Optimality of the coordinate-wise median mechanism for strategyproof facility location in two dimensions *

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Abstract

We consider the facility location problem in two dimensions. In particular, we consider a setting where agents have Euclidean preferences, defined by their ideal points, for a facility to be located in $\mathbb{R}^2$. We show that for the $p-norm$ ($p \geq 1$) objective, the coordinate-wise median mechanism (CM) has the lowest worst-case approximation ratio in the class of deterministic, anonymous, and strategyproof mechanisms. For the minisum objective and an odd number of agents n, we show that CM has a worst-case approximation ratio (AR) of $\sqrt{2}\frac{\sqrt{n^2+1}}{n+1}$. For the $p-norm$ social cost objective ($p \geq 2$), we find that the AR for CM is bounded above by $2^{\frac{3}{2}-\frac{2}{p}}$. We conjecture that the AR of CM actually equals the lower bound $2^{1-\frac{1}{p}}$ (as is the case for $p = 2$ and $p = \infty$) for any $p \geq 2$.

1 Introduction

We consider the problem of locating a facility on a plane where a set of strategic agents have private preferences over the facility location. Each agent's preference is defined by its ideal point so that the cost incurred by an agent equals the Euclidean distance between the facility location and the ideal point. A central planner wishes to locate the facility to minimize the social cost. Since agents may lie about their ideal points if it benefits them, the planner is constrained to choose a mechanism that is strategyproof. In this paper, we consider the problem of finding the strategyproof mechanism that best approximates the optimal social cost as measured by the worst-case approximation ratio (AR) and quantifying

its performance.

We find that for an odd number of agents n and the $p-norm$ objective with $p \geq 1$, the coordinate-wise median mechanism is optimal in the class of deterministic, anonymous, and strategyproof mechanisms. For the utilitarian social objective of minimizing the sum of individual costs ($p = 1$), we show that the coordinate-wise median mechanism has an AR of $\sqrt{2}\frac{\sqrt{n^2+1}}{n+1}$. For the general $p-norm$ objective ($p \geq 2$), we show that the asymptotic AR of the coordinate-wise median mechanism is bounded between $2^{1-\frac{1}{p}}$ and $2^{\frac{3}{2}-\frac{2}{p}}$. We conjecture that the asymptotic AR of the coordinate-wise median mechanism is actually equal to the lower bound $2^{1-\frac{1}{p}}$ (as is the case when $p = 2$ or $p = \infty$).

This problem has been extensively studied in the literature known as *Approximate Mechanism Design without money*. It was first introduced by Procaccia and Tennenholtz [28] who studied the setting of locating a single facility on a real line under the utilitarian (sum of individual costs) and egalitarian (maximum of individual costs) objectives. Since then, the problem has received much attention, with extensions to alternative objective functions, multiple facilities, obnoxious facilities, different networks, etc. Cheng and Zhou [8] and more recently, Chan et al. [7] provide surveys of results in the last decade in several of these settings. In the class of deterministic strategyproof mechanisms for locating a facility, the median mechanism has been shown to be optimal under various objectives and domains [Procaccia and Tennenholtz [28], Feigenbaum et al. [12], Feldman and Wilf [14], Feldman et al. [13]].

There has been some related work in extending the problem to multiple dimensions. Meir [23] shows that in the d-dimensional Euclidean space, the approximation ratio of the coordinate-wise median mechanism for the utilitarian objective is bounded above by $\sqrt{d}$. Sui et al. [31] propose percentile mechanisms for locating multiple facilities in Euclidean space which are further analysed in Sui and Boutilier [30] and Walsh [33]. Meir [23], using techniques different from ours, finds the AR of coordinate-wise median mechanism under the minisum objective for the case of 3 agents. Gershkov et al. [17] shows that for some natural priors on the ideal points (that include i.i.d. marginals), taking the coordinate-wise median after a judicious rotation of the orthogonal axes can lead to welfare improvements under the least-squares objective. In other related work, El-Mhamdi et al. [11] find that the mechanism choosing the minisum optimal location (geometric median) is approximately strategyproof in a large economy. Lee Brady and Chambers [21] find that the geometric median is Nash-implementable and in the case of three agents, it is the unique rule that satisfies anonymity, neutrality, and Maskin-Monotonicity. Durocher and Kirkpatrick [10] and Bespamyatnikh et al. [3] analyse approximations to geometric median due to its instability and computational difficulty.

There is also a large literature in social choice theory on characterizing the set of strategyproof mechanisms under different assumptions on preference domains [Gibbard [18], Sat-

terthwaite [29], Moulin [24]]. In multiple dimensions with Euclidean preferences, the characterizations typically include or are completely described by the coordinate-wise median mechanism [Kim and Roush [19], Border and Jordan [5], Peters et al. [25], Peters et al. [27]]. Our work augments this literature, which provides strong axiomatic foundations for the coordinate-wise median mechanism, by demonstrating its quantitative optimality.

The paper proceeds as follows. In section 2, we formally define the problem and state some characterisation results and approximation results from the literature that will be useful in our analysis. In section 3, we discuss the optimality of the coordinate-wise median mechanism. In sections 4 and 5, we discuss the problem of finding the approximation ratio of the coordinate-wise median mechanism for the utilitarian objective and the p-norm objective. Section 6 concludes.

2   Preliminaries

Suppose (X, d) is a metric space. There are n agents and each agent has an ideal point $x_i \in X$ for a facility to be located in X. The cost of locating the facility at $y \in X$ for agent i is $d(y, x_i)$. Let $\mathbf{x}$ be the profile of ideal points: $\mathbf{x} = (x_1, \ldots, x_n)$. The social cost of locating the facility at y under profile $\mathbf{x} \in X^n$ is given by the *social cost function* $sc : X \times X^n \to \mathbb{R}$. Let $OPT(sc, \mathbf{x})$ denote the set of minimizers for sc given $\mathbf{x}$:

$$OPT(sc, \mathbf{x}) = \text{argmin}_y sc(y, \mathbf{x}).$$

When $OPT(sc, \cdot)$ is singleton-valued, we will abuse notation and use $OPT(sc, \mathbf{x})$ to refer to the unique element contained therein. When sc is clear from context, we will suppress the first argument and write $OPT(sc, \mathbf{x})$ simply as $OPT(\mathbf{x})$.

A *mechanism* is a function $f : X^n \to X$. It is said to be *strategyproof* if no agent can benefit by misreporting her ideal point, regardless of the reports of the other agents. Formally:

Definition 2.1. A mechanism f is strategyproof if for all $i \in N$, $x_i, x_i' \in X$, $x_{-i} \in X^{n-1}$,

$$d(f(x_i, x_{-i}), x_i) \leq d(f(x_i', x_{-i}), x_i).$$

Definition 2.2. A mechanism f is anonymous if for any permutation $\pi : [n] \to [n]$,

$$f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$$

To measure how closely a mechanism approximates the optimal social cost for a given profile, we use the approximation ratio.

Definition 2.3. For a social cost function sc, the approximation ratio of a mechanism f at a profile $\mathbf{x}$ is given by

$$AR_f(\mathbf{x}) = \frac{sc(f(\mathbf{x}), \mathbf{x})}{sc(OPT(\mathbf{x}), \mathbf{x})}.$$

In the case that $sc(OPT(\mathbf{x}), \mathbf{x}) = 0$, we take $AR_f(\mathbf{x})$ to be 1 if $sc(f(\mathbf{x}), \mathbf{x}) = 0$ and ∞ otherwise.

To compare mechanisms, we will evaluate them by their worst-case approximation ratio.

Definition 2.4. The worst-case approximation ratio of a mechanism f is given by

$$AR(f) = \sup_{\mathbf{x}} AR_f(\mathbf{x}).$$

Given a metric space (X, d) and a social cost function sc, the problem is to find a strategyproof mechanism with the smallest worst-case approximation ratio.

In this paper, we consider the Euclidean metric space with $X = \mathbb{R}^2$ and the p-norm social cost function which is the L_p norm of the vector of Euclidean distances $sc(y, \mathbf{x}) = [\sum \|y - x_i\|^p]^{\frac{1}{p}}$ where $p \geq 1$. We refer to the coordinates of points in $\mathbb{R}^2$ by a and b. We refer to the sets $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}$ as the a-axis and the b-axis, respectively. We refer to the sets $\pm\mathbb{R}_{\geq 0} \times \{0\}$ and $\{0\} \times \pm\mathbb{R}_{\geq 0}$ as the $\pm a$-axes and $\pm b$-axes, respectively. We use the notation $[y, z]$ to denote the line segment joining y and z: $\{ty + (1 - t)z : t \in [0, 1]\}$. Similarly, we denote by (y, z) the set $[y, z] \setminus \{y, z\}$.

Our analysis makes use of some previous results regarding characterization of strategyproof mechanisms and bounds on approximation ratios in the Euclidean domain. We collect those results here.

2.1 Characterisation results in two dimensions

First, let's define an important class of mechanisms in this domain.

Definition 2.5. In the Euclidean metric space with $X = \mathbb{R}^m$, a mechanism f is called a *generalized coordinate-wise median mechanism* with k constant points if there exists a coordinate system and points $c_1, c_2, \ldots, c_k \in (\mathbb{R} \cup \{-\infty, \infty\})^m$ so that for every profile $\mathbf{x} \in (\mathbb{R}^m)^n$ and every dimension $j = 1, 2, \ldots, m$, the jth coordinate of f is given by

$$f^j(\mathbf{x}) := \text{med}\left(x_1^j, x_2^j, \ldots, x_n^j, c_1^j, \ldots, c_k^j\right)$$

where "med" denotes the median of the subsequent real numbers.

This class of mechanisms has strong axiomatic foundations in the literature as illustrated in the following lemma:

Lemma 1 (Kim and Roush [19], Peters et al. [27, 25]). *In the Euclidean metric space with $X = \mathbb{R}^2$ and an odd number of agents n, a mechanism $f : (\mathbb{R}^2)^n \to \mathbb{R}^2$ is*

- *(Kim and Roush [19]) continuous, anonymous, and strategyproof if, and only if, f is a generalized coordinate-wise median mechanism with $n + 1$ constant points.*

- *(Peters et al. [25]) unanimous, anonymous, and strategyproof if, and only if, f is a generalized coordinate-wise median mechanism with $n - 1$ constant points.*

- *(Peters et al. [27]) Pareto optimal, anonymous, and strategyproof if, and only if, f is a generalized coordinate-wise median mechanism with 0 constant points.*

We refer to the generalized coordinate-wise median mechanism with 0 constant points and the standard coordinate-system as the *coordinate-wise median mechanism* and denote it by $c(\mathbf{x})$.

One subclass of generalized coordinate-wise median mechanisms that will play an important role in demonstrating the optimality of the coordinate-wise median mechanism is the following:

Definition 2.6. In the Euclidean metric space with $X = \mathbb{R}^m$, a mechanism f is called a *coordinate-wise quantile mechanism* if it is a generalized coordinate-wise mechanism where all the k constant points $c_1, c_2, \ldots, c_k \in \{-\infty, \infty\}^m$.

Note that if $|\{i : c_i^j = -\infty\}| = \ell$, then $f^j(x_1, \ldots, x_n)$ is the $\frac{n+k+1}{2} - \ell$ order statistic of the (multi)set $\{x_1^j, \cdots, x_n^j\}$. Hence, given a profile $\mathbf{x}$, every coordinate-wise median quantile mechanism locates the facility by selecting, for each dimension j, some fixed quantile of the ordered projection of $\mathbf{x}$ in the jth dimension as the coordinate of the facility location.

2.2 Approximation results in two dimensions

For the case of $X = \mathbb{R}^2$ with the Euclidean metric, there has been some work in finding bounds on AR for the utilitarian objective $sc(y, \mathbf{x}) = \sum \|y - x_i\|$. We discuss those findings here.

A point minimizing the sum of distances from a finite set of points in $\mathbb{R}^2$ is known as a *geometric median* for that set of points. The geometric median is characterised by the following result:

Lemma 2. *Given $\mathbf{x} \in (\mathbb{R}^2)^n$, a point $y \in \mathbb{R}^2$ is a geometric median for $\mathbf{x}$ if and only if there are vectors $u_1, \ldots, u_n$ such that*

$$\sum_{i=1}^{n} u_i = 0$$

where for $x_i \neq y$, $u_i = \frac{x_i - y}{\|x_i - y\|}$ and for $x_i = y$, $\|u_i\| \leq 1$.

This characterisation yields conditions under which changing a profile of points does not change the geometric median, as summarized in the following corollary:

Corollary 1. *Let $\mathbf{x} \in (\mathbb{R}^2)^n$, and denote by y the geometric median of $\mathbf{x}$. For any i, if $x_i \neq y$ and if $x_i' \in \{y + t(x_i - y) \mid t \in \mathbb{R}_{\geq 0}\}$, then the geometric median for the profile (x_i', x_{-i}) is also y.*

Informally, moving a point directly away from or directly towards (but not past) the geometric median leaves the geometric median unchanged. We will use this observation repeatedly in the sequel and note here that in fact it will be the only characteristic of the geometric median that we use for much of the paper. We refer to the geometric median by $g(\mathbf{x})$.

It follows from Lemma 1 that the geometric median mechanism is not strategyproof. Meir [23] finds an upper bound on the AR of the coordinate-wise median mechanism in the $m-$dimensional problem:

Lemma 3 (Meir [23]). *For $X = \mathbb{R}^m$ and the utilitarian objective $sc(y, \mathbf{x}) = \sum \|y - x_i\|$, the coordinate-wise median mechanism has an approximation ratio of at most $\sqrt{m}$ for any number of agents n.*

Feigenbaum et al. [12] consider the facility location problem for $X = \mathbb{R}$ and $d(x_i, x_j) = |x_i - x_j|$ with the social cost function $sc(y, \mathbf{x}) = [\sum |y - x_i|^p]^{\frac{1}{p}}$.

Lemma 4 (Feigenbaum et al. [12]). *For $X = \mathbb{R}$ and the p-norm objective $sc(y, \mathbf{x}) = [\sum |y - x_i|^p]^{\frac{1}{p}}$ with $p \geq 1$, the median mechanism has an approximation ratio of $2^{1-1/p}$. Further, any deterministic strategyproof mechanism has approximation ratio of at least $2^{1-1/p}$.*

3 Optimality of the coordinate-wise median mechanism

Our first major finding is that the coordinate-wise median mechanism is optimal with respect to the worst-case approximation ratio for the class of social cost functions we study.

Theorem 1. *For $X = \mathbb{R}^2$, and the p-norm objective $sc(y, \mathbf{x}) = [\sum \|y - x_i\|^p]^{\frac{1}{p}}$ where $p \geq 1$, the coordinate-wise median mechanism has the lowest approximation ratio among all deterministic, anonymous, and strategyproof mechanisms.*

To prove Theorem 1, we will show that for every deterministic, anonymous, and strategyproof mechanism f, there is a coordinate-wise quantile mechanism Q such that $AR(f) \geq AR(Q) \geq AR(CM)$. In the case that f is not unanimous, $AR(f) = \infty > AR(CM)$. In the case that f is unanimous, it follows from Lemma 1 that f is a generalized coordinate-wise median mechanism with $n-1$ constant points. Thus, to prove the theorem, we will show that for every such mechanism there is a coordinate-wise quantile mechanism with a lower AR (Lemma 5) and that CM has the lowest AR among all coordinate-wise quantile mechanisms (Lemma 6).

Lemma 5. *Let f be a generalized coordinate-wise median mechanism with $n - 1$ constant points. Then for any p-norm objective sc, there is some coordinate-wise quantile mechanism Q such that $AR(f) \geq AR(Q)$.*

Proof. Let $c_1, \ldots, c_{n-1}$ be the constant points for f. Let Q be the coordinate-wise quantile mechanism with constant points $q_1, \ldots, q_{n-1}$, where for each i, $q_i^j = \infty$ if $c_i^j = \infty$ and $q_i^j = -\infty$ otherwise. Now we'll show that $AR(f) \geq AR(Q)$.

Let $z^1 \in \mathbb{R}$ such that for every i, either $c_i^1 < z^1$ or $c_i^1 = \infty$, and similarly, let $z^2 \in \mathbb{R}$ such that for every i, either $c_i^2 < z^2$ or $c_i^2 = \infty$. Then for any $\mathbf{x}$ such that $x_i^1 > z^1$ and $x_i^2 > z^2$ for all i, it follows immediately from the definition of the q_i that

$$med(x_1^j, \ldots, x_n^j, q_1^j, \ldots, q_{n-1}^j) = med(x_1^j, \ldots, x_n^j, c_1^j, \ldots, c_{n-1}^j).$$

Defining $T = ([z^1, \infty] \times [z^2, \infty])^n$, it hence follows that $f(\mathbf{x}) = Q(\mathbf{x})$ for all $\mathbf{x} \in T$, and thus that $AR_Q(\mathbf{x}) = AR_f(\mathbf{x})$ for all $\mathbf{x} \in T$.

In addition, we note that for any $\mathbf{x} \in (\mathbb{R}^2)^n$, if $\mathbf{x}'$ is a translation of $\mathbf{x}$ (i.e. there is some $\Delta x \in \mathbb{R}^2$ such that $x_i' = x_i + \Delta x$ for all i), then $AR_Q(\mathbf{x}) = AR_Q(\mathbf{x}')$.

Putting these observations together, it then follows that

$$\begin{aligned}
AR(Q) &= \sup_{\mathbf{x} \in (\mathbb{R}^2)^N} AR_Q(\mathbf{x}) \\
&= \sup_{\mathbf{x} \in T} AR_Q(\mathbf{x}) \\
&= \sup_{\mathbf{x} \in T} AR_f(\mathbf{x}) \\
&\leq \sup_{\mathbf{x} \in (\mathbb{R}^2)^N} AR_f(\mathbf{x}) \\
&= AR(f).
\end{aligned}$$

$\square$

Lemma 6. *Let Q be a coordinate-wise quantile mechanism. Then for any p-norm objective sc, $AR(Q) \geq AR(CM)$.*

Proof. Since $Q = Q_1$ is a coordinate-wise quantile mechanism, there exist order statistics $(k_1, k_2) \in [n] \times [n]$ such that Q locates the facility by selecting, for each dimension, the k_ith order statistic of the projection of $\mathbf{x}$ in the ith dimension as the coordinate of the facility in the ith dimension.

Now consider the coordinate-wise quantile mechanisms Q_2, Q_3, Q_4 defined by the order statistics $(k_1, n + 1 - k_2), (n + 1 - k_1, k_2), (n + 1 - k_1, n + 1 - k_2)$ respectively. As these mechanisms are isomorphic (each can be obtained from the others by composition with a series of reflections across axes), they all have the same approximation ratio; that is, $AR(Q_i) = AR(Q)$ for each i. Observe that for any profile $\mathbf{x} \in (\mathbb{R}^2)^n$, $CM(\mathbf{x})$ is in the convex hull of $Q_1(\mathbf{x}), Q_2(\mathbf{x}), Q_3(\mathbf{x}), Q_4(\mathbf{x})$. Hence, since $sc(\mathbf{x}, y)$ is quasi-convex as a function

of y,[1] $sc(\mathbf{x}, CM(x)) \leq sc(\mathbf{x}, Q_\ell(x))$ for some $\ell \in 1, 2, 3, 4$, and so $AR_{CM}(\mathbf{x}) \leq AR_{Q_\ell}(\mathbf{x}) \leq AR(Q_\ell) = AR(Q)$. Thus, $AR_{CM}(\mathbf{x}) \leq AR(Q)$ for all $\mathbf{x}$, and so $AR(CM) \leq AR(Q)$. $\qquad \square$

Remark 2. *The techniques used to prove Theorem 1, together with characterization results for the one-dimensional facility location problem (Moulin [24]), can also be used to prove that the median mechanism is also optimal for any n odd and any $p \geq 1$. This strengthens the result in Feigenbaum et al. [12] (Lemma 4), which demonstrates there is no mechanism that is asymptotically superior to the median mechanism.*

Remark 3. *Lemma 5 and Lemma 6 both hold for larger classes of social cost functions than p-norms. For Lemma 5, it is sufficient that sc depends only on the distances to the facility, and for Lemma 6, it is sufficient that sc is quasiconvex. It follows that Theorem 1 holds for a much more general class of social cost functions. For instance, it holds when the planner's objective is to minimize a weighted sum of distances $sc(y, \mathbf{x}) = \sum_{i=1}^{n} \lambda_i \|y - x_i\|$.*

4   The minisum objective

In this section, we quantify exactly the approximation ratio for the coordinate-wise median mechanism under the minisum ($p = 1$) objective $sc(y, \mathbf{x}) = \sum_{i=1}^{n} \|y - x_i\|$. By Theorem 1, it follows that this quantity provides a lower bound for the approximation ratio of any deterministic, anonymous, and strategyproof mechanism under the minisum objective.

Theorem 2. *For n odd, $X = \mathbb{R}^2$, and $sc(y, \mathbf{x}) = \sum_{i=1}^{n} \|y - x_i\|$,*

$$AR(CM) = \sqrt{2} \frac{\sqrt{n^2 + 1}}{n + 1}.$$

For n odd[2], the geometric median $g(\mathbf{x})$ is unique and $OPT(\mathbf{x}) = g(\mathbf{x})$. Hence, Theorem 2 amounts to finding how well the social cost of the coordinate-wise median mechanism approximates the social cost of the geometric median in the worst case.

The argument for obtaining the exact value of $AR(CM)$ is rather involved. We provide a full proof for the case that $n = 3$ as we find the approach taken in its proof to be simple enough to be digestible yet sufficiently similar to the more nuanced approach required for arbitrary odd n as to be illuminating. We then provide a sketch of the proof for all odd n, relegating the formal proof for this case to the appendix.

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[1]Since $sc(\mathbf{x}, y)^p$ is convex as a function of y and the composition of a convex function with a nondecreasing function is quasi-convex, $sc(\mathbf{x}, y) = (sc(\mathbf{x}, y)^p)^{\frac{1}{p}}$ is quasi-convex as a function of y.

[2]When $n = 2m$ is even, the version of the coordinate-wise median mechanism given by $c(\mathbf{x}) = (\text{median}(-\infty, \mathbf{a}), \text{median}(-\infty, \mathbf{b}))$ has worst-case approximation ratio *equal* to $\sqrt{2}$. This follows from the bound in Lemma 3 and the worst-case profile $\mathbf{x}$ where $x_1 = x_2 \ldots x_m = (1, 0)$ and $x_{m+1} = x_{m+2} \ldots x_{2m} = (0, 1)$.

In both the $n = 3$ case and the general case, the key to the proof is to reduce the search space for the worst-case profile from $(\mathbb{R}^2)^n$ to a much smaller space of profiles that have a simple structure. In many cases, this involves "transforming" one profile into another profile that has a higher approximation ratio and a simpler structure. One important transformation that helps in significantly reducing the search space involves moving a point x_i directly towards $g(\mathbf{x})$, getting as close as possible to $g(\mathbf{x})$ without changing $c(\mathbf{x})$. Because this transformation will be used repeatedly throughout this section, we provide here a proof that this transformation leads to a profile (x_i', x_{-i}) with a weakly higher approximation ratio.

Lemma 7 (Towards geometric median). *Let $\mathbf{x}$ be a profile and $i \in N$, and let $\mathbf{x}'$ be any profile such that*

1. *$x_i' \in [x_i, g(\mathbf{x})]$,*

2. *for all $j \neq i$, $x_j' = x_j$, and*

3. *$c(\mathbf{x}') = c(\mathbf{x})$.*

Then $AR(\mathbf{x}') \geq AR(\mathbf{x})$ where $AR(\mathbf{x}) = \frac{sc(c(\mathbf{x}), \mathbf{x})}{sc(g(\mathbf{x}), \mathbf{x})}$

Proof. By corollary 1, $g(\mathbf{x}') = g(\mathbf{x})$ and by definition, $c(\mathbf{x}') = c(\mathbf{x})$. The change in optimal social cost is given by $\|x_i - x_i'\|$ while the change in social cost with respect to coordinate-wise median is $\|c(\mathbf{x}) - x_i'\| - \|c(\mathbf{x}) - x_i\|$. By triangle inequality, $\|x_i - x_i'\| \geq \|c(\mathbf{x}) - x_i'\| - \|c(\mathbf{x}) - x_i\|$. Thus, the $sc(OPT(\cdot), \cdot)$ reduces by a greater amount than $sc(CM(\cdot), \cdot)$ as we move x_i to x_i'. Since the ratio is always at least 1, it follows that $AR(\mathbf{x}') \geq AR(\mathbf{x})$. $\qquad\square$

4.1 Proof for $n = 3$ case

Corollary 4. *For $n = 3$, the worst-case approximation ratio for the coordinate-wise median mechanism is given by:*

$$AR(CM) = \frac{\sqrt{5}}{2}.$$

Remark 5. *There is a more explicit characterisation of the geometric median when $n = 3$. In this case, if any angle of the triangle formed by the three points is at least 120°, $g(\mathbf{x})$ lies on the vertex of that angle; otherwise, it is the unique point inside the triangle that subtends an angle of 120° to all three pairs of vertices*

Proof of Theorem 2 for $n = 3$. Define the set of *Centered perpendicular (CP)* profiles as follows:

$$CP = \{\mathbf{x} \in (\mathbb{R}^2)^3 : c(\mathbf{x}) = (0, 0) \text{ and } \forall i, \text{ either } a_i = 0 \text{ or } b_i = 0\}.$$

In words, a profile is in CP if the coordinate-wise median is at the origin and all points in $\mathbf{x}$ are on the axes.

Define the set of *Isosceles-centered perpendicular (I-CP)* profiles as follows:

$$I - CP = \{\mathbf{x} \in CP : \exists t \text{ such that } \mathbf{x} = ((t,0),(-t,0),(0,1)) \text{ and } g(\mathbf{x}) = (0,1)\}$$

In words, a profile is in $I - CP$ if there are two points on the a-axis equidistant from the origin and the third point is at $(0,1)$, which is also the geometric median.

We first show that we can reduce the search space for the worst-case profile from $(\mathbb{R}^2)^3$ to CP.

Lemma 8. *For any profile $\mathbf{x} \in (\mathbb{R}^2)^3$, there is a profile $\chi \in CP$ such that $AR(\chi) \geq AR(\mathbf{x})$.*

Proof. Let $\mathbf{x} \in (\mathbb{R}^2)^3$ be a profile. Let $\mathbf{x}'$ be the profile where $x_i' = x_i - c(\mathbf{x})$. Then $\mathbf{x}'$ has the same approximation ratio as $\mathbf{x}$ and $c(\mathbf{x}') = (0,0)$. Denote $A = \{i : a_i = 0\}$ and $B = \{i : b_i = 0\}$. Note that since $c(\mathbf{x}') = (0,0)$, it follows from the definition of $c(\mathbf{x}')$ that $A \neq \emptyset$ and $B \neq \emptyset$. For each i, define x_i'' as follows. Let $\Gamma = \{(a,b) \in \mathbb{R}^2 : a = 0 \text{ or } b = 0\}$. If $i \in A \cup B$, let $x_i'' = x_i'$; otherwise, let x_i'' be the point in $[x_i', g(\mathbf{x}')] \cap \Gamma^3$ that is closest to x_i'. Then $x_i'' \in \Gamma$ for all i and $c(\mathbf{x}'') = (0,0)$, so $\mathbf{x}'' \in CP$. Further, it follows from Lemma 7 that $AR(\mathbf{x}'') \geq AR(\mathbf{x}') = AR(\mathbf{x})$; hence, taking $\chi = \mathbf{x}''$ completes the proof. $\qquad\square$
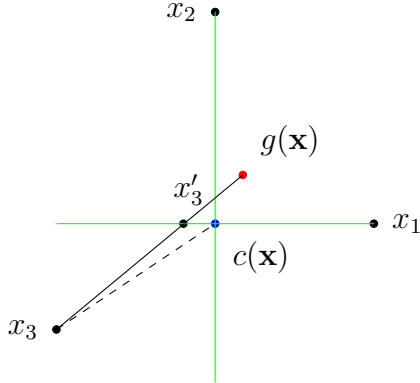

Figure 4.1: Towards geometric median

Now we show that we can further reduce the search space from CP to $I - CP$.

Lemma 9. *For any profile $\mathbf{x} \in CP$, there exists a profile $\chi \in I - CP$ such that $AR(\chi) \geq AR(\mathbf{x})$.*

Proof. Let $\mathbf{x}$ be a profile in CP.

Without loss of generality, we may assume that all x_i are weakly above the a-axis and there are at least two x_i on the a-axis, since reflecting a profile in CP across the a-axis, the b-axis, or the line $a = b$ gives a profile in CP with the same approximation ratio. Hence, we can label the points such that $x_1 = (-a,0)$, $x_2 = (b,0)$, and $x_3 = (0,c)$, for some $a,b,c \geq 0$.

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[3]The set $[x_i', g(\mathbf{x}')] \cap \Gamma$ is non-empty because $g(\mathbf{x}')$ cannot be in the same quadrant as x_i'. Any point in the same quadrant as x_i' subtends an angle of less than 90° with the other two points and hence it cannot be the geometric median.

If $c = 0$, then $AR(\mathbf{x}) = 1$, and so every profile has approximation ratio weakly greater than $\mathbf{x}$. Hence, we may further assume that $c > 0$.

Since x_1 and x_2 are on the a-axis, it follows from the characterization of the geometric median for three points given in remark 5 that $-a \le a_g(\mathbf{x}) \le b$ and $0 < b_g(\mathbf{x}) \le c$. Hence, moving x_3 to $g(\mathbf{x})$ then (if necessary) translating all points by the same vector so that the coordinate-wise median is at the origin yields a profile in CP which has higher approximation ratio. Hence, we may further assume that $g(\mathbf{x}) = x_3$.

Let $\mathbf{x}'$ be the profile where $x_1' = (-(a + b)/2, 0)$, $x_2' = ((a + b)/2, 0)$, and $x_3' = (0, c)$. By definition, $sc(g(\mathbf{x}'), \mathbf{x}') \le sc(g(\mathbf{x}), \mathbf{x}')$ and by an argument that exploits the convexity of the distance function, $sc(g(\mathbf{x}), \mathbf{x}') \le sc(g(\mathbf{x}), \mathbf{x})$. Combining these inequalities gives $sc(g(\mathbf{x}'), \mathbf{x}') \le sc(g(\mathbf{x}), \mathbf{x})$, and a simple calculation shows that $sc(c(\mathbf{x}'), \mathbf{x}') = sc(c(\mathbf{x}), \mathbf{x})$. Thus, $AR(\mathbf{x}') \ge AR(\mathbf{x})$.

Note that under $\mathbf{x}'$, $g(\mathbf{x}') = (0, k)$ for some $k \le c$. Define $\mathbf{x}''$ to be the profile with $x_1'' = x_1'$, $x_2'' = x_2'$, and $x_3'' = g(\mathbf{x}')$. Then, by Lemma 7, $AR(\mathbf{x}'') \ge AR(\mathbf{x}')$.

Finally, define $\mathbf{x}'''$ such that $x_i''' = \frac{1}{c}x_i''$ for each i. Then since $AR(\cdot)$ is homogeneous of degree 0, $AR(\mathbf{x}''') = AR(\mathbf{x}'')$, and so $AR(\mathbf{x}'') \ge AR(\mathbf{x})$. Further, $c(\mathbf{x}''') = (0, 0)$, $x_1''' = (-t, 0)$, $x_2''' = (t, 0)$, and $x_3''' = (0, 1)$ for some $t \ge 0$; in fact, it follows from the characterisation of the geometric median that $t \ge \sqrt{3}$. Hence, $\mathbf{x}''' \in I - CP$, and so taking $\chi = \mathbf{x}'''$ completes the proof.

$\square$

Denote by $\eta_t = ((t, 0), (-t, 0), (0, 1))$. It follows from the arguments in the proof of Lemma 9 that $I - CP = \{\eta_t : t \ge \sqrt{3}\}$. Let $\alpha(t) = \frac{2t+1}{2\sqrt{t^2+1}}$. A simple calculation shows that for $t \ge \sqrt{3}$, $AR(\eta_t) = \alpha(t)$. In particular, it follows that the approximation ratio of coordinate-wise median mechanism is equal to $\sup_{t \ge \sqrt{3}} \alpha(t)$. Since $\alpha(t)$ achieves its global maximum at $t^* = 2 > \sqrt{3}$, the ratio is $AR(\eta_2) = \alpha(2)$. Since $\alpha(2) = \sqrt{2}\dfrac{\sqrt{3^2 + 1}}{3 + 1}$, the result follows.

$\square$

4.2 Proof sketch for general n

Proof sketch. We now consider the case of $n = 2m + 1$ agents. We begin by defining classes of profiles analogous to those used in the proof for $n = 3$.

We define the class of Centered Perpendicular (CP) profiles as all profiles $\mathbf{x} \in (\mathbb{R}^2)^n$ such that

- $c(\mathbf{x}) = (0, 0)$

- for all i, either $a_i = 0$ or $b_i = 0$ or $x_i = g(\mathbf{x})$

- if $x_i' \in (x_i, g(\mathbf{x}))$, then $c(x_i', x_{-i}) \ne (0, 0)$

Since the last condition is slightly more subtle than the others and will be important in the sequel, we describe it now in words. This condition says that *any* (nonzero) movement of *any* x_i towards the geometric median would result in a change in the coordinate-wise median.

We define the class of Isosceles-Centered Perpendicular (I-CP) profiles as all $\mathbf{x} \in CP$ for which there exists $t \geq 0$ such that

- $x_1 = \cdots = x_m = (t, 0)$

- $x_{m+1} = (-t, 0)$

- $x_{m+2} = \cdots = x_{2m+1} = (0, 1)$

- $g(\mathbf{x}) = (0, 1)$.

The proof proceeds much as in the proof for $n = 3$. We first show that for every profile, there is some profile in CP with weakly higher approximation ratio. The approach used in the $n = 3$ case extends naturally here: first, translate the profile $\mathbf{x} \in (\mathbb{R}^2)^n$ so that coordinate-wise median moves to the origin; then, starting from $i = 1$ and going to $i = n$, move x_i directly towards the geometric median until either it reaches the geometric median or moving it further would move the coordinate-wise median. The resulting profile is in CP and has an approximation ratio that is weakly greater than $\mathbf{x}$'s.
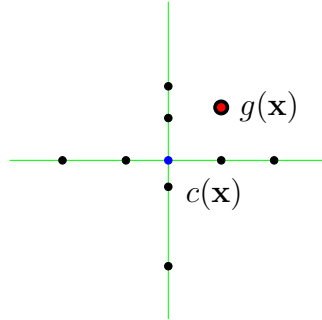

Figure 4.2: A CP profile

Next, we show that for any profile in CP, there is some profile in $I - CP$ with weakly higher approximation ratio. The approach used in the $n = 3$ case for this step *does not* extend in a straightforward manner to the general case—the main obstruction arises from the fact that for a profile $\mathbf{x}$ in CP, there may be $i \in N$ such that $x_i = g(\mathbf{x})$, which may not be on either axis. The next subsection is devoted to giving an overview of the procedure used to transform a profile in CP to one in $I - CP$ with weakly higher approximation ratio.

Finally, the approach used to calculate the worst-case approximation ratio for profiles in $I - CP$ has much the same structure as in the $n = 3$ case. We define $\eta_t = (x_1^t, \ldots, x_{2m+1}^t)$, where

$$
x_i^t = \begin{cases} (t, 0), & i = 1, \ldots, m \\ (-t, 0), & i = m + 1 \\ (0, 1), & i = m + 2, \ldots, 2m + 1 \end{cases}
$$

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and we show that $I - CP = \left\{ \eta_t : t \geq \sqrt{\frac{2m+1}{2m-1}} \right\}$. Defining $\alpha(t) = \frac{(m+1)t+m}{(m+1)\sqrt{t^2+1}}$, we show that for $t \geq \sqrt{\frac{2m+1}{2m-1}}$, $AR(\eta_t) = \alpha(t)$, and that $\alpha(t)$ has a global maximum at $t^* = \frac{m+1}{m} > \sqrt{\frac{2m+1}{2m-1}}$, from which it follows that

$$AR(CM) = \alpha\left(\frac{m+1}{m}\right) = \sqrt{2}\frac{\sqrt{(2m+1)^2+1}}{(2m+1)+1} = \sqrt{2}\frac{\sqrt{n^2+1}}{n+1}.$$
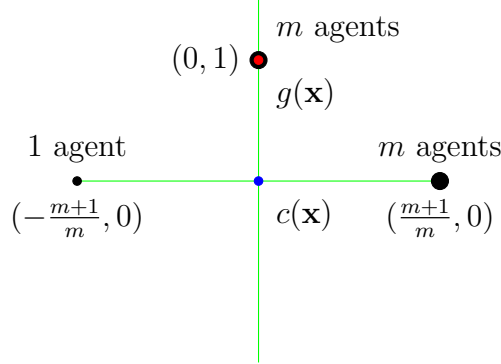

Figure 4.2: Worst case profile

$\square$

4.3 Reduction from CP to ICP

In this subsection, we discuss informally some transformations that allow us to deal with the profiles in CP. Without loss of generality (using reflections if necessary as in the $n = 3$ case), we may restrict consideration to profiles $\mathbf{x} \in CP$ with $g(\mathbf{x}) = (a_g, b_g)$ such that $a_g \geq 0$, $b_g \geq 0$, and $b_g \geq a_g$.

1. **Reducing axes**: In this step, we move all points on $-b$-axis to $-a$-axis while keeping them equidistant from $c(\mathbf{x}) = (0,0)$. This works because the $sc(c(\cdot), \cdot)$ remains the same while $sc(g(\cdot), \cdot)$ reduces, as the points move closer to the old geometric median. Thus, we get a profile in which all points are either on one of the $+a$-, $+b$-, or $-a$-axes or at $g(\mathbf{x})$.

2. **Convexity**: Consider a profile obtained after applying step 1. Transform the profile so that all points on the $+a$-, $+b$-, and $-a$-axes are at their mean coordinates on the $+a$-, $+b$-, and $-a$-axes respectively. Again, $sc(c(\cdot), \cdot)$ remains the same while $sc(g(\cdot), \cdot)$ falls because of convexity of the distance function. Thus, we get a profile with weakly higher approximation ratio which has k points at $(-b, 0)$, $m + 1 - k$ points at $(0, c)$, $m + 1 - k$ points at $(a, 0)$ and $k - 1$ points at $g(x)$. Note that we are able to pin down the exact cardinalities of these sets because of the third condition in the definition of CP, which requires that if any of the points were to move towards $g(\mathbf{x})$, then $c(\mathbf{x})$ would change.

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3. **Double Rotation**: Consider a profile obtained after applying step 2. Transform the profile by moving the $k-1$ points at $g(\mathbf{x})$ to $(0, \alpha)$, where $\alpha = d(c(\mathbf{x}), g(\mathbf{x}))$, and moving $k-1$ of the k points at $(-b, 0)$ to $(\beta, 0)$, where β is the unique positive number such that $d(g(\mathbf{x}), (\beta, 0)) = d(g(\mathbf{x}), (-b, 0))$. In this case, one can show that the increase in $sc(c(\cdot), \cdot)$ is at least $\sqrt{2}$ times the increase in $sc(g(\cdot), \cdot)$ and therefore, by Lemma 3, it follows that the approximation ratio weakly increases. Applying convexity again, we get a profile such that there is one point at $(-b, 0)$, m points at $(0, c)$ and m points at $(a, 0)$. Note that $g(\mathbf{x})$ may still not be on the axes.

4. **Geometric to axis**: Consider a profile obtained after applying step 3. In the case that $g(\mathbf{x})$ is not on the axes, we show that moving the m points at $(0, c)$ directly towards or away from $g(\mathbf{x})$ strictly increases the ratio. It follows then that there must be a worst-case profile where one point is at $(-b, 0)$, m points are at $(0, c)$, m points are at $(a, 0)$ and $g(\mathbf{x}) = (0, c)$.

From here, we apply a transformation similar to step 2 to get a profile in $I - CP$. Note that we have suppressed some details (especially when the same transformation must be used repeatedly) in order to make the exposition as clear as possible—see the appendix for a rigorous proof.

5 p-norm objective

In this section, we consider the problem of quantifying the approximation ratio for the coordinate-wise median mechanism under the p-norm objective $sc(y, \mathbf{x}) = (\sum_{i=1}^{n} \|y - x_i\|^p)^{\frac{1}{p}}$ for $p \geq 2$. While we do not exactly quantify the AR for arbitrary n in this case, we are able to obtain bounds on the asymptotic AR of the coordinate-wise median mechanism.

Theorem 3. *For $X = \mathbb{R}^2$ and the p-norm objective with $p \geq 2$,*

$$2^{1-\frac{1}{p}} \leq \sup_{n \in \mathbb{N}} AR(CM) \leq 2^{\frac{3}{2}-\frac{2}{p}}$$

The lower bound follows directly from Lemma 4, since restriction of the coordinate-wise median mechanism to profiles on the a-axis corresponds to the median mechanism in one dimension. [4]

For the upper bound, we again use Lemma 4 and note that, if a_c is the median of $(a_1, a_2, \ldots, a_n)$ and $OPT(a)$ is the optimal location, then

$$\sum_{i=1}^{n} \|a_c - a_i\|^p \leq 2^{p-1} \sum_{i=1}^{n} \|OPT(a) - a_i\|^p.$$

---

[4]The lower bound actually holds more generally in that if f is a deterministic, strategyproof mechanism defined for all n, then $\sup_{n \in N} AR(f) \geq 2^{1-\frac{1}{p}}$. If f is anonymous as well, the bound is a corollary of Theorem 3 due to the optimality of Coordinate-wise median (Theorem 1) for any n. For any f, the argument in Feigenbaum et al. [12] to show Lemma 4 extends to this setting as well and is in the appendix proof.

The upper bound is then obtained by using the following inequalities, together with Lemma 4:

$$(\alpha^2 + \beta^2)^{\frac{p}{2}} \geq (\alpha^p + \beta^p) \qquad \alpha^p + \beta^p \geq 2^{1-\frac{p}{2}}(\alpha^2 + \beta^2)^{\frac{p}{2}}.$$

The full proof is relegated to the appendix.

For $p = 2$, the upper and lower bound in Theorem 3 coincide and we get the following:

Corollary 6. *For $X = \mathbb{R}^2$ and $sc(y, \mathbf{x}) = [\sum \|y - x_i\|^2]^{\frac{1}{2}}$ $(p = 2)$,*

$$\sup_{n \in \mathbb{N}} AR(CM) = \sqrt{2}$$

For $p = \infty$, any deterministic strategyproof mechanism has $AR \geq 2$. Also, any Pareto optimal mechanism has $AR \leq 2$. Together, we get

Corollary 7. *For $X = \mathbb{R}^2$ and $sc(y, \mathbf{x}) = \max_i \|y - x_i\|$ $(p = \infty)$,*

$$AR(CM) = 2$$

.

The last corollary suggests that the upper bound in Theorem 3 is not tight. In fact, the AR of CM is actually equal to its lower bound in both cases $p = 2$ and $p = \infty$. This leads us to conjecture that:

Conjecture 8. *For $X = \mathbb{R}^2$, and the p- norm objective $sc(y, \mathbf{x}) = [\sum \|y - x_i\|^p]^{\frac{1}{p}}$ where $p \geq 2$,*

$$\sup_{n \in \mathbb{N}} AR(CM) = 2^{1-\frac{1}{p}}$$

.

6 Conclusion

In this work, we demonstrate that the coordinate-wise median mechanism is the optimal deterministic, anonymous, and strategyproof mechanism for a large, natural class of social cost functions. We show that the utilitarian cost of the coordinate-wise median is always within $\sqrt{2}\frac{\sqrt{n^2+1}}{n+1}$ of the utilitarian cost obtained under the optimal mechanism. For the p- norm objectives, we find that the worst-case approximation ratio for the coordinate-wise median mechanism is bounded above by $2^{\frac{3}{2}-\frac{2}{p}}$ for $p \geq 2$. For the case of $p = 2$ and $p = \infty$, the coordinate-wise median mechanism has AR equal to $\sqrt{2}$ and 2, respectively. This leads us to conjecture that the AR of coordinate-wise median mechanism is actually equal to $2^{1-\frac{1}{p}}$ for any $p \geq 2$.

We hope that the results and methods in this paper will encourage further research in this fundamental domain. The question of how well a randomized mechanism might approximate

the social cost of the geometric median remains open. A potentially good candidate is the mechanism that chooses a coordinate-wise median after a uniform rotation of the orthogonal axes. While its analysis seems hard in general, finding its AR on the worst-case profile in Theorem 2 might give a useful lower bound. Another question is to close the gap between the upper bound on AR of the coordinate-wise median mechanism and the lower bound on AR of any deterministic strategyproof mechanism for the p norm objective. The analysis for more general single-peaked preferences in multi-dimensional domains also remains open.

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A Proofs for Section 4 (The minisum objective)

Theorem 2. *For n odd, $X = \mathbb{R}^2$, and $sc(y, \mathbf{x}) = \sum_{i=1}^n \|y - x_i\|$,*

$$AR(CM) = \sqrt{2} \frac{\sqrt{n^2 + 1}}{n + 1}.$$

Proof. Define Centered Perpendicular (CP) profiles as all profiles $\mathbf{x} \in (\mathbb{R}^2)^n$ such that

- $c(\mathbf{x}) = (0, 0)$

- for all i, either $a_i = 0$ or $b_i = 0$ or $x_i = g(\mathbf{x})$

- if $x_i' \in (x_i, g(\mathbf{x}))$, then $c(x_i', x_{-i}) \neq (0, 0)$

Lemma 10 (CP). *For any profile $\mathbf{x} \in (\mathbb{R}^2)^n$, there exists a profile $\chi \in CP$ such that $AR(\chi) \geq AR(\mathbf{x})$.*

Proof. Let $\mathbf{x} \in (\mathbb{R}^2)^n$ be a profile. Let $\mathbf{x}'$ be the profile where $x_i' = x_i - c(\mathbf{x})$. Then $\mathbf{x}'$ has the same approximation ratio and $c(\mathbf{x}') = (0, 0)$. Denote $A = \{i : a_i = 0\}$ and $B = \{i : b_i = 0\}$. Note that since $c(\mathbf{x}') = (0, 0)$, it follows from the definition of $c(\mathbf{x}')$ that $A \neq \emptyset$ and $B \neq \emptyset$. Let $\Gamma = \{(a, b) : a = 0 \text{ or } b = 0\} \cup g(\mathbf{x}')$. Starting from $i = 1$ and going till n, define x_i'' to be the point in $[x_i', g(\mathbf{x}')] \cap \Gamma$ that is closest to $g(\mathbf{x}')$ under the constraint that $c(x_1'', x_2'', \ldots, x_i'', x_{i+1}, x_n) = (0, 0)$. Then $\mathbf{x}'' \in CP$. Further, by lemma 7 $AR(\mathbf{x}'') \geq AR(\mathbf{x}') = AR(\mathbf{x})$; hence, taking $\chi = \mathbf{x}''$ completes the proof. $\square$

Define Isosceles-Centered Perpendicular (I-CP) profiles as all $\mathbf{x} \in CP$ for which there exists $t \geq 0$ such that

- $x_1 = \cdots = x_m = (t, 0)$

- $x_{m+1} = (-t, 0)$

- $x_{m+2} = \cdots = x_{2m+1} = (0, 1)$

- $g(\mathbf{x}) = (0, 1)$.

Next, we prove some lemmas that will be useful in reducing the search space for the worst-case profile from CP to $I - CP$.

First, we show that we can reduce the number of half-axes that the points lie on from (at most) four to (at most) three.

Lemma 11 (Reduce axes). *Suppose $\mathbf{x}$ and $\mathbf{x}'$ are profiles which differ only at i where for some $a > 0$, $x_i = (0, -a)$ and $x_i' = (-a, 0)$, and for which $c(\mathbf{x}) = c(\mathbf{x}') = (0, 0)$ and $b_g(\mathbf{x}) \geq a_g(\mathbf{x}) \geq 0$. Then $AR(\mathbf{x}') \geq AR(\mathbf{x})$.*

Proof. Again $c(\mathbf{x}') = c(\mathbf{x})$ and $sc(c(\mathbf{x}'), \mathbf{x}') = sc(c(\mathbf{x}), \mathbf{x})$. Thus, it is sufficient to show that $sc(g(\mathbf{x}'), \mathbf{x}') \leq sc(g(\mathbf{x}), \mathbf{x})$. For this, we just need to show that $d(x_i', g(\mathbf{x})) \leq d(x_i, g(\mathbf{x}))$. This follows from the following simple calculation:

$$\begin{aligned}
d(x_i', g(\mathbf{x}))^2 &= (a_g(\mathbf{x}) + a)^2 + b_g(\mathbf{x})^2 \\
&= a_g(\mathbf{x})^2 + 2a_g(\mathbf{x})a + a^2 + b_g(\mathbf{x})^2 \\
&\leq a_g(\mathbf{x})^2 + b_g(\mathbf{x})^2 + 2ab_g(\mathbf{x}) + a^2 \\
&= a_g(\mathbf{x})^2 + (b_g(\mathbf{x}) + a)^2 \\
&= d(x_i, g(\mathbf{x}))^2.
\end{aligned}$$

$\square$

Next, we show that we can combine points on each of the three half-axes while weakly increasing the approximation ratio.

Lemma 12 (Convexity). *Let $\mathbf{x} \in CP$ and let $S \subseteq N$ be such that for all $i \in S$, $a_i > 0$ and $b_i = 0$. Let x_S be the mean of the x_i across $i \in S$. Let $\mathbf{x}'$ be the profile where*

1. $x_j' = x_j$ for $j \notin S$ and

2. $x_j' = x_S$ for $j \in S$.

Then $AR(\mathbf{x}') \geq AR(\mathbf{x})$.

Proof. It is immediate that $c(\mathbf{x}') = c(\mathbf{x})$. Hence, it will be sufficient to show that AR for $\mathbf{x}'$ with $c(\mathbf{x})$ and $g(\mathbf{x})$ instead of $c(\mathbf{x}')$ and $g(\mathbf{x}')$ is at least as big as $AR(\mathbf{x})$. Indeed, $sc(c(\mathbf{x}), \mathbf{x}') = sc(c(\mathbf{x}), \mathbf{x}))$ and $sc(g(\mathbf{x}'), \mathbf{x}') < sc(g(\mathbf{x}), \mathbf{x}') < sc(g(\mathbf{x}), \mathbf{x})$ where the last inequality follows from convexity of the distance function. $\square$

The same argument applies for any of the other strict half axes.

Next, we show that we can move all the points that are on the geometric median to the axis in a way that weakly increases the approximation ratio.

Lemma 13 (Double Rotation). *Let $\mathbf{x}$ and $\mathbf{x}'$ be profiles that differ only at i_1 and i_2, such that for some $a \geq 0$*

- $c(\mathbf{x}) = (0,0)$,

- $b_g(\mathbf{x}) \geq a_g(\mathbf{x}) > 0$,

- $x_{i_1} = (-a, 0)$,

- $x_{i_1}' = (a + 2a_g(\mathbf{x}), 0)$,

- $x_{i_2} = g(\mathbf{x})$, *and*

- $x_{i_2}' = (0, d(g(\mathbf{x}), (0, 0)))$.

Then $c(\mathbf{x}') = (0,0)$ and $AR(\mathbf{x}') \geq AR(\mathbf{x})$.

Proof. The first claim is immediate.

For the second claim, let

$$A = \sum_{i \neq i_1} d(x_i, c(\mathbf{x}))$$

$$B = \sum_{i \neq i_2} d(x_i, g(\mathbf{x})).$$

By a previous result,

$$A + d(x_{i_1}, c(\mathbf{x})) \leq \sqrt{2}B.$$

Hence, it follows that

$$[A + d(x_{i_1}, c(\mathbf{x}))]d(x'_{i_2}, g(\mathbf{x})) \leq \sqrt{2}Bd(x'_{i_2}, g(\mathbf{x})).$$

But since $b_g(\mathbf{x}) \geq a_g(\mathbf{x})$, it follows that $d(x'_{i_2}, g(\mathbf{x})) \leq \sqrt{2}a_g(\mathbf{x})$. Hence,

$$[A + d(x_{i_1}, c(\mathbf{x}))]d(x'_{i_2}, g(\mathbf{x})) \leq 2Ba_g(\mathbf{x})$$
$$= B(d(x'_{i_1}, c(\mathbf{x})) - d(x'_{i_2}, c(\mathbf{x}))).$$

From this it follows that

$$(A + d(x_{i_1}, c(\mathbf{x})))(B + d(x'_{i_2}, g(\mathbf{x}))) = AB + Bd(x_{i_1}, c(\mathbf{x})) + [A + d(x_{i_1}, c(\mathbf{x}))]d(x'_{i_2}, g(\mathbf{x}))$$
$$\leq AB + Bd(x'_{i_1}, c(\mathbf{x}))$$
$$= (A + d(x'_{i_1}, c(\mathbf{x})))B$$

and hence

$$AR(\mathbf{x}) = \frac{A + d(x_{i_1}, c(\mathbf{x}))}{B}$$
$$\leq \frac{A + d(x'_{i_1}, c(\mathbf{x}))}{B + d(x'_{i_2}, g(\mathbf{x}))}$$
$$= \frac{A + d(x'_{i_1}, c(\mathbf{x}'))}{B + d(x'_{i_2}, g(\mathbf{x}))}$$
$$\leq AR(\mathbf{x}').$$

$\square$

Lemma 14 (Geometric to axis). *Suppose that $\mathbf{x}$ is a profile such that there are $a \geq 0$ and $b, c > 0$ and subsets $L, R, U \subseteq N$ with $L \cap R = L \cap U = R \cap U = \emptyset$, $L \cup R \cup U = N$, $|L| = 1$, $|U| = |R| = m$, and*

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- $x_i = (-a, 0)$ *for* $i \in L$

- $x_i = (0, b)$ *for* $i \in U$

- $x_i = (c, 0)$ *for* $i \in R$

and so that $a_g(\mathbf{x}) > 0$.

Then, there exists another profile $\mathbf{z}$ such that $AR(\mathbf{z}) > AR(\mathbf{x})$.

Proof. We'll consider two separate cases.

First assume that $a > 0$. Let $\mathbf{x}(t)$ be the profile which is the same as $\mathbf{x}$ for $i \notin U$ and which has $\mathbf{x}_i(t) = g(\mathbf{x}) + t\left(-a_g(\mathbf{x}), b - b_g(\mathbf{x})\right)$ for $i \in U$. Then there exists $\epsilon > 0$ such that for $t \in [0, 1 + \epsilon]$,

$$AR(\mathbf{x}(t)) = \frac{(a + (1-t)a_g(\mathbf{x})) + m(c - (1-t)a_g(\mathbf{x})) + mb_g(\mathbf{x}) + mt(b - b_g(\mathbf{x}))}{d((-a,0), g(\mathbf{x})) + md((c,0), g(\mathbf{x})) + mtd((0,b), g(\mathbf{x}))}$$

$$= \frac{t\left((m-1)a_g(\mathbf{x}) + m(b - b_g(\mathbf{x}))\right) + a + a_g(\mathbf{x}) + m(c - a_g(\mathbf{x}) + b_g(\mathbf{x}))}{tmd((0,b), g(\mathbf{x})) + d((-a,0), g(\mathbf{x})) + md((c,0), g(\mathbf{x}))}$$

Note that $\mathbf{x}(1) = \mathbf{x}$. Now since the denominator of $AR(\mathbf{x}(t))$ is strictly positive for $t \geq 0$ and since both the numerator and the denominator are linear in t, $AR(\mathbf{x}(t))$ is monotonic on $[0, 1 + \epsilon]$. There are three possibilities.

If $AR(\mathbf{x}(t))$ is strictly increasing, then $AR(\mathbf{x}(1 + \epsilon)) > AR(\mathbf{x})$.
If $AR(\mathbf{x}(t))$ is strictly decreasing, then $AR(\mathbf{x}(0)) > AR(\mathbf{x})$.
If $AR(\mathbf{x}(t))$ is constant, consider the profile $\mathbf{z}'$ obtained by putting t such that $-a = (1-t)a_g(\mathbf{x})$. Then, under $\mathbf{z}'$, we have 1 agent at $(-a, 0)$, m agents at $(-a, bt + (1-t)b_g(\mathbf{x}))$ and m agents at $(c, 0)$. Also, $AR(\mathbf{x}) = AR(\mathbf{z}')$. We can translate this profile by a to the right and get a profile of points in the case where essentially we have $a = 0$. We'll deal with this case now.

Now let's consider the case where $\mathbf{x}$ is such that $a = 0$. To begin, note that since $g(\mathbf{x})$ must be in the convex hull of $(0,0)$, $(0, b)$ and $(c, 0)$, if $a_g(\mathbf{x}) \geq \frac{c}{2}$ and $b_g(\mathbf{x}) \geq \frac{b}{2}$, then $g(\mathbf{x}) = (\frac{c}{2}, \frac{b}{2})$. But then $\sum_{i=1}^{n} \frac{x_i - g(\mathbf{x})}{\|x_i - g(\mathbf{x})\|} = \frac{(c, b)}{\sqrt{c^2 + b^2}} \neq 0$, contradicting the characterization of the geometric median in Lemma 2. Hence, it must be that either $a_g(\mathbf{x}) < \frac{c}{2}$ or $b_g(\mathbf{x}) < \frac{b}{2}$.

Suppose that $a_g(\mathbf{x}) < \frac{c}{2}$. Let $\mathbf{z}$ be the profile obtained from $\mathbf{x}$ by moving the point at $(0,0)$ to $(a_g(\mathbf{x}) - \frac{c}{2}, 0)$ and moving one of the points at $(c, 0)$ to $(\frac{c}{2} + a_g(\mathbf{x}), 0)$. This transformation leaves coordinate-wise median unchanged, as well as leaving the sum of distances to the coordinate-wise median unchanged. However, the sum of distances to $g(\mathbf{x})$ strictly decreases, since for the unaltered points the distance to $g(\mathbf{x})$ remains the same, and the sum of the distances from the altered points to $g(\mathbf{x})$ is

$$d((0,0), g(\mathbf{x})) + d((c,0), g(\mathbf{x})) = d((2a_g(\mathbf{x}), 0), g(\mathbf{x})) + d((c,0), g(\mathbf{x}))$$

$$> 2d((a_g(\mathbf{x}) + \frac{c}{2}), g(\mathbf{x}))$$

$$= d((a_g(\mathbf{x}) - \frac{c}{2}), g(\mathbf{x})) + d((a_g(\mathbf{x}) + \frac{c}{2}), g(\mathbf{x})),$$

where the inequality follows from convexity of $d(\cdot, g(\mathbf{x}))$. Since $AR(\mathbf{z})$ is bounded below by the ratio of the sum of distances to the coordinate-wise median to the sum of distances to $g(\mathbf{x})$, it follows that $AR(\mathbf{z}) > AR(\mathbf{x})$.

Next, suppose that $b_g(\mathbf{x}) < \frac{b}{2}$. Let $\mathbf{z}$ be the profile obtained from $\mathbf{x}$ by moving the point at $(0,0)$ to $(0, b_g(\mathbf{x}) - \frac{b}{2})$ and moving one of the points at $(0,b)$ to $(0, \frac{b}{2} + b_g(\mathbf{x}))$. By essentially the same argument just given, $AR(\mathbf{z}) > AR(\mathbf{x})$.

$\square$

Finally, the following lemma shows that we can use convexity to make the triangle formed by the three groups of points isosceles.

Lemma 15 (Isosceles)**.** *Let $\mathbf{x}$ be a profile such for which are m points at $(a, 0)$, 1 point at $(-b, 0)$ and m points at $(0, c)$, and for which $g(\mathbf{x}) = (0, c)$ and $c(\mathbf{x}) = (0, 0)$. Let $\mathbf{x}'$ be the profile where there are m points at $\left(\dfrac{ma + b}{m+1}, 0 \right)$, 1 point at $\left(-\dfrac{ma + b}{m+1}, 0 \right)$, and m points at $(0, c)$. Then, $AR(\mathbf{x}') \geq AR(\mathbf{x})$.*

Proof. Note that $c(\mathbf{x}) = c(\mathbf{x}') = (0, 0)$. Since $m * a + b = m * \frac{(ma+b)}{m+1} + \frac{ma+b}{m+1}$, we get that the numerator in $AR(\mathbf{x})$ and $AR(\mathbf{x}')$ remains the same. Thus, we only need to argue that the denominator goes down as we go from $AR(\mathbf{x})$ to $AR(\mathbf{x}')$.

Even though $g(\mathbf{x}')$ may not be equal to $g(\mathbf{x})$ we have that $sc(g(\mathbf{x}), \mathbf{x}') \leq sc(g(\mathbf{x}), \mathbf{x})$ by the convexity of the distance function which would imply $sc(g(\mathbf{x}'), \mathbf{x}') \leq sc(g(\mathbf{x}), \mathbf{x})$ by definition of $g(\mathbf{x})$. Thus, we have that $AR(\mathbf{x}') \geq AR(\mathbf{x})$.

$\square$

Now, we use above lemmas to reduce the search space to I-CP.

Lemma 16 (ICP)**.** *For every $\mathbf{x} \in CP$, there exists $\chi \in I - CP$ such that $AR(\chi) \geq AR(\mathbf{x})$.*

Proof. Without loss of generality, consider any profile $\mathbf{x} \in CP$ such that $b_g(\mathbf{x}) \geq a_g(\mathbf{x}) \geq 0$. Applying Lemma 11 to all points on the negative b axis gives a profile $\mathbf{x}'$ with a weakly higher approximation ratio. In $\mathbf{x}'$, we have all points on positive a, negative a, positive b and the geometric median. Using lemma 12, we can combine the points on positive a, negative a, positive b to some $(a, 0), (0, b), (-c, 0)$ while weakly increasing AR. Let this profile be $\mathbf{x}''$. Now, we use lemma 13 to move points on the geometric median to $+b$-axis. Using 12 again, we get a profile $\mathbf{x}'''$ with m points on some $(a, 0)$, 1 point on $(-c, 0)$ and m points on $(0, b)$.

So we know that there must be a worst-case profile that takes this form. From Lemma 14, we can say that if the geometric median of such a profile is not on the b-axis, it cannot be a worst-case profile. Thus, there must be a worst-case profile $\mathbf{z}$ with m points on some $(a, 0)$, 1 point on $(-c, 0)$ and m points on $(0, b)$ and $a_g(\mathbf{z}) = 0$. Further, such a profile must have $b_g(\mathbf{z}) = b$, since otherwise, the profile with m points on $(a, 0)$, 1 point on $(-c, 0)$, and m

points on $(0, b_g(\mathbf{z}))$ would have a strictly higher approximation ratio than $\mathbf{z}$. By Lemma 15, since $\mathbf{z}$ is a worst-case profile, it must be that $c = a$.

Now, since $\mathbf{z}$ is a worst-case profile, the profile χ with $\chi_i = \frac{1}{b}\mathbf{z}_i$ is also a worst-case profile, and since $\chi \in I - CP$, the result follows. $\qquad\square$

Using Lemma 16, we can now restrict attention to profiles in $I - CP$. Define

$$\eta_t = (x_1^t, \ldots, x_{2m+1}^t),$$

where

$$x_i^t = \begin{cases} (t, 0) & i = 1, \ldots, m \\ (-t, 0) & i = m + 1 \\ (0, 1) & i = m + 2, \ldots, 2m + 1 \end{cases}$$

Then, $I - CP = \left\{ \eta_t \ : \ t \geq \sqrt{\frac{2m+1}{2m-1}} \right\}$. Defining $\alpha(t) = \frac{(m+1)t+m}{(m+1)\sqrt{t^2+1}}$, we get that for $t \geq \sqrt{\frac{2m+1}{2m-1}}$, $AR(\eta_t) = \alpha(t)$, and that $\alpha(t)$ is maximized at $t^* = \frac{m+1}{m} > \sqrt{\frac{2m+1}{2m-1}}$, from which it follows that

$$\text{Approximation ratio of CM} = \alpha\left(\frac{m+1}{m}\right) = \sqrt{2}\frac{\sqrt{(2m+1)^2+1}}{(2m+1)+1} = \sqrt{2}\frac{\sqrt{n^2+1}}{n+1}.$$

Thus, we get that the worst case approximation ratio is $\sqrt{2}\frac{\sqrt{n^2+1}}{n+1}$ as required. $\qquad\square$

B Proofs for Section 5 (p-norm objective)

Theorem 3. *For $X = \mathbb{R}^2$ and the p-norm objective with $p \geq 2$,*

$$2^{1-\frac{1}{p}} \leq \sup_{n \in \mathbb{N}} AR(CM) \leq 2^{\frac{3}{2}-\frac{2}{p}}$$

Proof. We'll prove that the lower bound actually holds for any deterministic, strategyproof mechanism (defined for all $n \in \mathbb{N}$) and hence, it holds for the coordinate-wise median mechanism. So suppose f is any deterministic, strategyproof mechanism. With $n = 2m + 1$ agents[5], for any profile $\mathbf{x}$ such that m agents have ideal point α, $m + 1$ agents have ideal point $\beta \neq \alpha$, and $f(\mathbf{x}) = \beta$,

$$AR_f(\mathbf{x}') = \frac{sc(f(\mathbf{x}), \mathbf{x})}{sc(OPT(\mathbf{x}), \mathbf{x})}$$

$$\geq \frac{(m * \|\alpha - \beta\|^p)^{\frac{1}{p}}}{((2m+1)(\|\alpha - \beta\|/2)^p)^{\frac{1}{p}}}$$

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[5]The same argument applies if we just take $n = 2$ but we wanted to illustrate that the result holds even under the restriction to odd number of agents

$$= 2^{1-\frac{1}{p}} \cdot \left(\frac{m}{m+1/2}\right)^{\frac{1}{p}}.$$

To see that such a profile always exists, consider the profile $\mathbf{x}$ where agents 1 through m have ideal point $(-1,0)$ and agents $m+1$ through $2m+1$ have ideal point $(1,0)$. If $f(\mathbf{x}) = (1,0)$, then $\mathbf{x}$ is such a profile; if not, let $\mathbf{x}'$ be the profile where agents 1 through $m+1$ have ideal point $f(\mathbf{x})$ and agents $m+2$ through $2m+1$ have ideal point at $(1,0)$. Since $\mathbf{x}'$, every agent's ideal point is either the same as under $\mathbf{x}$ or equal to $f(\mathbf{x})$, it follows from strategyproofness that $f(\mathbf{x}') = f(\mathbf{x})$, and so $\mathbf{x}'$ is such a profile.

Thus, for any $n = 2m+1$,

$$AR(f) \geq 2^{1-\frac{1}{p}} \cdot \left(\frac{n-1}{n}\right)^{\frac{1}{p}},$$

so

$$\sup_n AR(f) \geq 2^{1-\frac{1}{p}}.$$

Now we show that the asymptotic AR of coordinate-wise median mechanism is bounded above by $2^{\frac{3}{2}-\frac{2}{p}}$. Consider any profile $\mathbf{x} = (a_i, b_i) \in (\mathbb{R}^2)^n$. Let $g(\mathbf{x}) = (a_g(\mathbf{x}), b_g(\mathbf{x}))$ and $c(\mathbf{x}) = (a_c(\mathbf{x}), b_c(\mathbf{x}))$. Then, we have that

$$
\begin{aligned}
sc(g(\mathbf{x}), \mathbf{x})^p &= \sum_{i=1}^{n} \|g(\mathbf{x}) - x_i\|^p \\
&\geq \left(\sum_{i=1}^{n} \|a_g(\mathbf{x}) - a_i\|^p + \sum_{i=1}^{n} \|b_g(\mathbf{x}) - b_i\|^p\right) \\
&\geq \left(\sum_{i=1}^{n} \|OPT(a) - a_i\|^p + \sum_{i=1}^{n} \|OPT(b) - b_i\|^p\right) \\
&\geq \frac{1}{2^{p-1}} \left(\sum_{i=1}^{n} \|c_a - a_i\|^p + \sum_{i=1}^{n} \|c_b - b_i\|^p\right) \\
&\geq \frac{2^{1-\frac{p}{2}}}{2^{p-1}} \sum_{i=1}^{n} \|c(\mathbf{x}) - x_i\|^p \\
&= 2^{2-\frac{3p}{2}} sc(c(\mathbf{x}), \mathbf{x})^p
\end{aligned}
$$

Thus, we get $AR(CM) \leq 2^{\frac{3}{2}-\frac{2}{p}}$ for $p \geq 2$ as required. $\qquad\square$