PS 12 NOTES: WEEK 3

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1. Constant-sum games

Recall the matching pennies game from the week 2 notes:

Matching pennies

		Player 2	
		H	T
Player 1	H	$1, 0$	$0, 1$
	T	$0, 1$	$1, 0$

Notice that if you add up the payoffs in each cell, you always get 1. A game like this, where the sum of the players' payoffs is always the same no matter what they do, is called a *constant-sum game.*

To examine this game, let us imagine for the moment that player 1 will announce her strategy first, then player 2 will choose a strategy in response. For any $0 \leq p \leq 1$, let's denote by σ_p the strategy for player 1 that assigns probability p to H and probability $1 - p$ to T. How would player 2 respond if player 1 played σ_p? If player 2 plays H, her expected payoff is $1 - p$, and if she plays T, her expected payoff is p. So

- if $p < 1/2$, player 2 will play H
- if $p = 1/2$, player 2 is indifferent (no matter what she does, even if she randomizes, her expected payoff will be $1/2$)
- if $p > 1/2$, player 2 will play T

Anticipating player 2's responses, player 1 knows what her payoff will be:

- if $p < 1/2$, her expected payoff will be p
- if $p = 1/2$, her expected payoff will be $1/2$
- if $p > 1/2$, her expected payoff will be $1 - p$

So then what should player 1 do? In the $p < 1/2$ and $p > 1/2$ cases, her payoff will be less than $1/2$, whereas in the $p = 1/2$ case, her payoff will be $1/2$. Hence, player 1 will play $\sigma_{\frac{1}{2}}$.

1.1. Maximin strategy.

When considering what player 2 would do in response to player 1's choice, we looked for the response that would give player 2 the highest payoff. Now, since the payoffs always add to 1, finding the response that would give player 2 the highest payoff is the same as finding the response that would give player 1 the lowest payoff. So when considering what her payoff would be for playing σ_p, player 1 just needed to find the lowest possible payoff she could get when playing that strategy. Hence, when choosing a strategy, player 1 should find the strategy σ_p whose lowest possible payoff is as high as possible. In other words, player 1's optimal strategy is the one that maximizes her minimum possible expected payoff (where the minimum is taken over all responses player 2 could play). A strategy like this, which maximizes a player's minimum possible expected payoff, is called a *maximin strategy.* The observations that we have just made tell us that in any equilibrium, player 1 must be using a maximin strategy:

Theorem 1. *In any equilibrium of a two-player constant-sum game, every player must be using a maximin strategy. Moreover, every strategy profile in which both players are playing maximin strategies is an equilibrium.*

This means that in equilibrium, player 1's strategy must be $\sigma_{\frac{1}{2}}$. But since the game is symmetric (the game is the same from both player's points of view), the same analysis implies that in equilibrium, player 2's strategy must be $\sigma_{\frac{1}{2}}$. So the only equilibrium of this game is the strategy profile where both players use the strategy $\sigma_{\frac{1}{2}}$.

1.2. **Zero-sum games.** An important special case of constant-sum games is *zero-sum games* (constant-sum games where the player's payoffs always add to 0), especially symmetric two-player zero-sum games. Here we record an important observation later use: it is an equilibrium for both player to use the same maximin strategy, and since their expected payoff in this equilibrium must be the same and add to 0, it must be the case that both players get expected payoff 0 in equilibrium.

Theorem 2. *In a symmetric two-player zero-sum game, both players have expected payoff* 0 *in every equilibrium.*

2. Repeated games

When we analyze normal form games, we imagine that players interact a single time and act simultaneously. However, political and economic actors often engage in similar interactions repeatedly. We will use *repeated games* to model this kind of repeated interaction. A repeated game has

- a base (normal form) game
- a number of periods T that can be either finite or ∞
- a discount factor $0 < \delta < 1$.

The repeated game then works as follows. In each time period $t \leq T$, the players play the base game. At the "end", the players add up the payoffs they got in each period weighted by δ^t.

Consider, for example, the repeated game with $T = 3$, $\delta = .9$, and with the Prisoner's Dilemma as the base game:

Prisoner's Dilemma

		Player 2: C	Player 2: D
Player 1	C	$2, 2$	$-1, 3$
	D	$3, -1$	$0, 0$

If player 1 plays

- C in period 1
- D in period 2
- C in period 3

and player 2 plays

- D in period 1
- D in period 2
- C in period 3

then the period payoffs for the players would be

- -1 for player 1 and 3 for player 2 in period 1
- 0 for player 1 and 0 for player 2 in period 2
- 2 for player 1 and 2 for player 2 in period 3

So player 1's overall payoff would be

$$-1 \cdot (.9)^0 + 0 \cdot (.9)^1 + 2 \cdot (.9)^2 = .62$$

and player 2's overall payoff would be

$$3 \cdot (.9)^0 + 0 \cdot (.9)^1 + 2 \cdot (.9)^2 = 4.62.$$

2.1. **Repeated Prisoner's Dilemma.** Let's consider what subgame perfect equilibria are possible in the repeated Prisoner's Dilemma games (games with Prisoner's Dilemma as the base game). As we noted in week 2, the only equilibrium of Prisoner's Dilemma is for both players to play D. Now, since the very last period of repeated Prisoner's Dilemma is always identical to Prisoner's Dilemma, it follows that in any subgame perfect equilibrium of repeated Prisoner's Dilemma, both players must play D in the last period. Now, given this, the players know that in the second to last period of the game, their actions cannot affect what happens in the last period. Hence, the second to last period is now also identical to Prisoner's Dilemma, so again both players must play D. Working backwards, we can deduce:

Proposition 3. *In any finitely repeated Prisoner's Dilemma, the only subgame perfect equilibrium is for both players to play D every period no matter what.*

What about when the game is infinitely repeated (ie $T = \infty$)? Consider the following strategy, called the "grim trigger" strategy:

- In the first period, play C
- In all subsequent periods, if the other player played D in some previous period, play D; otherwise, play C

In words, the strategy is to play C as long as the other player plays C, then switch to playing D forever if the other player ever plays D. Intuitively, the idea is to cooperate, and to sustain the other player's cooperation by threatening to punish them by defecting forever if they ever defect. Can this work?

Let's analyze what happens if both players play this strategy. In this case, player 1's payoff is

$$\sum_{t=1}^{\infty} 2 \cdot \delta^{t-1} = \frac{2}{1-\delta}$$

If player 1 plays D in the first period, then she should also play D in all subsequent periods, since she knows that player 2 will play D in all subsequent periods regardless of what she does. Her payoff from doing this is

$$3 + \sum_{t=2}^{\infty} 0 \cdot \delta^{t-1} = 3.$$

So if $\delta \geq 1/3$, playing D in the first period can only decrease player 1's payoff. Moreover, if one of the players ever decides to play D forever, the other player's unique best response is to play D forever as well.

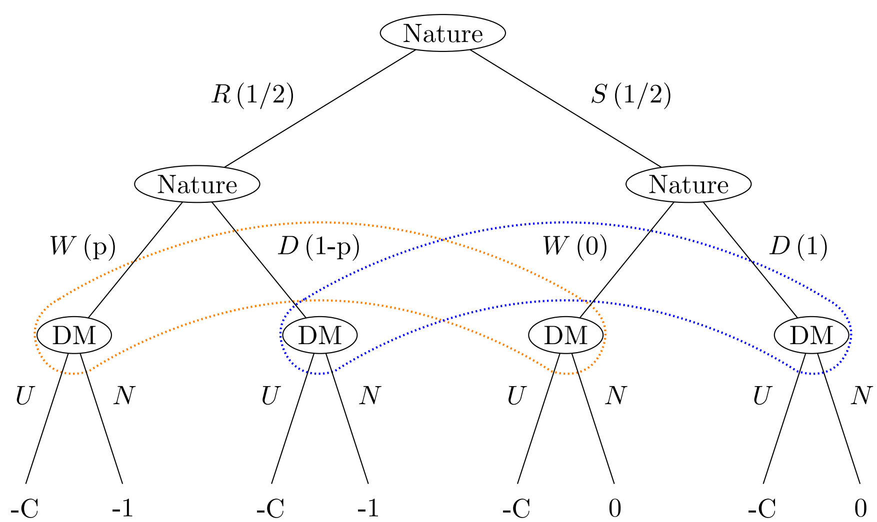
What have we learned? The strategy profile where both players play the grim trigger strategy is a subgame perfect equilibrium (as long as $\delta \geq 1/3$); moreover, it is a subgame perfect equilibrium in which both players always play C!

3. Games with incomplete information

In the first week, we discussed imperfect information as a model for the fact that agents do not always know the actions already taken by other agents. However, it is not just the actions taken previously that agents do not always know. It is possible that agents could know all of the actions taken by agents previously and still lack important information. For example, when playing poker, all actions taken by all players are publicly observed; nevertheless, players do not know each other's cards. We call games like this *games with incomplete information.*

3.1. Decision problems with incomplete information.

Here is a simple example of a decision problem involving incomplete information. It will be rainy (R) or sunny (S), and the probability of each is $1/2$. The decision maker (DM) will see a forecast, which will either be wet (W) or dry (D). If it is going to be rainy, the forecast will be wet with probability p and dry with probability $1-p$. If it is going to be sunny, the forecast will be dry with probability 1. Once the decision maker sees the forecast, she decides whether to bring her umbrella with her. If she brings the umbrella (U), she gets a payoff $-C$, where $0 < C < 1$. If she does not bring the umbrella (N), then she gets a payoff of -1 if it is rainy and 0 if it sunny. Here is an extensive form to represent the situation:



So what should the decision maker do? Clearly, the answer depends on how likely it is to rain, so we need to know how likely the decision maker finds rain given the forecast she sees.

If the forecast is wet, then the decision maker should definitely bring the umbrella, since the forecast wet only occurs when it is rainy! What if the forecast is dry? We need to figure out what probability the decision maker assigns rainy after seeing a dry forecast: $\mathbb{P}(R \mid D)$. To do this, we will need to make use of Bayes' rule. As a reminder, for any events A and B, Bayes' rule is:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A) \cdot \mathbb{P}(A)}{\mathbb{P}(B)}$$

So using Bayes' rule,

$$\mathbb{P}(R \mid D) = \frac{\mathbb{P}(D \mid R) \cdot \mathbb{P}(R)}{\mathbb{P}(D)}$$

We know $\mathbb{P}(D \mid R) = 1 - p$ and $\mathbb{P}(R) = 1/2$. What about $\mathbb{P}(D)$, the total probability of the dry forecast? The paths that lead to the information set where the decision maker sees the dry forecast are RD and SD, so

$$\mathbb{P}(D) = \mathbb{P}(R,\, D) + \mathbb{P}(S,\, D) = \frac{1}{2} \cdot (1 - p) + \frac{1}{2} \cdot 1 = \frac{2 - p}{2}$$

So

$$\mathbb{P}(R \mid D) = \frac{(1 - p) \cdot \frac{1}{2}}{\frac{2-p}{2}} = \frac{1 - p}{2 - p}$$

and

$$\mathbb{P}(S \mid D) = 1 - \mathbb{P}(R \mid D) = \frac{1}{2 - p}$$

So what should the decision maker do if the forecast is dry? If she brings the umbrella, her expected payoff will be

$$\frac{1 - p}{2 - p} \cdot -C + \frac{1}{2 - p} \cdot -C = -C$$

If she does not bring the umbrella, her expected payoff will be

$$\frac{1 - p}{2 - p} \cdot -1 + \frac{1}{2 - p} \cdot 0 = -\frac{1 - p}{2 - p}$$

So she should bring the umbrella if

$$-C > -\frac{1 - p}{2 - p}$$

or equivalently, if

$$p > 2C - 1$$

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