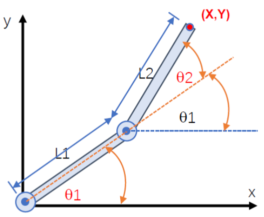
Kinematics of 2D Manipulator



- Forward (Simple geometry & trigo.)
  - $x = l_1 cos(\theta_1) + l_2 cos(\theta_1 + \theta_2)$
  - $y = l_1 sin(\theta_1) + l_1 sin(\theta_1 + \theta_2)$
- Inverse
  - $\theta_1 = arctan\left(\frac{y}{x}\right) - \arcsin\left(\frac{l_2 \sin(\theta_2)}{\sqrt{x^2+y^2}}\right)$
  - $\theta_2 = \arccos\left(\frac{x^2+y^2-l_1^2-l_2^2}{2l_1l_2}\right)$

Vector

- Inner product (Dot product)
- Outer product
  - $A \otimes B = AB^T$
  - $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{bmatrix}$
- Eigenvalues & Eigenvectors
  - $Ax = \lambda x;\ A\ is\ a\ matrix, \lambda\ is\ a\ number$
  - $x\ is\ eigenvector\ of\ A,$
    $\lambda\ is\ eigenvalue\ of\ A$
  - **TRUE** when $|A - \lambda I| = 0$

$\dot{x}_1 = x_1 + x_2$ (1)
$\dot{x}_2 = 4x_1 + x_2$

$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Try $x(t) = Ve^{\lambda t}$
$AV = \lambda V$

det$(\mathbf{A} - \lambda I) = \lambda^2 - 2\lambda - 3 = 0$
$(\lambda - \mathbf{3})(\lambda + \mathbf{1}) = 0$
$\lambda_1 = -\mathbf{1}$
$\lambda_2 = \mathbf{3}$

$\lambda_1 = -\mathbf{1}$;
$(\mathbf{A} - \lambda_1 I)V_1 = 0$
$\left(\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$2V_{11} + V_{21} = 0$
$2V_{11} = -V_{21}$ if $V_{11} = 1, V_{21} = -2$
$V_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda_2 = \mathbf{3}$;
$(\mathbf{A} - \lambda_2 I)V_2 = 0$
$\left(\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)\begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$-2V_{12} + V_{22} = 0$
$2V_{12} = V_{22}$ if $V_{12} = 1, V_{22} = 2$
$V_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\boxed{\begin{gathered} x(t) = C_1\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-t} + C_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{3t} \\ x_1(t) = C_1e^{-t} + C_2e^{3t}, \quad x_2(t) = -2C_1e^{-t} + 2C_2e^{3t} \end{gathered}}$

Transformation Matrix (Rotation)

- Notation (T_{AB}/ ${}^AT_B$)
  - $P_A = {}^AT_B \cdot P_B$
- ${}^BR_A = ({}^AR_B)^{-1} = ({}^AR_B)^T$
- counter-clockwise (+), clockwise (-)

2D
- $R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

3D
- $R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$

  $R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$

  $R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- P1 is rotated … by $\theta \rightarrow {}^0R_1(\theta)$

Composition of matrix

- Just multiply directly
- About current frame ($\times\ back, POST$)
  - $R_x(\theta) \rightarrow R_x(\theta)R_y(\emptyset)$
- About fixed frame ($\times\ front, PRE$)
  - $R_x(\theta) \rightarrow R_y(\emptyset)R_x(\theta)$

$$\boldsymbol{atan2} = \begin{cases} atan\left(\frac{y}{x}\right); x > 0 \\ \text{atan}\left(\frac{y}{x}\right) + \pi; x < 0, y \geq 0 \\ \text{atan}\left(\frac{y}{x}\right) - \pi; x < 0, y < 0 \end{cases}$$

$\left.\begin{matrix} *y \geq 0 \rightarrow (+) \\ y < 0 \rightarrow (-) \end{matrix}\right\}\ x = 0 \rightarrow \pm 90$

*Take the sign in $\frac{y}{x} \rightarrow \frac{-0.766}{0.684}$

Minimal representation (Euler angle)

- Represents as $ZX'Z''$ about current axis.
  * $Post\ Product$
  - $(\phi,\ \theta,\ \psi) \rightarrow R_z(\phi)R_x(\theta)R_z(\psi)$
- Inverse
  - $\theta = \begin{cases} atan2(\sqrt{r_{13}^2 + r_{23}^2}, r_{33});\quad Q1, Q2 \\ atan2(-\sqrt{r_{13}^2 + r_{23}^2}, r_{33}); Q3, Q4 \end{cases}$
    $\psi = atan2(r_{31}/S\theta, r_{32}/S\theta)$
    $\phi = atan2(r_{13}/S\theta, -r_{23}/S\theta)$

  $*\ S\theta = \sin\theta \neq 0$

Minimal representation (Roll-Pitch-Yaw)

- Represents as $Roll \cdot Pitch \cdot Yaw$ about fixed axis.
  *$Pre\ product$
  - $(\ \psi,\ \theta,\ \phi) \rightarrow R_z(\phi)R_y(\theta)R_x(\psi)$
- Inverse
  - $\theta = \begin{cases} atan2(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2});\ Q1, Q4 \\ atan2(-r_{31}, -\sqrt{r_{32}^2 + r_{33}^2}); Q2, Q3 \end{cases}$
    $\psi = atan2(r_{32}/C\theta, r_{33}/C\theta)$
    $\phi = atan2(r_{21}/C\theta, r_{11}/C\theta)$

  $*\ C\theta = \cos\theta \neq 0$

Quaternion

- Consists of scalar (w) and imaginary parts (x, y, z)
  - $Q = [w, x, y, z] = [w, \epsilon] = \left[\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)i, \sin\left(\frac{\theta}{2}\right)j, \sin\left(\frac{\theta}{2}\right)k\right]$
- Properties
  - $Q_1 + Q_2 = [w_1 + w_2, \epsilon_1 + \epsilon_2]$
  - $Q^* = [w, -\epsilon]$
  - $||Q|| = \sqrt{w^2 + \epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2} \rightarrow is\ 1\ for\ unit\ quarnion$
  - $Q^{-1} = \frac{Q^*}{||Q||^2}$
  - $Q_1 \circ Q_2 = [w_1w_2 - \epsilon_1^T\epsilon_2, w_1\epsilon_2 + w_2\epsilon_1 + (\epsilon_1 \times \epsilon_2)]$
- Application on vector v
  - Convert v to $(0, v)$ – quaternion.
  - Apply rotation quaternion $\vec{v}_{rot} = Q \circ \vec{v} \circ Q^*$
  - Answer $\vec{v}_{rot} = (0, v_{rot})$
- Convert to rotation matrix.
  - $R(w, \epsilon) = \begin{bmatrix} 2(w^2 + x^2) - 1 & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & 2(w^2 + y^2) - 1 & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & 2(w^2 + z^2) - 1 \end{bmatrix}$
- Convert back to quaternion.
  - $w = \frac{1}{2}\sqrt{r_{11} + r_{22} + r_{33} + 1}$
  - $\epsilon = \frac{1}{2}\begin{bmatrix} sgn(r_{32} - r_{23})\sqrt{r_{11} - r_{22} - r_{33} + 1} \\ sgn(r_{13} - r_{31})\sqrt{r_{22} - r_{33} - r_{11} + 1} \\ sgn(r_{21} - r_{12})\sqrt{r_{33} - r_{11} - r_{22} + 1} \end{bmatrix} \rightarrow sgn(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$

Homogenous Transformation Matix

- ${}^AT_B = \begin{bmatrix} {}^AR_B & {}^At_B \\ 0 & 1 \end{bmatrix}$, $transformation\ from\ B\ to\ A.$
- ${}^At_B$ is translation matrix $\rightarrow \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$
  - $is\ a\ vector\ pointing\ from\ A\ to\ B$
- ${}^BT_A = ({}^AT_B)^{-1} = \begin{bmatrix} {}^AR_B & {}^At_B \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^Tt \\ 0 & 1 \end{bmatrix}$
- ${}^CT_A = {}^CT_B\,{}^BT_A;\ pre\ multiplication\ A \rightarrow B\ then\ B \rightarrow C.$
- Keyword:
  - With respect to A
  - Related to A

  $\}\ {}^AT_{any\ axis}$
- Position $p \rightarrow [p, 1]^T$

DH Matrix

- $\theta: rotation\ about\ z\ axis$
  $d: translation\ along\ z\ axis$
  $\alpha: rotation\ about\ x\ axis$
  $a: translation\ along\ x\ axis$
- Reference all movement to the previous axis.
  - Find θ_1, so find the rotation of frame 1 ref to frame 0 about z axis.
- Each row of the table gives you the translation matrix from i to $i - 1$

Joint	θ_i	d_i	a_i	α_i
1	θ_1	d_1	a_1	α_1
2	θ_2	d_2	a_2	α_2
⋮	⋮	⋮	⋮	⋮

$\rightarrow {}^0T_1$

- ${}^{i-1}T_i = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & a_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & a_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$

NOTES

- p_0 is rotated in counterclockwise by θ about y axis. $\rightarrow p_1 = {}^1R_0(\theta) \cdot p_0$
- p_0 is the result from rotating in clockwise by θ about x axis. $\rightarrow p_0 = {}^0R_1(\theta) \cdot p_1$

From the given rotation matrix equivalent to the Roll-Pitch-Yaw angles,
Find rotated angle in $X(\psi)$ $Y(\theta)$ $Z(\phi)$ when $Theta\ (\theta)'s$ in Quadrant 3.

$$R = R_z(\phi)R_y(\theta)R_x(\psi) = \begin{bmatrix} 0 & -0.866 & 0.5 \\ -0.766 & -0.3214 & -0.5567 \\ 0.6428 & -0.383 & -0.6634 \end{bmatrix}$$

a) $X(\psi) = 30\ deg,\ Y(\theta) = -140\ deg,\ Z(\phi) = 90$ deg.

b) $X(\psi) = 30\ deg,\ Y(\theta) = -140\ deg,\ Z(\phi) = -90$ deg.

c) $X(\psi) = 30\ deg,\ Y(\theta) = 220\ deg,\ Z(\phi) = 90$ deg.

d) a) and c) are correct.

$$\theta = \arctan 2\left(-r_{31},\ -\sqrt{r_{32}^2 + r_{33}^2}\right)$$
$$= \arctan 2(-0.6428,\ -0.766)$$
$$= \arctan\left(\frac{-0.6428}{-0.766}\right) - \pi$$
$$= -140$$

$$\psi = \arctan 2\left(r_{32}/\cos\theta,\ r_{33}/\cos\theta\right)$$
$$= \arctan 2(0.5,\ 0.866)$$
$$= \arctan\left(\frac{0.5}{0.866}\right)$$
$$= 30^\circ$$

$$\phi = \arctan 2\left(r_{21}/\cos\theta,\ r_{11}/\cos\theta\right)$$
$$= \arctan 2(1,\ 0/-0.766)$$
$$= 90^\circ$$

) a)

) b)

) c)

) d)

Quaternion

- **Product of Quaternions**

$$Q_1{}^\circ\, Q2 = [w_1 \quad (x_1 \quad y_1 \quad z_1)][w_2 \quad (x_2 \quad y_2 \quad z_2)]$$
$$= [w_1 \quad v_1][w_2 \quad v_2]$$
$$= [w_1 w_2 - v_1 \cdot v_2 \quad w_1 v_2 + w_2 v_1 + v_1 \times v_2]$$
$$= \begin{bmatrix} w_1w_2 - x_1x_2 - y_1y_2 - z_1z_2 \\ \begin{pmatrix} w_1x_2 + x_1w_2 + y_1z_2 - z_1y_2 \\ w_1y_2 - x_1z_2 + y_1w_2 + z_1x_2 \\ w_1z_2 + x_1y_2 - y_1x_2 + z_1w_2 \end{pmatrix} \end{bmatrix}$$

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$
$$= (A_yB_z - A_zB_y)\hat{\imath} + (A_zB_x - A_xB_z)\hat{\jmath} + (A_xB_y - A_yB_z)\hat{k}$$