Notes

Assignments

- **4.5**
- **4.21 (b) (g) (h)**

Review

DT Fourier Series Pair $\left(\omega_o = \dfrac{2\pi}{N}\right)$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} \qquad \text{(Analysis equation)}$$

$\sum_{k=<N>}$ = Sum over *any* N consecutive values of k

$$x[n] = x[n+N]$$

$$a_{k+N} = a_k$$

LTI System, system function and frequency response

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0) a_k e^{jk\omega_0 t}$$

$$a_k \longrightarrow \underbrace{H(jk\omega_0)}_{\text{"gain"}} a_k$$

$$\boxed{H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt}$$

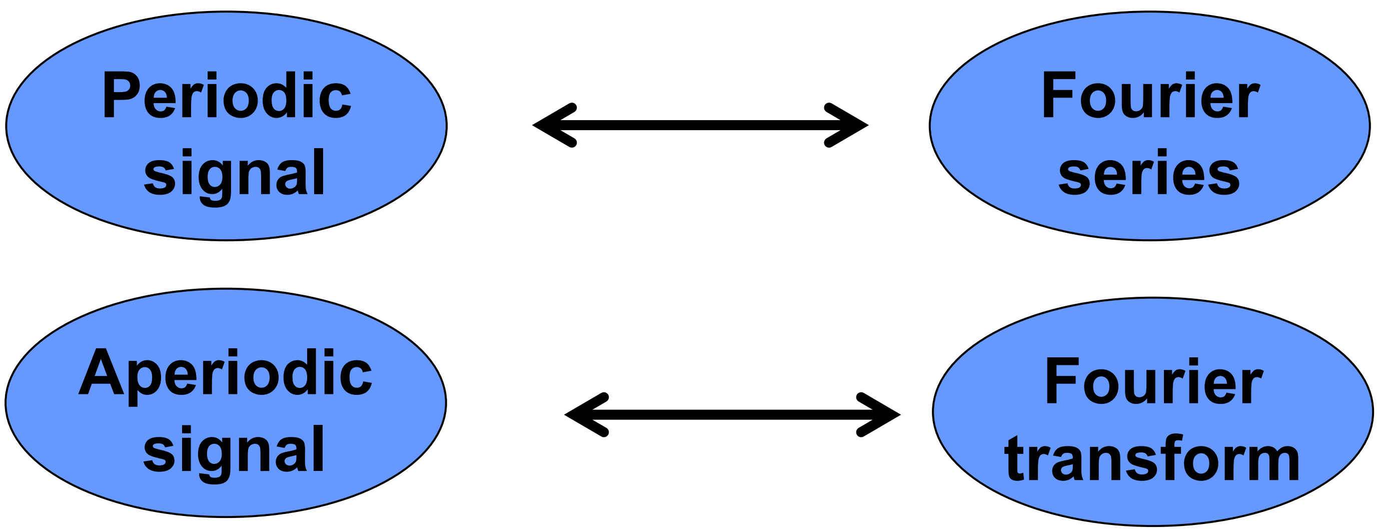
$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} H(e^{jk\omega_o}) a_k e^{jk\omega_0 n}$$

$$a_k \longrightarrow \underbrace{H(e^{jk\omega_o})}_{\text{"gain"}} a_k$$

$$\boxed{H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n}}$$

The effect of the LTI system is to modify each a_k through multiplication by the value of the frequency response at $k\omega_0$.

Chapter 4
The Continuous-Time Fourier Transform

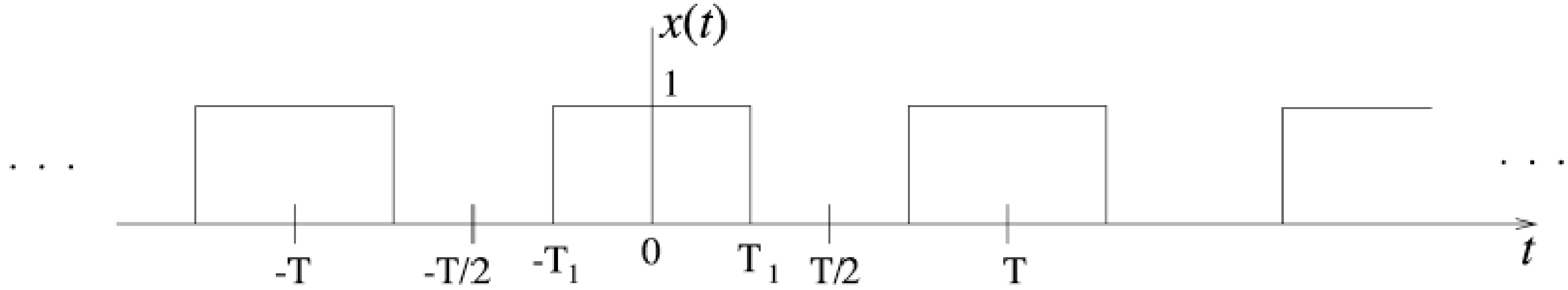


Fourier Transform

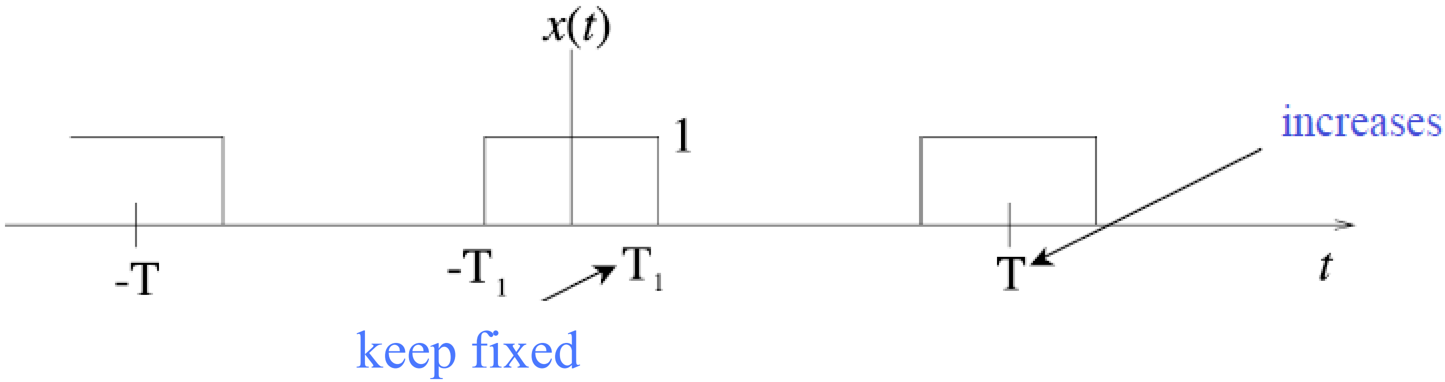
- We have shown that Fourier series are useful in analyzing periodic signals, but many (most) signals are aperiodic. Need a more general tool — *Fourier transform*.

Fourier's own derivation of the CT Fourier transform

- $x(t)$ – an aperiodic signal
  – view it as the limit of a periodic signal as $T \to \infty$



Motivating Examples: Square wave

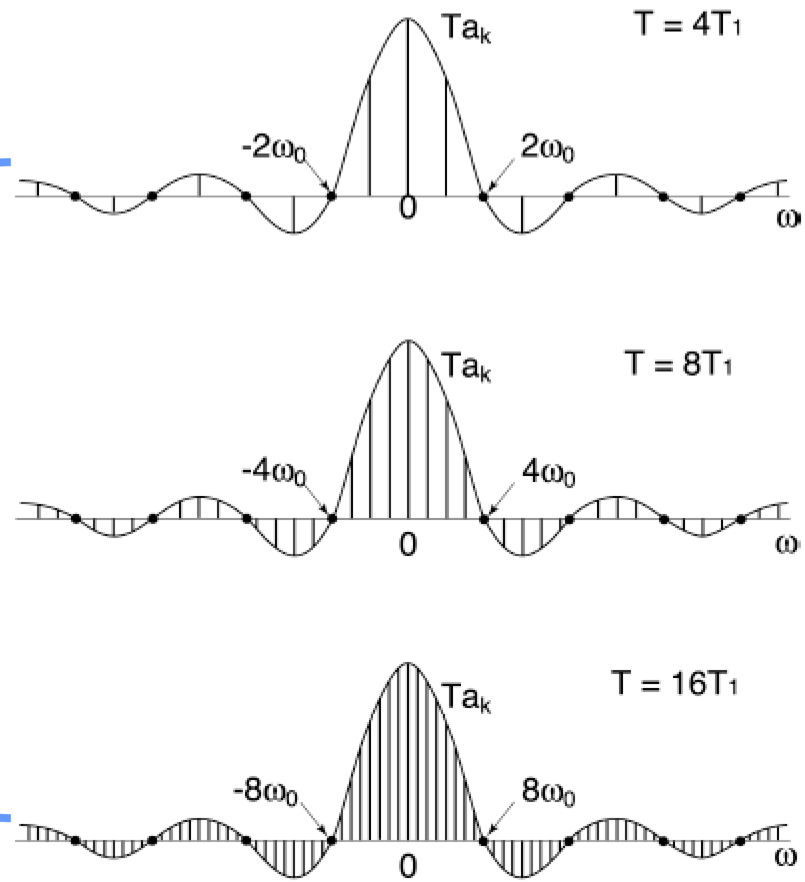


$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

$$Ta_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0}$$

Let $X(\omega) = \frac{2\sin(\omega T_1)}{\omega}$,

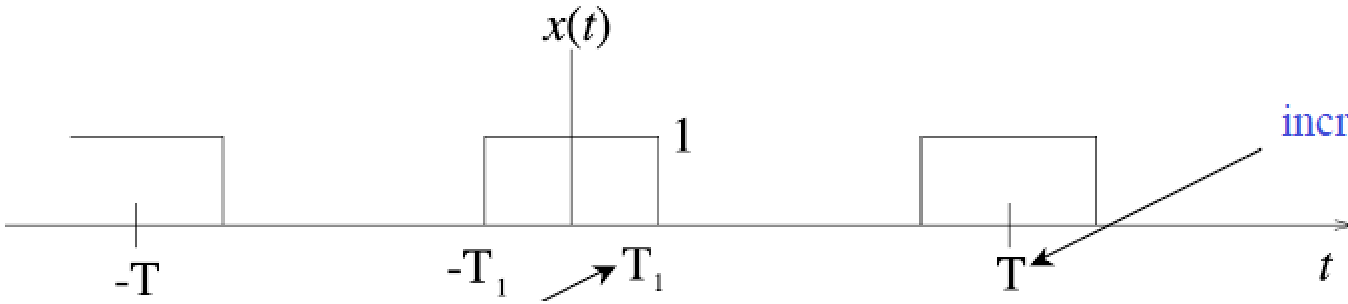
we have $Ta_k = X(k\omega_0)$

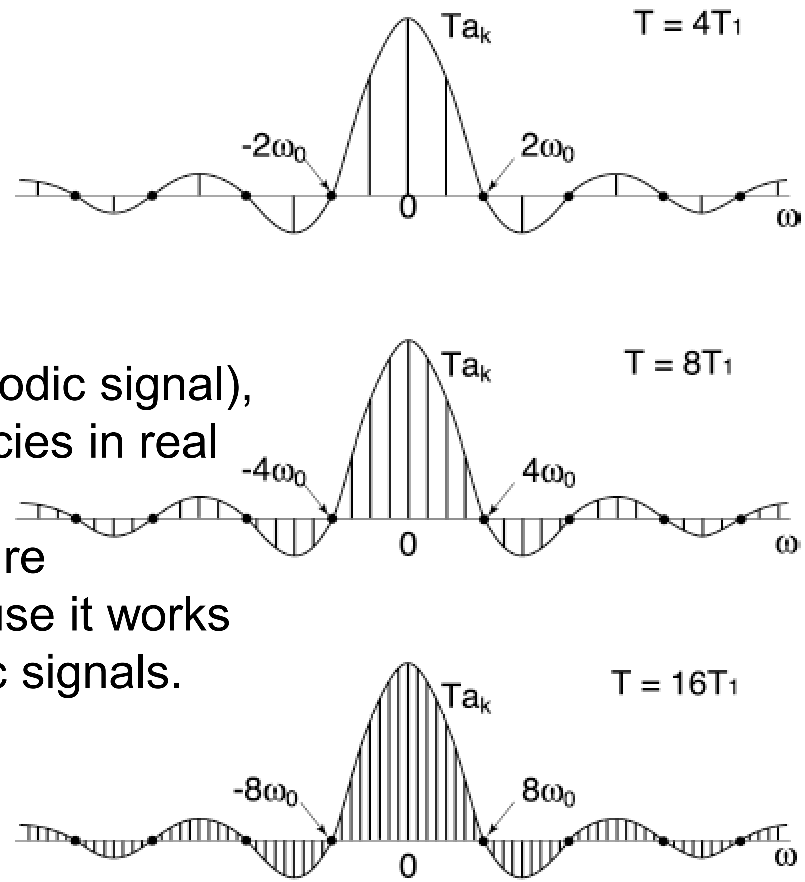


$(\omega_0 = \frac{2\pi}{T})$

Become denser in ω as T increases

Motivating Examples: Square wave





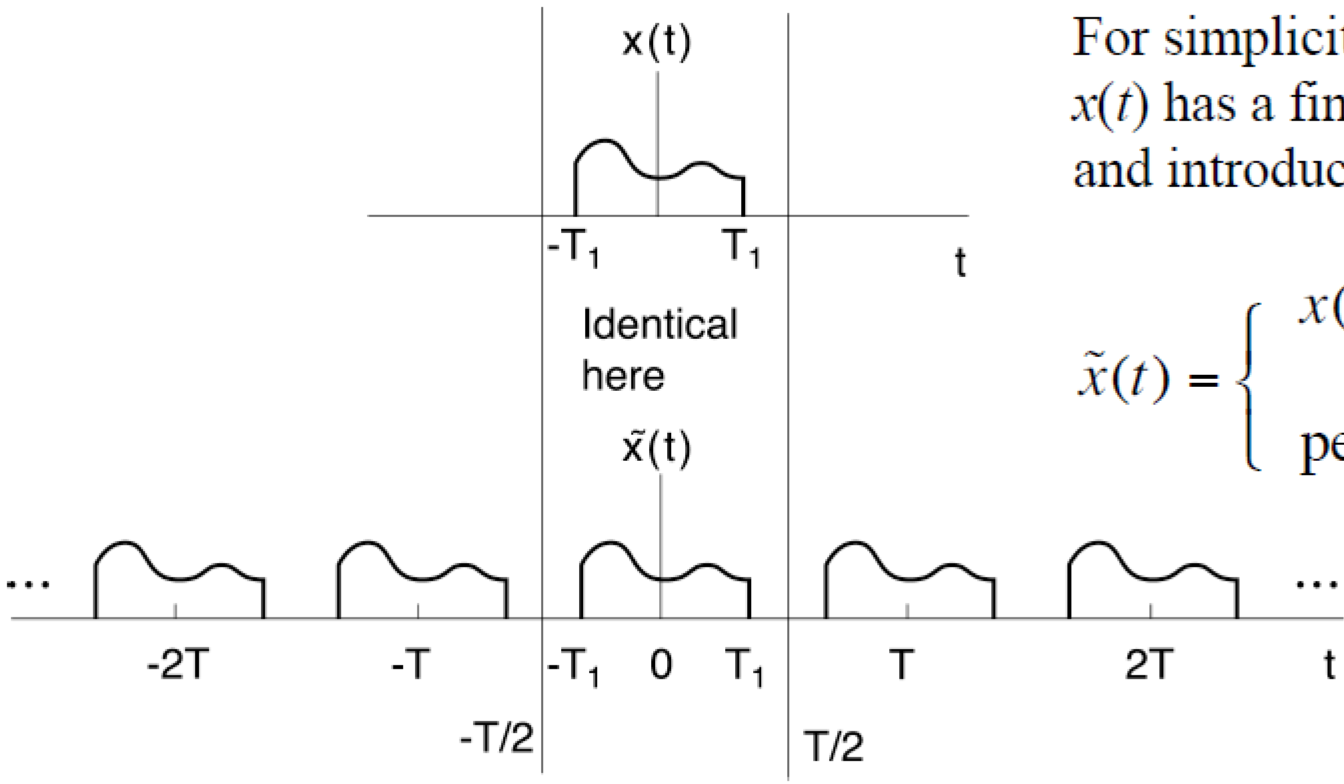
$(\omega_0 = \frac{2\pi}{T})$

Become denser in ω as T increases

Observations:

1. When T tends to infinity (aperiodic signal), x(t) consists of all the frequencies in real line.
2. Ta_k is a good metric to measure frequency components, because it works for both periodic and aperiodic signals.

So, on the derivation of FT ...



For simplicity, assume $x(t)$ has a finite duration, and introduce a periodic $\tilde{x}(t)$

$$\tilde{x}(t) = \begin{cases} x(t) & \frac{-T}{2} < t < \frac{T}{2} \\ \text{periodic} & |t| > \frac{T}{2} \end{cases}$$

$$\text{As } T \rightarrow \infty, \qquad x(t) = \tilde{x}(t) \ \text{ for all } t$$

Derivation (cont.): Analysis equation

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \qquad \omega_o = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt$$

$\uparrow$

$\tilde{x}(t) = x(t)$ in this interval

$$= \frac{1}{T}\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}dt \qquad (1)$$

If we define $\boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt}$

then, Eq. (1) $\Rightarrow$ $T a_k = X(jk\omega_0) = X(j\omega)|_{\omega = k\omega_0}$

Derivation (cont.): Synthesis equation

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \underbrace{\frac{1}{T} X(jk\omega_0)}_{a_k} e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} \omega_0 X(jk\omega_0) e^{jk\omega_0 t}$$

$$\Downarrow$$

As $T \to \infty$, $\omega_o \to 0$, $\sum \omega_o \to \int d\omega$, and $k\omega_o = \omega$, we get the CT FT pair

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \quad \text{Synthesis equation}$$

— "sum" of $e^{j\omega t}$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \text{Analysis equation}$$

The CT Fourier Transform Pair

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \qquad \text{— } FT$$

Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \qquad \text{— Inverse } FT$$

Inverse Fourier Transform

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\mathcal{F}(x(t)) = X(j\omega)$$

$$x(t) = \mathcal{F}^{-1}(X(j\omega))$$

Comparison with CT Fourier Series

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Harmonically related

- **Frequency components of periodic signals: $k\omega_0$**
- **Frequency components of aperiodic signals: all the real frequencies**
- **Observation: the spectra of periodic signals are discrete, but the spectra of aperiodic signals are continuous**

For what kinds of signals can we do FT?

It works also even if $x(t)$ is infinite duration, but satisfies:

a) Finite energy $\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$

In this case, there is *zero* energy in the error

$$e(t) = x(t) - \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega \quad \text{Then} \quad \int_{-\infty}^{\infty} |\, e(t)\,|^2\, dt = 0$$

b) Dirichlet conditions

1) absolutely integrable

2) finite number of maxima and minima within any finite interval

3) finite number of discontinuities with finite values within any finite interval

Example 4.3 Impulse function

(a) $$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = \quad ?$$

$$\Downarrow$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\omega t} d\omega$$ — Synthesis equation for $\delta(t)$

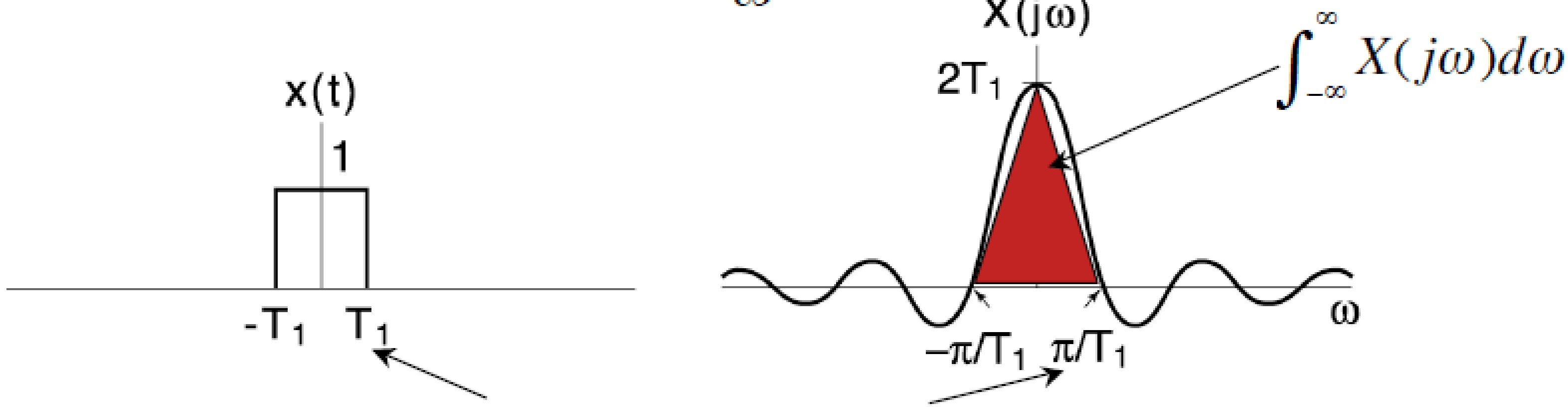
(b) $$x(t) = \delta(t - t_0)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t - t_0) e^{-j\omega t} dt$$

$$= e^{-j\omega t_o}$$ — Linear phase shift in ω

Example 4.4 A square pulse in the time-domain

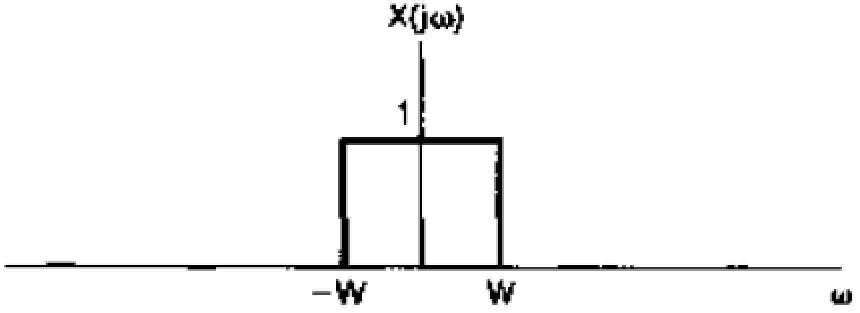
$$X(j\omega)=\int_{-T_1}^{T_1} e^{-j\omega t}dt = \frac{2\sin\omega T_1}{\omega}$$



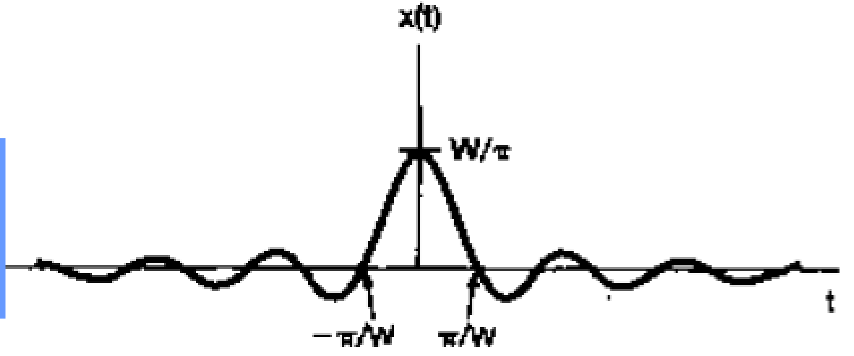
Note the inverse relation between the two widths ⇒ Uncertainty principle

Example 4.5 A square pulse in the frequency domain

$$X(j\omega)=\begin{cases}1, & |\omega|<W\\ 0, & |\omega|>W\end{cases}$$



$$x(t)=\frac{1}{2\pi}\int_{-W}^{W}e^{j\omega t}d\omega=\frac{\sin(Wt)}{\pi t}$$



How about $X(j\omega)=\delta(\omega)$?

CT Fourier Transforms of Periodic Signals

Suppose $$X(j\omega) = \delta(\omega - \omega_0)$$

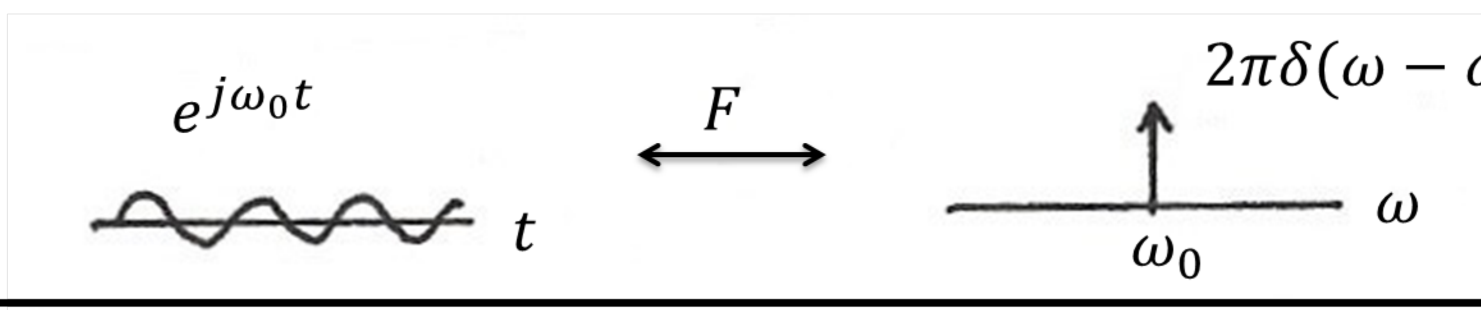
$$\Downarrow$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

— periodic in t with frequency ω_o

That is $$\boxed{e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega - \omega_0)}$$

— All the energy is concentrated in one frequency — ω_o



Properties of the CT Fourier Transform

1) Linearity $x(t) \xleftrightarrow{F} X(j\omega),\ y(t) \xleftrightarrow{F} Y(j\omega)$

$$ax(t) + by(t) \xleftrightarrow{F} aX(j\omega) + bY(j\omega)$$

2) Time Shifting $x(t - t_0) \longleftrightarrow e^{-j\omega t_o} X(j\omega)$

$$\boxed{\text{Proof: } \int_{-\infty}^{\infty} x(\underbrace{t - t_o}_{t'}) e^{-j\omega t} dt = e^{-j\omega t_o} \underbrace{\int_{-\infty}^{\infty} x(t') e^{-j\omega t'} dt'}_{X(j\omega)}}$$

FT magnitude unchanged

$$\left| e^{-j\omega_0 t} X(j\omega) \right| = \left| X(j\omega) \right|$$

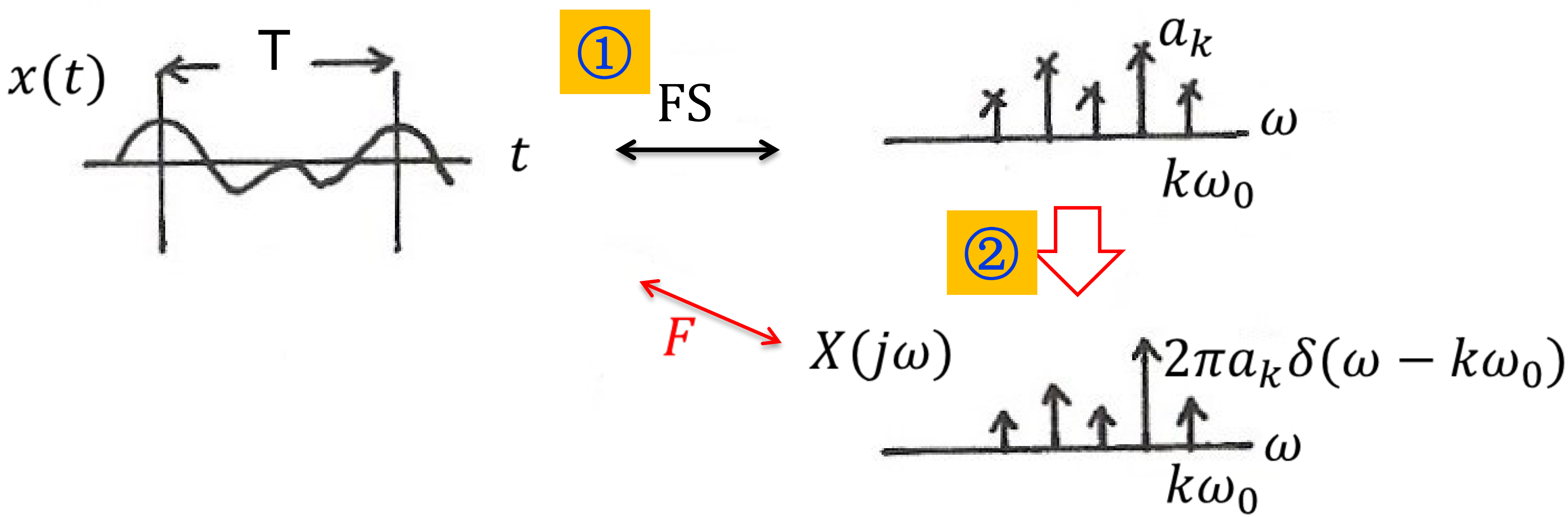
Linear change in *FT* phase

$$\angle(e^{-j\omega_0 t} X(j\omega)) = \angle X(j\omega) - \omega t_0$$

Fourier Transform for Periodic Signals – Unified Framework

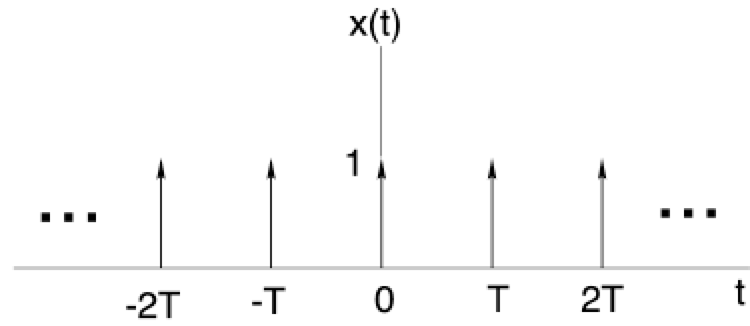
More generally, if $x(t) = x(t+T)$, then

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \longleftrightarrow X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)$$ Discrete spectra



Example 4.8

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \quad \text{— sampling function}$$

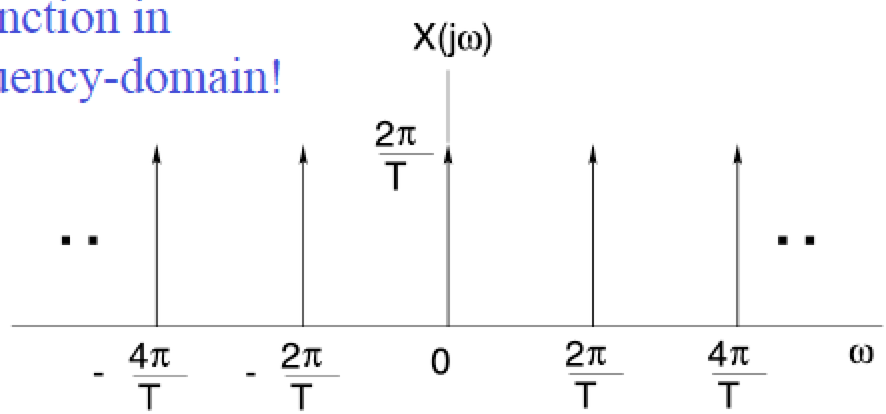


$$x(t) \overset{FS}{\longleftrightarrow} a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_o t}\,dt = \frac{1}{T}$$

$$\Downarrow\ x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} = \sum_{k=-\infty}^{+\infty} \frac{1}{T} e^{jk\omega_o t}$$

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T}\delta(\omega - \underbrace{\frac{k2\pi}{T}}_{k\omega_o})$$

Same function in the frequency-domain!



Note in this case, periodic in both time domain (with a period T) and frequency domain (with a period $2\pi/T$)

CTFT Properties (cont.)

3) Conjugation & Conjugate Symmetry

- **Conjugation** $x^*(t) \overset{F}{\longleftrightarrow} X^*(-j\omega)$

- **Conjugate Symmetry** $x(t)$ real $\longleftrightarrow X(-j\omega) = X^*(j\omega)$

$|X(-j\omega)| = |X(j\omega)|$ *Even* Or $\mathrm{Re}\{X(-j\omega)\} = \mathrm{Re}\{X(j\omega)\}$ *Even*

$\angle X(-j\omega) = -\angle X(j\omega)$ *Odd* $\mathrm{Im}\{X(-j\omega)\} = -\mathrm{Im}\{X(j\omega)\}$ *Odd*

When $x(t)$ is real (all the physically measurable signals are *real*), the negative frequency components do *not* carry any additional information from the positive frequency components. $\omega \geq 0$ will be sufficient.

CT Fourier Series Property

- Conjugate Symmetry

$$x(t) \quad real \quad \Rightarrow \quad a_{-k} = a_k{}^*$$

Proof: $$a_{-k} = \frac{1}{T}\int_T x(t)e^{jk\omega_0 t}dt = \left[\frac{1}{T}\int_T x^*(t)e^{-jk\omega_0 t}dt\right]^* = a_k^*$$

$\because$

$$a_k = \mathrm{Re}\{a_k\} + j\,\mathrm{Im}\{a_k\}$$

$\therefore$

$$\boxed{\mathrm{Re}\{a_{-k}\} + j\,\mathrm{Im}\{a_{-k}\}} = \boxed{\mathrm{Re}\{a_k\} - j\,\mathrm{Im}\{a_k\}}$$

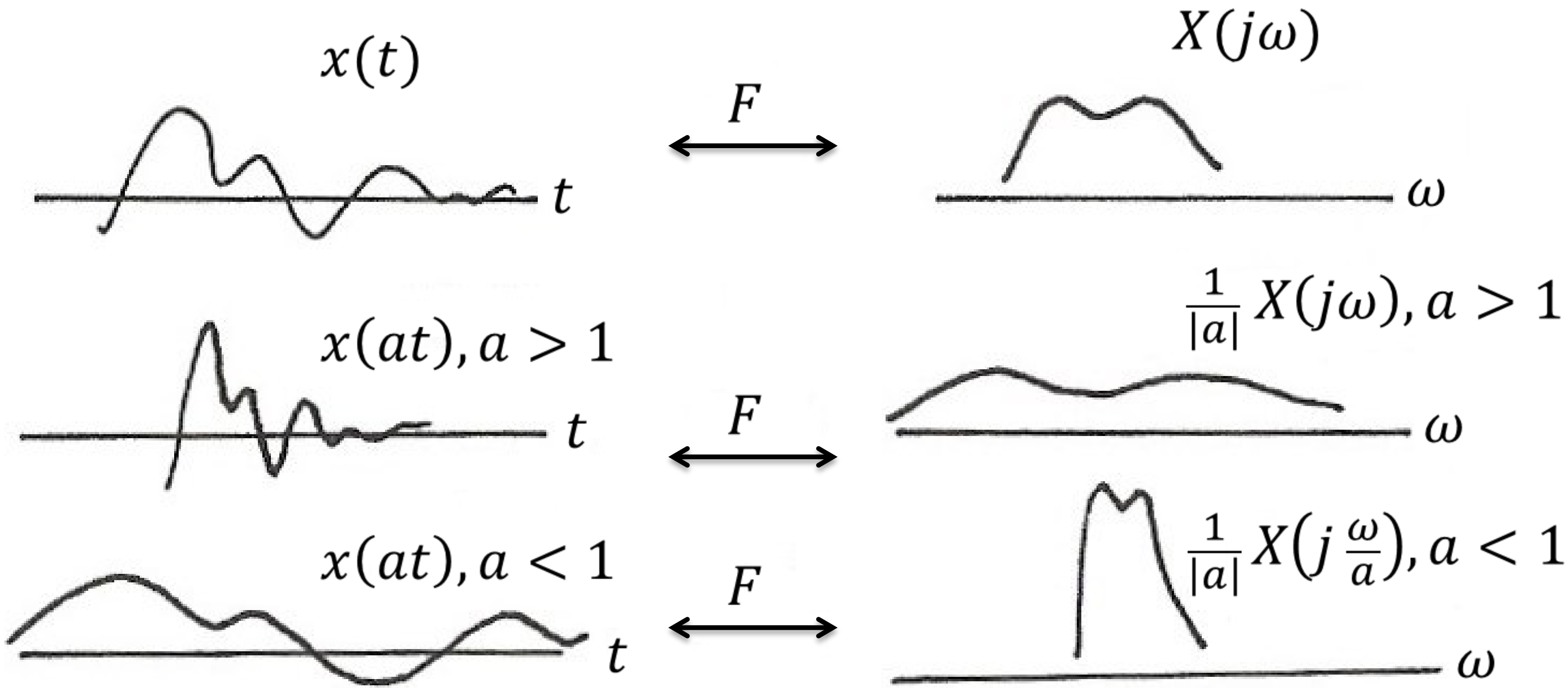
$\therefore$

$\mathrm{Re}\{a_k\}$ is even, $\mathrm{Im}\{a_k\}$ is odd

CTFT Properties (cont.)

$\boxed{\text{4) Time/Frequency Scaling}}$ $x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$

$E.g.\ a > 1 \rightarrow at > t$
compressed in time $\leftrightarrow$
stretched in frequency



CTFT Properties (cont.)

$\boxed{\text{4) Time/Frequency Scaling}}$ $x(at) \longleftrightarrow \dfrac{1}{|a|} X\left(j\dfrac{\omega}{a}\right)$

E.g. $a > 1 \rightarrow at > t$
compressed in time $\leftrightarrow$
stretched in frequency

$$\Downarrow a = -1$$

$$x(-t) \longleftrightarrow X(-j\omega)$$

$\boxed{\text{Time reversal}}$

$$\Downarrow$$

a) $x(t)$ real and even $\quad x(t) = x(-t) = x^*(t)$

$$\Rightarrow X(j\omega) = X(-j\omega) = X^*(-j\omega) \text{ — Real \& even}$$

b) $x(t)$ real and odd $\quad x(t) = -x(-t) = x^*(t)$

$$\Rightarrow X(j\omega) = -X(-j\omega) = X^*(-j\omega) \text{ — Purely imaginary \& odd}$$

c) $X(j\omega) = \text{Re}\{X(j\omega)\} + j\,\text{Im}\{X(j\omega)\}$

For real $x(t) = Ev\{x(t)\} + Od\{x(t)\}$

CTFT Properties (cont.)

5) Differentiation/Integration

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(j\omega) + \pi X(j0)\delta(\omega)$$

DC term

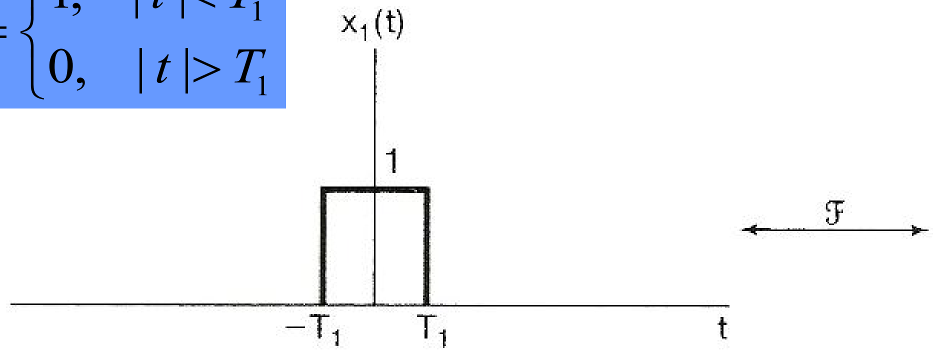
Example:

What is the Fourier transform for unit step function $u(t)$?

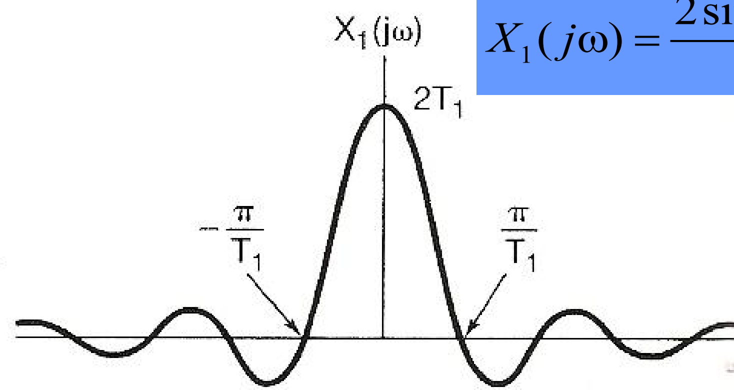
CTFT Properties (cont.)

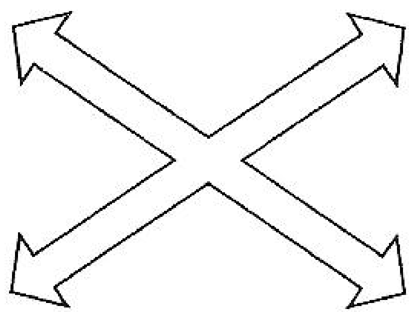
6) Duality

$$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$

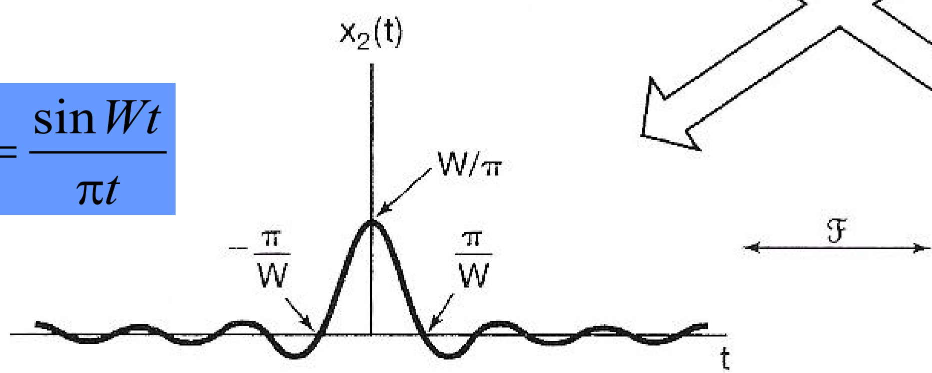


$$X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}$$

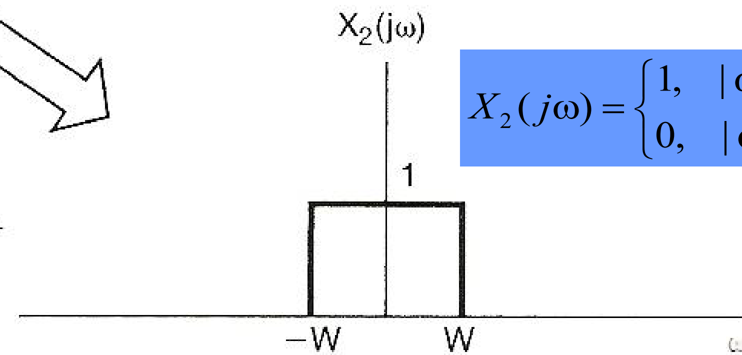




$$x_2(t) = \frac{\sin Wt}{\pi t}$$



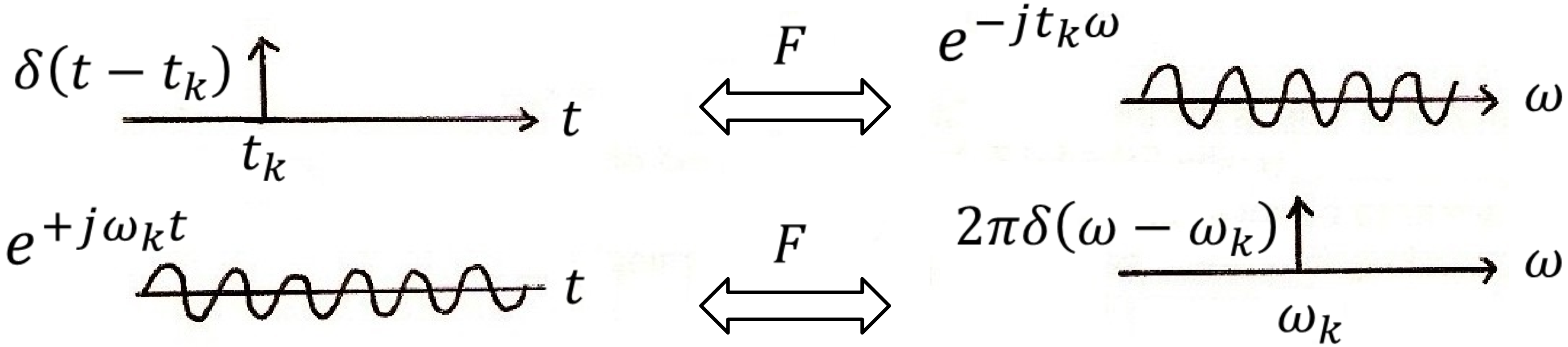
$$X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$



- Time/frequency domains are kind of "symmetric".
- If there are characteristics of a function of time that have implications with regard to the Fourier transform, then the same characteristics associated with a function of frequency will have ***dual*** implications in the time domain.

<u>Example:</u>

$\{\delta(t-t_k), -\infty < t_k < \infty\}$ $\qquad$ $\{2\pi\delta(\omega-\omega_k), -\infty < \omega_k < \infty\}$



CTFT Properties (cont.)

$\boxed{\text{7) Parseval's Relation}}$

$$\underbrace{\int_{-\infty}^{+\infty} |x(t)|^2 dt}_{\text{Total energy in the time-domain}} = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega}_{\text{Total energy in the frequency-domain}}$$

$\boxed{\frac{1}{2\pi}|X(j\omega)|^2 \text{ — spectral density}}$

Table 4.2
Basic Fourier Transform Pairs

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1$, $a_k = 0$, $k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Summary

- **Understand CT Fourier transform**
  - **Synthesis and analysis equations**
  - **Difference with CT Fourier series**
  - **Fourier transform for periodic signal**
  - **Properties of CT Fourier transform**