Homework and Tutorial

Assignments

- 4.22 (c) (e), 4.27, 4.14, 4.25, 4.31

Tutorial problems

- Basic Problems wish Answers 4.8, 4.9

- Basic Problems 4.23

- Advanced Problems 4.39, 4.40

Chapter 4
# The Continuous-Time Fourier Transform

(cont.)

Outline

- Interesting properties of CTFT
  - ➢ Real signal
  - ➢ Convolution property – LTI systems
  - ➢ Multiplication property – Modulation & Sampling

- Linear-constant-coefficient differential equation of LTI systems

Problem 4.24 (a)

(a)

- Determine which, if any, of the real signals in (a)-(f) have Fourier transforms that satisfy each of the following condition:

  (b)

  ➤ $Re\{X(j\omega)\} = 0$

  (c)

  ➤ $Im\{X(j\omega)\} = 0$

  (d)

  ➤ **There exists a real a such that $e^{ja\omega}X(j\omega)$ is real**

  ➤ $\int_{-\infty}^{\infty} X(j\omega)d\omega = 0$

  (e)

  ➤ $\int_{-\infty}^{\infty} \omega X(j\omega)d\omega = 0$

  (f)

  ➤ $X(j\omega)$ **is periodic**
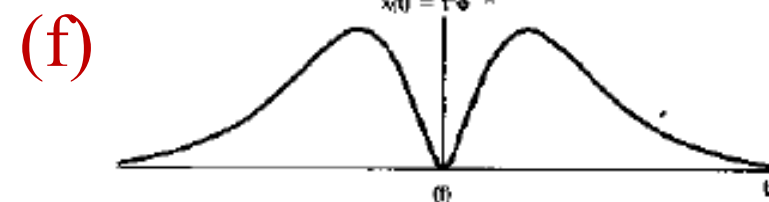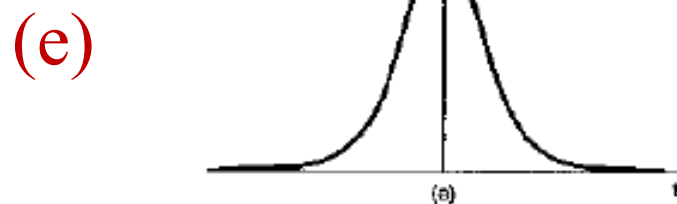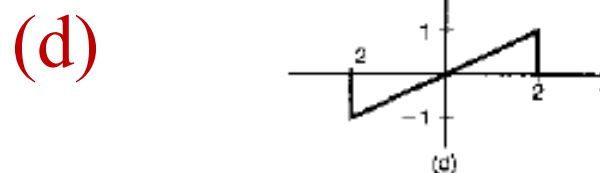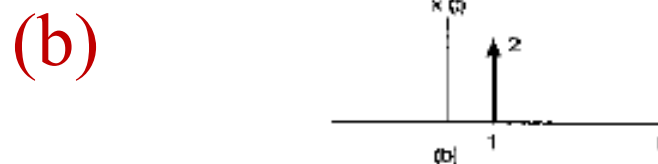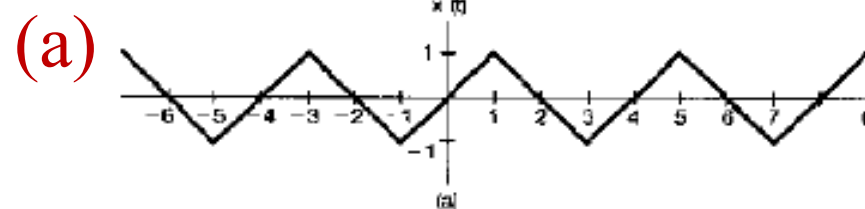
Impulse & Frequency Responses

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \rightarrow \boxed{h(t)} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt$$

Impulse response $\overset{\mathcal{F}}{\longleftrightarrow}$ Frequency response
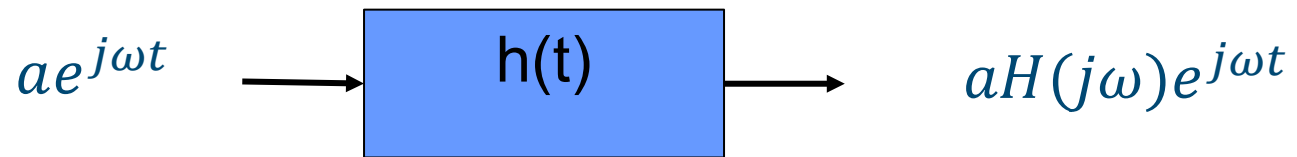
Remark:
Not every LTI system has frequency response; but according to Dirichlet conditions, most of stable LTI systems have.

Convolution Property

$$y(t) = h(t) * x(t) \longleftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

An intuitive interpretation:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega \Rightarrow x(t) = \int_{-\infty}^{+\infty} \underbrace{\frac{1}{2\pi}X(j\omega)d\omega}_{coefficient} e^{j\omega t}$$

$$ae^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow aH(j\omega)e^{j\omega t}$$

$$y(t) = \int_{-\infty}^{+\infty} \underbrace{H(j\omega)\frac{1}{2\pi}X(j\omega)d\omega}_{new\ coefficient} e^{j\omega t} \Rightarrow y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(j\omega)X(j\omega)e^{j\omega t}d\omega$$

$$Y(j\omega) = H(j\omega)\ X(j\omega)$$
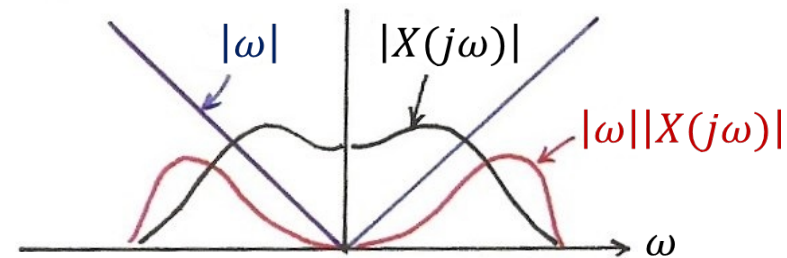
Frequency Response Examples

Example 4.16 A differentiator

$$y(t) = \frac{dx(t)}{dt} \qquad \text{--- an LTI system}$$

From differentiation property $\Rightarrow$ $\dfrac{d}{dt} \xleftarrow{\ FT\ } j\omega$

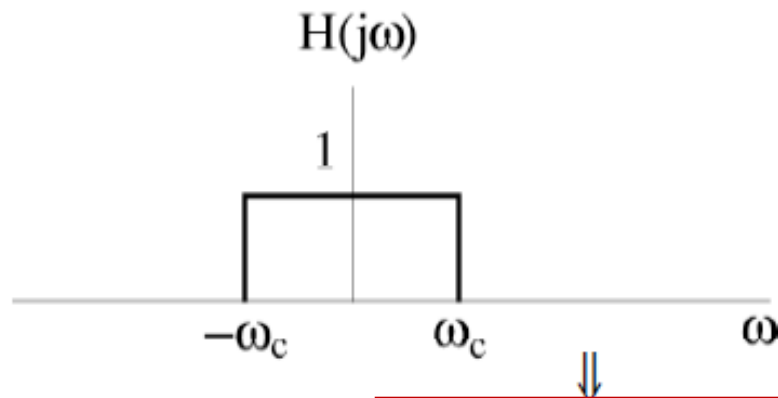$$\Downarrow$$

$$H(j\omega) = j\omega$$

1) Amplifies high frequencies (enhances sharp edges)

2) $+\pi/2$ phase shift ($j = e^{j\pi/2}$)

Larger at high ω_o phase shift

$$\frac{d}{dt}\sin\omega_0 t = \omega_0 \cos\omega_0 t = \omega_0 \sin(\omega_0 t + \pi/2)$$

$$\frac{d}{dt}\cos\omega_0 t = -\omega_0 \sin\omega_0 t = \omega_0 \cos(\omega_0 t + \pi/2)$$

Example 4.18 : Impulse Response of an *Ideal* Lowpass Filter

H(jω)

1

$-\omega_c$ ω_c ω
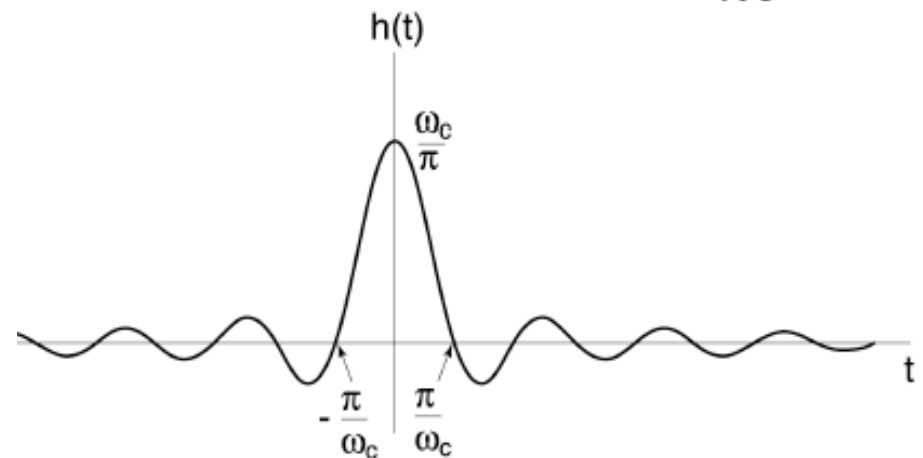
$$h(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}\, d\omega = \frac{\sin \omega_c t}{\pi t} = \frac{\omega_c}{\pi}\operatorname{sinc}\left(\frac{\omega_c t}{\pi}\right)$$

Questions:

1) Is this a causal system?
2) What is $h(0)$?

$$\sin c(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

h(t)

$\frac{\omega_c}{\pi}$

$-\dfrac{\pi}{\omega_c}$ $\dfrac{\pi}{\omega_c}$ t

Convolution Calculation Examples

Example 4.20

$$\underbrace{\frac{\sin 4\pi t}{\pi t}}_{x(t)} * \underbrace{\frac{\sin 8\pi t}{\pi t}}_{h(t)} = ?$$
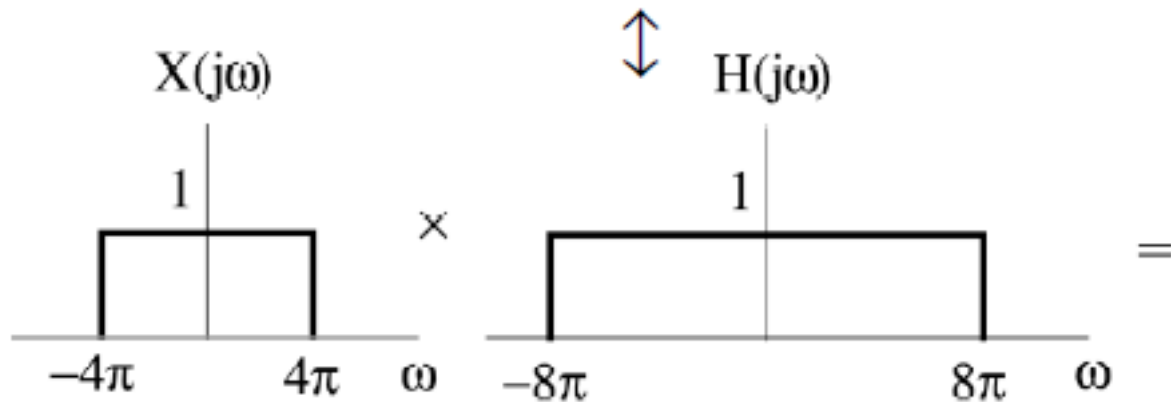
Example 4.19

$$h(t) = e^{-t}u(t) \ , \ x(t) = e^{-2t}u(t)$$

$$y(t) = h(t) * x(t) = ?$$

$$\Downarrow$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{1}{(1+ j\omega)} \cdot \frac{1}{(2 + j\omega)}$$

$$\Downarrow \boxed{\text{Partial fraction expansion}}$$

$$Y(j\omega) = \frac{1}{1+ j\omega} \ \mathbf{\textit{a=1}} - \frac{1}{2 + j\omega} \ \mathbf{\textit{a=2}}$$

$$\Downarrow \text{inverse } FT$$

$$y(t) = [e^{-t} - e^{-2t}]u(t)$$

Multiplication Property

Since FT is highly symmetric,

$$x(t) = \mathcal{F}^{-1}\{X(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)\,e^{j\omega t}d\omega, \quad X(j\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty}x(t)\,e^{-j\omega t}dt$$

thus if
$$x(t)*y(t) \longleftrightarrow X(j\omega)\cdot Y(j\omega)$$

then the other way around is also true

$$x(t)\cdot y(t) \longleftrightarrow \frac{1}{2\pi}X(j\omega)*Y(j\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$$

$$\frac{1}{2\pi}$$

— A consequence of *Duality*

Example: Frequency Shifting

$$e^{j\omega_0 t} x(t) \xleftarrow{\quad F \quad} X(j(\omega - \omega_0))$$

$$e^{j\omega_0 t} \cdot x(t) \longleftrightarrow \frac{1}{2\pi}\left[2\pi\delta(\omega - \omega_0) * X(j\omega)\right]$$
$$= X(j(\omega - \omega_0))$$

①   ②   ③

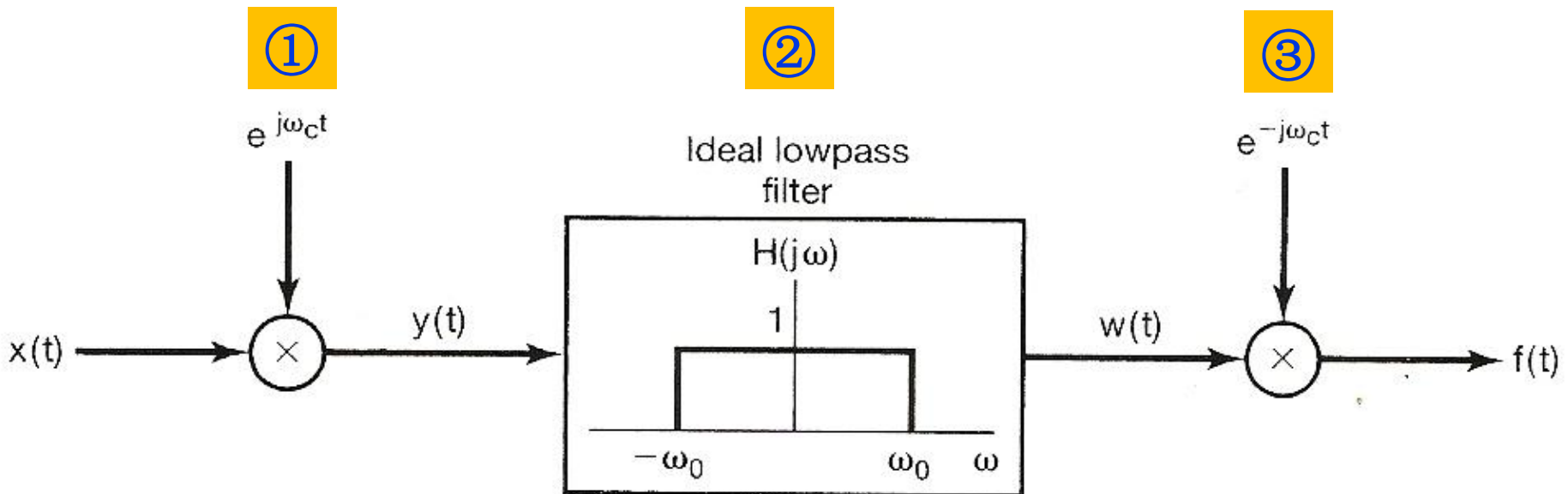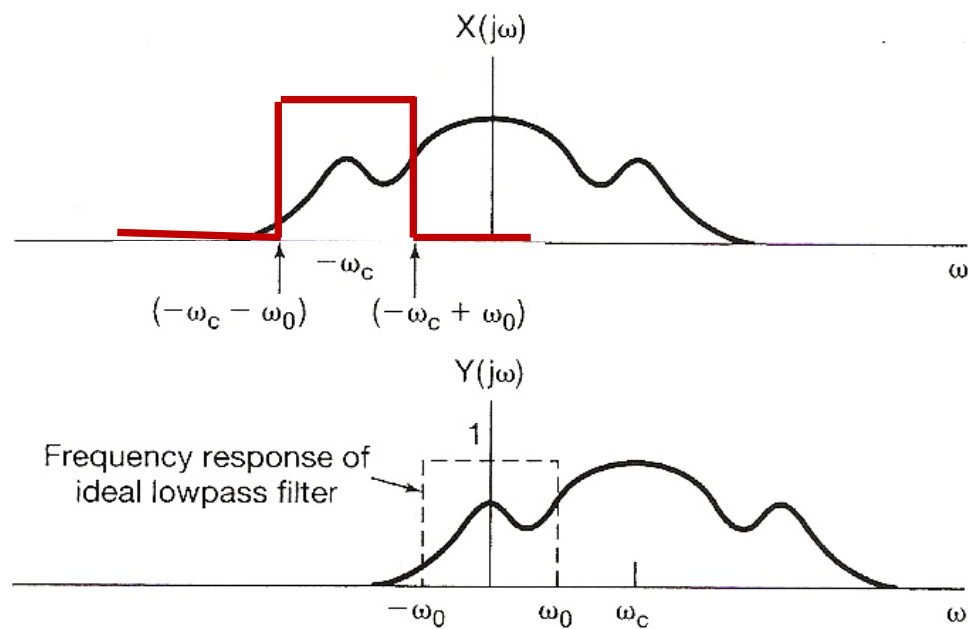**Figure 4.26** Implementation of a bandpass filter using amplitude modulation with a complex exponential carrier.
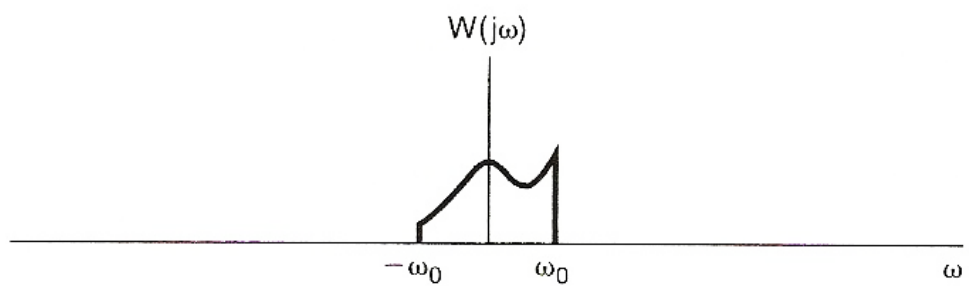
13
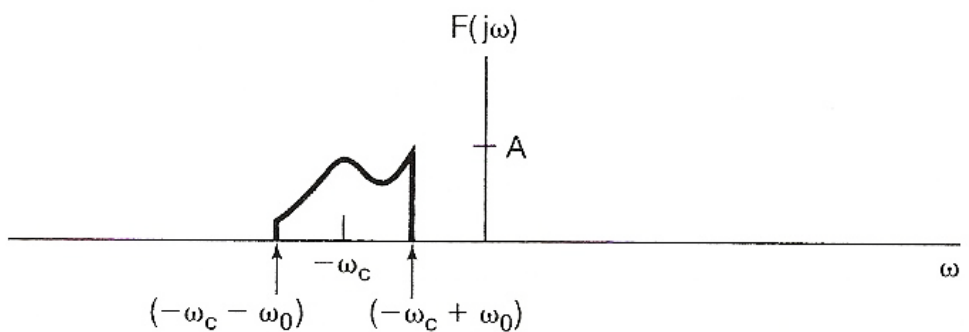
Figure 4.27 Spectra of the signals in the system of Figure 4.26.

Example: Amplitude Modulation

Example 4.21

$$r(t) = s(t) \cdot p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}\left[S(j\omega) * P(j\omega)\right]$$

$$\text{For} \quad p(t) = \cos\omega_0 t \longleftrightarrow P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$R(j\omega) = \frac{1}{2}S(j(\omega - \omega_0)) + \frac{1}{2}S(j(\omega + \omega_0))$$

(cont.)

ω_1 : bandwidth

S(jω)

A

−ω₁   ω₁   ω

$r(t) = s(t) \cdot \cos(\omega_o t)$
Amplitude modulation (*AM*)

π   P(jω)   π

−ω₀   ω₀   ω
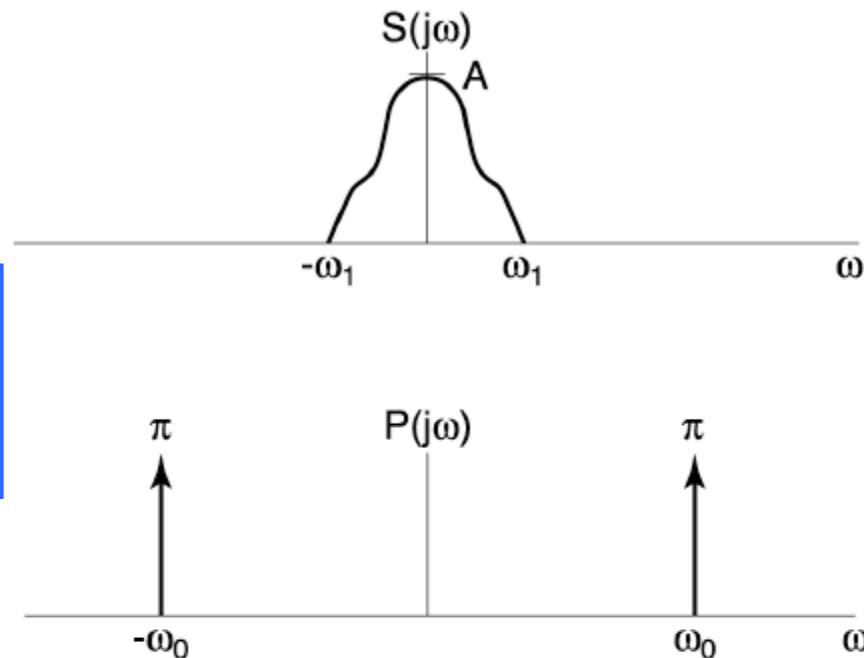
$$R(j\omega) = \frac{1}{2}[S(j(\omega - \omega_o)) + S(j(\omega + \omega_o))]$$

$$R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$

A/2

−ω₀   ω₀   ω

$(-\omega_0 - \omega_1)$ $(-\omega_0 + \omega_1)$ $(\omega_0 - \omega_1)$ $(\omega_0 + \omega_1)$
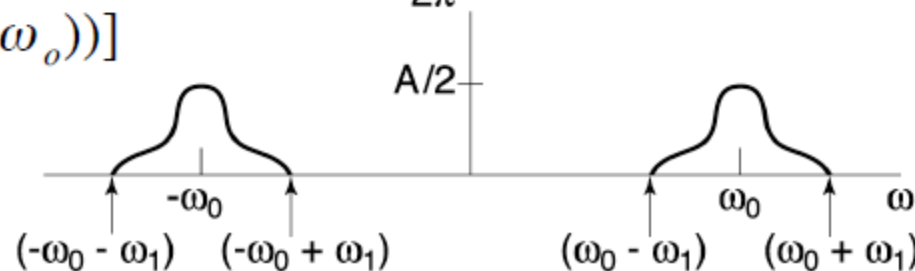
Drawn assume
$\omega_0 - \omega_1 > 0$
i.e. $\omega_0 > \omega_1$

(cont.)

$p(t)$
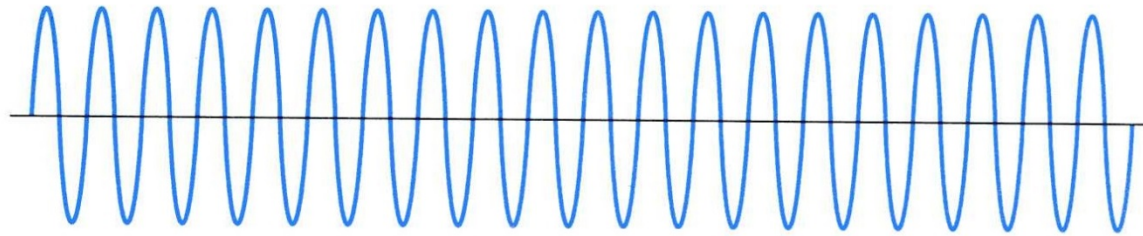
Carrier Signal

$s(t)$

Modulating Sine Wave Signal

$r(t)$
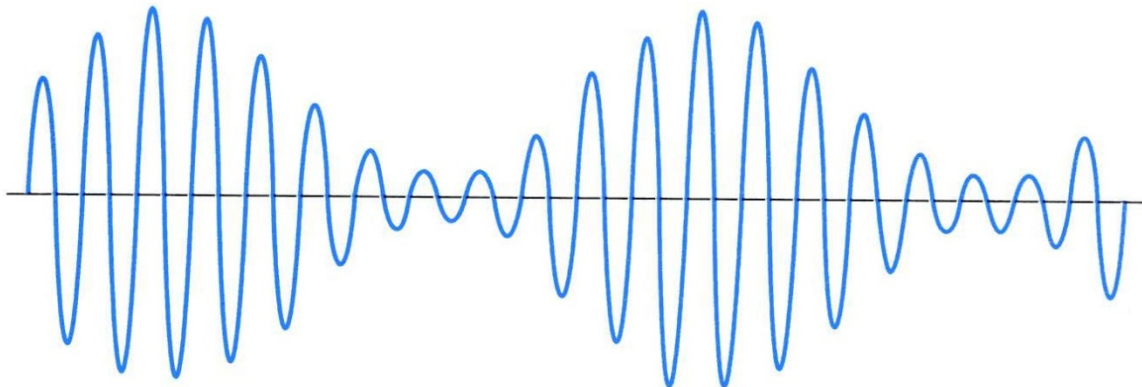
Amplitude Modulated Signal

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Review

Example 4.8

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \quad \text{— sampling function}$$

$$x(t) \overset{\text{FS}}{\longleftrightarrow} a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_o t} dt = \frac{1}{T}$$

$$\Downarrow x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} = \sum_{k=-\infty}^{+\infty} \frac{1}{T} e^{jk\omega_o t}$$
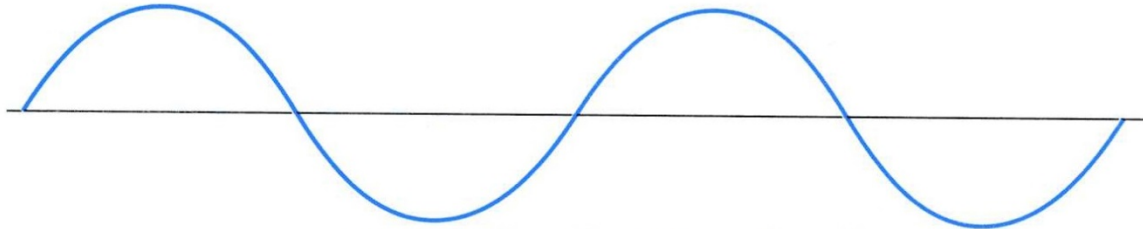
$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T} \delta(\omega - \underbrace{\frac{k2\pi}{T}}_{k\omega_o})$$
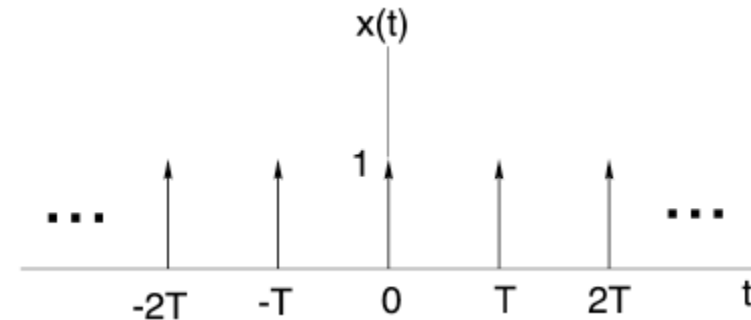
$\omega_s = 2\pi / T$: sampling frequency

x(t)

...   1   ...

-2T   -T   0   T   2T   t

Same function in the frequency-domain!

X(jω)

$\frac{2\pi}{T}$

..   ..

$-\frac{4\pi}{T}$ $-\frac{2\pi}{T}$ 0 $\frac{2\pi}{T}$ $\frac{4\pi}{T}$ ω
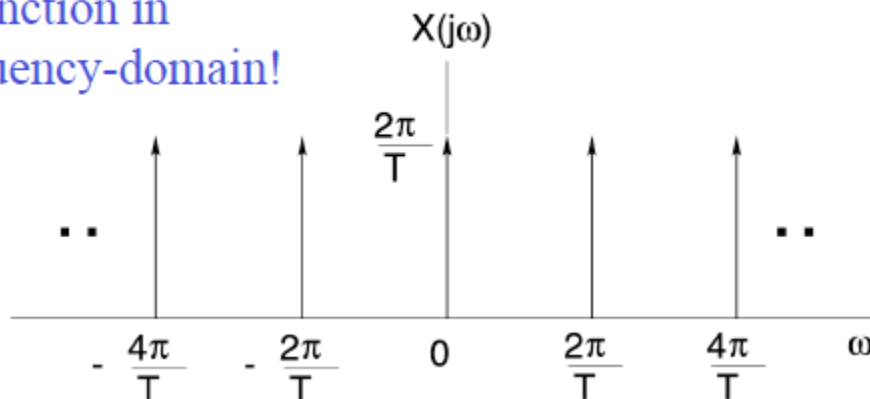
Note in this case, periodic in both time domain (with a period T) and frequency domain (with a period $2\pi/T$)

Example: Sampling

- **Sample a continuous signal**
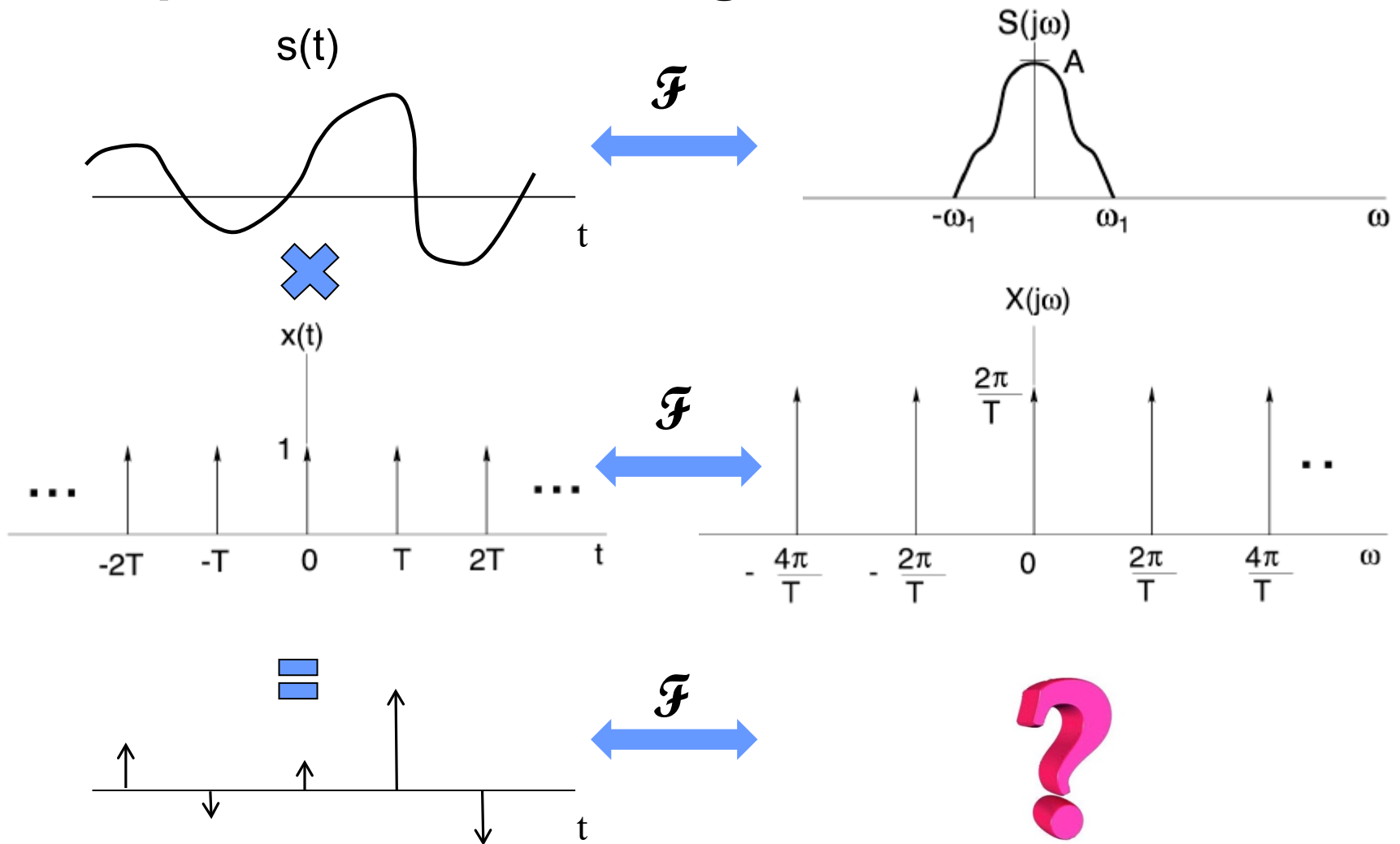
LTI Systems by LCCDE

(Linear-constant-coefficient differential equations)

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Using the Differentiation Property

$$\frac{d^k x(t)}{dt^k} \longleftrightarrow (j\omega)^k X(j\omega)$$

⇓ Transform both sides of the equation

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

⇓

$$Y(j\omega) = \underbrace{\left[\frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} \right]}_{H(j\omega)} X(j\omega)$$

$$H(j\omega) = \left[\frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} \right]$$

Partial Fraction Expansion

Partial Fraction Expansion (No identical poles):

$$H(j\omega) = \frac{b_M(j\omega + z_1)(j\omega + z_2)\dots(j\omega + z_M)}{a_N(j\omega + p_1)(j\omega + p_2)\dots(j\omega + p_N)}$$

$$= \frac{A_1}{j\omega + p_1} + \frac{A_2}{j\omega + p_2} + \dots + \frac{A_N}{j\omega + p_N}$$

Partial Fraction Expansion (with identical poles):

$$H(j\omega) = \frac{b_M(j\omega + z_1)(j\omega + z_2)\dots(j\omega + z_M)}{a_N(j\omega + p_1)^{k_1}(j\omega + p_2)^{k_2}\dots(j\omega + p_n)^{k_n}} =$$

$$= \frac{A_{1,1}}{(j\omega + p_1)^{k_1}} + \frac{A_{1,2}}{(j\omega + p_1)^{k_1-1}} + \dots \frac{A_{1,k_1}}{(j\omega + p_1)}$$

$$+ \dots +$$

$$\frac{A_{n,1}}{(j\omega + p_n)^{k_n}} + \frac{A_{n,2}}{(j\omega + p_n)^{k_n-1}} + \dots \frac{A_{n,k_n}}{(j\omega + p_n)}$$

- A stable LTI system S has the frequency response

$$H(j\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega}$$

  - Determine a differential equation relating the input $x(t)$ and output $y(t)$ of S
  - Determine the impulse response $h(t)$ of S
  - What is the output of S when the input is $\quad x(t) = e^{-4t}u(t) - te^{-4t}u(t)$

① $\dfrac{Y(j\omega)}{X(j\omega)} = H(j\omega) = \dfrac{j\omega + 4}{6 - \omega^2 + 5j\omega}$

$Y(j\omega) \cdot (6 - \omega^2 + 5j\omega) = X(j\omega) \cdot (j\omega + 4)$

$\Updownarrow$

$\dfrac{d^2 y(t)}{dt^2} + 5\dfrac{dy(t)}{dt} + 6y(t) = \dfrac{dx(t)}{dt} + 4x(t)$

② $H(j\omega) = \dfrac{j\omega + 4}{(2 + j\omega)(3 + j\omega)} = \dfrac{A}{2 + j\omega} - \dfrac{B}{3 + j\omega} = \dfrac{2}{2 + j\omega} - \dfrac{1}{3 + j\omega}$

$\because e^{-at}u(t) \longleftrightarrow \dfrac{1}{a + j\omega} \quad \therefore h(t) = 2e^{-2t}u(t) - e^{-3t}u(t)$

③ $\because te^{-at}u(t) \longleftrightarrow \dfrac{1}{(a + j\omega)^2}$

$\therefore X(j\omega) = \dfrac{1}{4 + j\omega} - \dfrac{1}{(4 + j\omega)^2}$

$\therefore Y(j\omega) = H(j\omega) \cdot X(j\omega) = \dfrac{1}{(4 + j\omega)(2 + j\omega)} = \dfrac{A}{4 + j\omega} - \dfrac{B}{2 + j\omega}$

Fourier Transform

Table 4.1 Properties of the Fourier Transform

TABLE 4.1   PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$	$X(j\omega)$
		$y(t)$	$Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\dfrac{1}{|a|}X\left(\dfrac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega - \theta))d\theta$
4.3.4	Differentiation in Time	$\dfrac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\displaystyle\int_{-\infty}^{t}x(t)dt$	$\dfrac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\dfrac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ |X(j\omega)| = |X(-j\omega)| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ $[x(t)$ real$]$ $x_o(t) = \mathcal{O}d\{x(t)\}$ $[x(t)$ real$]$	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

4.3.7   Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Table 4.2
Basic Fourier Transform Pairs

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0,$ otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0,$ otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0,$ otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \|t\| < T_1 \\ 0, & T_1 < \|t\| \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \|t\| < T_1 \\ 0, & \|t\| > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \|\omega\| < W \\ 0, & \|\omega\| > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \ \mathcal{R}e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \ \mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t), \ \mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

$$e^{-at}u(t) \longleftrightarrow \frac{1}{a + j\omega}$$

$$te^{-at}u(t) \longleftrightarrow \frac{1}{(a + j\omega)^2}$$