On a certain continuity property of the residues of the poles of $\sum_{n\geq 1} \Lambda(n) e^{-\pi i q n} n^{-s}$ with respect to $q\in \mathbb{Q}\cap (0,2)$

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Abstract

The purpose of this article is to present some result concerning the poles of the series $\sum_{n\geq 1} \Lambda(n) e^{-\pi i q n} n^{-s}$, where q is rational and Λ is the arithmetical Mangoldt Λ -function. A certain continuity property of the residues of the poles of M(s,q) with respect to $q \in \mathbb{Q} \cap (0,2)$ is discussed.

MSC 2010: 11M26

Keywords: Fourier transform; Gamma function; Mangoldt function; Riemann zeta-function.

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1 Introduction

The main topic of the article is to study on the distribution of poles of the function

$$M(s,q) \equiv \sum_{n \ge 1} \Lambda(n) e^{-\pi i q n} n^{-s},$$

where $\Lambda(n)$ is the arithmetical Mangoldt Λ -function and $q \in \mathbb{Q} \cap (0,2)$. Roughly speaking, we shall prove the following proposition.

Proposition 1. Suppose that M(s,1) has a pole at $s = \rho$ with residue R_0 , $1/2 < Re(\rho) < 1$ and $Im(\rho)$ some large positive real number. Then to each q = 1+a/b (a/b small) sufficiently close to q = 1, there corresponds a pole (or several of those) of M(s,a/b) whose residue is close to R_0 . The formula for residues and poles of M(s,a/b) are expressed by Dirichlet L-functions; specifically, as s approaches a pole of M(s,1+a/b), we have

$$M(s, 1+a/b) \sim M(s, a/b) \sim \sum_{\chi} A(a, b; \chi) \frac{L'}{L}(s, \chi).$$

We note that the poles of M(s,1) are exactly those of $(\zeta'/\zeta)(s)$, where ζ denotes the Riemann zeta-function.

Besides, $e^{-\pi i(1+a/b)p} = e^{-\pi ip - \pi iap/b} = -e^{-\pi iap/b}$ for p odd primes.

For the objective of providing a supporting argument for this proposition, we will first prove the following theorem.

Theorem 1. Let c > 0, $c + \delta + \kappa > 1$, $\kappa - c' > 1$, $\delta + c' > 1$, and $-\pi/2 < \arg(p) \le 0$. Then we have

$$\frac{1}{2\pi i} \int_{(c)} p^{-s-\delta} M(s+\delta+\kappa,p) \Gamma(s) \Gamma(s+\delta) \Gamma(1-s) \pi^{-s} e^{-\pi i s/2} ds$$

$$= -\frac{\zeta'}{\zeta} (\kappa) \Gamma(1+\delta) p^{-\delta} \int_0^1 e^{-\pi i p r} \Phi(p,r,1+\delta) dr$$

$$-\Gamma(1+\delta) p^{-\delta} \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} -\frac{\zeta'}{\zeta} (\kappa-w) \int_0^1 \Phi(p,r,1+\delta+w) e^{-\pi i p r} dr \frac{dw}{w}.$$
(1)

We will present a supporting argument for Proposition 1 in Section 3. The following formula for the Γ -function [3, pp.405]

$$\Gamma(\sigma + it) \approx |\sigma + it|^{\sigma - 1/2} e^{-\pi|t|/2} \tag{2}$$

as $|t| \to \infty$, will be often used.

2 The proof of Theorem 1

We define [3]

$$I_1(x) \equiv \frac{1}{2\pi i} \int_{(c)} \Gamma(s) x^{-s} ds = e^{-x}$$
(3)

and

$$I_2(x) \equiv \frac{1}{2\pi i} \int_{(c)} \Gamma(s+q-1)\Gamma(1-s)x^{-s}ds,$$

where 0 < q < 1 with 1 - q < c < 1.

Shifting the path in I_2 to the left, we find out that

$$I_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+q) x^{n+q-1}}{n!} = \Gamma(q) x^{q-1} (1+x)^{-q};$$
 (4)

express $(1+x)^{-q}$ as a power series in x at x=0 and use $\Gamma(s)s=\Gamma(s+1)$. We use these two formulas (3) and (4) to evaluate

$$I(x) \equiv \frac{1}{2\pi i} \int_{(c)} \Gamma(s) \Gamma(s+q-1) \Gamma(1-s) x^{-s} ds.$$
 (5)

Here, we recall that [4]

$$\frac{1}{2\pi i} \int_{(c)} \mathcal{F}(s) \mathcal{G}(s) x^{-s} ds = \int_0^\infty f(z) g(\frac{x}{z}) \frac{dz}{z},$$

where $\frac{1}{2\pi i} \int_{(c)} \mathcal{F}(s) x^{-s} ds = f(x)$ and $\frac{1}{2\pi i} \int_{(c)} \mathcal{G}(s) x^{-s} ds = g(x)$.

We associate \mathcal{F} and f with $\Gamma(s)$ and e^{-x} (by (3)), and \mathcal{G} and g with $\Gamma(s+q-1)\Gamma(1-s)$ and $\Gamma(q)x^{q-1}(1+x)^{-q}$ (by (4)), respectively.

This gives

$$I(x) = \Gamma(q)x^{q-1} \int_0^\infty e^{-z} (z+x)^{-q} dz.$$
 (6)

Thus choosing $x \mapsto xpn$ with $-\pi/2 < \arg(p) \le 0$, multiplying by

$$\Lambda(n)e^{-\pi ipn}p^{-\delta}n^{-\delta-\kappa}$$
,

and summing over $n \geq 2$, we have for $1 < \text{Re}(\kappa) < 2$ and $\delta > 0$ (so that $c + \delta + \kappa > 1$),

$$\frac{1}{2\pi i} \int_{(c)} \sum_{n\geq 2} \frac{\Lambda(n)e^{-\pi ipn}}{p^{s+\delta}n^{s+\delta+\kappa}} \Gamma(s) \Gamma(s+q-1) \Gamma(1-s) x^{-s} ds$$

$$= \sum_{n=1}^{\infty} \frac{\Lambda(n)e^{-\pi ipn}}{p^{\delta}n^{\delta+\kappa}} I(xpn)$$

$$= \Gamma(q) x^{q-1} \sum_{n=1}^{\infty} \frac{\Lambda(n)e^{-\pi ipn}}{p^{1-q+\delta}n^{1-q+\delta+\kappa}} \int_{0}^{\infty} e^{-z} (z+xpn)^{-q} dz,$$
(7)

or rearranging,

$$\frac{1}{2\pi i} \int_{(c)} \sum_{n\geq 2} \frac{\Lambda(n)e^{-\pi ipn}}{p^{s+\delta}n^{s+\delta+\kappa}} \Gamma(s)\Gamma(s+q-1)\Gamma(1-s)x^{-s}ds$$
$$= \Gamma(q)x^{-1} \sum_{n=1}^{\infty} \frac{\Lambda(n)e^{-\pi ipn}}{p^{1-q+\delta}n^{1-q+\delta+\kappa}} \int_{0}^{\infty} e^{-z}(z/x+pn)^{-q}dz.$$

From here on, we keep $1 < \text{Re}(\kappa) < 2$ unless otherwise mentioned. We use the change of variables z = xw, let $x = \pi e^{\pi i/2}$, and obtain

$$\frac{1}{2\pi i} \int_{(c)} \sum_{n\geq 2} \frac{\Lambda(n)e^{-\pi ipn}}{p^{s+\delta}n^{s+\delta+\kappa}} \Gamma(s) \Gamma(s+q-1)\Gamma(1-s)\pi^{-s}e^{-\pi is/2} ds$$

$$= \Gamma(q) \sum_{n=1}^{\infty} \frac{\Lambda(n)e^{-\pi ipn}}{p^{1-q+\delta}n^{1-q+\delta+\kappa}} \int_{0}^{\infty} e^{-\pi iw} (w+pn)^{-q} dw. \tag{8}$$

We choose $q = 1 + \delta$.

Next, we rewrite the integral \int_0^∞ on the right side of (8) as

$$\int_{0}^{\infty} e^{-\pi i w} (pn+w)^{-1-\delta} dw
= p^{-\delta} \int_{0}^{\infty} e^{-\pi i p w'} (n+w')^{-1-\delta} dw'
= p^{-\delta} \sum_{k \ge 0} \int_{k}^{k+1} e^{-\pi i p w'} (n+w')^{-1-\delta} dw'
= p^{-\delta} \sum_{k \ge 0} \int_{0}^{1} e^{-\pi i p (k+r)} (n+k+r)^{-1-\delta} dr
= p^{-\delta} e^{\pi i p n} \int_{0}^{1} \sum_{j \ge n} e^{-\pi i p (j+r)} (j+r)^{-1-\delta} dr,$$
(9)

where in obtaining the first equality, we used Cauchy's theorem on contour integrals with the paths $\{0 \le w \le R\}$, $\{w = Re^{it} : \arg(p) \le t \le 0\}$, and $\{w = qe^{i\arg(p)} : 0 \le q \le R\}$, letting $R \to \infty$, and then made the change of variables q = |p|w'.

By

$$\sum_{j \ge n} e^{-\pi i p(j+r)} (j+r)^{-1-\delta}$$

$$= \sum_{j \ge 1} e^{-\pi i p(j+r)} (j+r)^{-1-\delta} - \sum_{1 \le j < n} e^{-\pi i p(j+r)} (j+r)^{-1-\delta}$$

$$\equiv e^{-\pi i p r} \Phi(p, r, 1+\delta) - \sum_{1 \le j < n} e^{-\pi i p(j+r)} (j+r)^{-1-\delta},$$

(9) becomes

$$\begin{split} & \int_0^\infty e^{-\pi i w} (pn+w)^{-1-\delta} dw \\ & = p^{-\delta} e^{\pi i pn} \int_0^1 \sum_{j \geq n} e^{-\pi i p (j+r)} (j+r)^{-1-\delta} dr \\ & = p^{-\delta} e^{\pi i pn} \int_0^1 \left(e^{-\pi i p r} \Phi(p,r,1+\delta) - \sum_{1 \leq j < n} e^{-\pi i p (j+r)} (j+r)^{-1-\delta} \right) dr, \end{split}$$

and so (8) is rewritten as

$$\frac{1}{2\pi i} \int_{(c)} \sum_{n\geq 2} \frac{\Lambda(n)e^{-\pi ipn}}{p^{s+\delta}n^{s+\delta+\kappa}} \Gamma(s)\Gamma(s+\delta)\Gamma(1-s)\pi^{-s}e^{-\pi is/2} ds$$

$$= \Gamma(1+\delta)p^{-\delta} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\kappa}}$$

$$\times \int_{0}^{1} \left(e^{-\pi ipr}\Phi(p,r,1+\delta) - \sum_{1\leq j< n} e^{-\pi ip(j+r)}(j+r)^{-1-\delta}\right) dr$$

$$\equiv \Gamma(1+\delta)p^{-\delta}(Y_{1}-Y_{2}).$$
(10)

Here, it is plain that with the dominated convergence theorem,

$$Y_2 = \int_0^1 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\kappa}} \sum_{1 \le j \le n} e^{-\pi i p(j+r)} (j+r)^{-1-\delta} dr.$$

In order to analyze Y_2 , we use a variation of the following relation [5, pp.60]

$$\sum_{j < x} \frac{a_j}{j^s} = \frac{1}{2\pi i} \int_{c-iU}^{c+iU} f(s+w) \frac{x^w}{w} dw + O(x^c U^{-1} (\sigma + c - 1)^{-\alpha})$$

$$+ O(U^{-1} \psi(2x) x^{1-\sigma} \log x) + O(U^{-1} \psi(N) x^{1-\sigma} |x - N|^{-1}),$$
(11)

where x is not an integer, N is the integer nearest to x, c > 0, $\sigma + c > 1$, $a_n \ll \psi(n)$ for some non-decreasing ψ , and the series

$$f(s) = \sum_{n>1} \frac{a_n}{n^s}, \quad s = \sigma + it,$$

converges absolutely for $\sigma > 1$ with

$$\sum_{n>1} \frac{|a_n|}{n^{\sigma}} \ll (\sigma - 1)^{-\alpha}.$$

In the proof of (11) available in [5], we replace "n" and "x" by j + r and n, respectively, and obtain

$$\sum_{j < n} \frac{a_j}{(j+r)^s} = \frac{1}{2\pi i} \int_{c-iU}^{c+iU} f(s+w,r) \frac{n^w}{w} dw + O(n^c U^{-1} (\sigma + c - 1)^{-\alpha})$$

$$+ O(U^{-1} \psi(2n) n^{1-\sigma} \log n) + O(U^{-1} \psi(n) n^{1-\sigma} r^{-1}),$$
(12)

where $r \in (0,1)$ and

$$f(s,r) \equiv \sum_{n} \frac{a_n}{(n+r)^s}.$$

Choosing $s \mapsto 1 + \delta$ and $a_j \mapsto e^{-\pi i p j}$ in (12), we have

$$\sum_{j < n} \frac{e^{-\pi i p j}}{(j+r)^{1+\delta}} = \frac{1}{2\pi i} \int_{c-iU}^{c+iU} \Phi(p,r,1+\delta+w) \frac{n^w}{w} dw + O(U^{-1} (\log n)^2 (n+r)^c r^{-1} (\sigma+c-1)^{-\alpha}).$$

Furthermore, multiplying both sides by $\Lambda(n)n^{-\kappa}$ and summing all over the positive integers $n \geq 2$, we get

$$\sum_{n\geq 2} \frac{\Lambda(n)}{n^{\kappa}} \sum_{1\leq j < n} \frac{e^{-\pi i p j}}{(j+r)^{1+\delta}}
= \frac{1}{2\pi i} \int_{c-iU}^{c+iU} -\frac{\zeta'}{\zeta} (\kappa - w) \frac{\Phi(p, r, 1+\delta+w)}{w} dw
+ O_r (U^{-1} H(\sigma + c - 1)^{-\alpha}),$$
(13)

where

$$H \equiv \sum_{n>2} (\log n)^2 \Lambda(n) n^{-\kappa+c}.$$

By (13), we see that

$$\frac{1}{2\pi i} \int_{c-iU}^{c+iU} -\frac{\zeta'}{\zeta} (\kappa - w) \frac{\Phi(p, r, 1 + \delta + w)}{w} dw$$

$$\rightarrow \sum_{n \ge 2} \frac{\Lambda(n)}{n^{\kappa}} \sum_{1 \le j < n} \frac{e^{-\pi i p j}}{(j+r)^{1+\delta}} \tag{14}$$

uniformly in $r \in [1/N, 1-1/N]$ for any fixed N as $U \to \infty$.

Hence, the integral in Y_2 is rewritten as

$$\lim_{N \to \infty} \int_{1/N}^{1-1/N} \sum_{n \ge 2} \frac{\Lambda(n)}{n^{\kappa}} \sum_{1 \le j < n} \frac{e^{-\pi i p j}}{(j+r)^{1+\delta}} e^{-\pi i p r} dr$$

$$= \lim_{N \to \infty} \int_{1/N}^{1-1/N} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{\zeta'}{\zeta} (\kappa - w) \frac{\Phi(p, r, 1 + \delta + w)}{w} dw e^{-\pi i p r} dr \qquad (15)$$

$$= \lim_{N \to \infty} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{\zeta'}{\zeta} (\kappa - w) \int_{1/N}^{1-1/N} \Phi(p, r, 1 + \delta + w) e^{-\pi i p r} dr \frac{dw}{w};$$

the first equality is by pointwise convergence of (14), and the second by uniform convergence of (14) for each N.

But if c and κ are sufficiently large so that $\kappa - c > 1$ and $1 + \delta + c > 2$, then with integration by parts, it is easy to show that

$$\int_{1/N}^{1-1/N} \Phi(p, r, 1+\delta+w) dr \ll (|t|+1)^{-1}, \quad w = c+it, \tag{16}$$

and so the last integral $\int_{(c)}$ in (15) converges absolutely. This in turn enables us to put the limit $N \to \infty$ inside the integral symbol $\int_{(c)}$ (use the dominated convergence theorem); we get

$$Y_2 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{\zeta'}{\zeta} (\kappa - w) \int_0^1 \Phi(p, r, 1 + \delta + w) e^{-\pi i p r} dr \frac{dw}{w}.$$
 (17)

By (17), (10) becomes

$$\frac{1}{2\pi i} \int_{(c)} \sum_{n\geq 2} \frac{\Lambda(n)e^{-\pi ipn}}{p^{s+\delta}n^{s+\delta+\kappa}} \Gamma(s)\Gamma(s+\delta)\Gamma(1-s)\pi^{-s}e^{-\pi is/2}ds$$

$$= \Gamma(1+\delta)p^{-\delta} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\kappa}} \int_{0}^{1} e^{-\pi ipr} \Phi(p,r,1+\delta)dr$$

$$- \Gamma(1+\delta)p^{-\delta} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{\zeta'}{\zeta} (\kappa-w) \int_{0}^{1} \Phi(p,r,1+\delta+w)e^{-\pi ipr}dr \frac{dw}{w}.$$
(18)

This completes the proof of Theorem 1.

3 A supporting argument for Proposition 1

Suppose that $\rho_{n_1} = 1/2 + \eta_{n_1} + i\gamma_{n_1}$ is a nontrivial zero of the Riemann zeta-function with $\eta_{n_1} > 0$.

We first insert $p_1 = 1$ in (1). Note that for $p_1 = 1$, we have

$$M(s,1) = \sum_{n} \frac{\Lambda(n)e^{-\pi i n}}{n^s} = \Lambda(2)2^{-s} - \sum_{n \ge 3} \frac{\Lambda(n)}{n^s},$$

and so the poles of M(s,1) are exactly the nontrivial zeros of the Riemann zeta-function.

Using the residue theorem in (1), we shift the path of the integral on the left side to $\sigma = 1/2$; then it becomes

$$\frac{1}{2\pi i} \int_{(c)} M(s+\delta+\kappa,1)\Gamma(s)\Gamma(s+\delta)\Gamma(1-s)\pi^{-s}e^{-\pi i s/2}ds$$

$$= \pi^{-\rho_{n_1}+\delta+\kappa}e^{-\pi i(\rho_{n_1}-\delta-\kappa)/2}$$

$$\times \Gamma(\rho_{n_1}-\delta-\kappa)\Gamma(\rho_{n_1}-\kappa)\Gamma(1-\rho_{n_1}+\delta+\kappa)$$

$$+ E + S_{\delta,\kappa}, \tag{19}$$

where E is the collection of all the terms associated with poles other than ones related to zeros of the zeta-function and

$$S_{\delta,\kappa} \equiv \frac{1}{2\pi i} \int_{(1/2)} M(s+\delta+\kappa,1) \Gamma(s) \Gamma(s+\delta) \Gamma(1-s) \pi^{-s} e^{-\pi i s/2} ds.$$

On the right side of (1), we apply the residue theorem to the second integral and get

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{\zeta'}{\zeta} (\kappa - w) \int_{0}^{1} \Phi(1, r, 1 + \delta + w) e^{-\pi i r} dr \frac{dw}{w}
= -\sum_{\rho_{j}} (\kappa - \rho_{j})^{-1} \int_{0}^{1} \Phi(1, r, 1 + \delta + \kappa - \rho_{j}) e^{-\pi i r} dr
+ (\kappa - 1)^{-1} \int_{0}^{1} \Phi(1, r, \delta + \kappa) e^{-\pi i r} dr
+ \frac{1}{2\pi i} \int_{2-h-i\infty}^{2-h+i\infty} -\frac{\zeta'}{\zeta} (\kappa - w) \int_{0}^{1} \Phi(1, r, 1 + \delta + w) e^{-\pi i r} dr \frac{dw}{w},$$
(20)

with h > 0 arbitrary.

Thus, by (19) and (20), we have

$$\pi^{-\rho_{n_1} + \delta + \kappa} e^{-\pi i (\rho_{n_1} - \delta - \kappa)/2}$$

$$\times \Gamma(\rho_{n_1} - \delta - \kappa) \Gamma(\rho_{n_1} - \kappa) \Gamma(1 - \rho_{n_1} + \delta + \kappa)$$

$$+ E + S_{\delta, \kappa}$$

$$= \Gamma(1 + \delta)$$

$$\times \left(-\sum_{\rho_j} (\kappa - \rho_j)^{-1} \int_0^1 \Phi(1, r, 1 + \delta + \kappa - \rho_j) e^{-\pi i r} dr + D \right),$$
(21)

where

$$\begin{split} D &\equiv (\kappa-1)^{-1} \int_0^1 \Phi(1,r,\delta+\kappa) e^{-\pi i r} dr - \frac{\zeta'}{\zeta}(\kappa) \int_0^1 e^{-\pi i r} \Phi(1,r,1+\delta) dr \\ &+ \frac{1}{2\pi i} \int_{2-h-i\infty}^{2-h+i\infty} -\frac{\zeta'}{\zeta}(\kappa-w) \int_0^1 \Phi(1,r,1+\delta+w) e^{-\pi i r} dr \frac{dw}{w}. \end{split}$$

Here, we can show the convergence of the sum on the right side of (21) as follows.

It is easy to show that (see [1] for the bound on Φ)

$$\int_{0}^{1} \Phi(1, r, 1 + \delta + \kappa - \rho_{j}) e^{-\pi i r} dr$$

$$\ll \gamma_{j}^{-1} \int_{0}^{1} \Phi(1, r, \delta + \kappa - \rho_{j}) e^{-\pi i r} dr$$

$$\ll \gamma_{j}^{\eta_{j} - \delta - \kappa}, \tag{22}$$

with $\beta_j = 1/2 + \eta_j$ and $\delta, \kappa > 0$ small.

Let $N(\sigma, T)$ be the number of the nontrivial zeros of the zeta-function in the rectangle $\{s : \sigma < \text{Re}(s) < 1, 0 < \text{Im}(s) < T\}$.

Combining (22) with the well-known result $N(\sigma,T) \ll T^{3/2-\sigma}(\log T)^5$ [5], the convergence of the sum

$$\sum_{\rho_j} (\kappa - \rho_j)^{-1} \int_0^1 \Phi(1, r, 1 + \delta + \kappa - \rho_j) e^{-\pi i r} dr \ll \sum_{\rho_j} \gamma_j^{\eta_j - \delta - \kappa - 1}$$
 (23)

follows readily.

Once we have (21), we can let $\kappa = 0$ by analytic continuation.

Now, we put $\delta = \nu + i\gamma_{n_1}$ with $\nu > 0$ and then multiply both sides of (21) by $e^{\pi\gamma_{n_1}/2}$. Then it is easy to see that the left side of (21) is approximately

(LHS of (21))
$$e^{\pi \gamma_{n_1}/2} \simeq \gamma_{n_1}^{\eta_{n_1}};$$
 (24)

in particular, we have

$$|e^{\pi\gamma_{n_1}/2}S_{\nu+i\gamma_{n_1},0}| = e^{\pi\gamma_{n_1}/2} \left| \int_{(1/2)} n_1^{-s-\nu} M(s+\nu,1) \Gamma(s-i\gamma_{n_1}) \right|$$

$$\times \Gamma(\nu+s) \Gamma(1-s+i\gamma_{n_1}) \pi^{-s+i\gamma_{n_1}} e^{-\pi(s-i\gamma_{n_1})i/2} ds$$

$$\ll \int_{-\infty}^{\infty} (|t|+1)^{\nu+\epsilon} e^{-\pi|t-\gamma_{n_1}|} dt$$

$$\ll \gamma_{n_1}^{\nu+\epsilon},$$

with $\epsilon > 0$ arbitrary and $0 < \nu < \eta_{n_1}$. Therefore, (21) reduces to

$$\gamma_{n_1}^{\eta_{n_1}} \simeq \Gamma(1 + \nu + i\gamma_{n_1})e^{\pi\gamma_{n_1}/2} \times \left(-\sum_{\rho_j} (\kappa - \rho_j)^{-1} \int_0^1 \Phi(1, r, 1 + \delta + \kappa - \rho_j)e^{-\pi i r} dr + D\right).$$
 (25)

The completion of our argument for Proposition 1 depends on derivation of a more generalized form of (25), with p being any number in $\mathbb{Q} \cap (0,2)$, in a similar way. That is, we ought to establish a relationship of the form

$$c_{\rho_{m,p_{1}}}[\operatorname{Im}(\gamma_{m,p_{1}})]^{\eta_{m,p_{1}}} \sim \Gamma(1+\delta)p_{1}^{-\delta} \times \left(-\sum_{\rho_{i}}(\kappa-\rho_{j})^{-1}\int_{0}^{1}\Phi(p_{1},r,1+\delta+\kappa-\rho_{j})e^{-\pi ip_{1}r}dr+D_{p_{1}}\right),$$
(26)

where ρ_{m,p_1} is a pole of $M(s,p_1)$, $c_{\rho_{m,p_1}}$ is the residue of $M(s,p_1)$ at $s=\rho_{m,p_1}$, and other symbols are counterparts for associated ones in (21).

There are mainly two topics left to be discussed for achieving our goal based on (26).

First, we know that if $f_n(x)$ is continuous and uniformly convergent to f(x), then f(x) is continuous. Hence (23) shows that the right side of (25) is continuous in $p \in (0,2)$ (there is a difficulty at p=2) for each $\nu > 0$.

Second, we need to show that M(s,q) is a meromorphic function for rational q in the region Re(s) > 1/2. The rest of the argument focuses on this goal.

We set

$$F(s,q) \equiv \prod_{p} (1 + e^{-\pi i q p} p^{-s}).$$

The logarithmic derivative of F is

$$\frac{F'}{F}(s, a/b) = M(s, a/b) + O(\sum_{p} \frac{\log p}{p^{2s}})$$
 (27)

where we put q = a/b.

On the other hand, we can express F(s,a/b) in terms of Dirichlet L-functions as follows. We note that

$$e^{-\pi i a p/b} = \sum_{\chi: (\mathbb{Z}/b)^* \to \mathbb{C}^*} \chi(p) \frac{\sum_{x \in (\mathbb{Z}/b)^*} \overline{\chi}(x) e^{-\pi i a x/b}}{\phi(b)}$$

$$\equiv \sum_{\chi} \chi(p) A(a, b; \chi),$$
(28)

which can be easily shown with the well-known identity concerning Dirichlet characters [2]

$$\frac{\sum_{\chi \in (\mathbb{Z}/m)^*} \overline{\chi}(a)\chi(n)}{\phi(m)} = \begin{cases} 1 & n \equiv a \pmod{m}, \\ 0 & \text{otherwise.} \end{cases}$$

With (28), F(s, a/b) becomes

$$F(s, a/b) = \prod_{p} \left(1 + \sum_{\chi} \chi(p) A(a, b; \chi) p^{-s} \right),$$

and its logarithm is

$$\log F(s, a/b) = \sum_{p} \log \left(1 + \sum_{\chi} \chi(p) A(a, b; \chi) p^{-s} \right)$$

$$= \sum_{p} \left(\sum_{\chi} \chi(p) A(a, b; \chi) p^{-s} - \left[\sum_{\chi} \chi(p) A(a, b; \chi) \right]^{2} p^{-2s} + \cdots \right)$$

$$= \sum_{\chi} A(a, b; \chi) \sum_{p} \chi(p) p^{-s} - \sum_{p} e^{-2\pi i a p/b} p^{-2s} + \cdots$$

$$= \sum_{\chi} A(a, b; \chi) [\log L(s, \chi) - \sum_{k>2} \frac{1}{k} \sum_{p} \frac{\chi(p^{k})}{p^{ks}}] + O(\sum_{p} p^{-2s}),$$

where we used (28) and the complete multiplicativity of Dirichlet characters in obtaining the third and fourth equliaties, respectively.

Here, using (28) again, we can rewrite

$$\sum_{k \ge 2} \frac{1}{k} \sum_{p} \sum_{\chi} A(a, b; \chi) \frac{\chi(p^k)}{p^{ks}} = \sum_{k \ge 2} \frac{1}{k} \sum_{p} \frac{e^{-\pi i a p^k/b}}{p^{ks}} = O(\sum_{p} p^{-2s}).$$

Taking the derivative with respect to s, we get

$$\frac{F'}{F}(s, a/b) = \sum_{\chi} A(a, b; \chi) \frac{L'}{L}(s, \chi) + O(\sum_{p} p^{-2s} \log p)$$
$$\equiv \sum_{\chi} A(a, b; \chi) \frac{L'}{L}(s, \chi) + Q.$$

Inserting this into (27), we find out that

$$\sum_{\chi} A(a,b;\chi) \frac{L'}{L}(s,\chi) + Q = M(s,a/b). \tag{29}$$

By (29) and the following result [2]

$$\frac{L'}{L}(s,\chi) = \sum_{|t-\gamma|<1} \frac{1}{s-\rho} + O(\log[q(2+|t|)]),$$

where q is any positive integer and χ is any Dirichlet character modulo q, the formula (26) has become virtually available to us for $b \ll \text{Im}(\rho_{m,a/b})$ (so that the error terms in its left side becomes minor).

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