Tutorial 8

Ch. 7-a

Tutorial 8

- Problems: 7.25，7.37

Sampling Theorem

Sampling Theorem:
Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \pm 2, \ldots$, if

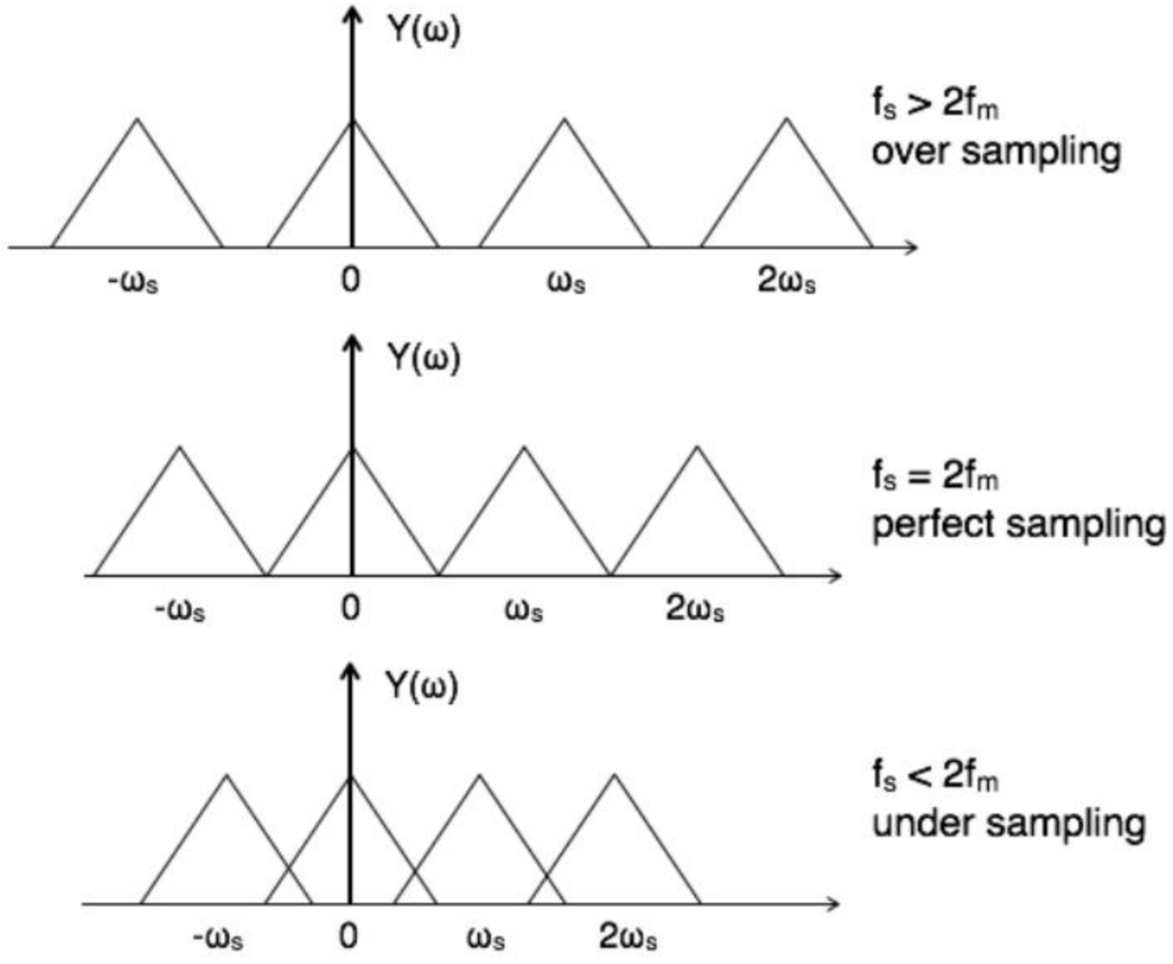
$$\omega_s > 2\omega_M,$$

where

$$\omega_s = \frac{2\pi}{T}.$$

Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain T and cutoff frequency greater than ω_M and less than $\omega_s - \omega_M$. The resulting output signal will exactly equal $x(t)$.

Sampling


Y(ω)
$f_s > 2f_m$
over sampling
-ωs
0
ωs
2ωs
Y(ω)
$f_s = 2f_m$
perfect sampling
-ωs
0
ωs
2ωs
Y(ω)
$f_s < 2f_m$
under sampling
-ωs
0
ωs
2ωs

Problem 7.25

7.25. In Figure P7.25 is a sampler, followed by an ideal lowpass filter, for reconstruction of $x(t)$ from its samples $x_p(t)$. From the sampling theorem, we know that if $\omega_s = 2\pi/T$ is greater than twice the highest frequency present in $x(t)$ and $\omega_c = \omega_s/2$, then the reconstructed signal $x_r(t)$ will exactly equal $x(t)$. If this condition on the bandwidth of $x(t)$ is violated, then $x_r(t)$ will *not* equal $x(t)$. We seek to show in this problem that if $\omega_c = \omega_s/2$, then for any choice of T, $x_r(t)$ and $x(t)$ will always be equal at the sampling instants; that is,

$$x_r(kT) = x(kT),\ k = 0, \pm 1, \pm 2, \ldots.$$

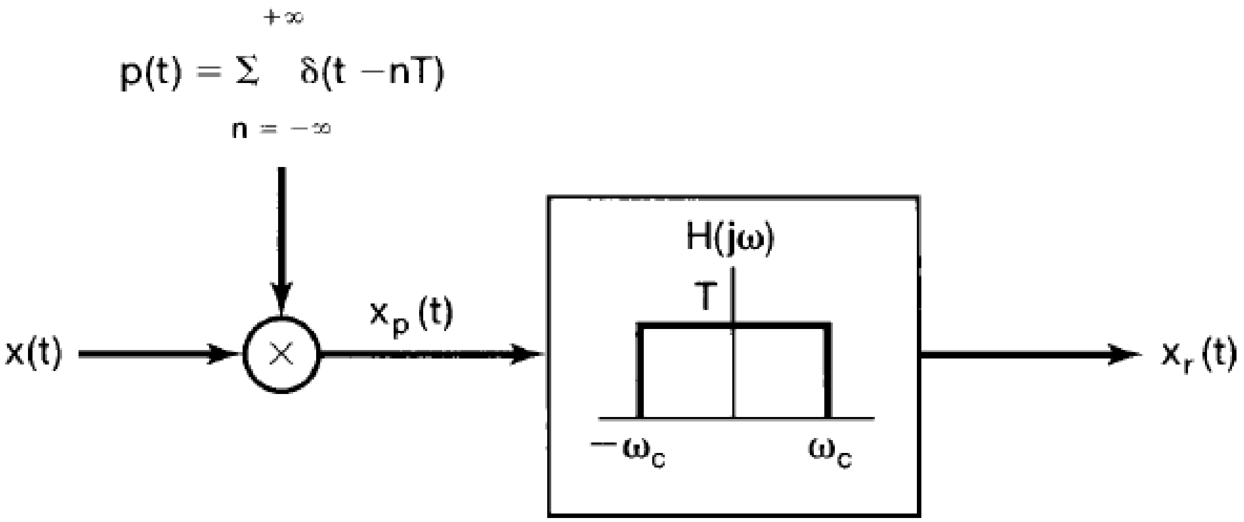


Figure P7.25

Problem 7.25

To obtain this result, consider eq. (7.11), which expresses $x_r(t)$ in terms of the samples of $x(t)$:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)T\frac{\omega_c}{\pi}\frac{\sin[\omega_c(t-nT)]}{\omega_c(t-nT)}.$$

With $\omega_c = \omega_s/2$, this becomes

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{\sin\left[\frac{\pi}{T}(t-nT)\right]}{\frac{\pi}{T}(t-nT)}. \qquad \text{(P7.25–1)}$$

By considering the values of α for which $[\sin(\alpha)]/\alpha = 0$, show from eq. (P7.25–1) that, without any restrictions on $x(t)$, $x_r(kT) = x(kT)$ for any integer value of k.

Answer 7.25

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\frac{\sin\left[\frac{\pi}{T}(t-nT)\right]}{\frac{\pi}{T}(t-nT)}$$

$$x_r(kT) = \sum_{n=-\infty}^{\infty} x(nT)\,\frac{\sin \pi(k-n)}{\pi(k-n)}$$

when $n \neq k$, k integer n integer

$$\frac{\sin \pi(k-n)}{\pi(k-n)} \equiv 0$$

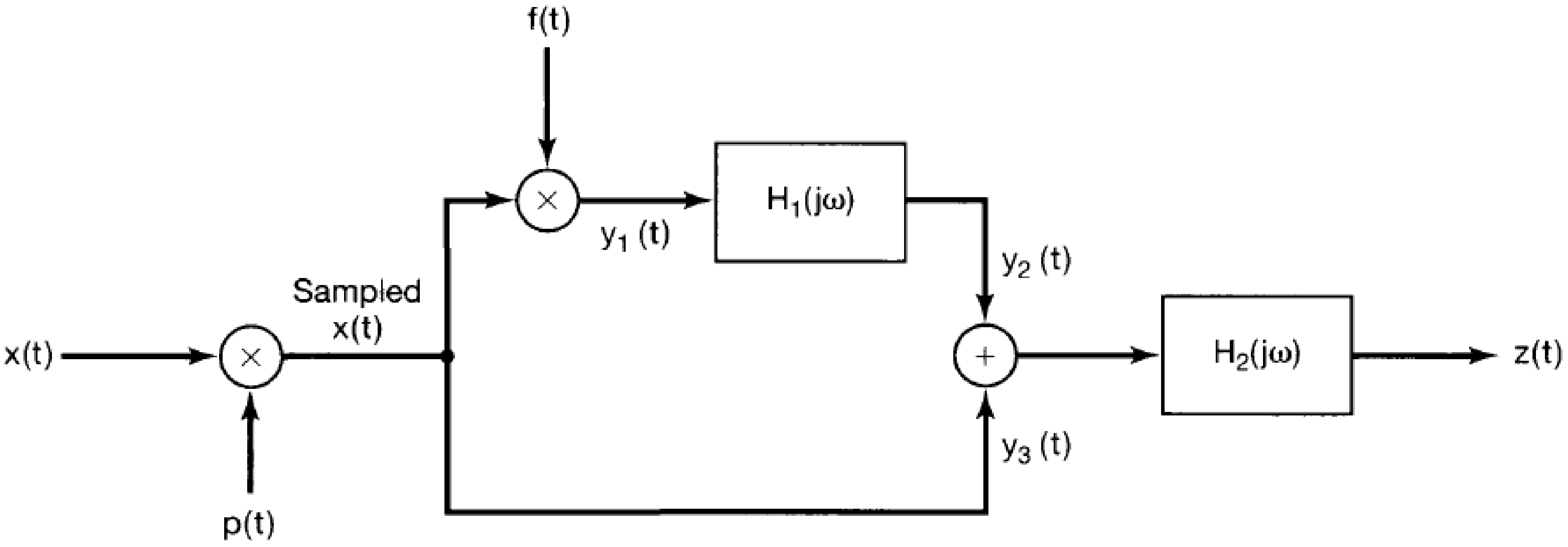
when $n = k$

$$\lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1$$

$$x_r(kT) = x(kT).$$

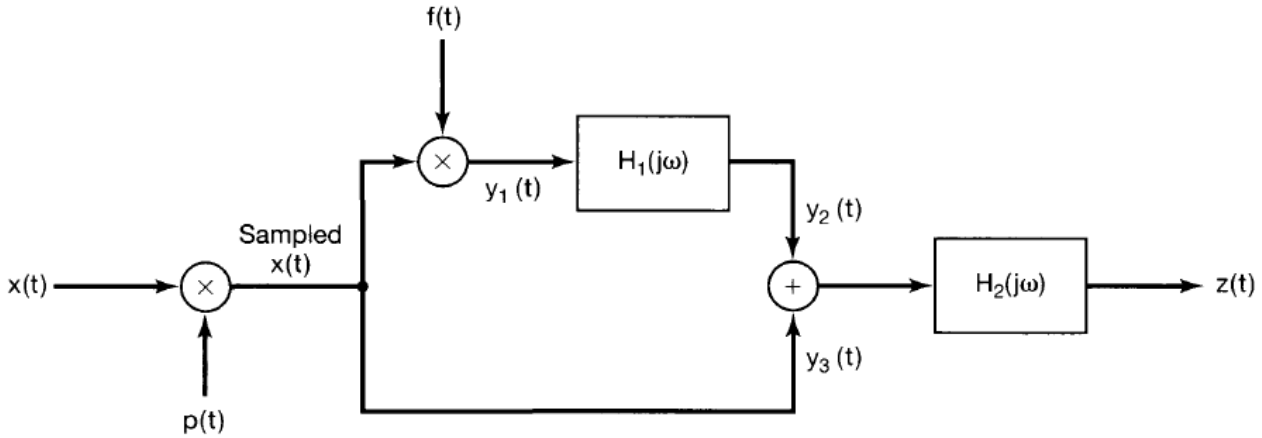
Problem 7.37

7.37. A signal limited in bandwidth to $|\omega| < W$ can be recovered from nonuniformly spaced samples as long as the average sample density is $2(W/2\pi)$ samples per second. This problem illustrates a particular example of nonuniform sampling. Assume that in Figure P7.37(a):

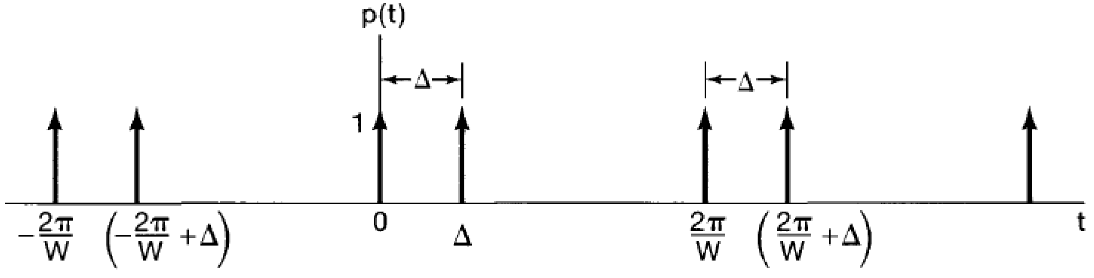


(a)

Problem 7.37



(a)



(b)

1. $x(t)$ is band limited; $X(j\omega) = 0, |\omega| > W$.
2. $p(t)$ is a nonuniformly spaced periodic pulse train, as shown in Figure P7.37(b).
3. $f(t)$ is a periodic waveform with period $T = 2\pi/W$. Since $f(t)$ multiplies an impulse train, only its values $f(0) = a$ and $f(\Delta) = b$ at $t = 0$ and $t = \Delta$, respectively, are significant.
4. $H_1(j\omega)$ is a 90° phase shifter; that is,

$$H_1(j\omega) = \begin{cases} j, & \omega > 0 \\ -j, & \omega < 0 \end{cases}.$$

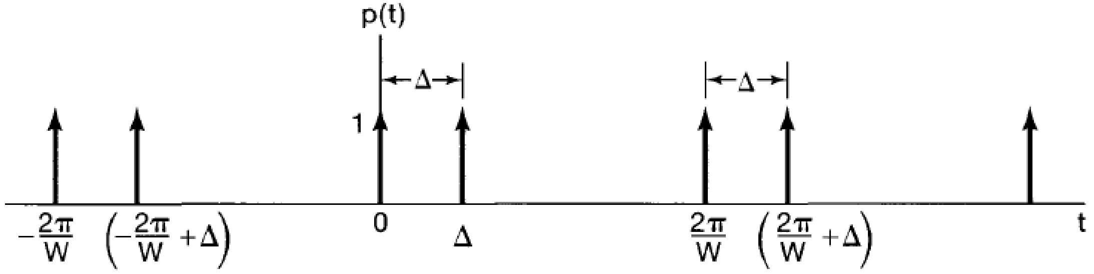
5. $H_2(j\omega)$ is an ideal lowpass filter; that is,

$$H_2(j\omega) = \begin{cases} K, & 0 < \omega < W \\ K^*, & -W < \omega < 0 \\ 0, & |\omega| > W \end{cases}$$

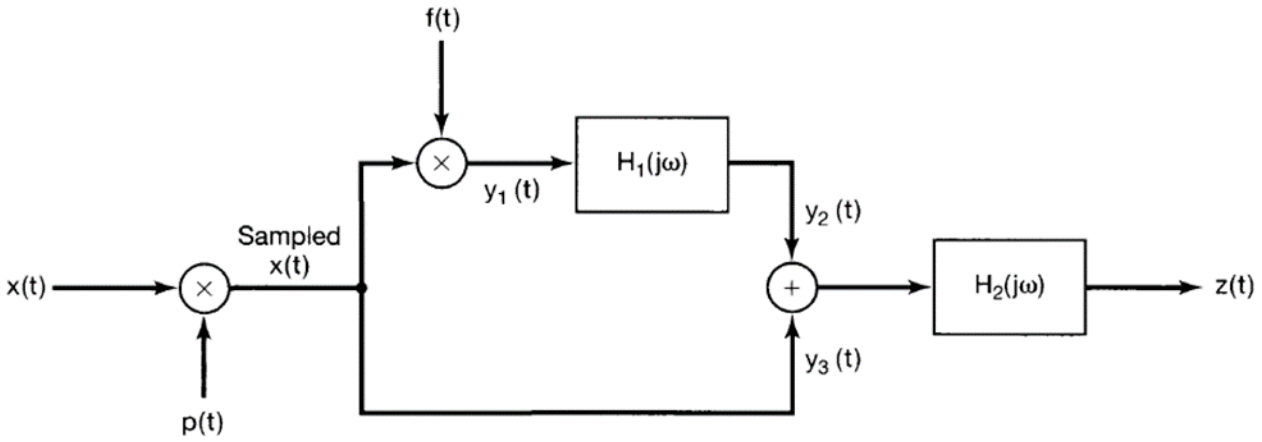
where K is a (possibly complex) constant.

Problem 7.37 (a)

(a) Find the Fourier transforms of $p(t)$, $y_1(t)$, $y_2(t)$, and $y_3(t)$.

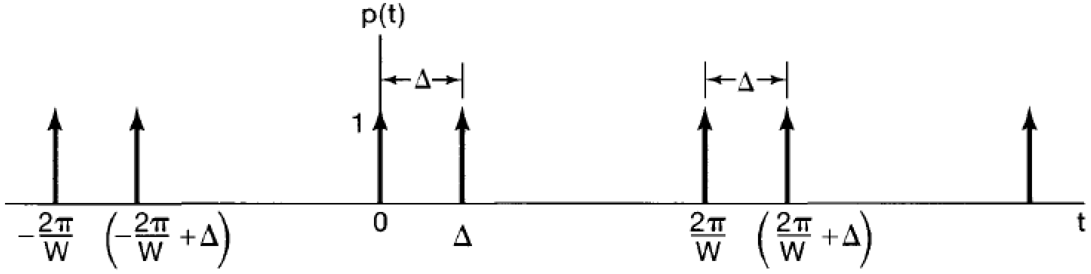


(b)



(a)

Answer 7.37 (a)



(b)

$$x(t-t_0) \longleftrightarrow e^{-j\omega t_o} X(j\omega)$$

We may write $p(t)$ as

$$p(t) = p_1(t) + p_1(t-\Delta),$$

where

$$p_1(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2\pi k/W).$$

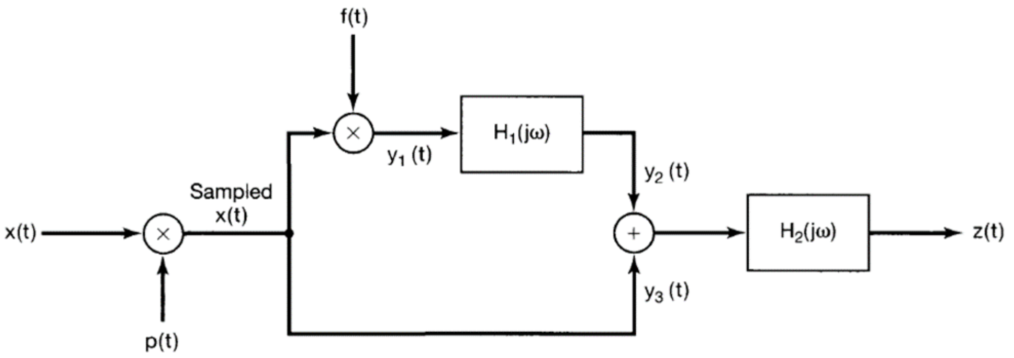
Therefore,

$$P(j\omega) = (1 + e^{-j\Delta\omega})P_1(j\omega),$$

where

$$P_1(j\omega) = W \sum_{k=-\infty}^{\infty} \delta(\omega - kW).$$

Answer 7.37 (a)



(a)

$f(t)$ is a periodic waveform with period $T = 2\pi/W$. Since $f(t)$ multiplies an impulse train, only its values $f(0) = a$ and $f(\Delta) = b$ at $t = 0$ and $t = \Delta$, respectively, are significant.

$$f(t)\delta(t) = a\delta(t) \qquad f(t)\delta(t-\Delta) = b\delta(t-\Delta)$$

$$x(t-t_0) \longleftrightarrow e^{-j\omega t_0}X(j\omega)$$

Let us denote the product $p(t)f(t)$ by $g(t)$. Then,

$$g(t) = p(t)f(t) = p_1(t)f(t) + p_1(t-\Delta)f(t).$$

This may be written as

$$g(t) = ap_1(t) + bp_1(t-\Delta).$$

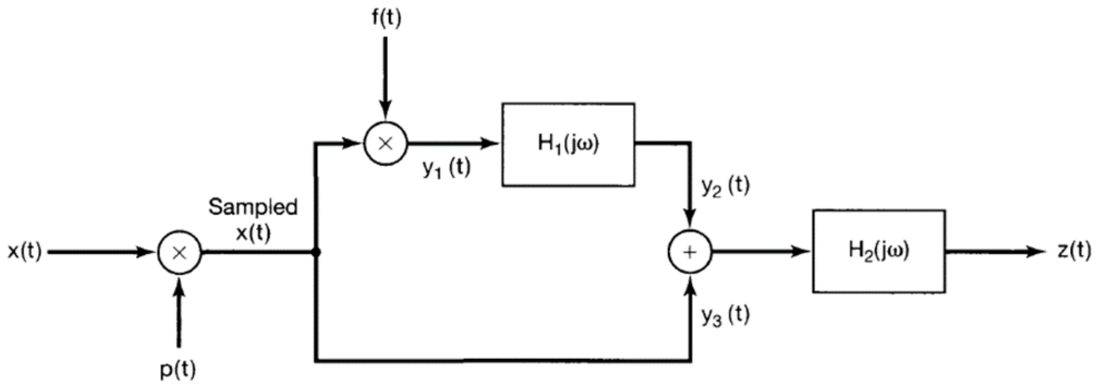
Therefore,

$$G(j\omega) = (a + be^{-j\omega\Delta})P_1(j\omega),$$

with $P_1(j\omega)$ is specified in eq. (S7.37-1). Therefore,

$$G(j\omega) = W\sum_{k=-\infty}^{\infty}[a + be^{-jk\Delta W}]\delta(\omega - kW).$$

Answer 7.37 (a)



(a)

$$r(t) = s(t)p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta$$

Let us denote the product $p(t)f(t)$ by $g(t)$. Then,

$$g(t) = p(t)f(t) = p_1(t)f(t) + p_1(t-\Delta)f(t).$$

$$G(j\omega) = W\sum_{k=-\infty}^{\infty}[a + be^{-jk\Delta W}]\delta(\omega - kW).$$

We now have

$$y_1(t) = x(t)p(t)f(t).$$

Therefore,

$$Y_1(j\omega) = \frac{1}{2\pi}[G(j\omega) * X(j\omega)].$$

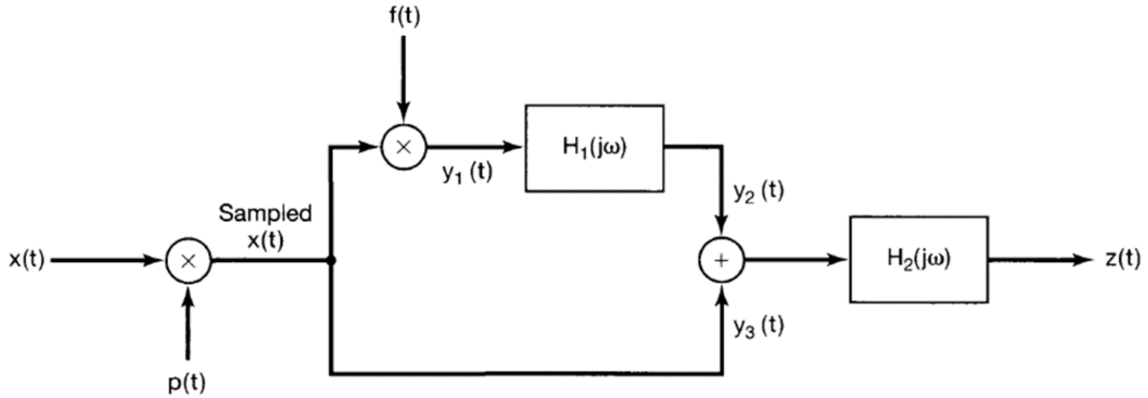
This gives us

$$Y_1(j\omega) = \frac{W}{2\pi}\sum_{k=-\infty}^{\infty}[a + be^{-jk\Delta W}]X(j(\omega - kW)).$$

In the range $0 < \omega < W$, we may specify $Y_1(j\omega)$ as

$$Y_1(j\omega) = \frac{W}{2\pi}\left[(a+b)X(j\omega) + (a + be^{-j\Delta W})X(j(\omega - W))\right].$$

Answer 7.37 (a)



(a)

$H_1(j\omega)$ is a 90° phase shifter; that is,

$$H_1(j\omega) = \begin{cases} j, & \omega > 0 \\ -j, & \omega < 0 \end{cases}.$$

Since $Y_2(j\omega) = Y_1(j\omega)H_1(j\omega)$, in the range $0 < \omega < W$ we may specify $Y_2(j\omega)$ as

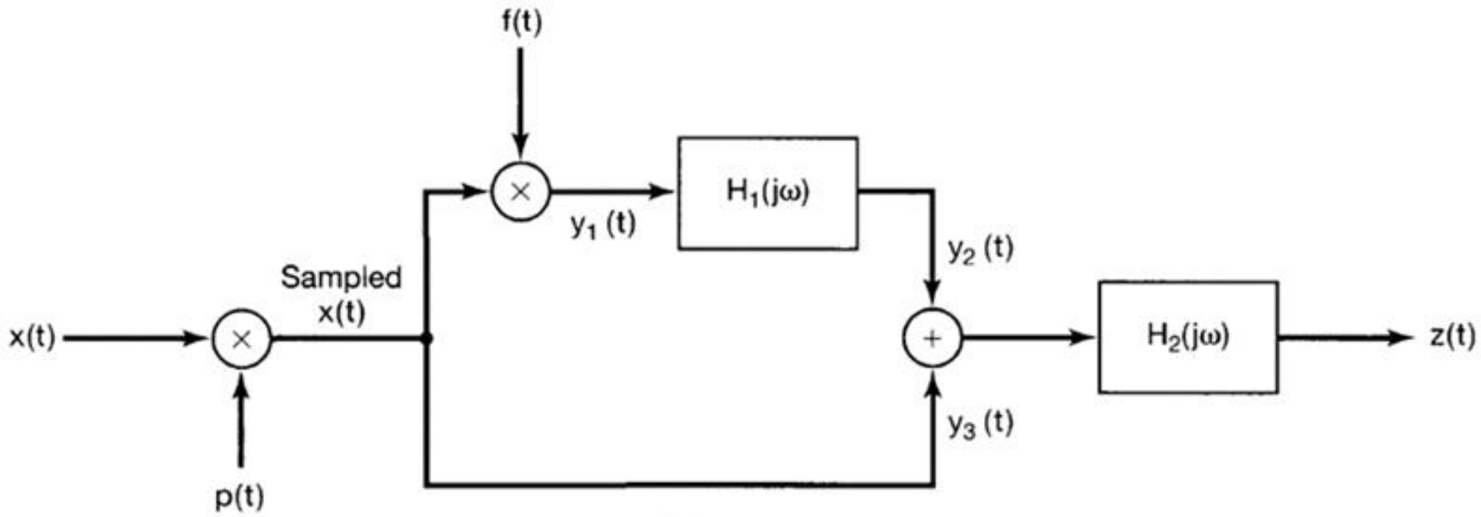
$$Y_2(j\omega) = \frac{jW}{2\pi}\left[(a+b)X(j\omega) + (a+be^{-j\Delta W})X(j(\omega - W))\right].$$

Since $y_3(t) = x(t)p(t)$, in the range $0 < \omega < W$ we may specify $Y_3(j\omega)$ as

$$Y_3(j\omega) = \frac{W}{2\pi}\left[2X(j\omega) + (1+e^{-j\Delta W})X(j(\omega - W))\right].$$

Problem 7.37 (b)

(b) Specify the values of a, b, and K as functions of Δ such that $z(t) = x(t)$ for any band-limited $x(t)$ and any Δ such that $0 < \Delta < \pi/W$.



(a)

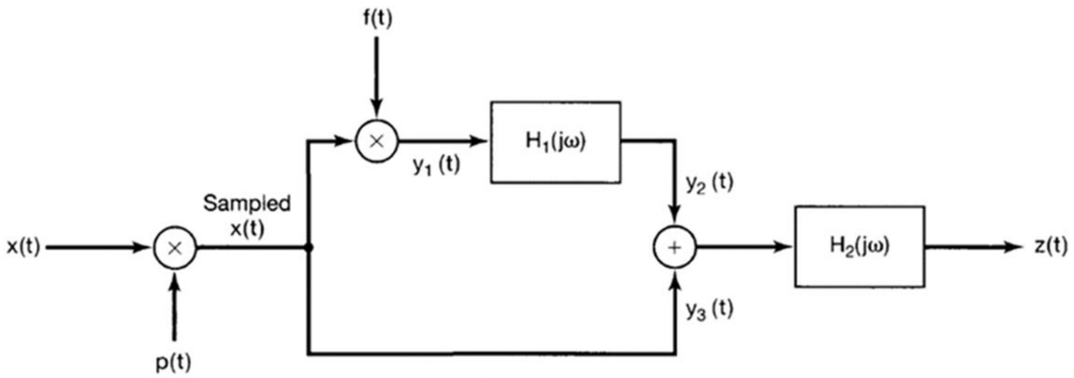
$f(t)$ is a periodic waveform with period $T = 2\pi/W$. Since $f(t)$ multiplies an impulse train, only its values $f(0) = a$ and $f(\Delta) = b$ at $t = 0$ and $t = \Delta$, respectively, are significant.

$H_2(j\omega)$ is an ideal lowpass filter; that is,

$$H_2(j\omega) = \begin{cases} K, & 0 < \omega < W \\ K^*, & -W < \omega < 0 \\ 0, & |\omega| > W \end{cases}$$

where K is a (possibly complex) constant.

Answer 7.37 (b)



(a)

$H_2(j\omega)$ is an ideal lowpass filter; that is,

$$H_2(j\omega) = \begin{cases} K, & 0 < \omega < W \\ K^*, & -W < \omega < 0 \\ 0, & |\omega| > W \end{cases}$$

where K is a (possibly complex) constant.

Given that $0 < W\Delta < \pi$, we require that $Y_2(j\omega) + Y_3(j\omega) = KX(j\omega)$ for $0 < \omega < W$. That is,

$$\frac{W}{2\pi}\left[(2 + ja + jb)X(j\omega)\right] + \frac{W}{2\pi}\left[(1 + e^{-j\Delta W} + ja + jbe^{-j\Delta W})X(j(\omega - W))\right] = KX(j\omega).$$

This implies that

$$1 + e^{-j\Delta W} + ja + jbe^{-j\Delta W} = 0.$$

Answer 7.37 (b)

Given that $0 < W\Delta < \pi$, we require that $Y_2(j\omega) + Y_3(j\omega) = KX(j\omega)$ for $0 < \omega < W$. That is,

$$\frac{W}{2\pi}\left[(2 + ja + jb)X(j\omega)\right] + \frac{W}{2\pi}\left[(1 + e^{-j\Delta W} + ja + jbe^{-j\Delta W})X(j(\omega - W))\right] = KX(j\omega).$$

This implies that

$$1 + e^{-j\Delta W} + ja + jbe^{-j\Delta W} = 0.$$

Solving this we obtain

$$a = 1, \qquad b = -1,$$

when $W\Delta = \pi/2$. More generally, we get

$$a = \sin(W\Delta) + \frac{(1 + \cos(W\Delta))}{\tan(W\Delta)} \quad \text{and} \quad b = -\frac{1 + \cos(W\Delta)}{\sin(W\Delta)}, \quad \text{except when } W\Delta = \pi/2.$$

$$\mathrm{K} = \frac{W}{2\pi}(2 + ja + jb)$$

Answer 7.37 (b)

$$1+e^{-j\Delta w}+ja+jbe^{-j\Delta w}\overset{set}{=}0$$

$$1+\cos(-\Delta w)+j\sin(-\Delta w)+ja+$$

$$jb\cos(-\Delta w)+(-b)\sin(-\Delta w)=0$$

$$\begin{cases}1+\cos(-\Delta w)-b\sin(-\Delta w)=0 & \text{实}\\ \sin(-\Delta w)+a+b\cos(-\Delta w)=0 & \text{虚}\end{cases}$$

$$b=\frac{1+\cos(-\Delta w)}{\sin(-\Delta w)}=-\frac{1+\cos(\Delta w)}{\sin(\Delta w)}$$

$$a=\sin(w\Delta)+\frac{1+\cos(w\Delta)}{\tan(w\Delta)}$$