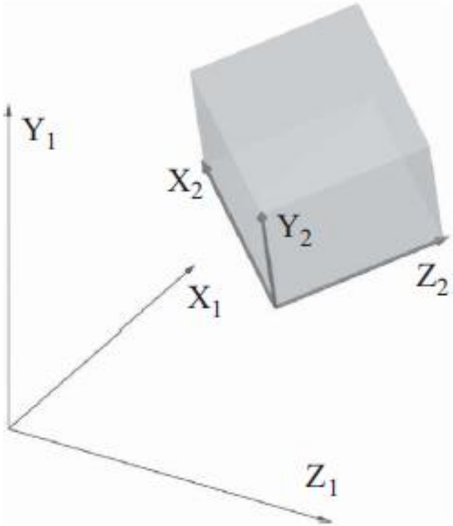
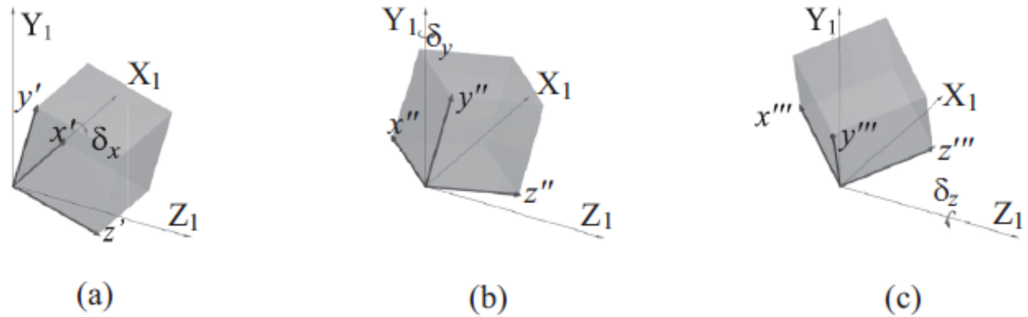
CHAPTER 3

Position and Orientation (Rotation matrix)

Pose = Position + Orientation

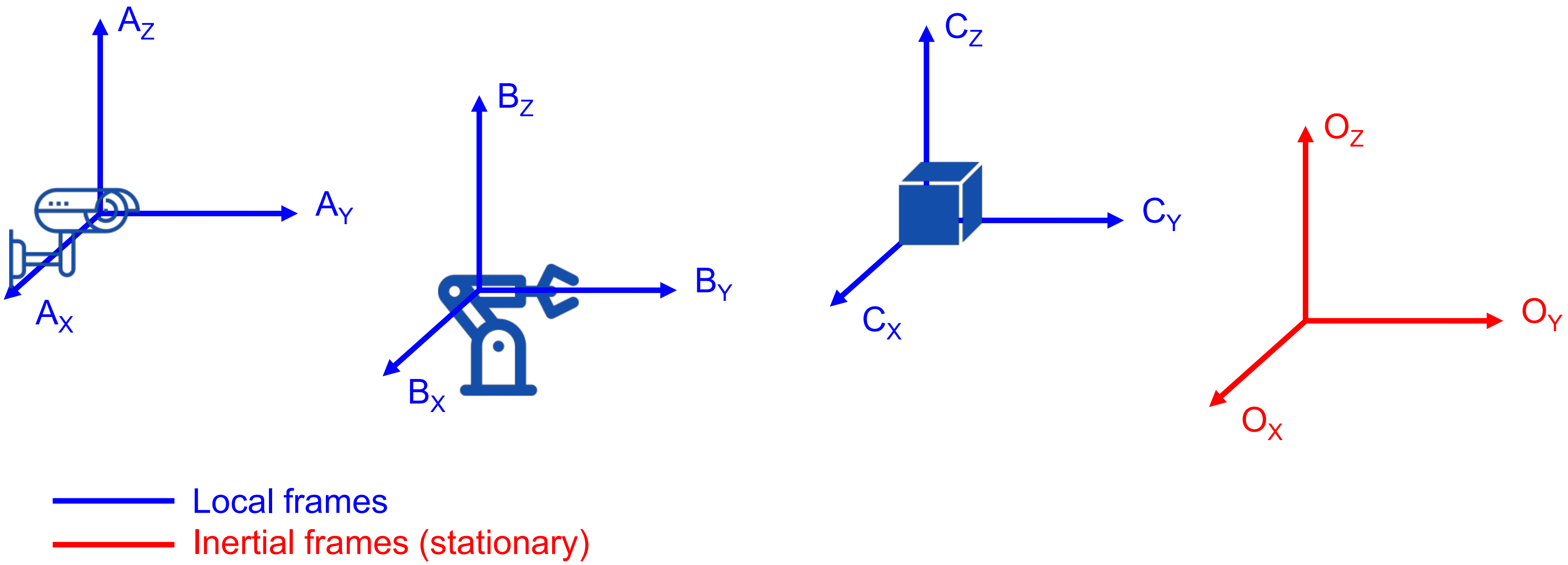




A *rigid body* is completely described in space by its *position* and *orientation* (in brief *pose*) with respect to a reference frame.

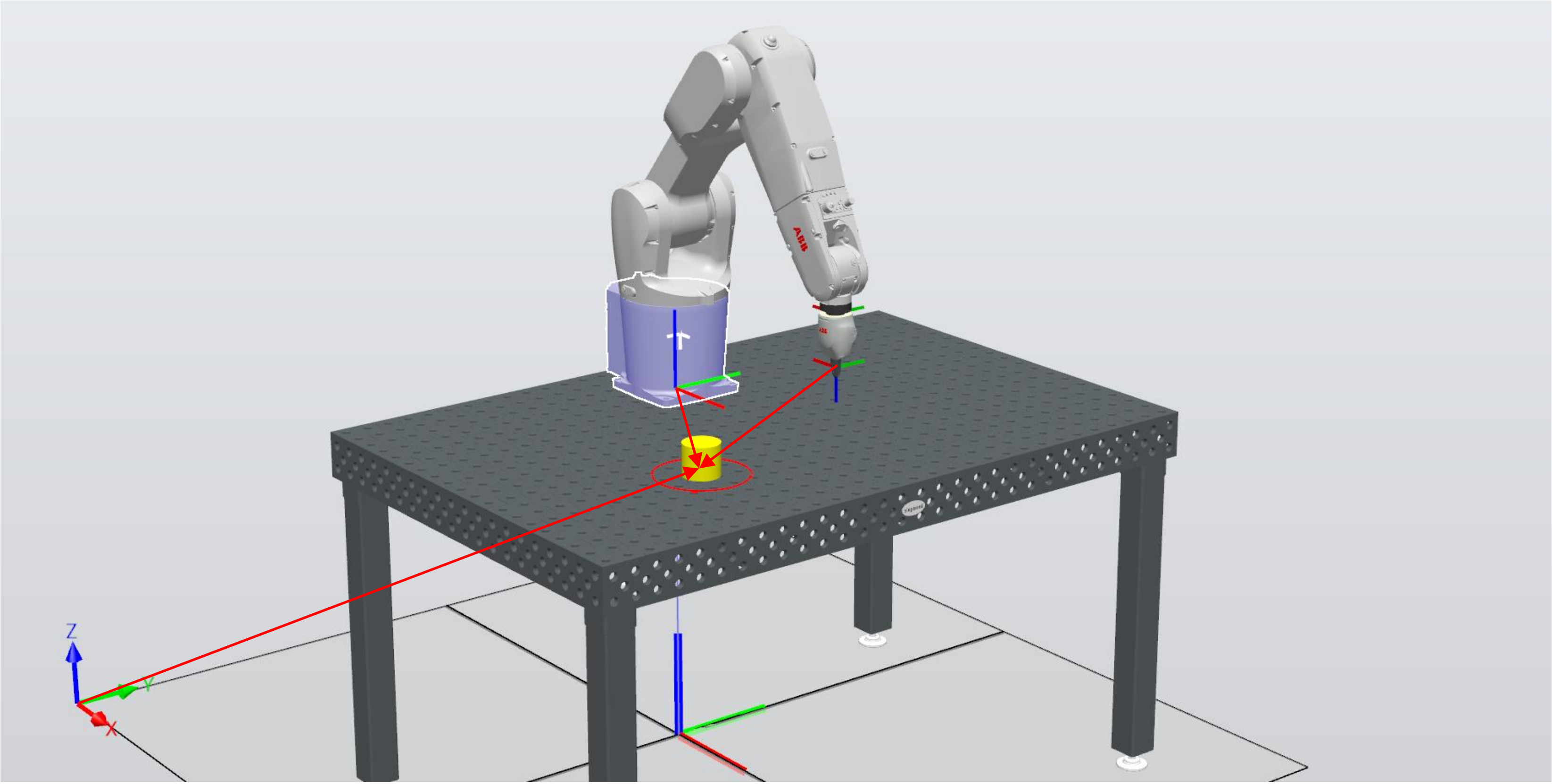
In this part, we will learn how to transform the position and orientation between current and reference frame using "Rotation Matrix"

3D Coordinate Rotation; Why?



1. To get absolute values of some vectorial quantities (position, velocity, acceleration and so on …)
2. In order to do calculations on different vectorial quantities by expressing them on the same frame.

3D Coordinate Rotation; Why?

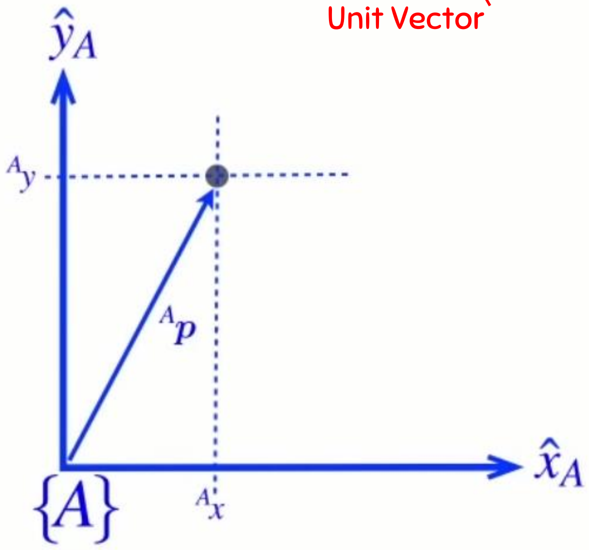


2D Rotation

Vector P on frame A

$$ {}^{A}\boldsymbol{p} = {}^{A}x\,\hat{\boldsymbol{x}}_A + {}^{A}y\,\hat{\boldsymbol{y}}_A $$

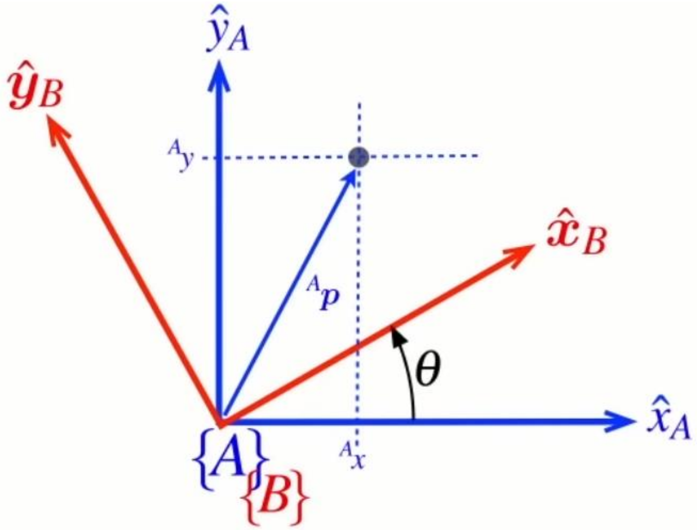
Unit Vector



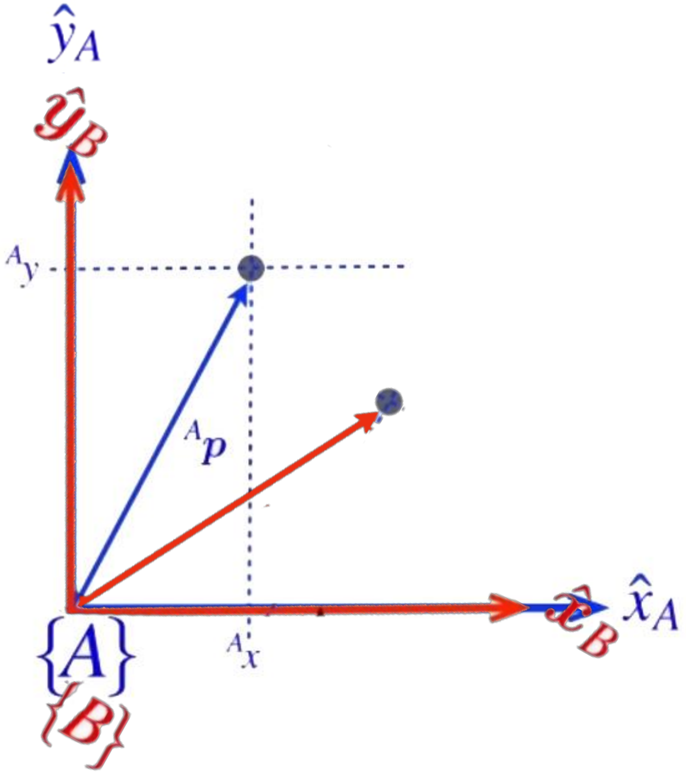
New coordinate frame B
(from rotate frame A by angle = θ)

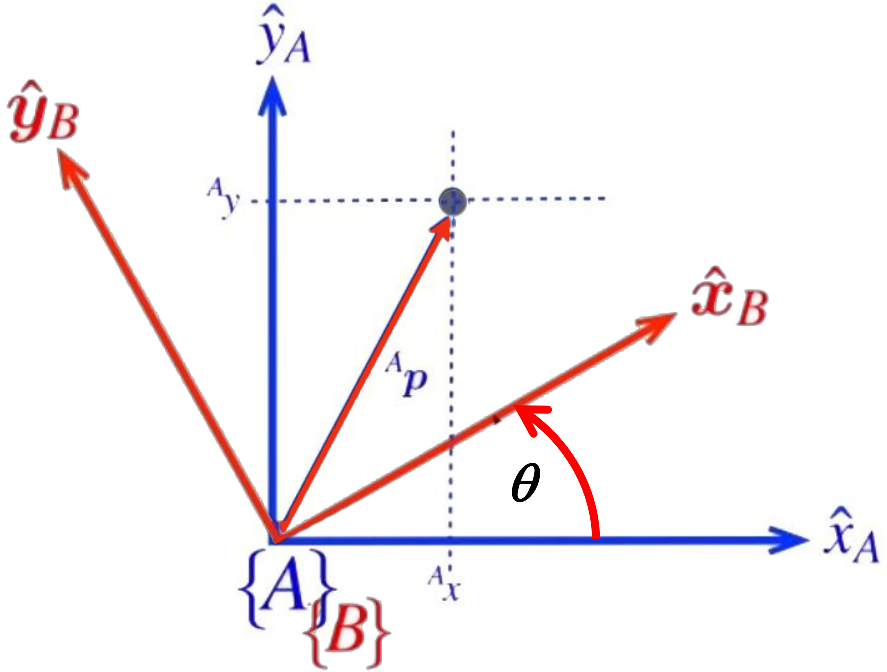
Vector P on frame B

$$ {}^{B}\boldsymbol{p} = {}^{B}x\,\hat{\boldsymbol{x}}_B + {}^{B}y\,\hat{\boldsymbol{y}}_B $$



2D Rotation


$\hat{y}_A$
$\hat{y}_B$
${}^A y$
${}^A p$
$\hat{x}_A$
$\hat{x}_B$
$\{A\}$
$\{B\}$
${}^A x$

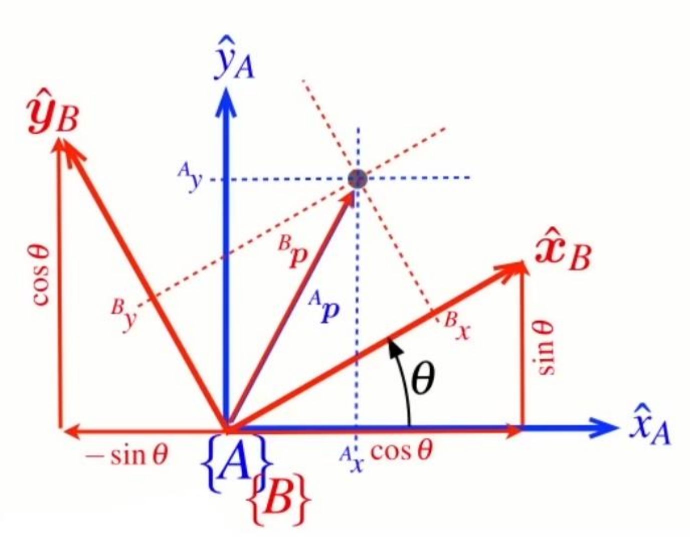

$\hat{y}_A$
$\hat{y}_B$
${}^A y$
${}^A p$
$\hat{x}_B$
θ
$\hat{x}_A$
$\{A\}$
$\{B\}$
${}^A x$

2D Rotation

Write Unit Vector in term of cosine and sine

$$\hat{\boldsymbol{x}}_B = \cos\theta\,\hat{\boldsymbol{x}}_A + \sin\theta\,\hat{\boldsymbol{y}}_A$$

$$\hat{\boldsymbol{y}}_B = -\sin\theta\,\hat{\boldsymbol{x}}_A + \cos\theta\,\hat{\boldsymbol{y}}_A$$



Then

$$^B\boldsymbol{p} = {}^B x\,\hat{\boldsymbol{x}}_B + {}^B y\,\hat{\boldsymbol{y}}_B$$

$$^B\boldsymbol{p} = {}^B x\,(\cos\theta\,\hat{\boldsymbol{x}}_A + \sin\theta\,\hat{\boldsymbol{y}}_A) + {}^B y\,(-\sin\theta\,\hat{\boldsymbol{x}}_A + \cos\theta\,\hat{\boldsymbol{y}}_A)$$

$$^B\boldsymbol{p} = \left({}^B x\cos\theta - {}^B y\sin\theta\right)\hat{\boldsymbol{x}}_A + \left({}^B x\sin\theta + {}^B y\cos\theta\right)\hat{\boldsymbol{y}}_A$$

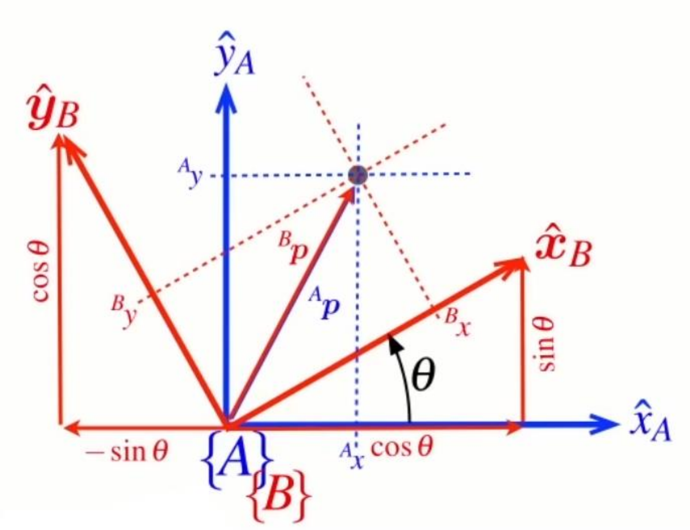
From Bp = Ap so we can compare their coefficient

$$^A\boldsymbol{p} = {}^A x\,\hat{\boldsymbol{x}}_A + {}^A y\,\hat{\boldsymbol{y}}_A \qquad {}^B\boldsymbol{p} = \left({}^B x\cos\theta - {}^B y\sin\theta\right)\hat{\boldsymbol{x}}_A + \left({}^B x\sin\theta + {}^B y\cos\theta\right)\hat{\boldsymbol{y}}_A$$

2D Rotation

$$ {}^{A}\boldsymbol{p} = {}^{A}x\,\hat{\boldsymbol{x}}_A + {}^{A}y\,\hat{\boldsymbol{y}}_A $$

$$ {}^{B}\boldsymbol{p} = \left({}^{B}x\cos\theta - {}^{B}y\sin\theta\right)\hat{\boldsymbol{x}}_A + \left({}^{B}x\sin\theta + {}^{B}y\cos\theta\right)\hat{\boldsymbol{y}}_A $$



So

$$ {}^{A}x = {}^{B}x\cos\theta - {}^{B}y\sin\theta $$

$$ {}^{A}y = {}^{B}x\sin\theta + {}^{B}y\cos\theta $$

In term matrix

transform matrix

$$ \begin{pmatrix} {}^{A}x \\ {}^{A}y \end{pmatrix} = \boxed{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}} \begin{pmatrix} {}^{B}x \\ {}^{B}y \end{pmatrix} = {}^{A}\boldsymbol{p} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} {}^{B}\boldsymbol{p} $$

2D Rotation

$$\begin{pmatrix} {}^{A}x \\ {}^{A}y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} {}^{B}x \\ {}^{B}y \end{pmatrix} \quad = \quad {}^{A}\mathbf{p} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} {}^{B}\mathbf{p}$$

Now, the rotation matrix

$${}^{A}\mathbf{p} = {}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{p}$$

$${}^{A}\mathbf{R}_{B} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$${}^{A}\mathbf{R}_{B}$$

R is rotation matrix that rotate the vector from frame B to frame A (${}^{B}\mathbf{p}$ to ${}^{A}\mathbf{p}$)

2D Rotation

Properties of rotation matrix

$$^{A}\mathbf{R}_{B} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- An orthogonal (orthonormal) matrix
  - Each column is a unit length vector
  - Each column is orthogonal to all other columns
- The inverse is the same as the transpose $\mathbf{R}^{-1} = \mathbf{R}^{T}$
- The determinant is +1 $det(\mathbf{R}) = 1$
  - the length of a vector is unchanged by rotation
- Rotation matrices belong to the Special Orthogonal group of dimension 2 $\mathbf{R} \in SO(2)$

2D Rotation

Inverse rotation

$$^{A}\mathbf{p} = {}^{A}\mathbf{R}_{B}\, {}^{B}\mathbf{p}$$

- To rotate a vector from frame {A} to frame {B} we use the inverse rotation matrix
  - The inverse is simply the transpose

$$^{B}\mathbf{p} = {}^{A}\mathbf{R}_{B}^{-1}\, {}^{A}\mathbf{p}$$

Then

$$^{B}\mathbf{p} = {}^{B}\mathbf{R}_{A}\, {}^{A}\mathbf{p}$$

$$^{A}\mathbf{R}_{B} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\boxed{^{B}\mathbf{R}_{A} = {}^{A}\mathbf{R}_{B}^{-1}}$$

2D Rotation

$$ {}^{A}\mathbf{R}_{B} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} $$

2D Rotation Matrix

```
rot2(0)
rot2(0.2) %0.2 radian
rot2(30,"deg") % degree angle
```

Properties of Rotation Matrix

```
R=rot2(30,"deg")
c1=R(:,1) % 1st column of matrix
c2=R(:,2) % 2nd column of matrix
inv(R)
R'
det(R)
dot(c1,c2)
% Plot in coordinate frame
trplot2(R)
axis("equal")
```

$$ rot2(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} $$

$$ rot2(0.2) = \begin{bmatrix} 0.9801 & -0.1987 \\ 0.1987 & 0.9801 \end{bmatrix} $$

$$ rot2(30,"deg") = \begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix} $$

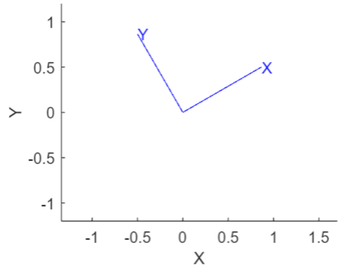
$$ c1 = \begin{bmatrix} 0.866 \\ 0.5 \end{bmatrix}, c2 = \begin{bmatrix} -0.5 \\ 0.866 \end{bmatrix} $$

$$ R^{-1} = R^{T} \begin{bmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{bmatrix} $$

$$ \det(R) = -(-0.5 * 0.5) + (0.866 * 0.866) = 1 $$

$$ \overrightarrow{c1} \cdot \overrightarrow{c2} = \left|\overrightarrow{c1}\right| \left|\overrightarrow{c2}\right| cos(90^{\circ}) = 0 $$

trplot2(R)



2D Rotation : Exercises

1. Find 2D rotation matrix with angle (π/4 Radian)

$$^{A}R_{B} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

2. Find 2D rotation matrix with angle (45 Deg.)

$$^{A}R_{B} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

3. Find 2D rotation matrix with angle (−15 Deg.)

$$^{A}R_{B}(-15) = \begin{bmatrix} 0.97 & 0.26 \\ -0.26 & 0.97 \end{bmatrix}$$

2D Rotation : Exercises

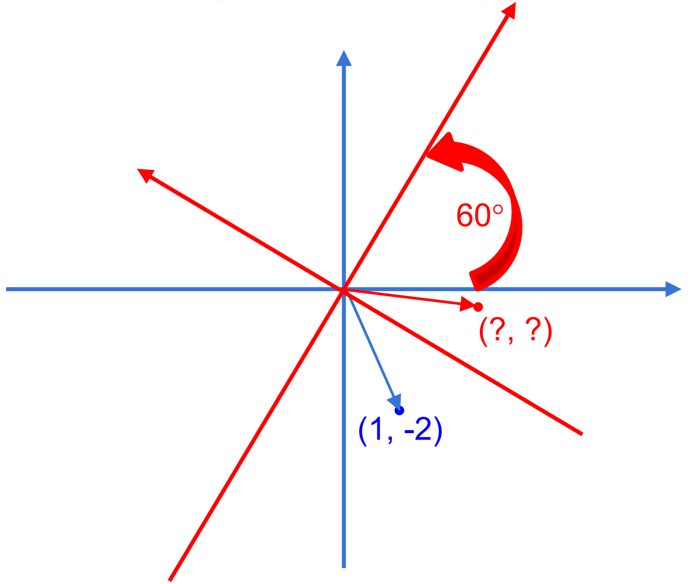
4. If Vector Bp(1, −2) is rotated in counter-clockwise direction by 60 deg
Find The rotation matrix and the coordinate value Ap ?

$$^{A}\mathbf{R}_{B} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$A_p = {}^{A}R_{B} \cdot B_p$$

$$= \begin{bmatrix} 0.5 & -0.87 \\ 0.87 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2.23 \\ -0.13 \end{bmatrix}$$



2D Rotation : Exercises

4. If Vector Bp(1, −2) is rotated in counter-clockwise direction by 60 deg
Find The rotation matrix and the coordinate value Ap ?

$$^{A}\boldsymbol{p} = {}^{A}\mathbf{R}_{B}\,{}^{B}\boldsymbol{p}$$

$$^{A}p = {}^{A}R_{B}\,{}^{B}p$$

60°

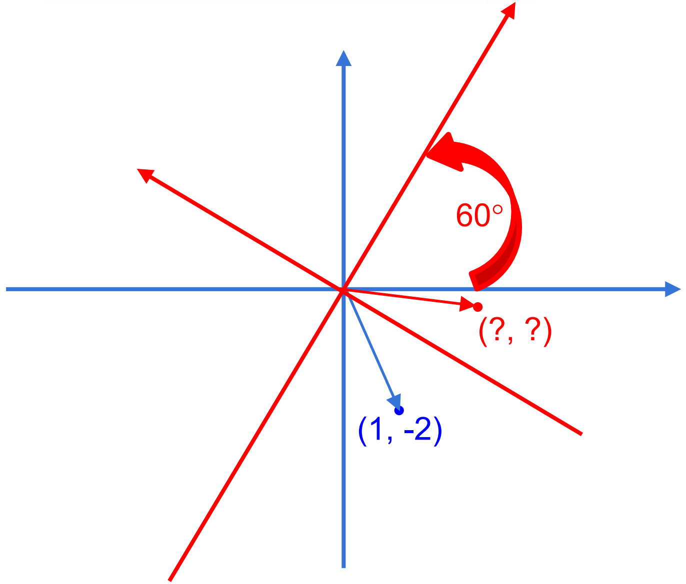
$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} + \sqrt{3} \\ \frac{\sqrt{3}}{2} - 1 \end{bmatrix}$$

$$\therefore {}^{A}p = \left(\frac{1}{2} + \sqrt{3} \,,\ \frac{\sqrt{3}}{2} - 1 \right)$$

$$^{A}\mathbf{R}_{B} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



2D Rotation : Exercises

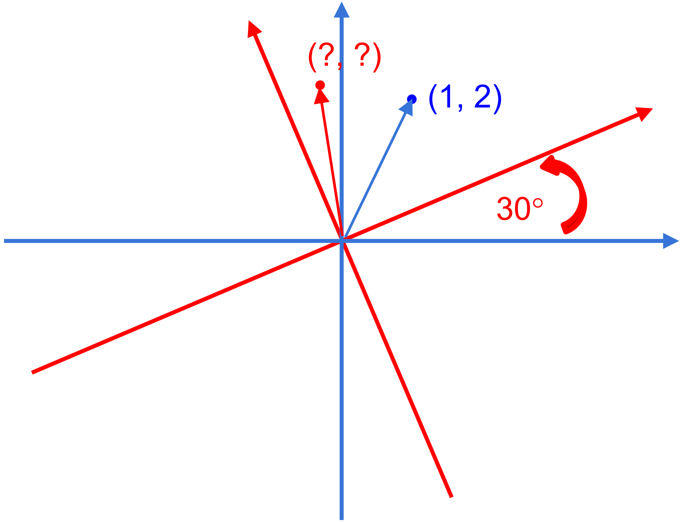
5. If Vector Bp(1, 2) is rotated in counter-clockwise direction by 30 deg
Find The rotation matrix and the coordinate value Ap ?

$$A_p = {}^A R_B \cdot B_p$$

$$= \begin{bmatrix} 0.87 & -0.5 \\ 0.5 & 0.87 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.13 \\ 2.23 \end{bmatrix}$$

$${}^A\mathbf{R}_B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



2D Rotation : Exercises

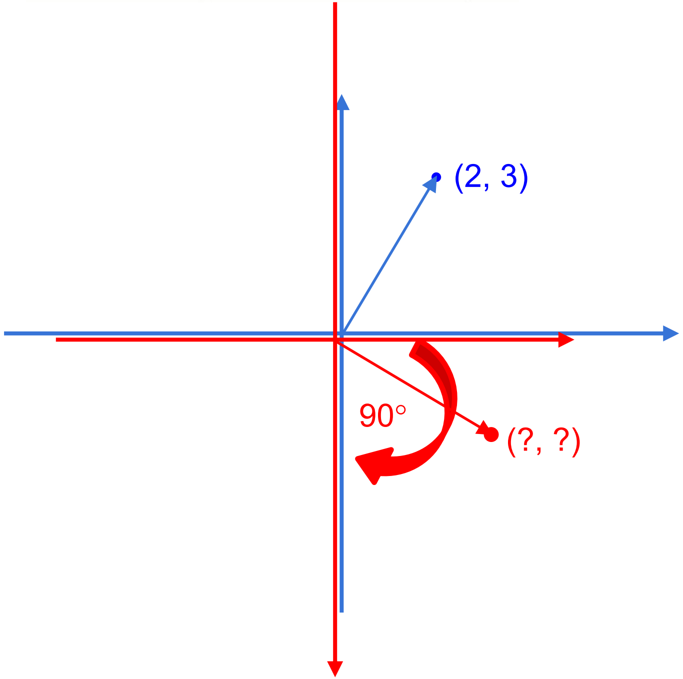
6. If Vector Bp(2, 3) is rotated in clockwise direction by 90 deg
Find The rotation matrix and the coordinate value Ap ?

$$^{A}\mathbf{R}_B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$A_p = {}^{A}R_B \cdot B_p$

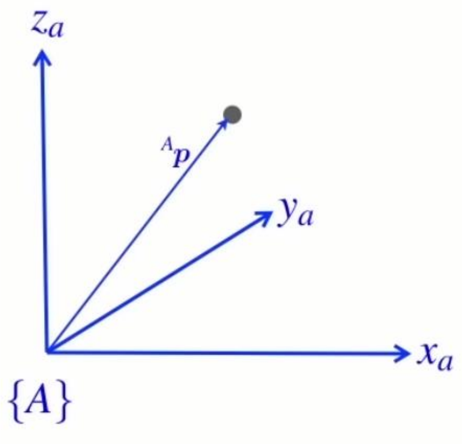
$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

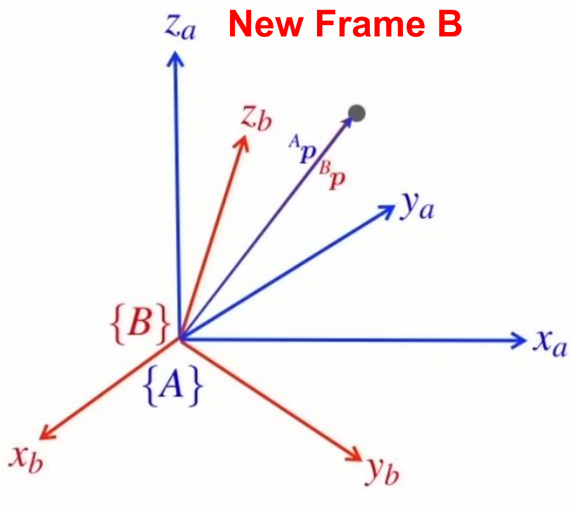
$= \begin{bmatrix} 3 \\ -2 \end{bmatrix}$



3D Rotation

Old Frame A





- For the 3D case we also describe the new axes {B} in terms of the old axes {A}

$$\begin{pmatrix} {}^A x \\ {}^A y \\ {}^A z \end{pmatrix} = \begin{array}{c} \begin{array}{ccc} \text{new x-axis} & \text{new y-axis} & \text{new z-axis} \end{array} \\ \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \end{array} \begin{pmatrix} {}^B x \\ {}^B y \\ {}^B z \end{pmatrix}$$

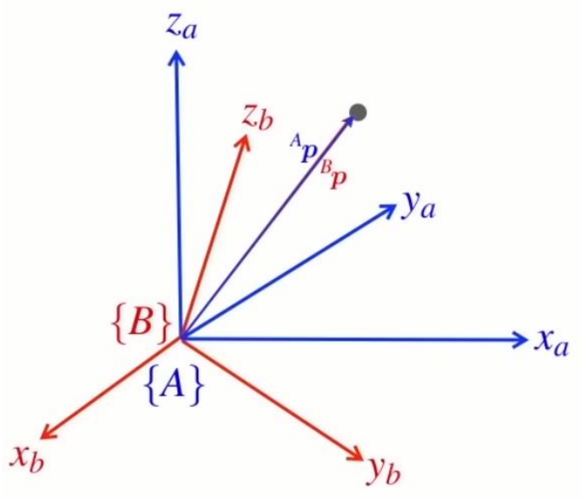
- Which transforms vectors from the new frame {B} to the old frame {A}

3D Rotation

Properties of rotation matrix

$$ {}^{A}\mathbf{R}_{B} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} $$

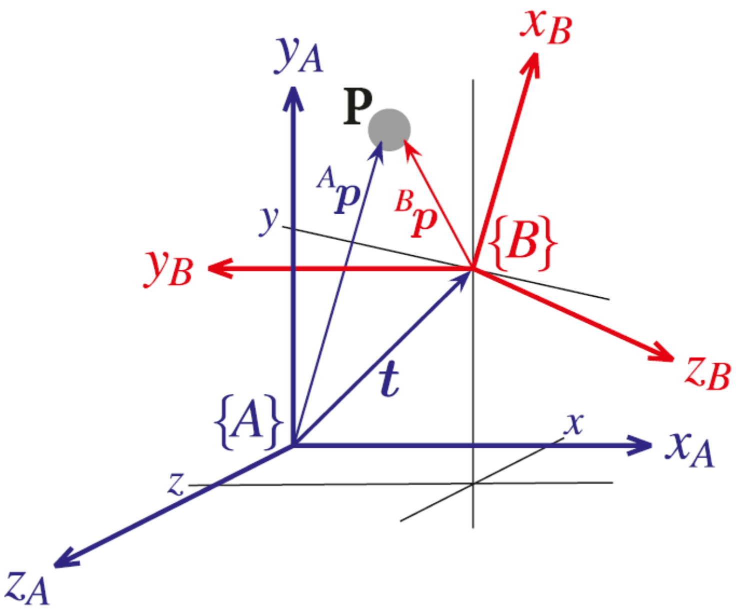
- An orthogonal (orthonormal) matrix
- Each column is a **unit** length **vector**
- Each column is **orthogonal** to all other columns
- The inverse is the same as the transpose $\mathbf{R}^{-1} = \mathbf{R}^{T}$
- The determinant is +1 $det(\mathbf{R}) = 1$
  - ➡ the length of a vector is unchanged by rotation
- Rotation matrices belong to the Special Orthogonal group of dimension 3 $\mathbf{R} \in SO(3)$



3D Rotation

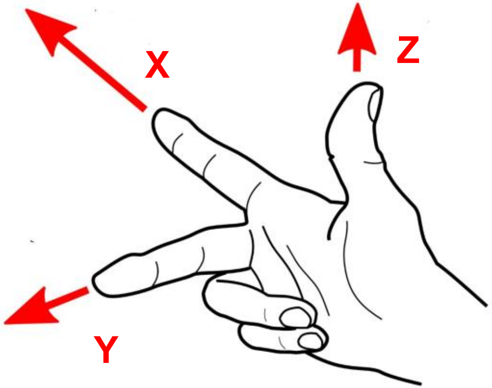
In mechanics and geometry, the 3D rotation group, often denoted SO(3), is the group of all rotations about the origin of three-dimensional Euclidean space R^3 under the operation of composition.

The group SO(3) is used to describe the possible rotational symmetries of an object, as well as the possible orientations of an object in space. Its representations are important in physics, where they give rise to the elementary particles of integer spin.



3D Rotation

Elementary of rotation matrix



Rotation about x axis by θ

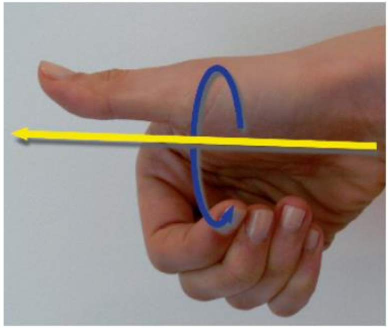
$$\mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

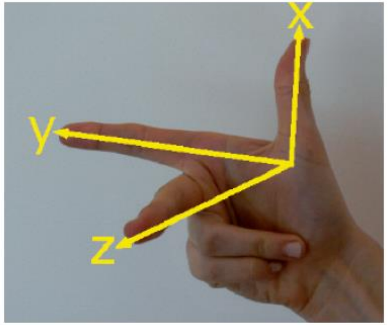
Rotation about y axis by θ

$$\mathbf{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Rotation about z axis by θ

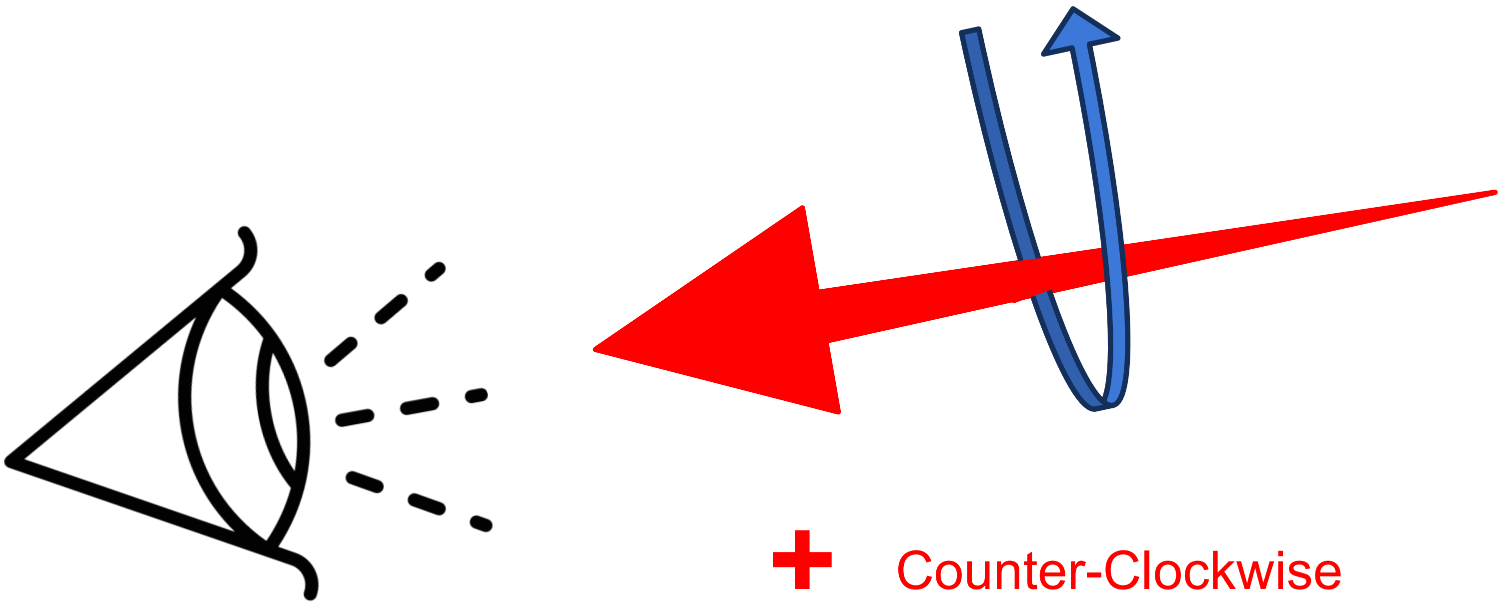
$$\mathbf{R}_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



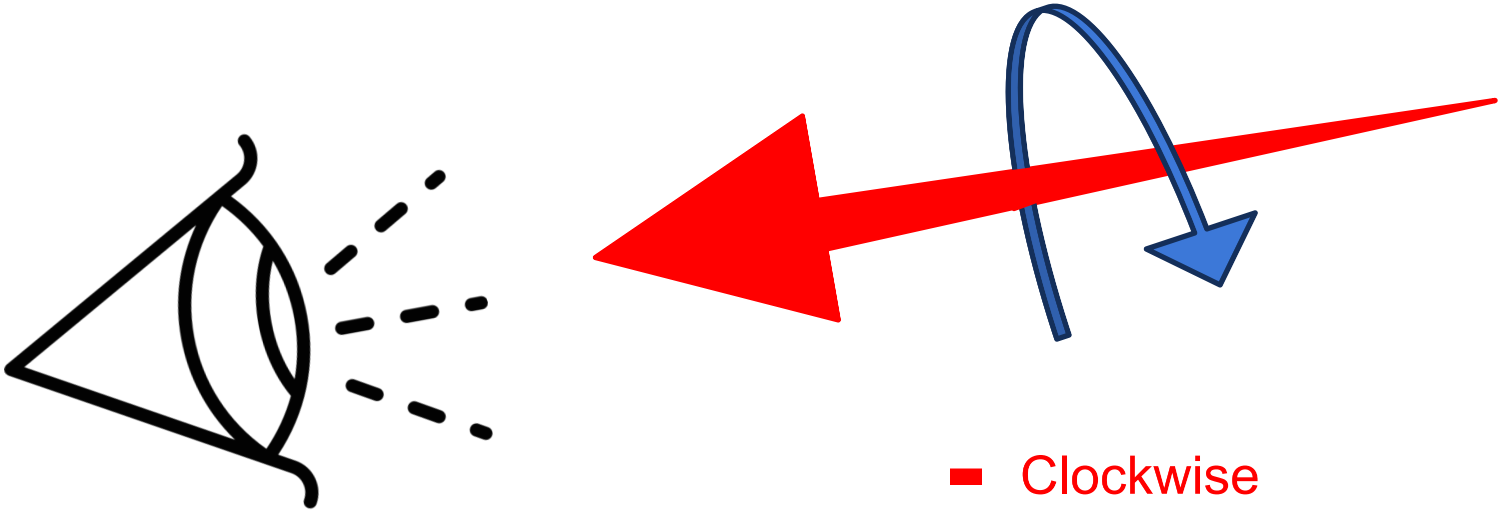


"The curl of your fingers indicates the direction of increasing angle."

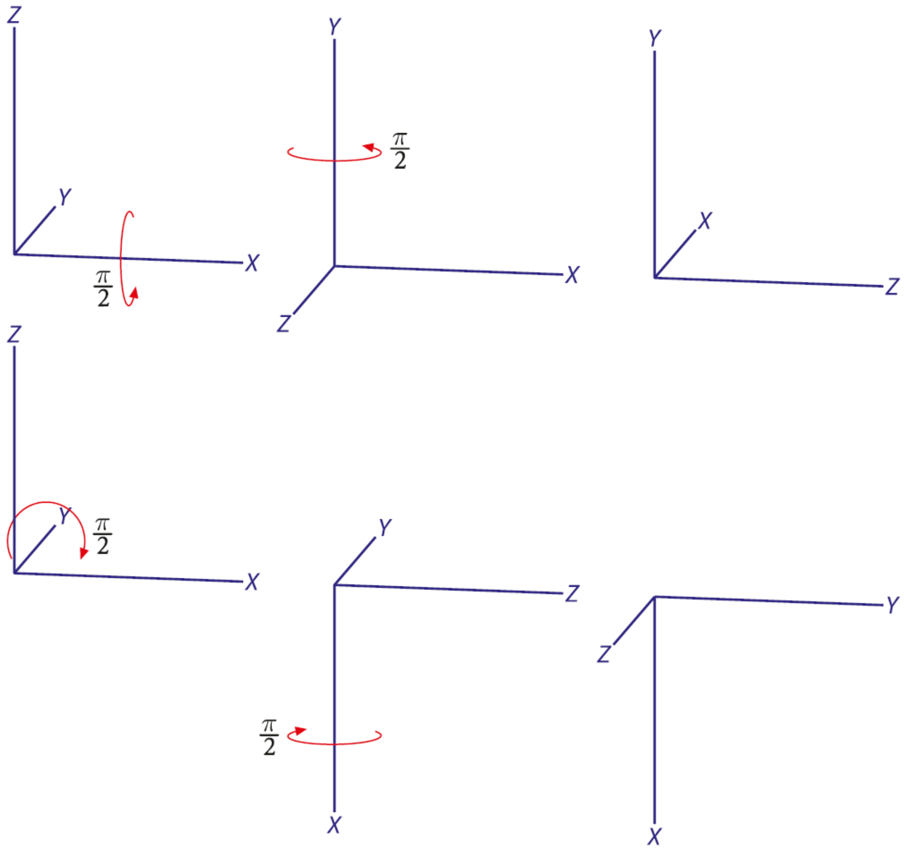
3D Rotation



3D Rotation



3D Rotation



Example showing the noncommutativity (can't change the process order) of rotation.

- In the top row the coordinate frame is rotated by π /2 about the *x*-axis and then π /2 about the *y*-axis.
- In the bottom row the order of rotations has been reversed. The results are clearly different

3D Rotation

3D Rotation Matrix

```
R=rotx(pi/6)
```

```
R1=rotx(30,"deg")
det(R1)
inv(R1)
R1'
c1=R1(:,1)
c2=R1(:,2)
c3=R1(:,3)
dot(c1,c2)
dot(c1,c3)
dot(c2,c3)
norm(c1)
norm(c2)
norm(c3)
% Plot rotation in 3D coordinate
trplot(R1)
```

$$\mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

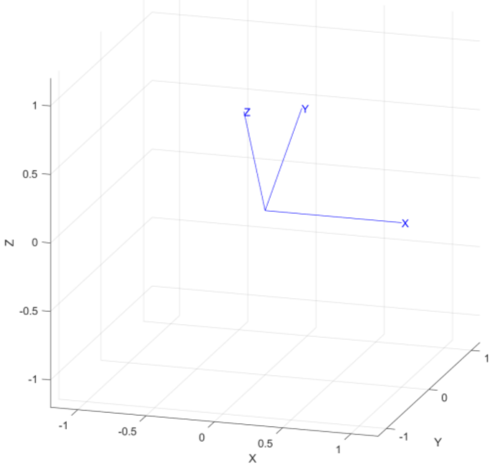
$$\mathbf{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\mathbf{R}_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
R = 3x3
    1.0000         0         0
         0    0.8660   -0.5000
         0    0.5000    0.8660
```

```
R1 = 3x3
    1.0000         0         0
         0    0.8660   -0.5000
         0    0.5000    0.8660

ans = 1
ans = 3x3
    1.0000         0         0
         0    0.8660    0.5000
         0   -0.5000    0.8660

ans = 3x3
    1.0000         0         0
         0    0.8660    0.5000
         0   -0.5000    0.8660

c1 = 3x1
     1
     0
     0

c2 = 3x1
         0
    0.8660
    0.5000

c3 = 3x1
         0
   -0.5000
    0.8660
```

```
ans = 0
ans = 0
ans = 0
ans = 1
ans = 1
ans = 1
```

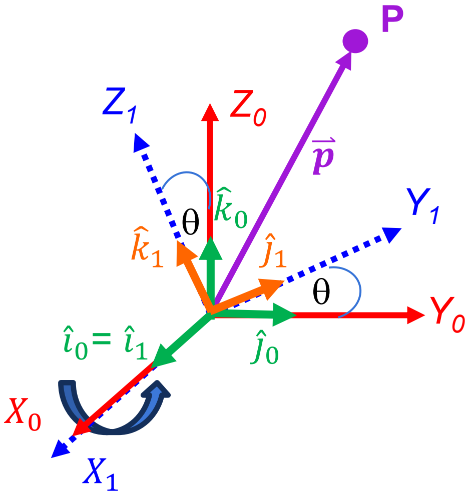


3D Rotation (Prove)

Rotation about x axis by θ deg.

$$\mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$${}^{0}\mathbf{P} = {}^{0}_{1}\mathbf{R}_x \; {}^{1}\mathbf{P}$$



"Unit vector"

$$\hat{\imath}_0 = \hat{\imath}_1$$
$$\hat{j}_0 = \cos(\theta)\hat{j}_1 - \sin(\theta)\hat{k}_1$$
$$\hat{k}_0 = \sin(\theta)\hat{j}_1 + \cos(\theta)\hat{k}_1$$

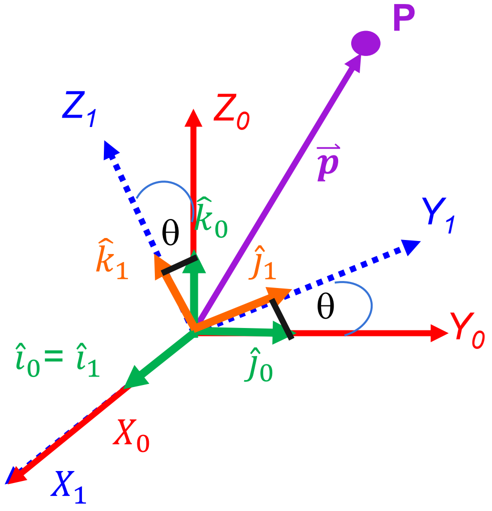
* Calculate by vector projection (sine, cosine)

$$x_0 = x_1$$
$$y_0 = y_1\cos(\theta) - z_1\sin(\theta)$$
$$z_0 = y_1\sin(\theta) + z_1\cos(\theta)$$

Write in matrix form…

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

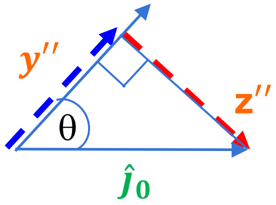
$${}^{0}\boldsymbol{R}_1 \; (elementary \;\; x \;\; rotation\, matrix)$$



$$\hat{\imath}_0 = \hat{\imath}_1$$
$$\hat{j}_0 = \cos(\theta)\hat{j}_1 - \sin(\theta)\hat{k}_1$$
$$\hat{k}_0 = \sin(\theta)\hat{j}_1 + \cos(\theta)\hat{k}_1$$

$$\begin{bmatrix} \hat{\imath}_0 \\ \hat{j}_0 \\ \hat{k}_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} \hat{\imath}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{bmatrix}$$

$$\mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

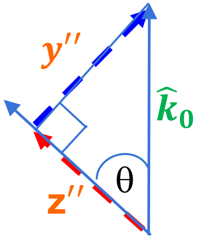


$$\cos(\theta) = \frac{y''}{\hat{j}_0}, thus\ y'' = cos(\theta)$$

$$\sin(\theta) = \frac{z''}{\hat{j}_0}, thus\ z'' = sin(\theta)$$

Vector "Unit vector"

$$\hat{j}_0 = \cos(\theta)\hat{j}_1 + \sin(\theta)(-\hat{k}_1)$$



$$\cos(\theta) = \frac{z''}{\hat{k}_0}, thus\ z'' = cos(\theta)$$

$$\sin(\theta) = \frac{y''}{\hat{k}_0}, thus\ y'' = sin(\theta)$$

Vector "Unit vector"

$$\hat{k}_0 = \cos(\theta)\hat{k}_1 + \sin(\theta)(\hat{j}_1)$$
$$= \sin(\theta)(\hat{j}_1) + \cos(\theta)\hat{k}_1$$

3D Rotation (Prove)

Rotation about x axis by θ

$$\mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

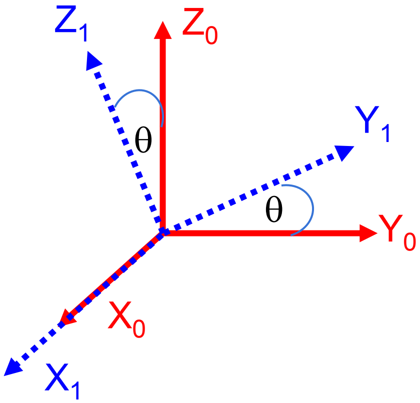
therefore

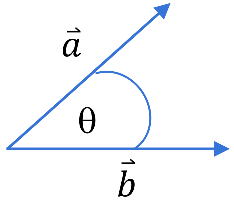
$$x_1 \cdot x_0 = 1 \cdot 1 \cdot \cos(0) = 1$$
$$x_1 \cdot y_0 = 1 \cdot 1 \cdot \cos\left(\frac{\pi}{2}\right) = 0$$
$$x_1 \cdot z_0 = 1 \cdot 1 \cdot \cos\left(\frac{\pi}{2}\right) = 0$$
$$y_1 \cdot x_0 = 1 \cdot 1 \cdot \cos\left(\frac{\pi}{2}\right) = 0$$
$$y_1 \cdot y_0 = 1 \cdot 1 \cdot \cos(\theta) = \cos(\theta)$$
$$y_1 \cdot z_0 = 1 \cdot 1 \cdot \cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$$
$$z_1 \cdot x_0 = 1 \cdot 1 \cdot \cos\left(\frac{\pi}{2}\right) = 0$$
$$z_1 \cdot y_0 = 1 \cdot 1 \cdot \cos\left(\frac{\pi}{2} + \theta\right) = -\sin(\theta)$$
$$z_1 \cdot z_0 = 1 \cdot 1 \cdot \cos(\theta) = \cos(\theta)$$

* Calculate by Dot product (x1,x0, y1,y0, z1, z0 are unit vector)

Projection ⬌ Dot product

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$





$$\vec{a} \cdot \vec{b} = |\vec{a}|\left|\vec{b}\right|\cos(\theta)$$

$${}^{0}\mathbf{R}_{x_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

3D Rotation (Prove)

Rotation about y axis by θ

$$\mathbf{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Let's Prove It

$x_1 \cdot x_0 =?$ $\cos\theta$

$x_1 \cdot y_0 =?$ 0

$x_1 \cdot z_0 =?$ $\cos(90+\theta) = -\sin\theta$

$y_1 \cdot x_0 =?$ 0

$y_1 \cdot y_0 =?$ 1

$y_1 \cdot z_0 =?$ 0

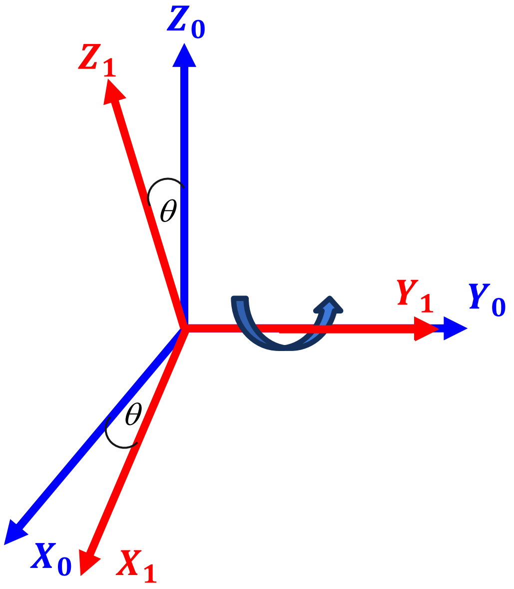
$z_1 \cdot x_0 =?$ $\cos(90-\theta) = \sin\theta$

$z_1 \cdot y_0 =?$ 0

$z_1 \cdot z_0 =?$ $\cos\theta$

* Calculate by Dot product (x1,x0, y1,y0, z1, z0 are unit vector)

$${}^{0}\mathbf{R}_{y_1} = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix} = ?$$



3D Rotation (Prove)

Rotation about z axis by θ

$$\mathbf{R}_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let's Prove It

$x_1 \cdot x_0 =?$ cos θ

$x_1 \cdot y_0 =?$ cos (90 - θ) = sin θ

$x_1 \cdot z_0 =?$ 0

$y_1 \cdot x_0 =?$ cos (90 + θ) = - sin θ

$y_1 \cdot y_0 =?$ cos θ

$y_1 \cdot z_0 =?$ 0

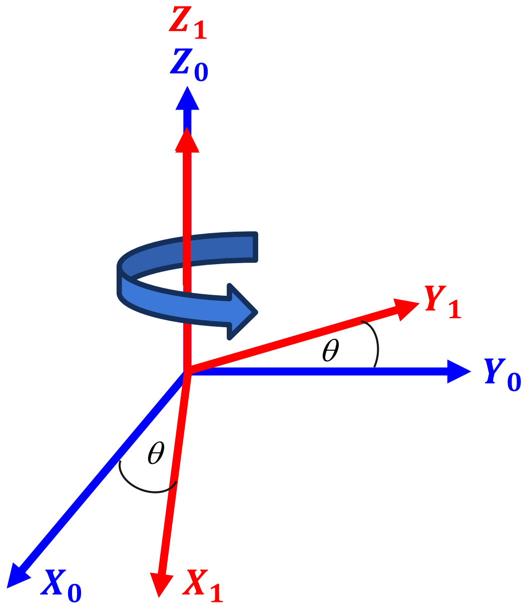
$z_1 \cdot x_0 =?$ 0

$z_1 \cdot y_0 =?$ 0

$z_1 \cdot z_0 =?$ 1

* Calculate by Dot product (x1,x0, y1,y0, z1, z0 are unit vector)

$${}^{0}\mathbf{R}_{z_1} = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix} = ?$$



3D Rotation : Exercises

1. Find 3D rotation matrix rotated in counter-clockwise direction by $\pi/8$ radian about y-axis

$$\begin{bmatrix} 0.92 & 0 & 0.38 \\ 0 & 1 & 0 \\ -0.38 & 0 & 0.92 \end{bmatrix}$$

2. Find 3D rotation matrix rotated in counter-clockwise direction by 45 Deg. about z-axis

$$\begin{bmatrix} 0.71 & -0.71 & 0 \\ 0.71 & 0.71 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Find 3D rotation matrix rotated in clockwise direction by 15 Deg. about x-axis

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.97 & 0.26 \\ 0 & -0.26 & 0.97 \end{bmatrix}$$

3D Rotation : Exercises

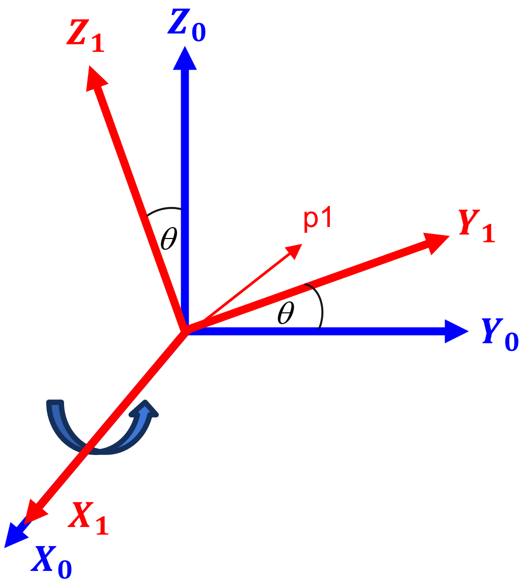
4. If Vector p1(3, 7, 4) is rotated in counter-clockwise direction by 180 deg about x-axis, Find the rotation matrix and the coordinate value p0(?, ?, ?).

$$p_0 = R_x(180) \cdot p_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -7 \\ -4 \end{bmatrix}$$

$${}^{0}\mathbf{R}_{\mathbf{x}\,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

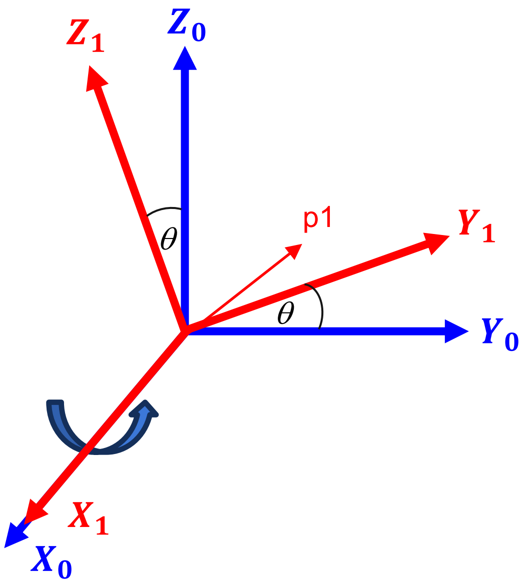


4. If Vector p1(3, 7, 4) is rotated in counter-clockwise direction by 180 deg about x-axis, Find the rotation matrix and the coordinate value p0(?, ?, ?).

$$^{0}R_{x1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(180) & -\sin(180) \\ 0 & \sin(180) & \cos(180) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$^{0}P = {}^{0}R_{x1}{}^{1}P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \\ -4 \end{bmatrix}$$

$$^{0}_{1}\mathbf{R_x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$



3D Rotation : Exercises

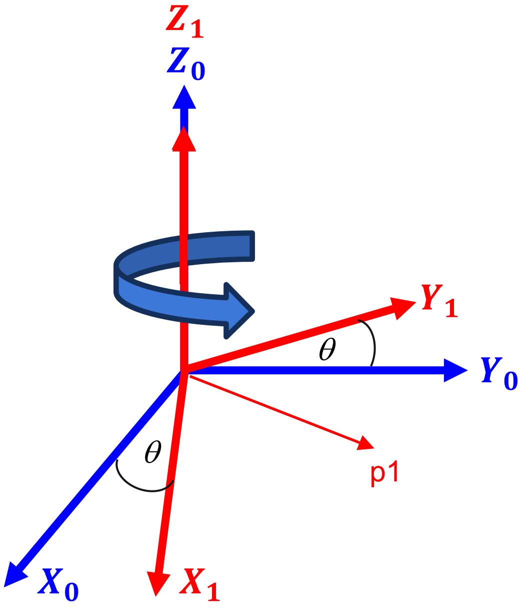
5. If Vector p1(5, 10, −2) is rotated in counter-clockwise direction by $\pi/4$ rad. About z-axis, Find the rotation matrix and the coordinate value p0(?, ?, ?).

$$p0 = R_z\left(\frac{\pi}{4}\right) p1$$

$$= \begin{bmatrix} 0.7 & -0.7 & 0 \\ 0.7 & 0\ 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -3.5 \\ 10.6 \\ -2 \end{bmatrix}$$

$${}^{0}\mathbf{R}_{\mathbf{z}\,1} = \begin{bmatrix} \cos(\theta) & -sin(\square) & 0 \\ sin(\square) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Recall Lecture#2

3D Rotation Matrix

Derive "How to get the rotation matrix"

- Projection
- Dot product

Determine new position after rotate for "One time"

$$\mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbf{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\mathbf{R}_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3D Rotation

Inverse rotation

$$ {}^{0}\mathbf{P} = {}^{0}\mathbf{R}_{1x}\, {}^{1}\mathbf{P} $$

$$ \mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} $$

$$ \boxed{{}^{1}\mathbf{P} = {}^{0}\mathbf{R}_{1x}^{-1}\, {}^{0}\mathbf{P}} $$

Refer to properties of rotation matrix

$$ {}^{0}\mathbf{R}_{1x}^{-1} = {}^{0}\mathbf{R}_{1x}^{T} = {}^{1}\mathbf{R}_{0x} $$

$$ \boxed{{}^{1}\mathbf{P} = {}^{0}\mathbf{R}_{1x}^{-1}\, {}^{0}\mathbf{P} = {}^{0}\mathbf{R}_{1x}^{T}\, {}^{0}\mathbf{P}} $$

3D Rotation : Example

1. If Vector p0(4, 5, 6) is the results from rotation in counter-clockwise direction by 30 deg about x-axis, Find the rotation matrix and the coordinate value p1(?, ?, ?).

$$p_0 = R_x^{-1} \, p_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.87 & 0.5 \\ 0 & -0.5 & 0.87 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 7.3 \\ 2.7 \end{bmatrix}$$

$${}^{0}R_{1x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$${}^{0}P = {}^{0}R_{1x} \, {}^{1}P$$

$$\boxed{{}^{1}P = {}^{0}R_{1x}^{-1} \, {}^{0}P}$$

3D Rotation : Example

1. If Vector p0(4, 5, 6) is the results from rotation in counter-clockwise direction by 30 deg about x-axis, Find the rotation matrix and the coordinate value p1(?, ?, ?).

$$ {}^{0}\mathbf{P} = {}^{0}\mathbf{R}_{1x}\, {}^{1}\mathbf{P} $$

$$\begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 0.866 & -0.5\\0 & 0.5 & 0.866\end{bmatrix}\begin{bmatrix}P_{1x}\\P_{1y}\\P_{1z}\end{bmatrix}$$

$$ {}^{1}\mathbf{P} = {}^{0}\mathbf{R}_{1x}^{-1}\, {}^{0}\mathbf{P} $$

$$\begin{bmatrix}P_{1x}\\P_{1y}\\P_{1z}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 0.866 & 0.5\\0 & -0.5 & 0.866\end{bmatrix}\begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}4\\7.3301\\2.6962\end{bmatrix}$$

$$ {}^{0}\mathbf{R}_{1x} = \begin{bmatrix}1 & 0 & 0\\0 & \cos(\theta) & -\sin(\theta)\\0 & \sin(\theta) & \cos(\theta)\end{bmatrix}$$

$$ {}^{0}\mathbf{P} = {}^{0}\mathbf{R}_{1x}\, {}^{1}\mathbf{P} $$

$$ \boxed{{}^{1}\mathbf{P} = {}^{0}\mathbf{R}_{1x}^{-1}\, {}^{0}\mathbf{P}} $$

Exercises (20 Mins)

Wrap Up

1. **Position and Orientation (Rotation matrix)**
   - 2D Rotation
   - 3D Rotation
   - Prove & Find Value of P vector transform from Frame1 to Frame0
2. **Exercises (20 Mins)**

References

1. Bruno Siciliano and et.al. , Robotics: Modelling, Planning and Control, Robotics Modelling, Planning and Control,
2. Prof. Alessandro De Luca, Robotic1
3. https://robotacademy.net.au/lesson/describing-rotation-in-2d/
4. Robotics, Vision and Control Fundamental Algorithms in MATLAB, Peter Corke