2019 MATH CAMP LECTURE NOTES

WADE HANN-CARUTHERS

1. Point Set Topology

1.1. **Fixing ideas.** The first definition of continuity most people are introduced to is the pencil one: a function is continuous if you can draw it without lifting your pencil off of the paper. This is capturing the idea that nearby x values should map to nearby y values. For a function $f : \mathbb{R} \to \mathbb{R}$, this is formalized as follows.

Definition 1. *f is continuous at x if for every $\epsilon > 0$ there is a $\delta > 0$ such that*

$$|x' - x| < \delta \implies |f(x') - f(x)| < \epsilon.$$

What does this definition say? It says that for any choice of which values are close to $f(x)$ (ϵ) we can make a choice of which values are close to x (δ) such that every value which is close to x maps to a value that is close to $f(x)$.

This reading of the definition of continuity gives the jumping off point for topology. The only structure that we needed on $\mathbb{R}$ was a notion of which sets are sets that contain all points "sufficiently close" to a given value.

1.2. **Topological spaces.** Let X be a set and $\mathcal{O} \subseteq 2^X$ be a set of subsets of X. Then (X, Ω) is a *topological space* if

- $\emptyset, X \in \mathcal{O}$
- closed under pairwise intersection
- closed under arbitrary union

We call Ω the *open sets*.

Exercise 1. *Let X be a set and let $\mathcal{O} = 2^X$ (that is, $\mathcal{O}$ is the set of all subsets of X). Show that $(X, \mathcal{O})$ is a topological space.*

Exercise 2. *Let X be a set and let $\mathcal{O} = \{\emptyset, X\}$. Show that $(X, \mathcal{O})$ is a topological space.*

Exercise 3. *Let $\mathcal{O}$ be the set of subsets U of $\mathbb{R}$ such that for every $x \in \mathbb{R}$, $x \in U$ implies there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq U$. Prove that $(\mathbb{R}, \mathcal{O})$ is a topological space.*

This is called the *standard topology* on $\mathbb{R}$.

1.3. **Families of open subsets.** Given a topological space $(X, \mathcal{O})$ and a family of open sets $\Lambda \subseteq \mathcal{O}$, we say that Λ is

- an *open cover* if $\bigcup_{U \in \Lambda} U = X$
- a *generating set* if $\mathcal{O}$ is the smallest subset of 2^X which contains Λ and is closed under union and finite intersection
- a *base* if for every $U \in \mathcal{O}$ there is a family $\Gamma \subseteq \Lambda$ such that $\bigcup_{V \in \Gamma} V = U$

Let T be $\mathbb{R}$ with the standard topology. Let $\Lambda = \{(p - q, p + q) : p, q \in \mathbb{Q},\ q > 0\}$.

Exercise 4. *Prove that Λ is an open cover for T.*

Exercise 5. *Prove that Λ is a generating set for T.*

Exercise 6. *Prove that Λ is a base for T.*

1.4. **Compactness.** Let $(X, \mathcal{O})$ be a topological space, and let $Y \subseteq X$. We say that a family of open sets $\Lambda \subseteq \mathcal{O}$ is an *open cover* for Y if

$$Y \subseteq \bigcup_{U \in \Lambda} U.$$

If Λ and Λ' are open covers of Y with $\Lambda' \subseteq \Lambda$, we say that Λ' is a *subcover* of Λ.

Definition 2. *Let Y be a subset of X. We say that Y is compact if every open cover of Y has a finite subcover (that is, a subcover containing only finitely many sets).*

Exercise 7. *Using the least upper bound property of the real numbers*[1]*, show that $[0, 1]$ is compact in $\mathbb{R}$ with the standard topology.*

[1]As a reminder, the least upper bound property says the following:
For any $A \subseteq \mathbb{R}$, we say that $u \in \mathbb{R}$ is an *upper bound* for A if $u \geq a$ for all $a \in A$. Let Ω be the set of upper bounds for A. If Ω is nonempty (so A has at least one upper bound), then there is some $\tilde{u} \in \Omega$ such that $\tilde{u} \leq u$ for all $u \in \Omega$.

1.5. **Closed sets and limits.** Let $(X, \mathcal{O})$ be a topological space.

Definition 3. *We say that $V \subseteq X$ is closed if $X \setminus V \in \mathcal{O}$; that is, if the complement of V is open.*

Definition 4. *Let $U, U' \in \mathcal{O}$. We say that U' is a closure of U if*

- *$U \subseteq U'$*
- *U' is closed*
- *For any $V \subseteq X$, $U \subseteq V$ and V closed implies $U' \subseteq V$.*

Exercise 8. *Let $A \subseteq X$. Prove that A has exactly one closure (that is, show that the closure of A exists and is unique).*

Given this result, we call the unique closure of A *the* closure of A and refer to it by $\bar{A}$.

Exercise 9. *Let $A \subseteq X$. Prove that if A is closed, then $\bar{A} = A$. Prove that (even if A is not closed), $\bar{\bar{A}} = \bar{A}$.*

Definition 5. *Let $A \subseteq X$, and let $x \in X$. We say that x is a limit point of A if for every $U \in \mathcal{O}$ with $x \in U$, $U \cap A \neq \emptyset$.*

Exercise 10. *Let $A \subseteq X$, and let B be the set of limit points of A. Prove that $B = \bar{A}$.*

1.6. **Continuity.** Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces.

Definition 6. *We say that $f : X \to Y$ is continuous if for every $U \in \mathcal{O}_Y$, $f^{-1}(U) \in \mathcal{O}_X$.*

The way this is usually put is that f is continuous if the preimage of every open set is open.

Exercise 11. *Show that for a function $f : \mathbb{R} \to \mathbb{R}$, definitions 1 and 6 are equivalent (when $\mathbb{R}$ is endowed with the standard topology).*

California Institute of Technology