2019 MATH CAMP LECTURE NOTES

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1. Linear Algebra

Let's begin by noting some facts about ordered pairs of numbers.

(1) You can add two of them together (coordinate-wise) and get another ordered pair.
(2) Multiplying both entries by the same number gives another ordered pair.
(3) If you multiply both numbers by 1, you get the same ordered pair.
(4) Multiplication by a scalar commutes with scalar addition and vector addition (how do I say this better?)

We could instead ask for a set and operations which satisfy these properties. WE call anything like that a *vector space*. What is very nice about thinking about things abstractly like this is that lots of facts about ordered pairs of numbers (or ordered n-tuples of numbers) hold because they form a vector space. Figuring out facts about vector spaces is beneficial both because those facts will hold about the vector spaces that we usually see and because they tell us which features will be portable to other vector spaces.

To begin our investigation of vector spaces, we start with what turns out the be the most important structural definition.

We say that vectors $v_1, \ldots, v_m$ are *linearly dependent* if there exist $c_1, \ldots, c_m$ with not all c_i equal to 0 such that

$$c_1 v_1 + \cdots + c_m v_m = 0.$$

(Keep in mind, 0 here means the zero vector.) We say that vectors are *linearly independent* if they are not linearly dependent.

We say that a vector space is *finite dimensional* if there exists a $d \in \mathbb{N}$ such every set of size $d + 1$ is linearly dependent. We define the *dimension* to be the smallest such d.

Exercise 1. *Show that if V is d-dimensional, then there exist d vectors $v_1, \ldots, v_d$ such that every vector can be written as a linear combination of these d vectors.*

Such a set is called a *basis*.

Exercise 2. *Show that a set of d vectors is a basis if and only if the vectors are not linearly dependent.*

Exercise 3. *Show that for any such set of vectors as above and for any other vector, there is a unique way to write it as a linear combination.*

Often what we care about (especially in Econometrics) are "structure-preserving transformations" of vector spaces. This happens a lot in math – once you have objects of some kind, you want a notion of "functions" from one object to another that "preserve the structure". In the context of vector spaces, these are called *linear maps*. Formally, a function $T : V \to W$ is a linear map if $T(\lambda v + v') = \lambda T(v) + T(v')$ for all $\lambda \in \mathbb{R}$ and $v, v' \in V$.

We'll begin our study of linear maps by looking at *endomorphisms*: linear maps from a vector space to itself. As it turns out, endomorphisms have any inherently combinatorial structure that is exploited **constantly**. Until otherwise noted, let V be a vector space of dimension n.

First, just to ground ourselves a little bit and get a feeling for why understanding endomorphisms will be so useful, let's relate them to $n \times n$ matrices. Fix a basis $v_1, \ldots, v_n$ of V.

Exercise 4. *Let M be any $n \times n$ matrix. Show that, if every vector is identified with the ordered tuple of coefficients in the expression as a linear combination of the v_i, the normal multiplication by M is an endomorphism.*

Exercise 5. *Let T be an endomorphism. Show that there is an $n \times n$ matrix M such that T is equal to (the stuff above).*

This connection between endomorphisms and matrices is easy to establish, but as we will see, the practical value in the connection will come in the ability to prove facts about endomorphisms and get corresponding facts about matrices.

Note that we can "add" and "scale" endomorphisms by simply applying the corresponding operation to the results. We can also "multiply" endomorphisms by composing them. Define "plugging in" an endomorphism into a polynomial by formally replacing each power of x with the corresponding power of T and replacing the constant coefficient with the endomorphism which is multiplication by that constant coefficient.

Exercise 6. *Let $v \in V$. Denote $v^0 = v$ and for $1 \leq i \leq n$, define $v^i = T(v^{i-1})$. Show that are $c_0, \ldots, c_n$ not all 0 such that*

$$c_0 v^0 + c_1 v^1 + \cdots + c_n v^n = 0.$$

Note that this is equivalent to saying that for any $v \in V$, there is some polynomial $p(x)$ such that $(p(T))(v) = 0$. It is not obvious that this alone tells us much about T, since we may need a different polynomial for every v. However, it turns out that it is possible to find a single polynomial that works for all v.

Exercise 7. *Show that there is a polynomial $p(x)$ such that for all $v \in V$,*

$$(p(T))(v) = 0.$$

This is a much more striking result, and it has deep structural implications. To get at these implications, we will need a few more definitions.

Define the complexification V^* of V to be the set of functions from $\{v_1, \dots, v_n\}$ to $\mathbb{C}$. Notice that if we identify V with the set of functions with image contained in $\mathbb{R}$ in the natural we, there is a unique (complex) endomorphism T^* which extends T. We say that $\lambda \in \mathbb{C}$ is an eigenvalue for an endomorphism T^* if there is some $v \in V^*$ such that $T^*(v) = \lambda v$. We say that λ is an eigenvalue of T if it is an eigenvalue of T^*[1].

We say that $p(x)$ is a minimal polynomial for T if $p(T) = 0$ and for every polynomial $q(x)$ with $deg(q) < deg(p)$, $q(T) \neq 0$.

Exercise 8. *Show that any root of any minimal polynomial for T is an eigenvalue of T.*

We say that a subset W of V is a *subspace* of V if W is a vector space. (Note that this amounts to showing that that it is closed under taking linear combinations.) Let W', W'' be subspaces of V. We say that V is a direct sum of W' and W'', and write $V = W' \oplus W''$, if $W' \cap W'' = \{0\}$ and for every $v \in V$, there are $w' \in W'$ and $w'' \in W''$ such that $v = w' + w''$. We say that W is in a T-invariant subspace of V if $T(W) \subseteq W$. For any T-invariant subspace, we define $T|_W$ to be the endomorphism of W such that for all $w \in W$, $T|_W(w) = T(w)$.

[1] It may seem surprising that the complex numbers are entering here. After all, when will we need complex numbers in Social Sciences? Isn't this a bit esoteric (at least for our purposes)? In fact, essentially all of the (matrices corresponding to) endomorphisms in the Social Sciences have enough extra properties that complexification is completely unnecessary. However, the facts that we will establish here hold on those settings for exactly the same reasons – that is, you can copy the proofs nearly line for line. The only fact about the complex numbers that we care about and that we will make use of here is that they are "algebraically closed". This means that for every polynomial $p(x)$ with complex coefficients there is some $r \in \mathbb{C}$ such that $p(r) = 0$.

We say that a linear map $R : V' \to V''$ is an *isomorphism* if there is a linear map $S : V'' \to V'$ such that for all $v' \in V'$ and all $v'' \in V''$, $S(R(v')) = v'$ and $R(S(v'')) = v''$. In this case, we say that R is *invertible* and call S the *inverse* of R, written R^{-1}. We say that an endomorphism T is an *automorphism* if it is invertible.

Exercise 9. *Let T be an endomorphism of V. Show that there is a largest T-invariant subspace W of V such that $T|_W$ is an automorphism. Show that there is a T-invariant subspace W' of V such that $V = W \oplus W'$ and $T|_{W'}^n = 0$.*

Let I denote the identity endomorphism. (It will generally be clear from context what space I is an endomorphism of; when it is not, we will subscript I with the name of the associated space to make it clear. Hence, for example, we write I_U for the identity endomorphism of U.)

Exercise 10. *Let T be an automorphism of V and let λ be an eigenvalue of T. Show that there is a largest T^*-invariant subspace W^* of V^* such that $(T^* - \lambda I)|_{W^*}$ is an automorphism. Show that there is a T^*-invariant subspace W'^* such that $V^* = W^* \oplus W'^*$ and $T^*|_{W'^*}^n = 0$.*

Exercise 11. *Let T be an endomorphism with eigenvalues $\lambda_1, \dots, \lambda_r$. Show that there are T^*-invariant subspaces $V_1^*, \dots, V_r^*$ such that...(direct sum, restriction minus lambda torsion or invertible)*

Say that T^* is a λ-escalator if there is a basis $v_1^*, \dots, v_n^*$ of V^* such that for $1 \leq i < n$, $T^*(v_i^*) = \lambda v_i^* + v_{i+1}^*$ and $T(v_n^*) = \lambda v_n^*$.

Exercise 12. *Suppose T^* is an automorphism with unique eigenvalue λ. Show that V^* is a direct sum of subspaces for which the restriction of T^* to each is a λ escalator.*

Putting the previous results together, we get the main structure theorem for endomorphisms.

Exercise 13. *Let T be an endomorphism. Show that V^* is a direct sum of subspaces for which the restriction of T^* to each is a λ-escalator for some λ.*

We say that $\tau : \mathbb{C} \times \mathbb{N}_{>0} \to \mathbb{N}$ is a *type* if $\tau(c, n) = 0$ for all but finitely many (c, n) pairs. Given a decomposition as above, we say that τ is the type of the decomposition if for each $c \in \mathbb{C}$ and each $n \in \mathbb{N}_{>0}$, $\tau(c, n)$ is equal to the number of c-escalators of dimension n in the decomposition.

Exercise 14. *Show that any two decompositions for T have the same type.*

We say that τ is the type of T if it is the type of any decomposition for T.

Exercise 15. *Let T and T' be endomorphisms of V. Show that there is an automorphism B of V such that $BT = T'B$ if and only if T and T' have the same type.*

We say that T is *diagonalizable* if there exists a direct sum decomposition as in the exercise in which every summand has dimension 1.

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