CHAPTER 1

ALGEBRA

Exercise: 1.1

1. (a) (i) $f(0) = 4 \times 0 + 5 = 5$ (ii) $f(6) = 4 \times 6 + 5 = 29$

(iii) $f(-2) = 4 \times (-2) + 5 = -3$ (iv) $f(2a) = 4 \times 2a + 5 = 8a + 5$

(v) $f(2 + a) = 4(2 + a) + 5 = 13 + 4a$

(vi) $f(a - 3) = 4 \times (a - 3) + 5 = 4a - 12 + 15 = 4a - 7$

(vii) $f(x + h) - f(x) = 4x + 4h + 5 - 4x - 5 = 4h$

(viii) $\dfrac{f(x + h) - f(x)}{h} = \dfrac{4h}{h} = 4$

(b) (i) $f(0) = 4 \times 0 = 0$ (ii) $f(1) = 4 + 1^2 = 4 + 4 = 8$

(iv) $f(5) = 2 \times 5 + 6 = 16$ (v) $f(-5) = 4 \times (-5) = 20$

(vi) $f(2 + h) = 2$(Negative number) $= 4(2 + h) = 8 + 4h$

(vii) $f(3 + h)$ $[3 + h \geq 0] = f(0)$ minimum, $f(2) =$ Minimum $= 4 \times 0 = 0$

or $4(3 + h)$ or $4 + (3 + h)^2$

2. (a) $f^{-1} = \{(1, 1), (8, 2), (27, 3)\}$ (b) $f^{-1} = \{(-1, 1)\ (2, 2), (-3, 3)\}$

3. (a) Let, $y = f(x) \Leftrightarrow x = f^{-1}(y)$. Now, $\frac{1}{2}(y + 3) = x$. i.e. $f^{-1}(y) = \frac{1}{2}(y + 3)$.

i.e. $f^{-1}(x) = \frac{1}{2}(x + 3)$

(b) Let, $y = h(x) \Leftrightarrow x = f^{-1}(y)$. Now, $\frac{1}{3}(y + 6) = x$. i.e. $h^{-1}(y) = \frac{1}{3}(y + 6)$.

i.e. $h^{-1}(x) = \frac{1}{3}(x + 6)$

(c) Inverse of a function exist iff the function is bijective. $f^{-1}(x) = (x - 5)^{1/3}$.

4. (a) Here domain of f = {1, 4, 9, 16}. Range of g = {3, 4, 9, 8}. Since, Range of g is not subset of domain of g, so fog does not exist.

Similarly, domain of g = {1, 3, 4, 5}, Range of f = {1, 3, 4} i.e. Range of f is subset of domain of g, so gof exist. gof = {(1, – 2), (4, 4), (9, – 6), (16, 8)}

(b) Domain of f = {3, 9, 12}. Range of f = {1, 3, 4}.

Domain of g = {1, 3, 4, 5}. Range of g = {3, 9}.

Since, Range of f $\subseteq$ Domain of g and Range of g $\subseteq$ Domain of f, so both fog and gof exist.

fog = {(3, 3), (9, 3), (12, 9)}. gof = {(1, 1), (3, 1), (4, 3), (5, 3)}.

5. (a) $f(x) = 5x - 3$, $g(x) = x - 2$. $f(g(x)) = f(x - 2) = 5(x - 2) - 3 = 5x - 10 - 3 = 5x - 13$. $f(g(1)) = 5 \times 1 - 13 = -8$.

$g(f(x)) = g(5x - 3) = 5x - 3 - 2 = 5x - 5$. $g(f(1)) = 5 \times 1 - 5 = 0$

(b) $f(g(x)) = f(x^2 + 2) = 2(x^2 + 2) - 3 = 2x^2 + 1$

$g(f(x)) = g(2x - 3) = (2x - 3)^2 + 2 = 4x^2 - 12x + 9 + 2 = 4x^2 - 12x + 11$

6. (a) $gof = g(x^2 + 3x + 1) = 2(x^2 + 3x + 1) - 3 = 2x^2 + 6x - 1$

(b) $fog = f(2x - 3) = (2x - 3)^2 + 3(2x - 3) + 1 = 4x^2 - 6x + 1$

(c) $fof = f(x^2 + 3x + 1) = (x^2 + 3x + 1)^2 + 3(x^2 + 3x + 1) + 1$

(d) $gog = g(2x - 3) = 2(2x - 3) - 3 = 4x - 9$

7. (a) $f(x) = x^2 + 1 \Rightarrow y = x^2 + 1 \Rightarrow (y - 1)^{1/2} = x \Rightarrow f^{-1}(x) = (x - 1)^{1/2}$.

$(gof)(x) = g(f(x)) = g(x^2 + 1) = (x^2 + 1)^5$

$(fog)(x) = f(g(x)) = f(x^5) = (x^5)^2 + 1 = x^{10} + 1$.

8. (a) $(fog)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2)^2 - 3 = 2(9x^2 + 12x + 4) - 3 = 18x^2 + 24x + 5$.

(b) $(gof)(x) = g(f(x)) = g(2x^2 - 3) = 3(2x^2 - 3) + 2 = 6x^2 - 7$.

(c) (fog) (x) and (gof) (x) are not one-to-one because different per-images have not different images. $[6x_1^2 = 6x_2^2 \Rightarrow x_1 = \pm x_2]$

9. (a) $(fog)(x) = f(g(x)) = f(4x - 1) = (4x - 1)^3 + 2$

(b) $(gof)(x) = g(f(x)) = g(x^3 + 2) = 4(x^3 + 2) - 1 = 4x^3 + 7$

(c) Since, $(fog)(x) \neq (gof)(x)$, so the composite function is not commutative.

10. (a) $f(x) = x^3 - 1$ i.e. $(x + 1)^{1/3} = f^{-1}(x)$

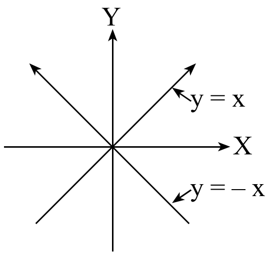
$g(x) = 2x - 3$ i.e. $\frac{1}{2}(x + 3) = g^{-1}(x)$.

So, (a) $(f^{-1}og)(2) = f^{-1}(g(2)) = f^{-1}(2 \times 2 - 3) = f^{-1}(1) = 2^{1/3}$

(b) $(fog^{-1})(1) = f(g^{-1}(1)) = f\left(\frac{1}{2}(1 + 3)\right) = f(2) = 2^3 - 1 = 7$.

11. (a) Slope of a line = tan 45° = 1

(b) The slope of two lines equally inclined with co-ordinate axes = (+ 1) and (– 1). $y = \pm x$.



12. (a) Slope of the line $= \dfrac{16-(-2)}{-3-3} = \dfrac{18}{-6} = -3$

(b) Slope of the line $= \dfrac{b-0}{0-a} = -\dfrac{b}{a}$.

(c) $(2-x)^2 + (2-5)^2 = 5^2$

i.e. $(2-x)^2 = 4^2$

i.e. $2 - x = \pm\sqrt{4^2} = \pm 4$, $x = -2$ and 6

So, slope $= \dfrac{2-5}{2+2}$ or $\dfrac{2-5}{2-6} = -\dfrac{3}{4}$ or $\dfrac{3}{4}$.

13. (a) $-7 \le 2x + 5 < 7$,

i.e. $-7-5 \le 2x < 7-5$,

i.e. $\dfrac{-12}{2} \le x < \dfrac{2}{2}$

i.e. $-6 \le x < 1$ Proved.

(b) $-5 \le 2x + 3 \le 7$

or, $-5-3 \le 2x \le 7-3$ i.e. $-\dfrac{8}{2} \le x \le \dfrac{4}{2}$, i.e. $-4 \le x \le 2$.

(c) $-14 < 3x - 8 < -2$, i.e. $-14 + 8 < 3x < -2 + 8$.

i.e. $-6 < 3x < 6$, i.e. $-\dfrac{6}{3} < \dfrac{3x}{3} < \dfrac{6}{3}$, i.e. $-2 < x < 2$. Proved.

14. (a) $|-9| + |4| - |-2| = 9 + 4 - 2 = 11$.

(b) $|-6| + |-1| + |7| = 6 + 1 + 7 = 14$

(c) $|3| - |-5| + |-8| = 3 - 5 + 8 = 6$

15. (a) $\left|\dfrac{x}{y}\right| = \left|\dfrac{4}{(-2)}\right| = |-2| = 2$. $\dfrac{|x|}{|y|} = \dfrac{|4|}{|-2|} = \dfrac{4}{2} = 2$

(b) $|x + y| = |-2-1| = 3$. $|x| + |y| = |-2| + |-1| = 3$.

$\therefore$ $|x + y| \le |x| + |y|$.

(c) $|x - y| = \left|\dfrac{1}{2} + \dfrac{3}{2}\right| = |2| = 2$. $|x| - |y| = \left|\dfrac{1}{2}\right| - \left|-\dfrac{3}{2}\right| = \dfrac{1}{2} - \dfrac{3}{2} = -1$

$\therefore \quad |x - y| \geq |x| - |y|.$

16. (a) $-7 \leq 2x + 5 \leq 7$. i.e. $-7 - 5 \leq 2x \leq 7 - 5$. i.e. $-12 \leq 2x \leq 2$. i.e. $-6 \leq x \leq 1$.

(b) $-14 < 3x - 8 < -2$ i.e. $-14 + 8 < 3x < -2 + 8$, i.e. $-\frac{6}{3} < x < \frac{6}{3}$ i.e. $-2 < x < 2$.

17. (a) $|3x + 5| \leq 5$

i.e. $-5 \leq 3x + 5 \leq 5$

i.e. $-5 - 5 \leq 3x \leq 5 - 5$

i.e. $\frac{-10}{3} \leq x \leq 0$

(b) $|2x + 3| < 2$ i.e. $-2 < 2x + 3 < 2$

i.e. $-2 - 3 < 2x < 2 - 3$

i.e. $\frac{-5}{2} < x < \frac{-1}{2}$

(c) $|3 - 5x| \leq 2x$ i.e. $-2 \leq 3 - 5x \leq 2$

i.e. $-2 - 3 \leq -5x \leq 2 - 3$

i.e. $\frac{-5}{(-5)}\, x \geq \frac{-1}{(-5)}$

i.e. $1 >, x \geq, \frac{1}{5}$

i.e. $\frac{1}{5} \leq x \leq 1$

18. (a) $-1 \leq x \leq 5$ |i.e. $-2 \leq 2x \leq 10$| [Mean of -1 and $5 = \frac{-1+5}{2} = 2$]

So, $-1 - 2 \leq x - 2 \leq -5 - 2$ i.e. $-3 \leq x - 2 \leq 3$ i.e. $|X - 2| \leq 3$.

(b) Mean of -5 and $-2 = \frac{-7}{2}$. So, multiply each by 2.

$-10 \leq 2x \leq -4$ i.e. $-10 + 7 \leq 2x + 7 \leq -4 + 7$. i.e. $-3 \leq 2x + 7 \leq 3$

i.e. $|2x + 7| \leq 3$

(c) Mean of -4 and $7 = \frac{1}{2}(7 - 4) = \frac{3}{2}$ So, Multiple each them by 2. $-8 \leq 2x \leq 14$

i.e. $-8 - 3 \leq 2x - 3 \leq 14 - 3$. i.e. $-11 \leq 2x - 3 \leq 11$. $|2x\ \ 3| \leq 11$.

19. (i) Let f: R$\rightarrow$ R. (i) $f(x) = \frac{1}{x}$ [$x \neq 0$] then, $f^{-1}(x) = \frac{1}{x}$. $|2x - 3| \leq 11$,.

(ii) $f(x) = \frac{2}{x}$ then $f^{-1}(x) = \frac{2}{x}$. $f(x) = \frac{3}{x}$ then $f^{-1}(x) = \frac{3}{x}$. [we can give other examples also.]

20. $(f\,0g)\,(x) = f(g(x)) = f\,(2x + 2) = 3\,(2x + 2) + 4 = 6x + 10.$

$(g0f)\,(x) = g(f(x)) = g(3x + 4) = 3\,(2(x + 1) + 4 = 6x + 6 + 4 = 6x + 10.$

Exercise 1.2

1. Concern introductory part of the book.

2. (a) Here, $Q = -5p + 35$, slope $= -5$

(i) At $P = 3$, $Q = 35 - 15 = 20$.

So, elastically of demand $= \frac{P}{Q} \times m = \frac{3}{20} \times (-5) = \frac{-3}{4}$. $|Ed| = \frac{3}{4}$.

(ii) At $P = 4$, $Q = 35 - 20 = 15$. $|Ed| = |\frac{4}{15} \times (-5)| = \frac{4}{3}$.

(b) $Q = 3P + 20$. Slope (m) $= 3$ (i) At $P = 10$. $Q = 3- + 20 = 50$.

$|Ed| = \left|\frac{P}{Q} \times M\right| = \left|\frac{10}{50} \times 3\right| = \frac{3}{5}$.

(ii) At $P = 15$, $Q = 65$ $|Ed| = \left|\frac{15}{65} \times 3\right| = \frac{15}{13}$

(c) $P = -4Q + 40$, $Q = \frac{-1}{4}P + 10$, $M = \frac{-1}{4}$

When $Q = 4$, $P = 2$ $|Ed| = \left|\frac{26}{4} \times \left(\frac{-1}{4}\right)\right| = \frac{13}{8}$

3. (a) Elastically of commodity demand $= \frac{\text{Percentage change in quantity}}{\text{Percentage change in price}} = \frac{30}{6}$

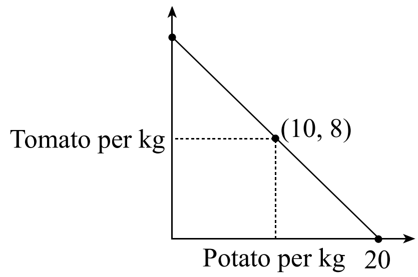
(b) Elastically of commodity demand $= \frac{40}{12} = 333.$

4. (a)

Potato (Per kg)	Tomato (Per kg)	Total cost
20	0	20 × 40 = Rs 800
10	8	10 × 40 + 850 = Rs 800
0	16	16 × 50 = Rs 800

∴ Consumer income = Rs 800

(b)



(c)

Potato (Kg)	Tomato (kg)	Total cost (Rs)
40	0	800
20	8	800
0	16	800

(d)

Potato (Kg)	Tomato (kg)	Total cost (Rs)
40	0	1600
20	18	1600
0	32	1600

Total income = Rs 1600

5. (a) Percentage changed in income = $\frac{\text{Rs } 500}{\text{Rs } 1500} \times 100\ \% = \frac{100}{3}\ \%$

Percentage changed in demand = $\frac{5}{10} \times 100\% = 50\%$

So, income elastically of demand = $\frac{\text{Percentage change in quantity}}{\text{Percemtage change in income}}$

So, 1% increase in income results 1.5% increases in demand

(b) Percentage change in income = $\frac{50000 - 40000}{40000} \times 100\% = 25\%$

Percentage change in demand = $\frac{32 - 25}{25} \times 100\% = 28\%$

$\therefore$ Income elastically of demand = $\frac{28}{25} = 1.12$

Comment: 1% increase in income results 1.12% increase in demand

6. (a) Elastically of demand $= \dfrac{(\text{change quantity demand}) \times (\text{Inital Income})}{(\text{Initial quantity demanded}) \times (\text{Change in Income})}$

$$= \frac{2 \times 20000}{3 \times 5000} = 0.8$$

(b) 8% changes in quantity changes 0.8% changes in income.

Elasticity of demand $= \dfrac{2 \times 750}{3 \times 450} = 1.11\%$

$\boxed{\text{1\% increase in price changes 1.11\% increase in demand}}$

7. Concern 1.2.3 or find in Google search.

Exercise 1.3

1. Concern introductory part of the book.

2. Slope $= \dfrac{-a}{b}$, Y - intercept $= \dfrac{-c}{a}$

3. (a) 2 (b) 3 (c) $3x - 3 + 6 = 0, 3x = -3, [x = -1]$

4. (a) $x + 2y = 7 \Rightarrow x + 2y = 7$

$3x + y = 35 \Rightarrow 6x + 2y = 70$

Adding: $7x = 77, x = 11y = -2, [(x, y) = (11, -2)]$

(b) $12x + 85y + 49 \Rightarrow 2 \times 12x + 2\ 85y + 2 \times 49 = 0 \Rightarrow 24x + 170y + 98 = 0$... (i)

$19x - 34y - 91 = 0 \Rightarrow 5 \times 19 + 5 \times 34y - 5 \times 91 = 0 \Rightarrow 95x + 170y - 455 = 0$ ... (ii)

Adding equation (i) and (ii), $119x = + 357 \Rightarrow x = 3, y = 1$ $[(x, y) = (3, 1)]$

(c) $1 \times [2x - 4y = 3] \Rightarrow 2x = 4y = 3$ ... (i)

$2 \times [3x - 2y = -4] \Rightarrow 6x - 4y = 8$ ... (ii)

(–) (+) (–)

Subtracting $-4x = -5 \Rightarrow x = \dfrac{5}{4}$

$2 \times \dfrac{5}{4} - 3 = 4y, y = -\dfrac{1}{8}$

$\left[\therefore (x, y) = \left(\dfrac{5}{4}, \dfrac{-1}{8}\right)\right]$

(d) $5 \times [7x - 3y = -17] \Rightarrow 35x + 15y = -85$ ... (i)

$3 \times [2x + 5y = 1]\ 6x + 15y = 3$ ... (ii)

Adding (i) and (ii) $41x = -82, x = -2, y = 1$ $[\therefore (x, y) = (-2, 1)]$

(e) $3 \times [3x + 4y = 27] \Rightarrow 9x + 12y = 81$ ... (i)

$4 \times [5x - 3y = 16] \Rightarrow 20x - 12y = 64$ ... (ii)

Adding (i) and (ii): 29x = 145, x = 5, y = 3

(f) $9 \times [14x + 15y = 116] \Rightarrow 126x - 12x = 81$ ... (i)

$5 \times [32x - 27y = 20] \Rightarrow 160x - 135y = 100$ ...(ii)

Adding (i) and (ii): $286x = 1144, \Rightarrow x = 4, y = 4$ [$\therefore$ (x, y) = (4, 4)]

5. (a) $\frac{1}{3} \times \left[\frac{x}{3} - \frac{y}{2} = 2\right] \Rightarrow \frac{x}{9} - \frac{y}{6} = \frac{2}{3}$ ... (i)

$\frac{1}{2} \times \left[\frac{x}{4} - \frac{y}{3} = 2\right] \Rightarrow \frac{x}{8} - \frac{y}{6} = 1$ ... (ii)

(i) – (ii): $\frac{x}{9} - \frac{x}{8} = \frac{2}{3} - 1$

i.e. x = 2y, y – 12 [$\therefore$ (x, y) = (24, 12)]

(b) $\frac{1}{4} \times \left[\frac{2}{3}x + y = 18\right] \Rightarrow \frac{1}{6}x + \frac{1}{4}y = \frac{9}{2}$ ... (i)

$1 \times \left[\frac{1}{4}x + \frac{1}{4}y = 12\right] \Rightarrow \frac{1}{4}x + \frac{1}{4}y$ ... (ii)

(i) – (ii): $\frac{1}{6}x - \frac{1}{4}x = \frac{9}{12} - 12$ i.e. x = 90, y = – 42 [$\therefore$ (x, y) = (90, – 42)]

(c) $1 \times \left[\frac{x}{9} - \frac{y}{3} = 2\right] \Rightarrow \frac{x}{9} - \frac{y}{3} = 2$ ... (i)

$\frac{1}{9} \times [2x - 3y = -3] \Rightarrow \frac{2}{9}x - \frac{y}{3} = \frac{-1}{3}$ ... (ii)

(i) – (ii): $\frac{x}{9} - \frac{2x}{9} = 2 + \frac{1}{3}$ i.e. $\frac{-x}{9} = \frac{7}{3}$, x = – 21, y = – 123 [$\therefore$ (x, y) = (– 21, – 13)]

(d) $\frac{2}{3x + y} = \frac{17}{3} \Rightarrow [51x + 17y = 6] \times 2 \Rightarrow 102x + 34y = 12$ ... (i)

$\frac{5}{x + 2y} = \frac{17}{3} \Rightarrow [17x + 34y = 15] \times 1 \Rightarrow 17x + 34y = 16$ ... (ii)

(–) (–) (–)

$85x = -3, x = \left(\frac{-3}{85}\right), y = \frac{39}{85}$

$\left[\therefore (x, y) = \left(\frac{-3}{95}, \frac{39}{85}\right)\right]$

(e) $\frac{x + 1}{10} = \frac{3y - 5}{8} \Rightarrow 2x + 2 = 30y - 50 \Rightarrow x - 15y = -26$ ... (i)

$\frac{x + 1}{10} = \frac{x - y}{8} \Rightarrow 8x + 8 = 10x - 10y \Rightarrow x - 5y = 4$ ... (ii)

(i) – (ii): – 15y + 5y = – 26 – 4 i.e. – 10y = – 30, y – 3, x = 19

[$\therefore$ (x, y) = (19, 3)]

(f) $\frac{y+9}{5} = \frac{2x+1}{3} \Rightarrow [10x + 3y = + 22] \times 1 \Rightarrow 10x - 3x = + 22$ ... (i)

$\frac{y+9}{5} = \frac{x+y+2}{4} \Rightarrow [5x + y = 26] \times 3 \Rightarrow 15x + 13y = 78$ ... (ii)

Adding (i) and (ii): $25x = 100$, $x = \frac{100}{25} = 4$, $y = 6$, [$\therefore$ (x, y) = (4, 6)]

(g) $\frac{x+1}{8} = \frac{y+3}{5} \Rightarrow 5x - 8y = 19$ ... (i)

$\frac{x+1}{8} = \frac{x-y}{4} \Rightarrow 4x - 8y = 4$ ... (ii)

(i) and (ii) $5x - 4x = 19 - 4$ i.e. $x = 15$, $y = 7$ [$\therefore$ (x, y) = (15, 7)]

6. (a) $\frac{1}{2} \times \left[\frac{x}{2} + \frac{y}{3} = 1\right] \Rightarrow \frac{x}{4} + \frac{y}{6} = \frac{1}{2}$ ... (i)

$\frac{1}{3} \times \left[\frac{x}{3} + \frac{y}{2} = 1\right] \Rightarrow \frac{x}{9} + \frac{y}{6} = \frac{1}{3}$ ... (ii)

(i) – (ii): $\frac{x}{4} - \frac{x}{9} = \frac{1}{2} - \frac{1}{3}$ i.e. $\frac{5x}{63} = \frac{1}{6}\left[\therefore (x, y) = \left(\frac{6}{5}, \frac{6}{5}\right)\right]$

(b) $\frac{2x-3y}{21} = \frac{x-11}{8} \Rightarrow [5x + 24y = 231] \times 16 \Rightarrow 80x + 384y = 3696$ ... (i)

$\frac{2x-3y}{21} = \frac{y+3\frac{1}{2}}{20} \Rightarrow [80x - 162y = 147] \times 1 \Rightarrow 80x - 162y = 147$ ... (ii)

(i) – (ii): $546y = 3549$, $x = 15$, $y = \frac{13}{2}\left[\therefore (x, y) = \left(15, \frac{13}{2}\right)\right]$

(c) $\left[\frac{x}{5} + y = 4\right] \times 3 \Rightarrow \frac{3x}{5} + 3y = 12$ ... (ii)

$\left[\frac{x}{2} + 3y = 1\right] \times 1 \Rightarrow \frac{x}{2} + 3y = 1$ ... (ii)

(i) – (ii): $\frac{3x}{5} - \frac{x}{2} = 12 - 1$

i.e. $\frac{x}{10} = 11$

i.e. $x = 110$, $y = -18$ [$\therefore$ (x, y) = (110, – 18)

(d) $\frac{x-5}{3} = \frac{3y+11}{2} \Rightarrow 2x - 9y = 43$ ... (i)

$\frac{x = 4y - 1}{3} = \frac{x-5}{3} \Rightarrow 4y = -4$ ... (ii)

From (ii), $y = -1$ and from (i) $x = 17$

$\therefore \quad (x, y) = (17, -1)$

(i) – (ii): $7x = 7$, i.e. $x = 1, y = 2$ [$\therefore (x, y) = (1, 2)$]

7. (a) $x + 3 = \frac{8}{y}$, i.e. $\left[x - \frac{8}{y} = -3 \text{ ... (i)}\right] \times 1 \Rightarrow x \frac{8}{y} = -3$

$5 - \frac{4}{y} = 3x$, i.e. $\left[3x + \frac{4}{y} = 5 \text{ ... (ii)}\right] \times 2 \Rightarrow 6x + \frac{8}{y} = 10$

(i) – (ii): $7x = 7$, i.e. $x = 1, y = 2$ [$\therefore (x, y) = (1, 2)$]

(b) $\left[\frac{2}{x} + \frac{3}{y} = 2\right] \times 3 \Rightarrow \frac{6}{x} + \frac{9}{y} = 6$ ... (i)

$\left[\frac{8}{x} + \frac{9}{y} = 7\right] \times 1 \Rightarrow \frac{8}{x} + \frac{9}{y} = 7$ ... (ii)

(i) – (ii): $\frac{6}{x} - \frac{8}{x} = 6 - 7$ i.e. $x , y = 3$ [$\therefore (x, y) = (2, 3)$]

(c) Same as 7 (a)

(d) $\left[\frac{2}{x} + \frac{3}{y} = 2\right] \times 10 \Rightarrow \frac{20}{x} + \frac{30}{y} = 20$ ... (i)

$\left[\frac{5}{x} + \frac{10}{y} = 5\frac{5}{6}\right] \times 3 \Rightarrow \frac{5}{x} + \frac{30}{y} = \frac{35}{2}$ ... (ii)

(i) – (ii): $\frac{20}{x} - \frac{15}{x} = 20 - \frac{35}{2}$ i.e. $\frac{5}{x} = \frac{5}{2}$, $x = 2, y = 3$ [$\therefore (x, y) = (2, 3)$]

(e) $\left[\frac{3}{4}x + \frac{4}{5} \times \frac{1}{y} = \frac{31}{20}\right] \times \frac{5}{4} \Rightarrow \frac{15}{16}x + \frac{1}{y} = \frac{31}{16}$ ... (i)

$\left[\frac{4}{5}x + \frac{5}{6} \times \frac{1}{y} = \frac{49}{30}\right] \times 3 \Rightarrow \frac{15}{x} + \frac{30}{y} = \frac{35}{2}$ ... (ii)

(i) – (ii): $\frac{20}{x} - \frac{24}{25}x = \frac{31}{16} - \frac{49}{25}$, $x = 1, y = 1$ [$\therefore (x, y) = (1, 1)$]

(f) $\left[\frac{2}{x} + \frac{3}{y} = 1\right] \times 4 \Rightarrow \frac{8}{x} + \frac{12}{y} = 4$ ... (i)

$\left[\frac{7}{x} + \frac{4}{y} = 1\frac{7}{8}\right] \times 3 \Rightarrow \frac{21}{x} + \frac{21}{y} = \frac{45}{8}$ ... (ii)

(i) – (ii): $\frac{8}{x} - \frac{21}{x} = 4 - \frac{45}{8}$, i.e. $x = 8, y = 4$ [$\therefore (x, y) = (8, 4)$]

8. (a) $[4x - 9y = 5xy] \times 5 \Rightarrow 20x - 45y = 25xy$ ... (i)

$[41x + 15y = 2xy] \times 3 \Rightarrow 42x + 45y = 6xy$ ... (ii)

(i) + (ii): $62x = 31xy$

i.e. $\frac{62}{y} = 31$, y = 2, x = – 3 [$\therefore$ (x, y) = (– 3, 2)]

(b) $3y + 2x = xy \Rightarrow \left[\frac{3}{x} + \frac{2}{y} = 1\right] \times 2 \Rightarrow \frac{6}{x} + \frac{4}{y} = 2$ ... (i)

$5y + 4x = \frac{13}{8}x \Rightarrow \left[\frac{5}{x} + \frac{4}{y} = \frac{13}{8}\right] \times 1 \Rightarrow \frac{5}{x} + \frac{4}{y} = \frac{13}{8}$ ... (ii)

(i) – (ii): $\frac{6}{x} - \frac{5}{x} = 2 - \frac{13}{8}$, $x = \frac{8}{3}$, $y = -16\left[\therefore (x, y) = \left(\frac{8}{3}, -16\right)\right]$

9. (a) $\left[\frac{25}{x-y} + \frac{33}{x+} = 8\right] \times 7 \Rightarrow \frac{120}{x-y} + \frac{231}{x+y} = 56$ ... (i)

$\left[\frac{40}{x-y} + \frac{77}{x+y} = 15\right] \times 3 \Rightarrow \frac{120}{x-y} + \frac{231}{x+y} = 56$ ... (i)

(i) – (ii): $\frac{175}{x-y} - \frac{120}{x-y} = 56 - 45$ i.e. x – y= 5, x + y = 11

$\therefore$ (x, y) = (8, 3)

(b) $5 \times \left[\frac{2}{x+3y} + \frac{3}{3x-y} = 2\right] \Rightarrow \frac{10}{x+3y} + \frac{3}{3x-y} = 10$ ... (i)

$2 \times \left[\frac{5}{x+3y} + \frac{3}{2(3x-y)} = 3\right] \Rightarrow \frac{10}{x+y} - \frac{24}{x-y} = 6$ ... (ii)

(i) – (ii): $\frac{12}{3x-y} = 4 \Rightarrow 3x - y = 3$, x = 3y = 2 solving both $\left[\therefore (x, y) = \left(\frac{11}{10}, \frac{-9}{10}\right)\right]$

(c) $3 \times \left[\frac{12}{x+y} + \frac{8}{x-y} = 8\right] \Rightarrow \frac{36}{x+y} + \frac{24}{x-y} = 24$ ... (i)

$2 \times \left[\frac{27}{x+y} - \frac{12}{x-y} = 3\right] \Rightarrow \frac{54}{x+y} - \frac{24}{x-y} = 6$ ... (ii)

(i) + (ii): $\frac{90}{x+y} = 30$, x + y = 3, x – y = 2 $\left[\therefore (x, y) = \left(\frac{5}{2}, \frac{1}{2}\right)\right]$

10. (a) x + y + z = 9. 9i) 2x + 5y + 7z + 52, (ii) 2x + y – z = 0 ... (iii)

(i) + (iii): 3x + 2y = 9 and, (ii) + 7 (iii): 16x + 12y = 52 [$\therefore$ (x, y, z) = (1, 3, 5)]

(b) x + 2y + 3z = 10 ... (iii)

13x + 6y + z = 52 ... (i)

5x + 7y + 9z = 38 ... (ii)

9 × equation (i) – equation (ii): 112x + 47y = 430 [$\therefore$ (x, y, z) = (3, 2, 1)]

3 × equation (i) – equation (ii): 19x + 8z = 73

(c) 2x + 3y – 4z = – 1 ... (i)] (i) + 2 × (ii) – 8x – y = 27

3x – 2y + 2z = 14 ... (ii)] (i) + 4 × (iii) – 38 x + 35y = 23

– 10x + 8y + z = 6

[$\therefore$ (x, y, z) = (4, 5, 6)]

8(8x – y = 27) + (– 38x + 35y = 23)

i.e. x = 4, y = 5, z = 6.

11. (a) 2 (4x + 9y) = 7(2y + z) i.e. 8x + 4y – 7z = 0, (i) 7(x + 2y) = 8(y + z) i.e. 7x + 6z – 8z = 0 (ii) 3x + 4y + 5z = 38 (iii) [By 8 × equation (i) – 7 × equation (ii)] $\Rightarrow$ 8x – 10y = 0, i.e. 15x = 10y, 3x = 2y [5 × equation (i) + 7 × equation (iii)] $\Rightarrow$ 61x + 48y = 266 i.e. 61x + 48 × $\frac{3x}{2}$ = 266 i.e. x = 2, y = 3, z = 4 [$\therefore$ (x, y, z) = (2, 3, 4)]

(b) 2 (2x + y) + 3y = 61 i.e. 4x + 2y + 3z = 61 ... (i)

4x + 2y + 3z –+ 2x + y + 3z = 61 + 30 i.e. 2x + 3y = 31 ... (ii)

(i) + 3 × (ii): 13x + 8y = 121

Now, (i) – (iii): 4x + 2y + 3z – 2x + y – 3z = 61 – 30 i.e. 2x + 3y = 31

131x + 8y = 121 and 2x + 3y = 31

3 × [13x + 8y – 121] – 8 [2x + 3y – 31] = 0

i.e. 39x – 363 – 16x + 248 = 0 $\Rightarrow$ x = 5, y = 7, z = 9 [$\therefore$ (x, y, z) = (5, 7, 9)]

(c) 3x – 4y – 6z = – 16 ... (i0 4x – y – z = 5 ... (ii) x – 3y – 2z = – 2 ... (ii)

(i) – 6 × (ii): (3x – 24x) + (– 4y + 6y) = – 16 – 30 i.e. – 21x + 2y = – 46

(i) – 3 × (iii): (3x – 3x) + (– 4y + 9y) = – 16 + 6 i.e. 5y = – 10, i.e. y = – 2, z = 5, x = 2

[$\therefore$ (x, y, z) = (2, – 2, 5)]

12. (a) Here, x = 4k, y = 2k and z = 3k (say) x + 3y + z = 26

i.e. 4k + 6k + 3k = 26, i.e. k = 2. So, x = 8, y = 4 and z = 6

(b) $\left[\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 6\right] \times 3 \Rightarrow \frac{3}{x} + \frac{3}{y} + \frac{3}{z} = 18$ ... (i)

$\left[\frac{2}{x} - \frac{3}{y} + \frac{4}{z} = 8\right] \times 1 \Rightarrow \frac{2}{x} - \frac{3}{y} + \frac{4}{z} = 8$ ... (ii)

Adding (i) and (ii) $\frac{5}{x} + \frac{7}{z} = 26$

Again,

$\left[\frac{2}{x} - \frac{3}{y}\frac{4}{z} = 11\right] \times 7 = \frac{14}{x} - \frac{21}{y} + \frac{28}{z} = 56$ ... (iii)

$\left[\frac{3}{x} - \frac{7}{x} + \frac{5}{z} = 4\right] \times 3 \Rightarrow \frac{9}{x} - \frac{21}{y} + \frac{15}{z} = 12$ ... (iv)] $\frac{5}{x} + \frac{13}{z} = 4$

Now,

$$\frac{5}{x}+\frac{13}{y}-\left(\frac{5}{x}+\frac{7}{z}\right)=44-26 \text{ i.e. } \frac{6}{z}=18 \text{ i.e. } \left[z=\frac{1}{3}, y=\frac{1}{z}, x=1\right]$$

(c) $\left[\frac{4}{x}+\frac{2}{y}+\frac{1}{z}=11\right]\times 3 \Rightarrow \frac{12}{x}+\frac{6}{y}+\frac{3}{z}=33$ ... (i)

$\left[\frac{3}{x}-\frac{3}{y}+\frac{4}{z}=8\right]\times 2 \Rightarrow \frac{4}{x}-\frac{6}{y}+\frac{8}{z}=16$ ... (ii)

Adding (i) and (ii) $\frac{16}{x}+\frac{11}{z}=49$

$\left[\frac{4}{x}+\frac{2}{y}+\frac{1}{z}=11\right]\times 2 \Rightarrow \frac{8}{x}+\frac{4}{y}+\frac{2}{z}=22$ ... (iii)

$\left[\frac{3}{x}-\frac{4}{y}+\frac{5}{z}=10\right]\times 1 \Rightarrow \frac{3}{x}-\frac{4}{y}+\frac{5}{z}=10$ ... (iv)

Adding (iii) and (iv) $\frac{11}{x}+\frac{17}{z}=32$

$7\times\left[\frac{16}{x}+\frac{11}{z}=49\right]\Rightarrow \frac{112}{x}+\frac{77}{z}=343$ ... (v)

$11\times\left[\frac{11}{x}+\frac{7}{z}=32\right]\Rightarrow \frac{121}{x}+\frac{77}{x}=352$ ... (vi)

(vi) – (v): $\frac{9}{x}=9$ i.e. $x=1$ $\left[\therefore x=1, y=\frac{1}{2}, z=1\frac{1}{3}\right]$

13. (a) x + y = 25, y + z = 27, z + x = 32. Adding all of there
2 (x + y + z) = 84, i.e. x + y + z = 42 from (i) x + y + z – (x + y) = 42 – 25 i.e. z = 17, x + y + z – (y + z) = 42 – 27, x = 15, y = 10. [∴ (x, y, z) = (15, 10, 17)]

(b) x + y = 3, y + z = 5, z + x = 4 Adding all of these equations: 2 (x + y + z) = 12, x + y + z – (x + z) = 6 – 3 i.e. z = 3, x + y + z – (y + z) = 6 – 5, x = 1, y = 2 [∴(x, y, z) = (1, 2, 3)]

(c) $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=3$

$\frac{2}{x}-\frac{2}{y}+\frac{3}{z}=0$

Adding: $\frac{3}{x}=3$, i.e. $x=1$, $\frac{3}{x}-\frac{4}{y}=1$, i.e. $\frac{4}{y}=2$, i.e. $y=2$

[∴ (x, y, z) = (1, 2, 3)]

Exercise 1.4

Q.N. 1, 2 and 3: Concern theoretical portion or introductory part of the book.

4. (a) $Qd = Qs$ i.e. $10000 (12 – 2p) = 1000 (20p)$

i.e. $12 – 2p = 2p$, $p = 3$, $Q = 10000 (12 – 6) = 60000$

(b) $500 – 5p = 110 + 8p$ i.e. $13p = 390$, $p = 30$, $q = 350$

(c) $3p – 20 = 220 – 5p$ i.e. $8p = 240$, $p = 30$, $q = 70$

(d) $80 – 2p = 3p – 20$ i.e. $100 = 5p$, $p = 20$, $q = 40$

5. (a) $Qd = \frac{1}{2}(120 – p)$, $Qs = \frac{1}{3}(p – 60)$, $Qd = Qs$ i.e. $\frac{1}{2}(120 – p) = \frac{1}{3}(p, 60)$

i.e. $360 – 3p = 2p – 120$ i.e. $480 = 5p$, $[p = 96, q = 12]$

(b) $195 – 8p = 12 – 5$ i.e. $200 = 20p$, $[p = 10, Q = 115]$

(c) $400 – 4p = 6p – 10$ i.e. $410 = 10p$, $[p = 41, Q = 236]$

6. (a) $D_1 = S_1$ i.e. $10 + p_2 – p_1 = 6 + p_1 + 2p_2$ i.e. $2p_1 + p_2 = 4$ ... (i)

$D_2 = S_2$ i.e. $12 + 2p_1 – p_2 = 19 = 3p_1 – 5p_2$ i.e. $p_1 – 4p_2 = 7$ ... (ii)

$4 \times$ equation (i) + equation (ii): $9p_1 = 9$, $p_1 = 1$ $p_2 = 2$ $[\therefore (p_1, p_2) = (1, 2)]$

(b) $Qd_1 = Qs_1$ i.e. $4p_1 – 3p_2 + 2 = 8$, i.e. $4p_1 – 3p_2 = 6$ ... (i)

$Qd_2 = Qs_2$ i.e. $p_1 + 2p_2 + 5 = 12$, i.e. $p_1 + 2p_2 = 7$ ... (ii)

$2 \times$ equation (i) + $3 \times$ equation (ii): $8p_1 + 3p_1 = 12 + 21$ i.e. $p_1 = 3$, $p_2 = 2$

$[\therefore (p_1\ p_2) = (3, 2)]$

(c) $Qd_1 = Qs_1$ i.e. $410 – 5p_1 – 2p_1 = 3p_1 – 60$ i.e. $8p_1 + 2p_2 = 470$ ... (i)

$Qd_2 = Qs_2$ i.e. $295 + p_1 – 3p_2 = 2p_2 – 120$ i.e. $p_1 – 5p_2 = – 415$ ... (ii)

$5 \times$ equation (i) + $2 \times$ equation (ii): $40p_1 + 2p_1 = 2350 – 830$,

i.e. $42p_1 = 1520m$ $p_1 = \frac{760}{21}$, $p_2 = \frac{9475}{105}$

7. (a) $D_A = S_A$ i.e. $100 – 2P_A – P_B – 2P_C = 40$ i.e. $2P_A + P_B + 2P_C = 60$ ... (i)

$D_B = S_B$ i.e. $200 – 10P_A – 2P_B – 3P_C = 70$ i.e. $10P_A + 2P_B + 3P_C = 130$ ... (ii)

$D_C = S_C$ i.e. $150 – 2P_A – 3P_B – 5P_C = 10$ i.e. $2P_A + 3P_B + 5P_C = 140$ ... (iii)

(i) $5 \times$ equation (i) – $2 \times$ equation (iii) $6P_A – P_B = 20$ and $3 \times$ equation (i) – $2 \times$ equation (ii): $14P_A + P_B = 80$

Adding both of new equations: $20P_A = 100$, $P_A = 5$, $P_B = 10$, $P_C = 20$

(b) $D_A = S_A$ i.e. $114 – P_A – 2P_B – 3P_C = 100$ i.e. $P_A + 2P_B = 3P_C = 14$ ... (i)

$D_B = S_B$ i.e. $211 – 2P_A – 3P_B – P_C = 200$ i.e. $2P_A + 3P_B + P_C = 11$ ... (ii)

$D_C = S_C$ i.e. $67 – 3P_A – P_B – 4P_C = 50$ i.e. $3P_A + P_B + 4P_C = 17$ ... (iii)

equation (i) – $3 \times$ equation (ii): $5P_A + 7P_B = 19$ and equation (iii) – $4 \times$ equation (ii): $5P_A + 11P_B = 27$

Subtracting the second equation from first equation (new) we get: $– 4P_B = – 8$, $[P_B = 2, P_A = 1, P_C = 3]$

8. (a) Cost = Revenue: $1500 – Q^2 = PQ$, i.e. $1500 – Q^2 = 15Q – Q^2 = 15Q – Q^2$, Q = 10

(b) Cost = Revenue: $\frac{1}{2}(Q^2 – 160Q) = PQ$ i.e. $\frac{1}{2}(Q^2 – 160Q) = \frac{1}{2}(180Q – Q^2($, Q = 90 Profit = Revenue – cost i.e. Profit = 50x – (35x + 1000) = 15x – 1000, where x = 400, profit = 500 -units.

Exercise 1.5

1. Concern introductory part of the book.

2. (a) $x^2 – 4x + 3 = 0$ i.e. $x^2 – 3x – x = 3 = 0$ i.e. $(x – 3)(x – 1) = 0$ i.e. x = 1 and 3.

(b) $15x^2 – x – 28 = 0$ i.e. $15x^2 – 21x + 20x – 28 = 0$ i.e. $(3x + 4)(5x – 7) = 0$ i.e. x = $\frac{-3}{4}$ and $\frac{7}{5}$

(c) $3x^2 – 5x – 4 = 0$ i.e. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ i.e. $x = \frac{5 \pm \sqrt{25 + 48}}{6} = \frac{5 \pm \sqrt{73}}{24}$

(d) $x^2 + 8 = 55x – 11x^2 – 55$ i.e. $12x^2 – 55x + 63 = 0$ i.e. $x = \frac{55 \pm \sqrt{3205 - 3024}}{24} = \frac{55 \pm \sqrt{181}}{24}$

3. (a) $b^2 – 4ac = 0$ i.e. $k^2\ \ 36 = 0$ i.e. $k = \pm 6$

(b) $b^2 – 4ac = 0$ i.e. $4k^2 – 4(7k – 12) = 0$ i.e. k = 3 and 4

(c) $(2k + 4)^2 – 4(4 – k)(8k + 1) = 0$ or, $9k^2 – 27k = 0$, k = 0, 3

(d) $(x – a)(x – b) + (x – b)(x – c) + (x – c)(x – a) = 0$

i.e. $x^2 – (a + b)x + ab = x^2 – (b + c)x + bc = x^2 – (a + c)x + ac = 0$

i.e. $3x^2 – (2a = 2b + 2c)x + (ab + bc + ac0 = 0$ $[Ax^2 + Bx + C = 0 = 4AC]$

Now, $4(a + b + c)^2 = 4 \times 3\ 9ab + bc + bc)$

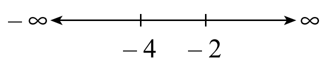
or, $(a + b + c)^2 – 3(ab + bc + ac) = 0$ i.e. $a^2 + b^2 + c^2 + 2ab + 2bc + 2ac – 3(ab = bc + ac) = 0$

i.e. $a^2 = b^2 + c^2 – ab – bc – ac = 0$ i.e. possible only if a = b = c

(c) $B^2 = 4AC$, i.e. $4m^2 = 4(1 + m^2)(c^2 – a^2)$

i.e. $4m^2 c^2 = 4(c^2 – a^2 – m^2c^2 – a^2m^2)$ i.e. $c^2 a^2 m^2 = 0$, i.e. $c^2 = a^2(1 + m^2)$

4. (a) $x^2 + 5x + 4 > 0$ i.e. $x^2 – 4x – x + 4 > 0$ i.e. $(x – 4)(x – 1) > 0$



in $(– \infty, 1)$: Let x = 0: $x^2 – 5x + 4 > 0$ (4 > 0) (True)

in (1, 4): Let x = 2, $x^2 – 5x + 4 > 0$ i.e. 4 – 10 + 4 > 0 (False) in $(4, \infty)$: Let x = 5: $x^2 - 5x + 4 > 0$ i.e. $25 - 25 + 4 > 0$ (True)

(b) $-(x^2 + 8) - 6x > 0$ i.e. $-1(x^2 + 6x + 8) > 0$, i.e. $-(x + 2)(x + 4) > 0$ i.e. $(x + 2)(x + 4) < 0$

$$-\infty \longleftarrow \underset{-4}{|} \quad \underset{-2}{|} \longrightarrow \infty$$

in $(-\infty, -4)$: $(x + 2)(x + 4) > 0$. In $(-4, -2)$: $(x + 2)(x + 4) < 0$, In $(-2, \infty)$: $(x + 2)(x + 4) < 0$ $\therefore$ required solution is $-4 < x < -2$.

(c) $\dfrac{x^2 + x + 1}{x^2 + 2} < \dfrac{1}{3}$ i.e. $3x^2 + x + 1 < x^2 + 2$ i.e. $2x^2 + x - 1 < 0$

i.e. $2x^2 + 2x - x - 1 < 0$ i.e. $(2x - 1)(x + 1) < 0$

in $(-\infty, -1)$, $(2x - 1)(x + 1) < 0$, in $\left(-1, \dfrac{1}{2}\right)$: $(2x - 1)(x + 1) < 0$, in $\left(\dfrac{1}{2}, \infty\right)$:

$(2x - 1)(x + 1) > 0$. $\therefore -1 < x , \dfrac{1}{2}$ is the required solution.

(d) $\dfrac{x^2 - 2x + 5}{3x^2 - 2x - 5} \geq \dfrac{1}{2}$ i.e. $\dfrac{x^2 - 2x + 5}{3x^2 - 2x - 5} - \dfrac{1}{2} \geq 0$ i.e. $\dfrac{x^2 + 2x - 15}{3x^2 - 2x - 5} \leq 0$

i.e. $\dfrac{(x + 5)(x - 3)}{(x + 1)(2x - 5)} \leq 0$

$$-\infty \longleftarrow \underset{-5}{|} \quad \underset{-1}{|} \quad \underset{\frac{5}{3}}{|} \quad \underset{3}{|} \longrightarrow +\infty$$

$[-5, -1] \cup \left[\dfrac{5}{3}, 3\right]$ or $-5 \leq x \leq -1$ or $\dfrac{5}{3} \leq x \leq 3$ satisfy the inequality.

(e) $\dfrac{x - 1}{4x + 5} < \dfrac{x - 3}{4x - 3}$ i.e. $\dfrac{x - 1}{4x + 5} - \dfrac{(x - 3)}{(4x - 3)} < 0$

i.e. $\dfrac{(x - 1)(4x - 3) - (x - 3)(4x + 5)}{(4x + 5)(4x - 3)} < 0$

i.e. $\dfrac{+12}{(4x + 5)(3 - 4x)} < 0$

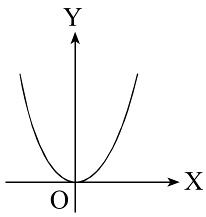
$$-\infty \longleftarrow \underset{-\frac{5}{3}}{|} \quad \underset{\frac{3}{4}}{|} \longrightarrow +\infty$$

In the interval $-\dfrac{5}{4} < x < \dfrac{3}{4}$, $\dfrac{12}{(4x + 5)(3 - 4x)} < 0$.

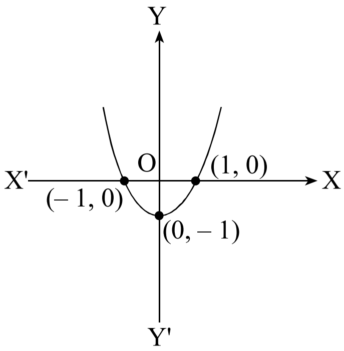
$\boxed{\text{Solution is } \dfrac{-5}{4} < x < \dfrac{3}{4}}$

5. (a) (i) For $y = x^2$ Domain = $(-\infty, \infty)$ Range = $[-0, \infty)$

(ii) Turning upward from (0, 0)



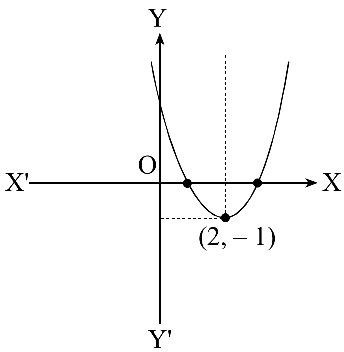
(b) $y = x^2 - 1$ turning point: $(0, -1)$. Turning upward X-intercepts: $(+1, 0)$, $(-1, 0)$
domain = $(-\infty, \infty)$ Range = $[-1, \infty]$



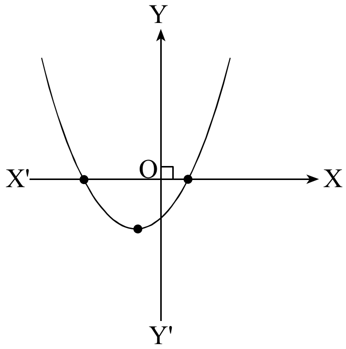
(c) $y = x^2 - 4x + 3 = (x^2 - 4x + 4) - 1 = (x - 2)^2 - 1$

y-intercept = $(0, 3)$ turning point: $(2, -1)$ turning upward.

Domain = $(-\infty, \infty)$. Range = $[-1, \infty]$ X- intercept: $(1, 0)$, $(3, 0)$

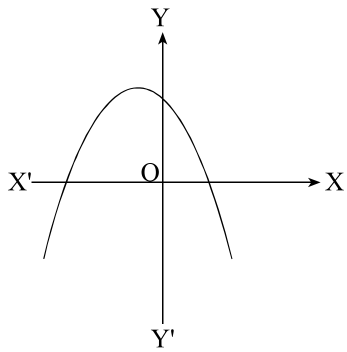


(d)



- $y = x^2 + x - 6 = \left(x + \frac{1}{2}\right)^2 - \frac{25}{4}$
- Turning point $\left(\frac{-1}{2}, \frac{-25}{4}\right)$
- Y-intercept = (0, – 6). X-intercept: (– 3, 0), (2, 0)

(e) $y = 4 - 5x - x^2 = \frac{41}{4} - \left(x + \frac{5}{4}\right)^2$ Turning point: $\left(\frac{-5}{4}, \frac{41}{4}\right)$ Turning downward X-intercept = $\left(\frac{5 \pm \sqrt{41}}{-2}, 0\right)$



(f) $y = \frac{65}{16} - \left(x - \frac{3}{4}\right)^2$ turning point = $\left(\frac{3}{4}, \frac{65}{16}\right)$ turning down ward.

Exercise 1.6

1. (a) $D_P = S_P$: $2Q - Q^2 = 3Q - 2Q^2 - 2$

i.e. $Q^2 - Q + 2 = 0$

i.e. $Q^2 - 2Q + Q + 2 = 0$

i.e. $(Q + 1)(Q - 2) = 0$

i.e. $Q = 2$

(b) $D_P = S_P$

i.e. $20Q - Q^2 = Q^2 + 8Q + 2$

i.e. $2Q^2 - 12Q + 2 = 0$

i.e. $Q^2 - 6Q + 1 = 0, Q = \frac{6 \pm \sqrt{36 - 4}}{2}$

i.e. $Q = \frac{6 + \sqrt{32}}{2} = \sqrt{8} + 3$

2. (a) At break ever point, Total cost = Revenue

i.e. $Q^2 - 16Q + 20 = 20 - 2Q$

i.e. $Q^2 - 14Q = 0$

i.e. $Q = 14$

(b) $C(Q) = R(Q)$

i.e. $Q^2 + 8Q + 2 = 20Q - Q^2$

i.e. $2Q^2 - 12Q + 2 = 0,$

i.e. $Q = \frac{6 \pm \sqrt{36 - 4}}{2} = 3 + 2\sqrt{2}$

3. (a) $R(Q) = C\,(Q)$

i.e. $260Q - 3Q^2 = 500 + 20Q$

i.e. $3Q^2 - 240Q + 500 = 0$

i.e. $Q = 240 \pm \sqrt{\frac{(240)^2 - 4 \times 3 \times 500}{2 \times 3}}$

i.e. $Q = \frac{240 \pm 20\sqrt{129}}{6}$

$\pi(Q) = R(Q) - C(Q) = 260Q - 3Q^2 - (500 + 20Q) = -3Q^2 + 240Q - 500$

Comparing with $y = ax^2 + bx + c$, we get

$$x = \frac{-b}{2a} = \frac{-240}{2 \times (-3)} = 40, \frac{4ac - b^2}{4a} = \frac{4 \times (-3) \times (-500) - (240)^2}{4(-3)}$$

$\therefore$ Required point (40, 4300)

(b) $R(Q) = C\,(Q)$

i.e. $5 - 4Q + 3Q^2 = 3 + 2Q$

$3Q^2 - 6Q + 2 = 0$

$$Q = \frac{6 \pm \sqrt{36 - 24}}{2 \times 3} = \frac{6 \pm \sqrt{12}}{6} = \frac{3 \pm \sqrt{3}}{3} = \frac{3 + \sqrt{3}}{3}$$

Now,

$\pi(Q) = R(Q) - C(Q) = 5 - 4Q + 3Q^2 - (3 + 2Q\,) = 3Q^2 - 6Q + 2$

Comparing with $y = ax^2 + bx + c. \frac{-b}{2a} = \frac{6}{2 \times 3} = 1$

$$\frac{4ac\ \ b^2}{4a} = \frac{4 \times 3 \times 2 - 36}{4 \times 3} = \frac{24 - 36}{12} = -1 \therefore \text{Point} = (1, -1)$$

4. Revenue = $P.Q = \frac{1}{8}(Q - 160)\ Q = \frac{1}{8}(Q^2 - 160Q)$

Cost = 4Q

Profit = Revenue – cost

i.e. $500 = \frac{1}{8}(Q^2 - 16Q) - 4Q$

or, $500 = \frac{1}{8}(Q^2 - 16Q - 32Q)$

Now,

$$Q^2 - 48Q - 4000 = 0$$

$$Q = 48 \pm \frac{\sqrt{(48)^2 - 4 \times 1 \times (-400)}}{2 \times 1}$$

$$= \frac{48 \pm 8\sqrt{286}}{2}$$

$$= 24 \pm 4\sqrt{286}$$

$$p = \frac{1}{8}(Q - 160) = \frac{1}{8}(24 \pm 4\sqrt{286} - 160) = \frac{1}{8}(\pm 4\sqrt{286} - 136)$$

Exercise 1.7

1. (a) $3^{2x+1} = 9^{2x-1}$

i.e. $3^{2x+1} = 3^{2(2x+1)}$

i.e. $2x + 1 = 4x - 2$

i.e. $2x = 3, \left[x = \frac{3}{2}\right]$

(b) $9^x = \frac{9}{3^x}$

i.e. $3^{2x} = 3^{2-x}$

i.e. $3x = 2, \left[x = \frac{2}{3}\right]$

2. (a) $3^{2x+1} = 9^{x+2} - 26$

i.e. $3^{2x+1}[3 - 81] = -26$

i.e. $3.3^{2x} - 3^4 . 3^{2x} = -26$

i.e. $3^{2x}[3-81] = -26$

i.e. $3^{2x} = 3^{-1}, 3^{2x} = 3^{-1}, \left[x = \frac{1}{2}\right]$

(b) $49 \times 7^x = (343)^{2x-5}$

i.e. $7^{2+x} = 7^{3(2x-5)}$

i.e. $2 + x = 6x - 15$

i.e. $\left[x = \frac{17}{5}\right]$

3. (a) $4^x + \frac{1}{4^x} = 16\frac{1}{16}$

i.e. $16. (4^x)^2 - 257. 4^x + 16 = 0$

i.e. $(4^x - 4^2)(16. 4^x + 1) = 0$ $[x = 2, -2]$

(b) $7^x + \frac{343}{7^x} = 56$

i.e. $(7^x)^2 - 56. 7^x + 343 = 0$

i.e. $(7^x)^2 - 49.7^x - 7.7^x + 343 = 0$

i.e. $(7^x - 7)(7^x - 49) = 0$, $[x = 1,2]$

4. (a) $5^{1-x} + 5^{x-1} = \frac{26}{5}$

i.e. $\frac{1}{5.5^x} + 5^x . \frac{1}{5} = \frac{26}{5}$

i.e. $(5^x)^2 - 26.5^x + 25 = 0$

i.e. $(5^x)^2 - 25. 5^x - 5^x + 25 = 0$

i.e. $(5^x - 1)(5^x - 25) = 0$

i.e. $[x = 0, 2]$

(b) $2^{x-1} + 2^{-x} = \frac{3}{2}$

i.e. $\frac{2^x}{2} + \frac{1}{2^x} = \frac{3}{2}$

i.e. $(2^x)^2 - 3.2 + 2 = 0$

i.e. $(2^x)^2 - 2.2^x - 2^x + 2 = 0$

i.e. $(2^x - 2)(2^x - 1) = 0$, $[x = 0, 1]$

5. (a) $3^{x-2} = 5$

i.e. $\log(3^{x-2}) = \log 5$

i.e. $(x-2)\log 3 = \log 5, (x, -2) = \frac{\log 5 + 2\log 3}{\log 3}$

$\therefore \quad x = \dfrac{\log 5 + 2 \log 3}{\log 3}$

(b) $5^{5-3x} = 2^{x+2}$

$(5 - 3x) \log 5 = (x + 2) \log 2$

$5^{5-3x} = 2^{x+2} x = \dfrac{\log 5 - 2 \log 2}{\log 2 + 3 \log 5}$

(c) $4^{3x-1} = 7 \times 3^{x+1}$

i.e. $(3x + 1) \log 4 = \log 7 + (x + 1) \log 3$

i.e. $3x \log 4 - \log 4 = \log 7 + x \log 3 + \log 3$

i.e. $x [3 \log 4 - \log 3] = \log 7 + \log 3 + \log 4x = \dfrac{\log 7 + \log 3 + \log 4}{3 \log 4 - \log 3}$

(d) $x^{25} = 7$

i.e. $25 \log x = \log 7$

i.e. $\log x = \dfrac{\log 7}{25}$, $x = 10^{0.0338}$ [x = 1.08]

6. (a) $1.0061 = \left(1 + \dfrac{r}{100}\right)^{12}$

i.e. $\log 10061 = 12 \log - \left(1 + \dfrac{r}{100}\right)$

i.e. $\log \left(1 + \dfrac{r}{100}\right) = \dfrac{\log 1.0061}{12}$

i.e. $\left(1 + \dfrac{r}{100}\right) = 10^{4.002641 \div 12}$, r = 15.5 (approx)

(b) $340 \left(1 + \dfrac{r}{100}\right)^{7} = 621$

i.e. $\left(1 + \dfrac{r}{100}\right)^{7} = \dfrac{621}{340}$

$7 \log \left(1 + \dfrac{r}{100}\right) = \log 621 - \log 340$

i.e. $\log \left(1 + \dfrac{r}{100}\right) = \dfrac{\log 621 - \log 340}{7}$

i.e. $\log \left(1 + \dfrac{r}{100}\right) = 0.03773$

i.e. $1 + \dfrac{r}{10} = 10^{0.037373}$, [r = 0.89] approx

(c) $\log \left(1 + \dfrac{r}{100}\right) = \dfrac{\log 1200 - \log 960}{3}$

i.e. $\left(1+\frac{r}{100}\right) = 10^{0.0323033}$, r = 7.71

(d) $151 \log\left(1+\frac{r}{100}\right) = \log 10065 - \log 2000,\ 1+\frac{r}{100} = 10^{0.046785}$, r = 11.37

7. $4^{x+3} = 3^{-x}$

i.e. $(x+3)\log 4 = x \log 3$

i.e. $x \log 4 + x \log 3 = -3 \log 4$

i.e. $x = \frac{-3 \log 4}{\log 4 + \log 3}$

8. $\log_3 (2x-1) - \log_3 (x-4) = 2$

i.e. $\log_3\left(\frac{2x-2}{x-4}\right) = 2$

i.e. $\frac{2x-1}{x-4} = 2^3$

i.e. $2x - 2 = 8x - 32$

i.e. $6x = 30$

$x = 5.$

9. $\log_e (4x+6) - \log_e (x = 5) = \log_e x$

i.e. $\log_e\left(\frac{4x+6}{x+5}\right) = \log_e x$

i.e. $4x + 6 = x^2 + 5x$

i.e. $[x = -3, 2]$

x = 2 is the solution x = – 3 is not solution because logarithm of negative number cannot be taken.

10. y(t) = 78 15 log (t + 1)

(i) When t = 1, y 91) = 78 – 15 log 2 = 73.48

(ii) When t = 4, y(4) = 78 – 15 log 5 = 67.57

11. (i) N(a) = 100 + 200ln a, when, a = 100, N (1000) = 100 + 200ln 1000 = 1482 .

(ii) N(5000) = 100 + 200in 5000 = 1804

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CHAPTER 2

CALCULUS

Exercise 2.1 (A)

1. **(a)** $\lim\limits_{x \to 1} \dfrac{x^2 + 3x - 1}{2x + 4}$ **(b)** $\lim\limits_{x \to 1} \dfrac{3x - 5}{2x - 4}$

(c) $\lim\limits_{x \to 3} \dfrac{x + 3}{x^2 + 5x + 6}$

Solution:

(a) $\lim\limits_{x \to 1} \dfrac{x^2 + 3x - 1}{2x + 4} = \dfrac{(1)^2 + 3 \times 1 + 1}{2 \times 1 + 4} = \dfrac{1 + 3 - 1}{6} = \dfrac{1}{2}$

(b) $\lim\limits_{x \to 1} \dfrac{3x - 5}{2x - 4} = \dfrac{3 \times 1 - 5}{2 \times 1 - 4} = \dfrac{-2}{-2} = 1$

(c) $\lim\limits_{x \to 3} \dfrac{x + 3}{x^2 + 5x + 6} = \dfrac{3 + 3}{3^2 + 5 \times 3 + 6} = \dfrac{6}{30} = \dfrac{1}{5}$

2. **(a)** $\lim\limits_{x \to 2} \dfrac{x^2 - 5x + 6}{x^2 - 4}$ **(b)** $\lim\limits_{x \to 1} \dfrac{x^2 + 4x - 5}{x - 1}$

(c) $\lim\limits_{x \to 1} \dfrac{x^2 + 3x - 4}{x^2 - x}$ **(d)** $\lim\limits_{x \to 1} \dfrac{x^2 - 3x + 2}{x^2 - 4x + 3}$

Solution:

(a) When x = 2, $f(2) = \dfrac{0}{0}$

So, $\lim\limits_{x \to 2} \dfrac{(x - 3)(x - 2)}{(x - 2)(x + 2)} = \lim\limits_{x \to 2} \dfrac{x - 3}{x + 2} = \dfrac{-1}{4}$

(b) When x = 1, $f(1) = \dfrac{0}{0}$ $\lim\limits_{x \to 1} \dfrac{(x + 5)(x - 1)}{(x - 1)} = \lim\limits_{x \to 1} (x + 5) = 6$

(c) $\lim\limits_{x \to 2} \dfrac{x^2 + 3x + 4}{x^2 - x} = \lim\limits_{x \to 1} \dfrac{(x + 4)(x - 1)}{x(x - 2)} = \lim\limits_{x \to 1} \dfrac{(x + 4)}{x} = \dfrac{1 + 4}{2} = 5$

(d) $\lim\limits_{x \to 1} \dfrac{x^2 - 3x + 2}{x^2 - 4x + 3} = \lim\limits_{x \to 1} \dfrac{x^2 - 2x - x + 2}{x^2 - 3x - x + 3} = \lim\limits_{x \to 1} \dfrac{(x - 2)(x - 1)}{(x - 3)(x - 1)}$

$= \lim\limits_{x \to 1} \dfrac{x - 2}{x - 3} = \dfrac{1 - 2}{1 - 3} = \dfrac{1}{2}$

3. **(a)** $\lim\limits_{x\to 3}\dfrac{x^3-27}{x^2-3x}$ **(b)** $\lim\limits_{x\to 1}\dfrac{x^3-1}{x^2-x}$

(c) $\lim\limits_{x\to 5}\dfrac{x^3-125}{x^2-25}$

Solution:

(a) $\lim\limits_{x\to 3}\dfrac{x^2-27}{x^2-3x}=\lim\limits_{x\to 3}\dfrac{(x-3)(x^2+3x+9)}{x(x-3)}=\lim\limits_{x\to 3}\dfrac{x^2+3x+9}{x}=\dfrac{9+9+9}{3}=9$

(b) $\lim\limits_{x\to 1}\dfrac{x^3-1}{x^2-3x}=\lim\limits_{x\to 1}\dfrac{(x-1)(x^2+x+1)}{x(x-1)}=\lim\limits_{x\to 1}\dfrac{x^2+x+1}{x}=\dfrac{1+1+1}{1}=3$

(c) $\lim\limits_{x\to 5}\dfrac{x^3-125}{x^2-25}=\lim\limits_{x\to 5}\dfrac{(x-5)(x^2+5x+25)}{(x-5)(x+5)}=\lim\limits_{x\to 5}\dfrac{x^2+5x+25}{x+5}=\dfrac{75}{10}=\dfrac{15}{2}$

4. **(a)** $\lim\limits_{x\to 3}\dfrac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$ **(b)** $\lim\limits_{x\to a}\dfrac{\sqrt{2x}-\sqrt{3x-a}}{\sqrt{x}-\sqrt{a}}$

(c) $\lim\limits_{x\to 2}\dfrac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$ **(d)** $\lim\limits_{x\to 1}\dfrac{x^2-1}{\sqrt{3x+1}-\sqrt{5x-1}}$

Solution:

(a) $\lim\limits_{x\to 3}\dfrac{x-3}{\sqrt{x-2}-\sqrt{4-x}}\times\dfrac{\sqrt{x-2}+\sqrt{4-x}}{\sqrt{x-2}+\sqrt{4-x}}=\lim\limits_{x\to 3}\dfrac{(x-3)(\sqrt{x-2}+\sqrt{4-x})}{x-2-4+x}$

$$=\lim_{x\to 3}\frac{(x-3)(\sqrt{x-2}+\sqrt{4-x})}{2(x-3)}$$

$$=\lim_{x\to 3}\frac{(\sqrt{x-2}+\sqrt{4-x})}{2}=\frac{2}{2}=1$$

(b) $\lim\limits_{x\to a}\dfrac{\sqrt{2x}-\sqrt{3x-a}}{\sqrt{x}-\sqrt{a}}=\lim\limits_{x\to a}\dfrac{\sqrt{2x}-\sqrt{3x-a}}{\sqrt{x}-\sqrt{a}}\times\dfrac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\times\dfrac{\sqrt{2x}+\sqrt{3x-a}}{\sqrt{2x}+\sqrt{3x-a}}$

$$=\lim_{x\to a}\frac{(a-x)(\sqrt{x}+\sqrt{a})}{-(a-x)(\sqrt{2x}+\sqrt{3x-9})}=\frac{-1}{\sqrt{2}}$$

(c) $\lim\limits_{x\to 2}\dfrac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}=\lim\limits_{x\to 2}\dfrac{(x-2)(x+2)}{\sqrt{3x-2}-\sqrt{x+2}}\times\dfrac{\sqrt{3x-1}+\sqrt{x+2}}{\sqrt{3x-2}+\sqrt{x+2}}$

$$=\lim_{x\to 2}\frac{(x-2)(x+2)\sqrt{3x-2}+\sqrt{x+2}}{2(x-2)}=8$$

(d) $\lim\limits_{x\to 1}\dfrac{x^2-1}{\sqrt{3x+1}-\sqrt{5x-1}}=\lim\limits_{x\to 1}\dfrac{(x-1)(x+1)}{\sqrt{3x+1}-\sqrt{5x-1}}\times\dfrac{\sqrt{3x+1}+\sqrt{5x-1}}{\sqrt{3x+1}+\sqrt{5x-1}}$

$$=\lim_{x\to 1}\frac{(x-1)(x+1)(\sqrt{3x+1}+\sqrt{5x-1})}{-2(x-1)}=-4$$

5. **(a)** $\lim\limits_{x \to -2} \dfrac{x^5 + 32}{x + 2}$ **(b)** $\lim\limits_{x \to 2} \dfrac{x^{\frac{1}{3}} - 2^{\frac{1}{3}}}{x^{\frac{1}{2}} - 2^{\frac{1}{2}}}$

(c) $\lim\limits_{x \to 64} \dfrac{x^{\frac{1}{6}} - 64^{\frac{1}{6}}}{x^{\frac{1}{3}} - 64^{\frac{1}{3}}}$ **(d)** $\lim\limits_{x \to a} \dfrac{x^5 - a^5}{x^4 - a^4}$

Solution:

(a) $$\lim_{x \to (-2)} \frac{x^5 + 32}{x + 2} = \lim_{x \to (-2)} \frac{(x)^5 - (-2)^5}{(x) - (-2)} = 5 \times (-2)^{5-1} = 5 \times 16 = 80$$

(b) $$\lim_{x \to 2} \frac{x^{\frac{1}{3}} - 2^{\frac{1}{3}}}{x^{\frac{1}{2}} - 2^{\frac{1}{2}}} \left[\frac{0}{0}\right] = \lim_{x \to 2} \left[\left(\frac{x^{\frac{1}{3}} - 2^{\frac{1}{3}}}{x - 2}\right) \div \left(\frac{x^{\frac{1}{2}} - 2^{\frac{1}{3}}}{x - 2}\right)\right]$$

$$= \frac{1}{3} \times 2^{\frac{1}{3} - 1} \div \frac{1}{2} \times 2^{\frac{1}{2} - 1} = \frac{1}{3} \times 2^{\frac{5}{6}}$$

(c) $$\lim_{x \to 64} \frac{x^{\frac{1}{6}} - 64^{\frac{1}{6}}}{x^{\frac{1}{3}} - 64^{\frac{1}{3}}} \left[\frac{0}{0}\right] = \lim_{x \to 64} \left[\frac{x^{\frac{1}{6}} - 64^{\frac{1}{6}}}{x - 64} \div \frac{x^{\frac{1}{3}} - 64^{\frac{1}{3}}}{x - 64}\right] = \frac{\frac{1}{6} \times 64^{\frac{1}{6} - 1}}{\frac{1}{3} \times 64^{\frac{1}{3} - 1}} = \frac{1}{4}$$

(d) $$\lim_{x \to a} \frac{x^5 - a^5}{x^4 - a^4} \left[\frac{0}{0}\right] = \lim_{x \to a} \left[\frac{x^5 - a^5}{x - a} \div \frac{x^4 - a^4}{x - a}\right] = \frac{5a^{5-1}}{4a^{4-1}} = \frac{5a}{4}$$

6. **(a)** $\lim\limits_{x \to 3} \left(\dfrac{1}{x - 3} - \dfrac{9}{x^3 - 3x^2}\right)$ **(b)** $\lim\limits_{x \to 3} \left(\dfrac{1}{x - 3} - \dfrac{3}{x^2 - 3x}\right)$

(c) $\lim\limits_{x \to 3} \left(\dfrac{x^2 + 9}{x^2 - 9} - \dfrac{3}{x - 3}\right)$ **(d)** $\lim\limits_{x \to 4} \left(\dfrac{1}{x - 4} - \dfrac{16}{x^3 - 4x^2}\right)$

Solution:

(a) $$\lim_{x \to 3} \left(\frac{1}{x - 3} - \frac{9}{x^3 - 3x^2}\right) [\infty - \infty] = \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x + 3)}{(x - 3)}$$

$$= \lim_{x \to 3} (x + 3) = 3 + 3 = 6$$

(b) $$\lim_{x \to 3} \left(\frac{1}{x - 3} - \frac{3}{x^2 - 3x}\right) [\infty - \infty] = \lim_{x \to 3} \left[\frac{x - 3}{x(x - 3)}\right] = \lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$$

(c) $$\lim_{x \to 3} \left(\frac{x^2 + 9}{x^2 - 9} - \frac{3}{x - 3}\right) [\infty - \infty] = \lim_{x \to 3} \left(\frac{x^2 + 9 - 3x - 9}{(x - 3)(x + 3)}\right)$$

$$= \lim_{x \to 3} \frac{x(x - 3)}{(x - 3)(x + 3)} = \lim_{x \to 3} \frac{x}{x + 3} = \frac{1}{2}$$

(d) $$\lim_{x \to 4} \left(\frac{1}{x - 4} - \frac{16}{x^3 - 4x^2}\right) [\infty - \infty] = \lim_{x \to 4} \frac{x^2 - 16}{x^2(x - 4)} = \lim_{x \to 4} \frac{x + 4}{x^2} = \frac{8}{16} = \frac{1}{2}$$

7. **(a)** $\lim\limits_{x \to \infty} \dfrac{2x^2 + 5x - 9}{3x^2 + 8x + 10}$ **(b)** $\lim\limits_{x \to \infty} \dfrac{5x^2 + 4x - 6}{3x^2 - 6x + 2}$

(c) $\lim\limits_{x \to \infty} \dfrac{6x^2 + 5x - 8}{8x^2 + 9x + 3}$ **(d)** $\lim\limits_{x \to \infty} \dfrac{x^2 + 4x + 3}{(x + 5)(3x + 7)}$

Solution:

(a) $\lim\limits_{x \to \infty} \dfrac{2x^2 + 5x - 9}{3x^2 + 8x + 10}\left(\dfrac{\infty}{\infty}\right)$

Let $y = \dfrac{1}{x}$, $\lim\limits_{y \to 0}\left[\left(\dfrac{2}{y^2} + \dfrac{5}{y} - 9\right) \div \left(\dfrac{3}{y^2} + \dfrac{8}{y} + 10\right)\right] = \lim\limits_{y \to 0} \dfrac{2 + 5y - 9y^2}{3 + 8y + 10y^2} = \dfrac{2}{3}$

(b) $\lim\limits_{x \to \infty} \dfrac{5x^2 + 4x - 6}{3x^2 - 6x + 2} = \lim\limits_{x \to 0}\left[\left(\dfrac{5}{y^2} + \dfrac{4}{y} - 6\right) \div \left(\dfrac{3}{y^2} - \dfrac{6}{y} + 2\right)\right]$

$= \lim\limits_{y \to 0} \dfrac{5 + 4y - 6y^2}{3 - 6y + 2y^2} = \dfrac{5}{3}$

(c) $\lim\limits_{x \to \infty} \dfrac{6x^2 + 5x - 8}{8x^2 + 9x + 3} = \lim\limits_{y \to 0}\left[\left(\dfrac{6}{y^2} + \dfrac{5}{y} - 8\right) \div \left(\dfrac{8}{y^2} - \dfrac{9}{y} + 3\right)\right]$

$= \lim\limits_{y \to 0} \dfrac{6 + 5y - 8y^2}{8 + 9y + 3y^2} = \dfrac{6}{8} = \dfrac{3}{4}$

(d) $\lim\limits_{x \to \infty} \dfrac{x^2 + 4x + 3}{(x + 5)(3x + 7)} = \lim\limits_{y \to 0} \dfrac{\left(\dfrac{1}{y^2} + \dfrac{4}{y} + 3\right)}{\left(\dfrac{1}{y} + 5\right)\left(\dfrac{3}{y} + 7\right)}$

$= \lim\limits_{y \to 0} \dfrac{1 + 4y + 3y^2}{(1 + 5y)(3 + 7y)} = \dfrac{1}{1 \times 3} = \dfrac{1}{3}$

8. **(a)** $\lim\limits_{x \to \infty} (\sqrt{x + 1} - \sqrt{x})$ **(b)** $\lim\limits_{x \to \infty} \dfrac{\sqrt{1 + 2x} - \sqrt{1 - 2x}}{x}$

(c) $\lim\limits_{x \to \infty} \sqrt{x}\,(\sqrt{x + 2} - \sqrt{x})$ **(d)** $\lim\limits_{x \to \infty} \sqrt{x^2 + 5x + 4} - \sqrt{x^2 - 3x + 4}$

Solution:

(a) $\lim\limits_{x \to \infty} (\sqrt{x + 1} - \sqrt{x})\ [\infty - \infty] = \lim\limits_{x \to \infty} \dfrac{\sqrt{x + 1} - \sqrt{x}}{(\sqrt{x + 1} + \sqrt{x})} \times \sqrt{x + 1} + \sqrt{x}$

$= \lim\limits_{x \to \infty} \dfrac{1}{\sqrt{x + 1} + \sqrt{x}} = \dfrac{1}{\infty} = 0$

(b) $\lim\limits_{x \to \infty} \dfrac{\sqrt{1 + 2x} - \sqrt{1 - 2x}}{x}\ [\infty - \infty]$

$= \lim\limits_{x \to \infty} \dfrac{(\sqrt{1 + 2x}) - (\sqrt{1 - 2x})}{x} \times \dfrac{\sqrt{1 + 2x} + \sqrt{1 - 2x}}{\sqrt{1 + 2x} + \sqrt{1 - 2x}}$

$= \lim\limits_{x \to \infty} \dfrac{4x}{x(\sqrt{1 + 2x} + \sqrt{1 - 2x})} = \dfrac{4}{\infty} = 0$

(c) $\lim\limits_{x \to \infty} \sqrt{x}\,(\sqrt{x+2} - \sqrt{x} = (\infty - \infty) = \lim\limits_{x \to \infty} \sqrt{x^2 + 2x} - x$

$$= \lim_{x \to \infty} \frac{x^2 + 2x - x^2}{\sqrt{x^2 + 2x} + x}$$

$$= \lim_{y \to 0} \frac{\frac{2}{y}}{\frac{\sqrt{1 + 2y} + 1}{y}} = 1$$

(d) $\lim\limits_{x \to \infty} \sqrt{x^2 + 5x + 4} - \sqrt{x^2 - 3x + 4}\ (\infty - \infty)$

$$= \lim_{x \to \infty} (\sqrt{x^2 + 5x + 4} - \sqrt{x^2 - 3x + 4)}\ \frac{(\sqrt{x^2 + 5x + 4} + \sqrt{x^2 - 3x + 4})}{(\sqrt{x^2 + 5x + 4} + \sqrt{x^2 - 3x + 4})}$$

$$= \lim_{x \to \infty} \frac{x^2 + 5x + 4 - x^2 + 3x - 4}{\sqrt{x^2 + 5x + 4} + \sqrt{x^2 - 3x + 4}} = \lim_{x \to \infty} \frac{8x}{\sqrt{x^2 + 5x + 4} + \sqrt{x^2 - 3x + 4}}$$

$$= \lim_{y \to \infty} \frac{\frac{8}{y}}{\sqrt{1 + 5y + 4y^2} + \sqrt{1 - 3y + 4y^2}}$$

$$= \lim_{y \to \infty} \frac{8}{\sqrt{1 + 5y + 4y^2} + \sqrt{1 - 3y + 4y^2}} = \frac{8}{2} = 4$$

9. (a) If $f(x) = \dfrac{ax + b}{x + 1}$, $\lim\limits_{x \to 0} f(x) = 2$ and $\lim\limits_{x \to \infty} f(x) = 1$, prove that $f(-2) = 0$.

(b) If $f(x) = \dfrac{x + 6}{cx - d}$, $\lim\limits_{x \to 0} f(x) = -6$ and $\lim\limits_{x \to \infty} f(x) = \dfrac{1}{3}$, prove that $f(13) = -\dfrac{1}{2}$.

(c) If $f(x) = \dfrac{px + q}{x - 3}$, $\lim\limits_{x \to 0} f(x) = -2$ and $\lim\limits_{x \to \infty} f(x) = 3$, prove that $f(2) = -12$

Solution:

(a) $\lim\limits_{x \to 0} \dfrac{ax + b}{x + 1} = b = 2,$

$$\lim_{x \to \infty} \frac{ax + b}{x + 1} = \lim_{y \to 0} \frac{\frac{a}{y} + b}{\frac{1}{y} + 1}$$

$$\lim_{y \to \infty} \frac{a + by}{1 + y} = a = 1.$$

So, $f(x) = \dfrac{x + 2}{x + 1}$

$$f(-2) = \frac{-2 + 2}{-2 + 1} = 0$$

(b) $\lim_{x \to 0} \frac{x + 6}{cx - d} = \frac{6}{-d} = -6,$

$$\lim_{x \to \infty} \frac{x + 6}{cx - d} = \lim_{y \to 0} \frac{\frac{1}{y} + 6}{\frac{c}{y} - d} = \lim_{y \to 0} \frac{1 + 6y}{c - dy} = \frac{1}{c} = \frac{1}{3}, c = 3.$$

$$f(x) = \frac{x + 6}{3x - 1}$$

$$f(13) = \frac{13 + 6}{3 \times 13 - 1} = \frac{19}{38} = \frac{1}{2} \text{ Proved.}$$

(c) $\lim_{x \to 0} \frac{px + q}{x - 3} = \frac{q}{-3} = -2, q = 6$

$$\lim_{x \to \infty} \frac{px + q}{x - 3}$$

$$\lim_{y \to 0} \frac{p + qy}{1 - 3y} = p = 3$$

$$f(x) = \frac{3x + 6}{x - 3}$$

$$f(2) = \frac{3 \times 2 + 6}{2 - 3} = \frac{12}{(-1)} = -12 \text{ Proved.}$$

10. Find the left limit and right limit at points mentioned:

(a) $\frac{x^2 - 4}{x - 2}$ **at** $x = 2$ **(b)** $\frac{x^2 - 9}{x - 3}$ **at** $x = 3$ **(c)** $\frac{x^3 - 8}{x - 2}$ **at** $x = 2$

Solution:

(a) Left limit $= \lim_{h \to 0} f(2 - h) = \lim_{h \to 0} \frac{(2 - h)^2 - 4}{2 - h - 2}$

$$= \lim_{h \to 0} \frac{4 - 4h + h^2 - 4}{-h} = \lim_{h \to 0} \frac{h(h - 4)}{(-h)} = 4$$

Right limit $= \lim_{h \to 0} f(2 + h) = \lim_{h \to 0} \frac{(2 + h)^2 - 4}{2 + h - 2}$

$$= \lim_{h \to 0} \frac{h(h + 4)}{h} = \lim_{h \to 0} h + 4 = 4$$

(b) left limit $= \lim_{h \to 0} f(3 - h) = \lim_{h \to 0} \frac{(3 - h)^2 - 9}{3 - h - 3}$

$$= \lim_{h \to 0} \frac{h^2 - 6h}{-h} = \lim_{h \to 0} \frac{h\,(6 - h)}{h} = 6$$

Right limit $= \lim_{h \to 0} f(3 + h) = \lim_{h \to 0} \frac{(3 + h)^2 - 9}{3 + h - 3}$

$$= \lim_{h \to 0} \frac{h^2 + 6h}{h} = \lim_{h \to 0} \frac{h\,(h + 6)}{h} = 6$$

(c) left limit $= \lim_{h \to 0} f(2-h) = \lim_{h \to 0} \frac{(2-h)^3 - 8}{2-h-2}$

$= \lim_{h \to 0} \frac{8 - 3 . 2^2 . h + 3 . 2 . h^2 - h^3 - 8}{-h} = \lim_{h \to 0} h^2 - 6h + 12 = (12)$

Right limit $= \lim_{h \to 0} f(2+h) = \lim_{h \to 0} \frac{(2+h)^3 - 8}{2+h-2}$

$= \lim_{h \to 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h} = \lim_{h \to 0} h^2 + 6h + 12 = 12$

11. Does the following limits exist?

(a) $f(x) = \begin{Bmatrix} x + 4 \text{ for } x > 1 \\ 2x + 3 \text{ for } x < 1 \end{Bmatrix}$ **at** $x = 1$ **(b)** $f(x) = \begin{Bmatrix} 2 - x \text{ for } x < 2 \\ 2 + x \text{ for } x > 2 \end{Bmatrix}$ **at** $x = 2$

(c) $f(x) = \begin{Bmatrix} 2x + 3 \text{ for } x < 1 \\ 2 + x \text{ for } x > 2 \end{Bmatrix}$ **at** $x = 1$

Solution:

(a) LHL $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1} 2x + 3 = 5$

RHL $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1} x + 4 = 1 + 4 = 5$

LHL = RHL, limit exist at x = 1

(b) LHL $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2} 2 - x = 2 - 2 = 0$

RHL $= \lim_{x \to 2^+} f(x) = \lim_{x \to 2} 2 + x = 2 + 2 = 4$

LHL ≠ RHL, limit does not exist at x = 2

(c) LHL $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1} 2x + 3 = 2x\ 1 + 3 = 5$

RHL $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1} 2 + x = 2 + 1 = 3$

LHL ≠ RHL, limit does not exist at x = 1

12. Explain $\lim_{x \to 2} x^2 = 4$ **with figure. Also write the difference between** $f(2)$ **if exists and** $\lim_{x \to 2} f(x)$ **for the function** $f(x) = \frac{4 - x^2}{2 - x}$**.**

Solution:

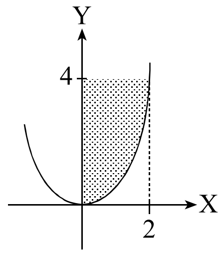
When $\lim_{x \to 2^+} f(x)$

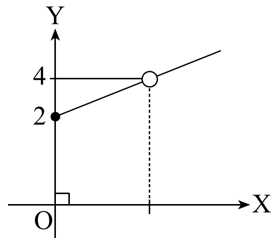
or, $\lim_{x \to 2^-} f(x)$ the values

or, area is meanly 4 (rectangle)

For $f(x) = \frac{4-x^2}{2-x}$, $f(2) = \frac{4-4}{2-2} = \frac{0}{0}$ does not exist

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{(2-x)(2+x)}{(2+x)} = 4$$





$f(x) = \frac{4-x^2}{2-x}$

f(2) = 4

i.e. point on the parabola.

Exercise 2.1 (B)

1. Examine the continuity or discontinuity of the following functions at the point mentioned:

(a) $f(x) = x^2$ **at** $x = 2$ **(b)** $f(x) = x^2 + 3x + 4$ **at** $x = 1$

(c) $f(x) = \frac{1}{x}$ **at** $x = 0$ **(d)** $f(x) = \frac{x^2-4}{x+2}$ **at** $x = 0$

(e) $f(x) = \frac{x^2-49}{x-7}$ **at** $x = 7$ **(f)** $f(x) = \frac{3x+2}{2x-1}$ **at** $x = 1$

Solution:

(a) $f(x) = x^2$

$$\text{LHL} = \lim_{h \to 0} (2-h^2) = 4 = \text{RHL} = \lim_{h \to 0} (1+h)^2 = 4 = f(2)$$

So, f(x) is continuous at x = 2.

(b) $\lim_{h \to 0} f(1-h) = 1 - 2h + h^2 + 3 - 3h + 4 = 8 = f(1+h) = \lim_{h \to 0} f(1+h)$ and $f(1) = 1 + 3 = 4 = 8$. So, f(x) is continuous at x = 1

(c) $f(x) = \frac{1}{x}$

LHL = RHL = f (0) (does not exist). So, f(x) is discontinuous at x = 0.

(d) $f(0) = -2$.

$$\text{LHL} = \lim_{h \to 0} \frac{h^2-4}{h+2} = \text{RHL} = -2$$

So, f(x) is continuous at x = 0.

(e) $f(x) = \dfrac{x^2 - 49}{x - 7}$ at $x = 7$

f(7) does not exist.

So, discontinuous at x = 7.

(f) f(1) = 5

$$\text{LHL} = \lim_{h \to 0} \frac{3(1 - h) + 2}{2(1 - h) - 1} = 5.$$

So, f(x) is continuous at x = 1

2. Discuss the continuity of:

(a) $f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, x \neq 3 \\ 6, x = 3 \end{cases}$ **at** $x = 3$

(b) $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, x \neq 3 \\ 5, x = 3 \end{cases}$ **at** $x = 3$

(c) $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, x \neq 2 \\ 4, x = 2 \end{cases}$ **at** $x = 2$

(d) $f(x) = \begin{cases} \dfrac{x^2 - 7x}{x - 7}, x \neq 7 \\ 5, x = 7 \end{cases}$ **at** $x = 7$

Solution:

(a) f(3) = 6

$$\text{LHL} = \text{RHL} = \lim_{x \to 3} \frac{(x - 3)(x + 3)}{(x - 3)} = \lim_{x \to 3} (x + 3) = 6. \text{ Continuous at } x = 3$$

(b) f(3) = 5

$$\text{LHL} = \text{RHL} = \lim_{x \to 3} \frac{x^2 - 3x + 2x - 6}{(x - 3)} = \lim_{x \to 3} \frac{(x - 3)(x + 2)}{(x - 3)} = \lim_{x \to 3} x + 2 = 5$$

So, f(x) is continuous at x = 3

(c) f(2) = 4

$$\lim_{x \to 7} \frac{x^2 - 4}{x - 2} = \lim_{x \to 7} \frac{(x - 2)(x + 2)}{(x - 7)} = \lim_{x \to 7} x + 2 = 4. \text{ Continuous at } x = 2.$$

(d) f(7) = 5

$$\lim_{x \to 7} \frac{x^2 - 7x}{x - 7} = \lim_{x \to 7} \frac{x(x - 7)}{x - 7} = \lim_{x \to 7} x = 7. \text{ Continuous at } x = 7$$

3. (a) A function $f(x)$ is defined by $f(x) = \begin{cases} \dfrac{5x^2 - 10x}{x - 2}, & x \neq 2 \\ a, & x = 2 \end{cases}$. Find the value of '$a$' so that the function $f(x)$ is continuous at $x = 2$.

(b) For what value of k, the following function $f(x)$ is continuous at $x = 1$?

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ k, & x = 1 \end{cases}$$

(c) A function $f(x)$ is defined as follows $f(x) = \begin{cases} \dfrac{x^2 - 2x}{x - 2}, & x \neq 3 \\ k, & x = 3 \end{cases}$, find the value of k so that the function $f(x)$ is continuous at $x = 3$.

Solution:

(a) $f(2) = a$

$$\lim_{x \to 2} \frac{5x^2 - 10x}{x - 2} = \lim_{x \to 2} \frac{5x^2 - 10x}{x - 2} = \lim_{x \to 2} \frac{5x(x-2)}{x - 2} = \lim_{x \to 2} 5x = 10 \ \boxed{a = 10}$$

(b) $f(1) = k$

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x \to 1} x + 1 = 1 + 1 = 2 \ \boxed{k = 2}$$

(c) $f(3) = k$

$$\lim_{x \to 3} \frac{x^2 - 2x}{x - 2} = \lim_{x \to 3} \frac{x(x-2)}{x - 2} = \lim_{x \to 3} x = 3. \text{ So, } \boxed{k = 3}$$

4. **Show that:**

(a) $f(x) = \begin{cases} 2x - 1, & \text{where } x < 1 \\ x, & \text{where } x \geq 1 \end{cases}$ **is continuous at** $x = 1$.

(b) $f(x) = \begin{cases} 3 - x, & \text{where } x \leq 0 \\ x^2, & \text{where } x > 0 \end{cases}$ **is continuous at** $x = 1$.

(c) $f(x) = \begin{cases} 2x, & \text{where } x < 2 \\ 2, & \text{where } x = 2 \\ x^2, & \text{where } x > 2 \end{cases}$ **is discontinuous** $x = 2$.

Solution:

(a) $f(1) = 1$

$$\text{LHL} = \lim_{x \to 1} 2x - 1 = 1$$

$$\text{RHL} = \lim_{x \to 1} x = 1 \text{ So, continuous at } f(1) = \text{LHL} = \text{RHL} \ \boxed{x = 1}$$

(b) $f(0) = 3 - 0 = 3$

LHL = $\lim_{x \to 0} 3 - x = 3$

RHL = $\lim_{x \to 0} x^2 = 0$. Discontinuous at $\boxed{x = 0}$

(c) $f(2) = 2$

LHL = $\lim_{x \to 2} 2x = 4$

RHL = $\lim_{x \to 2} x^2 = 4$. Discontinuous at $\boxed{x = 2}$

5. Find the points of discontinuity for:

(a) $\dfrac{4x^2 - 16}{2x - 4}$ **(b)** $f(x) = \dfrac{x^2}{x^2 - 8x - 20}$ **(c)** $f(x) = \dfrac{3x + 5}{x^2 - x - 6}$

Solution:

(a) Point of discontinuity: $2x - 4 = 0$ i.e. $\boxed{x = 2}$

(b) Point of discontinuity: $x^2 - 8x - 20 = 0$. $x^2 - 10x + 2x - 20 = 0$ $\boxed{x = 10, -2}$

(c) Point of discontinuity: $x^2 - x - 6 = 0$. $x^2 - 3x + 2x - 6 = 0$, $\boxed{x = 3, -2}$

Exercise 2.2 (A)

1. Find the approximate change (dy) from the following:

(a) $y = x^3, x = 5, x + \Delta x = 5.01$

(b) $y = x^4 - 10$ **and if** x **changes from 2 to 1.99**

Solution:

(a) $y = x^3$, $\Delta x = 5.01 - 5 = 0.01 = \Delta x$. $y + \Delta y = (x + \Delta x)^3$, $\Delta y = (5.01)^3 - 5^3 = 0.751501$, $dy = 3x^2\, dx = 3 \times 5^2 \times 0.01 = 0.75$

(b) $y = x^4 - 10$, $dy = 4x^3\, dx = 4 \times 2^3 \times (-0.01) = 32 \times (0.01) = (-0.32)$ $\boxed{\text{Modulus} = 0.16}$

2. Compute Δy, dy and $\Delta y - dy$ from the following:

(a) $y = 2x$ **when** $\Delta x = 0.1$ **(b)** $y = x^2 + 5x + 6$ **when** $x = 3, \Delta x = 0.1$

Solution:

(a) $y = 2x$. $y + \Delta y = 2(x + \Delta x)$

i.e. $\Delta y = 2\,\Delta x = 2x\ 0.1 = 0.2$, $dy = 2dx = 2\Delta x = 2 \times 0.1 = 0.2$ $\boxed{\text{So, } \Delta y - dy = 0}$

(b) $\Delta x = dx = 0.1$. $y + \Delta y = (x + \Delta x)^2 + 5(x + \Delta x) + 6$. Δy

$= 2.x.\Delta x + \Delta x^2 + 5\Delta x = \Delta x\,(2x + \Delta x + 5)$

i.e. $\Delta y = 0.1\,(2 \times 3 + 0.1 + 5) = 1.11$ $dy = (2x + 5)\,dx$

$= (2 \times 3 + 5) \times 0.1 = 1.1$ $\Delta y - dy = 1.11 - 1.1 = 0.01$

3. **The edge of the cube increases from 10 cm to 10.025 cm. Find the approximate increments in the volume and the surface area of the cube. Also, find the actual increments and the percentage error in the approximation.**

Solution:

$A = 6^2$ and $v = \ell^3$ $\Delta\ell = d\ell = 10.025 - 10 = 0.025$ cm

$dA = 12\ell\, d\ell = 12 \times 10 \times 0.025 = 3\ cm^2$

$dv = 3\ell^2\, d\ell = 3 \times 10^2 \times 0.025 = 7.5\ cm^3$.

So, $\boxed{\text{approximate increment area} = 3cm^2,\ \text{volume} = 7.5\ cm^3}$

$\Delta A = 6[(\ell + \Delta\ell^2) - \ell^2] = 6[(10.025)^2 - 10^2] = 3.00375,$

$\Delta v = (\ell + \Delta\ell)^3 - \ell^3 = (10.025)^3 - 10^3 = 7.518785.$

Actual increment in area and volume = 3.00375 cm^2 and 7.518785cm^3

% error is area $= \dfrac{\Delta A - dA}{A} \times 100\% = \dfrac{3.00375 - 3}{6 \times 10^2} \times 100\% = 6.25 \times 10^{-4}\%$

% error in volume $= \dfrac{\Delta v - dv}{V} \times 100\% = \dfrac{7.518785 - 7.5}{10^3} \times 100\% = 1.8785 \times 10^{-3}\%$

4. **The radius of a sphere was measured and found to be 21 cm with a possible errors in measurement of at most 0.05 cm. Use the differential to estimate approximate increase in volume of sphere. Also, find the percentage error in the estimate.**

Solution:

$dr = \Delta r = 0.05cm$

$A = 4\pi r^2 = 4\pi \times 21^2\ xm^2 = 1764\pi\ cm^2$

$v = \dfrac{4}{3}\pi r^3 = \dfrac{4}{3}\pi\,(21)^3\ cm^3 = 12348\ \pi cm^3.$

Approximate error:

$dA = 4\pi\ .\ 2rdr = 4\pi \times 2 \times 21 \times 0.05\ cm = 8.4\ \pi\ cm^2$

$dv = \dfrac{4}{3}\pi\ .\ 3r^2\ dr = 4\pi\ .(21)^2 \times 0.05 = 88.2\ \pi\ cm^3$

Actual error:

$\Delta A = 4\pi\,(r + \Delta r)^2 - 4\pi r^2 = 4\pi\,[(r + \Delta r)^2 - r^2] = 4\pi\,[(21.05)^2 - 21^2] = 8.4\pi\ cm^2$

$\Delta v = \dfrac{4}{3}\pi\,[(r + \Delta r)^3 - r^3] = 4\pi\,[(21.05)^3 - 21^3] = 265.2305\ \pi\ cm^2$

Percentage error:

$$\text{Area: } \frac{\Delta A - dA}{A} \times 100\% = 5.6689 \times 10^{-4}\%$$

$$\text{Volume: } \frac{\Delta v - dv}{V} \times 100\% = 1.43\%$$

5. **A circular copper plate is heated so that its radius increases from 5 cm to 5.06 cm. Find the approximate increase in area and also the actual increase in area.**

Solution:

$dr = \Delta r = 0.06$ cm, $A = \pi r^2 = 25\,\pi$ cm^2 $[A = 4\pi r^2]$

Approximate error: $dA = \pi.2rdr = 10\pi \times 0.06\text{cm}^2 = 0.6\pi$ cm^2

Actual error: $\Delta A = \pi\,[(r + \Delta r)^2 - r^2] = \pi\,[(5.06)^2 - 5^2]\text{cm}^2 = 0.6036$ cm^2

$$\text{Percentage: } \frac{\Delta A - dA}{A} \times 100\% = 0.0144\% \text{ (extra)}$$

6. **Find the percentage error allowed by differential approximation $y = 3x^3 + x - 1$ when x increases from 1 to 1.1.**

Solution:

$y(1) = 3 \times 1^3 + 1 - 1 = 3$, $dx = \Delta x = 1.1 - 1 = 0.1$

Approximate error: $dy = (3.3x^2 + 1)dx = (9 \times 1^2 + 1) \times 0.1 = 1$

Actual error: $\Delta y = 3[(x + \Delta x)^2 - x^2] + [(x + \Delta x) - x] = 0.63 + 0.1 = 0.73$

$$\text{Percentage error: } \left|\frac{\Delta y - dy}{y(1)} \times 100\%\right| = 9\%$$

Exercise 2.2 (B)

1. **Find, from the first principle, the derivative of:**

(a) $\frac{1}{2x+3}$ **(b)** $\frac{1}{3x-5}$ **(c)** $\frac{1}{3-4x}$

Solution:

(a) $$y = \frac{1}{2x+3}\;\; y \,.\, \Delta y = \frac{1}{2x + 2\Delta x + 3} \cdot \frac{\Delta y}{\Delta x} = \frac{2x + 2\Delta x + 3 - 2x - 3}{\Delta x\,(2x+3)\,.\,(2x + 2\Delta x + 3)}$$

$$= \frac{2}{(2x+3)\,(2x + 2\Delta x + 3)}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{2}{(2x+3)^2}$$

(b) $\frac{\Delta y}{\Delta x} = \frac{3x + 3\Delta x - 5 - (3x - 5)}{\Delta x\,(3x - 5)\,(3x + 3\Delta x - 5)} = \frac{3}{(3x - 5)\,(3x + 3\Delta x - 5)} \cdot \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

$= \frac{3}{(3x - 5)^2}$

(c) $\frac{\Delta y}{\Delta x} = \frac{3 - 4\Delta x - 4x - (3 - 4x)}{\Delta x\,(3 - 4x - 4\Delta x)\,(3 - 4x)} = \frac{-4}{(3 - 4x - 4\Delta x)\,(3 - 4x)} \cdot \frac{dy}{dx}$

$= \lim_{\Delta x \to 0} \cdot \frac{\Delta y}{\Delta x} = \frac{-4}{(3 - 4x)^2}$

2. Find the derivative of y with respect to *x* from the first principle:

(a) x^2 (b) $x^2 + 1$ (c) $2x^2 - 5x + 6$

Solution:

(a) $\frac{\Delta y}{\Delta x} = \frac{x^2 + 2.x.\Delta x + \Delta x^2 - x^2}{\Delta x} \cdot \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x$

(b) $\frac{\Delta y}{\Delta x} = \frac{x^2 + 2\,.\,x\,.\,\Delta x + \Delta x^2 + 1 - (x^2 + 1)}{\Delta x} \cdot \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x$

(c) $\frac{\Delta y}{\Delta x} = \frac{2(x^2 + 2.x.\Delta x + \Delta x^2 - x^2) - 5\,[x + \Delta x - x]}{\Delta x} \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 4x - 5$

3. Find from the first principle, the derivative of:

(a) $\sqrt{x + 2}$ (b) $\frac{1}{\sqrt{x}}$ (c) $\frac{1}{\sqrt{3x + 2}}$ (d) $x + \sqrt{x}$

Solution:

(a) $\frac{\Delta y}{\Delta x} = \frac{\sqrt{x + \Delta x + 2} - \sqrt{x + 2}}{\Delta x} \times \frac{\sqrt{x + \Delta x + 2} + \sqrt{x + 2}}{\sqrt{x + \Delta x + 2} + \sqrt{x + 2}} = \frac{1}{\sqrt{x + \Delta x + 2}\,\sqrt{x + 2}} \cdot \frac{dy}{dx}$

$= \frac{1}{2(x + 2)^{\frac{1}{2}}}$

(b) $\frac{\Delta y}{\Delta x} = \frac{\sqrt{x} - \sqrt{x + \Delta x}}{\Delta x.\sqrt{x}.\sqrt{x + \Delta x}} \times \frac{\sqrt{x} + \sqrt{x + \Delta x}}{\sqrt{x} + \sqrt{x + \Delta x}} = \frac{-1}{\sqrt{x}.\sqrt{x + \Delta x}\,(\sqrt{x} + \sqrt{x + \Delta x})} \frac{dy}{dx}$

$= \frac{-1}{x.2\sqrt{x}} = \frac{-1}{2x^{\frac{3}{2}}}$

(c) $\frac{\Delta y}{\Delta x} = \frac{\sqrt{3x + 2} - \sqrt{3x + 3\Delta x + 2}}{\Delta x.(\sqrt{3x + 2}).(\sqrt{3x + 3\Delta x + 2})} \times \frac{\sqrt{3x} + \sqrt{3x + 3\Delta x + 2}}{\sqrt{3x} + \sqrt{3x + 3\Delta x + 2}}$

$= \frac{-3}{(\sqrt{3x + 2})\,(\sqrt{3x + 3\Delta x + 2})\,(\sqrt{3x} + \sqrt{3x + 3\Delta x + 2})}$

$\therefore \frac{dy}{dx} = \frac{-3}{2(3x + 2)^{\frac{3}{2}}}$

(d) $\dfrac{\Delta y}{\Delta x} = \dfrac{(x + \Delta x - x)}{\Delta x} + \dfrac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \times \dfrac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \cdot \dfrac{dy}{dx} = 1 + \dfrac{1}{2\sqrt{x}}$

4. Find the derivative of following with respect to *x*:

(a) $\dfrac{6}{\sqrt[3]{x^2}}$ **(b)** $x^{\frac{5}{2}}$ **(c)** $\dfrac{2}{\sqrt{x^3}}$ **(d)** $\dfrac{3}{\sqrt[6]{x^2}}$

Solution:

(a) Let $y = 6.x^{\frac{-2}{3}}, \dfrac{dy}{dx} = 6 \times \left(\dfrac{-2}{3}\right). x^{\frac{-2}{3}-1} = -4.x^{\frac{-5}{3}} = \dfrac{4}{\sqrt[3]{x^5}}$

(b) Let $y = x^{\frac{5}{2}}, \dfrac{dy}{dx} = \dfrac{5}{2} x^{\frac{5}{2}-1} = \dfrac{5}{2} x^{\frac{3}{2}}$

(c) Let $y = 2x^{\frac{3}{2}}, \dfrac{dy}{dx} = 2.\dfrac{3}{2} x^{\frac{3}{2}-1} = 3x^{\frac{1}{2}}$

(d) Let $y = 3.x^{\frac{2}{6}}. \dfrac{dy}{dx} = 3 . \dfrac{1}{3} x^{\frac{1}{3}-1} = x^{\frac{-2}{3}} = \dfrac{2}{\sqrt[3]{x^2}}$

5. Find the derivative of following with respect to *x*:

(a) $x^2 \log x$ **(b)** $x^3 \log x$ **(c)** $e^x \log x$ **(d)** $x^3 (1 + \log x)$

Solution:

(a) Let, $y = x^2 \log x . \dfrac{dy}{dx} = x^2 \dfrac{d}{dx}(\log x) + \dfrac{d}{dx}(x^2)\log x = x^2 . \dfrac{1}{x} + 2x \log x = x(1 + 2\log x)$

(b) Let, $y = x^3 \log x . \dfrac{dy}{dx} = x^3 \dfrac{d}{dx}(\log x) + \dfrac{d}{dx}(x^3) \log x = x^3 . \dfrac{1}{x} + 3x^2 \log x$

$= x^2 (1 + 3\log x)$

(c) Let $y = e^x \log x . \dfrac{dy}{dx} = e^x \dfrac{d}{dx}(\log x) + \dfrac{d}{dx}(e^x) \log x = e^x. \dfrac{1}{x} + e^x \log x = e^x \left(\dfrac{1}{x} + \log x\right)$

(d) Let, $y = x^3 (1 + \log x) = x^3 + \log x.x^3, \dfrac{dy}{dx} = 3x^2 + x^2 + 3x^2 \log x = x^2 (4 + 3 \log x)$

6. Find the derivative of following with respect to *x*:

(a) $(2x^2 + x - 1)(3x^2 - 2)$ **(b)** $(2x + 3)(5x^2 - 7x + 1)$

(c) $(x^2 - x - 2)(x^2 + x + 3)$

Solution:

(a) Let, $y = (2x^2 + x - 1)(3x^2 - 2) . \dfrac{dy}{dx}$

$= (2x^2 + x - 1)\dfrac{d}{dx}(3x^2 - 2) + \dfrac{d}{dx}(2x^2 + x - 1) . (3x^2 - 2)$

$= (2x^2 + x - 1) . 6x + (4x + 1)(3x^2 - 2)$

$= 12x^3 + 6x^2 - 6x + 12x^3 + 3x^2 - 8x - 2$

$= 24x^3 + 9x^2 - 14x - 2$

(b) Let, $y = (2x + 3) . (5x^2 - 7x + 1) . \frac{dy}{dx}$

$= (2x + 3) \frac{d}{dx}(5x^2 - 7x + 1) + \frac{d}{dx}(2x + 3) . (5x^2 - 7x + 1)$

$= (2x + 3)(10x - 7) + 2.(5x^2 - 7x + 1)$

$= 2x^2 + 30x - 14x - 21 + 10x^2 - 14x + 2$

$= 30x^2 + 2x - 19$

(c) Let $y = (x^2 - x - 2)(x^2 + x + 3) . \frac{dy}{dx}$

$= (x^2 - x - 2)\frac{d}{dx}(x^2 + x + 3) + \frac{d}{dx}(x^2 - x - 2) . (x^2 + x + 3)$

$= (x^2 - x - 2)(2x + 1) + (2x - 1)(x^2 + x + 3)$

$= 2x^3 - 2x^2 - 4x + x^2 - x - 2 + 2x^3 + 2x^2 + 6x - x^2 - x - 3 = 4x^3 - 5$

7. Find the derivative of following with respect to *x*:

(a) $\frac{x^2}{x-1}$ **(b)** $\frac{x^2+3x+1}{x^2-x+1}$ **(c)** $\frac{x^2+3x+1}{x^2-1}$ **(d)** $\frac{e^x}{(1+x^2)}$

Solution:

(a) Let, $y = \frac{x^2}{x-1} . \frac{dy}{dx} = \frac{d}{dx}(x^2)(x-1) - \frac{d}{dx}(x-1).x^2/(x-1)^2$

$= \frac{2x.(x-1) - (1).x^2}{(x-1)^2}$

$= \frac{x^2 - 2x}{(x-1)^2}$

(b) Let, $y = \frac{x^2 + 3x + 1}{x^2 - x + 1} . \frac{dy}{dx}$

$= \frac{\frac{d}{dx}(x^2 + 3x + 1) . (x^2 - x + 1) - \frac{d}{dx}(x^2 - x + 1) . (x^2 + 3x + 1)}{(x^2 - x + 1)^2}$

$= \frac{4(1 - x^2)}{(x^2 - x + 1)^2}$

(c) Let, $y = \frac{x^2 + 3x + 1}{x^2 - 1} . \frac{dy}{dx} = \frac{(x^2 - 1)\frac{d}{dx}(x^2 + 3x + 1) - \frac{d}{dx}(x^2 - 1)(x^2 + 3x + 1)}{(x^2 - 1)^2}$

$= -\frac{(3x^2 + 4x + 3)}{(x^2 - 1)^2}$

(d) Let, $y = \frac{e^x}{1 + x^2} . \frac{dy}{dx} = \frac{(1 + x^2) . \frac{d}{dx}(e^x) + e^x . \frac{d}{dx}(1 + x^2)}{(1 + x^2)} = e^x\left(\frac{1 + x}{1 + x^2}\right)^2$

8. **Find $\frac{dy}{dx}$ if**

(a) $y = \frac{1}{\sqrt{a^2 - x^2}}$ (b) $y = \frac{1}{\sqrt[3]{x^2 - 2x + 1}}$

(c) $y = \sqrt{ax^2 + bx + c}$ (d) $y = \sqrt{\frac{1 + e^x}{1 - e^x}}$

Solution:

(a) $y = (a^2 - x^2)^{\frac{-1}{2}} \cdot \frac{dy}{dx} = \frac{d}{d(a^2 - x^2)} (a^2 - x^2)^{\frac{-1}{2}} \cdot \frac{d}{dx} (a^2 - x^2) = \frac{-1}{2} (a^2 - x^2)^{\frac{-1}{2} - 1} \cdot (-2x)$

$= \frac{x}{(a^2 - x^2)^{\frac{3}{2}}}$

(b) $y = (x^2 - 2x + 1)^{\frac{-1}{3}} = (x - 1)^{\frac{-2}{3}} \cdot \frac{dy}{dx} = \frac{d}{d(x - 1)} (x - 1)^{\frac{-2}{3}} \cdot \frac{d}{dx} (x - 1)$

$= \frac{-2}{3} (x - 1)^{\frac{-2}{3} - 1} = \left(\frac{-2}{3}\right) (x - 1)^{\frac{-5}{3}}$

(c) $y = (ax^2 + bx + c)^{\frac{1}{2}} \cdot \frac{dy}{dx} = \frac{d}{d(ax^2 + bx + c)} (ax^2 + bx + c)^{\frac{1}{2}} \cdot \frac{d}{dx} (ax^2 + bx + c)$

$= \frac{(2ax + b)}{2\sqrt{ax^2 + bx + c}}$

(d) $y = \left(\frac{1 + e^x}{1 - e^x}\right)^{\frac{1}{2}} \cdot \frac{dy}{dx} = \frac{d}{d\left(\frac{1 + e^x}{1 - e^x}\right)} \left(\frac{1 + e^x}{1 - e^x}\right)^{\frac{1}{2}} \cdot \frac{d}{dx} \left(\frac{1 + e^x}{1 - e^x}\right) = \frac{e^x}{\sqrt{(1 + e^x)(1 - e^x)^3}}$

9. **Find the derivative of:**

(a) $y = \frac{1}{\sqrt{3x - 2} - \sqrt{3x - 5}}$ (b) $y = \frac{1}{\sqrt{2x - 3} - \sqrt{2x - 5}}$

(c) $y = \frac{1}{\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2}}$

Solution:

(a) $y = \frac{1}{\sqrt{3x - 2} - \sqrt{3x - 5}} \times \frac{\sqrt{3x - 2} + \sqrt{3x - 5}}{\sqrt{3x - 2} + \sqrt{3x - 5}} = \frac{(3x - 2)^{\frac{1}{2}} + (3x - 5)^{\frac{1}{2}}}{3} \cdot \frac{dy}{dx}$

$= \frac{1}{3} \frac{1}{2} \left[3(3x - 2)^{\frac{-1}{2}} + 3.(3x - 5)^{\frac{-1}{2}}\right] = \left[\frac{1}{\sqrt{3x - 2}} + \frac{1}{\sqrt{3x - 5}}\right]$

(b) $y = \dfrac{(2x-3)^{\frac{1}{2}} + (2x-5)^{\frac{1}{2}}}{2} \cdot \dfrac{dy}{dx} = \dfrac{1}{2}\dfrac{1}{2}.2\left[(2x-3)^{\frac{-1}{2}} + (2x-5)^{\frac{-1}{2}}\right]$

$$= \frac{1}{2}\frac{1}{\sqrt{2x-3} + \dfrac{1}{\sqrt{2x-5}}}$$

(c) $y = \dfrac{(x^2+a^2)^{\frac{1}{2}} - (x^2+b^2)^{\frac{1}{2}}}{a^2-b^2} \cdot \dfrac{dy}{dx} = \dfrac{1}{a^2-b^2}\left[\dfrac{1}{2}(x^2+a^2)^{\frac{-1}{2}}.2x - \dfrac{1}{2}(x^2+b^2)^{\frac{-1}{2}}.2x\right]$

$$= \frac{x}{a^2-b^2}\left[\frac{1}{\sqrt{x^2+a^2}} - \frac{1}{\sqrt{x^2+b^2}}\right]$$

Exercise: 2.2 (C)

1. Find $\dfrac{dy}{dx}$ in each of the following cases.

(a) $xy = c^2$ **(b)** $y^3 - 3xy^2 = x^3 + 3x^2y$

(c) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ **(d)** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(e) $x^5 + y^5 = 5xy$ **(f)** $(x+y)^2 = 2axy$

(g) $(x^2+y^2)^2 = xy\backslash$

Solution:

(a) $xy = c^2. \dfrac{d}{dx}(xy) = \dfrac{d}{dx}(c^2)$ i.e. $x\dfrac{dy}{dx} + y.\dfrac{dy}{dx} = 0$ i.e. $\dfrac{dy}{dx} = \dfrac{-y}{x}$

(b) $y^3 - 3xy^2 = x^3 + 3x^2y$ i.e. $3y^2\dfrac{dy}{dx} - 3x^2.2y\dfrac{dy}{dx} - 3y^2 = 3x^2 + 3x^2\dfrac{dy}{dx} + 6xy$

i.e. $\dfrac{dy}{dx} = \dfrac{x^2+2xy+y^2}{y^2-2xy-x^2}$

(c) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}. \dfrac{2}{3}x^{\frac{2}{3}-1} + \dfrac{2}{3}y^{\frac{2}{3}-1}\dfrac{dy}{dx} = 0$

i.e. $\dfrac{2}{3}x^{\frac{-1}{3}} = \dfrac{-2}{3}y^{\frac{-1}{3}}\dfrac{dy}{dx}$

i.e. $\dfrac{dy}{dx} = -\dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$

(d) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \cdot \dfrac{2x}{a^2} + \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0$

i.e. $\dfrac{dy}{dx} = \dfrac{-2x}{a^2} \div \dfrac{2y}{b^2} = \dfrac{-b^2x}{a^2y}$

(e) $x^5 + y^5 = 5xy$

i.e. $5x^4 + 5y^4\dfrac{dy}{dx} = 5\left[x.\dfrac{dy}{dx} + y\right] \cdot \dfrac{dy}{bx} = \dfrac{x^4 - y}{x - y^4}$

(f) $(x + y)^2 = 2axy$

i.e. $2(x + y)\left[1 + \dfrac{dy}{dx}\right] = 2a\left[x.\dfrac{dy}{dx} + y\right]$

i.e. $(x + y - ay) = (ax - x - y)\dfrac{dy}{dx} \cdot \dfrac{dy}{dx} = \dfrac{x + y - ay}{ax - x - y}$

(g) $(x^2 + y^2)^2 = xy$

i.e. $2(x^2 + y^2) \cdot \left(2x + 2y\dfrac{dy}{dx}\right) = x \cdot \dfrac{dy}{dx} + y$

i.e. $4x\,(x^2 + y^2) + 4y\,(x^2 + y^2)\dfrac{dy}{dx} = x\dfrac{dy}{dx} + y$

i.e. $\dfrac{dy}{dx} = \dfrac{4x^3 + 4xy^2 - y}{x - 4x^2y - 4y^3}$

2. Find $\dfrac{dy}{dx}$ if

(a) $y = t^4 - 1, x = t^4 + 1$ **(b)** $x = t + \dfrac{1}{t}, y = t - \dfrac{1}{t}$

(c) $y = z^3 + 2z + 1, x = z^2 + 2$ **(d)** $x = 2at, y = at^2$

(e) $x = \dfrac{t}{1 + t}, y = \dfrac{t}{1 - t}$ **(f)** $x = \dfrac{3at}{1 + t^3}, y = \dfrac{3at^2}{1 + t^3}$

Solution:

(a) $y = t^4 - 1, \dfrac{dy}{dt} = \dfrac{d}{dt}(t^4 - 1) = 4t^3 \cdot \dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = \dfrac{4t^3}{4t^3} = 1,$

$x = t^4 + 1, \dfrac{dx}{dt} = \dfrac{d}{dt}(t^4 + 1) = 4t^3$

(b) $x = t + \dfrac{1}{t}, \dfrac{dx}{dt} = \dfrac{d}{dt}\left(t + \dfrac{1}{t}\right) = 1 - (-1)t^{-2} = \dfrac{t^2 - 1}{t^2} \cdot \dfrac{dy}{dx} = \dfrac{t^2 + 1}{t^2 - 1}$

$y = t - \dfrac{1}{t}, \dfrac{dy}{dt} = \dfrac{d}{dt}\left(t - \dfrac{1}{t}\right) = 1 - (-1)t^{-2} = \dfrac{t^2 + 1}{t^2} \cdot \dfrac{dy}{dx} = \dfrac{t^2 + 1}{t^2 - 1}$

(c) $y = z^3 + 2z + 1. \dfrac{dy}{dz} = 3z^2 + 2. \dfrac{dx}{dz} = 2z. \dfrac{dy}{dx} = \dfrac{dy}{dz} \div \dfrac{dx}{dz} = \dfrac{3z^2 + 2}{2z}$

(d) $x = 2at$. $\frac{dx}{dt} = 2a\frac{d}{dx}(t) = 2a$, $y = at^2$. $\frac{dy}{dt} = 2at$. $\frac{dy}{dx} = t = \frac{x}{2a}$

(e) $x = \frac{t}{1+t}$. $\frac{dx}{dt} = \frac{(t+1)\cdot\frac{d}{dt}(t) - t\cdot\frac{d}{dt}(t+1)}{(t+1)^2} = \frac{t+1-t}{(t+1)^2} = \frac{1}{(t+1)^2}$

$y = \frac{t}{1-t} = \frac{dy}{dt} = \frac{1.(1-t) - t(0-1)}{(1-t)^2} = \frac{1-t+t}{(1-t)^2} = \frac{1}{(t-1)^2}$. $\frac{dy}{dx} = \frac{1}{(t-1)^2} \div \frac{1}{(t+1)^2}$

$= \left(\frac{t+1}{t-1}\right)^2$

(f) $\frac{dx}{dt} = \frac{3a.(1+t^3) - 3at.3t^2}{(1+t^3)^2} = \frac{3a(1-2t^3)}{(1+t^3)^2}$

$\frac{dy}{dt} = \frac{3a.2t(1+t^3) - 3a.3t^2}{(1+t^3)^2} = \frac{3at[2-t^3]}{(1+t^3)^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{3at(2-t^3)}{(1+t^3)^2} \div \frac{3a(1-2t^3)}{(1+t^3)^2} = \frac{t(1-t^3)}{1-2t^3}$

Exercise 2.2 (D)

1. Find the second and third order derivatives of the following:

(a) $y = 2x^3 - 3x^2 + 4$ **(b)** $y = 3x^3 - 4x^2 + 6x$ **(c)** $y = x^3 \log x$

(d) $y = x^2 e^{2x}$ **(e)** $f(x) = \frac{2}{x^4} + \frac{1}{x^3} + \frac{1}{x^2}$

Solution:

(a) $y = 2x^3 - 3x^2 + 4$. $\frac{dy}{dx} = 6x^2 - 6x\frac{d^2y}{dx^2} = 12x - 6$

(b) $\frac{dy}{dx} = 9x^2 - 8x + 6$. $\frac{d^2y}{dx^2} = 18x - 8$

(c) $\frac{dy}{dx} = x^3.\frac{dy}{dx}(\log x) + \log x.\frac{d}{dx}(x^3) = x^3.\frac{1}{x} + \log x.3x^2 = x^2(1 + 3\log x)$

(d) $\frac{dy}{dx} = x^2\frac{d}{dx}(e^{2x}) + e^{2x}.\frac{d}{dx}(x^2) = x^2.2e^{2x} + e^{2x}.2x = 2xe^{2x}(x+1)$

(e) $f'(x) = -8x^{-5} - 3x^{-4} - 2x^{-3}$ $f''(x) = \frac{40}{x^6} + \frac{12}{x^5} + \frac{6}{x^4}$

2. **If $y = 4x^5 + 7x^4 + 3x + 10$, find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ at $x = (1, 2)$.**

Solution:

$\frac{dy}{dx} = \frac{d}{dx}(4x^5) + \frac{d}{dx}(7x^4) + \frac{d}{dx}(3x) + \frac{d}{dx}(10) = 20x^4 + 28x^3 + 3$. at(1, 2) $\frac{dy}{dx} = 51$

$\frac{d^2y}{dx^2} = \frac{d}{dx}(20x^4 + 28x^3 + 3) = 80x^3 + 84x^2$. At (1, 2) $\frac{d^2y}{dx^2} = 64$

$\frac{d^3y}{dx^3} = \frac{d}{dx}(80x^3 + 84x^2) = 240x^2 + 168x$. At (1, 2) $\frac{d^3y}{dx^3} = 408$

Exercise 2.3

1. (a) **The total cost function $C(x) = 50 + 3x + \sqrt{x}$, find the marginal cost at the output of 100 units.**

   (b) **The revenue function $R(Q) = \frac{1}{3}Q^3 - 10Q^2 + 75Q$, find the marginal revenue when $Q = 8$ unit.**

   (c) **If the total variable cost is given by $C(Q) = 200 + 5Q + 7Q^2$. Find the marginal cost at the production level 8.**

Solution:

(a) $C'(x) = 3 + \frac{1}{2\sqrt{x}}$. At x = 100, $C'(100) = 3 + \frac{1}{2\sqrt{100}} = 3 + \frac{1}{20} = \frac{61}{20}$

(b) $R'(Q) = \frac{1}{3} \cdot 3Q^2 - 10.2Q + 75.1 = Q^2 - 20Q + 75$. At Q = 8, $R'(8) = -21$

(c) $C'(Q) = \frac{d}{dQ}(200 + 5Q + 7Q^2) = 5 + 14Q$ $C'(8) = 5 + 14 \times 8 = 117$

2. **The demand function faced by a firm is $P = 500 - 0.2Q$ and its cost function is $C = 25Q + 1000$ (Where Q output or quantity). Find**

   (a) Marginal cost | **(b) Average cost**

   (c) Marginal revenue | **(d) Average revenue**

   (e) Profit function | **(f) Marginal profit.**

Solution:

$C(Q) = 25Q + 1000$. $R(Q) = P \times Q = 500Q - 0.2Q^2$

(a) $C'(Q) = \frac{d}{dQ}(25Q + 1000) = 25$

(b) Average cost $= \frac{C(Q)}{Q} = \frac{25Q + 1000}{Q} = 25 + \frac{1000}{Q}$

(c) Marginal Revenue = $R'(Q) = \frac{d}{dQ}(500Q - 0.2Q^2) = 500 - 0.4Q$

(d) Average Revenue = $\frac{R(Q)}{Q} = \frac{5000Q - 0.2Q^2}{Q} = 500 - 0.2Q$

(e) Profit = Revenue – Total cost = $500Q - 0.2Q^2 - (25Q + 1000)$

$$= 475Q - 0.2Q^2 - 1000$$

$$= 1000 + 475Q - 0.2Q^2.$$

(f) Marginal profit = $\pi'(Q) = \frac{d}{dQ}(1000 + 475Q - 0.2Q^2) = 475 - 0.4Q$

3. The average cost function (*AC*) for a product is given by

$AC = 0.0002x^2 - 0.05x + 7 + \frac{8000}{x}$, where x the output.

(a) Find the marginal cost function.

(b) What is the marginal cost when 100 units are produced?

Solution:

Total cost $C(x) = \left(0.001x^2 - 0.05x + 7 + \frac{8000}{x}\right)x = 0.0001x^3 - 0.05x^2 + 7x + 8000$

(a) Marginal cost function $= C'(x) = \frac{d}{dx}(0.0001x^3 - 0.05x^2 + 7x + 8000)$

$$= 0.0006x^2 - 0.1x + 7 = 3 - 10 + 7 = 0$$

(b) $C'(100) = 0.0006 \times (100)^2 - 0.1 \times 100 + 7 = 6 - 10 + 7 = 0$

4. The demand function and total cost function of a company are $Q = 150 - P$ and $C = \frac{Q^2}{2}$. Find the marginal profit at (*a*) level of production being 20 units. (*b*) level of production being 30 units.

Solution:

Profit function = Revenue function – Total cost function

$$\pi(Q) = P.Q. - \frac{Q^2}{2} = (150 - Q).\ Q - \frac{Q^2}{2} = 150Q - \frac{3}{2}Q^2\ \pi'(Q) = 150 - 3Q$$

(a) $\pi'(20) = 150 - 3 \times 20 = 90$ and

(b) $\pi'(30) = 150 - 3 \times 30 = 60$

Exercise 2.4

Evaluate the following integrals:

1. **(a)** $\int \frac{ax^3 + bx^2 + cx}{x}\,dx$ **(b)** $\int \frac{4x^3 + 5}{3x}\,dx$ **(c)** $\int \left(\frac{3}{x-2} + \frac{3x^{\frac{1}{2}}}{2}\right)dx$

Solution:

(a) $\int \frac{ax^3 + bx^2 + cx}{x} = a\int (x^2dx) + c\int (xdx) + c\int dx = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + \text{constant}$

(b) $\int \frac{4x^3 + 5}{3x}\,dx = \frac{4}{3}\int x^2dx + \frac{5}{3}\int \frac{1}{x}\,dx = \frac{4}{3}\cdot\frac{x^3}{3} + \frac{5}{3}\log x + \text{constant}$

(c) $\int \left(\frac{3}{x-2}\right)dx + \frac{3}{2}\int x^{\frac{1}{2}}\,dx = 3\log(x-2) + \frac{3}{2}\cdot\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \text{constant}$

$= 3\log(x-2) + x^{\frac{3}{2}} + \text{constant}$

2. **(a)** $\int \left(x - \frac{1}{x}\right)^2 dx$ **(b)** $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$ **(c)** $\int (3x+2)^2\,dx$

Solution:

(a) $\int \left(x - \frac{1}{x}\right)^2 dx = \int x^2dx - 2\int dx + \int x^{-2}\,dx = \frac{x^3}{3} - 2x + \frac{x^{-2+1}}{-2+1} + \text{constant}$

$= \frac{x^3}{3} - 2x - \frac{1}{x} + c$

(b) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx = \int xdx - 2\int dx + \int \frac{1}{x}\,dx = \frac{x^2}{2} - 2x + \log x + \text{constant}$

(c) $\int (3x+2)^2\,dx = 9\int x^2\,dx + 12\int x\,dx + 4\int dx = 9.\frac{x^3}{3} + 12.\frac{x^2}{2} + 4x + c$

$= 3x^3 + 6x^2 + 4x + c$

3. **(a)** $\int \frac{x+1}{x-1}\,dx$ **(b)** $\int \frac{3x+4}{6x+7}\,dx$ **(c)** $\int \frac{x+3}{x-3}\,dx$

Solution:

(a) $\int \frac{x+1}{x-1}\,dx = \int dx - \int \frac{2}{x-1}\,dx = x - 2\log(x-1) + c$

(b) $\int \frac{3x+4}{6x+7}\,dx = \frac{1}{2}\int \frac{6x+7}{6x+7}\,dx + \frac{1}{2}\int \frac{dx}{6x+7} = \frac{1}{2}x + \frac{1}{12}\log(6x+7) + \text{constant}$

(c) $\int \frac{x+3}{x-3}\,dx = \int \frac{x-3}{x-3}\,dx + 6\int \frac{1}{x-3}\,dx = x + 6\log(x-3) + \text{constant}$

4. **(a)** $\int \frac{1}{\sqrt{x-a}-\sqrt{x-b}}\,dx$ **(b)** $\int \frac{1}{\sqrt{2x+5}-\sqrt{2x-5}}\,dx$ **(c)** $\int \frac{1}{\sqrt{x+1}-\sqrt{x}}$

Solution:

(a) $$\int \frac{1}{\sqrt{x-a}-\sqrt{x-b}}\,dx = \int \frac{\sqrt{x-a}+\sqrt{x+b}}{x-a-x+b}\,dx$$

$$= \frac{1}{b-a}\left[\int (x-a)^{\frac{1}{2}}\,dx + \int (x-b)^{\frac{1}{2}}\,dx\right]$$

$$= \frac{1}{b-a}\left[\frac{1}{\frac{3}{2}}(x-a)^{\frac{3}{2}} + \frac{1}{\frac{3}{2}}(x-b)^{\frac{3}{2}}\right] + c$$

$$= \frac{2}{3(b-a)}\left[(x-a)^{\frac{3}{2}} + (x-b)^{\frac{3}{2}}\right] + c$$

(b) $$\int \frac{1}{\sqrt{2x+5}-\sqrt{2x-5}}\,dx = \int \frac{(2x+5)^{\frac{1}{2}}+(2x-5)^{\frac{1}{2}}}{2x+5-2x+5}\,dx$$

$$= \frac{1}{10}\left[\int (2x+5)^{\frac{1}{2}}\,dx + \int (2x-5)^{\frac{1}{2}}\,dx\right]$$

$$= \frac{1}{10}\left[\frac{1}{2}\frac{(2x+5)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{1}{2}\frac{(2x-5)^{\frac{3}{2}}}{\frac{3}{2}}\right] + c$$

$$= \frac{1}{10}\times\frac{1}{2}\times\frac{2}{3}\left[(2x+5)^{\frac{3}{2}} + (2x-5)^{\frac{3}{2}}\right] + c$$

$$= \frac{1}{30}\left[(2x+5)^{\frac{3}{2}} + (2x-5)^{\frac{3}{2}}\right] + c$$

(c) $$\int \frac{1}{\sqrt{x+1}-\sqrt{x}} \times \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}\,dx$$

$$= \int (x+1)^{\frac{1}{2}}\,dx + (x)^{\frac{1}{2}}\,dx$$

$$= \frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{2}{3}\left[(x+1)^{\frac{3}{2}} + x^{\frac{3}{2}}\right] + c$$

5. (a) $\int \frac{(2ax+b)}{\sqrt{ax^2+bx+c}}dx$ (b) $\int \frac{(2x-3)}{(x^2-3x+2)^4}dx$ (c) $\int \frac{(5x-3)}{\sqrt{5x^2-6x+3}}dx$

Solution:

(a) $\int \frac{(2ax+b)}{ax^2+bx+c}$.

Let, $ax^2+bx+c=t$, $(2ax+b)\,dx = dt$ $\int t^{\frac{-1}{2}}dt = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\sqrt{ax^2+bx+c} + c$

(b) $\int \frac{(2x-3)}{(x^2-3x+2)^4}dx$.

Let, $x^2-3x+2=t$, $(2x-3)\,dx = dt$ $\int t^{-4}\,dt = \frac{t^{-3}}{-3} = \frac{-1}{3(x^2-3x+1)^3} + c$

(c) $\int \frac{(5x-3)}{(5x^2-6x+3)^{\frac{1}{2}}}dx$.

Let, $5x^2-6x+2=t$

i.e., $(10x-6)\,dx = dt$, $(5x-3)\,dx$

$$= \frac{1}{2}dt.\ \frac{1}{2}\int \frac{dt}{t^{\frac{1}{2}}} = \frac{1}{2}\ \frac{t^{\frac{1}{2}+1}}{\frac{-1}{2}+1} + c$$

$$= \sqrt{t} + c = \sqrt{5x^2-6x+3} + c$$

6. (a) $\int \frac{(1+\log x)^2}{x}dx$ (b) $\int \frac{1}{x(1+\log x)}dx$ (c) $\int \frac{2+\frac{1}{x}}{(2x+\log x)}dx$

Solution:

(a) $\int \frac{(1+\log x)^2}{x}dx$. Let, $1+\log x = t$, $\frac{1}{x}dx = dt$. $\int t^2 dt = \frac{t^3}{3} + c = \frac{(1+\log x)^3}{3} + c$

(b) $\int \frac{1}{x(1+\log x)}dx$. Let $1+\log x = t$, $\frac{1}{x}dx = dt$. $\int \frac{1}{t}dt = \log t = \log(1+\log x) + c$

(c) $\int \frac{\left(2+\frac{1}{x}\right)}{(2x+\log x)}dx$ Let, $2x+\log x = t$, $\left(2+\frac{1}{x}\right)dx = dt$.

$\int \frac{1}{t}dt = \log t + c = \log(2x+\log x) + c$

7. **(a)** $\int e^{2x+3}\,dx$ **(b)** $\int e^{3x+7}\,dx$ **(c)** $\int e^{11-5x}\,dx$

Solution:

(a) $\int e^{2x+3}\,dx = e^{2x+3} \times \dfrac{1}{\frac{d}{dx}(2x+3)} + c = \dfrac{1}{2} e^{2x+3}$ + constant

(b) $\int e^{3x+7}\,dx = e^{3x+7} \times \dfrac{1}{\frac{d}{dx}(3x+7)} = \dfrac{1}{3} e^{3x+7}$ + constant

(c) $\int e^{11-5x}\,dx = e^{11-5x} \times \dfrac{1}{\frac{d}{dx}(11-5x)} = \left(-\dfrac{1}{3}\right) e^{11-5x}$ + constant

8. **(a)** $\int x\, e^{x^2}\,dx$ **(b)** $\int (2x+3)\, e^{x^2+3x+5}\,dx$ **(c)** $\int (4x-3) e^{2x^2-3x+5}\,dx$

Solution:

(a) $\int xe^{x2}\,dx =$ Let, $x^2 = t \Rightarrow 2x\,dx = dt$. $\dfrac{1}{2}\int e^t\,dt = \dfrac{1}{2} e^t + c = \dfrac{1}{2} e^{x^2} + c$

(b) $\int (2x+3)\, e^{x^2+3x+5}\,dx.$

Let, $x^2 + 3x + 5 = t \Rightarrow (2x+3)\,dx = dt$

$\int e^t\,dt = e^t + c = e^{x^2+3x+5} + c$

(c) $\int (4x-3)\, e^{2x^2-3x+5}\,dx$. Let $e^{2x^2-3x+5} = e^t$, $(4x-3)\,dx = dt$.

$\int e^t\,dt = e^t + c = e^{2x^2-3x+5} + c$

9. **(a)** $\int \dfrac{e^x+1}{e^x}\,dx$ **(b)** $\int \dfrac{e^{3x}+e^{2x}+e^x}{e^x}\,dx$ **(c)** $\int e^x + \dfrac{1}{e^x} + \dfrac{1}{e^{2x}}\,dx$

Solution:

(a) $\int \dfrac{e^x+1}{e^x}\,dx = \int \dfrac{e^x}{e^x}\,dx + \int \dfrac{1}{e^x}\,dx = x - e^{-x} + c$

(b) $\int e^{3x-x}\,dx + \int e^{2x-x}\,dx + \int dx = \dfrac{1}{2} e^{2x} + e^x + x + c$

(c) $\int e^x\,dx + \int e^{-x}\,dx + \int e^{-2x}\,dx = e^x + e^{-x}(-1) + (e^{-2x})\left(-\dfrac{1}{2}\right) + c$

10. **(a)** $\int x(3x+2)^4\,dx$ **(b)** $\int x\,(2x+3)^4\,dx$ **(c)** $\int x(4x+1)^{-\frac{1}{2}}\,dx$

Solution:

(a) $\int x(3x+2)^4\,dx = x\int (3x+2)^4\,dx - \int\left[\frac{d}{dx}(x)\int (3x+2)^4\,dx\right]dx$

$$= x\,.\,\frac{(3x+2)^5}{5}\cdot\frac{1}{3} - \frac{1}{15}\int (3x+2)^5\,dx$$

$$= \frac{x}{15}(3x+2)^5 - \frac{1}{270}(3x+2)^6 + c$$

(b) $\int x(2x+3)^4\,dx = x\int (2x+3)^4\,dx - \int\left[\frac{d}{dx}(x).\int (2x+3)^4 dx\right]dx$

$$= \frac{1}{10}\cdot x\,(2x+5)^5 - \frac{1}{120}(2x+3)^6 + c$$

(c) $\int x(4x+1)^{\frac{-1}{2}}\,dx = x\int (4x+1)^{\frac{-1}{2}}\,dx - \int\left[\frac{d}{dx}(x).\int (4x+1)^{\frac{-1}{2}}dx\right]dx$

$$= \frac{x\sqrt{4x+1}}{2} - \frac{(4x+1)^{\frac{3}{2}}}{12} + c$$

11. **(a)** $\int x\,e^x\,dx$ **(b)** $\int x\,e^{3x}\,dx$ **(c)** $\int x^2\,e^x\,dx$

Solution:

(a) $\int xe^x\,dx = x\int e^x dx - \int\left[\frac{d}{dx}(x).\int e^x\,dx\right]dx = xe^x - e^x + c$

(b) $\int x.e^{3x}\,dx = x\int e^{3x}\,dx - \int\left[\frac{d}{dx}(x).\int e^{3x}\,dx\right]dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$

(c) $\int x^2.e^x\,dx = x^2\int e^x\,dx - \int\left[\frac{d}{dx}(x^2).\int e^x\,dx\right] = x^2e^x - 2\int xe^x\,dx$

$$= x^2e^x - 2xe^x + e^x + c.$$

12. **(a)** $\int \log x\,dx$ **(b)** $\int \frac{1}{x}\log x\,dx$ **(c)** $\int \log(x+2)\,dx$

Solution:

(a) $\int \log x\,dx = \log x\,.\int dx - \int\left[\frac{d}{dx}(\log x).\int dx\right]dx = x\log x - \int \frac{1}{x}\,.\,x.dx$

$$= x\log x - x + c$$

(b) $\int \frac{1}{x}\log x\,dx \Rightarrow$ Let, $\log x = t \Rightarrow \frac{1}{x}dx = dt$. So, $\int t\,dt = \frac{t^2}{2} + c = \frac{(\log x)^2}{2} + c$

(c) $\int \log(x+2)\,dx = \log(x+2)\int dx - \int\left[\frac{d}{dx}(\log(x+2)).\int dx\right]dx$

$$= x\log(x+2) - \int \frac{x}{x+2}\,dx$$

$$= x\log(x+2) - \left[\int \frac{x+2}{x+2}\,dx - \int \frac{2}{x+2}\,dx\right]$$

$$= x\log(x+2) - x + 2\log(x+2) + c$$

13. (a) Let the marginal cost function of a firm be 100 – 10*x* + (0.1)*x*² where *x* is the output. Obtain the total cost function of the firm under the assumption that its fixed cost is Rs 520.

(b) If the marginal cost of product is given by 16 – 4*x* + 3*x*² and the initial cost is Rs 20. Find the total cost function.

(c) The marginal cost function *MC* = *x*² + *x* + 2, *x* being output produced. Find the total cost function, where fixed cost is Rs 50.

Solution:

(a) $\frac{dc}{dx} = 100 - 10x + 0.1x^2 \Rightarrow C = \int 100\,dx - 10\int x dx + 0.1\int x^2\,dx$

$C = 100x - 10\,.\frac{x^2}{2} + 0.1 \times \frac{x^3}{2} + 520 \Rightarrow C = 100x - 5x^2 + \frac{0.1}{3}x^3 + 520.$

(b) $\frac{dC}{dx} = 16 - 4x + 3x^2.\ C = \int 16\,dx - 4\int x\,dx + 3\int x^2 dx = 16x - \frac{4x^2}{2} + \frac{3x^3}{3} + 20$

$= 20 + 16x - 2x^2 + x^3$

(c) $C = \int (x^2 + x + 2)\,dx = \frac{x^3}{3} + \frac{x^2}{2} + 2x + 50$

14. (a) The marginal cost function for a firm is *Q*² + *Q* + 2, where *Q* is the output. Find the total cost function and average cost function, if the fixed cost is Rs. 50.

(b) If the marginal cost (*MC*) is 25 + 30*x* – 9*x*², where x is the number of units produced, and the total cost of producing one unit is Rs 40. Find the total cost function and average cost function.

(c) If the marginal cost function of a firms 2 + 3*e*^*x*, where x is the output. Find the total cost and average cost function if the fixed cost is Rs 500.

Solution:

(a) $C = \int (Q^2 + Q + 2)\ dQ = \frac{Q^3}{3} + \frac{Q^2}{2} + 2Q + 50$. Average cost $= \frac{Q^2}{3} + \frac{Q}{2} + 2 + \frac{50}{Q}$

(b) $C = \int (25 + 30x - 9x^2)\,dx = 25x + 30 \times \frac{x^2}{2} - 9 \times \frac{x^3}{3} + \text{constant}$

When x = 1, cost = 40, 40 = 25 + 15 – 3 + constant. Constant = 3.

$C = 3 + 25x + 15x^2 - 3x^3$. Average cost function $= \frac{3}{x} + 25 + 15x - 3x^2$.

(c) $C = \int (2 + 3\, e^x)\, dx = 2x + 3e^x + \text{constant}$

$$= 2x + 3e^x + 500\ .\ AC = 2 + \frac{3e^x}{x} + \frac{500}{x}$$

15. (a) A company suffers a loss of Rs 110, if one of its special product does not sell. If marginal revenue is approximated by *MR* = 20 – 3*x* and marginal cost by *MC* = 10 + 2*x*, find the total profit function.

(b) If *MR* = 5 – 4*Q* + 3*Q*², *MC* = 3 + 2*Q* and fixed cost is zero. Find the profit function and the total profit where *Q* = 4.

(c) The marginal cost function and revenue function of a firm are given as *MR* = 3 + 2*x* and *MC* = 5 – 4*x* – 3*x*². If fixed cost and revenue are each zero, find the profit function and the profit where the output is 2.

Solution:

(a) $R = \int (20 - 30x)\, dx = 20x - \frac{3x^2}{2} + \text{constant. When } x = (-1),\ R = -110$

i.e. $-110 = 20(-1) - \frac{3(-1)^2}{2} + \text{Constant}$

$\text{Constant} = -110 + 20 + \frac{3}{2} = -88.5$

$R = 20x - \frac{3x^2}{2} - 88.5.\ C = \int (10 + 2x)dx = 10x + \frac{2x^2}{2} + \text{Constant}$

$= 10x - \frac{5x^2}{2} + \text{constant. When } x = 1,\ \pi(x) = -110\ .\ -110 = 10 - 2.5 + \text{Constant}$

$\text{Constant} = -117.5\ \pi(x) = -117.5 + 10x - \frac{5x^2}{2}$

(b) $MC = 3 + 2Q\ .\ C = \int (3 + 2Q)\, dQ = 3Q + \frac{2Q^2}{2} + \text{constant} = Q^2 + 3Q.$

$MR\ = 5 - 4Q + 3Q^2.\ R = 5Q - \frac{4Q^2}{2} + \frac{3Q^3}{3} + \text{constant}$

$= 5Q - 2Q^2 + Q^3 + \text{constant.}$

Profit function $= Q^3 - 2Q^2 + 5Q - (Q^2 + 3Q) = Q^3 - 3Q^2 + 2Q + \text{constant.}$

When, constant is considered as zero, and Q = 4,

profit,

$4^3 - 3 \times 4^2 + 2 \times 4 = 24.$

Profit function $= \int (3 + 2x)\, dx - \int 5 - 4x - 3x^2\, dx$

$$= 3x + \frac{2x^2}{2} - 5x + \frac{4x^2}{2} + \frac{3x^3}{3} + \pi(x) = x^3 + 3x^2 - 2x\ .\ \pi(2)$$

$$= 2^3 + 3 \times 2^2 - 2 \times 2 = 16$$

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