Student Seminar #9 (2018/06/26)

Bayesian Data Analysis CHAPTER 17: Metric Predicted Variable with One Metric Predictor

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Contents

Overview of Chapter 17

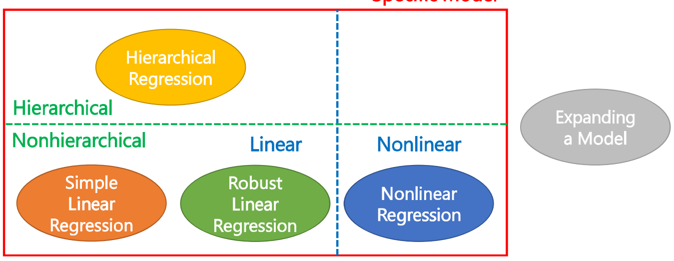
■ Purpose

✓ Predict one metric variable from one metric predictor

y x

ex.) weight height

■ Outline



17.1 Simple Linear Regression

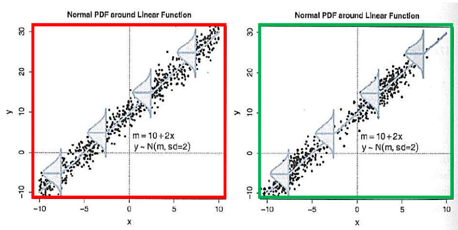
■ Step

✓ Generate any random x

✓ Compute the mean predicted value of y by $\mu = \beta_0 + \beta_1 x$

✓ Generate random variable for datum y from a normal distribution (μ : mean, σ : standard deviation)

■ Figure



Generate x from

Left : a uniform distribution

Right : a bimodal distribution

Both shows data from the same model

17.1 Simple Linear Regression

■ Assumption

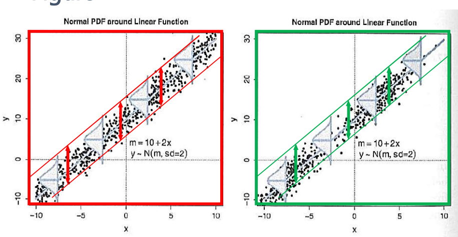
✓ *Homogeneity of variance*

At every value of x, the variance of y is the same

■ Note

✓ Simple Linear Regression describes **tendencies**, not **causality**

■ Figure



Both assumes *Homogeneity of variance*

But hard to see it on **the right figure** because there is an area which x is sparse

17. 2 Robust Linear Regression

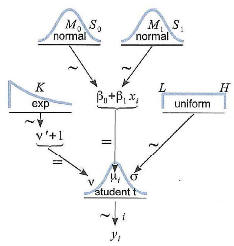
■ Object

✓ Data which have **outliers**

■ Assumption

✓ The datum y_i is a **t-distributed** random value around the central tendency $\mu_i = \beta_0 + \beta_1 x_i$

■ Diagram



17. 2 Robust Linear Regression

■ Goal

✓ Determine what combinations of β_0, β_1, σ, ν are credible, given the data

The answer (from Bayes' rule) :

$$p(\beta_0, \beta_1, \sigma, \nu | D) = \frac{p(D|\beta_0, \beta_1, \sigma, \nu)\, p(\beta_0, \beta_1, \sigma, \nu)}{\iiiint d\beta_0\, d\beta_1\, d\sigma\, d\nu\, p(D|\beta_0, \beta_1, \sigma, \nu)\, p(\beta_0, \beta_1, \sigma, \nu)}$$

→ Complicated…

Use JAGS or Stan!

■ What we have to do

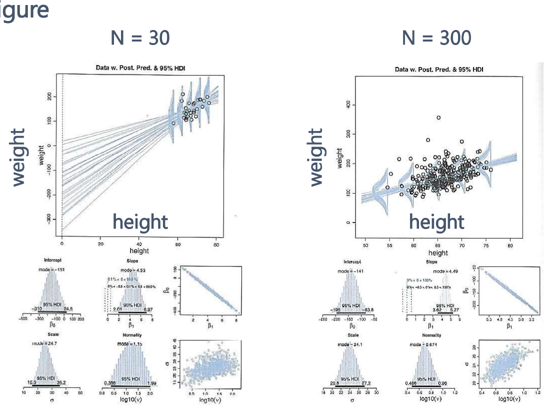
✓ Specify sensible priors

✓ Make sure that the MCMC process generates a trustworthy sample that is converged and well mixed

→ Talk about it later

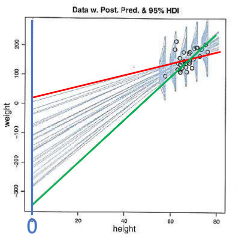
17. 2 Robust Linear Regression

■ Figure



17. 2 Robust Linear Regression 7

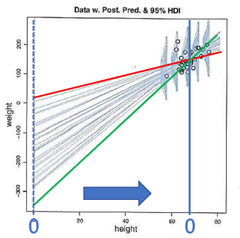
- Problem with using raw data
  - ✓ Parameter-correlation problem
    The credible slopes and intercepts trade off
    When the slope is small, the intercept is big
    When the slope is big, the intercept is small
    → MCMC sampling is difficult
    Two parameter values change slowly



- Ways to make the sampling more efficient
  - ✓ Change the sampling algorithm → Stan : HMC
  - ✓ Transform the data → **JAGS : Standardization**

17. 2 Robust Linear Regression 8

- Mean centering
  - ✓ Slide the axis
    so that zero falls under the mean
    → The Slope changes without any big changes on the intercept
    → Solve parameter-correlation problem(???)



- Standardize data
  - ✓ Re-scaling the data relative to their mean(***M***) and standard deviation(***SD***): $z_x = \frac{(x - M_x)}{\mathrm{SD}_x}$ and $z_y = \frac{(y - M_y)}{\mathrm{SD}_y}$
  - ✓ Linear Regression using standardized data

$$z_{\hat{y}} = \boxed{\zeta_0} + \boxed{\zeta_1} z_x \qquad \text{by definition of the model}$$

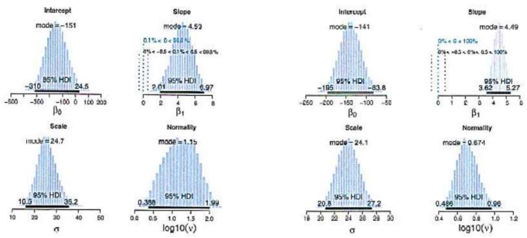
$$\frac{(\hat{y} - M_y)}{\mathrm{SD}_y} = \zeta_0 + \zeta_1 \frac{(x - M_x)}{\mathrm{SD}_x} \qquad \text{from Equation 17.1}$$

$$\hat{y} = \underbrace{\zeta_0 \mathrm{SD}_y + M_y - \zeta_1 M_x \mathrm{SD}_y/\mathrm{SD}_x}_{\beta_0} + \underbrace{\zeta_1 \mathrm{SD}_y/\mathrm{SD}_x}_{\beta_1} x \qquad (17.2)$$

$\boxed{\zeta_0}$: the slope with the data
$\boxed{\zeta_1}$: the intercept with the data

17. 2 Robust Linear Regression 9

- Interpreting the posterior distribution
  - ✓ Compare N=30 regression and N=300 one
  - ✓ **The slope, intercept and scale** are about the same
  - ✓ **The certainty** of the estimate for N=300 is tighter than for N=30
  - ✓ **The normality parameter** for N=300 is bigger than for N=30



N=30 N=300

17. 3 Hierarchical Regression on Individuals Within Groups 10

- Object
  - ✓ Data that each individual contributes multiple observations
    ex.) Reading-ability scores of children across several years
    Family income for different size of the family, for different regions
- Assumption
  - ✓ Each individual is representative of the group
    → Every individual informs the estimate of the group slope and intercept
    Get sharing of information across individuals
- Goal
  - ✓ describe each individual with a linear regression
  - ✓ Estimate the typical slope and intercept of the group overall

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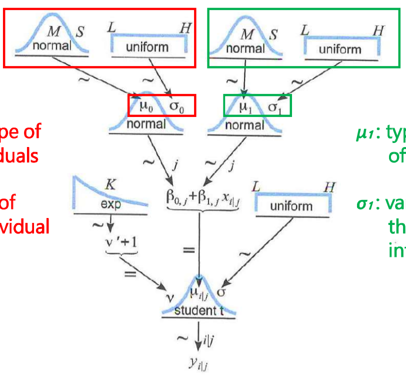
- The model and implementation in JAGS
  - ✓ Diagram



μ_0: typical slope of the individuals

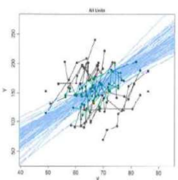
σ_0: variability of those individual slopes

μ_1: typical intercepts of the individuals

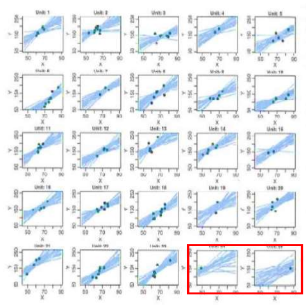
σ_1: variability of those individual intercepts

17. 3 Hierarchical Regression on Individuals Within Groups 12

- The posterior distribution : Shrinkage and prediction
  - ✓ Overall : Clearly positive by integrating each individual slope
  - ✓ Individual : Notable shrinkage of the estimates of the individuals
  - ✓ The estimates are tightly constrained by each data



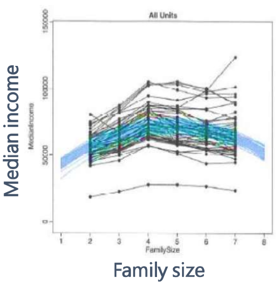
Overall



Individual

17. 4 Quadratic Trend and Weighted Data 13

- Object
  - ✓ Data that y appears to have a **nonlinear** trend as x increases
- Example
  - ✓ Family size and family income for each state in the U.S
- Model
  - ✓ $\mu_i = b_0 + b_1 x + b_2 x^2$
  - ✓ If $b_2 = 0$ → linear model
  - ✓ If $|b_2|$ is big
    → nonlinear model is reasonable



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- Nonlinear Regression using standardized data

$$z_{\hat{y}} = \boxed{\zeta_0} + \boxed{\zeta_1} z_x + \boxed{\zeta_2} z_x^2 \qquad \text{by definition of the model}$$

$$\frac{(\hat{y} - M_y)}{\mathrm{SD}_y} = \zeta_0 + \zeta_1 \frac{(x - M_x)}{\mathrm{SD}_x} + \zeta_2 \frac{(x - M_x)^2}{\mathrm{SD}_x^2} \qquad \text{from Equation 17.1}$$

$$\hat{y} = \underbrace{\zeta_0 \mathrm{SD}_y + M_y - \zeta_1 M_x \mathrm{SD}_y/\mathrm{SD}_x + \zeta_2 M_x^2 \mathrm{SD}_y/\mathrm{SD}_x^2}_{\beta_0}$$

$$+ \underbrace{\left(\zeta_1 \mathrm{SD}_y/\mathrm{SD}_x - 2\zeta_2 M_x \mathrm{SD}_y/\mathrm{SD}_x^2\right)}_{\beta_1} x + \underbrace{\zeta_2 \mathrm{SD}_y/\mathrm{SD}_x^2}_{\beta_2} x^2 \qquad (17.3)$$

$\boxed{\zeta_0}, \boxed{\zeta_1}, \boxed{\zeta_2}$ coefficients using the data

- Weighting data
  - ✓ The data report the median income
    based on different numbers of families at each size
    → every median has a different amount of sampling noise
  - ✓ Consider "margin of error"
    If margin of error is **high**, noise parameter should be **increased**
    If margin of error is **small**, noise parameter should be **decreased**

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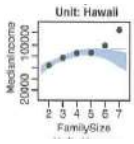
Results and interpretation

- ✓ The quadratic coefficient is -2200 ~ -1700
  - → Nonlinear model is reasonable
- ✓ Hawaii (the amount of data is not big)
  - → The trend is **upward**, but the curve is **downward curvature**
  - → Shrinkage from the group
- ✓ California
  - → a **narrow** spread at family size 2
    a **large** spread at family size 7
  - → the most of the data for large family sizes have **large standard errors**

Hawaii



California

17. 4 Quadratic Trend and Weighted Data 16

Further extensions

	An example of family income and family size	Extensions
Trend	linear quadratic	higher-order polynomial Sinusoidal exponential
Noise distribution	a single lying noise for all individuals	vary among individuals
Distribution for parameters	normal distribution	t distribution
Considering Covariation	No	Use a multivariate normal prior on the intercept and slope

17. 5 Procedure and Perils for Expanding a Model 17

Posterior predictive check

- ✓ Visualize the data and the posterior predictions
  - → If the prediction doesn't seem to fit the data, change the model
  - → New model should be both **meaningful** and **computationally tractable**
- ✓ Create a posterior predictive sampling distribution
  - → measure of discrepancy between the predictions and the data

Ways to extend a model

- ✓ Add a parameter
  ex.) $\mu_i = \beta_0 + \beta_1 x$ → $\mu_i = \beta_0 + \beta_1 x + \beta_2 x^2$
  You can check the validity of the model by considering β_2
- ✓ Try a completely different model
  - → Compare models by **Bayesian model comparison**
- ✓ "double dipping" : data are used to change **the prior distribution**

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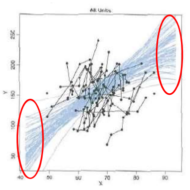
Steps to extend a JAGS or Stan model

- ✓ Carefully specify the model with its new parameters
  - → Draw a diagram
- ✓ Be sure all the new parameters have sensible priors
- ✓ Define initial values for all the new parameters
  - → You can let JAGS initialize parameters automatically
- ✓ Tell JAGS to track the new parameters
  - → Stan automatically tracks
- ✓ Modify the summary and graphics output to properly display the extended model
  - → You should write **R code** because graphics are displayed by R

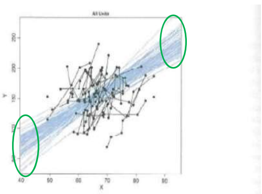
17. 5 Procedure and Perils for Expanding a Model 19

Perils of adding parameters

- ✓ Increase in uncertainty of a parameter estimate
  - → The curvature and slope trade-off strongly
  - → Even if curvature is 0, the certainty of slope decreases
- ✓ The one of the ways to solve the problem is standardizing the data



Quadratic trend



Linear trend