Notes

Assignments
- **4.5**
- **4.21 (b) (g) (h)**
- **4.22 (c) (e)**
- **4.27**

Tutorial problems
- **Basic Problems wish Answers 4.8, 4.9**
- **Basic Problems 4.23**
- **Advanced Problems 4.39, 4.40**

Review

DT Fourier Series Pair $\left(\omega_o = \frac{2\pi}{N}\right)$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} \qquad \text{(Analysis equation)}$$

$\sum_{k=<N>}$ = Sum over *any* N consecutive values of k

$$x[n] = x[n+N]$$

$$a_{k+N} = a_k$$

LTI System, system function and frequency response

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0) a_k e^{jk\omega_0 t}$$

$$a_k \longrightarrow \underbrace{H(jk\omega_0)a_k}_{\text{"gain"}}$$

$$\boxed{H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt}$$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} H(e^{jk\omega_o}) a_k e^{jk\omega_0 n}$$

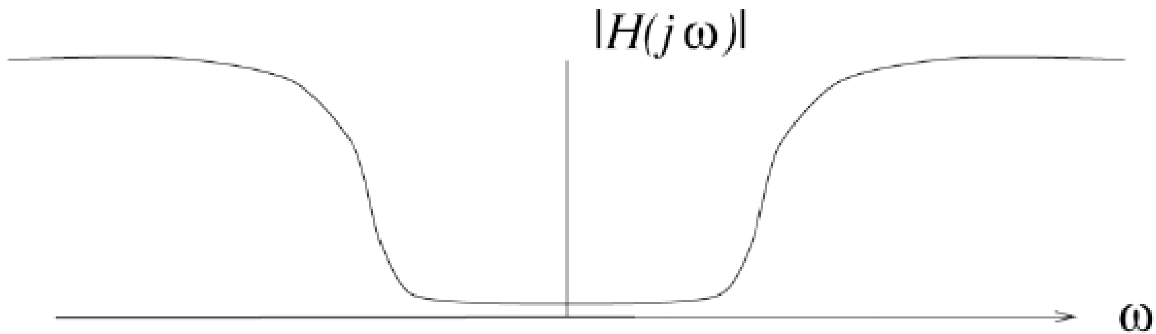
$$a_k \longrightarrow \underbrace{H(e^{jk\omega_o})a_k}_{\text{"gain"}}$$

$$\boxed{H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n}}$$

The effect of the LTI system is to modify each a_k through multiplication by the value of the frequency response at $k\omega_0$.

Highpass Filters

CT



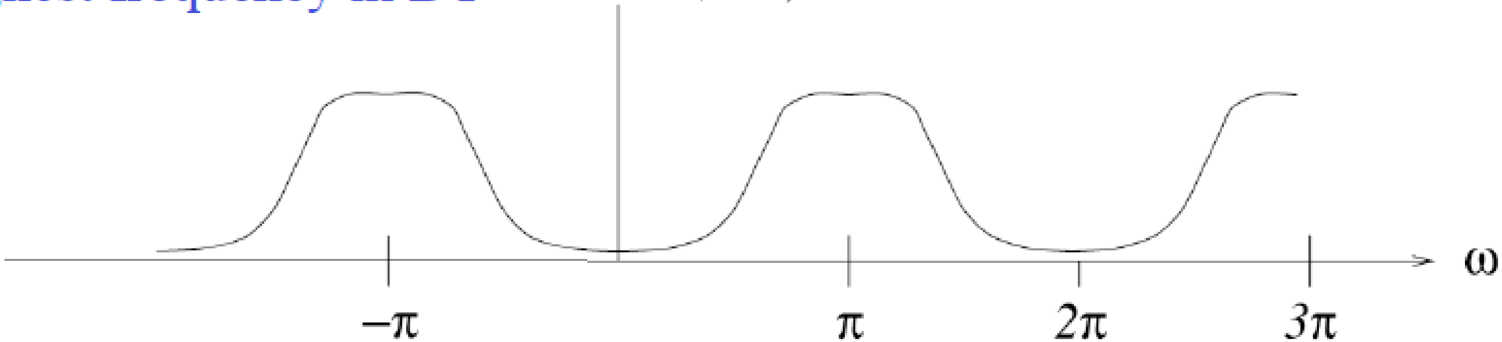
Remember:

$(-1)^n = e^{j\pi n}$

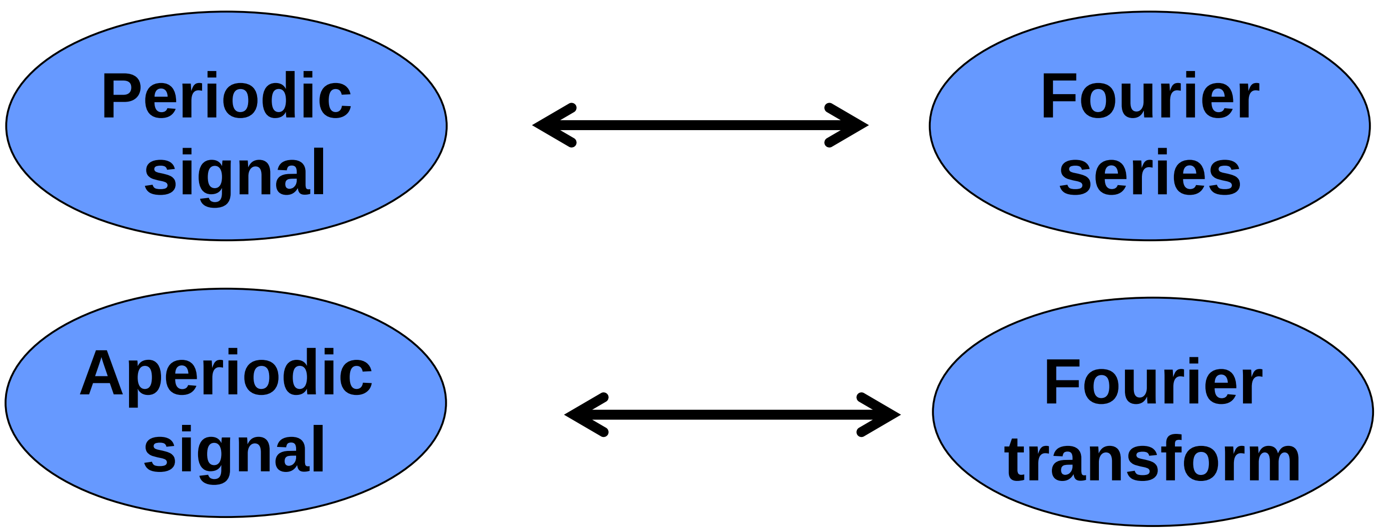
— π = highest frequency in DT

DT

$H(e^{j\omega})$



Chapter 4
The Continuous-Time Fourier Transform

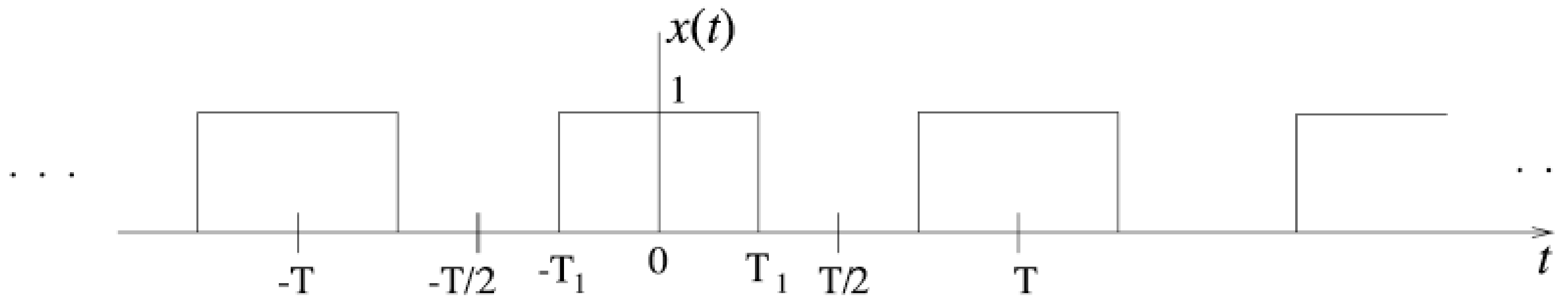


Fourier Transform

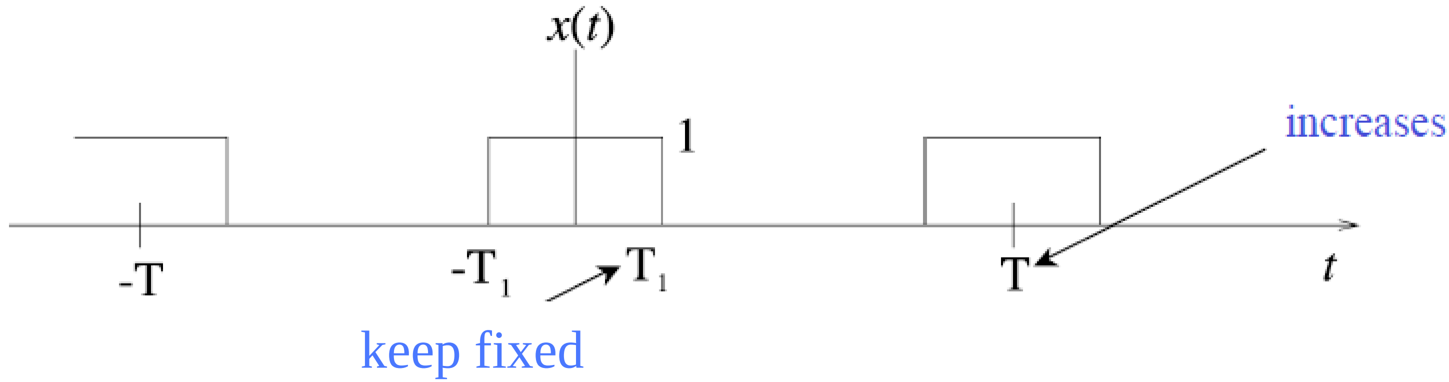
- We have shown that Fourier series are useful in analyzing periodic signals, but many (most) signals are aperiodic. Need a more general tool — *Fourier transform*.

Fourier's own derivation of the CT Fourier transform

- $x(t)$ – an aperiodic signal
  – view it as the limit of a periodic signal as $T \to \infty$



Motivating Examples: Square wave

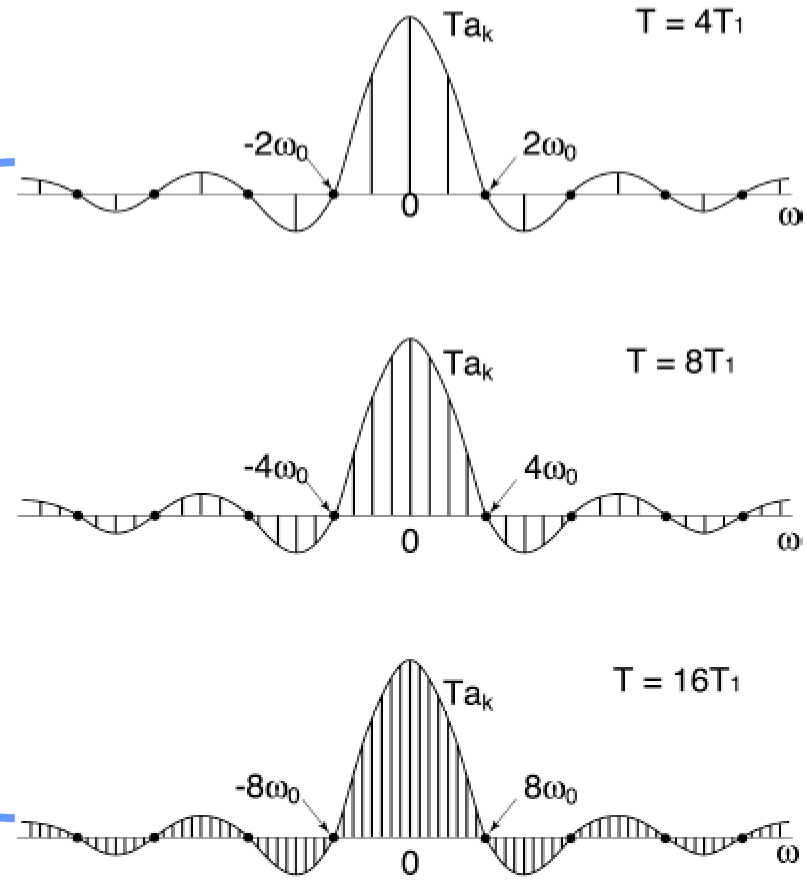


$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

$$Ta_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0}$$

Let $X(\omega) = \frac{2\sin(\omega T_1)}{\omega}$,

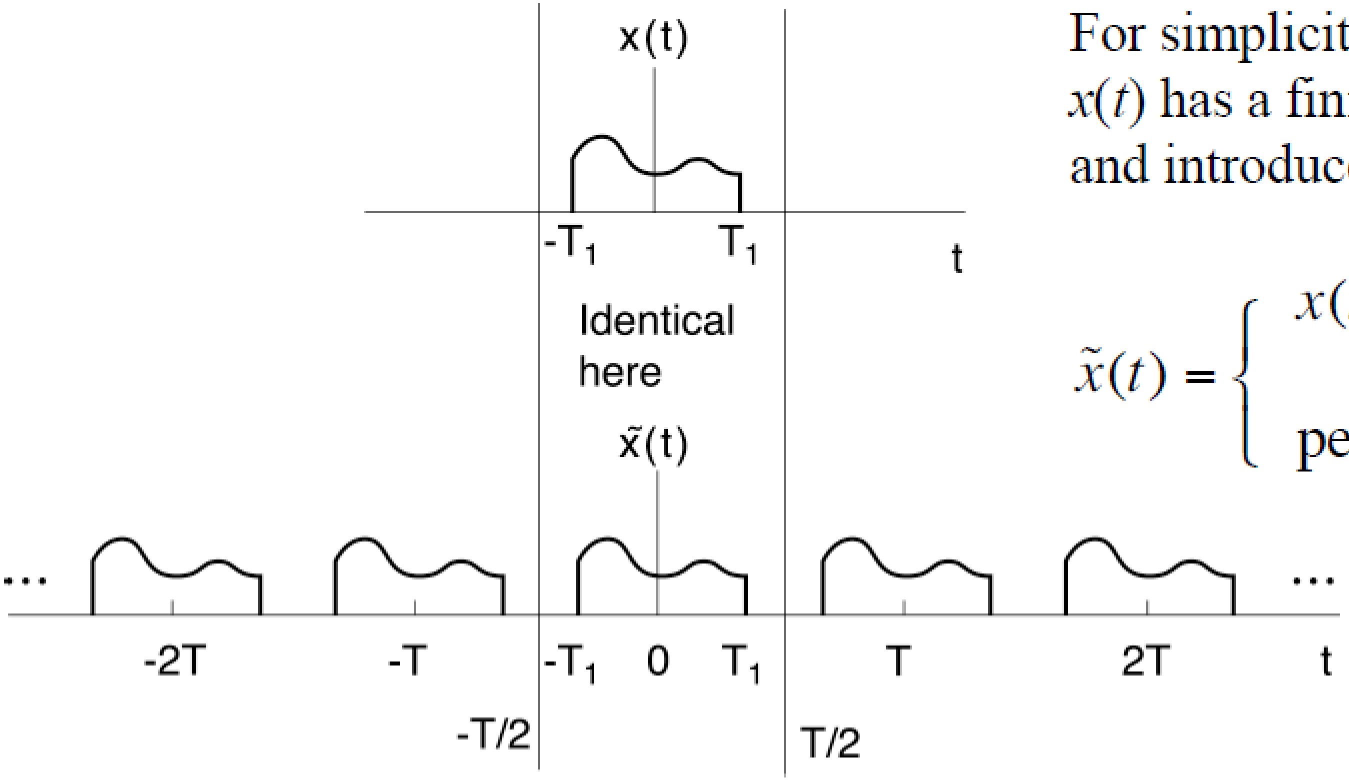
we have $Ta_k = X(k\omega_0)$



$(\omega_0 = \frac{2\pi}{T})$

Become denser in ω as T increases

So, on the derivation of FT ...



For simplicity, assume $x(t)$ has a finite duration, and introduce a periodic $\tilde{x}(t)$

$$\tilde{x}(t) = \begin{cases} x(t) & \frac{-T}{2} < t < \frac{T}{2} \\ \text{periodic} & |t| > \frac{T}{2} \end{cases}$$

$$\text{As } T \to \infty, \qquad x(t) = \tilde{x}(t) \ \text{ for all } t$$

Derivation (cont.): Analysis equation

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \qquad \omega_o = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jk\omega_0 t} dt$$

$$\uparrow$$

$\tilde{x}(t) = x(t)$ in this interval

$$= \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt \qquad (1)$$

If we define $\boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt}$

then, Eq. (1) $\Rightarrow$ $a_k = \frac{1}{T} X(jk\omega_0) = \frac{1}{T} X(j\omega)\big|_{\omega = k\omega_0}$

Derivation (cont.): Synthesis equation

Thus, for $-\frac{T}{2} < t < \frac{T}{2}$

$$x(t) = \tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \underbrace{\frac{1}{T} X(jk\omega_0)}_{a_k} e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} \omega_0 X(jk\omega_0) e^{jk\omega_0 t}$$

$$\Downarrow$$

As $T \to \infty$, $\omega_o \to 0$, $\sum \omega_o \to \int d\omega$, and $k\omega_o = \omega$, we get the CT FT pair

$$\boxed{\begin{array}{ll} x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega & \text{Synthesis equation} \\ & \text{— "sum" of } e^{j\omega t} \\ X(j\omega) = \displaystyle\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt & \text{Analysis equation} \end{array}}$$

The CT Fourier Transform Pair

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

— *FT*

Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

— Inverse *FT*

Inverse Fourier Transform

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

$$\mathcal{F}(x(t)) = X(j\omega)$$

$$x(t) = \mathcal{F}^{-1}(X(j\omega))$$

CT Fourier Transform Pair

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

CT Fourier Series Pair

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Harmonically related

For what kinds of signals can we do FT?

It works also even if $x(t)$ is infinite duration, but satisfies:

a) Finite energy $\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$

In this case, there is *zero* energy in the error

$$e(t) = x(t) - \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega \quad \text{Then} \quad \int_{-\infty}^{\infty} |\, e(t)\, |^2\, dt = 0$$

b) Dirichlet conditions

1) absolutely integrable $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

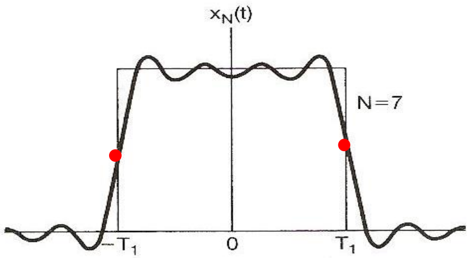
2) finite number of maxima and minima within any finite interval

3) finite number of discontinuities with finite values within any finite interval

(i) $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega = x(t)$ at points of continuity

(ii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega =$ midpoint at discontinuity

(iii) Gibb' s phenomenon



Example 4.3 Impulse function

(a) $$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = \quad ?$$

$$\Downarrow$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\omega t} d\omega$$ — Synthesis equation for $\delta(t)$

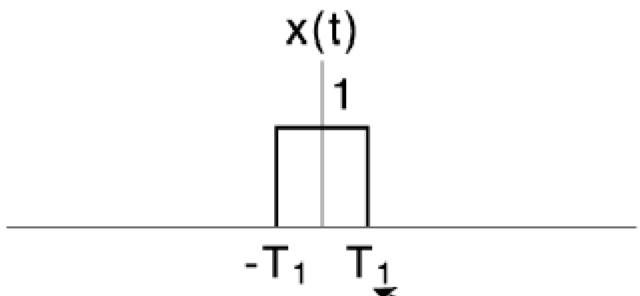
(b) $$x(t) = \delta(t - t_0)$$

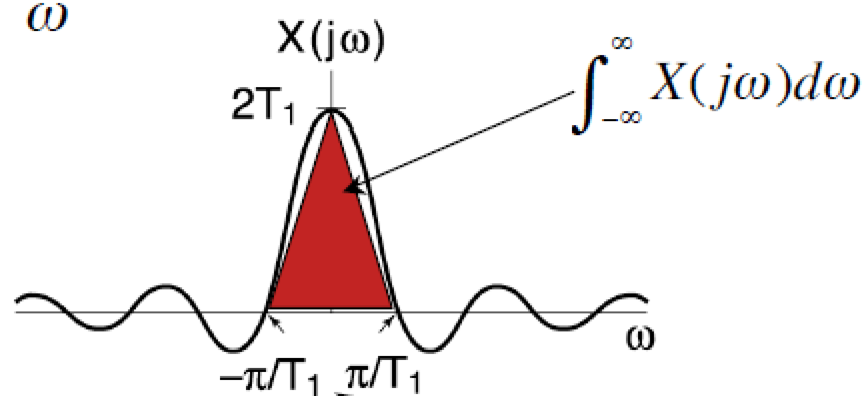
$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t - t_0) e^{-j\omega t} dt$$

$$= e^{-j\omega t_o}$$ — Linear phase shift in ω

Example 4.4 A square pulse in the time-domain

$$X(j\omega)=\int_{-T_1}^{T_1} e^{-j\omega t}\,dt = \frac{2\sin\omega T_1}{\omega}$$





Note the inverse relation between the two widths ⇒ Uncertainty principle

$$\boxed{\begin{aligned} X(j\omega) &= \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt \\ x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega \end{aligned}}$$

Useful facts about CTFT's

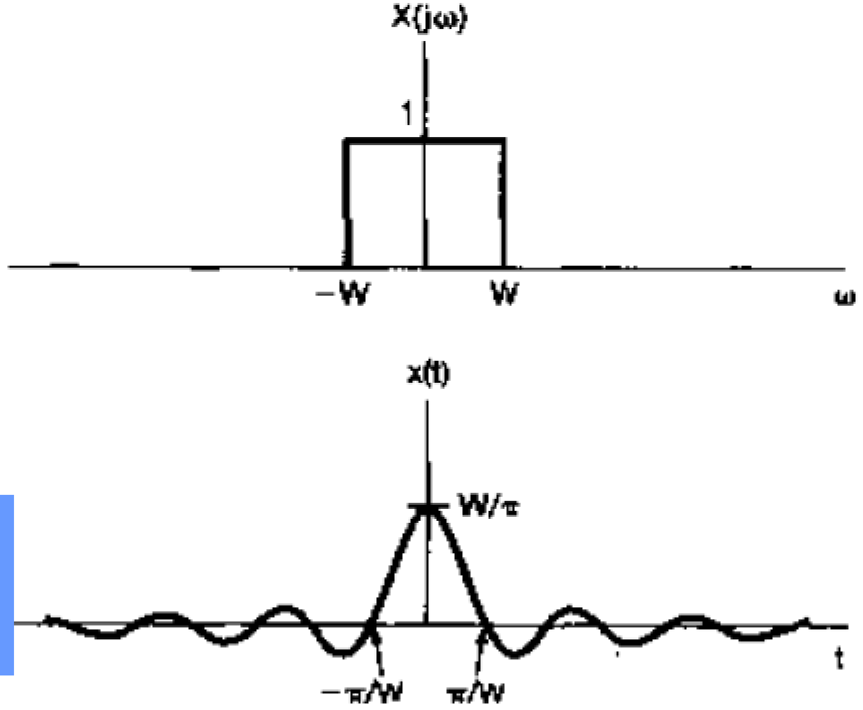
$$X(0)=\int_{-\infty}^{+\infty} x(t)\,dt$$

$$x(0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\,d\omega$$

Example 4.5 A square pulse in the frequency domain

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega = \frac{\sin(Wt)}{\pi t}$$



How about $X(j\omega)=\delta(\omega)$?

CT Fourier Transforms of Periodic Signals

Suppose $$X(j\omega) = \delta(\omega - \omega_0)$$

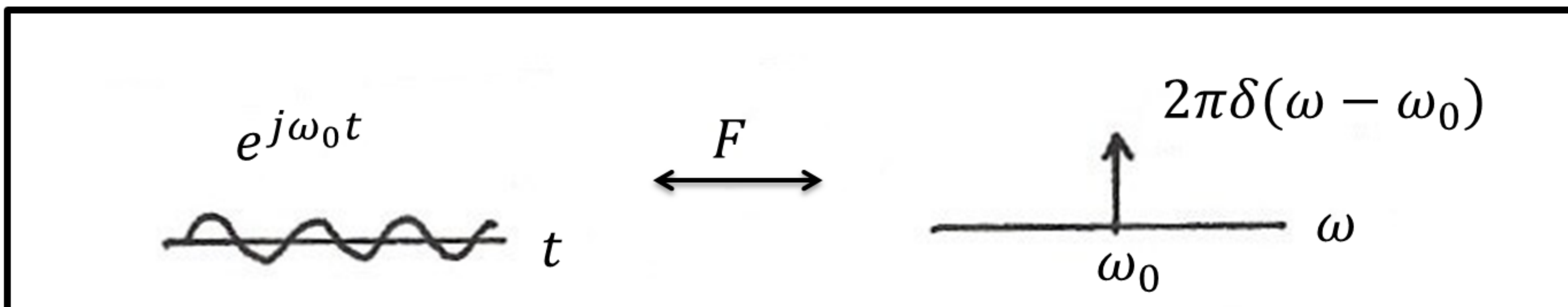
$$\Downarrow$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

— periodic in t with frequency ω_o

That is $$\boxed{e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega - \omega_0)}$$

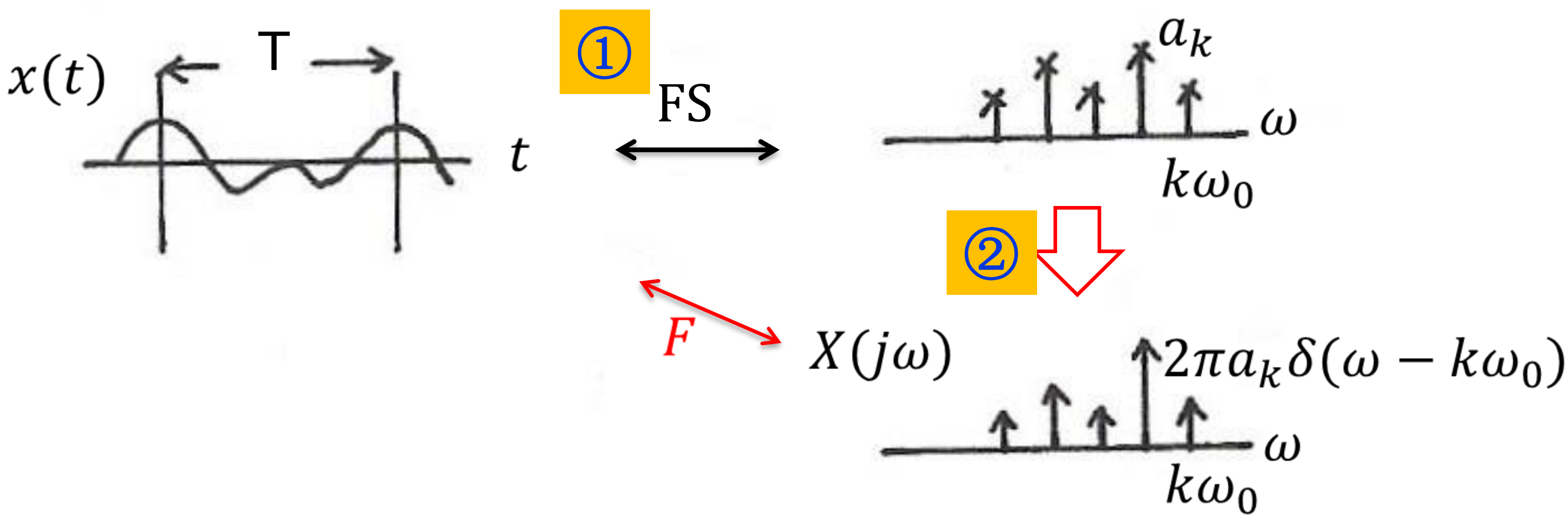
— All the energy is concentrated in one frequency — ω_o



Fourier Transform for Periodic Signals – Unified Framework

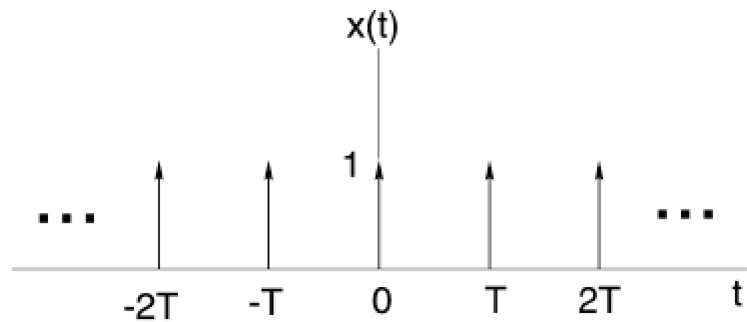
More generally, if $x(t) = x(t+T)$, then

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \longleftrightarrow X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)$$ Discrete spectra



Example 4.8

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \quad \text{— sampling function}$$

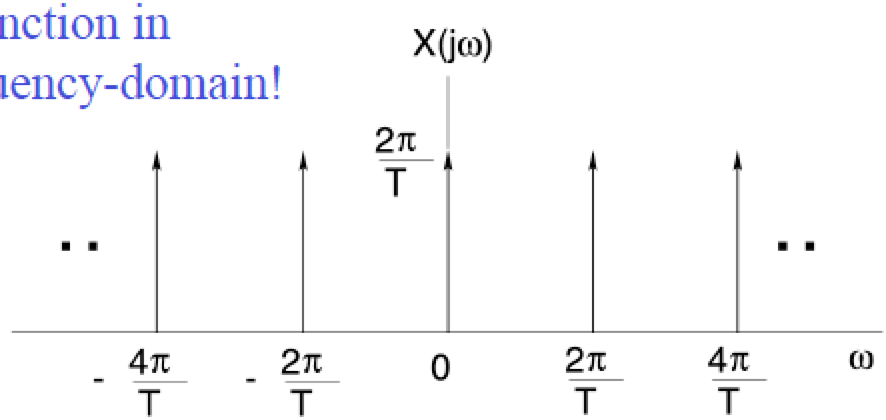


$$x(t) \xleftrightarrow{FS} a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_o t} dt = \frac{1}{T}$$

$$\Downarrow \; x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} = \sum_{k=-\infty}^{+\infty} \frac{1}{T} e^{jk\omega_o t}$$

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T} \delta(\omega - \underbrace{\frac{k2\pi}{T}}_{k\omega_o})$$

Same function in
the frequency-domain!



Note in this case, periodic in both time domain (with a period *T*) and frequency domain (with a period $2\pi/T$)

Properties of the CT Fourier Transform

1) Linearity

$$x(t) \xleftrightarrow{F} X(j\omega),\ y(t) \xleftrightarrow{F} Y(j\omega)$$

$$ax(t) + by(t) \xleftrightarrow{F} aX(j\omega) + bY(j\omega)$$

2) Time Shifting

$$x(t - t_0) \longleftrightarrow e^{-j\omega t_o} X(j\omega)$$

Proof: $\int_{-\infty}^{\infty} x(\underbrace{t - t_o}_{t'}) e^{-j\omega t} dt = e^{-j\omega t_o} \underbrace{\int_{-\infty}^{\infty} x(t') e^{-j\omega t'} dt'}_{X(j\omega)}$

FT magnitude unchanged

$$\left| e^{-j\omega_0 t} X(j\omega) \right| = \left| X(j\omega) \right|$$

Linear change in *FT* phase

$$\angle(e^{-j\omega_0 t} X(j\omega)) = \angle X(j\omega) - \omega t_0$$

CTFT Properties (cont.)

3) Conjugation & Conjugate Symmetry

- Conjugation $x^*(t) \xleftrightarrow{F} X^*(-j\omega)$

$$\left[\cdots a_{-1} e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + \cdots\right]^*$$

$$a_{-1}^* e^{j\omega_0 t}$$

- Conjugate Symmetry $x(t)$ real $\longleftrightarrow X(-j\omega) = X^*(j\omega)$

$\lvert X(-j\omega)\rvert = \lvert X(j\omega)\rvert$	*Even*	Or	$\mathrm{Re}\{X(-j\omega)\} = \mathrm{Re}\{X(j\omega)\}$	*Even*
$\angle X(-j\omega) = -\angle X(j\omega)$	*Odd*		$\mathrm{Im}\{X(-j\omega)\} = -\mathrm{Im}\{X(j\omega)\}$	*Odd*

When $x(t)$ is real (all the physically measurable signals are *real*), the negative frequency components do *not* carry any additional information from the positive frequency components. $\omega \geq 0$ will be sufficient.

CT Fourier Series Property

- Conjugate Symmetry

$$x(t) \quad real \quad \Rightarrow \quad a_{-k} = a_k{}^*$$

Proof: $$a_{-k} = \frac{1}{T}\int_T x(t)e^{jk\omega_0 t}dt = \left[\frac{1}{T}\int_T x^*(t)e^{-jk\omega_0 t}dt\right]^* = a_k^*$$

$\because$

$a_k = \mathrm{Re}\{a_k\} + j\,\mathrm{Im}\{a_k\}$ $\quad\therefore\quad \boxed{\mathrm{Re}\{a_{-k}\} + j\,\mathrm{Im}\{a_{-k}\}} = \boxed{\mathrm{Re}\{a_k\} - j\,\mathrm{Im}\{a_k\}}$

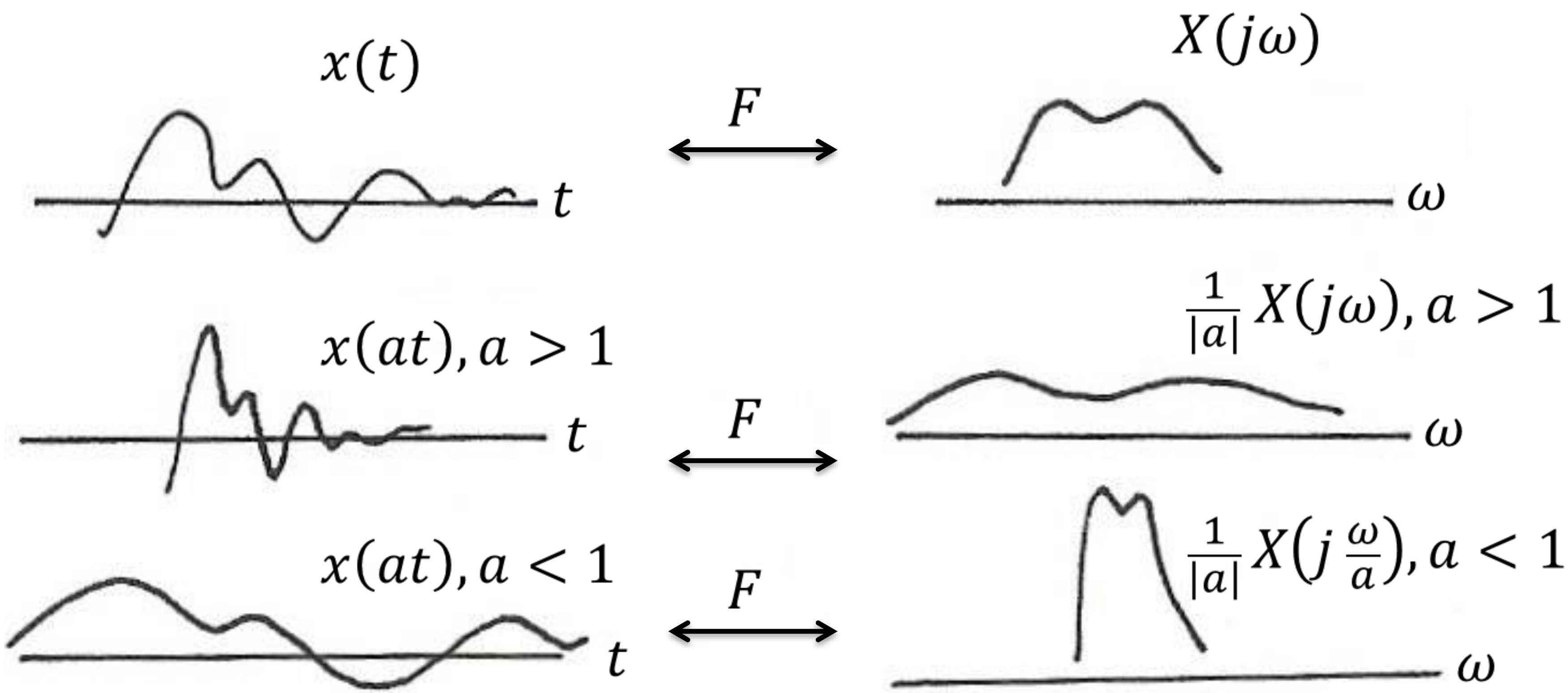
$\therefore$

$\mathrm{Re}\{a_k\}$ is even , $\mathrm{Im}\{a_k\}$ is odd

CTFT Properties (cont.)

4) Time/Frequency Scaling $x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$

E.g. $a > 1 \rightarrow at > t$
compressed in time $\leftrightarrow$
stretched in frequency



CTFT Properties (cont.)

4) Time/Frequency Scaling $x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$

E.g. $a > 1 \rightarrow at > t$
compressed in time $\leftrightarrow$
stretched in frequency

$$\Downarrow a = -1$$

$$x(-t) \longleftrightarrow X(-j\omega)$$

Time reversal

$$\Downarrow$$

a) $x(t)$ real and even $\quad x(t) = x(-t) = x^*(t)$

$\Rightarrow X(j\omega) = X(-j\omega) = X^*(-j\omega)$ — Real & even

b) $x(t)$ real and odd $\quad x(t) = -x(-t) = x^*(t)$

$\Rightarrow X(j\omega) = -X(-j\omega) = X^*(-j\omega)$ — Purely imaginary & odd

c) $X(j\omega) = \mathrm{Re}\{X(j\omega)\} + j\mathrm{Im}\{X(j\omega)\}$

For real $x(t) = Ev\{x(t)\} + Od\{x(t)\}$

CTFT Properties (cont.)

5) Differentiation/Integration

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(j\omega) + \underbrace{\pi X(j0)\delta(\omega)}_{\text{DC term}}$$

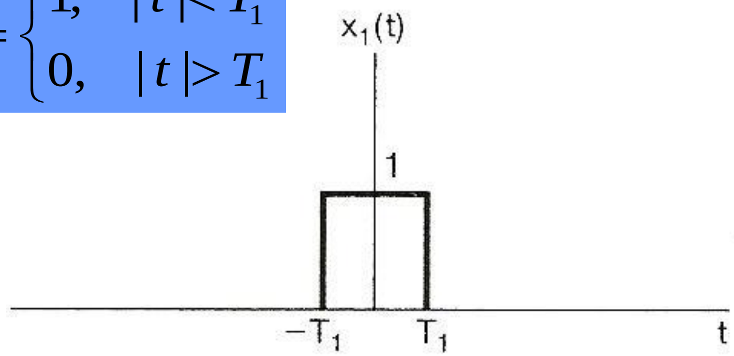
<u>Example:</u>

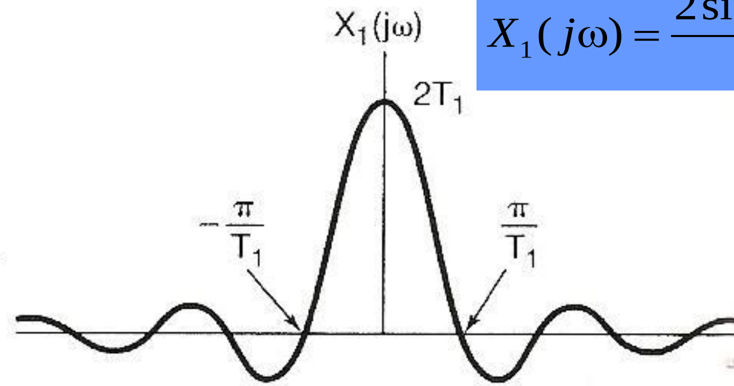
What is the Fourier transform for unit step function $u(t)$?

CTFT Properties (cont.)

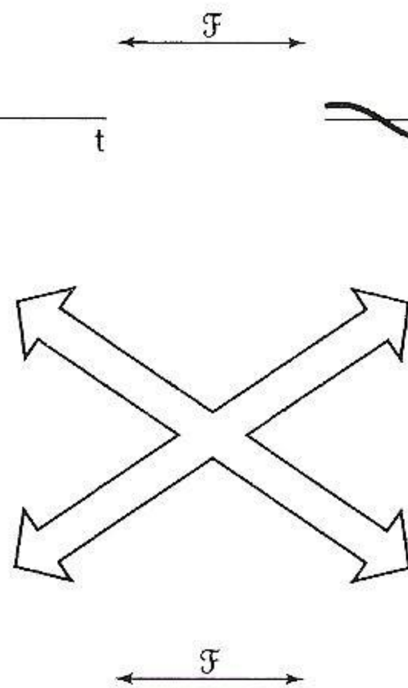
6) Duality

$$x_1(t)=\begin{cases}1, & |t|<T_1\\ 0, & |t|>T_1\end{cases}$$

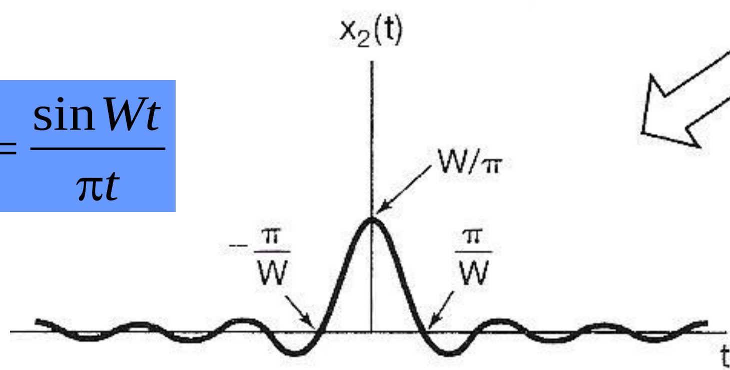


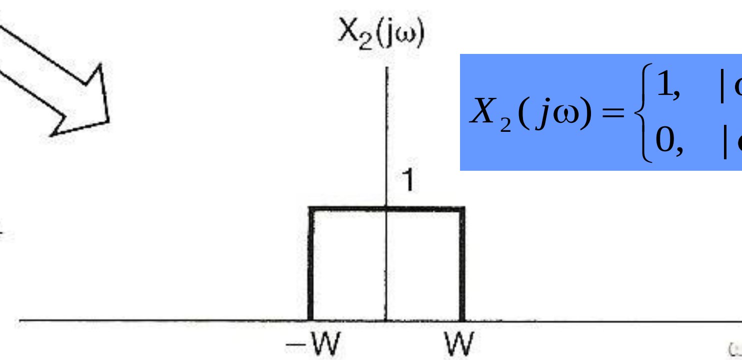


$$X_1(j\omega)=\frac{2\sin\omega T_1}{\omega}$$



$$x_2(t)=\frac{\sin Wt}{\pi t}$$



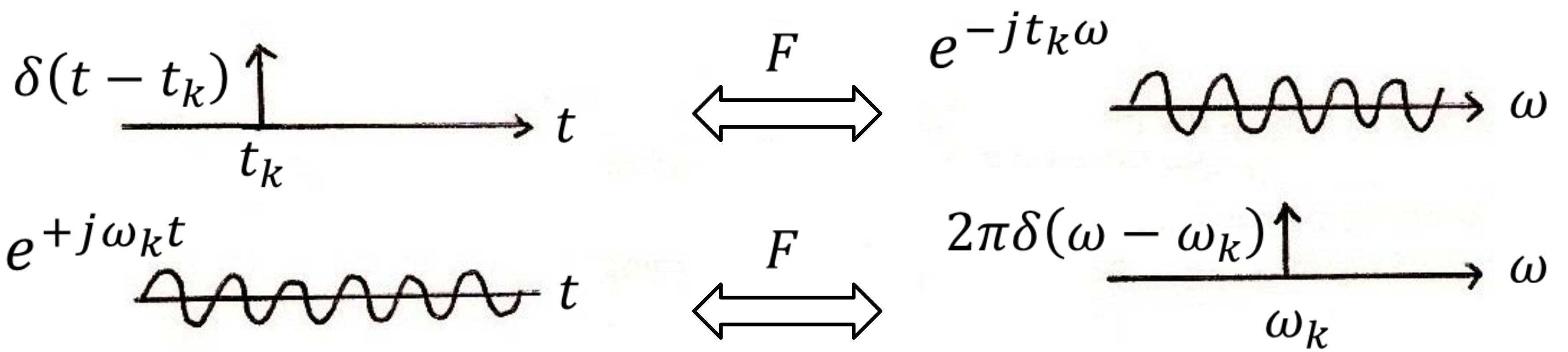


$$X_2(j\omega)=\begin{cases}1, & |\omega|<W\\ 0, & |\omega|>W\end{cases}$$

- Time/frequency domains are kind of "symmetric".
- If there are characteristics of a function of time that have implications with regard to the Fourier transform, then the same characteristics associated with a function of frequency will have ***dual*** implications in the time domain.

Example:

$\{\delta(t-t_k), -\infty < t_k < \infty\}$ $\{2\pi\delta(\omega-\omega_k), -\infty < \omega_k < \infty\}$



CTFT Properties (cont.)

7) Parseval's Relation

$$\underbrace{\int_{-\infty}^{+\infty} |x(t)|^2 dt}_{\text{Total energy in the time-domain}} = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega}_{\text{Total energy in the frequency-domain}}$$

$\frac{1}{2\pi}|X(j\omega)|^2$ — spectral density

Table 4.2
Basic Fourier Transform Pairs

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t \rvert < T_1 \\ 0, & T_1 < \lvert t \rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t \rvert < T_1 \\ 0, & \lvert t \rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Summary

- **Understand CT Fourier transform**
  - **Synthesis and analysis equations**
  - **Difference with CT Fourier series**
  - **Fourier transform for periodic signal**
  - **Properties of CT Fourier transform**