Homework: 7.21, 7.23
Tutorial Problems: 7.25, 7.37, 7.40

Chapter 7: Sampling

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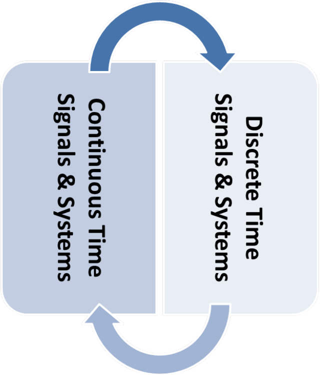
2022 - Fall
Last Update on: Tuesday 10^{th} May, 2022

Introduction

Sampling: to facilitate digital processing via computers or chips



Any lossless conversion?

Process CT signals with DT systems?

Interpolation: to present the output of digital processing

Video Recording

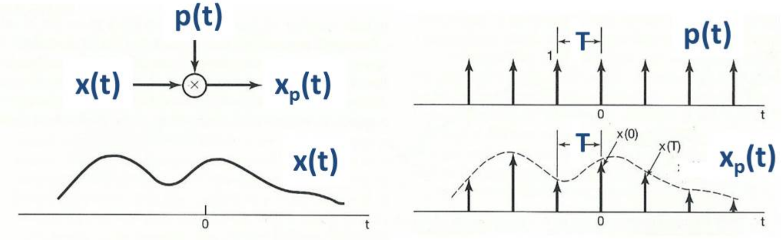
- Signal to be sampled: real scene (continuous-time signals)
- Sampling: record by camera with a rate of 24, 25 or 30 frames per second
- Sampled signal: video tapes, mp4 files, avi files and etc. (discrete-time signals)
- Reconstruction: watch via eyes and interpret in the brain
- In our consciousness, the real scene can be reconstructed without information loss

Outline

- Sampling is a general procedure to generate DT signals from CT signals, where information of the original signals can be kept
- Core sampling theory:
  - Impulse train, zero-order hold, first-order hold and etc
  - Analysis in frequency domain
  - Nyquist rate
- Undersampling: Aliasing
- Application: process continuous-time signals discretely
- More sampling techniques: decimation, downsampling and upsampling

Impulse-Train Sampling

- Mathematically, sampling can be represented by multiplication

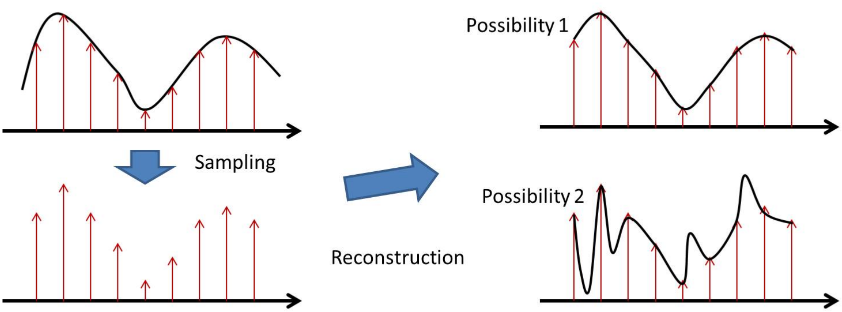


- Sampling function: $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$
- Sampling period: T
- Sampling:

$$x_p(t) = x(t) \times p(t) = x(t) \times \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

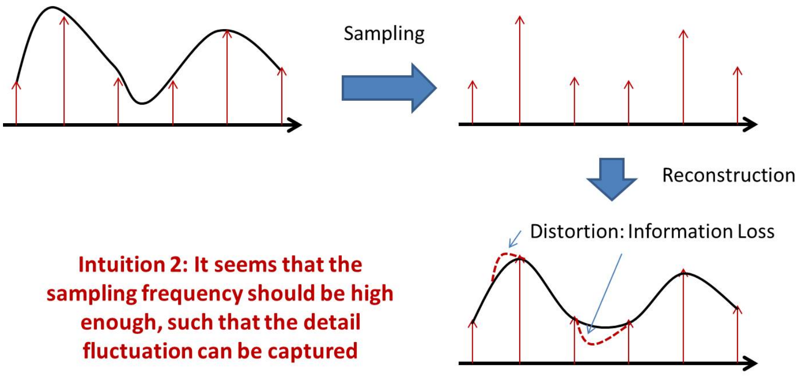
Sampling discards most of points in the original signals.
Is there any information loss in sampling?

Observation (1/2)



Intuition 1: It seems that we need a smooth interpolation

Observation (2/2)



- Sampling: the frequency should be high enough
- Reconstruction: the interpolation should be smooth enough

Frequency Analysis (1/2)

- Theoretical tool: continuous-time Fourier transform
- Principle:

$$x(t) \times p(t) \rightleftarrows \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

- Fourier series of $p(t)$:

$$\begin{aligned} a_k &= \frac{1}{T} \int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \delta(t - nT) e^{-jk\omega_s t} dt \quad where \quad \omega_s = \frac{2\pi}{T} \\ &= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T} \end{aligned}$$

- Fourier Transform of $p(t)$:

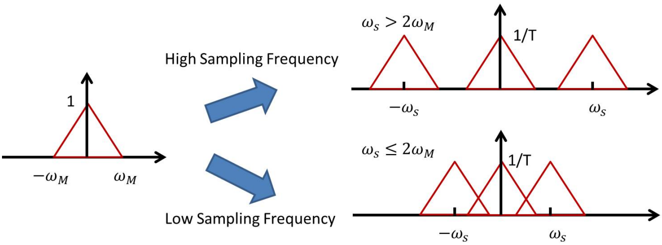
$$P(j\omega) = 2\pi a_k \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

Frequency Analysis (2/2)

- Fourier transform of sampled signal $x_p(t)$:

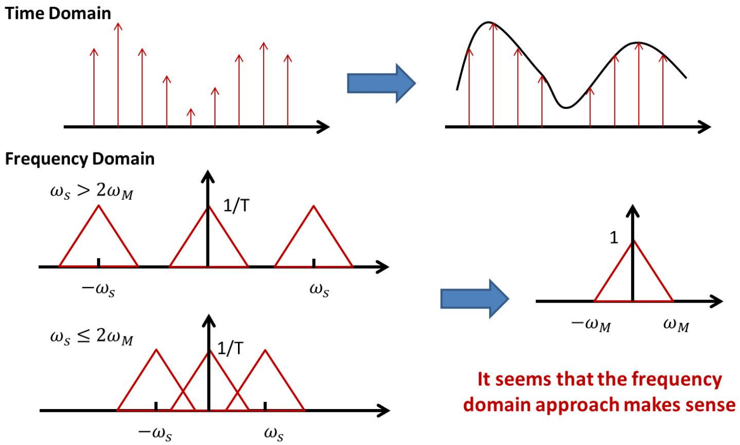
$$\begin{aligned} X_p(j\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))d\theta \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega-k\omega_s)) \end{aligned}$$

- Sampling: the Fourier transform of input signal is repeated with period ω_s



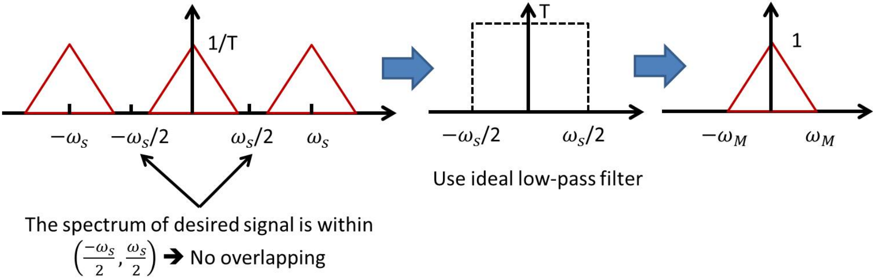
Reconstruction Problem

- Given the sampled signal, can we perfectly reconstruct the signal before sampling?



Reconstruction (1/2)

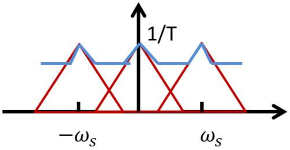
- Scenario of $\omega_s > 2\omega_M$



Remark: We use LPF with bandwidth $\frac{\omega_s}{2}$ here. In fact, the choice of bandwidth is not unique, i.e., $(\omega_M, \omega_s - \omega_M)$.

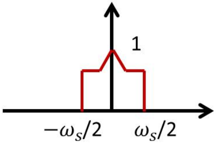
Reconstruction (2/2)

- Scenario of $\omega_s \leq 2\omega_M$

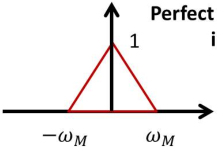


Since $\omega_s \leq 2\omega_M$, we don't know the frequency range of the desired signal

Sampling on the following signals can generate the same result:



OR



Perfect reconstruction is impossible

Observation: the original signal x(t) can be Uniquely and perfectly reconstructed from x(nT) only when $\omega_s > 2\omega_M$

Sampling Theorem

Let $x(t)$ be a band-limited signal with

$$X(j\omega) = 0 \text{ for } |\omega| > \omega_M.$$

Then, $x(t)$ is uniquely determined by its samples $x(nT)$ or $x_p(t)$ if

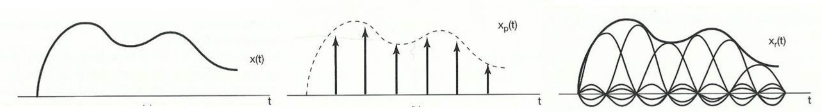
$$\omega_s = \frac{2\pi}{T} > 2\omega_M,$$

where $2\omega_M$ is referred to as the Nyquist rate.

- Questions:
  - How about $\omega_s = 2\omega_M$?
  - Sampling on band-pass signals

Signal Reconstruction: Interpolation

- If $\omega_s > 2\omega_M$, original signal can be perfectly reconstructed by ideal low-pass filter.
- Time domain interpretation of lowpass filtering



$$
\begin{aligned}
x_r(t) &= x_p(t) * h(t) = \sum_{n=-\infty}^{+\infty} x(nT)h(t-nT) \\
&= \sum_{n=-\infty}^{+\infty} x(nT)\frac{\sin\frac{\omega_s}{2}(t-nT)}{\frac{\omega_s}{2}(t-nT)} = \sum_{n=-\infty}^{+\infty} x(nT)sinc(\frac{t-nT}{T}),
\end{aligned}
$$

where $sinc(x) = sin(\pi x)/x$ and LPF with bandwidth $\frac{\omega_s}{2}$ is used.

- Ideal lowpass filtering: interpolation with sinc function

Problem 1

Problem (7.5)

Let $x(t)$ *be a signal with Nyquist rate* ω_0. *Also, let*

$$y(t) = x(t)p(t-1),$$

where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \quad \text{and} \quad T < \frac{2\pi}{\omega_0}.$$

Specify the constraints on the magnitude and phase of the frequency response of a filter that gives $x(t)$ *as its output when* $y(t)$ *is the input.*

Problem 2

Problem (7.36)

Let $x(t)$ be a band-limited signal such that $X(j\omega) = 0$ for $|\omega| \geq \pi/T$.
(a) If $x(t)$ is sampled using a sampling period T, determine an interpolating function $g(t)$ such that

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} x(nT)g(t-nT).$$

(b) Is the function $g(t)$ unique?