Homework & Tutorial Problems

- Tutorial
  - Property of DTFT: 5.24
  - Difference equation of LTI systems: 5.36
- Homework
  - 5.23, 5.29, 5.33

Recap: Definition of DTFT

- Therefore, we get the discrete-time Fourier transform pair

Discrete-Time Fourier Transform

$$\textit{Synthesis Equation:} \quad x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$\textit{Analysis Equation:} \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Observations
  - Continuous spectrum: Similar to CTFT
  - Periodic with period 2π: Different from CTFT
  - Low frequency: close to 0 and 2π; high frequency: close to π

Recap: DTFT of Periodic Signals

- According to the Fourier series, for a periodic signal with period N:

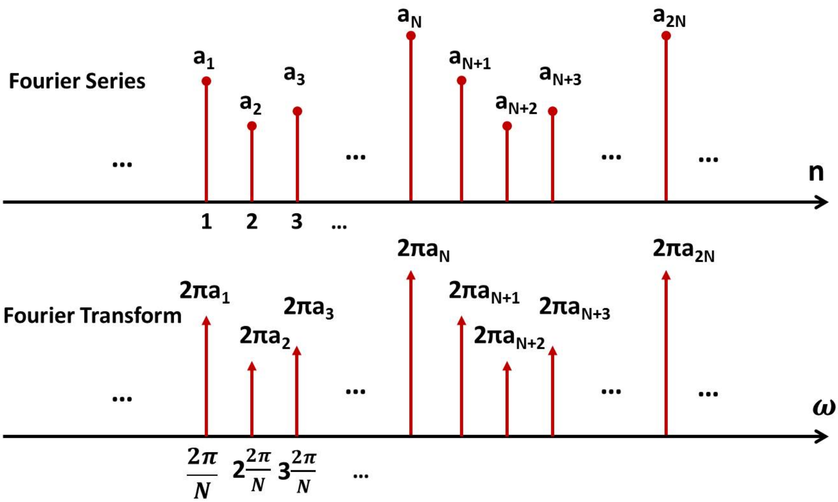
$$x[n] = a_0 + a_1 e^{j(2\pi/N)n} + \ldots + a_k e^{jk(2\pi/N)n} + \ldots + a_{N-1} e^{j(N-1)(2\pi/N)n}$$

- $e^{jk(2\pi/N)n} \longleftrightarrow \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - k(2\pi/N) - 2\pi l)$
- Then, due to the linearity of Fourier transform

$$\begin{aligned}\mathcal{F}\left\{x[n]\right\} = \sum_{k=0}^{N-1} a_k \mathcal{F}\left\{e^{jk(2\pi/N)n}\right\} &= \sum_{k=0}^{N-1} a_k \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - k(2\pi/N) - 2\pi l) \\ &= \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} a_k 2\pi\delta(\omega - k(2\pi/N) - 2\pi l)\end{aligned}$$

- Fourier transform of a periodic signal is a periodic sequence of impulses
  - What's the period? How many impulses within one period?

Recap: Fourier Series v.s. Fourier Transform



Recap: Periodicity, Linearity and Shifting

- Periodicity

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

  - How about CTFT? Why?

- Linearity

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

- Time Shifting and Frequency Shifting

$$\begin{aligned} x[n-n_0] &\longleftrightarrow e^{-j\omega n_0}X(e^{j\omega}) \\ e^{j\omega_0 n}x[n] &\longleftrightarrow X(e^{j(\omega-\omega_0)}) \end{aligned}$$

  - What's the physical meaning?

Conjugation, Differencing and Accumulation

- Conjugation

$$x^*[n] \longleftrightarrow X^*(e^{-j\omega})$$

  - $X(e^{j\omega}) = X^*(e^{-j\omega}) \Leftrightarrow x[n]$ is real
  - If $x[n]$ is real, then $\Re\{X(e^{j\omega})\}$ is even, $\Im\{X(e^{j\omega})\}$ is odd

- Differencing

$$x[n] - x[n-1] \quad \longleftrightarrow \quad (1 - e^{-j\omega})X(e^{j\omega})$$

  - High-pass or low-pass?

- Accumulation

$$\sum_{m=-\infty}^{n} x[m] \longleftrightarrow \frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

  - How to derive it via differencing?

Effect of Differencing

- $J(i,j) = |M(i,j) - M(i+1,j+1)| + |M(i+1,j) - M(i,j+1)|$

Fourier Transform of $u[n]$

- How to derive the Fourier transform of u[n]?
- Option 1: From definition of Fourier transform

$$\mathcal{F}\{u[n]\} = \sum_{n=-\infty}^{+\infty} u[n]e^{-j\omega n} = \sum_{n=0}^{+\infty} e^{-j\omega n} \quad Converge??$$

- Option 2: Since $u[n] = \sum_{m=-\infty}^{n} \delta[m]$, according to the property of accumulation

$$\mathcal{F}\{u[n]\} = \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

- Observation: Fourier transform of $u[n]$ does not exist according to the definition; however, it can be expressed in terms of $\delta(\cdot)$

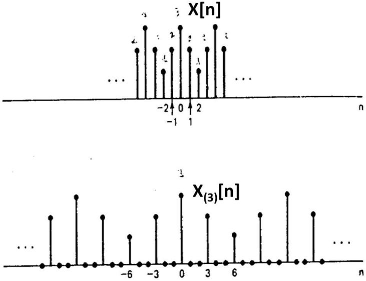
Time Reversal and Expansion

- Time Reversal

$$x[-n] \longleftrightarrow X(e^{-j\omega})$$

- Time Expansion
  - Define $x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of k} \\ 0, & \text{Otherwise} \end{cases}$, then

$$x_{(k)}[n] \longleftrightarrow X(e^{jk\omega}) \tag{1}$$



Differentiation and Parseval

- Differentiation in Frequency

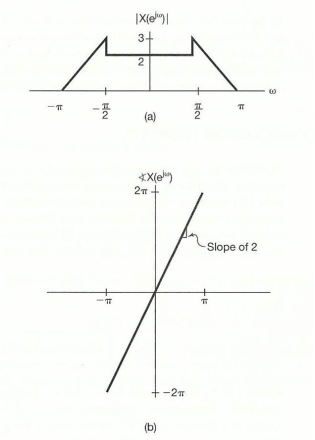
$$nx[n] \longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$$

- Parseval's Relation

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

  - Energy density spectrum — $\frac{|X(e^{j\omega})|^2}{2\pi}$

Problem



- See textbook, Example 5.10
- Spectrum within $[-\pi, \pi]$
- Is it periodic, real, even, of finite energy?

Problem

1. Let $X(e^{j\omega})$ be the Fourier transform of $x[n]$. Derive expressions in terms of $X(e^{j\omega})$ for the Fourier transforms of the following signals.
   - $\mathbf{Re}\{x[n]\}$
   - $x^*[-n]$
   - $\mathbf{Ev}\{x[n]\}$
   - $(n-2)^2 x[n]$
2. Please calculate the Fourier transform of $(n-1)(\frac{1}{4})^{|n|}$

Convolution Property & LTI Systems

Convolution Property

If $y[n] = x_1[n] * x_2[n]$, *then*

$$Y(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega})$$

- For general input signal $x[n]$:

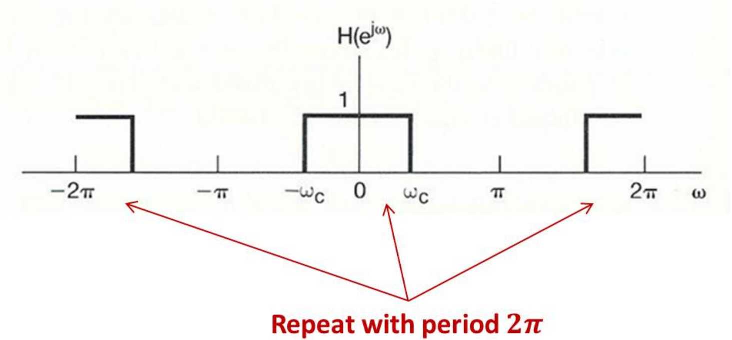
$$y[n] = x[n] * h[n] \longleftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- Observation: It's easier to evaluate LTI systems in frequency domain
- Drawback: Not every LTI system has frequency response
  - $h[n] = a^n u[n]$ $(a > 1)$
  - Stable LTI system has frequency response, because

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Example: Ideal Low-Pass Filter (1/2)

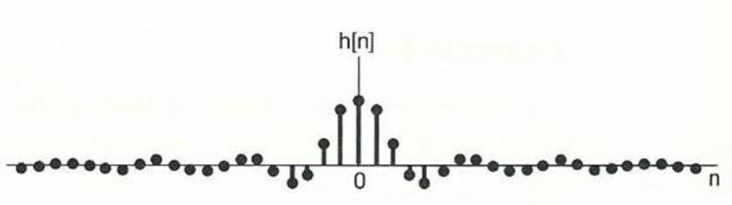
- What's ideal low-pass filter?
- Perfectly maintain the low-frequency component
- Perfectly cancel the high-frequency component



Example: Ideal Low-Pass Filter (2/2)

- Impulse response:

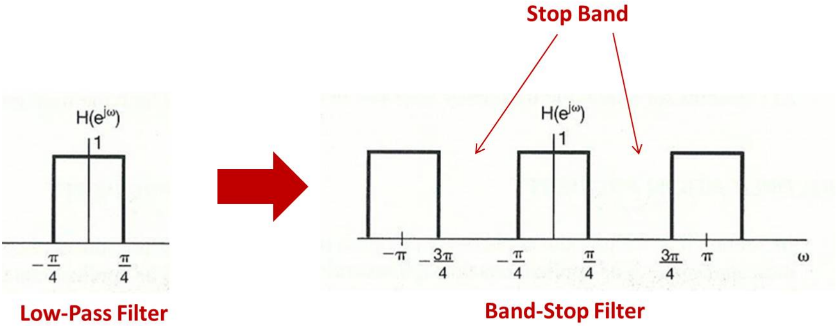
$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega = \frac{sin\omega_c n}{\pi n}$$



- Pros: no distortion in frequency domain
- Cons: non-causal
- See textbook, Example 5.12

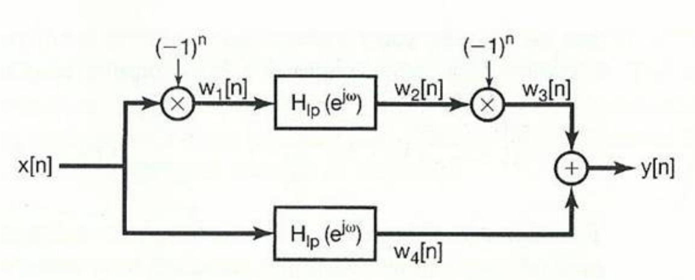
Example: Band-Stop Filter (1/2)

- Given low-pass filter $H_{lp}(e^{j\omega})$, how to generate band-stop effect from low-pass filter?



Example: Band-Stop Filter (2/2)

- Two branches: low-pass + high-pass



- $(-1)^n = e^{j\pi n} \Rightarrow W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$

$$\Rightarrow W_2(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j(\omega-\pi)})$$

$$\Rightarrow W_3(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j(\omega-2\pi)}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega})$$

- See textbook, Example 5.14

Multiplication Property

Multiplication Property

Let $y[n] = x_1[n]x_2[n]$*, then*

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta,$$

which is periodic convolution of $X_1(e^{j\omega})$ *and* $X_2(e^{j\omega})$

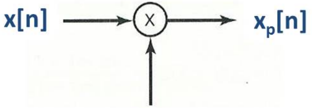
- Comparison: Multiplication of periodic signals

$$x_1[n] \longleftrightarrow a_k \quad \text{and} \quad x_2[n] \longleftrightarrow b_k$$

$$\Rightarrow x_1[n]x_2[n] \longleftrightarrow \sum_{k=<N>} a_k b_{n-k} \quad \text{discrete-time periodic convolution}$$

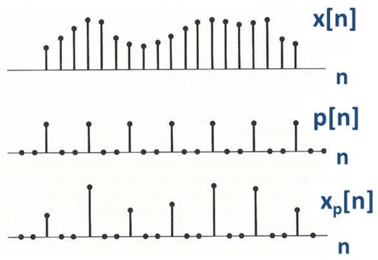
Example: Sampling on Discrete-Time Signals

- In digital signal processing, we may sample the discrete-time signals
- Example: image, audio, video compression
- Its mathematical model is give below:



$$\mathbf{p[n]} = \sum_{\mathbf{k}=-\infty}^{+\infty} \delta[\mathbf{n}-\mathbf{kN}]$$

$$\mathbf{x_p[n]} = \sum_{\mathbf{k}=-\infty}^{+\infty} x[kN]\delta[\mathbf{n}-\mathbf{kN}]$$



Review: Discrete-Time Impulse Chain

- What's the Fourier transform of $x[n] = \sum_{k=-\infty}^{+\infty} \delta[n-kN]$?
- First of all, we calculate the Fourier series:

$$\begin{aligned} a_k &= \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\frac{2\pi}{N}n} \\ &= \frac{1}{N} \sum_{n=<N>} \sum_{k=-\infty}^{+\infty} \delta[n-kN] e^{-jk\frac{2\pi}{N}n} \\ &= \frac{1}{N} \sum_{n=<N>} \delta[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} \end{aligned}$$

- Then, we have

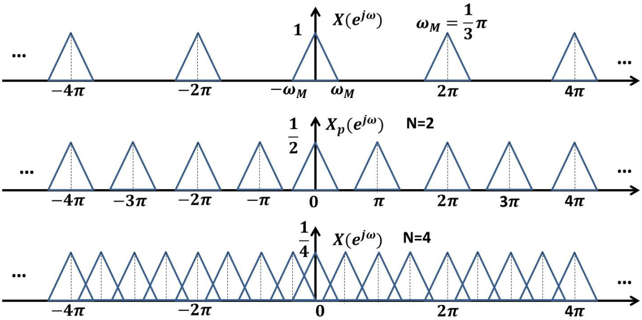
$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} \frac{2\pi}{N} \delta(\omega - k(2\pi/N) - 2\pi l) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{N})$$

- Time domain period $\times$ Frequency domain period $=$?

Example: Frequency Analysis

$$P(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s) \quad where\ \omega_s = 2\pi/N$$

$$X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})X(e^{j(\omega-\theta)})d\theta = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-k\omega_s)})$$



Summary of Duality

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$ discrete time aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ continuous frequency periodic in frequency

duality

duality

duality

- See textbook, Table 5.3

LTI by Difference Equation

- A number of DT LTI systems can be written as the following linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

What's the frequency response?

- Taking Fourier transform on both side, we have

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$$

$$\Rightarrow \quad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

What's the impulse response?

Problem: Difference Equation

Please calculate the frequency and impulse response of the following LTI systems

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n].$$

Solution

According to the last slide,

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}.$$

Moreover, by partial fraction expansion

$$H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}} \Rightarrow h[n] = 4(\frac{1}{2})^n u[n] - 2(\frac{1}{4})^n u[n]$$