2019 MATH CAMP LECTURE NOTES

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1. Advanced Analysis

1.1. **Correspondences.** A *correspondence* $F : X \rightrightarrows Y$ is a function $X \to 2^Y$. Let $E \subseteq Y$.

Definition 1. *The strong/upper preimage of E is the set of points in X whose image is contained in E.*

Think of this as universal quantification.

Definition 2. *The weak/lower preimage of E is the set of points in X whose image has nonempty intersection with E.*

Think of this as existential quantification.

Definition 3. *A correspondence is*

- *upper hemicontinuous if the strong preimage of every open set is open*
- *lower hemicontinuous if the weak preimage of every open set is open*
- *continuous if it is upper hemicontinuous and lower hemicontinuous*

Exercise 1. *Consider the correspondence $\mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ which maps (m, b) to the set of points (x, y) that satisfy $y = mx + b$. Prove that this correspondence is lower hemicontinuous but not upper hemicontinuous.*

Exercise 2. *Consider the correspondence $\mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ which maps (a, b) to the set of points (x, y) such that $ax \geq 0$ and $by \geq 0$. Prove that this correspondence is upper hemicontinuous but not lower hemicontinuous.*

For any property P, we say that a correspondence is P-valued if every point is mapped to a set with property P. We often care about correspondences that are convex-valued, compact-valued, and nonempty-valued.

1.2. **Theorem of the maximum.** Let $f : X \times \Theta \to \mathbb{R}$ be a continuous function.

Note 1. *We can interpret Θ as the parameter space, X as the space of all possible alternatives, and f as the payoff function.*

Let $C : \Theta \rightrightarrows X$ a compact and nonempty valued correspondence.

Note 2. *We can interpret C as giving the set of available alternatives given the parameter.*

Define $f^* : \Theta \to \mathbb{R}$ by

$$f^*(\theta) = \sup\{f(x, \theta) \,:\, x \in C(\theta)\}.$$

Note 3. *We can interpret $f^*(\theta)$ as the payoff an agent receives when she makes the optimal choice given that the parameter is θ.*

Define $C^* : \Theta \rightrightarrows X$ by

$$\begin{aligned} C^*(\theta) &= \arg\sup\{f(x, \theta) : x \in C(\theta)\} \\ &= \{x \in C(\theta) : f(x, \theta) = f^*(\theta)\} \end{aligned}$$

Note 4. *We can interpret $C^*(\theta)$ as the set of alternatives that are optimal among those that are available given that the parameter is θ.*

Theorem 3. *If C is continuous at θ, then f^* is continuous and C^* is upper hemicontinuous with nonempty and compact values.*

Note 5. *In words, the theorem says that, given a technical condition (C is continuous at θ), the value function f^* is continuous in the parameter and the set of optimal alternatives given the parameter is nonempty and compact (plus C^* is nice in the sense that it upper hemicontinuous.)*

Let $X = \mathbb{R}^2$, $\Theta = \mathbb{R}^2$, with $f : \Theta \times X \to \mathbb{R}$ and $C : \Theta \rightrightarrows X$ given by

$$\begin{aligned} f((x_0, y_0), (x_1, y_1)) &= -(x_1^2 + y_1^2) \\ C(x_0, y_0) &= \{(x, y) \,:\, (x - x_0)^2 + (y - y_0)^2 \leq 1\}. \end{aligned}$$

Exercise 4. *Verify that f and C satisfy the assumptions of Theorem 3.*

Exercise 5. *Compute f^* and C^*.*

Exercise 6. *Verify directly that the consequences of Theorem 3 hold for f^* and C^*.*

1.3. **Kakutani fixed point theorem.** Let $S \subseteq \mathbb{R}^n$ nonempty, compact, convex. Let $\phi : S \rightrightarrows S$ a correspondence.

Definition 4. *The graph of ϕ is*

$$Gr(\phi) = \{(v, w) \,:\, w \in \phi(v)\}.$$

Definition 5. *A fixed point of ϕ is an $s \in S$ such that $s \in \phi(s)$.*

Theorem 7. *If ϕ is nonempty and convex valued and $Gr(\phi)$ is closed, then ϕ has a fixed point.*

Exercise 8 (Brouwer fixed-point theorem)**.** *Let $S \subseteq \mathbb{R}^n$ nonempty, compact, convex. Let $f : S \to S$ be a continuous function. Prove that f has a fixed point.*

1.4. **Envelope Theorem.** Let

$$\begin{aligned} f :&[0,1] \times [0,1] \to \mathbb{R} \\ &(x,t) \mapsto 2xt - x^2. \end{aligned}$$

Define

$$V(t) = \sup_{x \in [0,1]} f(x,t)$$

and

$$x^*(t) = \arg\sup_{x \in [0,1]} f(x,t).$$

Note 6. *For this particular choice of f, $x^*(t)$ is actually a function, so in this case we have $V(t) = f(x^*(t), t)$.*

Exercise 9. *Calculate $V(t)$ and $x^*(t)$.*

Exercise 10. *Prove that*

$$V'(t) = \frac{\partial f(x,t)}{\partial t}\Big|_{(x^*(t),t)}.$$

Exercise 11. *Sketch a graph of $V(t)$ as well as $f(x,t)$ for several values of x.*

This simple exercise illustrates the core ideas behind the following result:[1]

Theorem 12 ((some of the) Envelope Theorem)**.** *Let*

$$f(x_1, \ldots, x_n, t_1, \ldots, t_m) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$$

be smooth. Let

$$x^*(t_1, \ldots, t_m) = \arg\sup_{x \in \mathbb{R}^n} f(x_1, \ldots, x_n, t_1, \ldots, t_m),$$

and let

$$V(t_1, \ldots, t_m) = f(x^*(t_1, \ldots, t_m), t_1, \ldots, t_m).$$

Then for $i = 1, \ldots, m$,

$$\frac{\partial V(t_1, \ldots, t_m)}{\partial t_i} = \frac{\partial f(x, t_1, \ldots, t_m)}{\partial t_i}\Big|_{(x^*(t_1,\ldots,t_m), t_1, \ldots, t_m)}.$$

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[1]We will give the statement of the full envelope theorem when we discuss constrained optimization.