2019 MATH CAMP LECTURE NOTES

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1. Univariate optimization

This will be a short, simple note on the basic ideas underlying optimization of a univariate function on an interval. The basic ideas here will carry over to optimization in more complex settings, albeit with more complex corresponding methods.

The first exercise demonstrates the fundamental limit to what optimization procedures can do for you.

Exercise 1. *Find the maximum of* $2^4/15$*,* $e^{.06}$*, and* $(\pi^2+1)/10$*.*

The important thing here is that there really is no clever technique at this point, other than calculating these values with enough precision to compare them. So the point is that once we are optimizing over a finite list, our problem is about as easy as it is going to get.

What if we need to optimize over an infinite set? In general, we need some structure, since otherwise we're stuck with our algorithm for finite sets, which isn't so good when the set is infinite.

Let's fix ideas. Let $f : [a, b] \to \mathbb{R}$. Our goal is to maximize f on $[a, b]$.

The following observations are really what univariate optimization is all about:

(1) If there is an $x' \in [a, b]$ such that $f(x') > f(x)$, then f is not maximized at $f(x)$
(2) If f is differentiable at $x \in (a, b)$ and $f'(x) \neq 0$, then there is an x' such that $f(x') > f(x)$

Exercise 2. *Prove statement* 2*.*

So what have we gained? We started with an infinite set of possibilities for places where f might be maximized. We have now managed to *remove* from that set all $x \in (a, b)$ such that $f'(x)$ is defined and nonzero. This means what remains on the list are the values of x for which f' is not defined, for which $f' = 0$, a, and b. In many, many contexts, this set is finite, and so we have reduced to the first kind of problem.

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