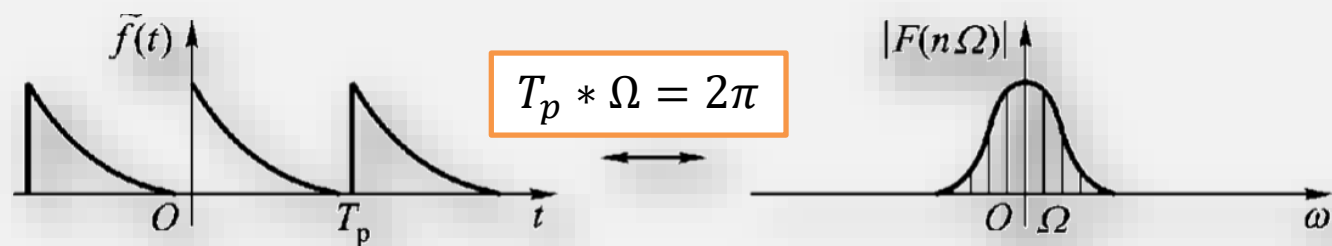
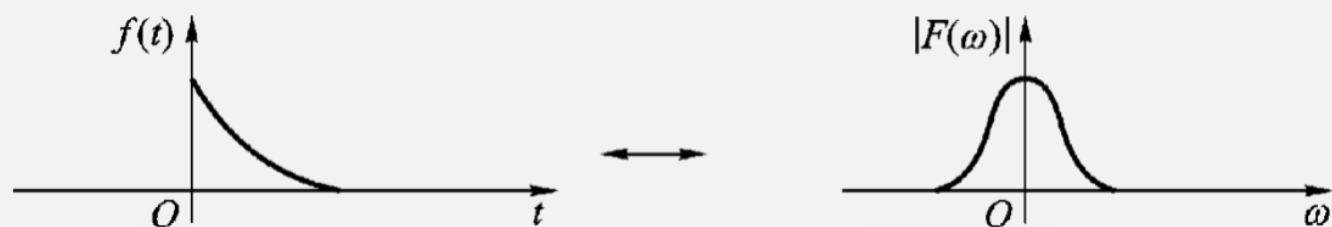
**Electrical & Electronic Engineering**

**Southern University of Science and Technology**

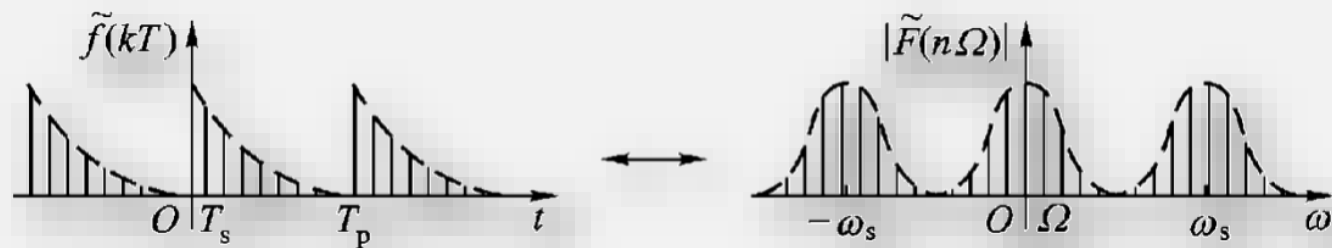
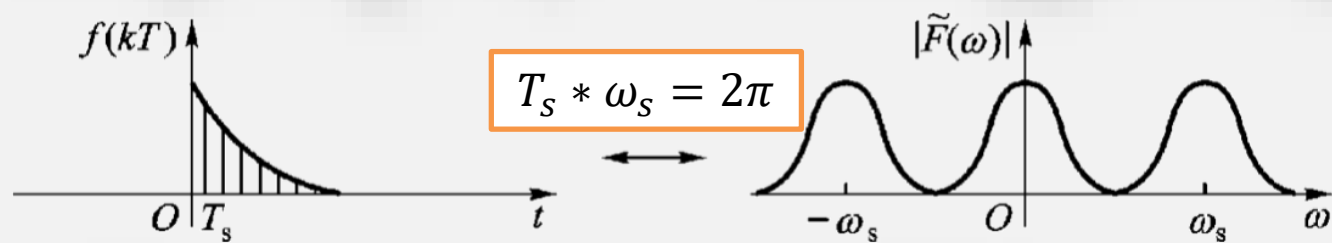
# Review

- ✓ 1. How to calculate the output of DT LTI system in **frequency domain**
- ✓ 2. How to calculate the output of CT LTI system
- ✓ 3. How to calculate the ***DTFS*** of signal via Matlab

FT

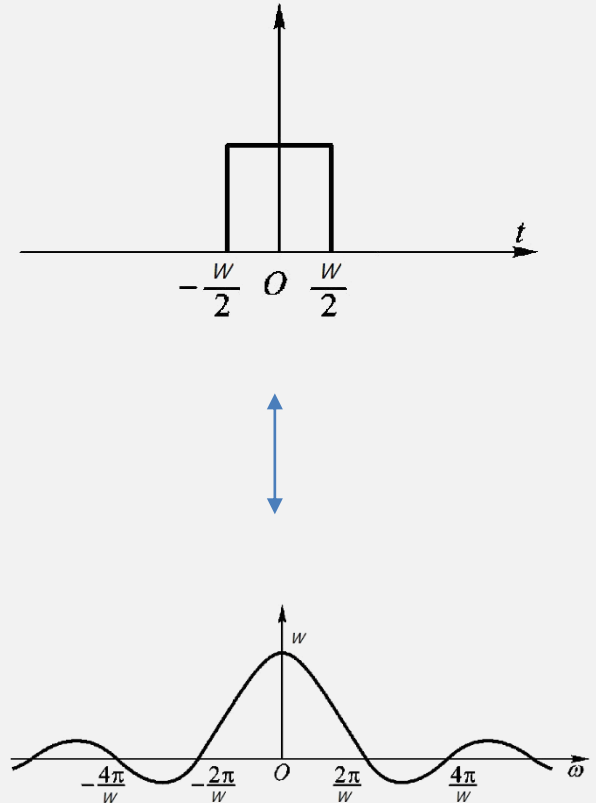


DFT



# Overview

- **CT Fourier Transform (CTFT):**
  - How to calculate via Matlab?
- **Frequency Response:**
  - How to calculate via Matlab ?
  - How to convert to impulse response
- **Application of CTFT:**
  - Analysis of amplitude modulation (AM)

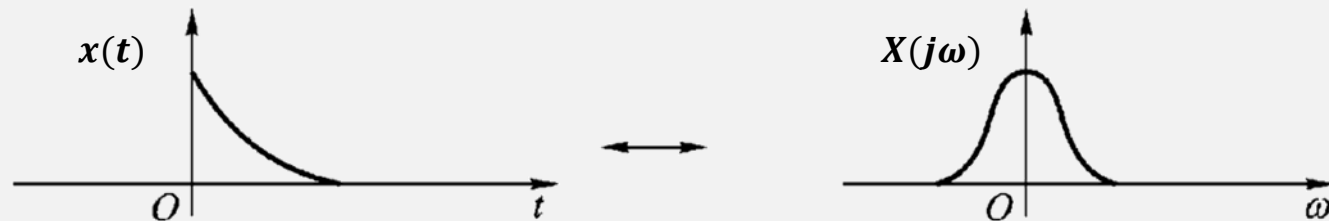


# CT Fourier Transform

- Definition of CT Fourier transform:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

- Function of **continuous** frequency  $\omega$
- Represent the “**spectrum**” of a signal



# Use `fft()` to calculate the CTFT

CT Fourier Transform:  $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$



DT Fourier Series:  $a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n}$

**Let  $t = n\tau$**

**CT Fourier Transform:**

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$
$$= \lim_{\tau \rightarrow 0} \tau \sum_{n=-\infty}^{\infty} x(n\tau) e^{-j\omega \tau n}$$



**DT Fourier Series:**

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

- Suppose the dominant region of  $x(t)$  is in  $[0,T]$ , then we can use the following approximation

$$\text{➤ } X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \approx \int_0^T x(t)e^{-j\omega t} dt \approx \sum_{n=0}^{N-1} x(n\tau)e^{-j\omega n\tau}\tau$$

- It could be observed that
  - Slice into  $N$  intervals, each with length  $\tau=T/N$

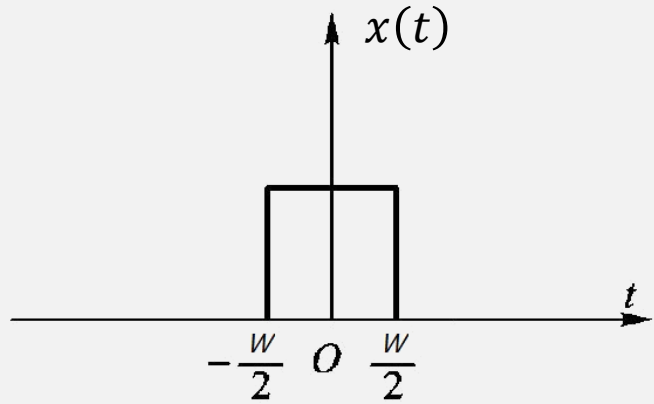
$$\text{➤ } X(j\frac{2\pi k}{N\tau}) \approx \tau \sum_{n=0}^{N-1} x(n\tau)e^{-j\frac{2\pi kn}{N}} \quad \text{➤ } k = \frac{1-N}{2}, \dots, -1, 0, 1, \dots, \frac{N-1}{2}$$

- $\text{fft}()$  of  $[x(0), x(\tau), x(2\tau), \dots, x((N-1)\tau)]$

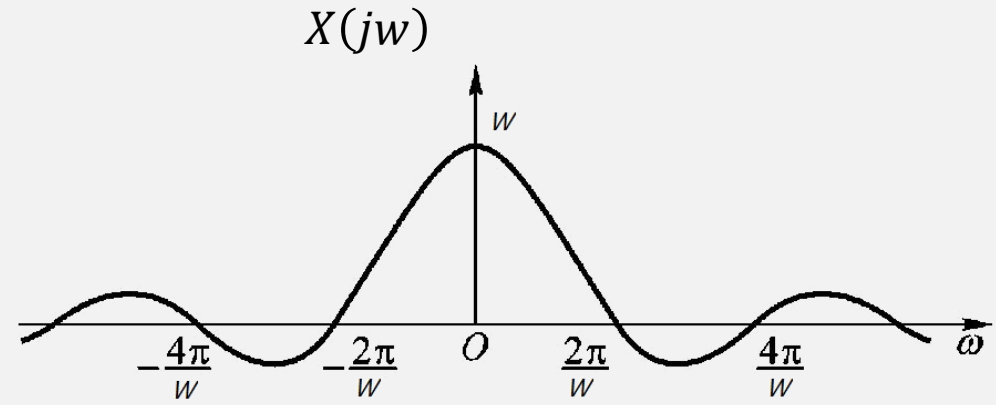
- **Conclusion:** we can use  $\text{fft}()$  to calculate CTFT approximately, which could reduce the computation complexity



# Example-sinc function



$$x(t) = \begin{cases} 1, & |t| < \frac{W}{2} \\ 0, & |t| > \frac{W}{2} \end{cases}$$



$$X(j\omega) = \left[ \frac{\sin\left(\frac{\omega W}{2}\right)}{\frac{\omega W}{2}} \right] = W \text{Sinc}\left(\frac{\omega W}{2}\right)$$

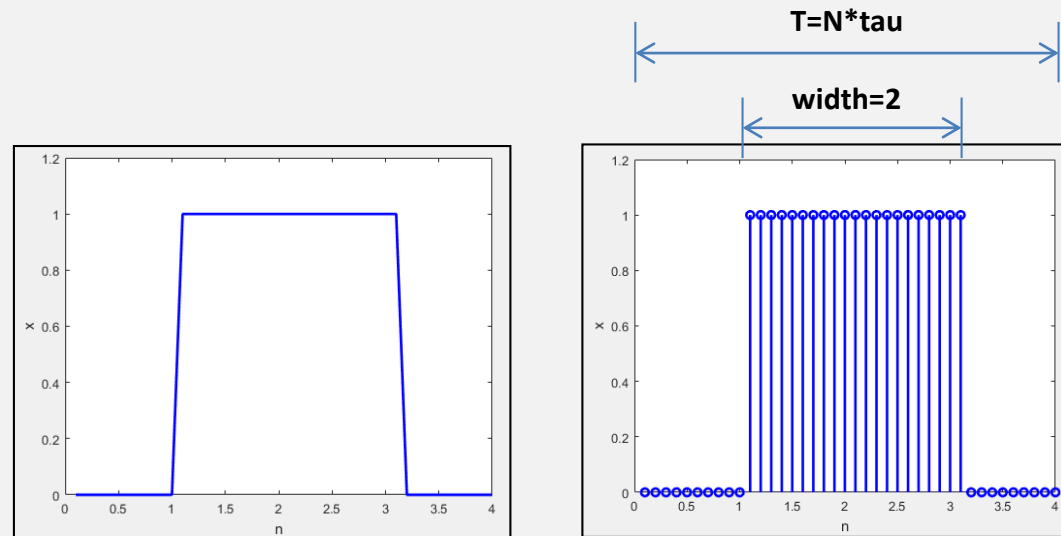
```

tau=0.5;
N=201; % N should be odd number,
N*tau=T>3
x=[zeros(1,1/tau),ones(1,2/tau+1),zeros(
1,N-3/tau-1)];
figure(1)
plot([1:N]*tau,x,'r-','LineWidth',2)
xlabel('n');ylabel('x');
axis([0 4 0 1.2])
legend('\tau=0.5,N=201')

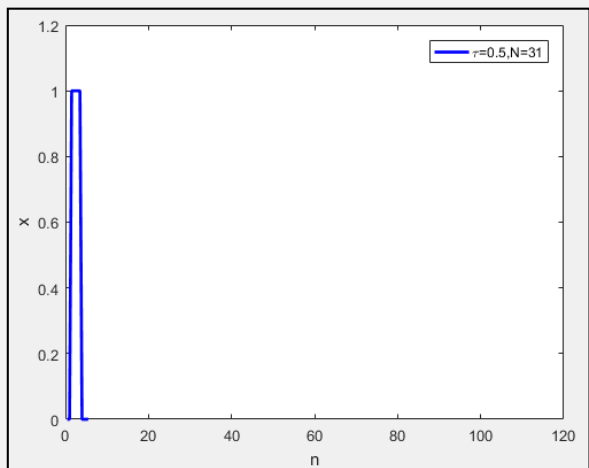
y=fftshift(fft(x));
X=tau*y;
lb = (1-N)*pi/N/tau;
ub = (N-1)*pi/N/tau;
step = 2*pi/N/tau;
figure(2)
plot(lb:step:ub, abs(X),'r-','LineWidth',2);
xlabel('\omega');ylabel('abs(X)');
legend('\tau=0.5,N=201')

```

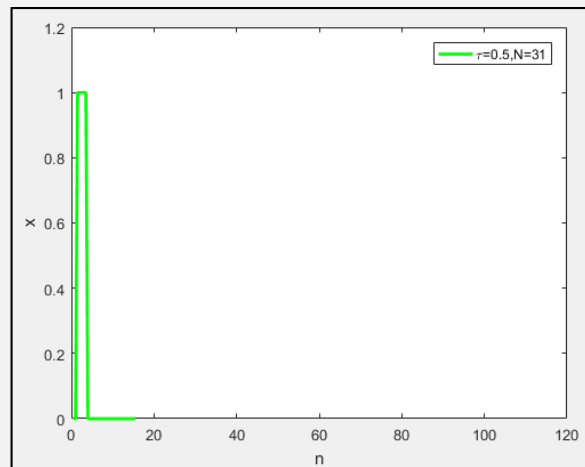
- Single rectangular wave with width=2
- T should be larger than 3



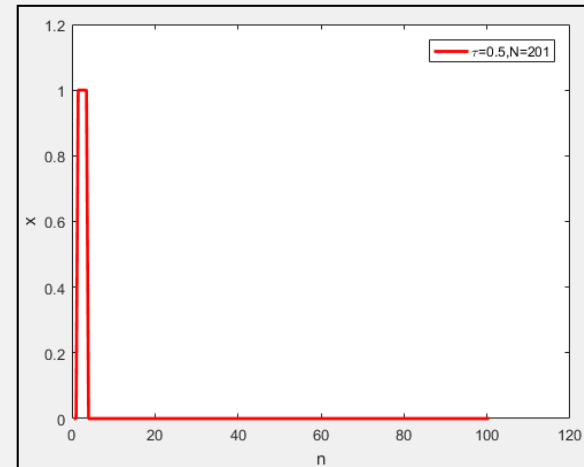
**What happen if we change tau and N? Why?**



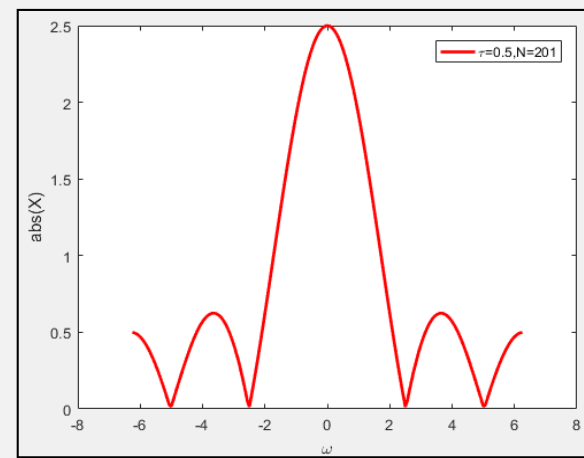
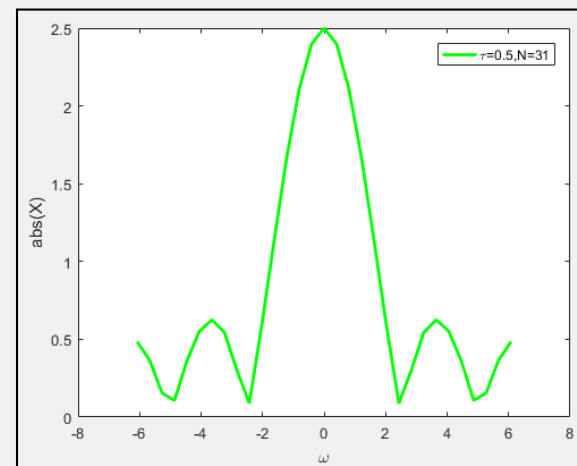
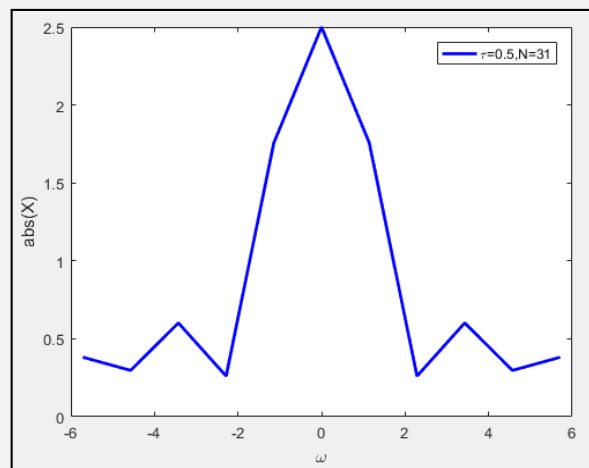
N=11

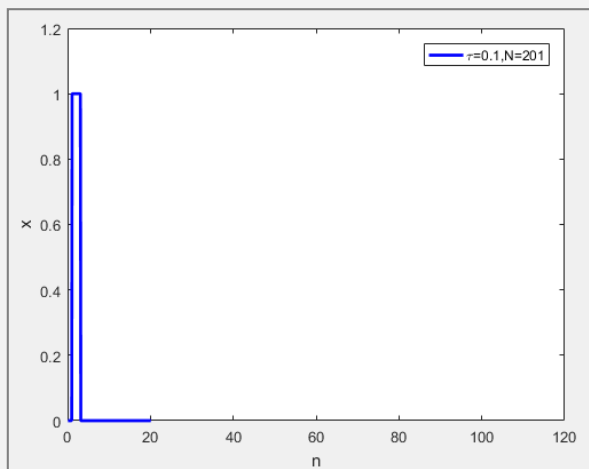


N=31

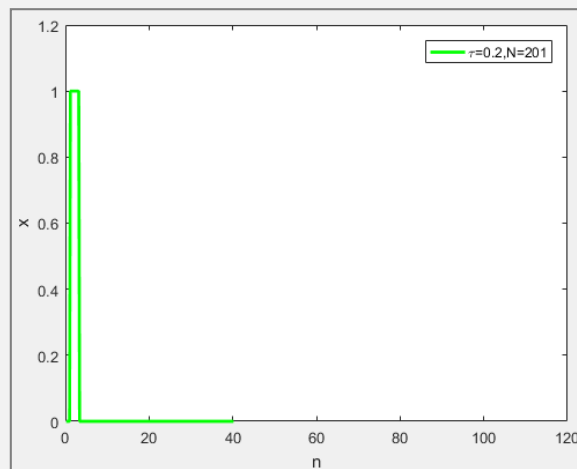


N=201

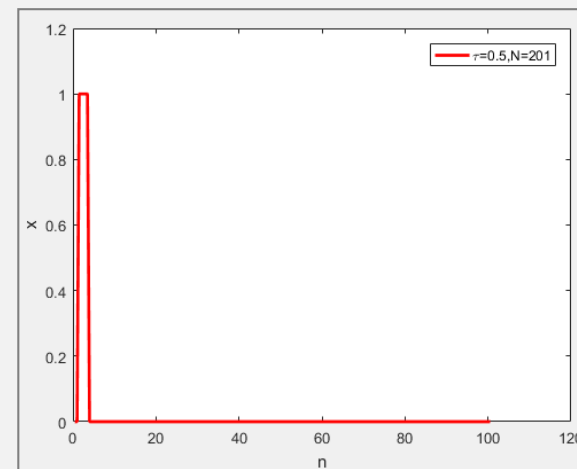




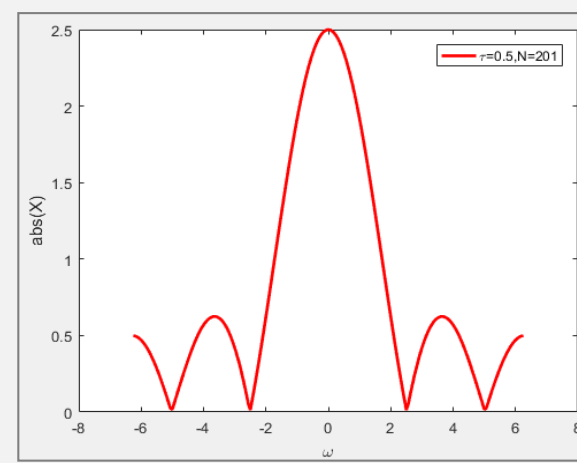
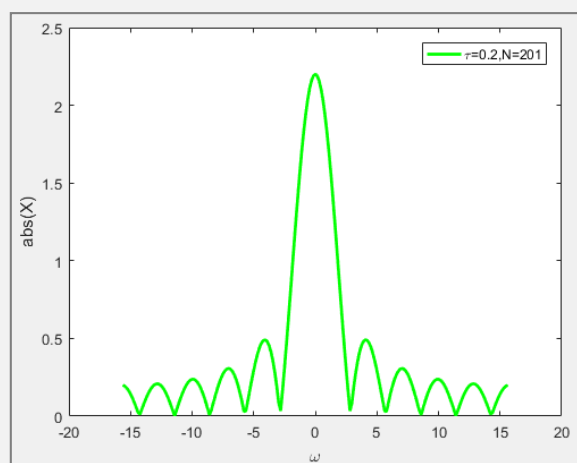
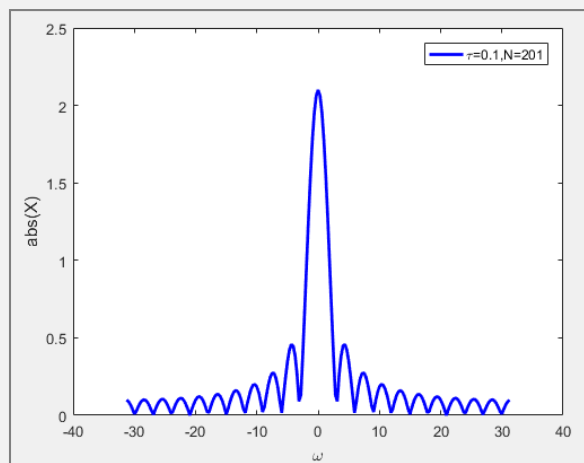
$\tau = 0.1$



$\tau = 0.2$

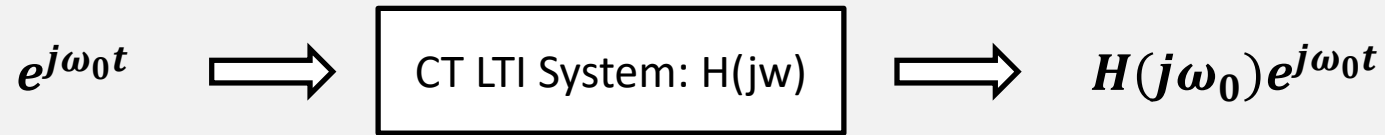


$\tau = 0.5$



# What's Frequency Response ?

- **Frequency response**
  - A function of frequency
  - System gain in frequency domain

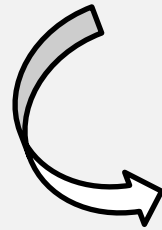


- **Math definition**
  - Fourier transform of impulse response

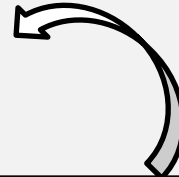
$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt = Y(j\omega)/X(j\omega)$$

# Differential Equation and Frequency Response

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$



$$H(j\omega) = \frac{b_M(j\omega)^M + b_{M-1}(j\omega)^{M-1} + \dots + b_1(j\omega) + b_0}{a_N(j\omega)^N + a_{N-1}(j\omega)^{N-1} + \dots + a_1(j\omega) + a_0}$$



**Proof:**

$$\begin{aligned} \mathcal{F} \left[ \sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} \right] &= \sum_{k=0}^N \mathcal{F} \left[ a_k \frac{d^k y(t)}{dt^k} \right] = \sum_{k=0}^N a_k (j\omega)^k Y(j\omega) \\ \mathcal{F} \left[ \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k} \right] &= \sum_{k=0}^M \mathcal{F} \left[ b_k \frac{d^k x(t)}{dt^k} \right] = \sum_{k=0}^M b_k (j\omega)^k X(j\omega) \end{aligned}$$

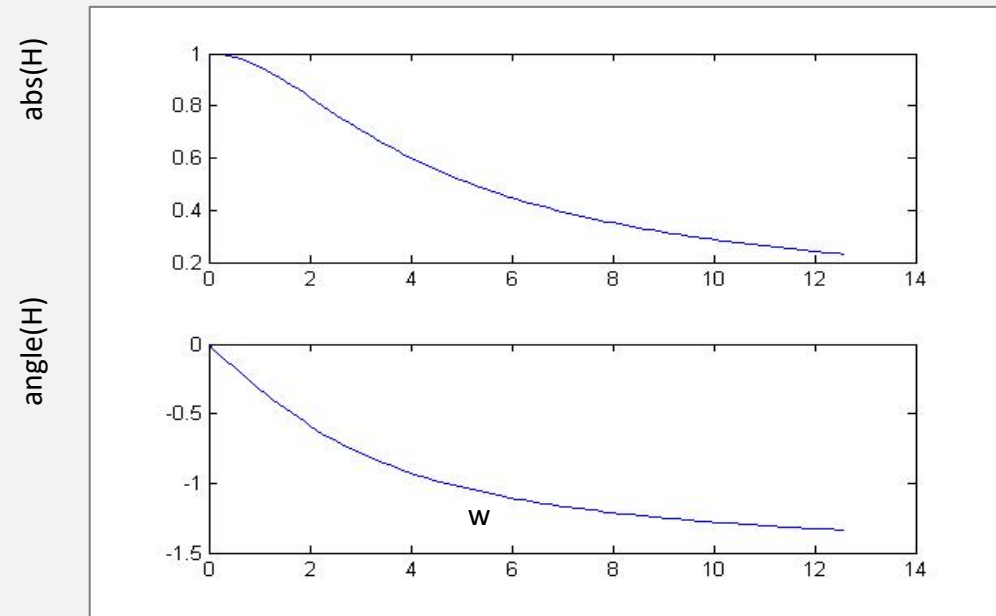
# Matlab Function: freqs()

- **Description:** generate the frequency response of LTI system
- **Syntax:** freqs(b,a,w)
- **Example:**

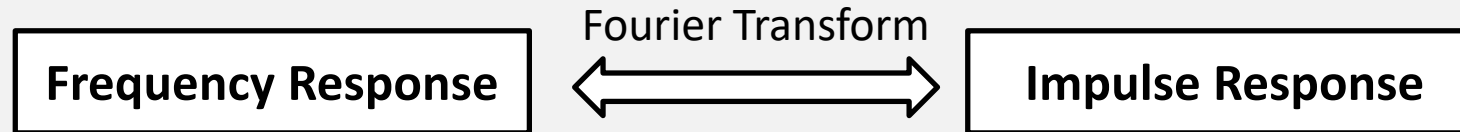
$$\frac{dy(t)}{dt} + 3y(t) = 3x(t)$$

```
a=[1 3];  
b=3;  
w=linspace(0,4*pi);  
H=freqs(b,a,w);  
subplot(2,1,1), plot(w,abs(H));  
subplot(2,1,2),  
plot(w,angle(H));
```

High-pass or Low-pass?



# Frequency Response and Impulse Response



➤ **Transform pair:**

➤  $e^{-at}u(t) \quad \text{Re}\{a\} > 0 \quad \Longleftrightarrow \quad \frac{1}{j\omega + a}$

➤ **Example:**

➤  $\frac{3}{j\omega + 3} \Longleftrightarrow 3e^{-3t}u(t)$

➤ **How about general fraction?**

$$H(j\omega) = \frac{b_M(j\omega)^M + b_{M-1}(j\omega)^{M-1} + \dots + b_1(j\omega) + b_0}{a_N(j\omega)^N + a_{N-1}(j\omega)^{N-1} + \dots + a_1(j\omega) + a_0}$$



# Partial Fraction Expansion

➤ Partial Fraction Expansion (No identical poles):

$$\begin{aligned} H(j\omega) &= \frac{b_M(j\omega + z_1)(j\omega + z_2) \dots (j\omega + z_M)}{a_N(j\omega + p_1)(j\omega + p_2) \dots (j\omega + p_N)} \\ &= \frac{A_1}{j\omega + p_1} + \frac{A_2}{j\omega + p_2} + \dots + \frac{A_N}{j\omega + p_N} \end{aligned}$$

➤ Partial Fraction Expansion (with identical poles):

$$\begin{aligned} H(j\omega) &= \frac{b_M(j\omega + z_1)(j\omega + z_2) \dots (j\omega + z_M)}{a_N(j\omega + p_1)^{k_1}(j\omega + p_2)^{k_2} \dots (j\omega + p_n)^{k_n}} = \\ &= \frac{A_{1,1}}{(j\omega + p_1)^{k_1}} + \frac{A_{1,2}}{(j\omega + p_1)^{k_1-1}} + \dots + \frac{A_{1,k_1}}{(j\omega + p_1)} \\ &\quad + \dots + \\ &\quad \frac{A_{n,1}}{(j\omega + p_n)^{k_n}} + \frac{A_{n,2}}{(j\omega + p_n)^{k_n-1}} + \dots + \frac{A_{n,k_n}}{(j\omega + p_n)} \end{aligned}$$

# Partial Fraction Expansion

Therefore, we should know the FT of

$$\frac{1}{(j\omega + a)^k}$$

We already know:

$$e^{-at}u(t) \quad \text{Re}\{a\} > 0 \quad \Leftrightarrow \quad \frac{1}{j\omega + a}$$

According to the FT property:

$$tx(t) \Leftrightarrow j \frac{d}{d\omega} X(j\omega)$$

We have:

$$te^{-at}u(t) \quad \text{Re}\{a\} > 0 \quad \Leftrightarrow \quad \frac{1}{(j\omega + a)^2}$$

...

# Matlab Function: residue()

- Convert between partial fraction expansion and polynomial coefficients
- $[r,p,k] = \text{residue}(b,a)$  ;  $[r,p] = \text{residue}(b,a)$

$$\frac{b(s)}{a(s)} = \frac{b_1 s^m + b_2 s^{m-1} + b_3 s^{m-2} + \dots + b_{m+1}}{a_1 s^n + a_2 s^{n-1} + a_3 s^{n-2} + \dots + a_{n+1}}$$

$$\frac{b(s)}{a(s)} = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \dots + \frac{r_n}{s - p_n} + k(s)$$

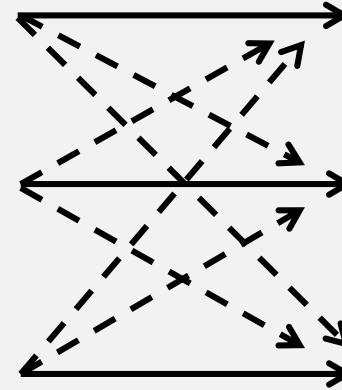
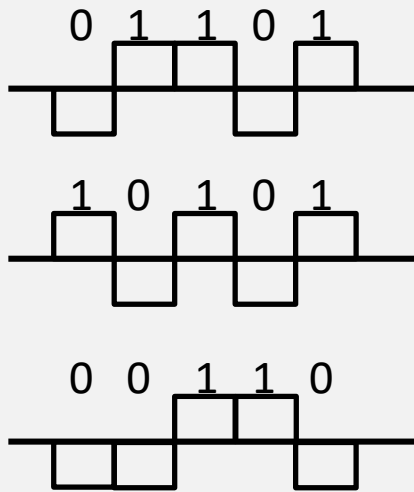
- Example:

$$\frac{b(s)}{a(s)} = \frac{5s^3 + 3s^2 - 2s + 7}{-4s^3 + 8s + 3}$$



```
b = [ 5 3 -2 7];  
a = [-4 0 8 3];  
[r, p, k] = residue(b,a);
```

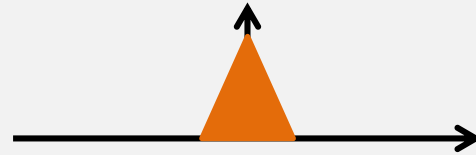
# What's Modulation



- Each phone wants to deliver information to its base station
- Therefore, there is cross-talk in the wireless channel
- **How can we solve this issue?**

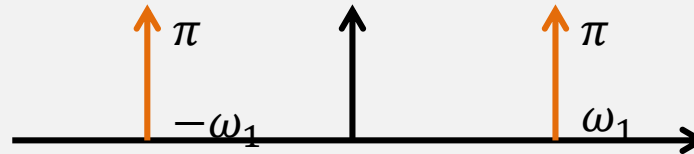
# Amplitude Modulation

- Signal:  $m(t)$



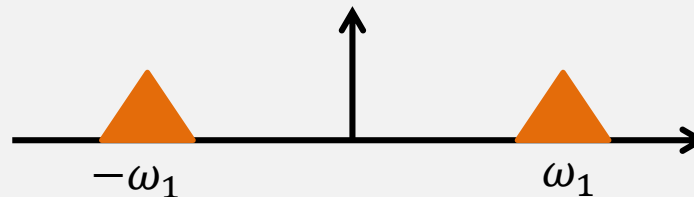
Spectrum of  $m(t)$

- Carrier:  $\cos(\omega_1 t) \Leftrightarrow \pi[\delta(\omega - \omega_1) + \delta(\omega + \omega_1)]$



Spectrum of carrier

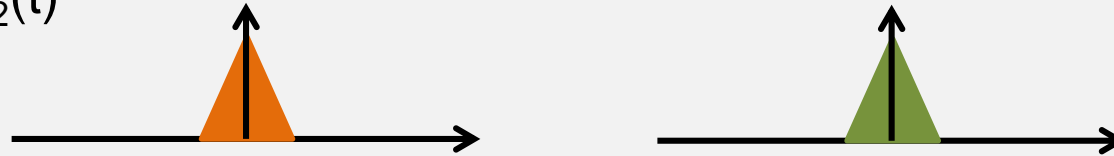
- Amplitude Modulation:  $y(t) = m(t) \times \cos(\omega_1 t)$



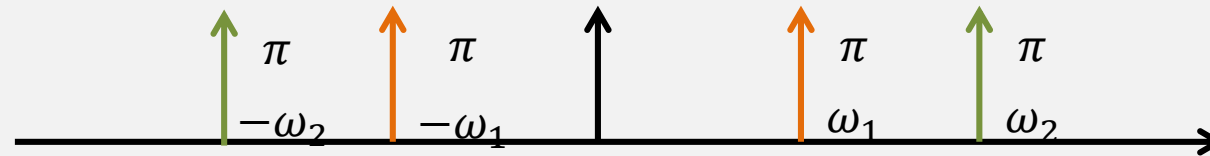
Shift by  $\omega_1$  and Scale by 1/2

**No information loss**

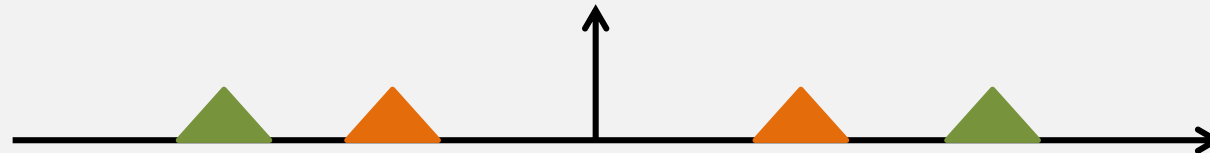
- Signals:  $m_1(t)$  and  $m_2(t)$



- Carriers:  $\cos(\omega_1 t)$  and  $\cos(\omega_2 t)$



- Amplitude Modulation:  $y(t) = m_1(t) \times \cos(\omega_1 t) + m_2(t) \times \cos(\omega_2 t)$



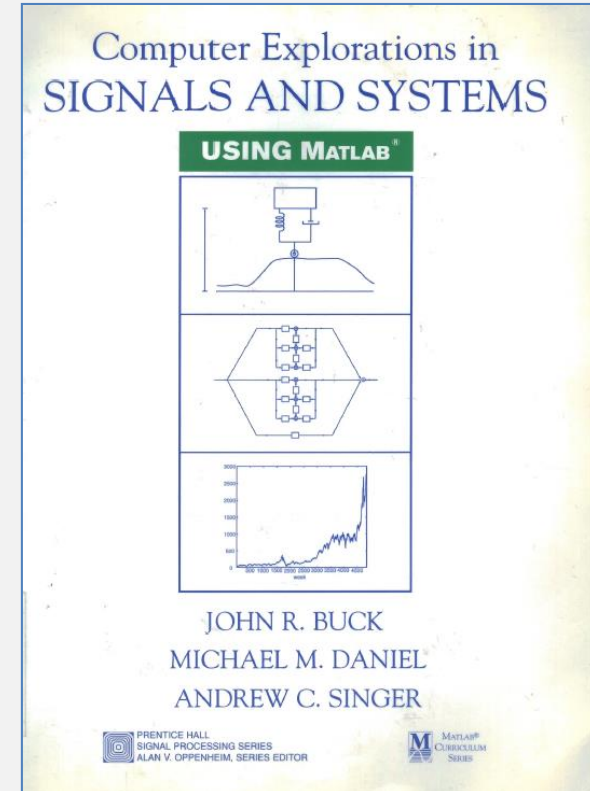
- Signals are distinguished in frequency domain !

# Lab Assignments

- Read tutorial 4.1 by yourself
- 4.2 & 4.6

$$x(t) = m_1(t) \cos(2\pi f_1 t) + m_2(t) \sin(2\pi f_2 t) + m_3(t) \sin(2\pi f_1 t)$$

- Submit your report.



- Question ?

