Homework & Tutorial Problems

- Tutorial
  - ▶ Property of DTFT: 5.24
  - ▶ Difference equation of LTI systems: 5.36
- Homework
  - ▶ 5.23, 5.29, 5.33

Convolution Property & LTI Systems (1/2)

- Let $h[n]$ be the impulse response of certain LTI system
- The output $y[n]$ of input $x[n]$ is given by $y[n] = x[n] * h[n]$
- For input signal $x[n] = e^{j\omega n}$,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} = e^{j\omega n} \underbrace{\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}}_{\text{Frequency Response } H(e^{j\omega})}$$

- For periodic input signal $x[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n}$:

$$y[n] = \sum_{k=<N>} a_k H(e^{jk(\frac{2\pi}{N})}) e^{jk(\frac{2\pi}{N})n}$$

$$b_k = a_k H(e^{jk(\frac{2\pi}{N})})$$

Convolution Property & LTI Systems (2/2)

Convolution Property

If $y[n] = x_1[n] * x_2[n]$, then

$$Y(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega})$$

- For general input signal $x[n]$:

$$y[n] = x[n] * h[n] \longleftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- Observation: It's easier to evaluate LTI systems in frequency domain
- Drawback: Not every LTI system has frequency response
  - $h[n] = a^n u[n]$ $(a > 1)$
  - Stable LTI system has frequency response, because

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Example: Ideal Low-Pass Filter (1/2)

- What's ideal low-pass filter?
- Perfectly maintain the low-frequency component
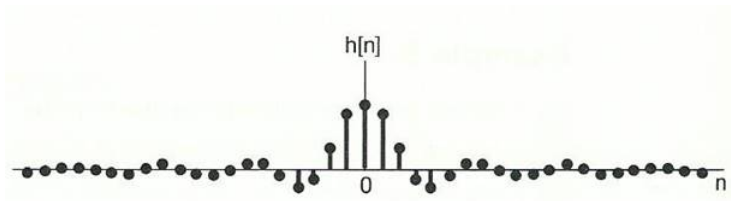- Perfectly cancel the high-frequency component
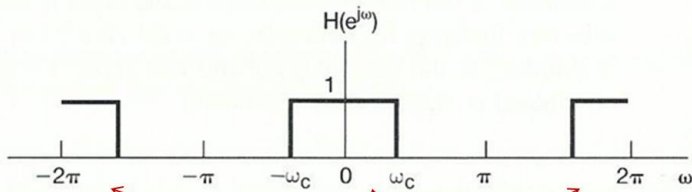
Repeat with period 2π

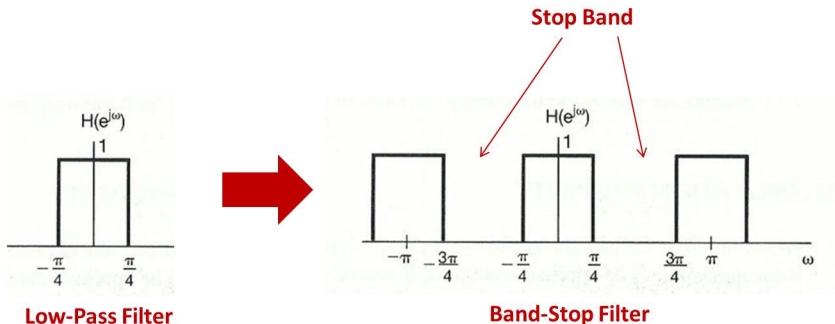# Example: Ideal Low-Pass Filter (2/2)

- Impulse response:

$$h[n] = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi} \int\limits_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{sin\omega_c n}{\pi n}$$

- Pros: no distortion in frequency domain
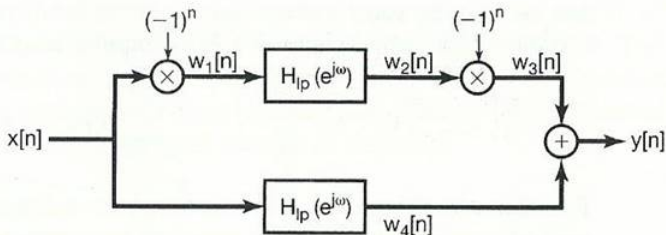- Cons: non-causal
- See textbook, Example 5.12

Example: Band-Stop Filter (1/2)

- Given low-pass filter $H_{lp}(e^{j\omega})$, how to generate band-stop effect from low-pass filter?

Stop Band

Low-Pass Filter

Band-Stop Filter

Example: Band-Stop Filter (2/2)

- Two branches: low-pass + high-pass

- $(-1)^n = e^{j\pi n} \Rightarrow W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$

$$\Rightarrow W_2(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j(\omega-\pi)})$$

$$\Rightarrow W_3(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j(\omega-2\pi)}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega})$$

- See textbook, Example 5.14

Multiplication Property

Multiplication Property

Let $y[n] = x_1[n]x_2[n]$, then

$$Y(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta,$$

which is *periodic convolution* of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$
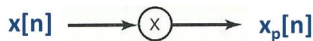
- Comparison: Multiplication of periodic signals

$$x_1[n] \longleftrightarrow a_k \text{ and } x_2[n] \longleftrightarrow b_k$$

$$\Rightarrow x_1[n]x_2[n] \longleftrightarrow \sum_{k=<N>} a_k b_{n-k} \text{ discrete-time periodic convolution}$$
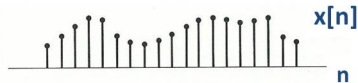
Example: Sampling on Discrete-Time Signals

- In digital signal processing, we may sample the discrete-time signals
- Example: image, audio, video compression
- Its mathematical model is give below:

$$p[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN]$$

$$x_p[n] = \sum_{k=-\infty}^{+\infty} x[kN]\delta[n - kN]$$

Review: Discrete-Time Impulse Chain

- What's the Fourier transform of $x[n] = \sum_{k=-\infty}^{+\infty} \delta[n-kN]$?

- First of all, we calculate the Fourier series:

$$
\begin{aligned}
a_k &= \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{N} \sum_{n=<N>} \sum_{k=-\infty}^{+\infty} \delta[n-kN] e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{N} \sum_{n=<N>} \delta[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}
\end{aligned}
$$

- Then, we have

$$
X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} \frac{2\pi}{N} \delta(\omega - k(2\pi/N) - 2\pi l) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{N})
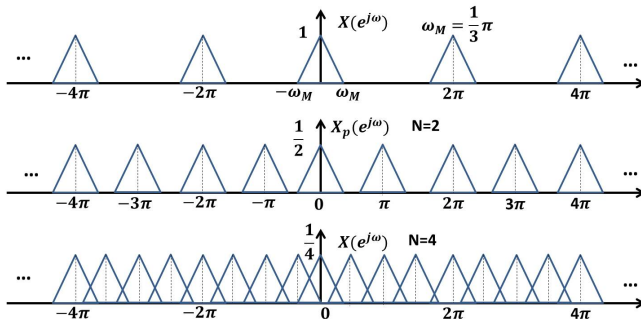$$

- Time domain period $\times$ Frequency domain period $=$?

Example: Frequency Analysis

$$P(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \quad where \; \omega_s = 2\pi/N$$

$$X_p(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} P(e^{j\theta}) X(e^{j(\omega-\theta)}) d\theta = \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j(\omega - k\omega_s)})$$

Duality in DTFS

- Analysis and synthesis equations of DTFS share a similar form:

$$\begin{aligned} x[n] &= \sum_{k=<N>} a_k e^{jk(2\pi/N)n} \\ a_k &= \frac{1}{N} \sum_{k=<N>} x[n] e^{-jk(2\pi/N)n} \end{aligned} \qquad (1)$$

- Every of DTFS has a dual (See Table 3.2)
  - ▸ Time shift v.s. Frequency shift
  - ▸ Time multiplication v.s. Frequency multiplication
- If $g[n] \longleftrightarrow f[k]$, then $f[n] \longleftrightarrow \frac{1}{N} g[-k]$

Dualtiy Between CTFS and DTFT

- DTFT transform pair is given by

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \\ X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \end{aligned}$$

- CTFS transform pair is given by

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\ a_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \end{aligned}$$

- Frequency Domain of DTFT $\Leftrightarrow$ Time Domain of CTFS
- Frequency Domain of CTFS $\Leftrightarrow$ Time Domain of DTFT
- Discrete-Time Low-pass Filter v.s. Continuous-Time Square Wave

Summary of Duality

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N} \sum_{k=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$ discrete time aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$ continuous frequency periodic in frequency

- See textbook, Table 5.3

LTI by Difference Equation

- A number of DT LTI systems can be written as the following linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

What's the frequency response?

- Taking Fourier transform on both side, we have

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$$

$$\Rightarrow \quad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

What's the impulse response?

Example: Difference Equation

Please calculate the frequency and impulse response of the following LTI systems

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n].$$

Solution

According to the last slide,

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}.$$

Moreover, by partial fraction expansion

$$H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}} \Rightarrow h[n] = 4(\frac{1}{2})^n u[n] - 2(\frac{1}{4})^n u[n]$$

Tutorial on Orthogonal Frequency Division Multiplexing (OFDM)

Discrete Fourier Transform (DFT)

- In practice, there is a huge demand on processing finite duration signals
- Given a finite duration signal $\{x[0], x[1], ..., x[N-1]\}$, its Fourier transform is

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

- Drawback: The spectrum of DTFT is continuous $\Rightarrow$ Cannot be handle by computer.
- Discrete Fourier Transfrom (DFT) is developed for digital processing of finite duration signals

$$\widetilde{X}[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk(2\pi/N)n} \quad k = <N>$$

- DFT: frequency sampling of DTFT

$$\widetilde{X}[k] = \frac{1}{N} X(e^{j2k\pi/N}) \quad k = <N>$$

Inverse DFT

- Equation system of DFT:

$$
\begin{pmatrix} \widetilde{X}[0] \\ \widetilde{X}[1] \\ \cdots \\ \widetilde{X}[N-1] \end{pmatrix} = \underbrace{\frac{1}{N} \begin{pmatrix} e^{-j0} & e^{-j0} & \cdots & e^{-j0} \\ e^{-j0} & e^{-j2\pi/N} & \cdots & e^{-j2(N-1)\pi/N} \\ \cdots & \cdots & \cdots & \cdots \\ e^{-j0} & e^{-j2(N-1)\pi/N} & \cdots & e^{-j2(N-1)(N-1)\pi/N} \end{pmatrix}}_{\mathbf{F}} \begin{pmatrix} x[0] \\ x[1] \\ \cdots \\ x[N-1] \end{pmatrix}
$$

- Transform matrix **F** is full rank.
- Observation: $\{\widetilde{X}[k]|k=<N>\}$ maintains all the information of $\{x[0], x[1], ..., x[N-1]\}$
- Inverse DFT is feasible:

$$
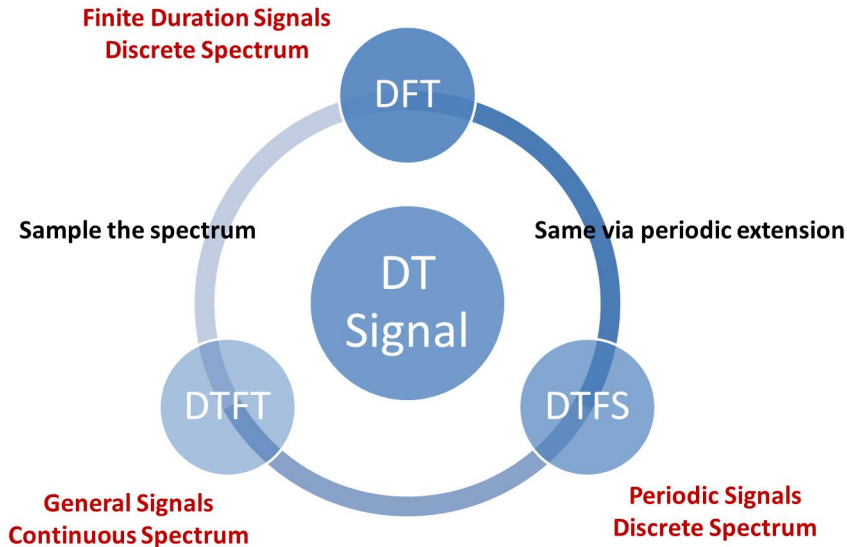x[n] = \sum_{k=0}^{N-1} \widetilde{X}[k] e^{jk(2\pi/N)n} \quad n = 0, 1, ..., N-1
$$

DFT and DTFS

- DFT is for finite duration signals; DTFS is for periodic signals
- Define $\widetilde{x}[n]$ as the periodic extension of $x[n]$: Repeat $x[n]$ with period N
- Fourier series of $\widetilde{x}[n]$:

$$\frac{1}{N}\sum_{n=0}^{N-1}\widetilde{x}[n]e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk(2\pi/N)n} \;\Rightarrow\; \text{DFT of } x[n]$$

- DFT of a finite-duration signal = DTFS of its periodic extension
- Reference on DFT:
  - Textbook: Problem 5.53, 5.54
  - http://en.wikipedia.org/wiki/Discrete_Fourier_transform

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Comparison

Finite Duration Signals
Discrete Spectrum

DFT

DT Signal

Sample the spectrum

Same via periodic extension

DTFT

DTFS

General Signals
Continuous Spectrum

Periodic Signals
Discrete Spectrum

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Periodic Convolution of Finite Duration Signals

Periodic Convolution

Let x and y be two finite duration signals with duration N, $\tilde{x}$ and $\tilde{y}$ be the associated periodic extension, then the periodic convolution of finite duration signals is defined as
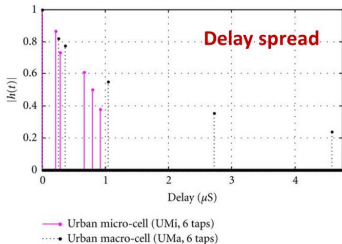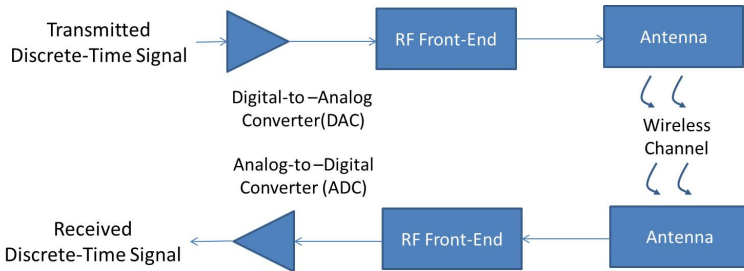
$$x[n] \circledast y[n] := \widetilde{x}[n] \circledast \widetilde{y}[n] = \sum_{k=<N>} \tilde{x}[k]\tilde{y}[n-k] \quad n = 0, 1, 2..., N-1$$

Convoluation Property of DFT

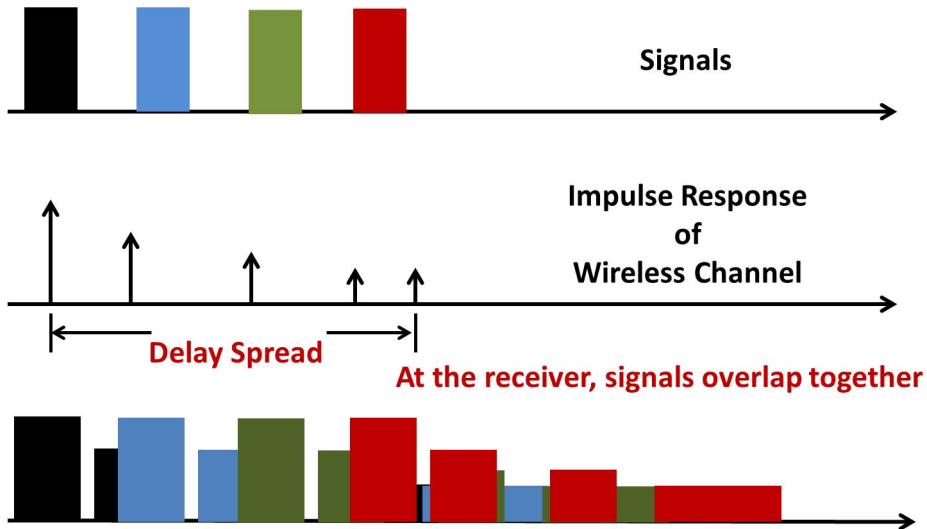*Time domain periodic convolution is equivalent to frequency domain multiplication, thus,*

- $x[n] \circledast y[n] \longleftrightarrow N\widetilde{X}[k]\widetilde{Y}[k]$

Wireless Systems

Delay spread causes inter-symbol interference

Inter-Symbol Interference (1/2)

Signals

Impulse Response of Wireless Channel

Delay Spread

**At the receiver, signals overlap together**

Inter-Symbol Interference (2/2)

- How to deliver signals without inter-symbol interference?
  - ▶ Suppose the duration of delay spread is ΔH seconds
- Approach 1: Send data on every ΔH seconds
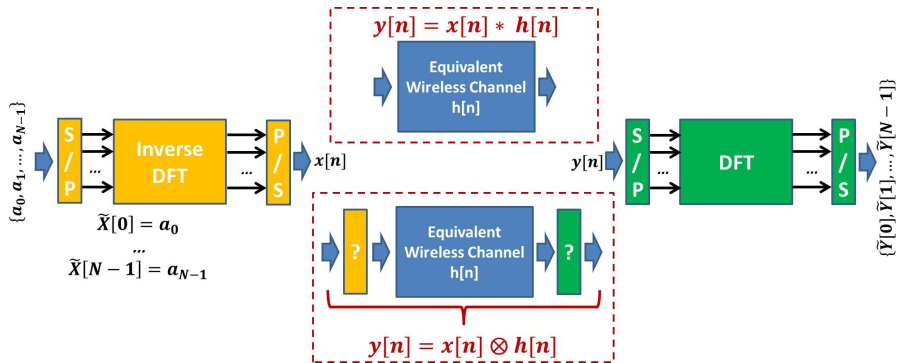- Approach 2: Channel equalizer
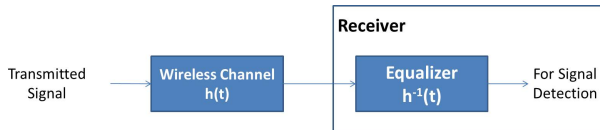
- Approach 3: Orthogonal Frequency Division Multiplexing
  - ▶ Pre-processing at the transmitter $+$ post-processing at the receiver

Discretetize Wireless Channel
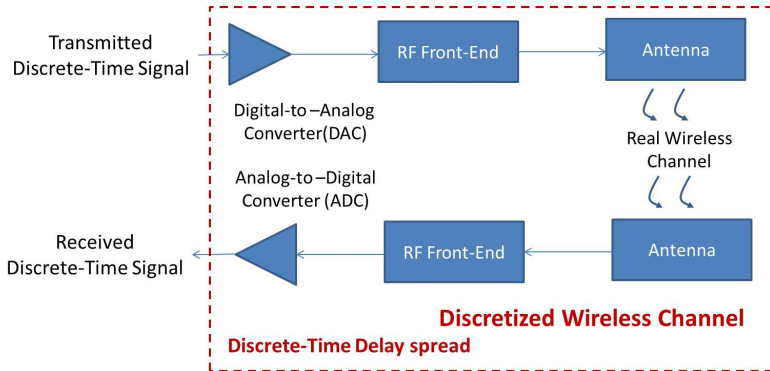
- Tx front-end + wireless channel + Rx front-end: approximately discrete-time LTI system
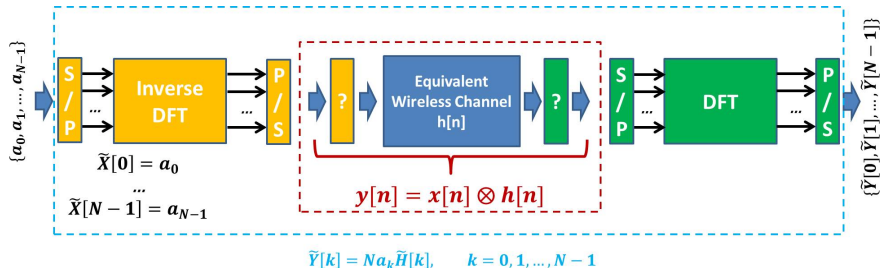- Denote the impulse response as $h[n]$

OFDM at First Glance

- Signals are loaded in frequency domain
- Some mechanism is necessary to generate the effect of periodic convolution

OFDM Analysis

- Let $\{a_0, a_1, ..., a_{N-1}\}$ be the sequence of bits to be delivered from the transmitter to the receiver.
- At the transmitter
  - we let
  $$\widetilde{X}[0] = a_0, \widetilde{X}[1] = a_1, ..., \widetilde{X}[N-1] = a_{N-1}$$
  - Take inverse DFT:
  $$\{x[0], x[1], ..., x[N-1]\} = DFT^{-1}\{\widetilde{X}[0], \widetilde{X}[1], ..., \widetilde{X}[N-1]\}$$
- With certain mechanism, the received signal becomes
$$y[n] = x[n] \circledast h[n]$$

- At the receiver, take DFT on $y[n]$:
$$\widetilde{Y}[k] = N\widetilde{X}[k]\widetilde{H}[k] = Na_k\widetilde{H}[k] \quad k = 0, 1, ..., N-1$$

So, ...

- How to detect $\{a_k | \forall k\}$ from $\{\widetilde{Y}[k] | \forall k\}$?
- How to design the blocks "?" ?

Channel Estimation & Signal Detection

- Channel estimation: Estimate the channel gain $\widetilde{H}[k]$ ($k = 0, 1, ..., N-1$)
  - ▶ The transmitter sends the common information to the receiver $\{a_0^c, a_1^c, ..., a_{N-1}^c\}$:

    $$\widetilde{Y}^c[k] = Na_k^c\widetilde{H}[k] \quad k = 0, 1, ..., N-1$$

  - ▶ Evaluate channel via

    $$\widetilde{H}[k] = \widetilde{Y}^c[k]/(Na_k^c) \quad k = 0, 1, ..., N-1$$

- Signal detection: Detect the transmitter's signal via the knowledge of $\widetilde{H}[k]$ ($k = 0, 1, ..., N-1$):
  - ▶ The transmitter sends the information to the receiver $\{a_0, a_1, ..., a_{N-1}\}$:

    $$\widetilde{Y}[k] = Na_k\widetilde{H}[k] \quad k = 0, 1, ..., N-1$$

  - ▶ Detect signal via

    $$a_k = \widetilde{Y}[k]/(N\widetilde{H}[k]) \quad k = 0, 1, ..., N-1$$
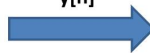
Convolution vs. Periodic Convolution

$$y[n] = x[n] \otimes h[n]$$

$$y[n] = \hat{x}[n] * h[n]$$
$$= x[n] \otimes h[n]$$
n=0,1,2,3,4
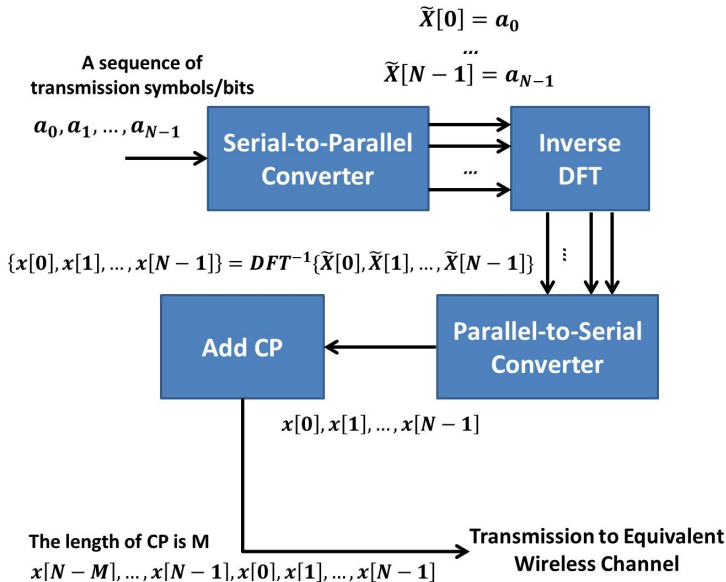
$\hat{x}[n]$

h[n]

y[0]

y[1]

...

y[4]
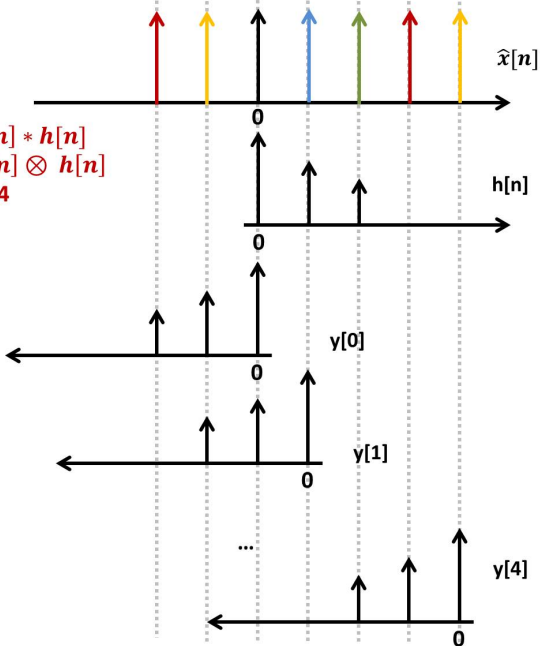
Observations

- Periodic convolution can be different from convolution
- In order to guarantee that

$$y[n] = x[n] \circledast h[n], \text{ for } n = 0, 1, 2, ..., N-1,$$
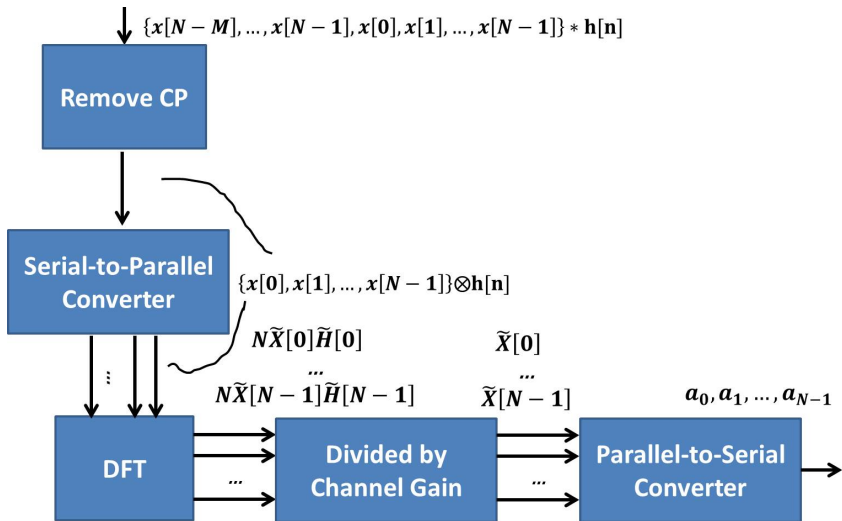
we should copy the last few samples of $x[n]$ to the beginning, which is named as cyclic prefix

  ▶ How many samples should we copy?

Transmitter Structure of OFDM

$$\widetilde{X}[0] = a_0$$
$$\vdots$$
$$\widetilde{X}[N-1] = a_{N-1}$$

A sequence of transmission symbols/bits

$a_0, a_1, \ldots, a_{N-1}$

Serial-to-Parallel Converter

Inverse DFT

$\{x[0], x[1], \ldots, x[N-1]\} = DFT^{-1}\{\widetilde{X}[0], \widetilde{X}[1], \ldots, \widetilde{X}[N-1]\}$

Add CP

Parallel-to-Serial Converter

$x[0], x[1], \ldots, x[N-1]$

The length of CP is M
$x[N-M], \ldots, x[N-1], x[0], x[1], \ldots, x[N-1]$

Transmission to Equivalent Wireless Channel

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Receiver Structure of OFDM

$$\{x[N-M], \ldots, x[N-1], x[0], x[1], \ldots, x[N-1]\} * \mathbf{h[n]}$$

Remove CP

Serial-to-Parallel Converter

$$\{x[0], x[1], \ldots, x[N-1]\} \otimes \mathbf{h[n]}$$

$$N\widetilde{X}[0]\widetilde{H}[0]$$

$$\widetilde{X}[0]$$

$$N\widetilde{X}[N-1]\widetilde{H}[N-1]$$

$$\widetilde{X}[N-1]$$

$$a_0, a_1, \ldots, a_{N-1}$$

DFT

Divided by Channel Gain

Parallel-to-Serial Converter

Reference

- Reference
  - www.gaussianwaves.com/2011/05/introduction-to-ofdm-orthogonal-frequency-division-multiplexing-2/
  - www.wirelesscommunication.nl/reference/chaptr05/ofdm/ofdmmath.htm