Signals and Systems Tutorial 5

Southern University of Science and Technology

Tutorial 1

Problems: 4.8,4.9,4.23.4.39,4.40

Tutorial 1

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

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Tutorial 1

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1,\quad a_k = 0,\ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t),\ \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t),\ \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t),$ $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

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The CT Fourier Transform Pair

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \text{— } FT$$

Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \quad \text{— Inverse } FT$$

Inverse Fourier Transform

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\mathcal{F}(x(t)) = X(j\omega)$$

$$x(t) = \mathcal{F}^{-1}(X(j\omega))$$

Linearity and Time Shifting

Linearity $x(t) \xleftrightarrow{F} X(j\omega),\ y(t) \xleftrightarrow{F} Y(j\omega)$

$$ax(t) + by(t) \xleftrightarrow{F} aX(j\omega) + bY(j\omega)$$

Time Shifting $x(t - t_0) \longleftrightarrow e^{-j\omega t_o} X(j\omega)$

Time/Frequency Scaling

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$

$$\Downarrow a = -1$$

$$x(-t) \longleftrightarrow X(-j\omega)$$

$\boxed{\text{Time reversal}}$

E.g. $a > 1 \rightarrow at > t$

compressed in time $\leftrightarrow$

stretched in frequency

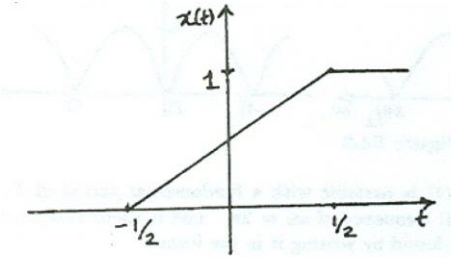
Differentiation/Integration

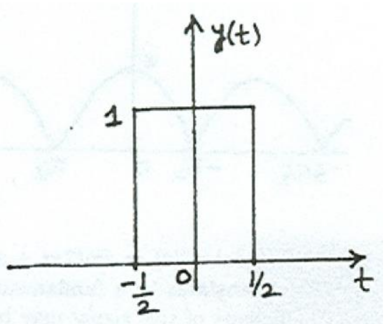
$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(j\omega) + \pi X(j0)\delta(\omega)$$

DC term

Problem 4.8





4.8. Consider the signal

$$x(t) = \begin{cases} 0, & t < -\frac{1}{2} \\ t + \frac{1}{2}, & -\frac{1}{2} \leq t \leq \frac{1}{2}. \\ 1, & t > \frac{1}{2} \end{cases}$$

(a) Use the differentiation and integration properties in Table 4.1 and the Fourier transform pair for the rectangular pulse in Table 4.2 to find a closed-form expression for $X(j\omega)$.

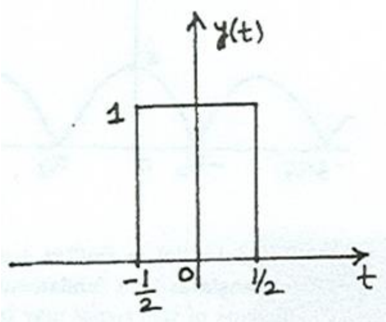
(b) What is the Fourier transform of $g(t) = x(t) - \frac{1}{2}$?

Problem 4.8 (a)

Differentiation/Integration

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{F} \frac{1}{j\omega}X(j\omega)+\pi \underset{\substack{\uparrow \\ \text{DC term}}}{X(j0)}\,\delta(\omega)$$



F

$$Y(j\omega) = \frac{2\sin(\omega/2)}{\omega}$$

Answer 4.8（a）

4.8 (a)The signal x(t) is as shown in Figure S4.8.

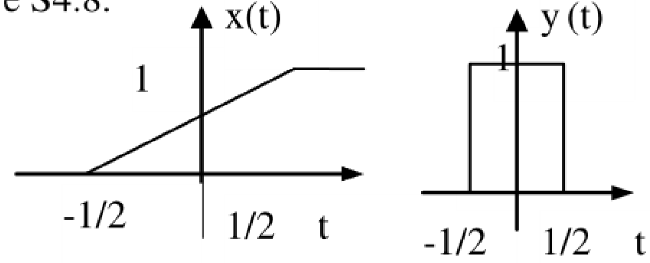


Figure S4.8

We may express this signal as

$$x(t) = \int_{-\infty}^{t} y(t)dt$$

Where y (t) is the rectangular pulse shown in S4.8 Using the integration property of FT we have

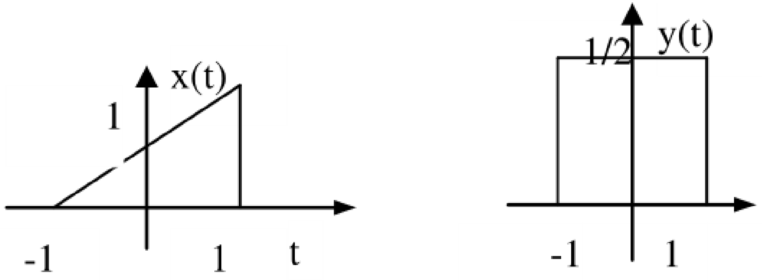
$$x(t) \xleftrightarrow{FT} X(j\omega) = \frac{1}{j\omega}Y(j\omega)+\pi Y(j0)\sigma(\omega)$$

we know from 4.2 that

$$Y(j\omega)=\frac{2\sin(w/2)}{w}$$

Therefore $X(j\omega)= \dfrac{2\sin(w/2)}{jw^2}+\pi\sigma(\omega)$

(b) if g(t)=x(t)-(1/2) $\pi\sigma(\omega)= \dfrac{2\sin(w/2)}{jw^2}$



Answer 4.8（b）

4.8. Consider the signal

$$x(t) = \begin{cases} 0, & t < -\frac{1}{2} \\ t + \frac{1}{2}, & -\frac{1}{2} \le t \le \frac{1}{2} \\ 1, & t > \frac{1}{2} \end{cases}.$$

(a) Use the differentiation and integration properties in Table 4.1 and the Fourier transform pair for the rectangular pulse in Table 4.2 to find a closed-form expression for $X(j\omega)$.

(b) What is the Fourier transform of $g(t) = x(t) - \frac{1}{2}$?

$$ax(t) + by(t) \xleftrightarrow{F} aX(j\omega) + bY(j\omega)$$

$$x(t) = 1 \xleftrightarrow{F} 2\pi\delta(\omega)$$

Problem 4.9

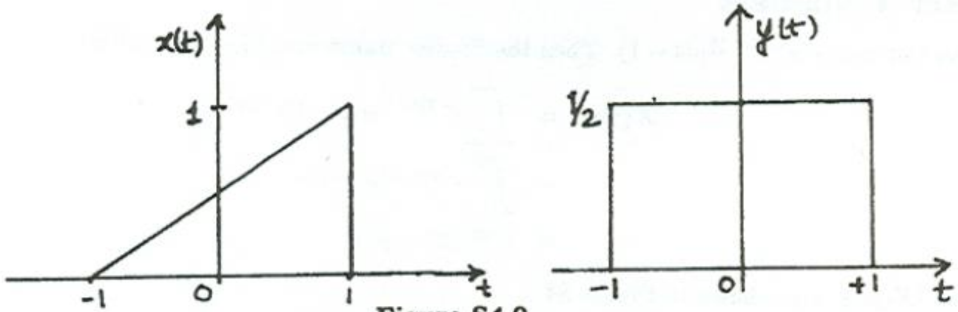


Figure S4.9

4.9. Consider the signal

$$x(t) = \begin{cases} 0, & |t| > 1 \\ (t+1)/2, & -1 \le t \le 1 \end{cases}.$$

(a) With the help of Tables 4.1 and 4.2, determine the closed-form expression for $X(j\omega)$.

(b) Take the real part of your answer to part (a), and verify that it is the Fourier transform of the even part of $x(t)$.

(c) What is the Fourier transform of the odd part of $x(t)$?

Answer 4.9 (a)

4.9 (a) the signal x(t) is plotted in figure

$$x(t)=\int_{-\infty}^{t} y(t)dt - u(t-1)$$

$$X(j\omega)=\frac{\sin\omega}{j\omega^2}-\frac{e^{-j\omega}}{j\omega}$$

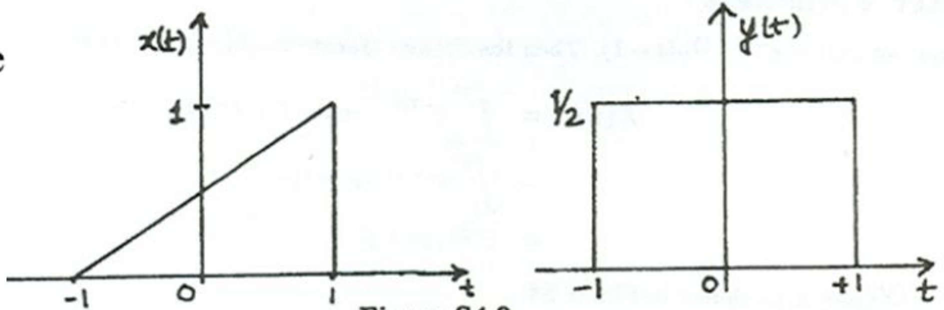


Figure S4.9

$$u(t) \xleftrightarrow{F} \frac{1}{j\omega}+\pi\delta(\omega)$$

$$x(t-t_0) \longleftrightarrow e^{-j\omega t_o} X(j\omega)$$

Answer 4.9 (b)

(b) Take the real part of your answer to part (a), and verify that it is the Fourier transform of the even part of $x(t)$.

Even part $\mathcal{E}v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$

$$X(j\omega) = \frac{\sin\omega}{j\omega^2} - \frac{e^{-j\omega}}{j\omega}$$

(b) the even part of x(t) is given by

$\varepsilon v\{x(t)\}$=(x(t)+x(-t))/2

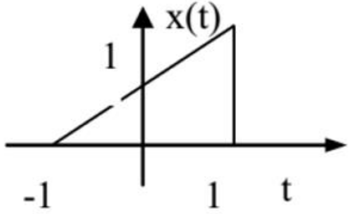
This is as shown in the 4.9

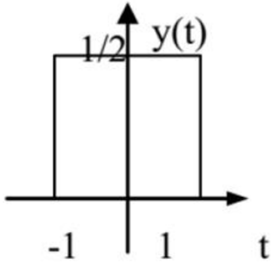
Therefore

$$FT\{\varepsilon v\{x(t)\}\} = \frac{\sin\omega}{\omega}$$

Now the real part of answer to part (a) is

$$\mathrm{Re}\{-\frac{e^{j\omega}}{j\omega}\} = \frac{1}{\omega}\mathrm{Re}\{j(\cos\omega - j\sin j\omega)\} = \frac{\sin\omega}{\omega}$$





Answer 4.9

(c) What is the Fourier transform of the odd part of $x(t)$?

(c) the FT of the odd part of x(t) is same as j times imaginary part of the answer to part (a),we have

$$\mathrm{Im}\{\frac{\sin\omega}{j\omega^2}-\frac{e^{-j\omega}}{j\omega}\}=-\frac{\sin\omega}{\omega^2}+\frac{\cos\omega}{\omega}$$

Therefore, the desired result is

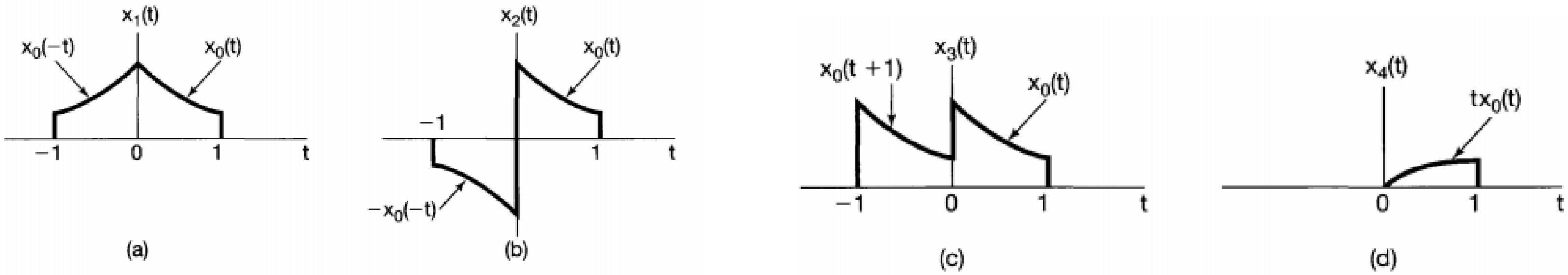
$$\mathcal{FT}\{\text{Odd part of} x(t)\}=\frac{\sin\omega}{j\omega^2}-\frac{\cos\omega}{j\omega}$$

Problem 4.23

4.23. Consider the signal

$$x_0(t) = \begin{cases} e^{-t}, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}.$$

Determine the Fourier transform of each of the signals shown in Figure P4.23. You should be able to do this by explicitly evaluating *only* the transform of $x_0(t)$ and then using properties of the Fourier transform.



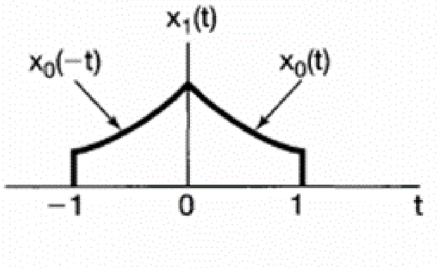
Problem 4.23

$$x_0(t) = \begin{cases} e^{-t}, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases} . \xleftrightarrow{F} X_0(j\omega) = \frac{1-e^{-(1+j\omega)}}{1+j\omega}$$

Answer 4.23 (a)

4.23. For the given signal $x_0(t)$, we use the Fourier transform analysis eq.(4.8) to evaluate the corresponding Fourier transform

$$X_0(j\omega) = \frac{1 - e^{-(1+j\omega)}}{1 + j\omega}$$



we know that

$$x_1(t) = x_0(t) + x_0(-t)$$

Using the linearity and time reversal properties of the Fourier transform we have

$$X_1(j\omega) = X_0(j\omega) + X_0(-j\omega) = \frac{2 - 2e^{-1}\cos\omega - 2\omega e^{-1}\sin\omega}{1 + \omega^2}$$

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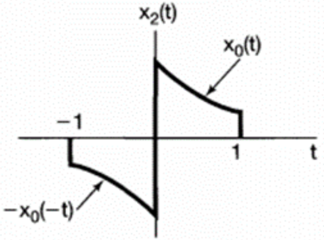
Answer 4.23 (b)



(b)

(ii) we know that

$$x_2(t) = x_0(t) - x_0(-t)$$

Using the linearity and time reversal properties of Fourier transform we have

$$X_2(j\omega) = X_0(j\omega) - X_0(-j\omega) = \frac{-2\omega + 2e^{-1}\sin\omega + 2\omega e^{-1}\cos\omega}{1+\omega^2}$$

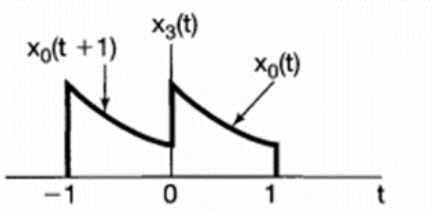
Answer 4.23（c）

（iii）we know that

$$x_3(t) = x_0(t) + x_0(t+1)$$

Using the linearity and time shifting properties of Fourier transform we have

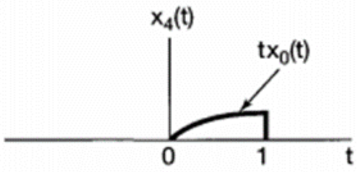
$$X_3(j\omega) = X_0(j\omega) + e^{j\omega} X_0(\quad j\omega)$$



(c)

$$x(t - t_0) \longleftrightarrow e^{-j\omega t_o} X(j\omega)$$

Answer 4.23 (d)



(d)

x4(t)=tx(t)

Using the differentiation frequency property $X_4(j\omega) = j\frac{d}{d\omega}X_0(j\omega)$

Therefore,

$$X_4(j\omega) = \frac{1 + j\omega e^{-1-j\omega}}{(1+j\omega)^2}$$

$$tx(t) \xleftrightarrow{F} j\frac{d}{d\omega}X(j\omega)$$

Problem 4.39

4.39. Suppose that a signal $x(t)$ has Fourier transform $X(j\omega)$. Now consider another signal $g(t)$ whose shape is the same as the shape of $X(j\omega)$; that is,

$$g(t) = X(jt).$$

(a) Show that the Fourier transform $G(j\omega)$ of $g(t)$ has the same shape as $2\pi x(-t)$; that is, show that

$$G(j\omega) = 2\pi x(-\omega).$$

(b) Using the fact that

$$\mathfrak{F}\{\delta(t + B)\} = e^{jB\omega}$$

in conjunction with the result from part (a), show that

$$\mathfrak{F}\{e^{jBt}\} = 2\pi\,\delta(\omega - B).$$

Answer 4.39 (a)

4.39. Suppose that a signal $x(t)$ has Fourier transform $X(j\omega)$. Now consider another signal $g(t)$ whose shape is the same as the shape of $X(j\omega)$; that is,

$$g(t) = X(jt).$$

(a) Show that the Fourier transform $G(j\omega)$ of $g(t)$ has the same shape as $2\pi x(-t)$; that is, show that

$$G(j\omega) = 2\pi x(-\omega).$$

4.39. (a) From the Fourier analyses equation. We have

$$G(j\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} X(jt)e^{-j\omega t}dt \qquad \text{(S4.39-1)}$$

Also from the Fourier transform equation, we have

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Switching the variables t and ω, we have

$$x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jt)e^{j\omega t}dt$$

We may also write this equation as

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(jt)e^{-j\omega t}dt$$

Substituting this equation in eq. (S4.39-1), we obtain

$$G(j\omega) = 2\pi x(-\omega)$$

Answer 4.39 (b)

4.39. Suppose that a signal $x(t)$ has Fourier transform $X(j\omega)$. Now consider another signal $g(t)$ whose shape is the same as the shape of $X(j\omega)$; that is,

$$g(t) = X(jt).$$

(a) Show that the Fourier transform $G(j\omega)$ of $g(t)$ has the same shape as $2\pi x(-t)$; that is, show that

$$G(j\omega) = 2\pi x(-\omega).$$

(b) Using the fact that

$$\mathfrak{F}\{\delta(t + B)\} = e^{jB\omega}$$

in conjunction with the result from part (a), show that

$$\mathfrak{F}\{e^{jBt}\} = 2\pi\, \delta(\omega - B).$$

(b) If in part (a) we have $x(t)=\delta(t+B)$, then we would have $g(t)=X(jt)=e^{jBt}$ and $G(j\omega)=2\pi x(-\omega)=2\pi\delta(-\omega+B)=2\pi\delta(\omega-B)$

Problem 4.40

4.40. Use properties of the Fourier transform to show by induction that the Fourier transform of

$$x(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t), \quad a > 0,$$

is

$$\frac{1}{(a + j\omega)^n}.$$

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Answer 4.40

Mathematical Induction

$$x(t) = \frac{t^{n-1}}{(n-1)!} e^{-at} u(t),\ a > 0, \overset{F}{\longleftrightarrow} \frac{1}{(a + j\omega)^n}$$

4.40. When n=1, $x_1(t) = e^{-at}u(t)$ and $X_1(j\omega) = 1/(a + j\omega)$

When n=2, $x_2(t) = e^{-at}u(t)$ and $X_2(j\omega) = 1/(a + j\omega)^2$

Now, let us assume that the given statement is true when n=m, that is,

$$x_m(t) = \frac{t^{m-1}}{(m-1)!} e^{-at}u(t) \overset{FS}{\longleftrightarrow} X_m(jw) = \frac{1}{(a+ j\omega)^m}$$

For n=m+1 we may use the differentiation in frequency property to write,

$$x_{m+1}(t) = \frac{t}{m} x_m(t) \overset{FS}{\longleftrightarrow} X_{m+1}(j\omega) = \frac{1}{m} j \frac{dX_m(j\omega)}{d\omega} = \frac{1}{(a+ j\omega)^{m+1}}$$

This shows that if we assume that the given statement is true for n=m, then it is true for n=m+1 .Since we also shown that the given statement is true for n=2, we may argue that it is true for n=2+1=3, n=3+1=4, and so on. Therefore, the given statement is true for any n.

Thank you for listening

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