Assignments (Week 3)

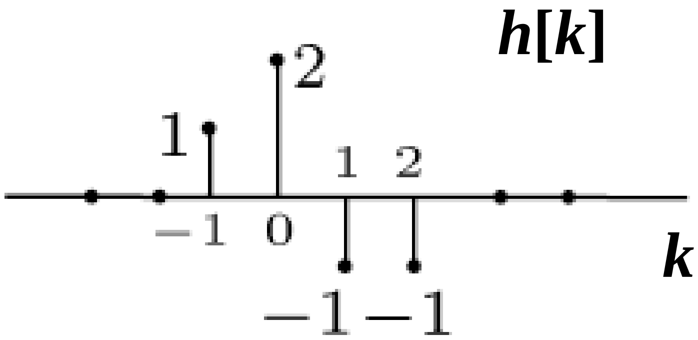
- 2.4
- 2.6
- 2.19
- 2.21 (c) (d)

Tutorial Problems (Week 3)

- Basic Problems with Answers 2.3, 2.7, 2.13
- Basic Problems 2.24, 2.26

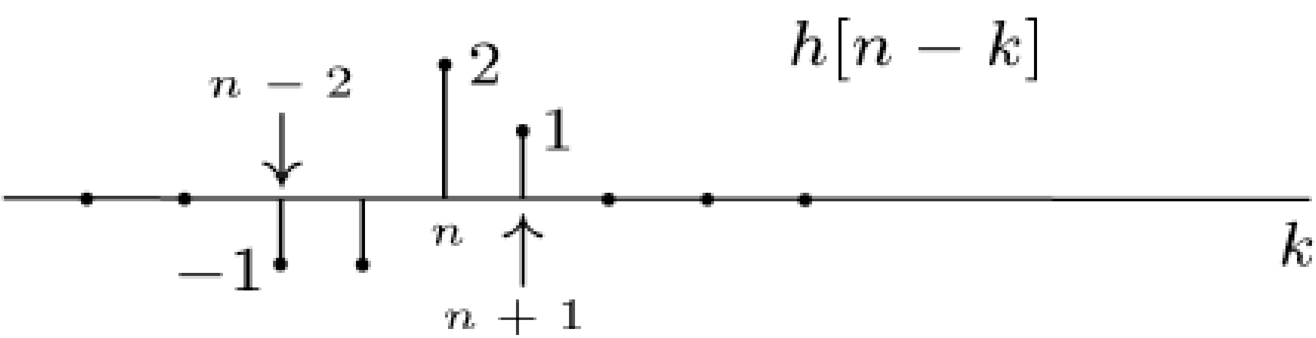
Ch 1 Review (1)

- **Time-shift and flip**



What is the plot for *h*[*n*-*k*]??
***n* is a constant**

$$h[k] \xrightarrow{\text{Flip}} h[-k] \xrightarrow{\text{Slide}} h[n-k]$$

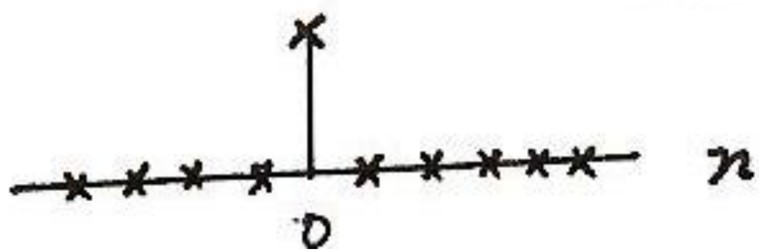


Ch 1 Review (2)

- **Unit impulse function (unit sample function)**

Discrete-time

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



- We can use unit impulse function to represent any other different signals, or it is a building function (or basic signal) --- it will be explained soon.

Ch 1 Review (3)

System properties:

1. **With memory or memoryless**

   $y[n]=f\,(\,x[n]\,)$

2. **Invertible**

   for a system $x \rightarrow y$, if $x_1 \neq x_2$, then $y_1 \neq y_2$

3. **Causal**

   … up to that time n …

4. **Stable**

   either prove the system is stable, or find a specific counterexample

Ch 1 Review (4)

5. Time-invariant

$$\text{If} \quad x[n] \to y[n]$$
$$\text{then} \quad x[n - n_0] \to y[n - n_0]\ .$$

6. Linear

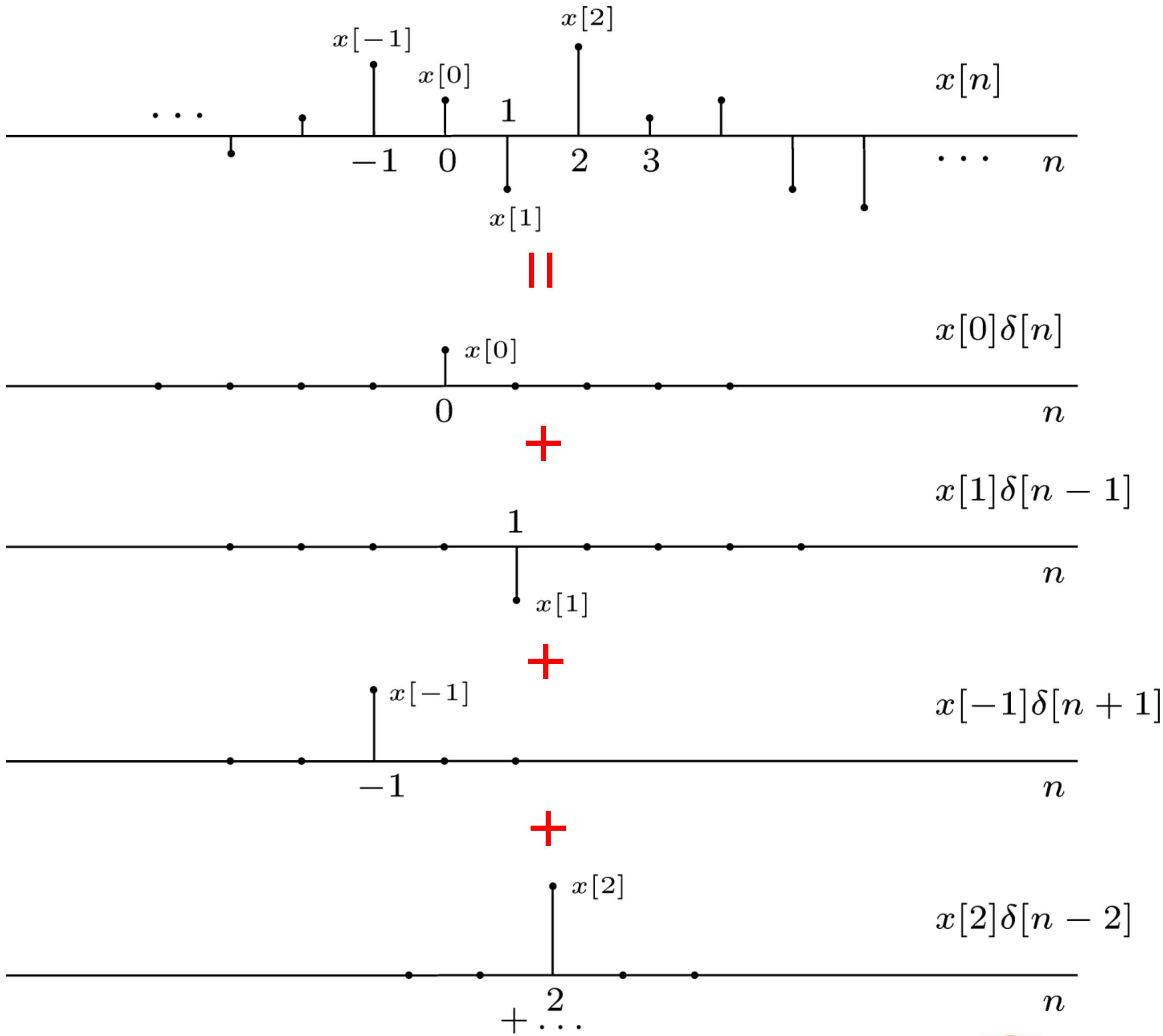
A (CT) system is linear if it has the $\boxed{\text{superposition property:}}$

$$\text{If} \quad x_1(t) \to y_1(t) \quad \text{and} \quad x_2(t) \to y_2(t)$$

$$\text{then} \quad ax_1(t) + bx_2(t) \quad \to \quad ay_1(t) + by_2(t)$$

Chapter 2: Linear Time-invariant (LTI) Systems

Representation of DT Signals Using Unit Samples



That is ...

$$x[n] = \ldots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \ldots$$

$$\Downarrow$$

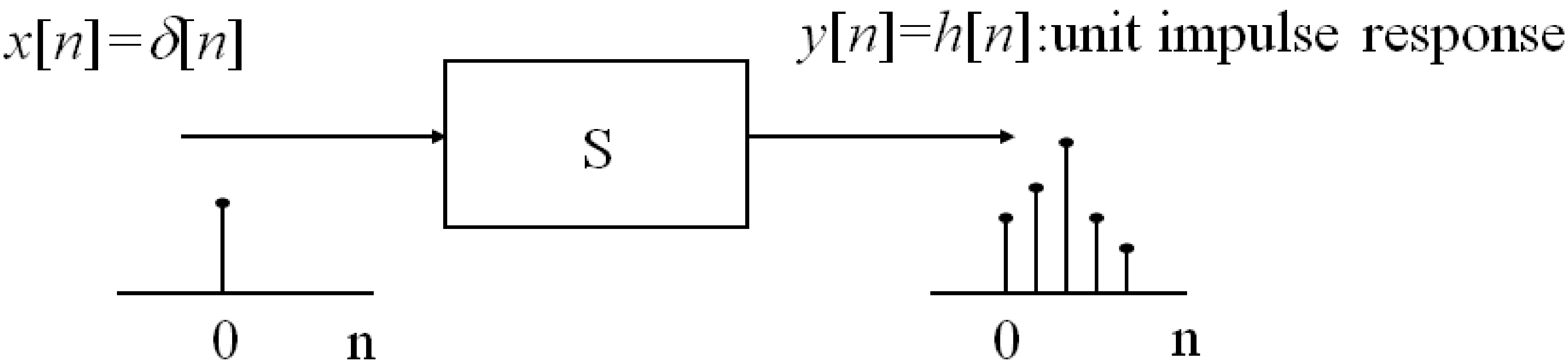
Important to note the "-" sign

$$x[n] = \sum_{k=-\infty}^{+\infty} \underbrace{x[k]}_{\text{Coefficients}} \underbrace{\delta[n-k]}_{\text{Basic Signals}}$$

Arbitrary DT signal can be written as a linear combination of impulse functions with different time shifting, i.e., linear combination of signals $\{\delta[n-k] | k = \cdots, -2, -1, 0, 1, 2, \ldots\}$.

Unit Impulse Response (Unit Sample Response)

- Define the output for a unit impulse input as the unit impulse response



Example: $y[n]=x[n]+2x[n-1]+4x[n-2]$

What is unit impulse response?

Example

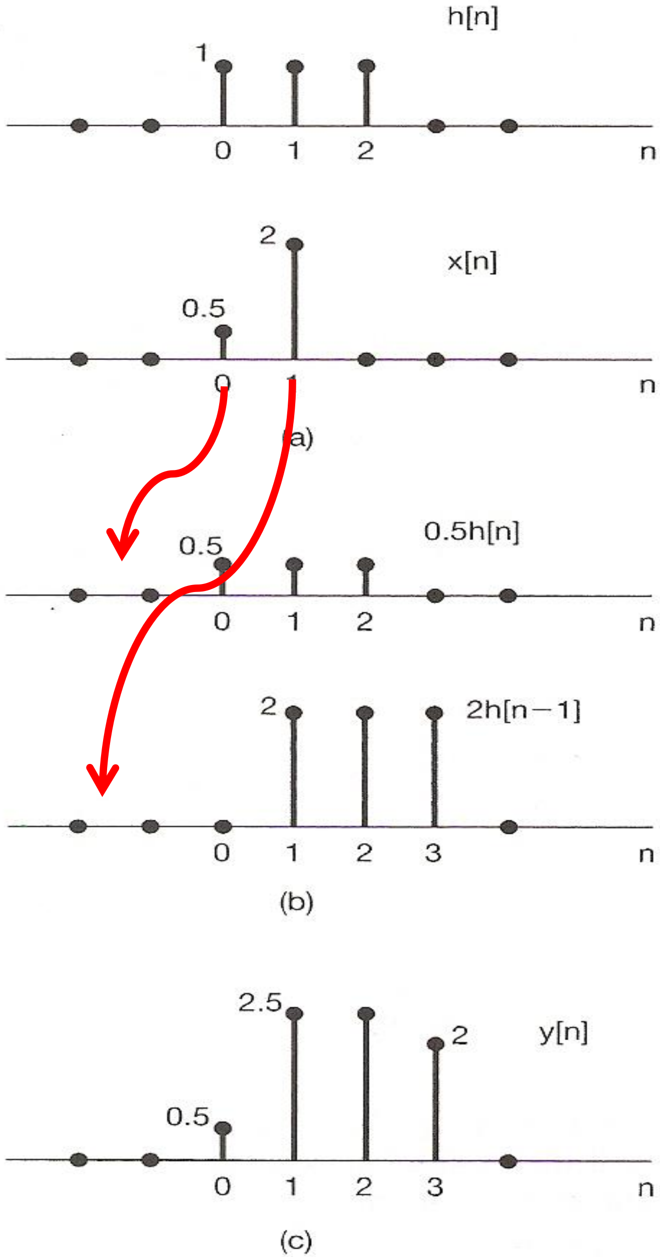


Figure 2.3 (a) The impulse response $h[n]$ of an LTI system and an input $x[n]$ to the system; (b) the responses or "echoes," $0.5h[n]$ and $2h[n-1]$, to the nonzero values of the input, namely, $x[0] = 0.5$ and $x[1] = 2$; (c) the overall response $y[n]$, which is the sum of the echos in (b).

$$x[n] = 0.5\delta[n] + 2\delta[n-1]$$

By linearity:

$$0.5\delta[n] \Rightarrow 0.5h[n]$$

By LTI:

$$2\delta[n-1] \Rightarrow 2h[n-1]$$

y[n]=0.5h[n]+2h[n-1]

Response of DT LTI Systems

$$x[n] \longrightarrow \boxed{\text{DT System}} \longrightarrow y[n]$$

- Now suppose the system is **LTI**, and define the *unit impulse response* $h[n]$: $\quad \delta[n] \longrightarrow h[n]$

$$\Downarrow$$

From Time-Invariance:

$$\delta[n-k] \longrightarrow h[n-k]$$

From Linearity:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k] \longrightarrow y[n] = \underbrace{\sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]}_{\textit{convolution sum}} = x[n] * h[n]$$

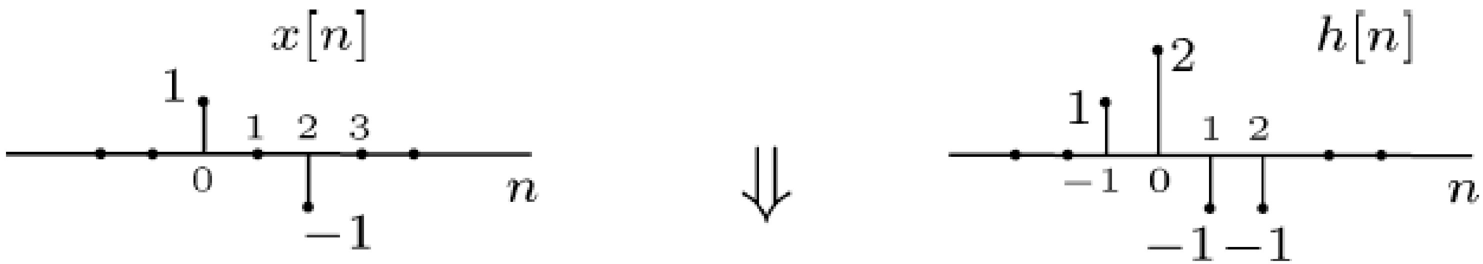
The output for an arbitrary input signal is the superposition of a series of "shifted, scaled unit impulse response"

Hence a Very Important Property of LTI Systems:

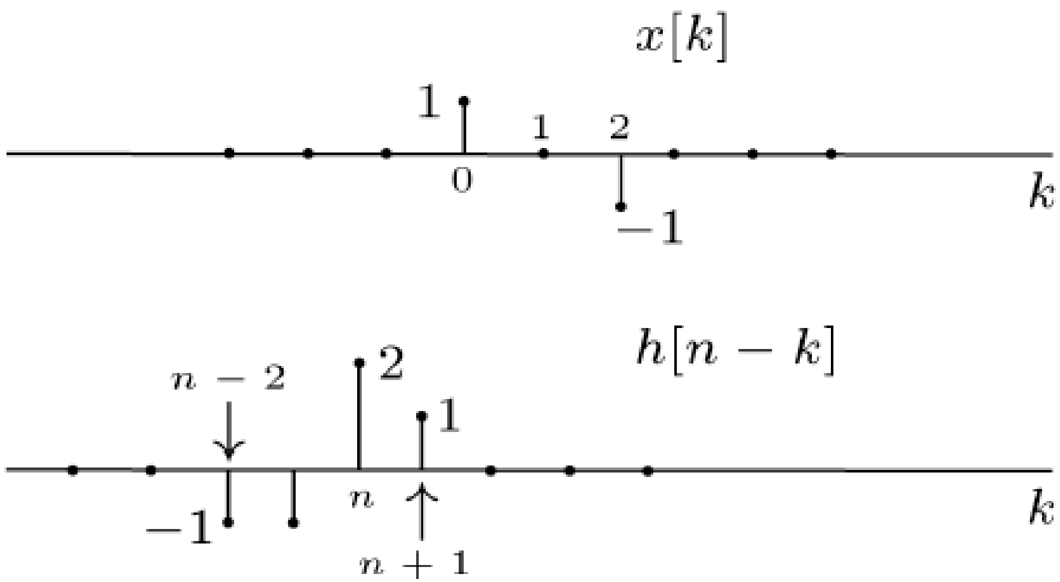
The output of any DT LTI system is a convolution of the input signal with the unit impulse response.

Any DT LTI system are completely characterized by its unit impulse response.

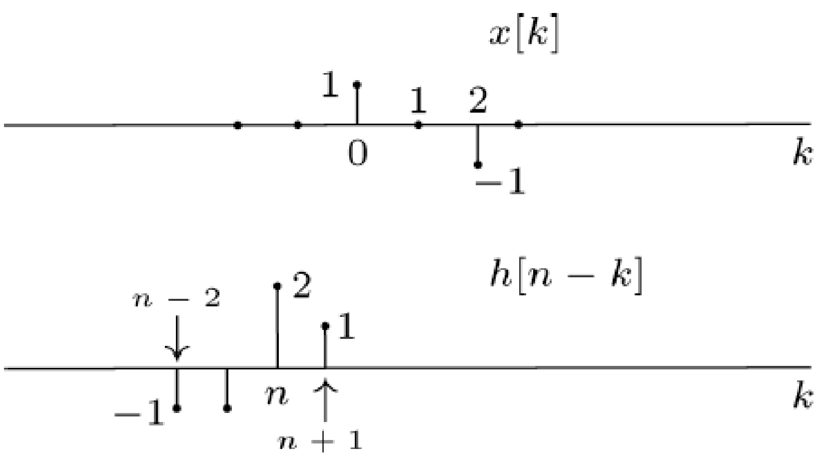
Example: Convolution Calculation



From $x[n]$ and $h[n]$ to $x[k]$ and $h[n\text{-}k]$



Calculating Successive Values: Shift, Multiply, Sum

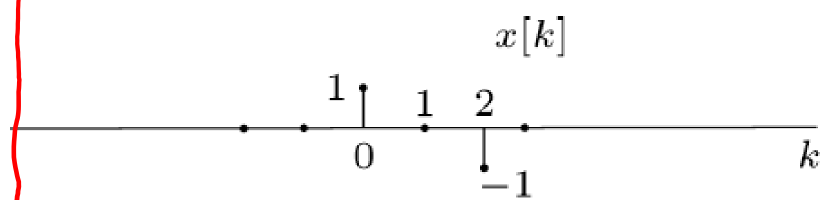


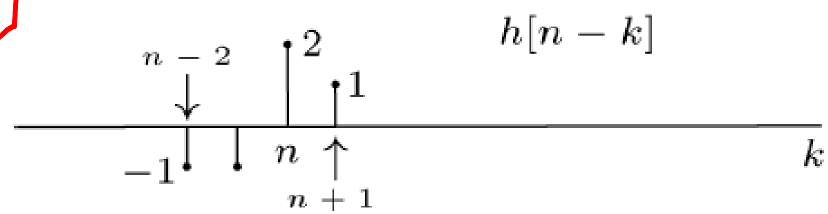
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$
\begin{aligned}
y[n] &= 0 \qquad \text{for } n < \\
y[-1] &= \\
y[0] &= \\
y[1] &= \\
y[2] &= \\
y[3] &= \\
y[4] &= \\
y[n] &= 0 \qquad \text{for } n >
\end{aligned}
$$

Calculating Successive Values: Shift, Multiply, Sum

$$\begin{bmatrix} -1 & -1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & -1 & 2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & -1 & -1 & 2 & 1 & 0 & \cdots \\ \cdots & & & & & & & \\ \cdots & & & & & & & \\ \cdots & & & & & & & \end{bmatrix} \begin{bmatrix} x[-3] \\ x[-2] \\ x[-1] \\ x[0] \\ x[1] \\ \vdots \\ x[5] \end{bmatrix} = \begin{bmatrix} y[-1] \\ y[0] \\ y[1] \\ \vdots \\ y[4] \end{bmatrix}$$





If $\{-1, -1, 2, 1\}$ $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$
are unknown.

then we have 4 unknown variables and 6 equations,
— Solvable!

$$\begin{aligned}
y[n] &= 0 \qquad \text{for } n < \\
y[-1] &= (-1)\times x[-3] + (-1)\times x[-2] + 2\times x[-1] + 1\times x[0] \\
y[0] &= (-1)\times x[-2] + (-1)\times x[-1] + 2\times x[0] + 1\times x[1] \\
y[1] &= (-1)\times x[-1] + (-1)\times x[0] + 2\times x[1] + 1\times x[2] \\
y[2] &= \quad \vdots \\
y[3] &= \quad \vdots \\
y[4] &= (-1)\times x[2] + (-1)\times x[3] + 2\times x[4] + 1\times x[5] \\
y[n] &= 0 \qquad \text{for } n >
\end{aligned}$$

Non-zero Region

$x[n]: \{0, 1, 2\}$ 3

$h[n]: \{-1, 0, 1, 2\}$ 4

$x[n] * h[n]: \{-1, 0, \dots, 4\}$ $3+4-1=6$

$x[n]: \{A, \cdots, B\}$ $M = B - A + 1$

$h[n]: \{C, \cdots, D\}$ $N = D - C + 1$

$x[n] * h[n]: \{A+C, \cdots, B+D\}$ $(B+D) - (A+C) + 1 = M + N - 1$

Convolution operation procedure:

$$h[k] \xrightarrow{\text{Flip}} h[-k] \xrightarrow{\text{Slide}} h[n-k] \xrightarrow{\text{Multiply}} x[k]h[n-k]$$

$$\xrightarrow{\text{Sum}} \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

F-S-M-S for every fixed n

Observations:

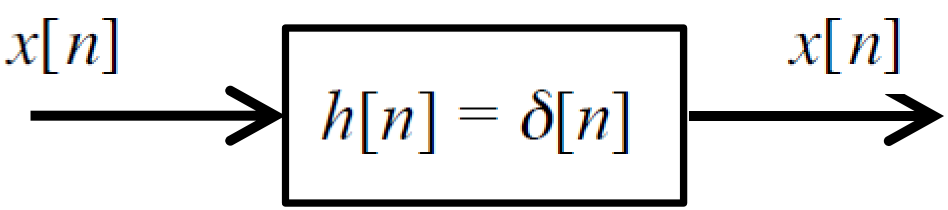
- Convolution of two finite signals leads to another finite signal
- What’s the relation on their non-zero duration?

Examples of Convolution and DT LTI Systems

Ex. #1: $h[n] = \delta[n]$

$$y[n] = x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$= x[n] \quad \text{— An Identity system}$$



Ex. #2: $h[n] = \delta[n - n_o]$

$$y[n] = x[n] * \delta[n - n_o] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - n_o - k]$$

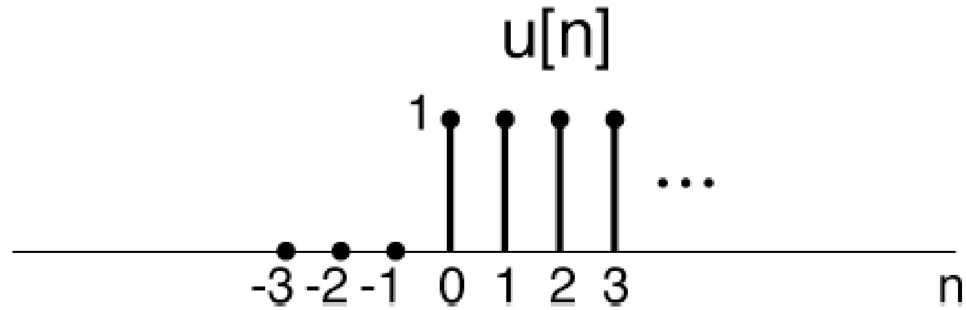
$$= x[n - n_o] \quad \text{— A Shift}$$

$$x[n] \longrightarrow \boxed{h[n] = \delta[n - n_o]} \longrightarrow x[n - n_o]$$

Ex. #3 $\quad y[n] = \sum_{k=-\infty}^{n} x[k]$ – An accumulator

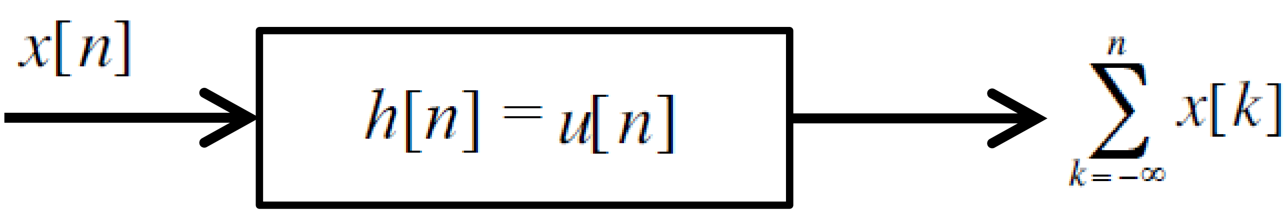
Unit Sample response

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n]$$



$$\Downarrow$$

$$x[n] * u[n] = \sum_{k=-\infty}^{n} x[k]$$



Ex. #4 (Example 2.3)

$0<\alpha<1$

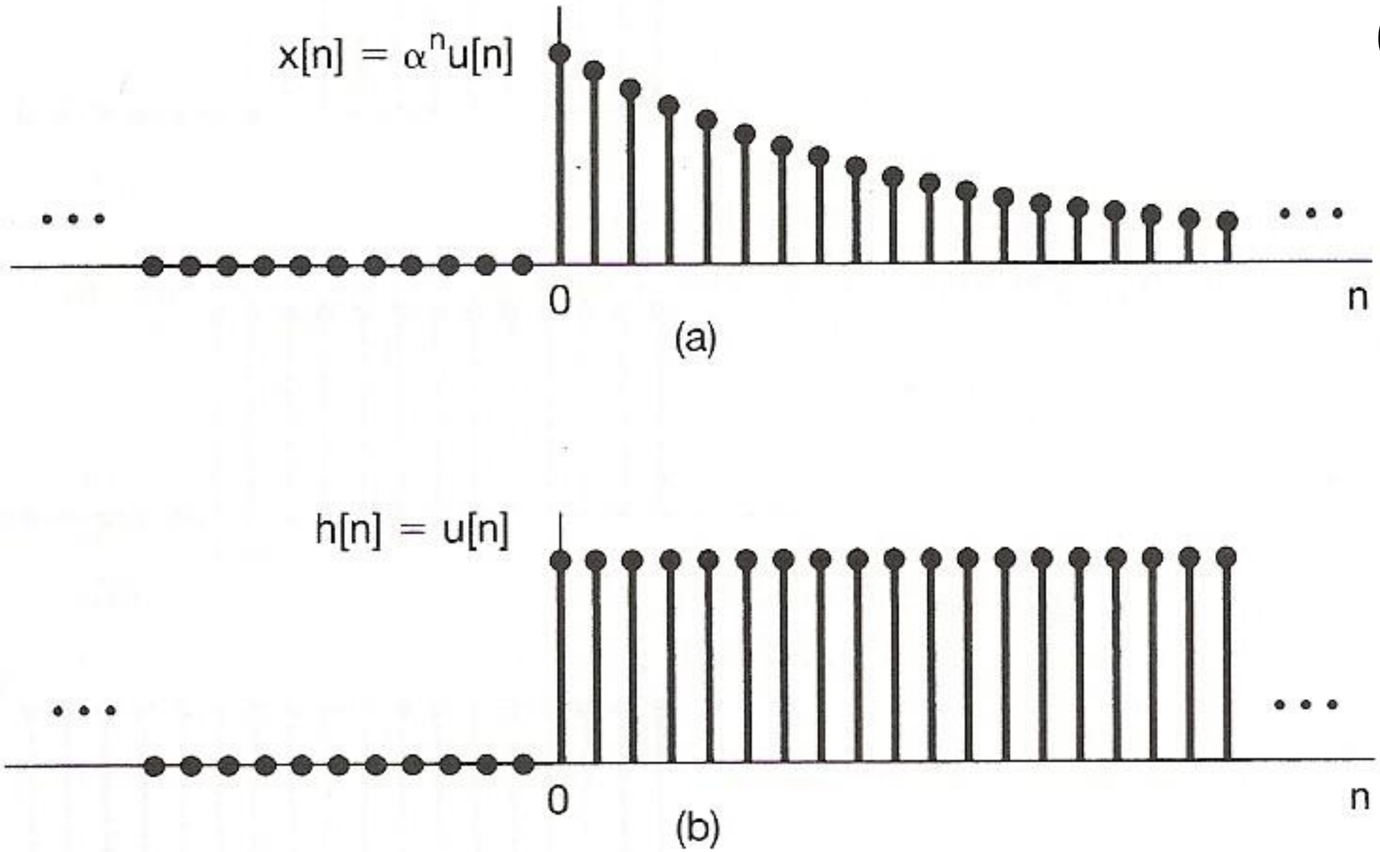


Figure 2.5 The signals $x[n]$ and $h[n]$ in Example 2.3.

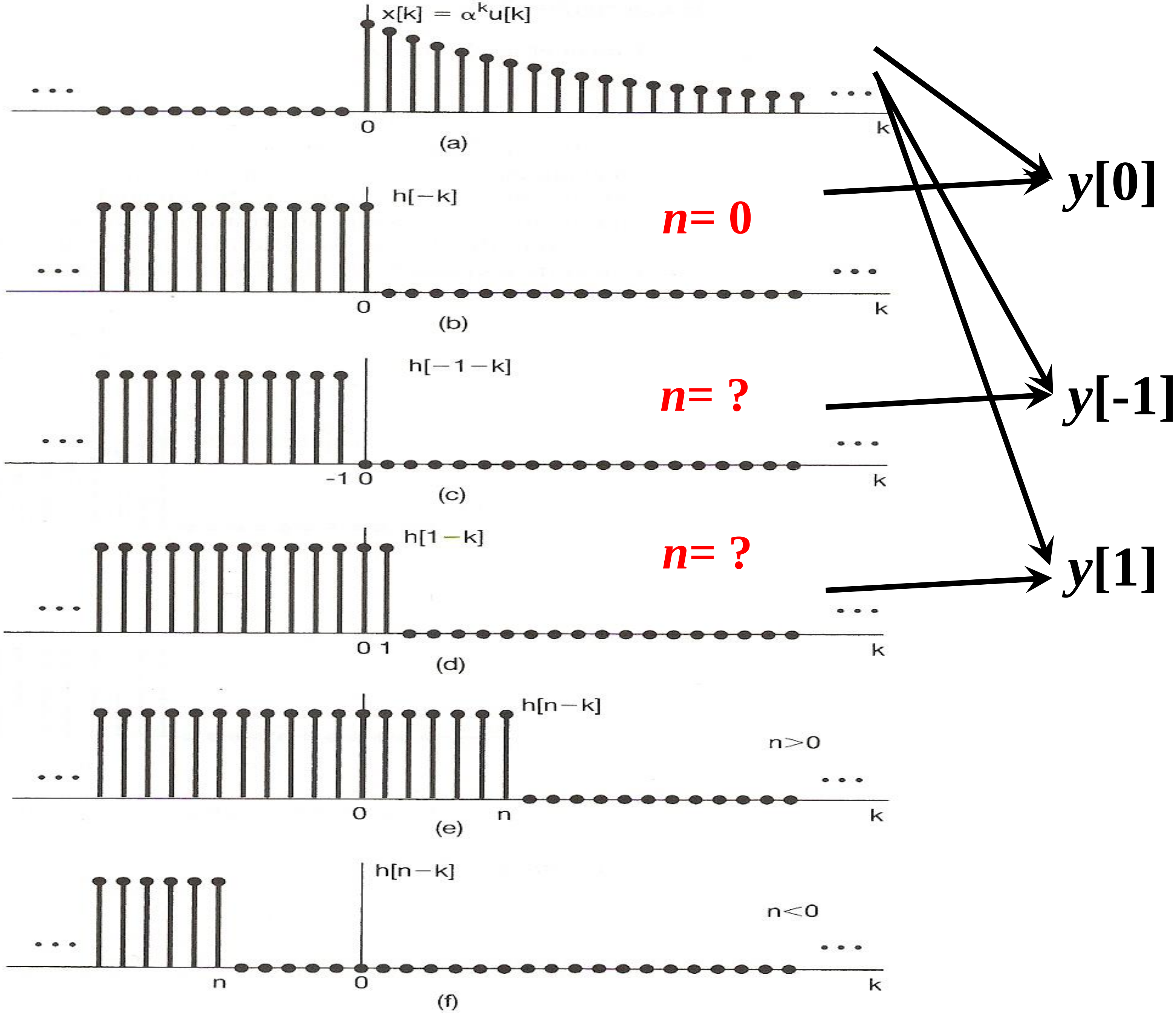


Figure 2.6 Graphical interpretation of the calculation of the convolution sum for Example 2.3.

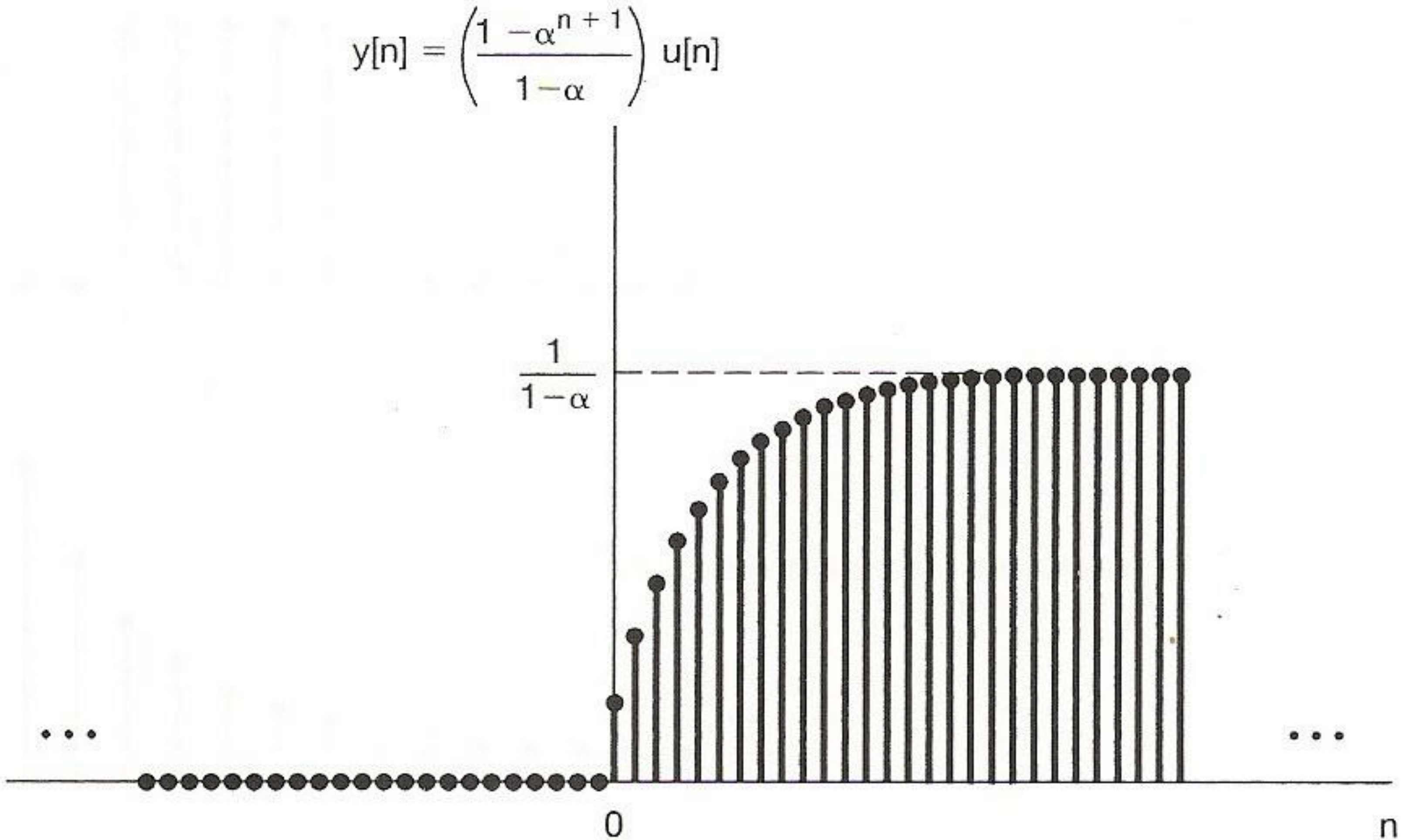


Figure 2.7 Output for Example 2.3.

Characteristics of an LTI system are completely determined by its impulse response.

- ***What if the system is nonlinear?***

Consider a discrete-time system with unit impulse response

$$h[n] = \begin{cases} 1 & n = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

If the system is LTI, the input/output relationship is

$$y[n] = x[n] + x[n-1].$$

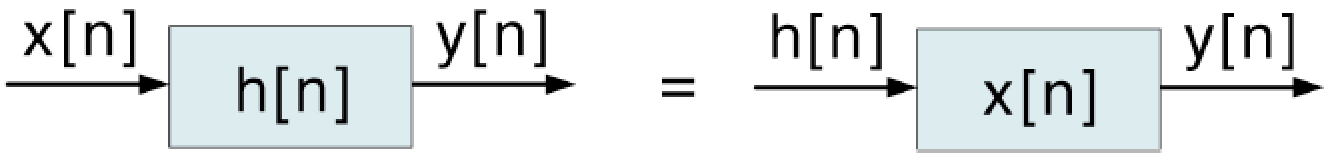
On the other hand, there are ***many*** nonlinear systems with the same response to the input $\delta[n]$.

$$y[n] = (x[n] + x[n-1])^2,$$

$$y[n] = \max(x[n], x[n-1]).$$

The Commutative Property of Convolution

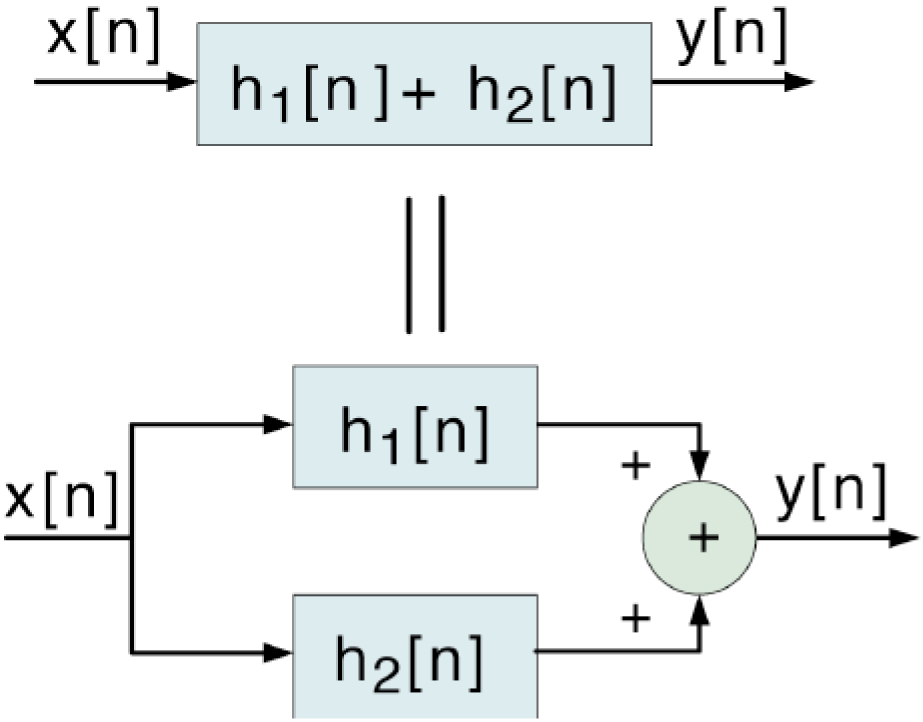
$$y[n] = x[n] * h[n] \quad = \quad h[n] * x[n]$$



The Distributive Property of Convolution

$$x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$$

Interpretation



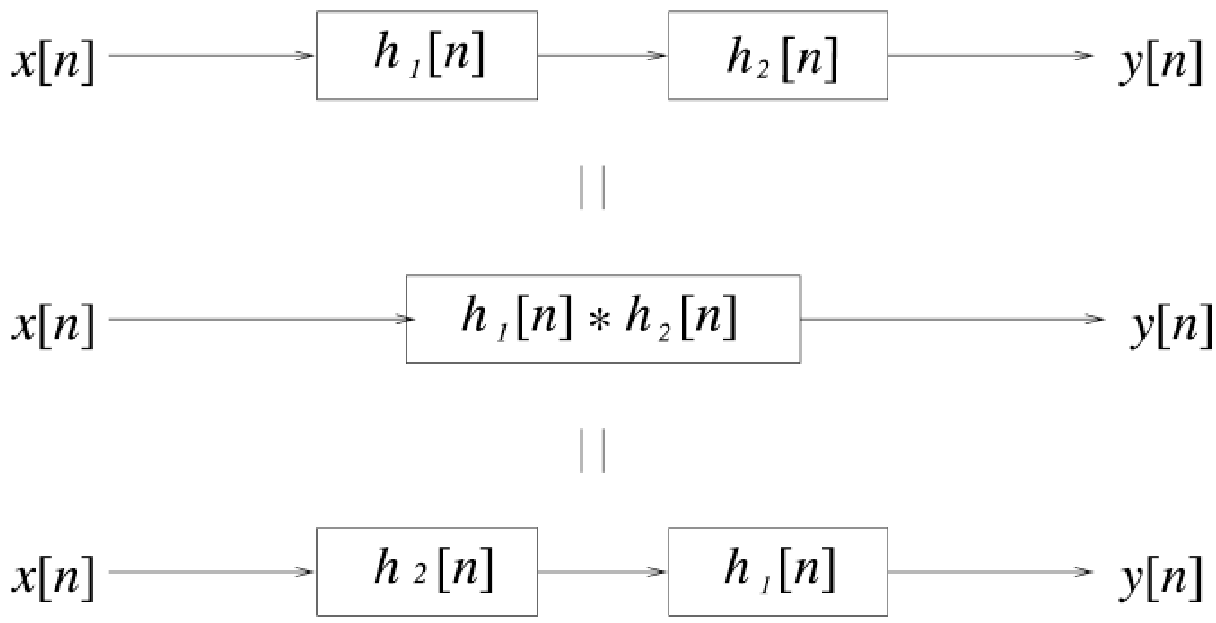
The Associative Property of Convolution

$$x[n]*(h_1[n]*h_2[n]) = (x[n]*h_1[n])*h_2[n]$$

(Commutativity) $\quad ||$

$$x[n]*(h_2[n]*h_1[n]) = (x[n]*h_2[n])*h_1[n]$$

Implication (Very special to LTI Systems)



Properties of Convolution

Combining the Commutative property,

$$y[n] = x[n] * h[n] \quad = \quad h[n] * x[n]$$

Distributive property,

$$x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$$

and Associative property,

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

symbolically, we can treat "$*$" as a "$\times$". Easy, piece of cake!

Some Useful Properties of LTI Systems

1) Causality $\Leftrightarrow$ $h[n] = 0$ for all $n < 0$

2) Stability $\Leftrightarrow$ $\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$

BIBO — Bounded Input $\Rightarrow$ Bounded Output

→ Sufficient condition: For $|x[n]| \leq x_{\max} < \infty$.

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} x[k]h[n-k]\right| \leq x_{\max}\left|\sum_{k=-\infty}^{\infty} h[n-k]\right| < \infty.$$

→ Necessary condition: If $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$

Let $x[n] = h^*[-n]/|h[-n]|$, then $|x[n]| \equiv 1$ bounded

But $y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k] = \sum_{k=-\infty}^{\infty} h^*[-k]h[-k]/|h[-k]| = \sum_{k=-\infty}^{\infty} |h[-k]| = \infty$

- Memoryless / with Memory
  - A linear, time-invariant, causal system is memoryless only

if $h[n] = K\delta[n]$ $\quad h(t) = K\delta(t)$

$y[n] = Kx[n]$ $\quad y(t) = Kx(t)$

if k=1 further, they are identity systems

$$y[n] = x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n] * \delta[n]$$

$$y(t) = x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t) * \delta(t)$$

Summary

Understand the following new concepts:

1. **Use unit impulse function to represent any function**
2. **Unit impulse response h[n]**
   - **Given the system input/output equation, how to decide the unit impulse response?**
3. **Convolution, its properties, and calculation steps (FSMS)**
   - **Understand the meaning of index 'k' and index 'n'**
4. **Decide LTI system property by using unit impulse response h[n]**