Unit 4 Mathematical Methods **Name:** _______________

Calculus – Practice SAC 2022 **Total marks: / 44**

Instructions

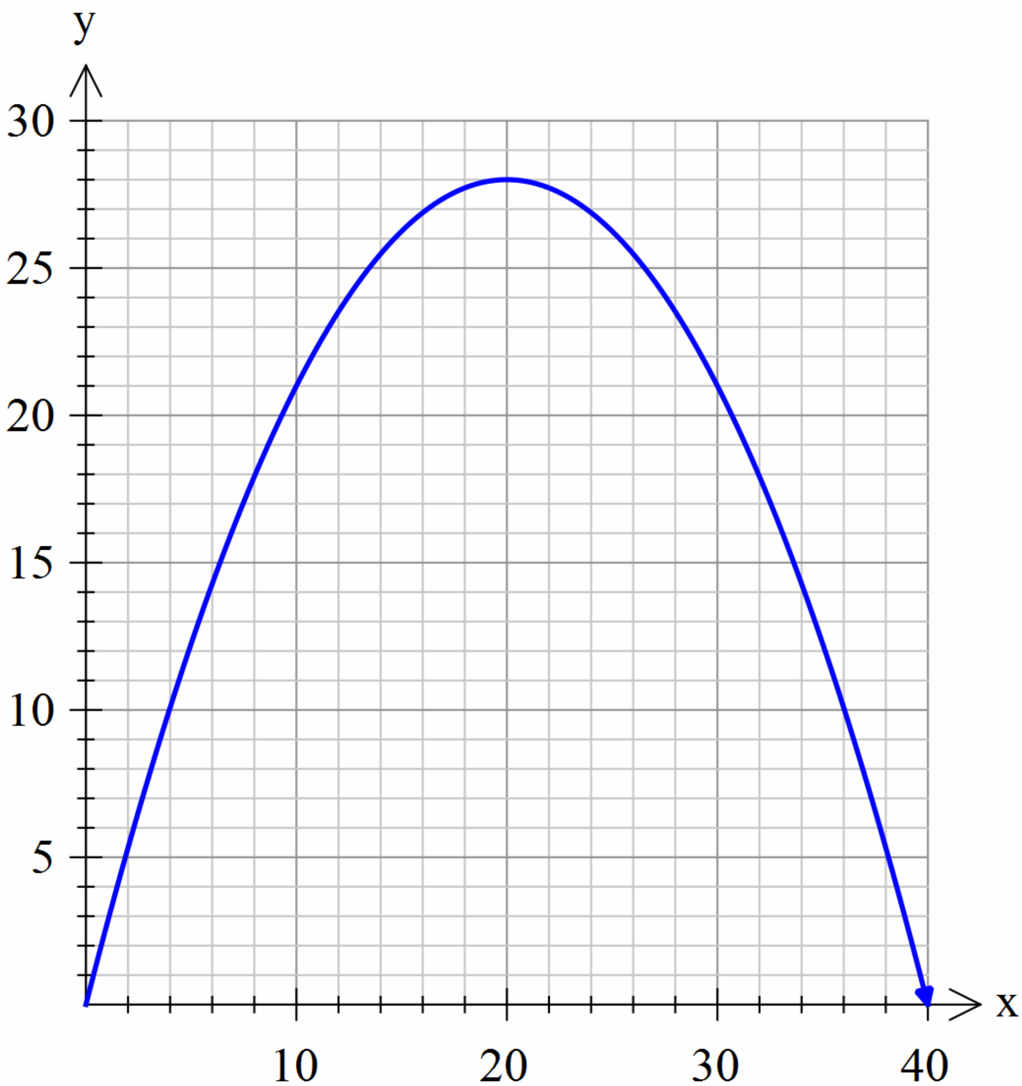
- All answers must given as exact values unless otherwise stated
- You should fully simplify all answers where relevant
- 90 minutes time allowed.

Part 1 (9 marks)

Daniel intends to build a hill out of clay for his model train set. The hill will eventually have tunnels cut through it for his trains to pass through.

The diagram below represents the vertical cross-section of the hill.

Measurements on both axes are in centimetres.



The hill's height, , is given by , where is the horizontal distance from the bottom-left corner of the hill (. The pronumerals , and are constants.

Question 1
The curve has a turning point , and it passes through the origin.

Show that the values of , and are given by , and respectively.

(3 marks)

Question 2
Use a *left*-endpoint rectangle approximation, with 4 equal-width rectangles, to find an approximation to the area of the right-hand side of the hill (from to). Answer to 1 decimal place.

(2 marks)

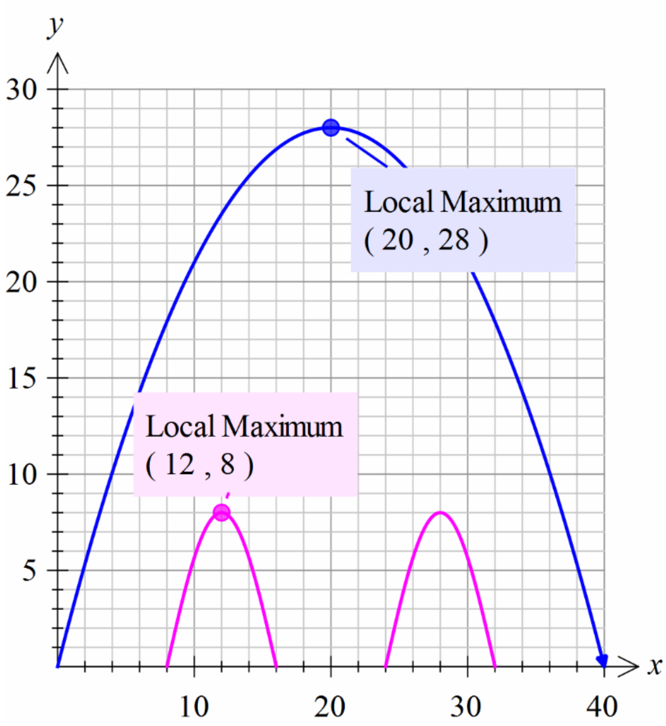
Question 3
Will your result from **Question 2** be an under-estimate, or an over-estimate of the exact area of the right-hand side of the hill? Explain your answer using a diagram. (2 marks)

Question 4
Determine the exact cross sectional area of the right-hand side of the hill in cm^2.
(2 marks)

End Part 1

Part 2 (15 marks)

Daniel now intends to create 2 tunnels through the hill as shown in the 2-D cross-section below.



The two tunnels are part of the graph of the function with rule .

Each tunnel is 8 cm high, and the -intercepts of the tunnels are 8, 16, 24 and 32 cm.

Question 5

a. Show that the values of and are and respectively. (2 marks)

b. State the domain of the sine function such that it shows only the outline of the two tunnels. (1 mark)

The tunnels are constructed by cutting the clay out of the original hill.

Question 6

a. Determine the exact cross-sectional area of clay that is to be *removed* to allow the two tunnels to be constructed.

(2 marks)

b. Hence, find the exact cross-sectional area of the hill left after the tunnels are constructed.

(1 mark)

Question 7
Let

and

a. Explain why that the cross sectional area left in the hill when the tunnel areas are removed can be given by the expression shown below:

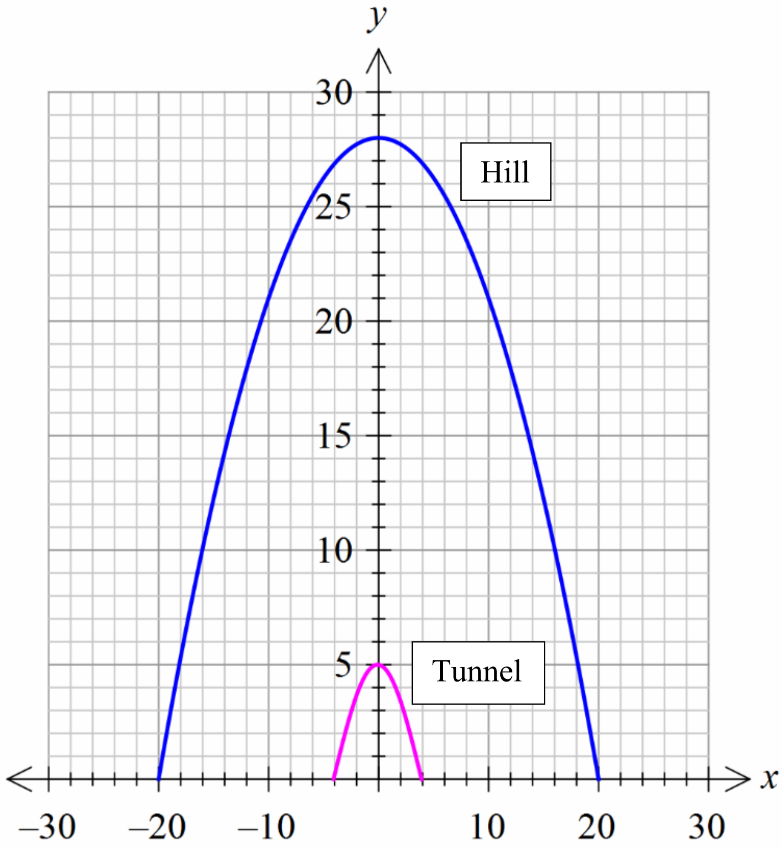
Your explanation must include a clear diagram (part of graphs and are drawn, to save you time).

(3 marks)

b. Evaluate the above expression.

(1 mark)

Question 8
Daniel decides to build another hill on a different section of track. The new hill is to have a single tunnel pass through the middle of it as shown on the diagram below.



The single tunnel, which is 5 cm high, can be modelled by the equation:

Note that the axes used by Daniel now have their origin at the *centre* of the base of the tunnel.

The -intercepts of the tunnel are and .
(Please note: although the graph shown suggests a value of 4 for , this is not its *exact* value.)

The turning point is for the parabolic hill and the -intercepts are and .

Note that . The value of **,** the amplitude of the sine graph, is 5.

Hence, show that the values of and are given by and respectively.

(3 marks)

Question 9
Write an expression using definite integral notation, in terms of , that could be used to evaluate the cross-sectional area of the *tunnel only* in the domain .

(1 mark)

Question 10
The rule for the hill in the graph on the previous page is .

Daniel has decided that the area of the tunnel above must be equal to exactly **5%** of the total area of the hill.

a. Calculate (correct to 3 decimal places) the value of .

(2 marks)

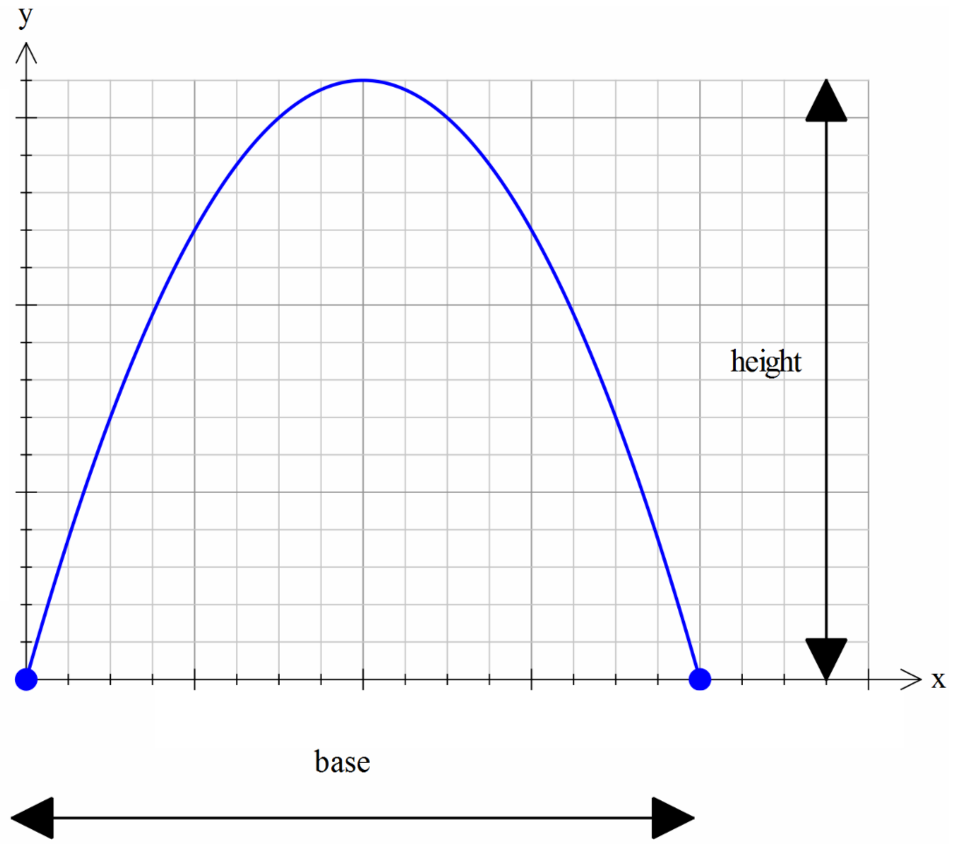
b. Hence, state the maximum width of the single tunnel (3 decimal places).

(1 mark)

Part 3 (10 marks)

Now we will consider the properties of parabolic arches more broadly.

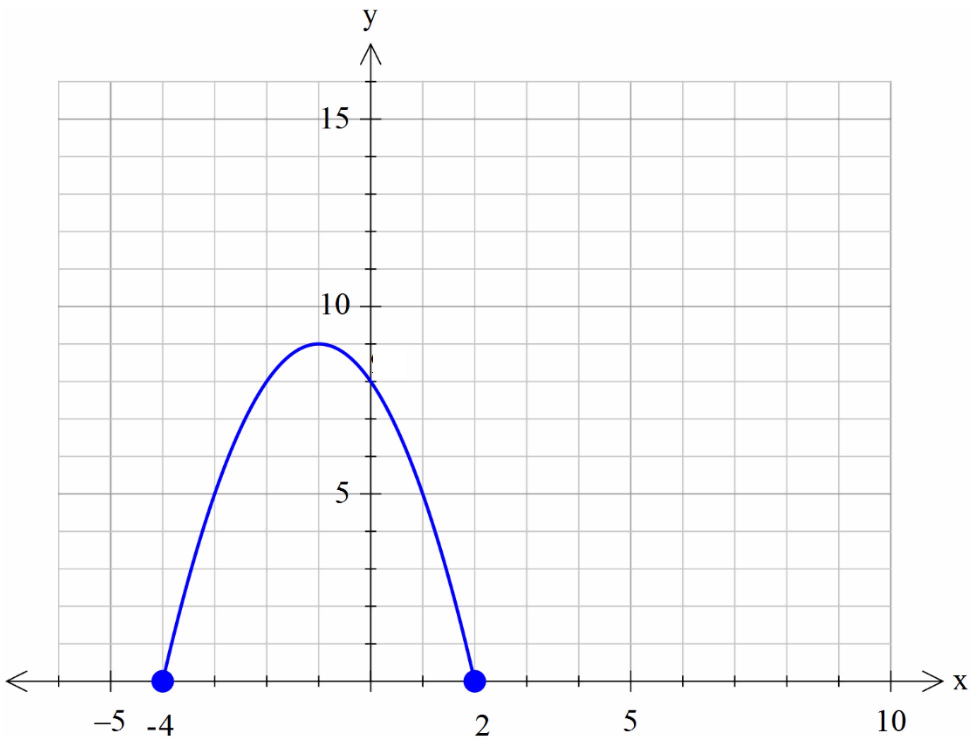
Throughout this Part, *base* and *height* for a parabolic arch are defined as shown below.



To clarify, we will use the following definitions for a *parabolic arch*:

- The *arch* is the part of a parabola lying above the -axis, as shown above.
- The *base*, , is the distance between the -intercepts
- The *height*, *,* is the vertical distance between the -axis and the maximum.

The parabolic arch shown below is for the equation



Question 11

a. What are the *base* and *height* of the parabola shown?

(1 mark)

b. Use an integral to determine the area between the curve and the x-axis.

(2 marks)

Archimedes (ancient Greek mathematician) proposed that the area bounded by a parabolic arch and the -axis is always equal to *the length of base* () *multiplied by the height of the arch* () *times* .

Question 12
Use Archimedes' rule to confirm your answer obtained for the area under the arch in Question 11, above. Does his rule work in this case?

(2 marks)

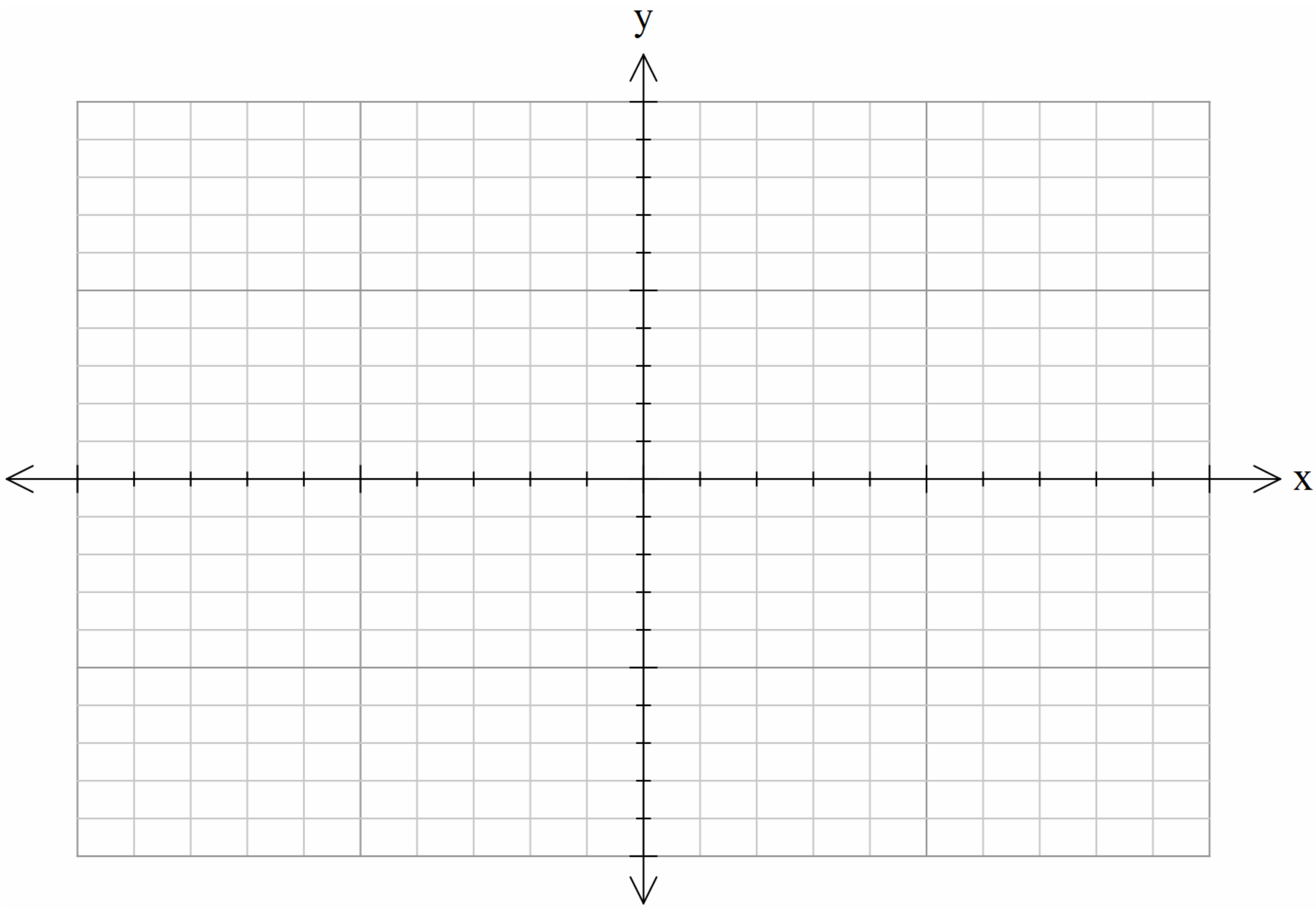
Continued over page

To finish this part we'll look at parabolic arches in general terms (ie., literal equations).

Question 13
On the axes below, sketch the graph of the parabola given by the literal equation

where *b* and *h* are positive constants. Clearly label the coordinates of the -intercepts and the turning point.

Sketch only the part of the graph above the -axis. (2 marks)



Question 14
Use a definite integral to find the area enclosed by the parabola

and the -axis. Clearly show all working.

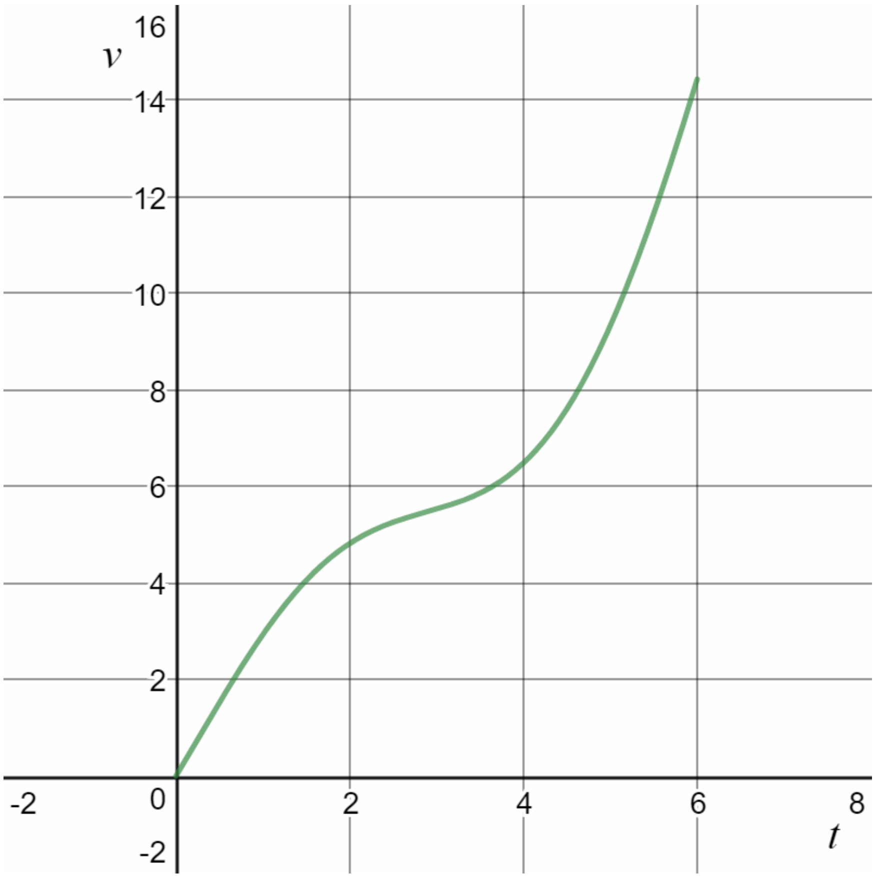
Comment on your result.

(3 marks)

Part 4 (7 marks)

Daniel's friend, Aiko, brings her train around to test in Daniel's new track. Aiko has a speedometer in her train which records its speed as it proceeds through the tunnel. *Assume the tunnel is a straight line*.

The speedometer captures the data about the train's speed over time in the following graph.



The rule for this graph of the train's velocity, in m/s, over the first 6 s of its journey is given by

Question 15

Determine an equation for the train's position, , in terms of .

The train's initial position is 0 m. (This means is its position compared to its starting point.)

(2 marks)

Question 16

a. Evaluate the expression .

(1 mark)

b. In the context of the situation being modelled, what does the above expression mean?

(1 mark)

c. Evaluate the expression . Compare this to your answer to **a.**, and explain your observations.

(1 mark)

Question 17
Aiko's train needs to be refuelled, so they pour some more fuel into its tank. The tank is initially empty.

The *rate* in mL per second of fuel being poured into the tank is given by

is the volume of fuel in the tank in mL and is the time in seconds since they starting pouring it in.

Use this information to determine how long it takes to fill the 8 mL tank, to two decimal places.

(3 marks)

End SAC
Have a great day!