PS 12 NOTES: WEEK 6

TA: WADE HANN-CARUTHERS

1. Spatial preferences in one dimension

1.1. **Single-peaked preferences.** A preference $\succ$ over $\mathbb{R}$ is *single-peaked* if there is some $x^* \in \mathbb{R}$ such that

- $x^* \succ x$ for all $x \neq x^*$
- if $x'' < x' < x^*$ then $x' \succ x''$
- if $x* > x' > x''$ then $x' \succ x''$.

1.2. **Majority rule and single-peaked preferences.** Suppose there are n (odd) individuals with preferences $\succ_1, \ldots, \succ_n$, and consider $A, B, C \in \mathbb{R}$. Suppose that $A \succ_{maj} B$ and $B \succ_{maj} C$. Assume that $A < B$. If $C < A$, then all individuals who prefer B to C must also prefer A to C, so $A \succ_{maj} C$. If $C > B$, then all individuals who prefer A to B also prefer A to C, so $A \succ_{maj} C$. Finally, it is impossible to have $A < C < B$. Thus, $A \succ_{maj} B$ and $B \succ_{maj} C$ implies $A \succ_{maj} C$. So the $\succ_{maj}$ is transitive and hence is actually a preference! So when preferences are single-peaked, it is possible to aggregate social preferences in a nontrivial way–use majority rule on a pairwise basis.

1.3. **Median voter theorem.** What is the most preferred alternative under $\succ_{maj}$? Consider an alternative x that is not equal to the ideal point of any individual. If there are strictly more individuals with ideal points to the left of x, then the closest ideal point to the left is majority preferred to x. Similarly, if there are strictly more individuals with ideal points to the right of x, then the closest ideal point to the right is majority preferred to x. Hence, the most preferred alternative must be some agent's ideal point. Now, suppose that strictly more than half of the voters' ideal points lie to one side of an agent's ideal point. Then the closest ideal point on that side is majority preferred. Hence, it must be that at most half of the individuals' ideal points lie on either side of the most preferred alternative–that is, the most preferred alternative under $\succ_{maj}$ must be the median of the ideal points.

Consider two candidates running for office who only care about winning and can only choose a policy position.[1] What are the equilibria of this game? Clearly, if either candidate does not choose the median ideal point, the other candidate can win for sure–so the only equilibrium is for both candidates to choose the median ideal point.

[1] A candidate wins if they get more votes than the other candidate and has a fifty percent chance of winning if they choose the same alternative as the other candidate.

2. Spatial preferences in two dimensions

2.1. Single-peaked preferences. A preference $\succ$ over $\mathbb{R}^2$ is *single-peaked* if there is some $x^* \in \mathbb{R}$ such that

- $x^* \succ x$ for all $x \neq x^*$
- $tx + (1-t)x^* \succ x$ for all $x \neq x^*$, $t \in (0,1)$

2.2. McKelvey 1: Chaos theorem. Suppose there are three individuals with ideal points $(-1,0)$, $(1,0)$, and $(0,1)$ and whose preferences are Euclidean (so an individual strictly prefers the alternative (a,b) to the alternative (a',b') if and only if it is (a,b) is strictly closer to their ideal point than (a',b'). Let (x,y) and (x',y') be two alternatives. Let $R \in \mathbb{N}$ be a natural number such (x',y') is within R of all three of the ideal points. Then

- $(0,y) \succ_{maj} (x,y)$
- $(0,0) \succ_{maj} (0,y)$
- $(.99,.99) \succ_{maj} (0,0)$
- $(0,1.01) \succ_{maj} (.99,.99)$
- $(0,-1) \succ_{maj} (0,1.01)$
- $(1.99,.99) \succ_{maj} (0,-1)$
- $(0,2.01) \succ_{maj} (1.99,.99)$
- $(0,-2) \succ_{maj} (0,2.01)$
- $\ldots$
- $(0,-R) \succ_{maj} (0,R+.01)$
- $(x',y') \succ_{maj} (0,-R)$.

In particular, it is possible to go from (x,y) to (x',y') by a sequence of majority improvements for any (x,y) and (x',y')!

2.3. McKelvey 2: 4r ball. Despite the chaos theorem, we can say something meaningful about what policy positions we should expect candidates to take.

A line is a *median line* if at least half of the ideal points are above and at least half of the ideal points are below the line. Any circle that contains all of the voters' ideal points will intersect all of the median lines. Consider the smallest circle C that intersects all of the median lines, and let C' be the circle that is concentric with C and whose radius is 4 times that of C. Then in fact every alternative that lies outside of C' is covered by an alternative that lies inside of C. So we should never expect candidates to take policy positions outside of C'.

2.4. Agenda voting with spatial preferences in two dimensions. [in progress]

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