EE205 Signals and Systems
Tutorial 1 (Week 3)

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Problem 2.3, 2.7, 2.13, 2.24, 2.26

2.3.

Problem:
Consider an input $x[n]$ and a unit impulse response h[n] given by

$$x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2],$$
$$h[n] = u[n+2].$$

Determine and plot the output $y[n] = x[n] * h[n]$.
Solution:
Hint:

- Shifting property of $\delta[n]$: $x[n-N] = x[n] * \delta[n-N]$.
- Definition of convolution: $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$
- Definition of unit step function $u[n]$:

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}.$$

According to the shifting property of $\delta[n]$, we can derive

$$x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2] = \left[\left(\frac{1}{2}\right)^{n} u[n]\right] * \delta[n-2],$$
$$h[n] = u[n+2] = u[n] * \delta[n+2].$$

Then the convolution can be obtained by

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \left[\left(\frac{1}{2}\right)^{n} u[n]\right] * \delta[n-2] * u[n] * \delta[n+2] \\ &= \left[\left(\frac{1}{2}\right)^{n} u[n]\right] * u[n] \\ &= \sum_{k=-\infty}^{+\infty} u[k] * \left[\left(\frac{1}{2}\right)^{n-k} u[n-k]\right] \\ &= \sum_{k=0}^{+\infty} \left(\frac{1}{2}\right)^{n-k} u[n-k] \\ &= \left(\frac{1}{2}\right)^{n} \sum_{k=0}^{+\infty} \left(\frac{1}{2}\right)^{-k} u[n-k]. \end{aligned}$$

$$\sum_{k=0}^{+\infty} \left(\frac{1}{2}\right)^{-k} u[n-k] = \begin{cases} 0, & n < 0 \\ \sum_{k=0}^{n} (\frac{1}{2})^{-k} = \frac{1(1-2^{n+1})}{1-2} = 2^{n+1} - 1, & n \geq 0 \end{cases}.$$

It can be simplified by $\sum_{k=0}^{+\infty}\left(\frac{1}{2}\right)^{-k}u[n-k]=(2^{n+1}-1)u[n]$.

$$
\begin{aligned}
y[n] &= \left(\frac{1}{2}\right)^n(2^{n+1}-1)u[n] \\
&= \left(2-\left(\frac{1}{2}\right)^n\right)u[n]
\end{aligned}
$$

2.7.

Problem:
A linear system S has the relationship

$$
y[n]=\sum_{k=-\infty}^{+\infty}x[k]g[n-2k]
$$

between its input $x[n]$ and its output y[n], where $g[n]=u[n]-u[n-4]$.
(a) Determine $y[n]$ when $x[n]=\delta[n-1]$.
(b) Determine $y[n]$ when $x[n]=\delta[n-2]$.
(c) Is S LTI?
(d) Determine $y[n]$ when $x[n]=u[n]$.
Solution:
Hint:

- Sampling property of $\delta[n]$: $\delta[n-k]x[n]=x[k]$.
- Given $y[n]=\sum_{-\infty}^{+\infty}x[k]g[n-2k]$, let $m[k]=g[n-2k]$.

(a)

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{+\infty}\delta[k-1]g[n-2k] \\
&= g[n-2\times 1] \\
&= g[n-2] \\
&= u[n-2]-u[n-6]
\end{aligned}
$$

(b)

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{+\infty}\delta[k-2]g[n-2k] \\
&= g[n-2\times 2] \\
&= g[n-4] \\
&= u[n-4]-u[n-8]
\end{aligned}
$$

(c)
Recall in (a) and (b), the input signal $x_2[n]$ in (b) is a delay version of the input signal $x_1[n]$ in (a), i.e.,

$$
x_2[n]=x_1[n-1].
$$

If it is LTI, the outputs shall satisfy

$$
y_2[n]=y_1[n-1]
$$

However, $y_2[n]=y_1[n-2]$. Therefore, it is not a LTI system.
(d)

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{+\infty}u[k]g[n-2k] \\
&= \sum_{k=0}^{+\infty}g[n-2k] \\
&= g[n]+g[n-2]+g[n-4]+\dots
\end{aligned}
$$

Recall that (it is recommended to plot $g[n]$)

$$\begin{aligned} g[n] &= u[n] - u[n-4] \\ &= \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] \end{aligned}$$

So that

$$\begin{aligned} y[n] &= g[n] + g[n-2] + g[n-4] + \dots \\ &= \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] \\ &+ \qquad\qquad \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5] \\ &+ \qquad\qquad\qquad\qquad \delta[n-4] + \delta[n-5] + \delta[n-6] + \delta[n-7] \\ &+ \qquad\qquad\qquad\qquad\qquad\qquad \dots \\ &= \delta[n] + \delta[n-1] + 2\Big(\delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5] + \dots\Big) \\ &= \delta[n] + \delta[n-1] + 2u[n-2] \end{aligned}$$

All in all, $y[n] = \delta[n] + \delta[n-1] + 2u[n-2]$.
Or it can be written as $y[n] = 2u[n] - \delta[n] - \delta[n-1]$

2.13.

Problem:
Consider a discrete-time system S_1 with impulse response

$$h[n] = \left(\frac{1}{5}\right)^n u[n].$$

(a) Find the integer A such that $h[n] - Ah[n-1] = \delta[n]$.
(b) Using the result from part (a), determine the impulse response $g[n]$ of an LTI system S_2 which is the inverse system of S_1.
Solution:
(a)

solution 1: The most simple way is putting $n = 1$ and solving A. We can get $A = \frac{1}{5}$.
solution 2: Note that $x[n]u[n] = x[0]\delta[n] + x[n]u[n-1]$.

$$\begin{aligned} h[n] - Ah[n-1] &= \delta[n] \\ \left(\frac{1}{5}\right)^n u[n] - A\left(\frac{1}{5}\right)^{n-1} u[n-1] &= \delta[n] \\ 1\delta[n] + \left(\frac{1}{5}\right)^n u[n-1] - A\left(\frac{1}{5}\right)^{n-1} u[n-1] &= \delta[n] \\ \left(\frac{1}{5}\right)^n u[n-1] - A\left(\frac{1}{5}\right)^{n-1} u[n-1] &= 0 \\ \left(\frac{1}{5}\right)^n u[n-1] &= A\left(\frac{1}{5}\right)^{n-1} u[n-1] \\ A &= \frac{1}{5} \end{aligned}$$

(b)

Determine the impulse response g[n] of the inverse LTI system S_2 so that

$$\delta[n] \xrightarrow{S_1, h[n]} h[n] \xrightarrow{S_2, g[n]} \delta[n]$$

$$h[n] * g[n] = \delta[n]$$

Recall that in part (a), we have $h[n] - \frac{1}{5}h[n-1] = \delta[n]$.

- Shifting property of $\delta[n]$: $x[n-N] = x[n] * \delta[n-N]$.

$$\begin{aligned} h[n] - \frac{1}{5}h[n-1] &= \delta[n] \\ h[n] * \delta[n] - \frac{1}{5}h[n] * \delta[n-1] &= \delta[n] \\ h[n] * \left(\delta[n] - \frac{1}{5}\delta[n-1]\right) &= \delta[n] \end{aligned}$$

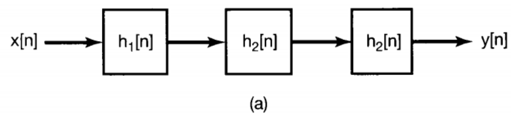
Therefore, $g[n] = \delta[n] - \frac{1}{5}\delta[n-1]$.

2.24.

Problem:
Consider the cascade interconnection of three causal LTI systems, illustrated in Figure P2.24(a). The impulse response $h_2[n]$ is

$$h_2[n] = u[n] - u[n-2],$$

and the overall impulse response is shown in Figure P2.24(b).



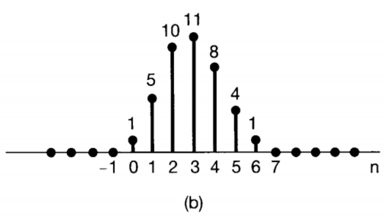


Figure P2.24

(a) Find the impulse response $h_1[n]$.
(b) Find the response of the overall system to the input

$$x[n] = \delta[n] - \delta[n-1].$$

Solution:
(a)

- Casual: Current output only depends on current or past input, not depends on future input, i.e., $h[n] = 0$ for $\forall n < 0$.

$$h_2[n] = u[n] - u[n-2] = \delta[n] + \delta[n-1]$$

$$\begin{aligned} h[n] &= h_1[n] * h_2[n] * h_2[n] \\ &= h_1[n] * (\delta[n] + \delta[n-1]) * (\delta[n] + \delta[n-1]) \\ &= (h_1[n] + h_1[n-1]) * (\delta[n] + \delta[n-1]) \\ &= h_1[n] + 2h_1[n-1] + h_1[n-2] \end{aligned}$$

Since we already know the figure of $h[n]$,

$$\begin{aligned} &h[0] = h_1[0] + 2h_1[-1] + h_1[-2] = h_1[0] = 1 \\ &h[1] = h_1[1] + 2h_1[0] + h_1[-1] = h_1[1] + 2h_1[0] = 5 \\ &h[2] = h_1[2] + 2h_1[1] + h_1[0] = h_1[0] = 10 \\ &\quad \ldots \\ &\quad \text{(Calculate by yourself)} \end{aligned}$$

(b)

$$\begin{aligned} y[n] &= (\delta[n] - \delta[n-1]) * h[n] \\ &= h[n] - h[n-1] \end{aligned}$$

The figure of $h[n]$ is given, so you can also plot the figure of $y[n]$ by yourself.

2.26.

Problem:
Consider the evaluation of

$$y[n] = x_1[n] * x_2[n] * x_3[n],$$

where $x_1[n] = (0.5)^n u[n]$, $x_2[n] = u[n+3]$, and $x_3[n] = \delta[n] - \delta[n-1]$.
(a) Evaluate the convolution $x_1[n] * x_2[n]$.
(b) Convolute the result of part (a) with $x_3[n]$ in order to evaluate $y[n]$.
(c) Evaluate the convolution $x_2[n] * x_3[n]$.
(d) Convolute the result of part (c) with $x_1[n]$ in order to evaluate $y[n]$.
Solution:

- Like "+,−,×,÷", convolution ($*$) also have the associative property, coomunicative property and distributive property.
- $x[n-N] = x[n] * \delta[n-N]$.

(a)

$$\begin{aligned} t_1[n] &= x_1[n] * x_2[n] \\ &= \Big((0.5)^n u[n]\Big) * u[n+3] \\ &= \Big((0.5)^n u[n]\Big) * u[n] * \delta[n+3] \\ &\overset{(a)}{=} \left(2 - \left(\frac{1}{2}\right)^n\right) u[n] * \delta[n+3] \\ &= \left(2 - \left(\frac{1}{2}\right)^{n+3}\right) u[n+3] \end{aligned}$$

where (a) can be derived in Problem 2.3.
(b)

$$\begin{aligned} y[n] &= t_1[n] * x_3[n] \\ &= t_1[n] * \Big(\delta[n] - \delta[n-1]\Big) \\ &= t_1[n] - t_1[n-1] \\ &= \left(2 - \left(\frac{1}{2}\right)^{n+3}\right) u[n+3] - \left(2 - \left(\frac{1}{2}\right)^{n+2}\right) u[n+2] \\ &\overset{(a)}{=} \left(2 - \left(\frac{1}{2}\right)^{-3+3}\right) \delta[n+3] + \left(2 - \left(\frac{1}{2}\right)^{n+3}\right) u[n+2] - \left(2 - \left(\frac{1}{2}\right)^{n+2}\right) u[n+2] \\ &= \delta[n+3] + \left(\left(\frac{1}{2}\right)^{n+2} - \left(\frac{1}{2}\right)^{n+3}\right) u[n+2] \\ &= \delta[n+3] + \left(\frac{1}{2}\right)^{n+3} u[n+2] \\ &\overset{(b)}{=} \left(\frac{1}{2}\right)^{n+3} u[n+3] \end{aligned}$$

where (a) and (b) are due to $x[n]u[n] = x[0]\delta[n] + x[n]u[n-1]$.
(c)

$$\begin{aligned} t_2[n] &= x_2[n] * x_3[n] \\ &= u[n+3] * \Big(\delta[n] - \delta[n-1]\Big) \\ &= u[n+3] - u[n+2] \\ &= \delta[n+3] \end{aligned}$$

(d)

$$\begin{aligned} y[n] &= x_1[n] * t_2[n] \\ &= x_1[n] * \delta[n+3] \\ &= x_1[n+3] \\ &= \left(\frac{1}{2}\right)^{n+3} u[n+3] \end{aligned}$$

Compared with the result of part (b), we can notice that results of (b) and (d) are the same.