Notes

- Assignment
  - 2.10
  - 2.11
  - 2.22 (b) (e)
  - 2.25
  - 2.28 (a) (c) (e) (g)

- Tutorial questions this week (Week 4)
  - Basic Problems with Answers 2.20
  - Basic Problems 2.29
  - Advanced Problems 2.40, 2.43, 2.47

That is ...

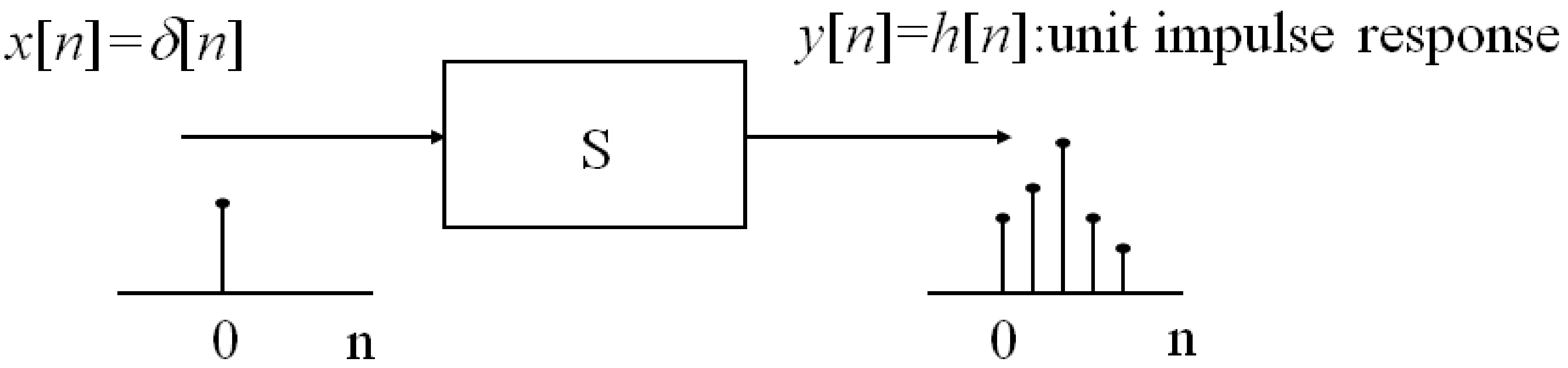
$$x[n] = \ldots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \ldots$$

$$\Downarrow$$

$$x[n] = \sum_{k=-\infty}^{+\infty} \underbrace{x[k]}_{\text{Coefficients}} \underbrace{\delta[n-k]}_{\text{Basic Signals}}$$

Important to note the "-" sign

Unit Impulse Response



Response of DT LTI Systems

$$x[n] \longrightarrow \boxed{\text{DT System}} \longrightarrow y[n]$$

- Now suppose the system is **LTI**, and define the *unit impulse response* $h[n]$: $\quad \delta[n] \longrightarrow h[n]$

$$\Downarrow$$

From Time-Invariance:

$$\delta[n-k] \longrightarrow h[n-k]$$

From Linearity:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k] \longrightarrow y[n] = \underbrace{\sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]}_{\textit{convolution sum}} = x[n] * h[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

input signal

Contribution to the output signal at time n

flipped version of $h[k]$ located at $k = n$

Convolution operation procedure:

$$h[k] \xrightarrow{\text{Flip}} h[-k] \xrightarrow{\text{Slide}} h[n-k] \xrightarrow{\text{Multiply}} x[k]h[n-k]$$

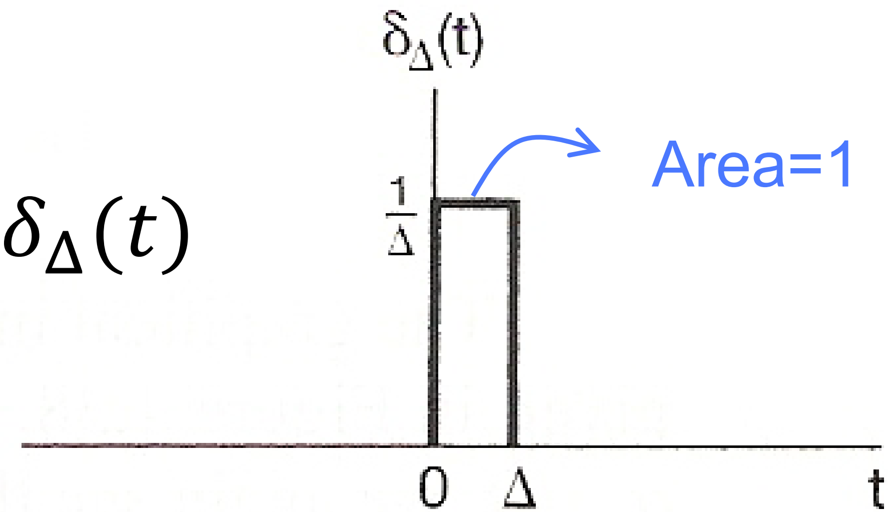
F S M S

$$\xrightarrow{\text{Sum}} \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Construction of the Unit-impulse function δ(*t*)

One of the simplest way — rectangular pulse, taking the limit $\Delta \rightarrow 0$.

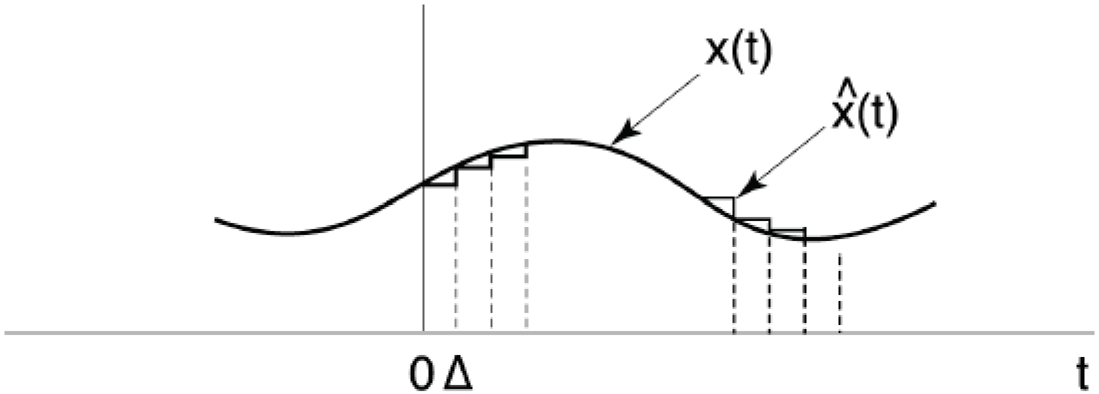
$$\delta(t) = \lim_{\Delta\rightarrow 0} \delta_\Delta(t)$$



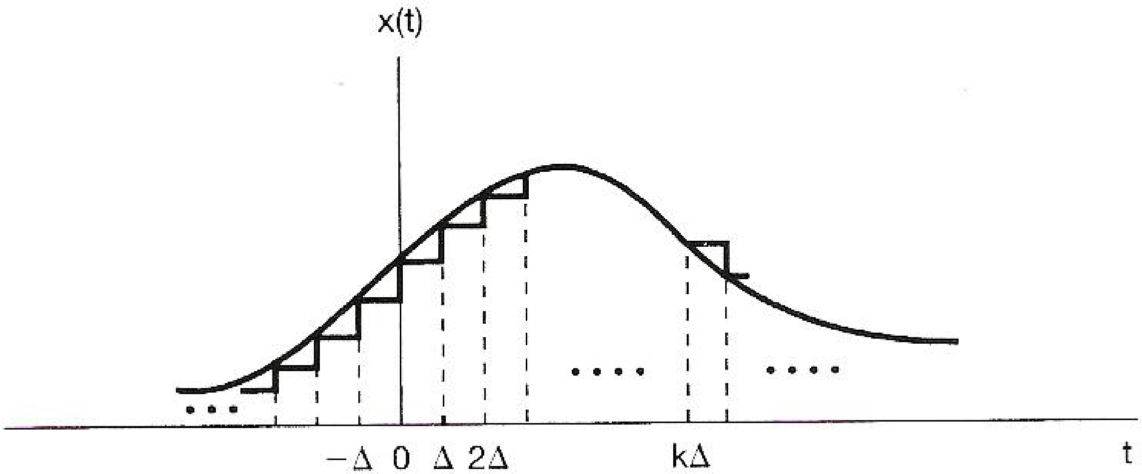
But this is by no means the only way. One can construct a δ(*t*) function out of many other functions, *Eg*. Gaussian pulses, triangular pulses, sinc functions, *etc*., as long as the pulses are short enough — much shorter than the characteristic time scale of the system.

Representation of CT Signals

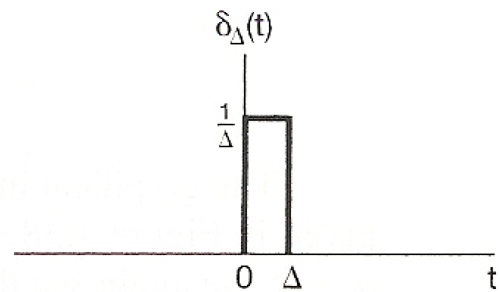
- Approximate any input $x(t)$ as a sum of shifted, scaled pulses (in fact, that is how we do integration)

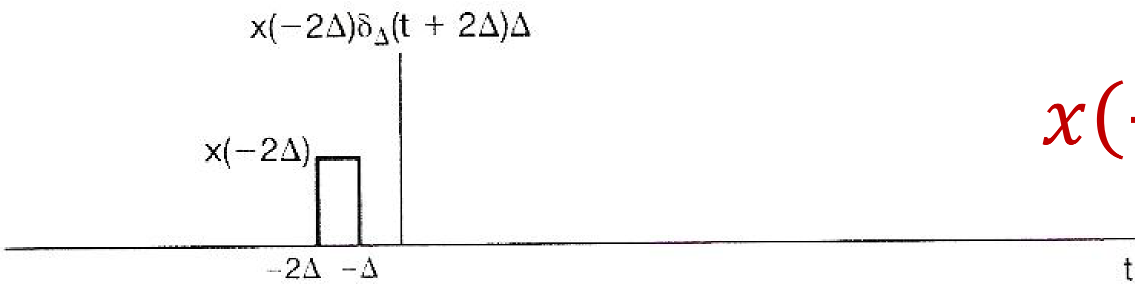


$$x(t) = \lim_{\Delta \to 0} \hat{x}(t)$$



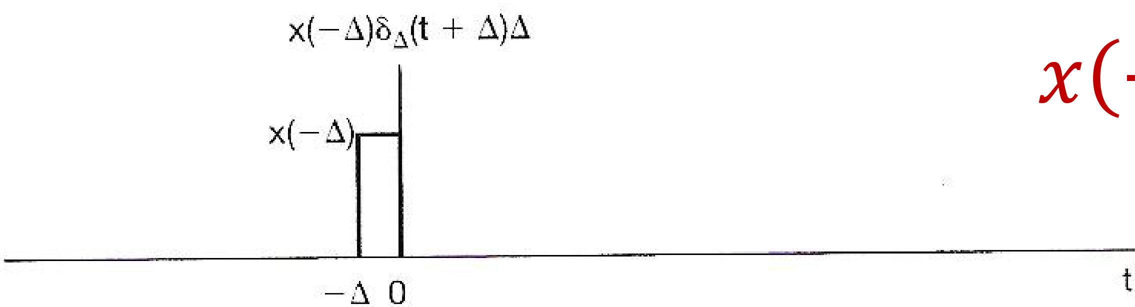
(a)





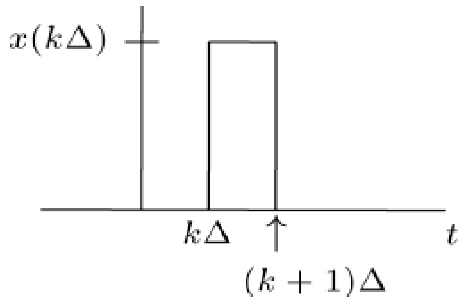
(b)

$$x(-2\Delta)\delta_{\Delta}(t+2\Delta)\Delta$$



$$x(-\Delta)\delta_{\Delta}(t+\Delta)\Delta$$

Representation of CT Signals (cont.)



$$= \quad x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$\Downarrow$$

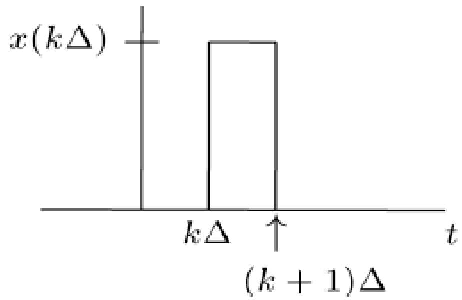
$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$\downarrow \quad \text{limit as } \Delta \to 0$$

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau}$$

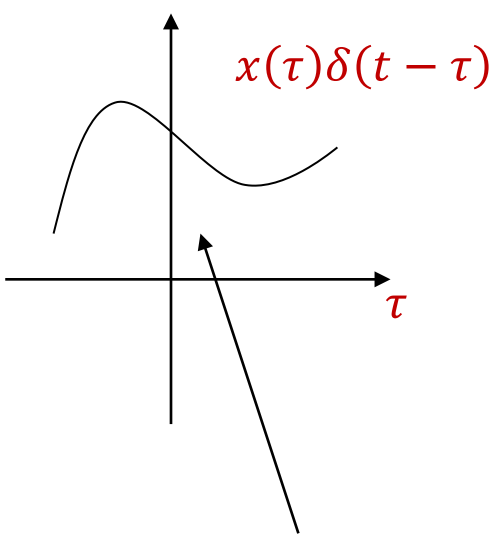
Sifting property of the unit impulse

Representation of CT Signals (cont.)



$$= \quad x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$\Downarrow$$



$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta \rightarrow \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta(t-k\Delta)\Delta$$

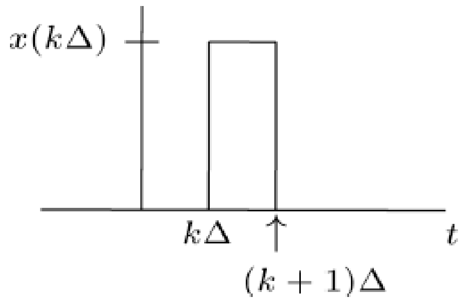
$$\downarrow \quad \text{limit as } \Delta \rightarrow 0$$

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau}$$

Sifting property of the unit impulse

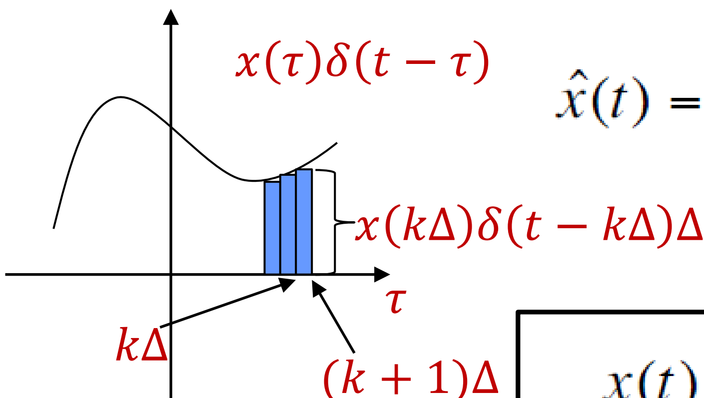
Area of $x(\tau)\delta(t-\tau)$

Representation of CT Signals (cont.)



$$= \quad x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$\Downarrow$$



$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta \;\rightarrow\; \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta(t-k\Delta)\Delta$$

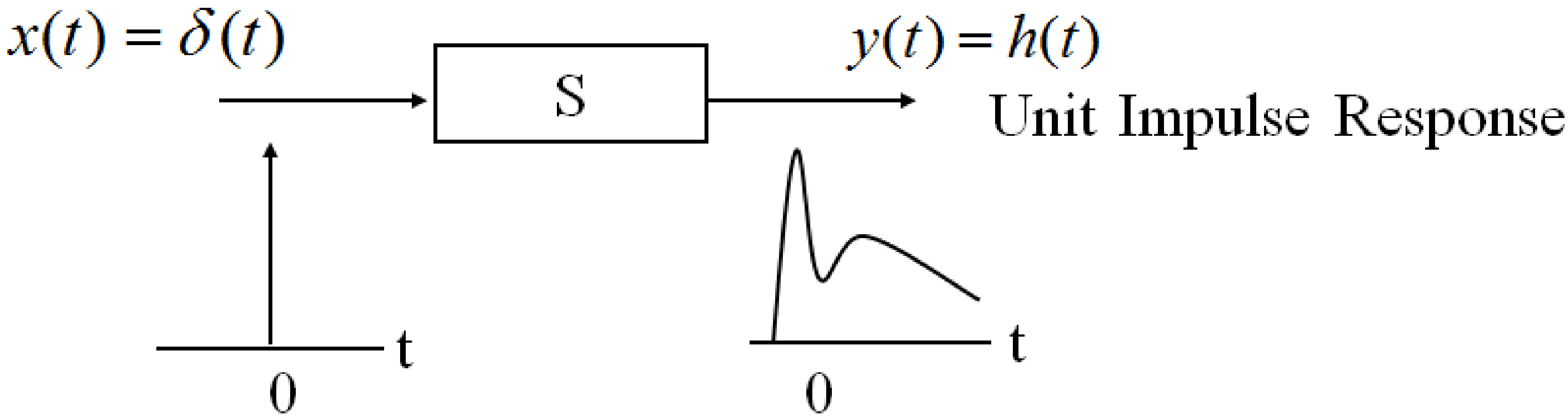
$$\downarrow \quad \text{limit as } \Delta \rightarrow 0$$

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau}$$

Sifting property of the unit impulse

$$\text{Area of } x(\tau)\delta(t-\tau) = \lim_{\Delta\rightarrow 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta(t-k\Delta)\Delta$$

Unit Impulse Response



Response of a CT LTI System

$$x(t) \longrightarrow \boxed{\text{CT LTI}} \longrightarrow y(t)$$

- Now suppose the system is **LTI**, and define the *unit impulse response* $h(t)$:

$$\delta(t) \longrightarrow h(t)$$

$$\Downarrow$$

From Time-Invariance:

$$\delta(t-\tau) \longrightarrow h(t-\tau)$$

From Linearity:

$$x(t) = \int_{-\infty}^{+\infty} \boxed{x(\tau)\delta(t-\tau)}\, d\tau \longrightarrow y(t) = \underbrace{\int_{-\infty}^{+\infty} \boxed{x(\tau)h(t-\tau)}\, d\tau}_{\textit{Convolution Integration}} = x(t) * h(t)$$

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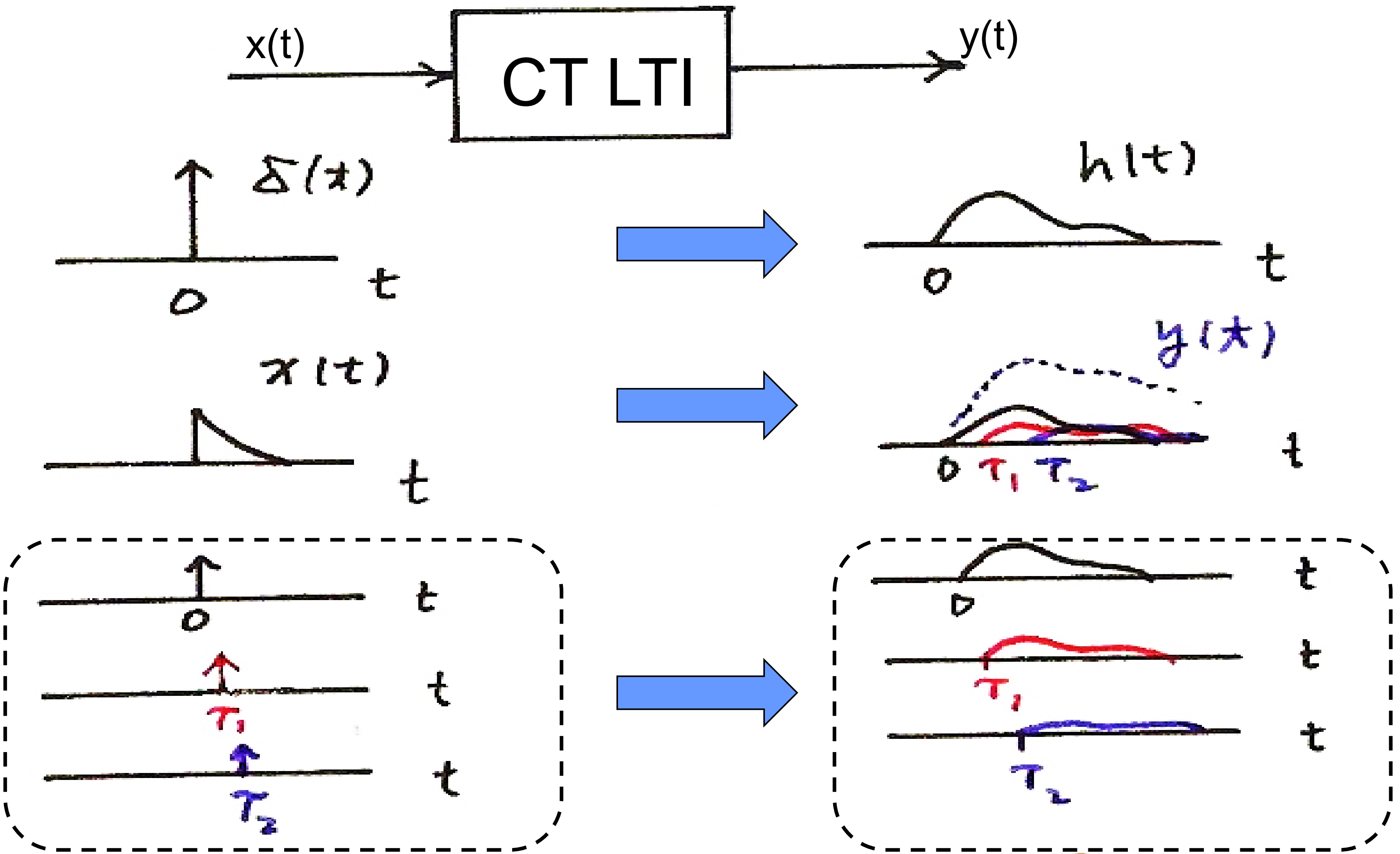
Response of a CT LTI System



Summary

Input: $x[n]=\sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$ ➔ $x(t)=\int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$

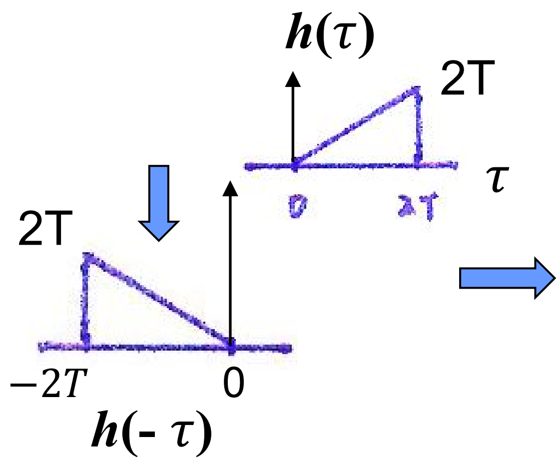
LTI: $\delta[n]\rightarrow h[n]$ ➔ $\delta(t)\rightarrow h(t)$

Output: $y[n]=\sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ ➔ $y(t)=\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$

$=x[n]*h[n]$ ➔ $=x(t)*h(t)$

$$h(\tau)\xrightarrow{Flip} h(-\tau)\xrightarrow{Slide} h(t-\tau)\xrightarrow{Multiply}$$

$$x(\tau)h(t-\tau)\xrightarrow{Integrate}\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$



$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

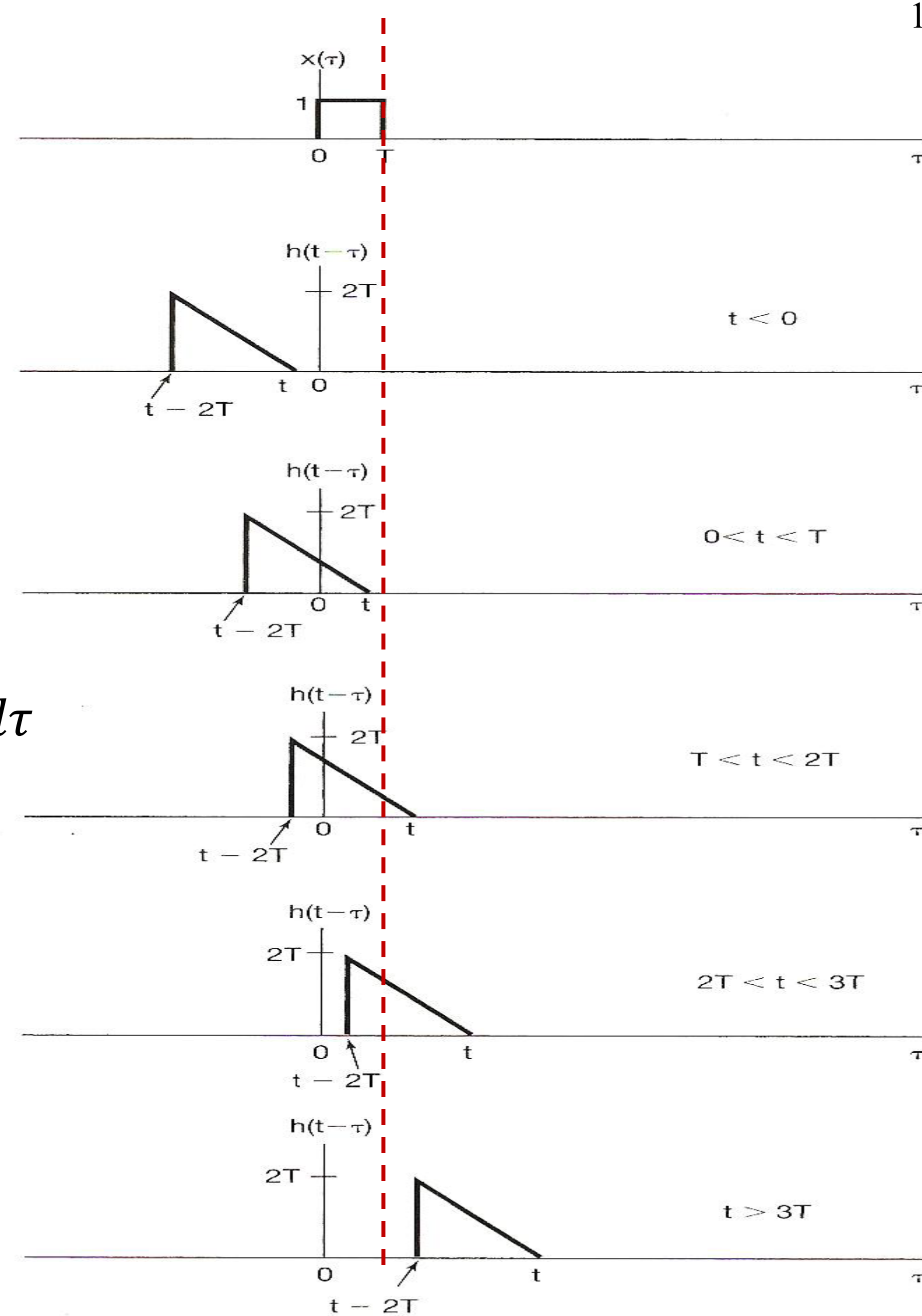


Figure 2.19 Signals $x(\tau)$ and $h(t-\tau)$ for different values of t for Example 2.7.

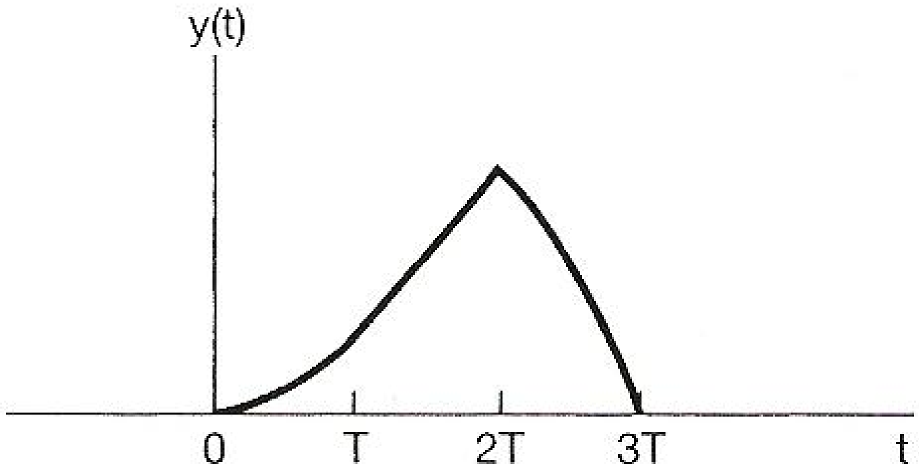


Figure 2.21 Signal $y(t) = x(t) * h(t)$ for Example 2.7.

Flip, slide, multiply, and integrate

One Important Convolution

$$x(t) * \delta(t-a)$$

$$= \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau-a)d\tau$$

$$= \int_{-\infty}^{+\infty} x(t-a)\delta(t-\tau-a)d\tau$$

$$= x(t-a)\int_{-\infty}^{+\infty} \delta(t-\tau-a)d\tau$$

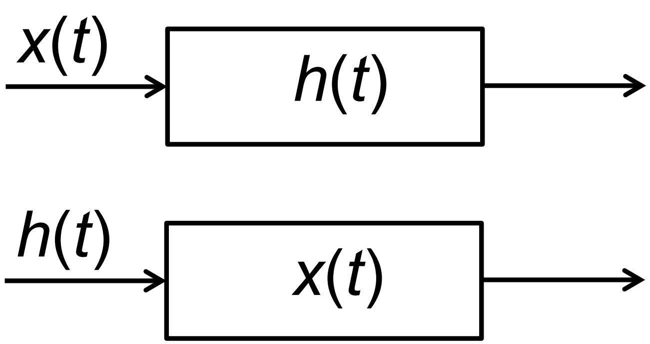
$$= x(t-a)$$

Similarly, $x[n] * \delta[n-k] = x[n-k]$

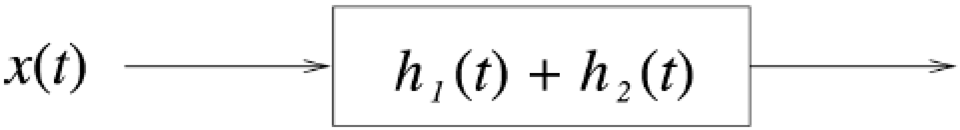
Property: Commutative

$$x(t) * h(t) = h(t) * x(t)$$

- The role of input signal and unit impulse response is interchangeable, giving the same output signal

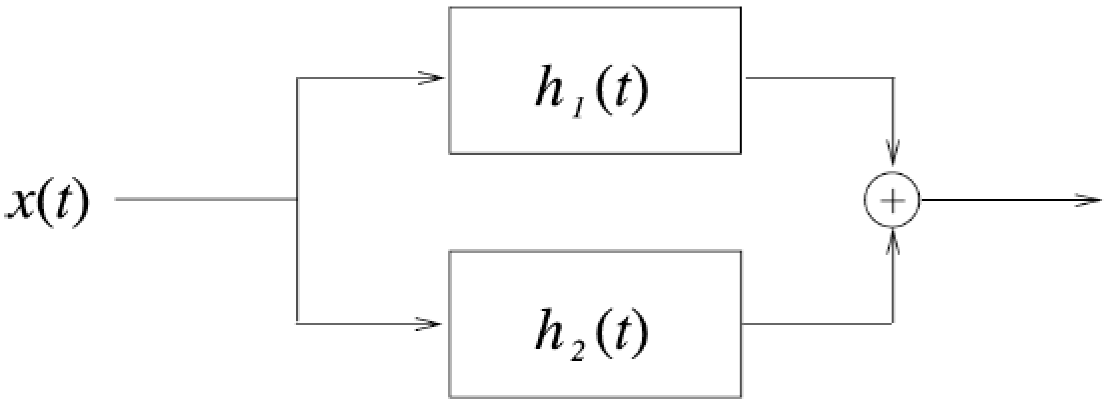


Property: Distributive



$$y(t) = x(t) * [h_1(t) + h_2(t)]$$

||



$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

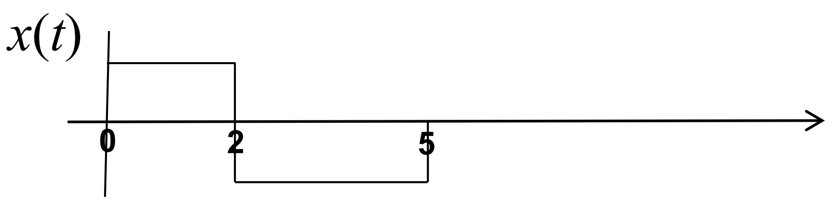
Property: Distributive (Cont.)

$$[x_1(t)+x_2(t)]*h(t)=x_1(t)*h(t)+x_2(t)*h(t)$$

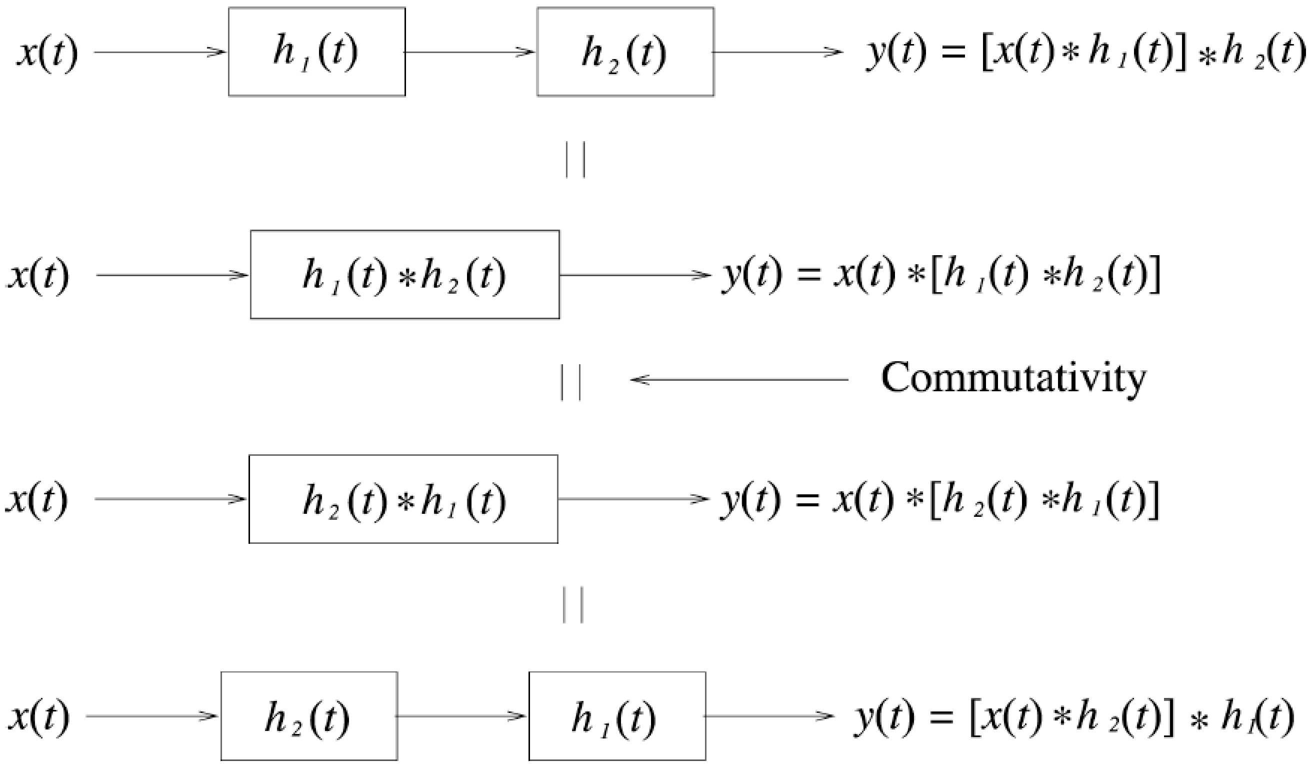
Problem 2.22 (b)

$$x(t)=u(t)-2u(t-2)+u(t-5)$$

$$h(t)=e^{2t}u(1-t)$$

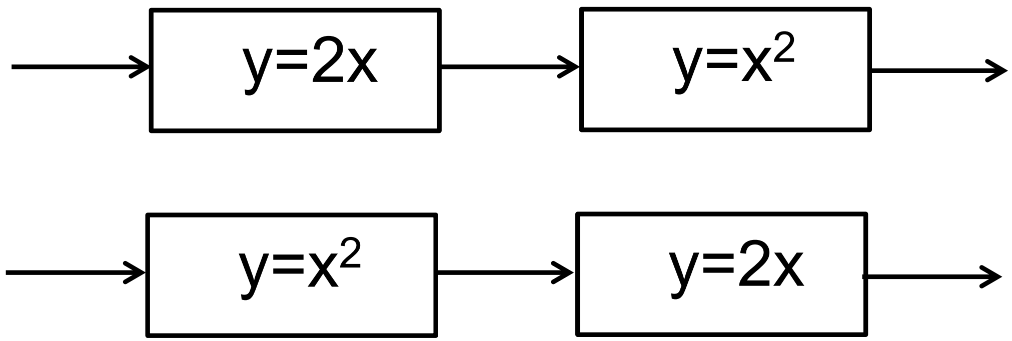


Properties: Associative



Properties: Associative (Cont.)

- The order in which non-linear systems are cascaded cannot be changed.
- e.g.



Property: Memory/Memoryless

– A linear, time-invariant, causal system is memoryless only

if $h[n] = K\delta[n]$ $\quad h(t) = K\delta(t)$

$y[n] = Kx[n]$ $\quad y(t) = Kx(t)$

if k=1 further, they are identity systems

$$y[n] = x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n] * \delta[n]$$

$$y(t) = x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t) * \delta(t)$$

Property: Invertibility

2.50. Consider the cascade of two systems shown in Figure P2.50. The first system, A, is known to be LTI. The second system, B, is known to be the inverse of system A. Let $y_1(t)$ denote the response of system A to $x_1(t)$, and let $y_2(t)$ denote the response of system A to $x_2(t)$.

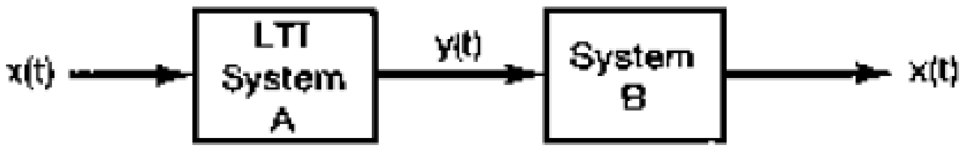


Figure P2.50

(a) What is the response of system B to the input $ay_1(t) + by_2(t)$, where a and b are constants?

(b) What is the response of system B to the input $y_1(t - \tau)$?

Property: Causality

Causality: CT LTI system is causal $\leftrightarrow$ $h(t) = 0$, at $t < 0$

- This is because that the input unit impulse function δ(t)=0 at t<0

$t - \tau \geq 0$, or $\tau \leq t$

As a result:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

$y(t)$ only depends on $x(\tau < t)$.

Property: Stability

BIBO Stability: CT LTI system is stable $\Leftrightarrow \int_{-\infty}^{+\infty} |h(\tau)|\, d\tau < \infty$

Sufficient condition: For $|x(t)| \le x_{\max} < \infty$.

$$|y(t)| = \left| \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau \right| \le x_{\max} \left| \int_{-\infty}^{+\infty} h(t-\tau) d\tau \right| < \infty.$$

Necessary condition: Suppose $\int_{-\infty}^{+\infty} |h(\tau)| d\tau = \infty$

Let $x(t) = h^*(-t)/|h^*(-t)|$, then $|x(t)| \equiv 1$ bounded

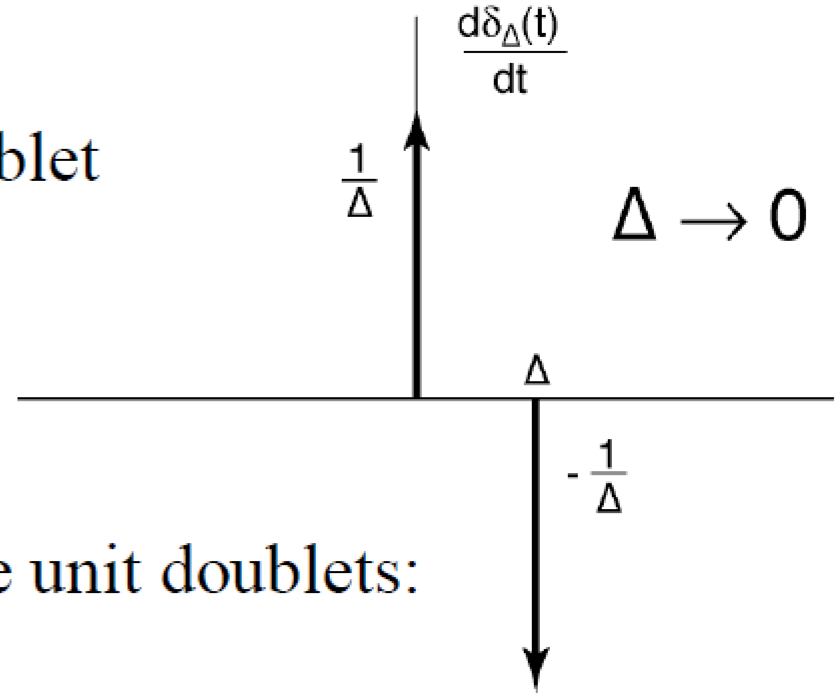
But $y(0) = \int_{-\infty}^{+\infty} x(\tau) h(-\tau) d\tau = \int_{-\infty}^{+\infty} \frac{h^*(-\tau) h(-\tau)}{|h(-\tau)|} d\tau = \int_{-\infty}^{+\infty} |h(-\tau)| d\tau = \infty$

Differentiator

$$x(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow y(t) = \frac{dx(t)}{dt}$$

Impulse response = unit doublet

$$\boxed{u_1(t) = \frac{d\delta(t)}{dt}}$$



The operational definitions of the unit doublets:

$$x(t) * u_1(t) = \frac{dx(t)}{dt}$$

Triplets and beyond!

$$x(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow \cdots \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow \frac{d^n x(t)}{dt^n}$$

$$n > 0$$

$$u_n(t) = \underbrace{u_1(t) * \cdots * u_1(t)}_{n\ times}$$

Operational definitions:

$$x(t) * u_n(t) = \frac{d^n x(t)}{dt^n} \qquad (n > 0)$$

Unit Step Response

unit step function → unit step response

Step response

$$s(t) = u(t) * h(t) = h(t) * u(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

$$x(t)=\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow$$ $h(t)$: unit impulse response

$$x(t)=u(t) \longrightarrow \boxed{h(t)} \longrightarrow$$ $s(t)$: unit step response

The relation between unit step function and unit impulse function

$$h(t) = \frac{ds(t)}{d(t)} = s'(t)$$

Integrators

$$x(t) \longrightarrow \boxed{\int} \longrightarrow y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

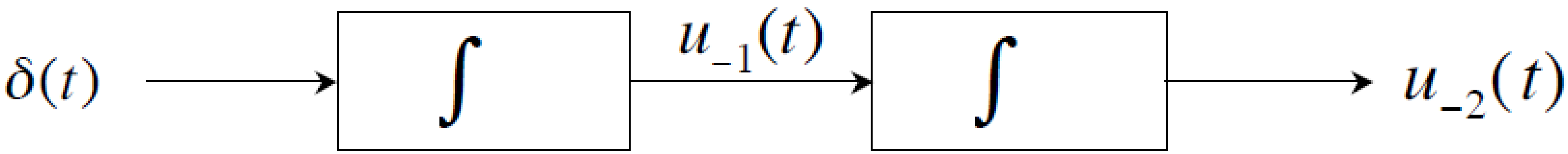
Impulse response: $\boxed{u_{-1}(t) \equiv u(t)}$

Operational definition: $x(t) * u_{-1}(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

Cascade of n integrators:

$$u_{-n}(t) = \underbrace{u_{-1}(t) * \cdots * u_{-1}(t)}_{n\ times} \qquad (n > 0)$$

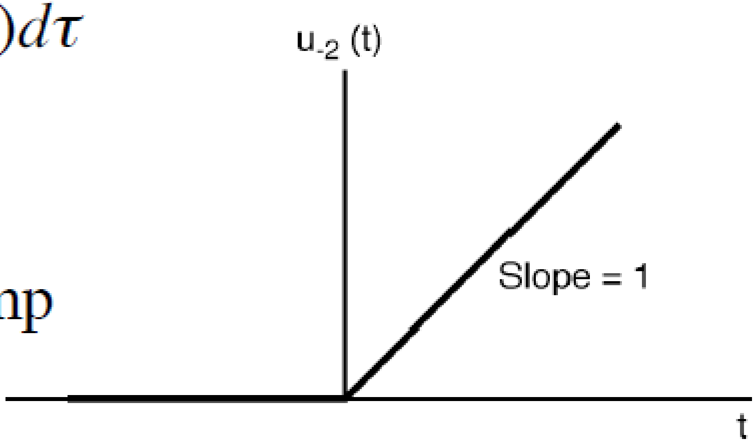
Integrators (Cont.)



$$u_{-2}(t) = \int_{-\infty}^{t} u_{-1}(\tau)d\tau = \int_{-\infty}^{t} u(\tau)d\tau$$

$$= u(t)\int_{0}^{t} d\tau$$

$$= t \cdot u(t) \qquad \text{the unit ramp}$$



More generally, for $n > 0$

$$u_{-n}(t) = \frac{t^{(n-1)}}{(n-1)!}u(t)$$

Notation

Define $\boxed{u_0(t) = \delta(t)}$

Then $$u_n(t) * u_m(t) = u_{n+m}(t)$$

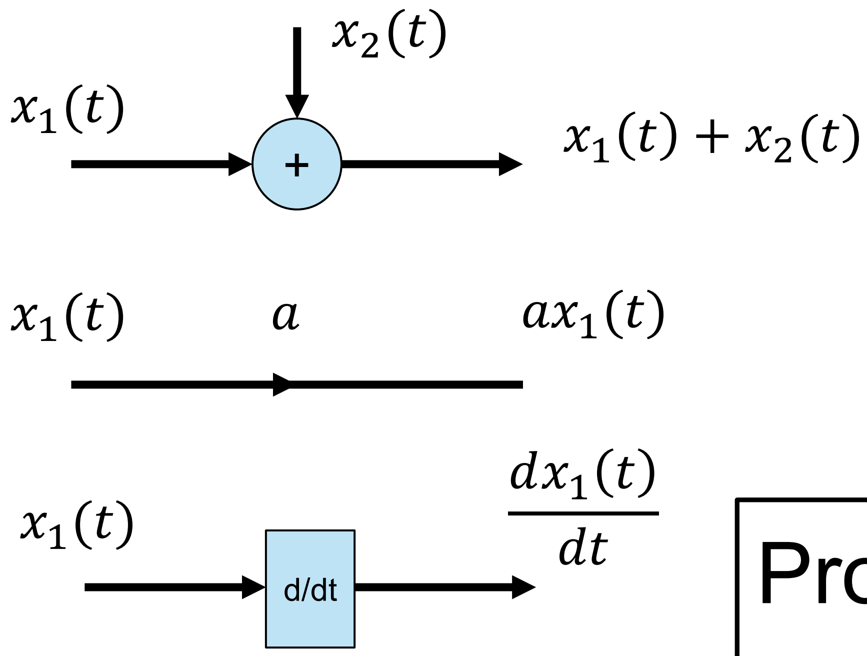
n and m can be $\pm$

E.g. $$u_1(t) * u_{-1}(t) = u_0(t)$$

$$||$$

$$\left(\frac{d}{dt}u(t)\right) = \delta(t)$$

Block diagram representation - CT

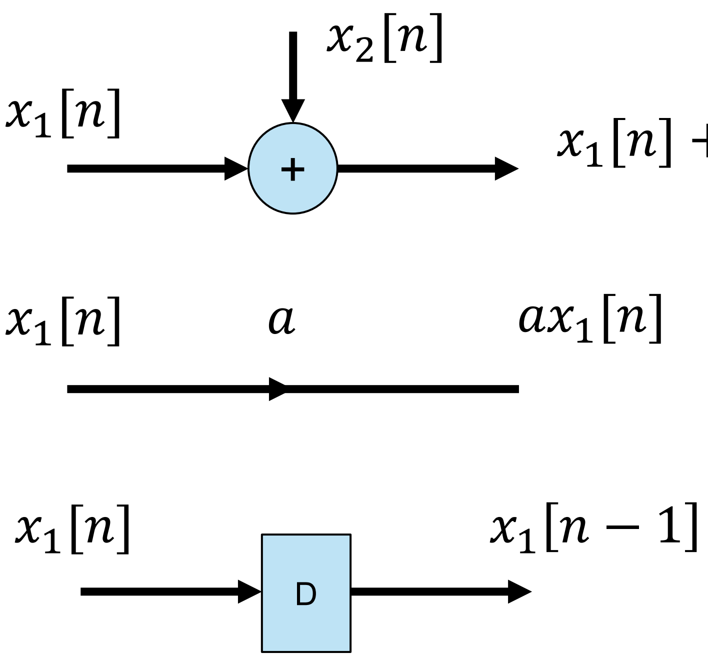


Problem 2.39

$$y(t) = -\frac{1}{2} dy(t)/dt + 4x(t)$$

$$dy(t)/dt + 3y(t) = x(t)$$

Block diagram representation - DT



Problem 2.38

$$y[n] = \frac{1}{3} y[n-1] + \frac{1}{2} x[n]$$

$$y[n] = \frac{1}{3} y[n-1] + x[n-1]$$

From block diagram to difference equation