On-Line and First Fit Colorings of Graphs

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ABSTRACT

A graph coloring algorithm that immediately colors the vertices taken from a list without looking ahead or changing colors already assigned is called "on-line coloring." The properties of on-line colorings are investigated in several classes of graphs. In many cases we find on-line colorings that use no more colors than some function of the largest clique size of the graph. We show that the first fit on-line coloring has an absolute performance ratio of two for the complement of chordal graphs. We prove an upper bound for the performance ratio of the first fit coloring on interval graphs. It is also shown that there are simple families resisting any on-line algorithm: no on-line algorithm can color all trees by a bounded number of colors.

1. INTRODUCTION

A *coloring* (or proper coloring) of a graph G is an assignment of positive integers called "colors" to the vertices of G so that adjacent vertices have different colors.

An *on-line coloring* is a coloring algorithm that immediately colors the vertices of a graph G taken from a list without looking ahead or changing colors already assigned. To be more precise, an on-line coloring of G is an algorithm that properly colors G by receiving its vertices in some order $v_1, \ldots, v_n$. The color of v_i is assigned by only looking at the subgraph of G induced by the set $\{v_1, \ldots, v_i\}$, and the color of v_i never changes during the algorithm.

Let A be an on-line coloring algorithm and consider the colorings of a graph G produced by A for all orderings of the vertices of G. The maximum number of colors used among these colorings is denoted by $\chi_A(G)$. Clearly, $\chi_A(G)$ measures the worst-case behavior of A on G.

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On-line coloring can be viewed as a two-person game on a graph G. In each step player I reveals the vertices of G and player II answers by immediately coloring the current vertex. The aim of II might be to use as few colors as possible and then the strategy of I against II consists in finding the "worst" order of vertices that forces as much color as possible.

The simplest on-line coloring is the first fit algorithm, for which we use the abbreviation "FF" throughout the paper. The *first fit algorithm* (FF) is an on-line coloring that works by assigning the smallest possible integer as color to the current vertex of the graph.

Obviously, FF produces a *maximal stable sequence partition* $V(G) = S_1 \cup \cdots \cup S_k$, where S_i is a maximal nonempty stable set in the subgraph induced by $S_i \cup \cdots \cup S_k$, for every i, $1 \leqq i \leqq k$. The converse is also true: every maximal stable sequence partition of G can be reproduced by FF if an appropriate ordering of the vertices is taken. Therefore, $\chi_{FF}(G)$ coincides with the canonical achromatic number of a graph G introduced in [8].

Our interest in on-line graph coloring algorithms is motivated by the fact that certain situations necessitate the performance of on-line operations. Such situations occur, for instance, in dynamic storage allocation [9]. In some cases these problems are formulated as two-dimensional packing problems (see [1 and 3]). Certain algorithms can be interpreted as on-line colorings in special families of graphs. In particular, the rectangle packing problem of M. Chrobak and M. Ślusarek ([4,5]) can be formulated as follows: how powerful are the on-line and FF colorings on the family of interval graphs? It is asked in [4] whether FF has a constant performance ratio in the family of interval graphs. This fascinating problem inspired the present paper.

Our main concern is to get upper bounds for $\chi_A(G)$, in particular for $\chi_{FF}(G)$, in terms of $\omega(G)$ for several classes of graphs. Here $\omega(G)$ denotes the *clique number* of G, i.e., the maximum number of vertices in a complete subgraph of G.

First we consider FF in the case of subfamilies of perfect graphs. We prove the following results:

$\chi_{FF}(G) \leqq \omega(G) + 1$	if G is split graph (Proposition 2.2);
$\chi_{FF}(G) \leqq \frac{3}{2} \cdot \omega(G)$	if G is the complement of a bipartite graph (Theorem 2.3);
$\chi_{FF}(G) \leqq 2 \cdot \omega(G) - 1$	if G is the complement of a chordal graph (Theorem 2.4).

For perfect graphs $\omega(G) = \chi(G)$, where $\chi(G)$ denotes the *chromatic number* of G, i.e., the minimum number of colors in a proper coloring of G. Therefore the (tight) bounds given above show that the performance ratio $\chi_{FF}(G)/\chi(G)$ of FF coloring is constant for the appropriate perfect graphs. The positive answer to the question of Chrobak and Ślusarek would imply the analogous result concerning interval graphs. The authors proved that for every n there is a constant $c(n)$ such that $\chi_{FF}(G) \leqq c(n) \cdot \omega(G)^{1+(1/n)}$ if G is interval graph. W. Just proved [personal communication] a better upper bound, $\chi_{FF}(G) \leqq c\omega(G)\log(\omega(G))$. We are grateful to him for allowing us to present his result as Theorem 3.4 in this paper.

Although no linear upper bound is known, the computer experiences of T. F. Liska [personal communication] with random interval families seems encouraging to accept FF coloring as a practical approximation algorithm for the on-line coloring problem of interval graphs. (The performance ratio obtained varies between 1.3 and 1.4). A result of Kierstead and Trotter ([10]) can be interpreted as an on-line algorithm to color interval graphs with at most $3\omega(G) - 2$ colors. This result is sharp, i.e., no on-line algorithm can color all interval graphs with less than $3\omega(G) - 2$ colors.

We should mention a result of C. McDiarmid in [8], which says that FF behaves quite well asymptotically, since $\chi_{FF}(G) \leqq (2 + \varepsilon) \cdot \chi(G)$ holds for almost all graphs G.

It is well known that FF coloring is ineffective on the family of bipartite graphs. We prove that bipartite graphs also resist any on-line algorithm: for all n there is a tree T_n such that $\chi_A(T_n) \geqq n$ for all on-line algorithms A (Theorem 2.5).

The power of on-line colorings depend to some extent on the absence of certain induced forests in the graphs to be colored. A graph G will be called "F-free" if G does not contain an induced subgraph isomorphic to F. It is easy to see, for example, that FF works "perfectly" on P_4-free graphs, that is, $\chi_{FF}(G) = \chi(G)$ if G is P_4-free (Proposition 2.1). On the other hand, no on-line algorithm can be effective on the family of P_6-free graphs: for all n there exists a P_6-free graph G_n such that $\chi(G_n) = 2$ and $\chi_A(G_n) \geqq n$ for all on-line algorithms A (Theorem 4.5).

Concerning small forests, we prove among other things that $\chi_{FF}(G)$ can be bounded in terms of $\omega(G)$ if G is claw-free (Theorem 4.1) or $2K_2$-free (Theorem 4.2). The family of all $(K_2 + 2K_1)$-free graphs G is an example where FF is ineffective but there exists a more sophisticated on-line coloring A for which $\chi_A(G)$ can be bounded in terms of $\omega(G)$ (Theorem 4.4).

2. FIRST FIT COLORINGS ON PERFECT GRAPHS

In this section $\chi_{FF}(G) \geqq k$ is interpreted as follows: the vertex set of G has a maximal stable sequence partition into k or more nonempty stable sets S_1, $S_2, \ldots$ satisfying that every S_i ($i = 1, 2, \ldots$) is a maximal stable set in the subgraph of G induced by $S_i \cup S_{i+1} \cup \ldots$.

For convenience, we also introduce the complementary notion of *first fit clique covering* as $\zeta_{FF}(G) = \chi_{FF}(\overline{G})$, where $\overline{G}$ denotes the complement of graph G, and we refer to the *stability number* $\alpha(G) = \omega(\overline{G})$.

Proposition 2.1. If G is a P_4-free graph (i.e., contains no path of four vertices as induced subgraph), then

$$\chi_{FF}(G) = \omega(G).$$

Proof. It is well-known (see [2]) that any maximal stable set of a P_4-free graph meets all maximal cliques of the graph. Thus the proposition follows by induction on $\omega(G)$. ∎

Concerning the role of P_4 subgraphs in on-line colorings, we note that $\chi_{FF}(G) = 2$ if and only if G is a P_4-free bipartite graph—i.e., the union of disjoint complete bipartite graphs—and has at least one edge.

Proposition 2.2. If G is a split graph (i.e., the union of a clique and a stable set with arbitrary edges between them), then $\chi_{FF}(G) \leqq \omega(G) + 1$. ∎

Since the complement of a split graph is also a split graph, by definition, we immediately obtain that $\zeta_{FF}(G) \leq \alpha(G) + 1$ holds for every split graph G.

Theorem 2.3. If G is bipartite graph, then $\zeta_{FF}(G) \leqq \frac{3}{2} \cdot \alpha(G)$, and this bound is tight.

Proof. A maximal clique sequence partition of G consists of a set F of independent edges and a stable set $X \cup Y$ of the nonsaturated vertices (X and Y are, respectively, in the first and second bipartition class of G). Obviously, $|F| + |X| \leqq \alpha(G)$, $|F| + |Y| \leqq \alpha(G)$, and $|X \cup Y| = |X| + |Y| \leqq \alpha(G)$. From these three inequalities $\zeta_{FF}(G) = |F| + |X \cup Y| \leqq \frac{3}{2} \cdot \alpha(G)$ follows.

To see that the bound is tight, let $V(G) = A \cup B \cup C \cup D$. Suppose that A, B, C, and D are pairwise disjoint stable sets of k vertices, and that $A \cup B$, $B \cup C$, and $C \cup D$ induce complete bipartite subgraphs. Then clearly, $\zeta_{FF}(G) = 3k$ and $\alpha(G) = 2k$. ∎

A graph is called *chordal* if it has no induced subgraph isomorphic to a cycle with at least four vertices. Note that the split graphs considered in Proposition 2.2 are chordal (in fact, as one can verify easily, G is a split graph if and only if both G and $\overline{G}$ are chordal graphs); furthermore, trees and interval graphs also belong to the family of chordal graphs.

Theorem 2.4. If G is chordal, then $\zeta_{FF}(G) \leq 2 \cdot \alpha(G) - 1$, and there are graphs satisfying equality.

Proof. To prove the upper bound we use induction on $|V(G)|$. The theorem is true if G is a complete graph. Let $C_1, C_2, \ldots, C_k$ be a first fit clique partition of G—that is, C_i is a maximal clique in the subgraph of G induced by $C_i \cup$

$\cdots \cup C_k$ for every i, $1 \leq i \leq k$. We shall show that $\alpha(G) \geq (k+1)/2$. Let $G^1 = G \backslash V(C_1)$. Obviously, $C_2, C_3, \ldots, C_k$ is a first fit clique partition of G^1.

If G^1 has more components than G, then the claim easily follows by using the inductive hypothesis. Otherwise, G has a simplicial vertex (a vertex whose neighborhoods induce a clique) in C_1. This vertex can be added to any maximum independent set of G^1 and again the proof follows by induction:

$$\alpha(G) \geq \alpha(G^1) + 1 \geq \frac{(k-1)+1}{2} + 1 > \frac{k+1}{2}.$$

The path $v_0, v_1, \ldots, v_{3n-1}$ with additional edges $v_{3i}v_{3i+2}$ for every $i = 0, 1, \ldots, n-1$ is an example of a graph G with $\zeta_{\mathrm{FF}}(G) = 2\alpha(G) - 1$. ∎

Theorem 2.4 says that χ_{FF} is bounded in terms of the clique number for the complement of chordal graphs. Our next result will show that this is not true for the family of chordal graphs, even more, there is no bounded on-line algorithm for trees.

Theorem 2.5. For every positive integer n there exists a tree T_n such that $\chi_A(T_n) \geqq n$ holds for every on-line algorithm A.

Proof. In the proof, on-line colorings are viewed as a two-person game; the vertices of a graph are revealed by player I and player II colors the current vertex. Suppose that I wins when II is forced to use at least n colors. We will show a winning strategy for I by defining "winning" trees T_n for every $n = 1, 2, \ldots$.

Let T_1 be the single vertex tree and assume that $T_1, \ldots, T_{n-1}$ have already been defined. Then T_n is obtained as follows: We take for every k, $1 \leqq k \leqq n-1$, $|V(T_k)|$ disjoint copies of T_k and in all copies we distinguish distinct vertices as roots. Tree T_n is formed as the union of all these rooted copies of $T_1, \ldots, T_{n-1}$ plus a new vertex x joined to every root.

Now we show how I has to play on T_n against II, who can apply an arbitrary on-line coloring algorithm A. One can assume that there are strategies for I that forces at least k distinct colors when playing on T_k $(1 \leqq k \leqq n-1)$.

We argue that I is able to obtain $n-1$ distinct colors at the roots. Assume that I has only revealed vertices from copies of $T_1, \ldots, T_{k-1}$ and forced $k-1$ distinct colors at the corresponding roots $(1 \leqq k \leqq n-1)$. Now continue the game with a copy of T_k until forcing a kth distinct color at some vertex v. Then I can freely identify T_k with its copy in T_n, which has the root corresponding to v.

Obtaining in this way $n-1$ distinct colors at the roots, I wins by revealing vertex x of T_n. This proves the theorem. ∎

3. FIRST FIT COLORING ON INTERVAL GRAPHS

The "rectangle problem" of M. Chrobak and M. Ślusarek [4] can be formulated as the question of whether FF achieves a constant performance ratio on the family of interval graphs. In more explicit form the problem is as follows:

Problem. Is there an absolute constant c such that $\chi_{FF}(G) \leqq c \cdot \omega(G)$ for every interval graph G?

For the family of unit interval graphs, M. Chrobak and M. Ślusarek proved $\chi_{FF} \leqq 2 \cdot \omega - 1$ in [5] and there are unit interval graphs (intersection graphs of unit intervals in the real line) for which the bound is tight.

For the family of all interval graphs it is easy to see that $\chi_{FF} \leqq \omega^2$. This was improved by A. Krawczyk [M. Chrobak, personal communication] to $\chi_{FF} \leqq \omega^2 - 2 \cdot \omega + 4$.

On the other hand, $\chi_{FF} \geqq 4\omega - 9$ has been proved in [5] and it was reported [M. Chrobak, personal communication] that this lower bound is improved by M. Ślusarek to $\chi_{FF} \geqq (22/5)\omega + c$ with some constant c.

The main result of this section (Theorem 3.4) is due to W. Just [personal communication]. It says that $\chi_{FF} \leqq c\omega \log \omega$, which improves the bound of Lemma 3.2.

Throughout this section $\mathcal{T}$ denotes a finite family of closed intervals of the real line R. (An interval can appear more than once in the family.) Let $G_{\mathcal{T}}$ denote the intersection graph of $\mathcal{T}$. It is natural to use the notation $\chi_{FF}(\mathcal{T}) = \chi_{FF}(G_{\mathcal{T}})$ and $\omega(\mathcal{T}) = \omega(G_{\mathcal{T}})$. We need some further notation. Let $\mathcal{T}$ be a family of intervals, $p \in R$ and $I \in \mathcal{T}$. Then we introduce

$$\rho(p, \mathcal{T}) = |\{I : I \in \mathcal{T}, p \in I\}|,$$
$$\rho(I, \mathcal{T}) = \min_{p \in I} \rho(p, \mathcal{T}),$$

and

$$\rho(\mathcal{T}) = \max_{I \in \mathcal{T}} \rho(I, \mathcal{T}).$$

It is easy to see that $\omega(\mathcal{T}) = \max_{p \in R} \rho(p, \mathcal{T})$ holds for every interval family $\mathcal{T}$. Finally, let

$$f(k) = \max\{\chi_{FF}(\mathcal{T}) : \omega(\mathcal{T}) = k\},$$

and

$$g(k) = \max\{\chi_{FF}(\mathcal{T}) : \rho(\mathcal{T}) = k\}.$$

Clearly, $f(k) \leqq g(k)$ follows from the definition. (Equality never holds; for example, $f(1) = 1, g(1) = 3$.)

In this section we interpret FF on $\mathcal{T}$ as a partition of $\mathcal{T}$. A partition $\mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_m$ is produced by FF if and only if every $\mathcal{T}_i$ contains pairwise disjoint intervals, and for all i and j, $1 \leqq i < j \leqq m$, $J \in \mathcal{T}_j$ implies $I \cap J \neq \emptyset$ for some $I \in \mathcal{T}_i$. Obviously, subfamilies $\mathcal{T}_1, \ldots, \mathcal{T}_m$ correspond to a maximal stable sequence partition of $G_{\mathcal{T}}$ produced by FF.

Proposition 3.1. For every family $\mathcal{T}$, $\rho(\mathcal{T}) \geqq \lceil \omega(\mathcal{T})/2 \rceil$.

Proof. Let $k = \omega(\mathcal{T})$, $\mathcal{H} = \{I_1, \ldots, I_k\}$ a set of pairwise intersecting intervals, and $p \in \bigcap_{j=1}^{k} I_j$. We choose the first $\lfloor (k-1)/2 \rfloor$ intervals of $\mathcal{H}$ in "increasing left end-point" order and then choose the first $\lfloor (k-1)/2 \rfloor$ intervals of $\mathcal{H}$ in "decreasing right end-point" order. At most $2 \cdot \lfloor (k-1)/2 \rfloor < k$ intervals are chosen and they cover any further $I_j \in \mathcal{H}$ separate $\lfloor (k-1)/2 \rfloor$-times. Thus $\rho(\mathcal{T}) \geqq \rho(I_j, \mathcal{H}) \geqq \lfloor (k-1)/2 \rfloor + 1 = \lceil k/2 \rceil$. ∎

The following lemma comes easily from Proposition 3.1.

Lemma 3.2. Let $1 \leqq t < k$, $m = \lceil (t+1)/2 \rceil$. Then

$$f(k) \leqq f(t) + f(k-m) + 2(k-m).$$

For small values of k, Lemma 3.2 gives $f(2) \leqq 4$, $f(3) \leqq 7$, $f(4) \leqq 12$, $f(5) \leqq 17$, etc. These upper bounds are tight only for $k = 2$ and $k = 3$ ($f(4) = 11$). The best upper bound that one can derive from Lemma 3.2 is $f(k) \leqq c(n)k^{1+1/n}$ for all n. We do not prove Lemma 3.2 since the following lemma of W. Just gives a better upper bound of $f(k)$:

Lemma 3.3 (W. Just). $g(2k) \leqq 2g(k) + 6k - 2$.

Proof. Assume that $g(2k) \geqq 2g(k) + 6k - 1 = n$. Then there exists a family $\mathcal{T}$ with $\rho(\mathcal{T}) = 2k$ such that $\mathcal{T}$ is partitioned into $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_n$ by FF. Consider the family $\mathcal{T}' = \bigcup_{i=m}^{n} \mathcal{T}_i$ where $m = g(k) + 6k - 1$. Clearly $\mathcal{T}'$ is partitioned into $n - m + 1 = g(k) + 1$ parts by FF; thus the definition of $g(k)$ implies the existence of $I \in \mathcal{T}'$ such that $\rho(I, \mathcal{T}') \geqq k + 1$.

Let A be defined as the set of those elements j from $\{1, \ldots, m-1\}$ for which $\mathcal{T}_j$ has an interval containing at least one end-point of I. Let $B = \{1, \ldots, m-1\} \backslash A$. For any fixed $j \in B$, let $\mathcal{T}_j^*$ denote the set of intervals in $\mathcal{T}_j$ that are in the interior of I. The definition of FF implies that $\mathcal{T}_j^* \neq \emptyset$ for all $j \in B$. Let $\mathcal{T}^* = \bigcup_{j \in B} \mathcal{T}_j^*$. It is obvious that $\mathcal{T}^*$ is partitioned into $|B|$ parts by FF.

On the other hand, $\rho(\mathcal{T}^*) \leqq 2k - \rho(I, \mathcal{T}') \leqq k - 1$, which implies $|B| < g(k)$. Therefore $|A| = m - 1 - |B| \geqq 6k - 1$, showing that at least $3k$ intervals of $\mathcal{T} \backslash \mathcal{T}'$ covers one of the end-points of I. Since $\rho(I, \mathcal{T}') \geqq k + 1$, that end-point of I is covered by at least $4k + 1$ intervals of $\mathcal{T}$ and Proposition 3.1 implies $\rho(\mathcal{T}) \geqq 2k + 1$, contradicting the assumption. ∎

The recursion of Lemma 3.3 clearly implies $g(k) \leqq ck \log k$. Recall that $f(k) \leqq g(k)$ and we have

Theorem 3.4 (W. Just). If G is an interval graph, $\chi_{FF}(G) \leqq c\omega(G) \log(\omega(G))$. ∎

4. ON-LINE COLORINGS OF GRAPHS WITH FORBIDDEN FORESTS

The results of this section concern the power of FF and on-line colorings in families of graphs that do not contain a fixed forest F as induced subgraph. In [6], it was conjectured that graphs in these families have proper colorings when the number of colors depends only on F and the clique number of the graph. We show that such proper colorings can be provided by FF (if F is one of the forests $K_{1,3}$, $2K_2$, and $K_{1,2} + K_1$) or by an on-line algorithm (when $F = K_2 + 2K_1$). On the other hand, $\chi(G)$ is bounded in terms of $\omega(G)$ if G is P_6-free ([7]) but no on-line algorithm can produce such a coloring of G.

Let $R(k, 3)$ denote the Ramsey function, i.e., $R(k, 3)$ is the smallest n for which any graph on n vertices contains either a complete subgraph on k vertices or three vertices inducing a stable set.

Proposition 4.1. If G is a claw-free graph, then

$$\chi_{FF}(G) \leqq R(\omega(G), 3).$$

Proof. Let $A_1, \ldots, A_m$ be the color classes defined by FF for some order of the vertices of G (which is a maximal stable sequence partition). Then for some fixed vertex $x \in A_m$ there exist vertices $x_i \in A_i$, $1 \leqq i \leqq m - 1$, such that $xx_i \in E(G)$. Since the subgraph of G induced by the set $\{x_1, \ldots, x_{m-1}\}$ contains neither a clique of $\omega(G)$ vertices nor a stable set of three vertices, $m \leqq R(\omega(G), 3)$ follows. ∎

Theorem 4.2. If G is a $2K_2$-free graph, then $\chi_{FF}(G)$ is bounded in terms of $\omega(G)$.

Proof. It is known that $\chi(G) \leqq \binom{\omega(G)+1}{2}$ if G is $2K_2$-free (see [11]). Let $A_1, \ldots, A_m$ denote the stable sets in a proper coloring of G such that $m \leqq \binom{\omega(G)+1}{2}$. Let $B_1, \ldots, B_k$ denote a maximal stable sequence partition formed by FF. We say that B_i is of type $(j_1, \ldots, j_t)$ for $1 \leqq j_1 < \cdots < j_t \leqq m$ when $B_i \cap A_j \neq \emptyset$ if and only if $j \in \{j_1, \ldots, j_t\}$.

We claim that all B_i are of different type.

Assume that B_1 and B_2 are of the same type, say type $(1, \ldots, t)$. We choose $q_i \in A_i \cap B_2$ for all i, $1 \leqq i \leqq t$. Clearly q_i is adjacent to some vertex of B_1, $1 \leqq i \leqq t$. Therefore we can choose an edge set of G in the form $\{q_{i1}p_{i2}, q_{i2}p_{i3},$

$q_{i3}p_{i4}, \ldots, q_{is}p_{i1}\}$ where $i1, \ldots,$ are distinct elements of $\{1, \ldots, t\}$ and $p_{ih} \in A_{ih} \cap B_1$ $(1 \leqq h \leqq s)$.

Suppose that this edge set is chosen so that s is as small as possible.

Consider the subgraph H induced by the vertices q_{i1}, p_{i2}, q_{is}, and p_{i1} in G. If $s = 2$, then H is isomorphic to $2K_2$, a contradiction. If $s > 2$, then $q_{is}p_{i2} \notin E(G)$ (from minimality of s), and again H is isomorphic to $2K_2$.

Thus we proved the claim that shows

$$\chi_{FF}(G) \leqq 2^m - 1 \leqq 2\binom{\omega(G)+1}{2} - 1 . \quad \blacksquare$$

We state our next theorem without proof.

Theorem 4.3. If G is a $(K_{1,2} + K_1)$-free graph, then $\chi_{FF}(G)$ is bounded in terms of $\omega(G)$. $\blacksquare$

Theorem 4.4. There exists an on-line algorithm A such that $\chi_A(G)$ is bounded in terms of $\omega(G)$ for all $(K_2 + 2K_1)$-free graphs G.

Proof. Assume that the vertices of G are given in the order $v_1, \ldots, v_n$. Let K denote a complete subgraph of G that may change during the algorithm. Initially $K = \{v_1\}$ and color v_1 with color one. We say that the algorithm is in stage i if $|K| = i$.

The algorithm A runs as follows: Assume that we are in stage i, $K = \{s_1, \ldots, s_i\}$. The vertices to be colored at state i may belong to $A^i_1, A^i_2, \ldots, A^i_i$ or to $B^i_{12}, B^i_{13}, \ldots, B^i_{i-1,i}$ according to the following rule:

If $y \in V(G)$ is adjacent to all vertices of $K\backslash\{s_j\}$ or $i = 1$, then $y \in A^i_j$. Assume that $1 \leqq j < k \leqq i$. The set $B^i_{j,k}$ denotes those vertices of $V(G)\backslash(\cup^i_{l=1} A^i_l \cup K)$ that are not adjacent to s_j and to s_k, but adjacent to all vertices of $\{s_1, \ldots, s_{k-1}\}\backslash\{s_j\}$. The definition of $A^i_1, \ldots, A^i_i$ implies that the sets A^i_j (for $j = 1, 2, \ldots, i$) and $B^i_{j,k}$ (for all pairs j, k satisfying $1 \leqq j < k \leqq i$) form a partition of $V(G)\backslash K$.

The algorithm A proceeds at stage i as follows: Let v_m be the next vertex of G to be colored. If $v_m \in B^i_{j,k}$ for some j and k, $1 \leqq j < k \leqq i$, then v_m gets a new color and A remains at stage i. If $v_m \in A^i_j$ for some j, $1 \leqq j \leqq i$, then color of v_m is the smallest color that was not used in $\{v_1, \ldots, v_m\}\backslash A^i_j$ and allowed by the previous colors assigned to $A^i_j\backslash\{v_m\}$ during stage i of A. If the vertices of $\{v_1, \ldots, v_m\}$ that belong to A^i_j during stage i of A determine a stable set in G, then A remains at stage i. Otherwise, there exists an $m' < m$ such that $v_{m'}v_m$ is an edge of G and $v_{m'} \in A^i_j$ during stage i of A. In this case stage i is finished and K is redefined as $(K\backslash\{s_i\}) \cup v_{m'}, v_m\}$. Clearly, K is a clique of G and $|K| = i + 1$.

We can change the notation to write K as $\{s_1, \ldots, s_i, s_{i+1}\}$ and A enters stage $i + 1$. Note that, for a fixed j, the vertices of A^i_j are colored with at most two colors during stage i of A. (In fact, a second color may come only at the last vertex of A_j^i during stage i of A.)

We prove that the number of colors used by A is bounded above by a function of $\omega(G)$.

It is clear that there are at most $\omega(G)$ stages during A, since at stage i there exists a clique of G with i vertices. For a fixed i ($1 \leqq j < k \leqq i$) the vertices of $B_{j,k}^i$ colored during stage i of A induce a clique in G, since G is $(K_2 + 2K_1)$-free. Therefore at most $\omega(G)$ colors are used on $B_{j,k}^i$ during stage i of A (for fixed i, j, and k). As we noted before, the vertices of A_j^i for fixed j and i, $1 \leqq j \leqq i$, are colored with at most two colors. Therefore, the number of colors used by A at stage i is at most $\omega(G) \cdot \binom{i}{2} + 2i$. The total number of colors used by A can be estimated as

$$\sum_{i=1}^{\omega(G)} \omega(G) \cdot \binom{i}{2} + 2i . \quad \blacksquare$$

It is worth noting that the idea used in the proof of Theorem 4.4 can be extended to get an on-line algorithm A for which $\chi_A(G)$ is bounded in terms of $\omega(G)$ and t, for all tK_2-free graphs G.

We also note that A cannot be changed to FF in Theorem 4.4. To see this, let G_n denote the graph obtained from the complete $n - n$ bipartite graph by removing n independent edges. Clearly, G_n is $(K_2 + 2K_1)$-free. Moreover, G_n is the standard example to demonstrate $\chi(G_n) = 2$ and $\chi_{\mathrm{FF}}(G_n) = n$ (see, e.g., [8]). We note that there is an obvious on-line coloring that uses three colors for all G_n.

Our final result in this section shows that on-line colorings are ineffective on P_6-free graphs.

Theorem 4.5. For every positive integer n there exists a bipartite P_6-free graph G_n such that $\chi_A(G_n) \geqq n$ holds for every on-line algorithm A.

Proof. This proof is similar to that of Theorem 2.5, where on-line colorings are viewed as a two-person game. The vertices of a graph are revealed by player I and player II colors the current vertex. Suppose that I wins when II is forced to use at least n colors. We give winning strategy for I by defining P_6-free bipartite graphs $G_1, G_2, \ldots$.

Let G_i be a single vertex and assume that $G_1, \ldots, G_{n-1}$ have already been given. Then G_n is defined as follows: We take for every k, $1 \leqq k \leqq n - 1$, disjoint copies of G_k, say G_k^1 and G_k^2. We add a new vertex x and join with every vertex of G_k^1 lying in the first bipartite class and with every vertex of G_k^2 lying in the second bipartite class, $1 \leqq k \leqq n - 1$. (For $k = 1$ the two classes coincide.)

One can easily check that G_n is bipartite and contains no induced P_6. Now we show how to play on G_n against II.

We argue that I is able to obtain $n - 1$ neighbors of x colored with distinct colors. Assume that I revealed only vertices from copies of $G_1, \dots, G_{k-1}$ and forced $k - 1$ distinct colors on the neighbors of x. Now I can continue the game with revealing vertices from a copy of G_k until forcing a new color at some vertex v. Let us identify this copy with G_k^1 or G_k^2 according to which v lies in the first or second bipartite class of G_k. Obtaining $n - 1$ distinct colors at the neighbors of x in this way, I wins by revealing vertex x. ∎

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