2019 MATH CAMP LECTURE NOTES

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1. Constrained Optimization

1.1. The linear case.

Exercise 1. *Compute*

$$\sup_{v \in S} v \cdot (1,1),$$

where

$$S = \{x \in \mathbb{R}^2 \, : \, x \cdot (1,2) = 3\}.$$

Exercise 2. *Compute*

$$\sup_{v \in S} v \cdot (-6,8)$$

where

$$S = \{x \in \mathbb{R}^2 \, : \, x \cdot (-3,4) = 2\}.$$

Note the stark contrast in these exercises. This dichotomy plays a very similar role in constrained optimization as the zero/nonzero derivative dichotomy in univariate optimization.

1.2. **Gradient interlude.** Let $f(x_1, \dots, x_n) : \mathbb{R}^n \to \mathbb{R}$. The *gradient* of f is the vector of partial derivatives,

$$\nabla f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}).$$

Fix $x^0, u \in \mathbb{R}^n$ and define

$$\tilde{f}(t) = f(x_1^0 + u_1 t, \dots, x_n^0 + u_n t).$$

Exercise 3. *Show that*

$$\tilde{f}'(0) = (\nabla f|_{x^0}) \cdot u.$$

As this exercise suggests, ∇f encodes how f behaves locally, and in particular yields the approximation

$$f \approx f(x_1^0, \dots, x_n^0) + \nabla f \cdot (x - x^0)$$

for x close to x^0.

The nonlinear case with a single constraint.

Let $f, g : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = x$$
$$g(x, y) = x^2 + y^2 - 1.$$

Exercise 4. *Calculate $\nabla f(\frac{3}{5}, \frac{4}{5})$ and $\nabla g(\frac{3}{5}, \frac{4}{5})$, and show that they are linearly independent.*

Let

$$Z = \{(x, y) \in \mathbb{R}^2 \,:\, g(x, y) = 0\}.$$

Exercise 5. *Sketch Z, along with the gradient vectors computed in the previous exercise at $(\frac{3}{5}, \frac{4}{5})$.*

Exercise 6. *Find a point (x', y') such that $g(x', y') = 0$ and $f(x', y') > f(\frac{3}{5}, \frac{4}{5})$.*

Keep the above exercises in mind when doing the following exercise.

Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be smooth functions, and suppose $x^0 \in \mathbb{R}^n$ is such that $g(x^0) = 0$.

Exercise 7. *Using the implicit function theorem,*[1] *show that if $\nabla f(x^0)$ and $\nabla g(x^0)$ are linearly independent, then there is an $x \in \mathbb{R}^n$ such that $g(x) = 0$ and $f(x) > f(x^0)$.*

Define

$$\mathcal{L}(x_1, \ldots, x_n, \lambda) = f(x_1, \ldots, x_n) + \lambda g(x_1, \ldots, x_n).$$

Exercise 8. *Using the result of the previous exercise, show that if there is no $\lambda^* \in \mathbb{R}$ such that*

$$\frac{\partial \mathcal{L}}{\partial x_1}\Big|_{(x_1^0, \ldots, x_n^0, \lambda^*)} = 0$$
$$\vdots$$
$$\frac{\partial \mathcal{L}}{\partial x_n}\Big|_{(x_1^0, \ldots, x_n^0, \lambda^*)} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda}\Big|_{(x_1^0, \ldots, x_n^0, \lambda^*)} = 0$$

then there is an $x \in \mathbb{R}^n$ such that $g(x) = 0$ and $f(x) > f(x^0)$.

[1] Add statement of implicit function theorem

The nonlinear case with multiple constraints.

Let $f, g_1, \dots, g_m : \mathbb{R}^n \to \mathbb{R}$ be smooth functions.

Define the *Lagrangian*

$$\mathcal{L}(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m) = f(x_1, \dots, x_n) + \lambda_1 g_1(x_1, \dots, x_n) + \cdots + \lambda_m g_m(x_1, \dots, x_n).$$

Exercise 9. *Suppose $g_1(x^0) = \cdots = g_m(x^0) = 0$. Prove that if there is no $(\lambda_1^*, \dots, \lambda_m^*)$ such that*

$$\frac{\partial \mathcal{L}}{\partial x_1}|_{(x_1^0,\dots,x_n^0,\lambda_1^*,\dots,\lambda_m^*)} = 0$$
$$\vdots$$
$$\frac{\partial \mathcal{L}}{\partial x_n}|_{(x_1^0,\dots,x_n^0,\lambda_1^*,\dots,\lambda_m^*)} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda_1}|_{(x_1^0,\dots,x_n^0,\lambda_1^*,\dots,\lambda_m^*)} = 0$$
$$\vdots$$
$$\frac{\partial \mathcal{L}}{\partial \lambda_m}|_{(x_1^0,\dots,x_n^0,\lambda_1^*,\dots,\lambda_m^*)} = 0$$

then there is some $x \in \mathbb{R}^n$ such that $g_1(x) = \cdots = g_m(x) = 0$ and $f(x) > f(x^0)$.

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