2019 MATH CAMP LECTURE NOTES

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1. Basic Calculus

1.1. **Lines (a la middle school).** (Almost) every line can be represented in the form

$$y = mx + b$$

where m is the *slope* and b is the *intercept*.

The slope tells us the rate at which y varies relative to the rate at which x varies.

Given two distinct points, there *exists* a *unique* line that contains both points. In particular, given any two points (x_0, y_0) and (x_1, y_1), there is a simple formula for the unique line containing these two points:

$$slope = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}.$$

1.2. **Nonlinear functions.** Imagine you drive down a straight road with the position of your car at time t given by $f(t)$, with time measured in seconds and position measured in meters.

Question: How fast are you going at time 100?

Preliminary question: What does that question even mean?

First, how does your car report speed at time 100? It calculates $f(100) - f(99)$ and reports that value [1]. In this setting, that seems reasonable, but why a one second interval? In fact, it kind of seems like any choice of interval would be somewhat arbitrary.

So here's an idea for defining the slope of a nonlinear function: find a number that is really close to the slopes for lines drawn between point of interest and other really close points. This idea gives rise to the following notion.

[1] This is a complete lie. In fact, speedometers are pretty cool mechatronic devices that do something closer to giving a noisy weighted average of recent values of speed. But for the duration of this note, let's suspend disbelief and pretend our cars measure speed in a less conceptually and technologically cool way.

1.2.1. *Secants.* Given a function f and two values x_1 and x_2, the *secant line* for (f, x_1, x_2) is the line that passes through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

In fact, secants already capture much of the structure that we care about in practice, so we'll spend a little while on these. In particular, since all of this was in pursuit of a notion of slope, we really only care about the slopes of secants. This leads to our next definition.

For any x and x' with $x' \neq x$, define $\Delta_x f(x')$ to be the slope of the secant line for (f, x, x'). Formally,

$$\Delta_x f(x') = \frac{f(x') - f(x)}{x' - x}.$$

To get comfortable with this definition (and to basically prove the facts we'll want to prove later on), we'll now prove some facts about Δ_x.

Exercise 1. *Prove that for a constant function f,*

$$\Delta_x f(x') = 0.$$

Exercise 2. *Prove that for a function f and for any $c \in \mathbb{R}$, if $h = cf$ then*

$$\Delta_x h(x') = c\Delta_x f(x').$$

Exercise 3. *Prove that for two functions f and g, if $h = f + g$ then*

$$\Delta_x h(x') = \Delta_x f(x') + \Delta_x g(x').$$

Exercise 4. *For functions f and g, show that if $h = f * g$ then*

$$\Delta_x h(x') = (\Delta_x f(x'))(g(x')) + (f(x))(\Delta_x g(x')).$$

Exercise 5. *For functions f and g, show that if $h = f \circ g$ then*

$$\Delta_x h(x') = (\Delta_{g(x)} f(g(x')))(\Delta_x g(x')).$$

Exercise 6. *For $n \in \mathbb{N}$, $n \geq 1$, show that if $f(x) = x^n$ then*

$$\Delta_x f(x') = \sum_{i=0}^{n-1} (x')^{n-1-i}(x)^i.$$

The main takeaway from these exercises (as we will see momentarily) is the following. Whenever you are thinking about derivatives, you should really think about there being two separate steps. The first is to establish some features of $\Delta_x f$. The next is to take a limit to then establish some features of derivatives.

1.2.2. *Derivatives.* Refresher on limits

We define the *derivative* of f at x, $f'(x)$, to be the limit of $\Delta_x f(x')$ as x' approaches x:

$$f'(x) = \lim_{x' \to x} \Delta_x f(x').$$

Conceptually, this is really the definition to keep in mind. For convenience (in terms of algebra), it is conventional to define $h = x' - x$ and to write this instead as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

(Take a moment to verify that this is just a different way of writing the same thing.) We now get the normal rules for taking derivatives almost for free.

Exercise 7. *Take limits of the equations involving $\Delta_x f$ to derive the usual derivative rules.*

1.3. **Taylor's theorem.** Now that we have derivatives at our disposal, let's put them to work. We'll assume for the rest of this note that any function we discuss is as nicely behaved as we could ask for[2] and that we only care about the function around $x = 0$[3].

We'll need a fact from topology for this part (one that we will prove ourselves in a few days):

Lemma 8. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then there is some $M \in \mathbb{R}$ so that $|f(x)| \leq M$ for all $-1 \leq x \leq 1$.*

Since we are only working with very nice functions, we can take the consequent as given going forward.

As a warm up (and because we need this fact anyways), let's prove (a version of) Rolle's theorem:

Exercise 9. *If $a < b$ and $f(a) = f(b) = 0$, then there is some c, $a < c < b$, such that $f'(c) = 0$.*

Now, let's get into Taylor's theorem.

Exercise 10. *Let $k \geq 1$. Suppose f has the following properties:*
- *$f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0$, and*
- *For all $x \in [-1, 1]$, $|f(x)| \leq 1$, $|f'(x)| \leq 1$, $\ldots$, $|f^{(k)}(x)| \leq 1$.*

Show that for $x \in [-1, 1]$, $|f(x)| \leq |x|^k$.

Exercise 11. *Prove Taylor's theorem.*

[2] Should state here (or somewhere) the specific assumptions.

[3] We do this for ease of exposition and digestion, but also because in cases where the facts that we establish don't hold, either (1) they *really* don't hold or (2) they hold for basically the same reasons as the ones we'll see here, but some technical modifications are required

1.4. **Integration.**

Definition 1. *$F(x)$ is an antiderivative for $f(x)$ if $F'(x) = f(x)$.*

The basic rules of differentiation can be used to make useful rules for finding antiderivatives.

Exercise 12. *Find an antiderivative for x^n.*

In many contexts, we will care about the area underneath a curve. We will work with Riemann integration, rather than Lebesgue integration, for a couple of reasons. First, when the Riemann integral exists, they coincide, and in these cases it is the Riemann formulation that gives a recipe for actually computing. Second, the Riemann integral will give us the opportunity to work with sums.

In general, Riemann sums should be independent of the net used. For simplicity, we will look only at the simple net that you are familiar with.

For our purposes, we will only look at the left Riemann sums:

Definition 2. *Define the n^{th} left Riemann sum for the function $f(x)$ for the interval $[a, b]$ to be*

$$S_n^{[a,b]}(f) = \frac{b-a}{n} \sum_{i=0}^{n-1} f\left(a + \frac{b-a}{n} i\right).$$

Define the integral of $f(x)$ from a to b to be

$$\int_a^b f(x)dx = \lim_{n\to\infty} S_n^{[a,b]}(f)$$

when the right hand side exists, and leave it undefined otherwise.

Exercise 13. *Calculate*

$$\int_0^y x^n dx.$$

Exercise 14. *Prove the fundamental theorem of calculus.*

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