PS 12 NOTES: WEEK 2

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1. Normal form games

An normal form game has:

- Players
- Action sets for each player
- Payoffs for each player for each action profile

Here is an example of a normal form game:

		Player 2	
		B_1	B_2
Player 1	A_1	$1, 7$	$-1, 4$
	A_2	$8, -2$	$13, 1$

So in this game, if (for example) player 1 played A_2 and player 2 played B_1, then player 1's payoff would be 8 and player 2's payoff would be -2.

When discussing extensive form games, we said that a strategy profile is an *equilibrium* if no agent has a strictly profitable unilateral deviation. Another way of saying this is that for every agent it is the case that, given the strategies all the other agents are using, the agent is choosing (one of) the best possible strategy. We will say that such a strategy is a *best response*. In normal form games, we will use this same notion of best response. In the game above, player 1's best responses are:

- A_2 if player 2 plays B_1 (because $8 > 1$)
- A_2 if player 2 plays B_2 (because $13 > -1$)

and player 2's best responses are:

- B_1 if player 1 plays A_1 (because $7 > 4$)
- B_2 if player 1 plays A_2 (because $1 > -2$)

Using this new terminology, we can say that a strategy profile is an equilibrium if every agent's action is a best response to the other agents' actions. Consider the action profile (A_1, B_1). This is not an equilibrium because player 1 is not best responding (since player 1's unique best response to B_1 is A_2). What about the action profile (A_2, B_2)? is this an equilibrium? Yes! As noted above, A_2 is a best response for player 1 when player 2 plays B_2 and B_2 is a best response for player 2 when player 1 plays A_2, so both players are best responding.

2. Mixed strategies

Unlike with extensive form games with perfect information, normal form games do not always have equilibria. Here is an (important) example of a game that has no equilibrium:

Matching pennies

		Player 2	
		H	T
Player 1	H	1, 0	0, 1
	T	0, 1	1, 0

We will enrich what players can do by giving them the ability to choose randomizations over actions instead of just choosing actions. A *mixed strategy* is a probability distribution over the set of actions available to an agent. Here is an example of a mixed strategy for player 1:

- with probability 2/3, play H
- with probability 1/3, play T

In general, if σ is a probability distribution over the set of actions, we say that a player's strategy is σ if the probability of each action a is $\sigma(a)$.

A *mixed strategy profile* is a choice of mixed strategy for each player. Suppose that player 1's strategy is σ_1 and player 2's strategy is σ_2, where

- $\sigma_1(H) = 2/3$
- $\sigma_1(T) = 1/3$
- $\sigma_2(H) = 1/5$
- $\sigma_2(T) = 4/5$

The interpretation is that players 1 and 2 will simultaneously choose an action, and the probabilities of their choices will be given by σ_1 and σ_2 respectively. So what is the probability, for example, that player 1 will end up playing T and player 2 will end up playing H? Since the players are randomizing independently, this will be

$$\mathbb{P}(\text{1 plays T, 2 plays H}) = \mathbb{P}(\text{1 plays T}) \cdot \mathbb{P}(\text{2 plays H}) = \sigma_1(T) \cdot \sigma_2(H) = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}.$$

Now, what will the payoff for player 1 be? It will be 1 if both players play H or both players play T, which happens with probability

$$\sigma_1(H) \cdot \sigma_2(H) + \sigma_1(T) \cdot \sigma_2(T) = \frac{2}{3} \cdot \frac{1}{5} + \frac{1}{3} \cdot \frac{4}{5} = \frac{2}{5}.$$

and it will be 0 if player 1 plays H and player 2 plays T or player 1 plays T and player 2 plays H, which happens with probability

$$\sigma_1(H) \cdot \sigma_2(T) + \sigma_1(T) \cdot \sigma_2(H) = \frac{2}{3} \cdot \frac{4}{5} + \frac{1}{3} \cdot \frac{1}{5} = \frac{3}{5}.$$

When analyzing mixed strategy profiles in normal form games, we assume that players care about maximizing their *expected payoff*. In this case, player 1's expected payoff will be

$$\frac{2}{5} \cdot 1 + \frac{3}{5} \cdot 0 = \frac{2}{5}.$$

A mixed strategy is a *best response* if there is no other mixed strategy that gives strictly higher payoff, given the mixed strategies the other players are using. In this case, it is easy to see that σ_1 is not a best response to σ_2. Why? Consider the strategy σ_1' where $\sigma_1'(H) = 0$ and $\sigma_1'(T) = 1$. Player 1's expected payoff when player 1's strategy is σ_1' and player 2's strategy is σ_2 is

$$(\sigma_1'(H) \cdot \sigma_2(H)) \cdot 1 + (\sigma_1'(H) \cdot \sigma_2(T)) \cdot 0 + (\sigma_1'(T) \cdot \sigma_2(H)) \cdot 0 + (\sigma_1'(T) \cdot \sigma_2(T)) \cdot 1 = \frac{4}{5}$$

which is higher than $\frac{2}{5}$.

A mixed strategy profile is an *equilibrium* if every agent's mixed strategy is a best response to the other agents' mixed strategies. An example of a mixed strategy equilibrium for this game is $(\tilde{\sigma}_1, \tilde{\sigma}_2)$, where

- $\tilde{\sigma}_1(H) = 1/2$
- $\tilde{\sigma}_1(T) = 1/2$
- $\tilde{\sigma}_2(H) = 1/2$
- $\tilde{\sigma}_2(T) = 1/2$

One of the most important facts about mixed strategy equilibria is that they always exist.

Theorem 1 (Nash 1951). *Every normal form game with finitely many players and finitely many actions for each player has a mixed strategy equilibrium.*

3. Strictly dominated strategies

We will now discuss an important game that comes up a lot called Prisoner's Dilemma. Here is a version of Prisoner's Dilemma:

Prisoner's Dilemma

		Player 2	
		C	D
Player 1	C	$2, 2$	$-1, 3$
	D	$3, -1$	$0, 0$

Observe that for any action player 2 could take, player 1 gets strictly higher payoff from taking action D than from taking action C. It follows that for any mixed strategy player 2 could take, player 1 gets strictly higher payoff from taking action D than from taking action C.

When an action A gives a player a strictly higher payoff than another action B no matter what the other players do, we say that A *strictly dominates* B and that B is *strictly dominated* by A. Using this language, D strictly dominates C for player 1; similarly, D strictly dominates C for player 2.

Note that it is impossible for a player to choose a mixed strategy that assigns nonzero probability to a strictly dominated action in equilibrium, since shifting that probability to the action that strictly dominates it would result in a strictly higher payoff for the player. As a result, we can easily see that in this game, the only equilibrium is (D, D).

4. Weakly dominated strategies

Now, consider the following game.

Variation on Prisoner's Dilemma

		Player 2		
		C	D	E
	C	$2, 2$	$-1, 3$	$10, 3$
Player 1	D	$3, -1$	$0, 0$	$10, 0$
	E	$3, 10$	$0, 10$	$10, 10$

Notice that in this game, E is weakly dominated (by D) for both players. Removing E for both players would result in the Prisoner's Dilemma game from above, and would clearly remove the equilibrium (E, E) (which gives both players the highest possible payoff!) Hence, unlike with strictly dominated strategies, we cannot remove weakly dominated strategies without potentially seriously altering the game.

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