Signals and Systems Tutorial 2 （Ch.2-2）

Southern University of Science and Technology

Tutorial 2-2

Problems: 2.20, 2.29, 2.40, 2.43, 2.47

Reviews

- CT LTI system
- Unit impulse $\delta(t)$ function
- Calculation of convolution-integral;
- Unite doublet and unite step response;
- Differentiator and integrators;

CT LTI

$$x(t) \longrightarrow \boxed{\text{CT LTI}} \longrightarrow y(t)$$

- Now suppose the system is **LTI**, and define the *unit impulse response* $h(t)$:

$$\delta(t) \longrightarrow h(t)$$

$$\Downarrow$$

From Time-Invariance:

$$\delta(t-\tau) \longrightarrow h(t-\tau)$$

From Linearity:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow y(t) = \underbrace{\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau}_{\textit{Convolution Integration}} = x(t) * h(t)$$

Unit Impulse

Sifting Property： $$\int_{-\infty}^{\infty} x(t)\delta(t)\,dt = x(0) \qquad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

Time Shift： $$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_o)\,dt = \int_{-\infty}^{\infty} x(\tau+t_o)\,\delta(\tau)\,d\tau = x(t_o)$$

Time Shift： $$\int_{-\infty}^{\infty} x(t)\,\delta(at)\,dt = \frac{1}{|a|}x(0), \quad a \neq 0 \qquad \delta(at) = \frac{1}{|a|}\delta(t), \quad a \neq 0$$

Unit Impulse

推广

1）、 $\delta(at-t_0)=\delta[a(t-\frac{t_0}{a})]=\frac{1}{a}\delta(t-\frac{t_0}{a})$

2）、 $\int_{-\infty}^{\infty}x(t)\delta(at)dt=\frac{1}{a}x(0)$

3）、 $\int_{-\infty}^{\infty}x(t)\delta(at-t_0)dt=\frac{1}{a}x(\frac{t_0}{a})$

Problem 2.20

2.20. Evaluate the following integrals:

(a) $\int_{-\infty}^{\infty} u_0(t)\cos(t)\,dt$

(b) $\int_{0}^{5} \sin(2\pi t)\,\delta(t+3)\,dt$

(c) $\int_{-5}^{5} u_1(1-\tau)\cos(2\pi\tau)\,d\tau$

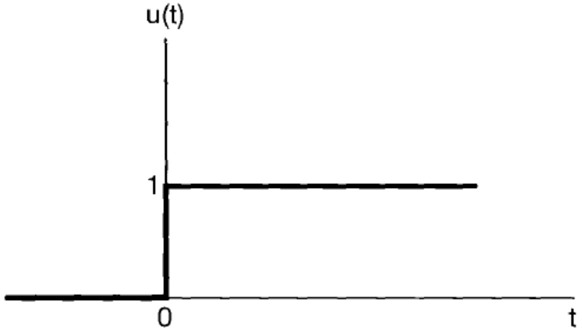
Notation

$$\boxed{u_0(t) = \delta(t)}$$

$$\boxed{u_1(t) = \frac{d\delta(t)}{dt}}$$

Problem 2.20

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



$$\delta(t) = \frac{du(t)}{dt}.$$

Notation

2.20. (a)

$$\boxed{u_0(t) = \delta(t)}$$

$$\int_{-\infty}^{\infty} u_0(t)\cos(t)dt = \int_{-\infty}^{\infty} \delta(t)dt = 1$$

(b)

$$\int_0^5 \sin(2\pi t)\delta(t+3)dt = \sin(6\pi) = 0$$

Problem 2.20

(c) In order to evaluate the integral

$$\int_{-5}^{5} u_1(1-\tau)\cos(2\pi\tau)d\tau,$$

consider the signal

$$x(t) = \cos(2\pi t)[u[(t+5) - u(t-5)]$$

We know that

$$\boxed{x(t) * u_1(t) = \frac{dx(t)}{dt}}$$

$$\frac{dx(t)}{dt} = u_1(t) * x(t) = \int_{-\infty}^{\infty} u_1(t-\tau)x(\tau)d\tau$$
$$= \int_{-5}^{5} u_1(t-\tau)\cos(2\pi\tau)d\tau$$

$$\boxed{u_1(t) = \frac{d\delta(t)}{dt}}$$

Now,

$$\left.\frac{dx(t)}{dt}\right|_{t=1} = \int_{-5}^{5} u_1(1-\tau)\cos(2\pi\tau)d\tau$$

which is the desired integral. We now evaluate the value of the integral as

$$\left.\frac{dx(t)}{dt}\right|_{t=1} = \sin(2\pi t)|_{t=1} = 0. \qquad -2\pi\sin(2\pi t)$$

SUSTech Southern University of Science and Technology

Problem 2.29

2.29. The following are the impulse responses of continuous-time LTI systems. Determine whether each system is causal and/or stable. Justify your answers.

(a) $h(t) = e^{-4t}u(t-2)$

(b) $h(t) = e^{-6t}u(3-t)$

(c) $h(t) = e^{-2t}u(t+50)$

(d) $h(t) = e^{2t}u(-1-t)$

(e) $h(t) = e^{-6|t|}$

(f) $h(t) = te^{-t}u(t)$

(g) $h(t) = (2e^{-t} - e^{(t-100)/100})u(t)$

Causality and Stability

1) Causality $\Leftrightarrow$ $h[n]=0$ for all $n<0$

2) Stability $\Leftrightarrow$ $\sum_{k=-\infty}^{+\infty}|h[k]|<\infty$

$$\int_{-\infty}^{+\infty}|h(t)|dt<\infty$$

SUSTech Southern University of Science and Technology

Answer 2.29

2.29. (a) causal because h(t)=0for t<0. stable because $\int_{\infty}^{\infty}|h(t)|dt = e^{-8}/4 < \infty$.

(b) Not causal because h(t) $\neq 0$ for t<0. Unstable because $\int_{\infty}^{\infty}|h(t)| = \infty$.

(c) Not causal because h(t) $\neq 0$ for t<0. a Stable because $\int_{\infty}^{\infty}|h(t)|dt = e^{100/2} < \infty$

(d) Not causal because h(t) $\neq 0$ for t<0. stable because $\int_{\infty}^{\infty}|h(t)|dt = e^{-2}/2 < \infty$

(e) Not causal because h(t) $\neq 0$ for t<0. stable because $\int_{\infty}^{\infty}|h(t)|dt = 1/3 < \infty$

(f) Causal because h(t)=0 for t<0. Stable because $\int_{\infty}^{\infty}|h(t)|dt = 1 < \infty$

(g) Causal because h(t)=0 for t<0. Unstable because $\int_{\infty}^{\infty}|h(t)|dt = \infty$

Problem 2.40

2.40. **(a)** Consider an LTI system with input and output related through the equation

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 2)\, d\tau.$$

What is the impulse response $h(t)$ for this system?

(b) Determine the response of the system when the input $x(t)$ is as shown in Figure P2.40.

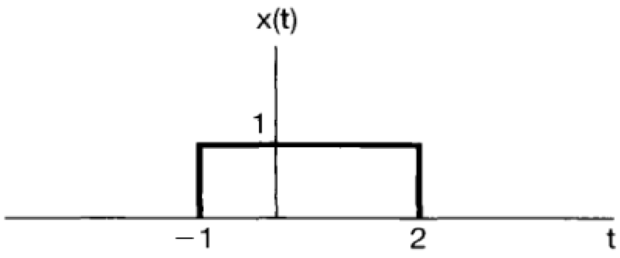


Figure P2.40

Problem 2.40 (a)

标准形式：$y(t)=\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x(\tau-2)d\tau = \int_{-\infty}^{t-2} e^{-(t-2-\tau')}x(\tau')d\tau'.$$

$$\int_{-\infty}^{\infty} e^{-(t-2-\tau')}u(t-2-\tau')x(\tau')\,d\tau'$$

$$h(t) = e^{-(t-2)}u(t-2).$$

$u(t-2-\tau') = (-\infty, t-2]$

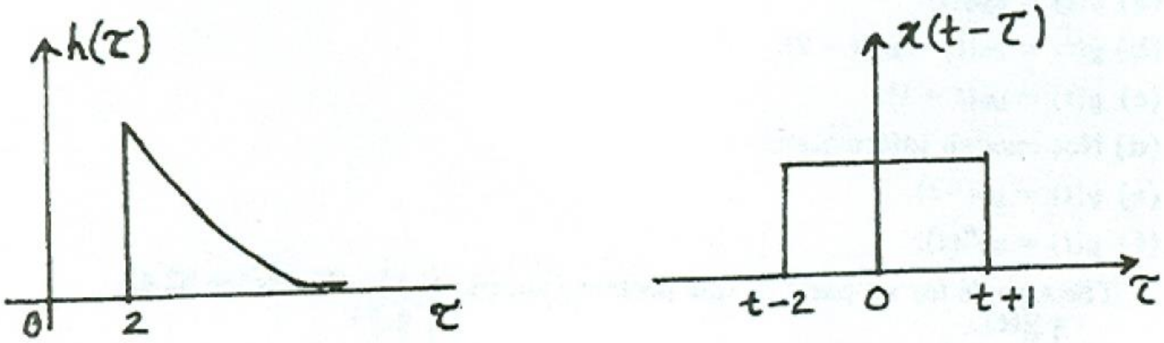


Figure S2.40

Problem 2.40 (b)

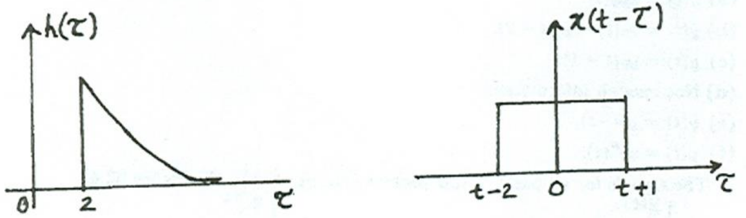


Figure S2.40

(b) We have

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$
$$= \int_{2}^{\infty} e^{-(\tau-2)}[u(t-\tau+1)-u(t-\tau-2)$$

$h(\tau)$ and $x(t-\tau)$ are as shown in the figure below.

Using this figure, we may write

$$y(t) = \begin{cases} 0, & t<1 \\ \int_2^{t+1} e^{-(\tau-2)}d\tau = 1-e^{-(t-1)}, & 1<t<4 \\ \int_{t-2}^{t+1} e^{-(\tau-2)}d\tau = e^{-(t-4)}[1-e^{-3}], & t>4 \end{cases}$$

SUSTech Southern University of Science and Technology

Convolution

1. 交换律

$$f_1(t)*f_2(t)=f_2(t)*f_1(t)$$

证明

2. 分配律

$$f_1(t)*[f_2(t)+f_3(t)]=f_1(t)*f_2(t)+f_1(t)*f_3(t)$$

系统并联运算

3. 结合律

$$[f(t)*f_1(t)]*f_2(t)=f(t)*[f_1(t)*f_2(t)]$$

系统级联运算

Convolution

已知 $g(t)=f(t)*h(t)$

则： $\boxed{g'(t)=f(t)*h'(t)=f'(t)*h(t)}$ 微分性

证明 $g(t)=f(t)*h(t)=\int_{-\infty}^{\infty}f(\tau)h(t-\tau)\,\mathrm{d}\tau$

两端对t 求导 交换律

$$\frac{\mathrm{d}\,g(t)}{\mathrm{d}t}=\int_{-\infty}^{\infty}f(\tau)\frac{\mathrm{d}\,h(t-\tau)}{\mathrm{d}t}\mathrm{d}\tau=\int_{-\infty}^{\infty}\frac{\mathrm{d}\,f(t-\tau)}{\mathrm{d}t}h(\tau)\,\mathrm{d}\tau$$

即

$$g'(t)=f(t)*h'(t)=f'(t)*h(t)$$

意义:卷积后求导和先对其任一求导再卷积的结果相同.

Convolution

设 X 和 Y 是两个独立的随机变量，考虑它们的和 $Z=X+Y$ 的分布. 首先, 我们推导当 X 和 Y 都是离散的情况下 Z 的分布列.

设 X 和 Y 是仅取整数值的独立随机变量, 它们的分布列分别为 p_X 和 p_Y. 则对于任意整数 z,

$$
\begin{aligned}
p_Z(z) &= \mathrm{P}(X+Y=z) \\
&= \sum_{\{(x,y)\mid x+y=z\}} \mathrm{P}(X=x, Y=y) \\
&= \sum_x \mathrm{P}(X=x, Y=z-x) \\
&= \sum_x p_X(x) p_Y(z-x).
\end{aligned}
$$

Convolution

Convolution

Convolution Kernel

卷积

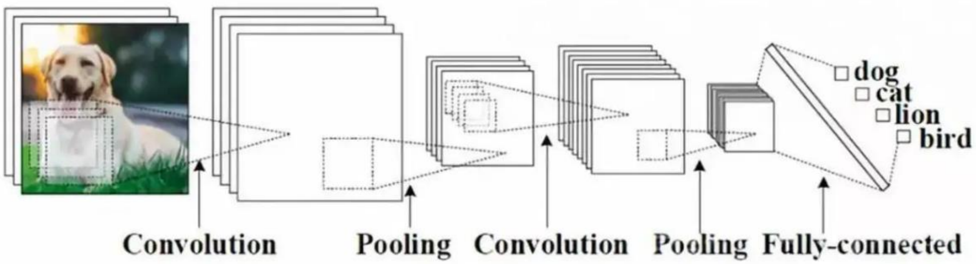
说起卷积，一般我们我们接触过的都是一维信号的卷积，也就是

$$y[n] = x[n] * h[n] = \sum_k x[k]h[n-k]$$

在信号处理中， $x[n]$ 是输入信号， $h[n]$ 是单位响应。于是输出信号 $y[n]$ 就是输入信号 $x[n]$ 的延迟响应的叠加。这也就是一维卷积本质：加权叠加/积分。

那么对于二维信号，比如图像，卷积的公式就是

$$y[m,n] = x[m,n] * h[m,n] = \sum_j \sum_i x[i,j]h[m-i,n-j]$$



Convolutional Neural Networks （CNN）

Problem 2.43

2.43. One of the important properties of convolution, in both continuous and discrete time, is the associativity property. In this problem, we will check and illustrate this property.

(a) Prove the equality

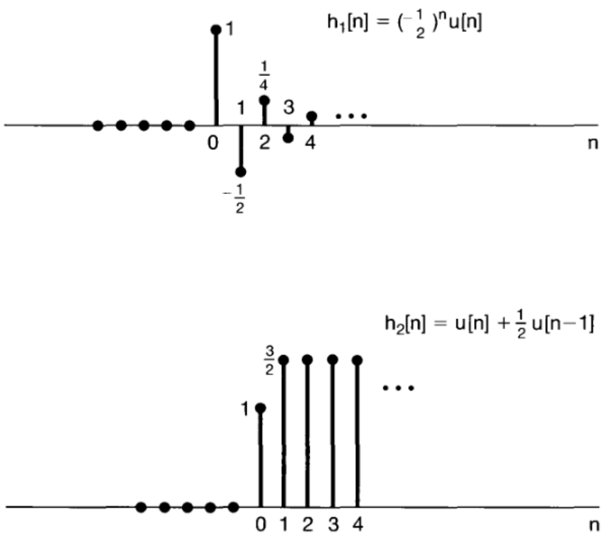
$$[x(t) * h(t)] * g(t) = x(t) * [h(t) * g(t)] \qquad \text{(P2.43–1)}$$

by showing that both sides of eq. (P2.43–1) equal

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(\tau)h(\sigma)g(t - \tau - \sigma)\, d\tau\, d\sigma.$$

Problem 2.43

(b) Consider two LTI systems with the unit sample responses $h_1[n]$ and $h_2[n]$ shown in Figure P2.43(a). These two systems are cascaded as shown in Figure P2.43(b). Let $x[n] = u[n]$.

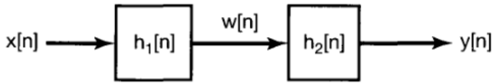


(a)

(i) Compute $y[n]$ by first computing $w[n] = x[n] * h_1[n]$ and then computing $y[n] = w[n] * h_2[n]$; that is, $y[n] = [x[n] * h_1[n]] * h_2[n]$.

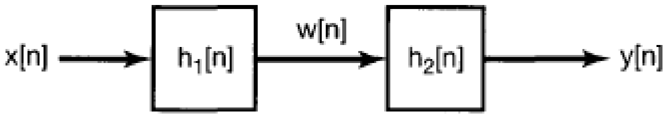
(ii) Now find $y[n]$ by first convolving $h_1[n]$ and $h_2[n]$ to obtain $g[n] = h_1[n] * h_2[n]$ and then convolving $x[n]$ with $g[n]$ to obtain $y[n] = x[n] * [h_1[n] * h_2[n]]$.

The answers to (i) and (ii) should be identical, illustrating the associativity property of discrete-time convolution.



(b)

Problem 2.43



(b)

(c) Consider the cascade of two LTI systems as in Figure P2.43(b), where in this case

$$h_1[n] = \sin 8n$$

and

$$h_2[n] = a^n u[n], \quad |a| < 1,$$

and where the input is

$$x[n] = \delta[n] - a\delta[n-1].$$

Determine the output $y[n]$. (*Hint:* The use of the associative and commutative properties of convolution should greatly facilitate the solution.)

Problem 2.43 (a)

We first have

$$
\begin{aligned}
[x(t) * h(t)] * g(t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\sigma' - \tau)g(t - \sigma')d\tau d\sigma' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\sigma)g(t - \sigma - \tau)d\tau d\sigma
\end{aligned}
$$

Also,

$$
\begin{aligned}
x(t) * [h(t) * g(t)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t - \sigma')h(\tau)g(\sigma' - \tau)d\sigma' d\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\sigma)h(\tau)g(t - \tau - \sigma)d\tau d\sigma \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\sigma)g(t - \sigma - \tau)d\tau d\sigma
\end{aligned}
$$

The equality is proved.

Problem 2.43 (b)

(i) We first have

$$w[n] = u[n] * h_1[n] = \sum_{k=0}^{n}\left(-\frac{1}{2}\right)^k = \frac{2}{3}\left[1 - (-\frac{1}{2})^{n+1}\right]u[n].$$

Now,

$$y[n] = w[n] * h_2[n] = (n+1)u[n].$$

(ii) We first have

$$g[n] = h_1[n] * h_2[n] = \sum_{k=0}^{n}\left(-\frac{1}{2}\right)^k + \frac{1}{2}\sum_{k=0}^{n-1}(-\frac{1}{2})^k = u[n]$$

Now,

$$y[n] = u[n] * g[n] = u[n] * u[n] = (n+1)u[n].$$

The same result was obtained in both parts (i) and (ii).

Problem 2.43 (c)

(c) Note that

$$x[n] * (h_2[n] * h_1[n]) = (x[n] * h_2[n]) * h_1[n].$$

Also note that

$$x[n] * h_2[n] = \alpha^n u[n] - \alpha^n u[n-1] = \delta[n].$$

Therefore,

$$x[n] * h_1[n] * h_2[n] = \delta[n] * \sin 8n = \sin 8n.$$

Problem 2.47

2.47. We are given a certain linear time-invariant system with impulse response $h_0(t)$. We are told that when the input is $x_0(t)$ the output is $y_0(t)$, which is sketched in Figure P2.47. We are then given the following set of inputs to linear time-invariant systems with the indicated impulse responses:

	Input $x(t)$	*Impulse response* $h(t)$
(a)	$x(t) = 2x_0(t)$	$h(t) = h_0(t)$
(b)	$x(t) = x_0(t) - x_0(t-2)$	$h(t) = h_0(t)$
(c)	$x(t) = x_0(t-2)$	$h(t) = h_0(t+1)$
(d)	$x(t) = x_0(-t)$	$h(t) = h_0(t)$
(e)	$x(t) = x_0(-t)$	$h(t) = h_0(-t)$
(f)	$x(t) = x_0'(t)$	$h(t) = h_0'(t)$

[Here $x_0'(t)$ and $h_0'(t)$ denote the first derivatives of $x_0(t)$ and $h_0(t)$, respectively.]

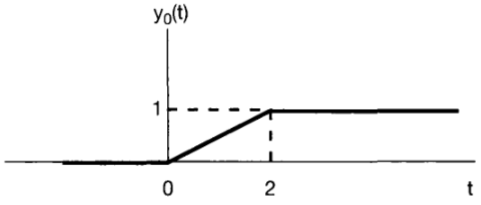


Figure P2.47

In each of these cases, determine whether or not we have enough information to determine the output $y(t)$ when the input is $x(t)$ and the system has impulse response $h(t)$. If it is possible to determine $y(t)$, provide an accurate sketch of it with numerical values clearly indicated on the graph.

Problem 2.47

The signals for all parts of this problem are plotted in the Figure S2.47.

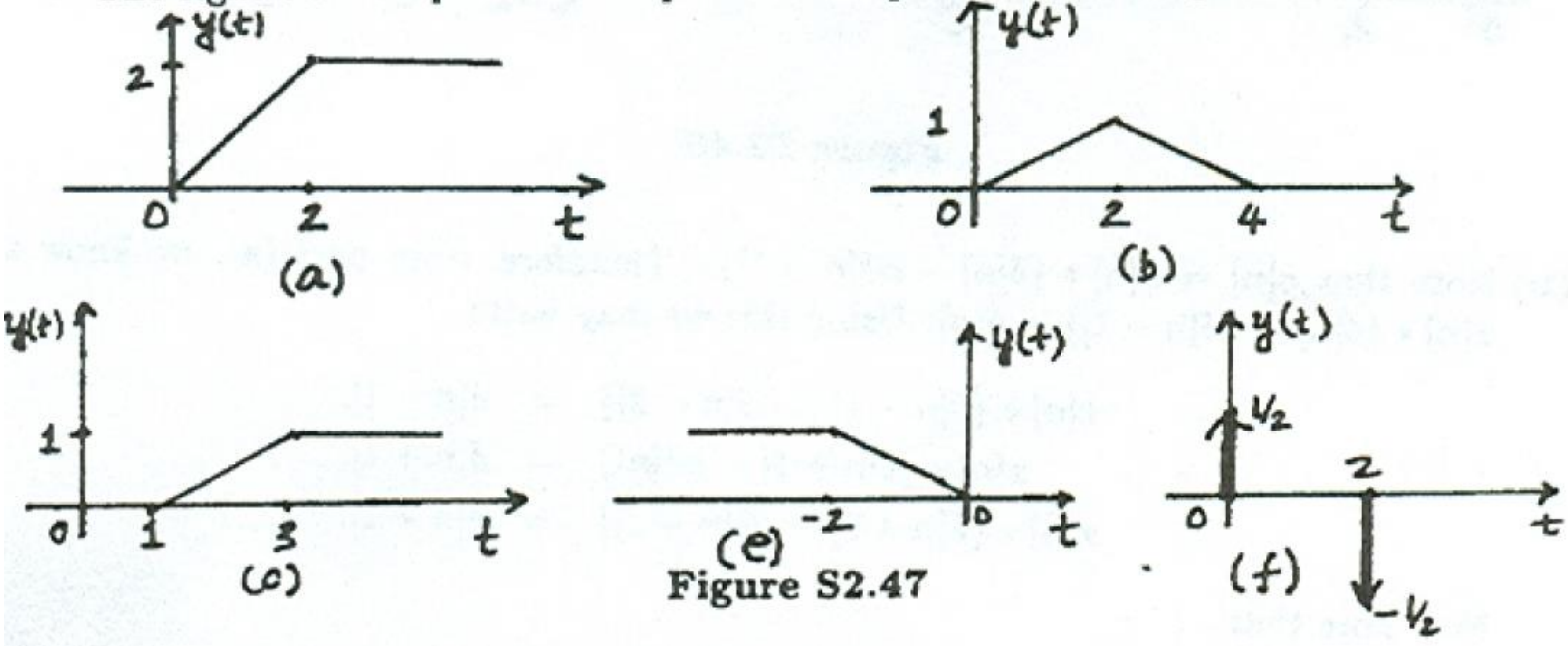


Figure S2.47

options

- Properties of unit impulse function

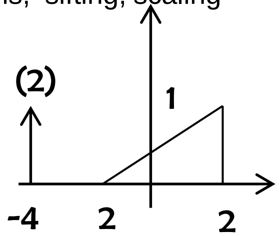
  multiply with normal functions, sifting, scaling

- Properties of unit doublet

  multiply with normal functions, sifting, scaling

- Signal scaling

  f(t) to f(4-2t)



- Differential and difference equation for casual LTI

options

- Differential and difference equation for casual LTI

CT: $y''(t) + 3y'(t) + 2y(t) = 2x'(t) + x(t)$

homogenous solution $y_h(t)$ and particular solution $y_p(t)$, Example 2.14

zero-state response $y_{zs}(t)$ and zero-input response $y_{zi}(t)$

DT: $y[k] + 2y[k-1] + y[k-2] = 2x[k] + x[k-1]$

iterative recursion, Example 2.15

homogenous solution $y_h[k]$ and particular solution $y_p[k]$

zero-state response $y_{zs}[k]$ and zero-input response $y_{zi}[k]$

Too complicated $\longrightarrow$ Frequency domain analysis, Fourier transform

Thank you for listening