Homework & Tutorial

Homework: 5.2, 5.5, 5.15, 5.21(a-f, h)
Tutorial Problems: 5.1, 5.3, 5.4, 5.41

Chapter 5: The Discrete-Time Fourier Transform

Department of Electrical & Electronic Engineering

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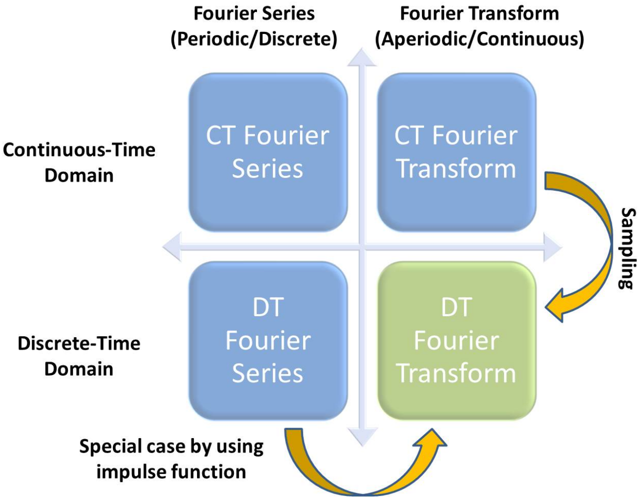
A Big Picture



Outline

- Definition of discrete time Fourier transform (DTFT)
  - From Fourier series to Fourier transform
  - Convergence issue
- Periodic signals' DTFT
- Properties of DTFT

Discrete-Time Fourier Transform

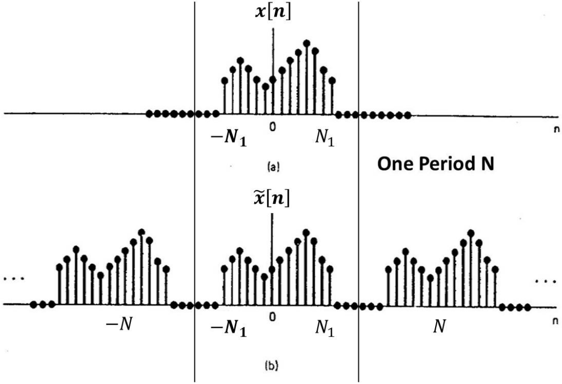
- Discrete-time Fourier series (for periodic signals)

$$\text{Synthesis Equation: } x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$\text{Analysis Equation: } a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk(2\pi/N)n}$$

- Aperiodic signals can be treated as periodic signals with period $N \to \infty$
  - $x[n]$ must be like $\sum_k b_k e^{jk(2\pi/N)n}$ or $\int_\omega b(\omega)e^{j\omega n}d\omega$
  - b_k or $b(\omega)$ can be calculated from $x[n]$

Derivation (1/3)



- Original signal: $x[n]$
- Define new periodic signal with period N: $\tilde{x}[n]$, such that

$$\tilde{x}[n] = x[n], \;\; n = -N/2, ..., N/2 - 1$$

- Notice: when $N \to \infty$, $\tilde{x}[n]$ becomes $x[n]$

Derivation (2/3)

- Look at the Fourier series of $\widetilde{x}[n]$:

$$a_k = \frac{1}{N}\sum_{n=-N/2}^{N/2+1} \widetilde{x}[n]e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-jk(2\pi/N)n}$$

- Define $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$, so that $a_k = X(e^{jk\omega_0})/N$ ($\omega_0 = 2\pi/N$)
- Look at the synthesis equation:

$$\begin{aligned}\widetilde{x}[n] &= \sum_{k=<N>} a_k e^{jk(2\pi/N)n} = \frac{1}{N}\sum_{k=<N>} X(e^{jk\omega_0})e^{jk\omega_0 n} \\ &= \frac{1}{2\pi}\sum_{k=<N>} X(e^{jk\omega_0})e^{jk\omega_0 n}\omega_0\end{aligned}$$

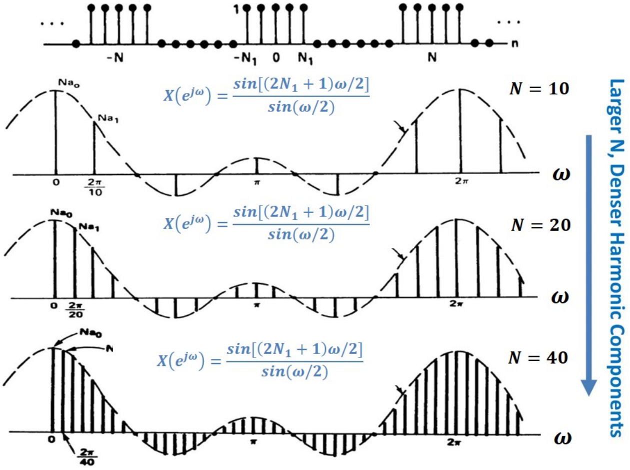
- When $N \to \infty$, $\omega_0 \to d\omega$ $\Sigma \to \int$ $k\omega_0 \to \omega$ $\widetilde{x}[n] \to x[n]$, thus,

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Example: From Periodic To Aperiodic

$$Na_k = \frac{\sin\frac{2\pi k\left(N_1+\frac{1}{2}\right)}{N}}{\sin(\frac{\pi k}{N})}$$

$$= X(e^{\frac{j2\pi k}{N}})$$



Derivation (3/3)

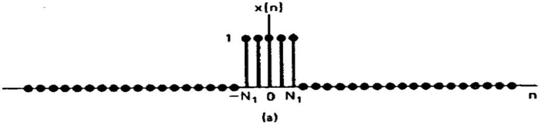
- Therefore, we get the discrete-time Fourier transform pair

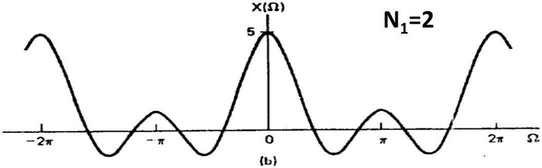
Discrete-Time Fourier Transform

$$\begin{aligned} \textit{Synthesis Equation:} \quad x[n] &= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \\ \textit{Analysis Equation:} \quad X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \end{aligned}$$

- Observations
  - Continuous spectrum: Similar to CTFT
  - Periodic with period 2π: Different from CTFT
  - Low frequency: close to 0 and 2π; high frequency: close to π

Fourier Transform Examples (1/2)

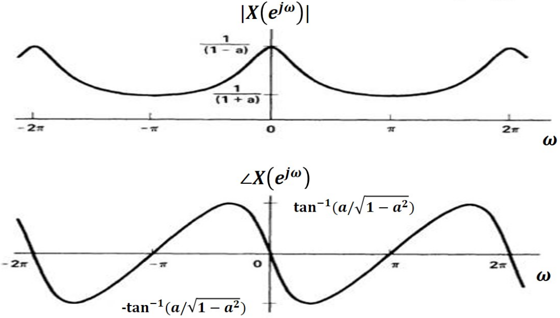




- $x[n] = 1 \ (n = -N_1, ..., 0, ..., N_1)$
- $X(e^{j\omega}) = \frac{sin\omega(N_1+1/2)}{sin(\omega/2)}$
- Width of $x[n]$: $W_t = 2N_1 + 1$; width of $X(e^{j\omega})$: $W_f = \frac{4\pi}{2N_1+1}$
- $W_t \times W_f = 4\pi$, which is a constant

Fourier Transform Examples (2/2)

$$x[n] = a^n u[n] \quad 0 < a < 1 \quad \Leftrightarrow \quad X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$



- See textbook, Example 5.1
- What's the shape of magnitude when $a \to 1$ or $a \to 0$?

Convergence Issue of Analysis Equation

Sufficient Condition of Convergence

The analysis equation $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$ *will converge either if* $x[n]$ *is absolutely summable or if the sequence has finite energy, thus,*

$$\sum_{-\infty}^{\infty} |x[n]| < \infty \ \textit{or} \ \sum_{-\infty}^{\infty} |x[n]|^2 < \infty$$

- Do the following signals have Fourier transform:
  - $a^n u[n]$ $(0 < a < 1)$
  - $\delta[n]$
  - $u[n]$
  - $e^{j\frac{2}{5}\pi n}$, $cos(\frac{2}{5}\pi n)$
  - $a^n u[n]$ $(a > 1)$

Can Periodic Signals Have DTFT?

- Definition of DTFT:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Justification of divergence:
  - Let $\omega = 2k\pi$, we have $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]$
  - Since $x[n]$ is periodic, the summation $\sum_{n=-\infty}^{\infty} x[n]$ will never converge unless $x[n] = 0$
- Conclusion: Most of periodic signals do NOT have DTFT according to the definition
- However, it's of significant engineering importance to extend Fourier transform to periodic signals

DTFT with Periodic Signals (1/2)

Fourier Transform of $e^{j\omega n}$

The following transform pair is actually NOT rigorously defined:

$$x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

- Synthesis:

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{2\pi} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)e^{j\omega n}d\omega = e^{j\omega_0 n}$$

- Analysis:

$$\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} e^{j(\omega_0-\omega)n} \quad \textit{converge??}$$

Remark

- We just believe the following equation is true:

$$\sum_{n=-\infty}^{\infty} e^{j\omega n} = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi l)$$

- When $\omega = 2k\pi$,

$$\sum_{n=-\infty}^{\infty} 1 = 2\pi\delta(0)$$

DTFT with Periodic Signals (2/2)

- According to the Fourier series, for a periodic signal with period N:

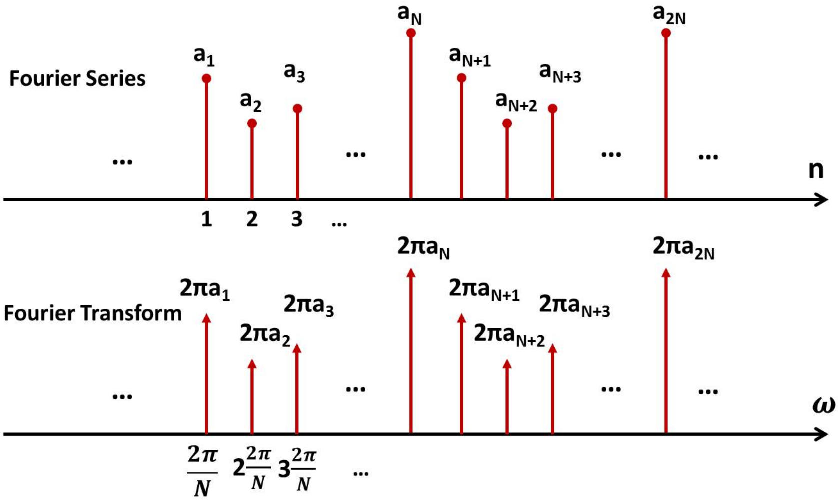
$$x[n] = a_0 + a_1 e^{j(2\pi/N)n} + \ldots + a_k e^{jk(2\pi/N)n} + \ldots + a_{N-1} e^{j(N-1)(2\pi/N)n}$$

- $e^{jk(2\pi/N)n} \longleftrightarrow \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - k(2\pi/N) - 2\pi l)$
- Then, due to the linearity of Fourier transform

$$\begin{aligned}\mathcal{F}\left\{x[n]\right\} = \sum_{k=0}^{N-1} a_k \mathcal{F}\left\{e^{jk(2\pi/N)n}\right\} &= \sum_{k=0}^{N-1} a_k \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - k(2\pi/N) - 2\pi l) \\ &= \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} a_k 2\pi\delta(\omega - k(2\pi/N) - 2\pi l)\end{aligned}$$

- Fourier transform of a periodic signal is a periodic sequence of impulses
  - What's the period? How many impulses within one period?

Fourier Series v.s. Fourier Transform



Example: Discrete-Time Impulse Chain

- What's the Fourier transform of $x[n] = \sum_{k=-\infty}^{+\infty} \delta[n-kN]$?
- First of all, we calculate the Fourier series:

$$
\begin{aligned}
a_k &= \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{N} \sum_{n=<N>} \sum_{k=-\infty}^{+\infty} \delta[n-kN] e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{N} \sum_{n=<N>} \delta[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}
\end{aligned}
$$

- Then, we have

$$
X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} \frac{2\pi}{N} \delta(\omega - k(2\pi/N) - 2\pi l) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{N})
$$

- Time domain period $\times$ Frequency domain period $=$?

Periodicity, Linearity and Shifting

- Periodicity

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

  - How about CTFT? Why?
- Linearity

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

- Time Shifting and Frequency Shifting

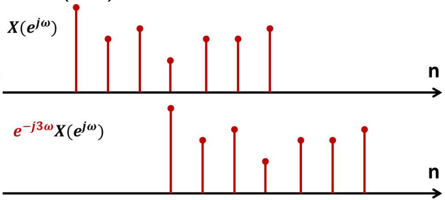
$$\begin{aligned} x[n-n_0] &\longleftrightarrow e^{-j\omega n_0}X(e^{j\omega}) \\ e^{j\omega_0 n}x[n] &\longleftrightarrow X(e^{j(\omega-\omega_0)}) \end{aligned}$$

  - What's the physical meaning?

Illustration on Shifting

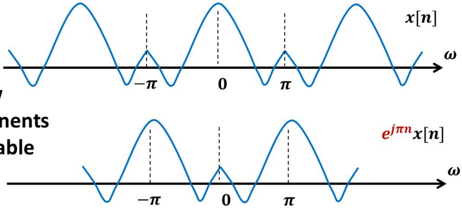
- $x[n - n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$

Delay raises linear phase shifting



- $e^{j\omega_0 n} x[n] \longleftrightarrow X(e^{j(\omega - \omega_0)})$

High and low frequency components are interchangeable

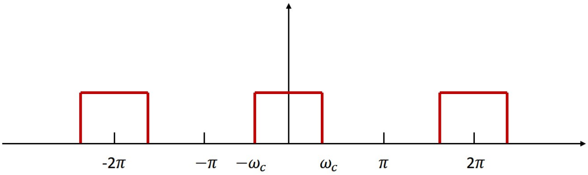


Example: Shifting

- Similar to CT, the DTFT of impulse response of a DT LTI system is frequency response.

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n}$$

- Given an ideal LPF with the following frequency response



- Suppose its impulse response is $h_{lpf}[n]$, please predict the behavior of another LTI system with impulse response $(-1)^n h_{lpf}[n]$.

Conjugation, Differencing and Accumulation

- Conjugation

$$x^*[n] \longleftrightarrow X^*(e^{-j\omega})$$

  - $X(e^{j\omega}) = X^*(e^{-j\omega}) \Leftrightarrow x[n]$ is real
  - If $x[n]$ is real, then $\Re\{X(e^{j\omega})\}$ is even, $\Im\{X(e^{j\omega})\}$ is odd

- Differencing

$$x[n] - x[n-1] \quad \longleftrightarrow \quad (1 - e^{-j\omega})X(e^{j\omega})$$

  - High-pass or low-pass?

- Accumulation

$$\sum_{m=-\infty}^{n} x[m] \longleftrightarrow \frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

  - How to derive it via differencing?

Problem 5.37

Problem

Let $X(e^{j\omega})$ be the Fourier transform of $x[n]$. Derive expressions in terms of $X(e^{j\omega})$ for the Fourier transforms of the following signals.

- $\mathbf{Re}\{x[n]\}$
- $x^*[-n]$
- $\mathbf{Ev}\{x[n]\}$

Effect of Differencing

- $J(i,j) = |M(i,j) - M(i+1,j+1)| + |M(i+1,j) - M(i,j+1)|$

Fourier Transform of $u[n]$

- How to derive the Fourier transform of u[n]?
- Option 1: From definition of Fourier transform

$$\mathcal{F}\{u[n]\} = \sum_{n=-\infty}^{+\infty} u[n]e^{-j\omega n} = \sum_{n=0}^{+\infty} e^{-j\omega n} \quad \textit{Converge??}$$

- Option 2: Since $u[n] = \sum_{m=-\infty}^{n} \delta[m]$, according to the property of accumulation

$$\mathcal{F}\{u[n]\} = \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

- Observation: Fourier transform of $u[n]$ does not exist according to the definition; however, it can be expressed in terms of $\delta(\cdot)$

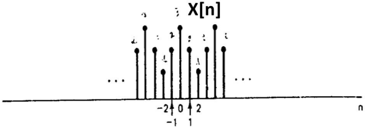
Time Reversal and Expansion

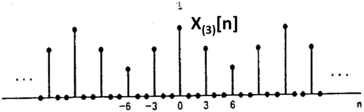
- Time Reversal

$$x[-n] \longleftrightarrow X(e^{-j\omega})$$

- Time Expansion
  - Define $x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of k} \\ 0, & \text{Otherwise} \end{cases}$, then

$$x_{(k)}[n] \longleftrightarrow X(e^{jk\omega}) \tag{1}$$





Differentiation and Parseval

- Differentiation in Frequency

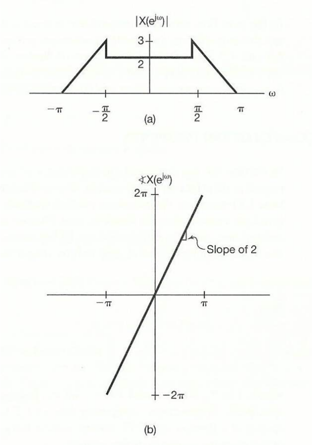
$$nx[n] \longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$$

- Parseval's Relation

$$\sum_{n=-\infty}^{+\infty}|x[n]|^2 = \frac{1}{2\pi}\int_{2\pi}|X(e^{j\omega})|^2 d\omega$$

  - Energy density spectrum — $\frac{|X(e^{j\omega})|^2}{2\pi}$

Example



- See textbook, Example 5.10
- Spectrum within $[-\pi, \pi]$
- Is it periodic, real, even, of finite energy?