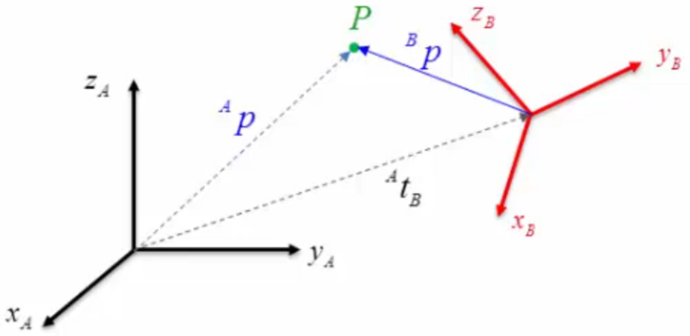
CHAPTER 5

Homogeneous transformations matrix

Homogenous Transformation Matrix

- **Homogeneous Transformation Matrices to Express Configurations in Robotics**

  a sixteen-entries, 4×4 matrix



^At_B : Origin of frame B with respect to frame A

- Point P of frame $\{B\}$ in frame $\{A\}$: $\quad {}^Ap = {}^At_B + {}^AR_B\,{}^Bp$
- In a more compact way:

$$\underbrace{\begin{bmatrix} {}^Ap \\ 1\end{bmatrix}}_{{}^A\tilde{p}} = \underbrace{\begin{bmatrix} {}^AR_B & {}^At_B \\ 0 & 1\end{bmatrix}}_{{}^AT_B}\underbrace{\begin{bmatrix} {}^Bp \\ 1\end{bmatrix}}_{{}^B\tilde{p}}$$

$\left.\begin{matrix}{}^A\tilde{p} \\ {}^B\tilde{p}\end{matrix}\right\}$ Homogeneous coordinates

Transformation Matrix

Homogenous Transformation Matrix

- They represent the position and orientation (pose) of a frame with respect to another frame
- Parts

$$T = \left[\begin{array}{ccc|c} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

Rotation (orientation)

translation (position)

- Mathematically, they belong to $SE(3)$: $\mathbb{R}^3 \times SO(3)$ (SE = Special Euclidean)
  - They can also be viewed from the perspective of projective geometry
- Pure transformations

$$T = \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$$

Pure rotation

$$T = \begin{bmatrix} I & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

Pure translation

Homogenous Transformation Matrix

Pure Transformation

Pure rotations:

$$Rot_x(\theta)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$Rot_y(\theta)=\begin{bmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & 0\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$Rot_z(\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Pure translations:

$$Trans_x(d)=\begin{bmatrix}1 & 0 & 0 & d\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$Trans_y(d)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & d\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

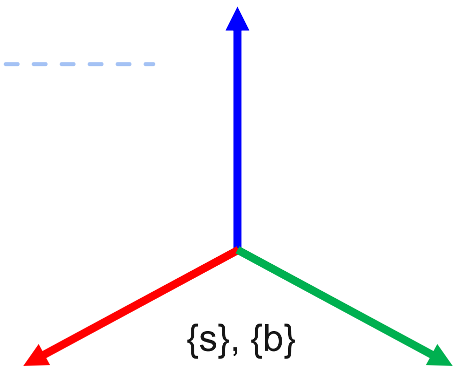
$$Trans_z(d)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & d\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Homogenous Transformation Matrix

- **No Rotation and No Translation**

  The body frame {b} is the same as the space frame {s}

$$^{s}T_b \ \ or \ \ Tsb = I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



- **Product of two Homogenous Transformation Matrix is also Transformation Matrix**

$T_1 T_2 \in SE(3)$; *special Euclidean group*

$$T_1 = \begin{bmatrix} R_1 & \mathbf{t}_1 \\ \mathbf{0} & 1 \end{bmatrix} \quad T_2 = \begin{bmatrix} R_2 & \mathbf{t}_2 \\ \mathbf{0} & 1 \end{bmatrix} \quad \Rightarrow \quad T_1 T_2 = \begin{bmatrix} R_1 R_2 & R_1 \mathbf{t}_2 + \mathbf{t}_1 \\ \mathbf{0} & 1 \end{bmatrix}$$

- **Associative property**

$$(T_1 T_2) T_3 = T_1 (T_2 T_3)$$

- **Noncommutative property**

$$T_1 T_2 \neq T_2 T_1$$

Homogenous Transformation Matrix

- **Inverse of Homogenous Transformation Matrix**

$$ {}^{s}T_b^{-1} \ \ or \ \ T_{sb}^{-1} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix} = {}^{b}T_s \ or \ T_{bs} $$

- **Decomposition**

Any homogenous transformation matrix can be decomposed in 2 components:

$$ T = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} I & t \\ 0 & 1 \end{bmatrix}}_{\substack{\textbf{Translation} \\ \boldsymbol{t}}} \underbrace{\begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}}_{\substack{\textbf{Rotation} \\ \boldsymbol{R}}} $$

- First, it applies a translation of *t* units.
- Then, it applies a rotation *R* with respect to current frame.

Homogenous Transformation Matrix

- **Subscript Cancellation Rule**

$$^{a}T_{b}\,^{b}T_{c} = {}^{a}T_{c} \quad or \quad TabTbc = T_{ac}$$

We can find the homogenous transformation matrix representing the pose of any frame relative to another frame. For instance, calculate the homogenous transformation matrix $T_{bc,}$ which represents the position and orientation of the frame {c} relative to the frame {b}, we can write:

$$^{b}T_{c} \;\; or \;\; Tbc = {}^{b}T_{s}\,^{s}T_{c} = {}^{s}T_{b}^{-1}\,^{s}T_{c}$$

Homogenous Transformation Matrix

Example Find Homogenous transformation matrix for rotation 30 degree about z-axis of global (space) frame.

$$T_{GB} = \begin{bmatrix} 0.87 & -0.5 & 0 & 0 \\ 0.5 & 0.87 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Homogenous Transformation Matrix

Example Find Homogenous transformation matrix for rotation 30 degree about z-axis of global (space) frame.

$$
{}^{G}T_{b} \ \ or \ \ TGb = \begin{bmatrix} cos30° & -sin30° & 0 & 0 \\ sin30° & cos30° & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0.866 & -0.5 & 0 & 0 \\ 0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Homogenous Transformation Matrix

Exercise A frame {A} is rotated 90° about x, and then it is translated a vector (6,-2,10) with respect to the fixed (initial) frame. Find the homogenous transformation matrix that describes {B} respect to {A}

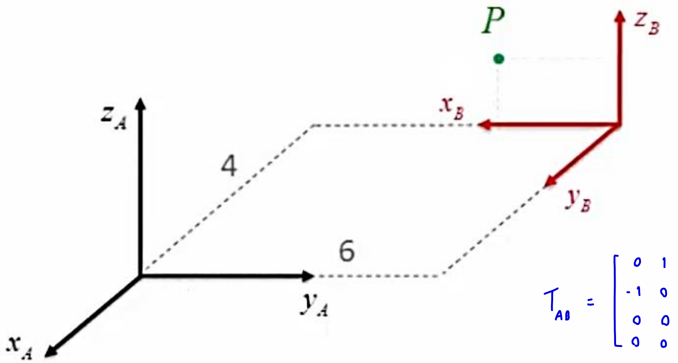
Homogenous Transformation Matrix

Exercise Rotation about and translation along a global axis.

A point P is located at (0,0,200) in a body coordinate frame. If the rigid body rotates 30 deg. About the global X-axis and the origin of the body frame translates to (X,Y,Z) = (500, 0, 600), then the coordinates of the point in the global frame are :

Homogenous Transformation Matrix

Example Consider frame $\{A\}$ and $\{B\}$. Point $\boldsymbol{P}$ in frame $\{B\}$ is given by $(2, 0, 1)$, find its coordinates with respect to frame $\{A\}$ using a homogeneous transformation matrix



$A \rightarrow B$

$$\begin{bmatrix} -4 \\ 6 \\ 0 \end{bmatrix}$$

$R_z(-90)$

$$T_{AB} = \begin{bmatrix} 0 & 1 & 0 & -4 \\ -1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow T_{BA} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

$$P_A = {}^{A}T_B \cdot T_B$$

↳ transform from B to A
but for the translation we consider
moving frame A to B pos

Homogenous Transformation Matrix

Exercise A frame $\{A\}$ is rotated 90° about x, and then it is translated a vector $(6, -2, 10)$ with respect to the fixed (initial) frame. Consider a point $\mathbf{p} = (-5, 2, -12)$ with respect to the new frame $\{B\}$. Determine the coordinates of that point with respect to the initial frame

$$T_{BA} = \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_A = T_{BA} \cdot P_B$$

$$= T_{BA} \cdot \begin{bmatrix} -5 \\ 2 \\ -12 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 10 \\ 12 \\ 1 \end{bmatrix}$$

Homogenous Transformation Matrix

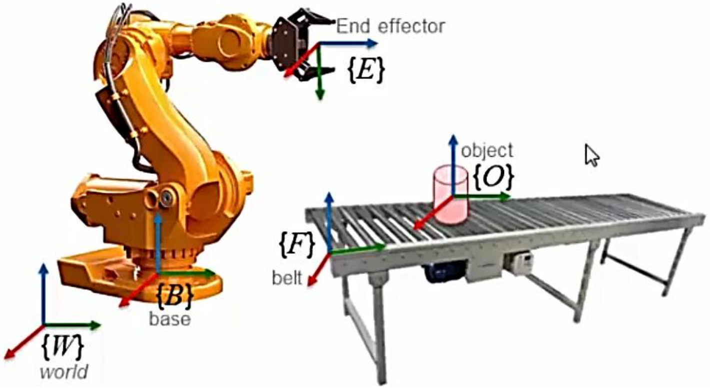
Example

Consider that the transformations of the belt and of the robot base with respect to a reference frame {W} are known. The transformation of the object with respect to the belt, as well as the transformation of the end effector with respect to the robot base are also known.

a) Find the pose of the object with respect to the base of the robot

b) Find the pose of the object with respect to the end effector

There are different frame attach to different part.



Homogenous Transformation Matrix

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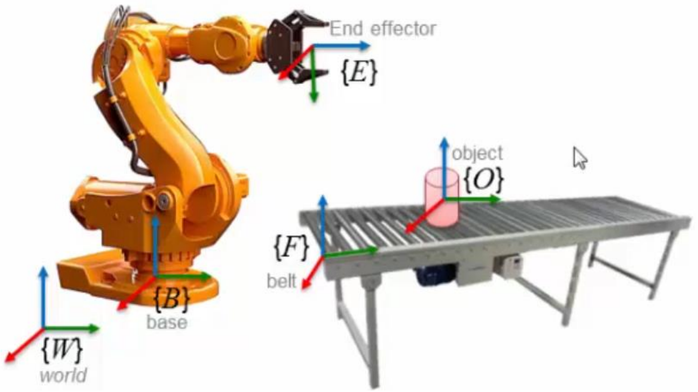


Solution

Belt(F) Resp.to World(W) frame | R-Base(B) Resp.to World(W) frame | Obj(O) Resp.to Belt(F) frame | End-Ef(E) Resp.to R-Base(B) frame

- Known transformations: ${}^{W}T_F, {}^{W}T_B, {}^{F}T_O, {}^{B}T_E$

a) Desired pose (in terms of the known transformations): ${}^{B}T_O$ Obj(O) Resp.to R-Base(B) frame = ?

$$ {}^{B}T_O = \left({}^{B}T_W\right)\left({}^{W}T_O\right) = \left({}^{W}T_B^{-1}\right)\left(\left({}^{W}T_F\right)\left({}^{F}T_O\right)\right) $$

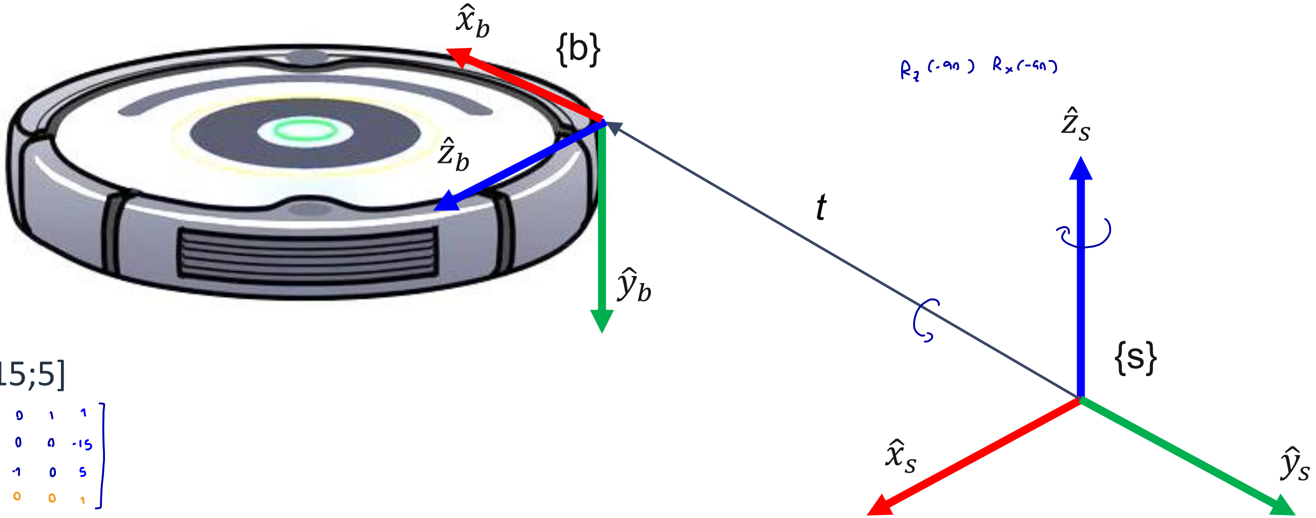
b) Desired pose (in terms of the known transformations): ${}^{E}T_O$ Obj(O) Resp.to End-Ef(E) frame = ?

$$ {}^{E}T_O = \left({}^{E}T_W\right)\left({}^{W}T_O\right) = \left({}^{W}T_E^{-1}\right)\left({}^{W}T_F\right)\left({}^{F}T_O\right) = \left(\left({}^{W}T_B\right)\left({}^{B}T_E\right)\right)^{-1}\left({}^{W}T_F\right)\left({}^{F}T_O\right) $$

$$ = \left({}^{B}T_E\right)^{-1}\left({}^{W}T_B\right)^{-1}\left({}^{W}T_F\right)\left({}^{F}T_O\right) $$

Homogenous Transformation Matrix

Example Expressing the configuration of a frame {B} relative to a space frame



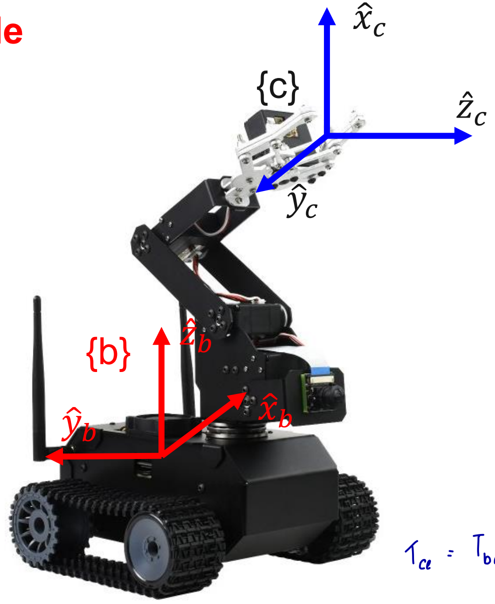
If t = [1;-15;5]

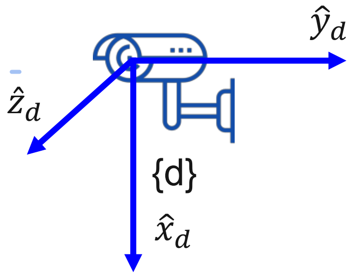
$^{S}T_b = ?$ $\begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & -15 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

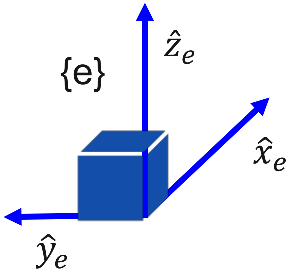
$T = R_z(-90)\ R_x(-9)\ t(1, -15, 5)$

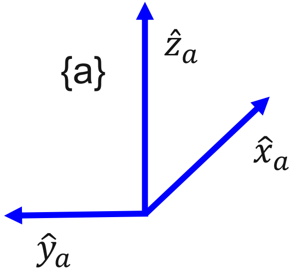
Homogenous Transformation Matrix

Example









$T_{ce} = T_{bc}^{-1} \cdot T_{db}^{-1} \; T_{de}$

$=$

$$T_{db} = \begin{bmatrix} 0 & 0 & -1 & 250 \\ 0 & -1 & 0 & -150 \\ -1 & 0 & 0 & 200 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_{de} = \begin{bmatrix} 0 & 0 & -1 & 300 \\ 0 & -1 & 0 & 100 \\ -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_{bc} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 30 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -40 \\ 1 & 0 & 0 & 25 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_{ad} = \begin{bmatrix} 0 & 0 & -1 & 400 \\ 0 & -1 & 0 & 50 \\ -1 & 0 & 0 & 300 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_{ce} = ?$$

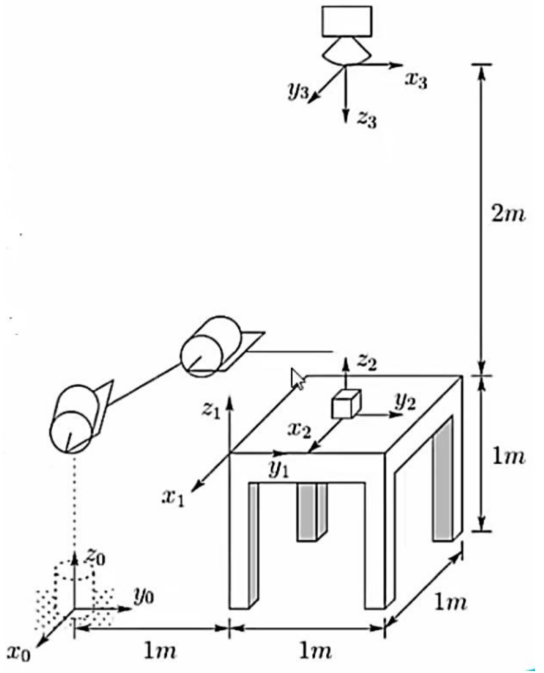
Homogenous Transformation Matrix

Example

The figure shows a robot whose base is 1m away from the base of the table. The table is 1m height and its surface is a square. Frame $\{1\}$ is fixed on a corner of the table. A 20 cm cube is located on the middle of the table, and it has frame $\{2\}$ attached to its center. A camera is located $2\mathrm{m}$ above the table, just over the cube, and it has frame $\{3\}$ attached to it.

a) Find the homogeneous transformations that relate each of these frames with the base system $\{0\}$

b) Find the homogeneous transformation that relates the cube frame $\{2\}$ with respect to the camera frame $\{3\}$



Exercises (20 Mins)

Exercise #5

Wrap Up

1. **Homogeneous transformations**
   - Pure Transformation
   - Composition
   - Decomposition
   - Examples
2. **Exercises (20 Mins)**

References

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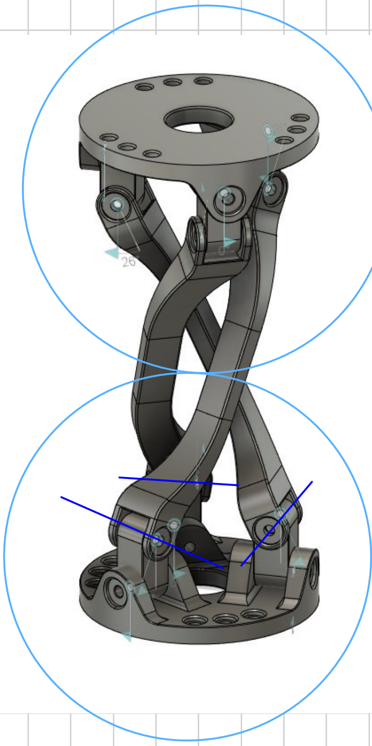
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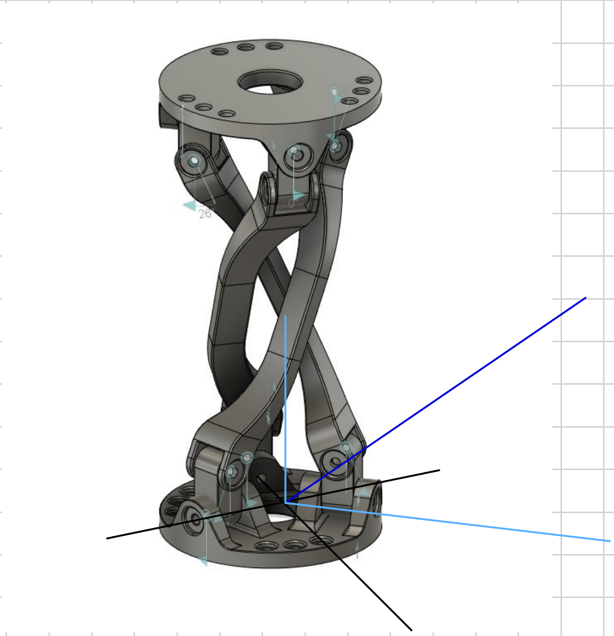

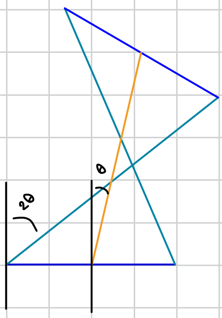

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